Extended Topological Field Theories and the Cobordism Hypothesis

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Chapter 1

Introduction

1.1 Topological field theories

In theoretical physics, a particle may be modelled as a physical field, which can be regarded as a smooth section of a vector bundle over space-time. A quantum field theory is a model for studying the interactions of particles through their underlying physical fields. Of particular interest is topological quantum field theories (or simply topological field theories), which are qunatum field theories that are invariant under the homotopy of the underlying space-time, and hence is insensitive to space-time warps.

Topological field theories turn out to have interesting applications in mathematics, for example in knot theory, the classification of 4-manifolds, and in the study of moduli spaces in algebraic geometry. Atiyah first axiomatised topological field theories in [Ati88] and gave some examples of known theories in dimension $n \leq 3$.

We give Atiyah's definition of topological field theories in modern language [Lur09c]:

Definition 1.1.1. The **bordism category** $\mathbf{Cob}(m)$ is defined as follows:

1. An object of $\mathbf{Cob}(m)$ is given by a compact oriented $(m-1)$-manifold.
2. For any pair of objects $M, N \in \mathbf{Cob}(m)$, a **bordism** from M to N is a compact oriented m-manifold B with an oriented boundary $\partial B = \partial B_0 \sqcup \partial B_1$ where $\partial B_0 \cong \overline{M}$ is the manifold M with the opposite orientation and $\partial B_1 \cong N$.
3. There is an equivalence relation on the set of bordisms from M to N given by orientation preserving bordisms $B \xrightarrow{\sim} B'$ such that it restricts to diffeomorphisms $\partial B_0 \xrightarrow{\sim} \partial B_0'$ and $\partial B_1 \xrightarrow{\sim} \partial B_1'$. Let $\mathrm{Hom}_{\mathbf{Cob}(m)}(M, N)$ be the equivalence classes of bordisms from M to N under this equivalence relation.
4. The identity morphism id_M is given by the class of bordism given by $M \times [0, 1]$.
5. Given two morphisms represented by bordisms $B : M \to M'$ and $B' : M' \to M''$, we can choose the representatives such that the composition $[B' \circ B] : M \to M'$ represented by the bordism $B \sqcup_{M'} B'$ is a smooth manifold. It is clear that the definition is independent of the choice of any such representatives.

A **symmetric monoidal category** (see Def. 2.6.1) is a category $\mathcal{C}$ equipped with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ satisfying the monoid axioms: there is a unit $\mathbf{1} \in \mathcal{C}$ and isomorphisms, for all objects $a, b, c \in \mathcal{C}$,

$$(a \otimes b) \otimes c \cong a \otimes (b \otimes c), \qquad \mathbf{1} \times a \cong a \cong a \times \mathbf{1}, \qquad a \otimes b \cong b \otimes a$$

satisfying some coherence properties.

For example, the category $\mathbf{Vect}(k)$ of vectorspaces over a field k can be regarded as a symmetric monoidal category with the usual tensor product.

$\mathbf{Cob}(m)$ can be endowed with a symmetric monoidal structure given by disjoint union of manifolds

$$\coprod : \mathbf{Cob}(m) \times \mathbf{Cob}(m) \to \mathbf{Cob}(m) : (M, N) \mapsto M \sqcup N.$$

A **symmetric monoidal functor** $F : \mathcal{C} \to \mathcal{D}$ between two symmetric monoidal categories is a functor satisfying, for all $a, b \in \mathcal{C}$, the isomorphisms

$$F(a \otimes b) \cong F(a) \otimes F(b), \qquad F(\mathbf{1}_{\mathcal{C}}) \cong \mathbf{1}_{\mathcal{D}}$$

up to some coherence properties.

Definition 1.1.2 (Atiyah)**.** Let k be a field. An **m-dimensional topological field theory** is a symmetric monoidal functor $Z : \mathbf{Cob}(m) \to \mathbf{Vect}(k)$.

Expanding out the definition, we see that a topological field theory Z encompasses the following set of data:

1. For each $(m-1)$-manifold M, a vectorspace $Z(M)$, such that $Z(\emptyset) = k$ and disjoint union of manifolds correspond to tensor product of vectorspaces: $Z(M \sqcup N) \cong Z(M) \otimes Z(N)$.
2. For each bordism $B : M \to N$, a k-linear map $Z(M) \to Z(N)$, satisfying the usual coherence axioms for categories.

More generally, we can replace $\mathbf{Vect}(k)$ in the definition of a topological field theory with an symmetric monoidal category $\mathcal{C}$. We call such a functor a **$\mathcal{C}$-valued topological field theory**.

Example 1.1.3. A 1-dimensional topological field theory Z can be explicitly described. The objects in $\mathbf{Cob}(1)$ are unions of positively and negatively oriented points $*_+$ and $*_-$ respectively. Let $A = Z(*_+)$ and $B = Z(*_-)$ be vectorspaces. Then, $Z(X)$ for any object $X \in \mathbf{Cob}(1)$ is given by the tensor product of A and B.

The morphisms of $\mathbf{Cob}(1)$ are given by 1-manifolds with boundaries. Since Z is a symmetric monoidal functor, to describe Z, it suffices to describe it on connected bordisms. The connected bordisms are precisely oriented line segments and circles. A line segment I can be viewed as a bordism in 4 different ways:

1. $I_1 : *_+ \to *_+$: since $I \cong \{*_+\} \times [0, 1]$, $Z(I) = \mathrm{id}_A$;
2. $I_2 : *_- \to *_-$: similarly $Z(I) = \mathrm{id}_B$;
3. $I_3 : \emptyset \to *_- \sqcup *_+$: let $Z(I) = \mathrm{coev} : k \to B \otimes A$;
4. $I_4 : *_+ \sqcup *_- \to \emptyset$: let $Z(I) = \mathrm{ev} : A \otimes B \to k$.

The circle S^1 can be seen as a composition $I_4 \circ I_3$, so $Z(I) \cong \mathrm{ev} \circ \mathrm{coev}$.

Figure 1.1: Compositions $(I_4 \sqcup I_1) \circ (I_1 \cup I_3)$ and $(I_2 \sqcup I_4) \circ (I_3 \sqcup I_2)$ in $\mathbf{Cob}(1)$.

The compositions shown in Figure 1.1 demonstrate that the two morphisms

$$A \xrightarrow{\mathrm{id}_A \otimes \mathrm{coev}} A \otimes B \otimes A \xrightarrow{\mathrm{ev} \otimes \mathrm{id}_A} A \qquad \text{and} \qquad B \xrightarrow{\mathrm{coev} \otimes \mathrm{id}_B} B \otimes A \otimes B \xrightarrow{\mathrm{id}_B \otimes \mathrm{ev}} B$$

are equivalent to the identity morphisms. This implies that $B \cong A^\vee$ is the dual of A, and they are finite dimensional vectorspaces, so $A \cong A^\vee$. Hence, we get a complete characterisation of the 1-dimensional topological field theory by specifying a finite dimensional vector space $A = Z(*_+)$. This is essentially the statement of the cobordism hypothesis in dimension 1.

Example 1.1.4. We can also give a general description of a 2-dimensional topological field theory Z. The objects of **Cob**(2) are compact oriented 1-manifolds, so they are disjoint unions of circles with positive or negative orientation. If $Z(S^1_+) = A$ is a k-vectorspace, then a similar argument to above replacing line segments with cylinders give $Z(S^1_-) = A^\vee$ to be the dual of A, so $A \cong A^\vee$. Indeed there is an orientation reversing diffeomorphism that send S^1_+ to S^1_-, so we may regard them as the same object S^1.

There is a morphism $P : S^1 \sqcup S^1 \to S^1$ given by "a pair of pants" (see Figure 1.2) and a morphism $S^1 \to \emptyset$ given by a closed disc D. They induce maps of vectorspaces

$$A \otimes A \xrightarrow{m} A \qquad \text{and} \qquad k \to A$$

which we view as a multiplication operation on A and identification of a unit. By considering various compositions of bordisms, we can show that the multiplication operation satisfy associativity, identity and commutativity axioms (this is a good drawing exercise; for a more algebraic discussion, see Section 4.1). Hence, this gives A a k-algebra structure.

Furthermore, let $\mathrm{tr} : A \to k$ be the morphism induced by $D : S^1 \to \emptyset$. Then, the composition of the pair of pants with the closed disc $D \circ P \cong S^1 \times [0,1] : S^1 \sqcup S^1 \to \emptyset$ gives a map

$$A \otimes A \xrightarrow{m} A \xrightarrow{\mathrm{tr}} k$$

which can be interpreted as a perfect pairing of A with itself. Hence, A is a commutative k-algebra with a non-degenerate trace map. Such an algebra is call a **Frobenius algebra**. Indeed, a 2-dimensional topological field is fully characterised by any choice of such A (for a detailed account, see [Koc04]).

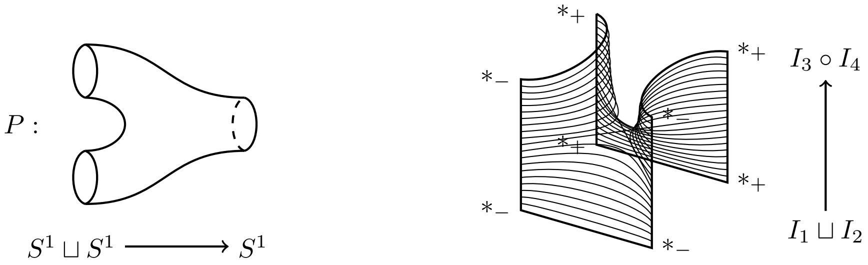


Figure 1.2: Left: "Pair of pants" in **Cob**(2). Right: A 2-bordism $I_1 \sqcup I_2 \to I_3 \circ I_4$ in $\mathbf{Co}_2(2)$.

1.2 Extended topological field theories and the cobordism hypothesis

However, topological field theories as defined above are too restrictive in many circumstances. For example, in **Cob**(2), the only objects are disjoint unions of circles and bordisms are oriented 2-manifolds with boundaries being disjoint union of circles. We will also like to include objects such as closed intervals and bordisms between such objects and the topological field theories associated to them. In this case, we can construct a bordism 2-category.

Definition 1.2.1. A **(strict) 2-category** $\mathcal{C}$ is a category enriched over categories, that is, it consists of a collection of objects, and for every pair of objects $a, b \in \mathrm{Ob}\,\mathcal{C}$, a category $\mathrm{Hom}_\mathcal{C}(a,b)$. The objects and morphisms in $\mathrm{Hom}_\mathcal{C}(a,b)$ are called 1- and 2-morphisms respectively. Composition in $\mathcal{C}$ is given by functor

$$\mathrm{Hom}_\mathcal{C}(a,b) \times \mathrm{Hom}_\mathcal{C}(b,c) \to \mathrm{Hom}_\mathcal{C}(a,c),$$

satisfying $f \circ \mathrm{id}_a = f = \mathrm{id}_b \circ f$ for all $f \in \mathrm{Hom}_\mathcal{C}(a,b)$ and the commutative diagram for associativity

$$\begin{tikzcd} \mathrm{Hom}_\mathcal{C}(a,b) \times \mathrm{Hom}_\mathcal{C}(b,c) \times \mathrm{Hom}_\mathcal{C}(c,d) \arrow[r] \arrow[d] & \mathrm{Hom}_\mathcal{C}(a,b) \times \mathrm{Hom}_\mathcal{C}(b,d) \arrow[d] \\ \mathrm{Hom}_\mathcal{C}(a,c) \times \mathrm{Hom}_\mathcal{C}(c,d) \arrow[r] & \mathrm{Hom}_\mathcal{C}(a,d) \end{tikzcd} . \tag{1.2.1}$$

Inductively, a **(strict)** n**-category** is a category enriched over $(n-1)$-categories.

We let $\mathbf{Cob}_2(m)$ be the 2-category whose objects are compact oriented $(m-2)$-manifolds and 1-morphisms are equivalence classes of compact oriented $(m-1)$-manifolds. For any two 1-morphisms $M, N : P \to Q$ where P, Q are objects of $\mathbf{Cob}_2(m)$, a bordism $B : M \to N$ is a m-manifold with boundary

$$\partial B = \overline{M} \coprod_{(P \sqcup \overline{Q}) \times \{0\}} \big((\overline{P} \sqcup Q) \times [0,1]\big) \coprod_{(\overline{P} \sqcup Q) \times \{0\}} N.$$

There is an equivalence relation on such bordisms which are given by diffeomorphisms which restrict to diffeomorphisms on each of $\overline{M}$, N, $\overline{P} \times [0,1]$ and $\overline{Q} \times [0,1]$. The 2-morphisms are equivalence classes of such bordisms. Composition is defined by choosing two suitable representatives and gluing them along the boundary as in Def. 1.1.1. An example of a 2-bordism of $\mathbf{Cob}_2(2)$ that is not contained in $\mathbf{Cob}(2)$ is shown in Figure 1.2.

It is possible to similarly define a k-category $\mathbf{Cob}_k(m)$ for $k \leq m$. However, as k grows, it is difficult to keep track of the diffeomorphism classes and thus to check the associativity axiom. A solution is to transfer the problem of tracking of such issues to a formal categorical problem. The construction of such a categorical solution is the main topic of this thesis. We will give an informal discussion here.

First, we introduce a notion of **weak n-categories**.

Definition 1.2.2 (sketch)**.** A weak 2-category $\mathcal{C}$ is the data of a collection of objects, and for each pair of objects $a, b \in \operatorname{Ob} \mathcal{C}$, there is a category $\operatorname{Hom}_{\mathcal{C}}(a, b)$. Composition is given by functors

$$\operatorname{Hom}_{\mathcal{C}}(a, b) \times \operatorname{Hom}_{\mathcal{C}}(b, c) \to \operatorname{Hom}_{\mathcal{C}}(a, c),$$

as in strict 2-categories, but are only required to satisfy the associativity and identity axioms up to natural isomorphisms, that is, there are natural isomorphisms $f \circ \mathrm{id}_a \cong f \cong \mathrm{id}_b \circ f$ for all $f \in \operatorname{Hom}_{\mathcal{C}}(a, b)$ and the square (1.2.1) is required to commute up to some natural isomorphism. The natural isomorphisms have to satisfy some further coherence properties (see [Lei98]).

More generally, a **weak n-category** $\mathcal{C}$ is the data of a collection of objects, and for each pair of objects $a, b \in \operatorname{Ob} \mathcal{C}$, there is a weak $(n-1)$-category $\operatorname{Hom}_{\mathcal{C}}(a, b)$. Composition is required to satisfy the associativity and identity axioms up to a notion of equivalences in weak $(n-1)$-categories. Inductively, we can obtain as a limit, a **weak ∞-category** which has n-morphisms for all $n \geq 1$.

With an appropriate construction of weak n-categories, we can define $\mathbf{Cob}_k(n)$ as follows.

Definition 1.2.3 (sketch)**.** Let $\mathbf{Cob}_k(n)$ be a weak k-category with the following set of data:

1. The objects are smooth compact oriented $(n-k)$-manifolds.
2. The 1-morphism is are bordisms between two objects.
3. For $1 < r \leq n$, for any $(r-1)$-morphisms $M, N : P \to Q$ where P, Q are $(r-2)$-morphisms, a r-morphism or a r-bordism is a smooth compact oriented $(n-k+1)$-manifold with boundary

   $$\partial B = (M \sqcup N) \coprod_{(P \sqcup Q) \times 0,1} \big((P \sqcup Q) \times [0,1]\big).$$

   The r-morphisms are equivalence classes of such bordisms.
4. The identity map is given by $\mathrm{id}_M = M \times [0,1]$ and composition is defined of two bordism $B : M \to M'$ and $B' : M' \to M''$ is done by choosing appropriate $\tilde{B} \cong B$ and $\tilde{B}' \cong B'$ such that $\tilde{B} \sqcup_{M'} \tilde{B}'$ is a smooth compact oriented $(n-k+1)$-manifold (the fact that this is always possible is proven in Section 3.1).

The checking of the associativity property can be avoided through higher categorical formalisms.

$\mathbf{Cob}_k(n)$ has a symmetric monoidal structure given by disjoint union of bordisms.

Definition 1.2.4. Let $\mathcal{C}$ be a symmetric monoidal weak n-category. A **$\mathcal{C}$-valued k-extended n-dimensional topological field theory** is a symmetric monoidal functor $Z : \mathbf{Cob}_k(n) \to \mathcal{C}$.

It is under this context that Baez and Dolan conjectured the cobordism hypothesis [BD95]. It is formulated for the n-category $\mathbf{Cob}_n^{\mathrm{fr}}(n)$ of framed bordisms (framing is a technical condition on the manifolds, see Def. 3.3.1). Phrased in the notations of this chapter, the theorem is as follows.

Theorem 1.2.5 (**(Baez-Dolan cobordism hypothesis)**). *Let $\mathcal{C}$ be a symmetric monoidal n-category. The evaluation functor $Z \mapsto Z(*)$ determines a bijection between the isomorphism classes of $\mathcal{C}$-valued extended framed topological field theories $Z : \mathbf{Cob}_n^{\mathrm{fr}}(n) \to \mathcal{C}$ and fully dualisable objects of $\mathcal{C}$.*

The theorem says that we can uniquely determine a topological field theory $Z : \mathbf{Cob}_n^{\mathrm{fr}}(n) \to \mathcal{C}$ if we know its value at the point $*$. Furthermore, each fully dualisable object of $\mathcal{C}$ (see Section 2.7) determines a topological field theory.

However, this is not the form of the cobordism hypothesis that we shall consider in this thesis. There are two main weaknesses in this formulation. Firstly, the notion of weak n-categories is particularly difficult to properly define and work with. Secondly, in the bordism categories $\mathrm{Cob}_k(n)$, we lose all information about the homotopy types of the diffeomorphism classes of bordisms.

Lurie reformulated the cobordism hypothesis in the context of (∞, n)-categories [Lur09c]. Very informally, an (∞, n)-category is a weak ∞-category in which all r-morphism for $r > n$ are invertible. There turns out to be several simple models of (∞, n)-categories based on the idea of simplicial sets (see, for example, [Ber10] and [BR12] for a survey of different models of $(\infty, 1)$- and (∞, n)-categories respectively).

The bordism (∞, n)-category $\mathbf{Bord}_n$ defined by Lurie (see Section 3.2) can roughly be interpreted as extending the weak n-category $\mathbf{Cob}_n(n)$ by defining $(n+1)$-morphisms to be diffeomorphisms of n-bordisms and r-morphisms for $r > n$ to be homotopies between the $(r-1)$-morphisms. For any pair of n-bordisms $B, B' : M \to N$, the mapping space $\mathrm{map}(B, B')$ is an $(\infty, 0)$-category whose objects are diffeomorphisms from B to B' and morphisms are homotopies. We can thus interpret $\mathrm{map}(B, B')$ as a topological space of diffeomorphisms from B to B' (or as a simplicial set encoding the homotopy type of the topological space, depending on the choice of the model of (∞, n)-categories).

In the language of (∞, n)-categories, Lurie formulated and proved the following form of the cobordism hypothesis.

Theorem 1.2.6 (**Cobordism hypothesis (Lurie)**). *Let $\mathcal{C}$ be a symmetric monoidal (∞, n)-category. The evaluation functor $Z \mapsto Z(*)$ induces an equivalence between the category $\mathbf{Fun}^{\otimes}(\mathbf{Bord}_n^{\mathrm{fr}}, \mathcal{C})$ of extended framed topological field theories and the subcategory of fully dualisable objects in $\mathcal{C}$.*

1.3 Summary

The purpose of this thesis is to give a formal and complete construction of all the machineries necessary to state the cobordism hypothesis as proven by Lurie in [Lur09c].

In Chapter 2, we define the notion of a symmetric monoidal (∞, n)-category with duals. The model for (∞, n)-categories we use is that of complete Segal n-spaces. Complete Segal 1-spaces was first defined by Rezk [Rez01], and has since been proven to be equivalent to other models of $(\infty, 1)$-categories [Ber07]. We recall this notion in Sections 2.1 to 2.4, while detailing and verifying the modifications that Lurie made to the definition for the context of bordism categories.

We then generalise to complete Segal n-spaces in Section 2.5, using the Δ^{op} construction of [Lur09c] and [Ber11], and prove the model structure theorem for complete Segal n-space. In Section 2.6, we generalise the construction of symmetric monoidal $(\infty, 1)$-categories given by Toën and Vezzosi [TV11] to (∞, n)-categories, including a construction of the symmetric monoidal (∞, n)-category $\mathbf{Fun}^{\otimes}(X, Y)$ of symmetric monoidal functors between symmetric monoidal (∞, n)-categories X and Y. We then follow [Lur09c] in defining a symmetric monoidal (∞, n)-category with duals and prove the existence of a fully dualisable subcategory. A consequence of this construction is a functorial way to define a maximal sub-(∞, r)-category of an (∞, n)-category for $r < n$.

In Chapter 3, we construct the bordism (∞, n)-category using the machinery from the previous chapter. We gather results from various papers and prove the topological background necessary for defining the bordism category in Section 3.1, largely following the methods sketched in [GMTW09, Gal11]. We formally define the bordism (∞, n)-category in the next section, and follow the steps as sketched in [Lur09c] to show that it is a symmetric monoidal (∞, n)-category with duals. Finally, with all the necessary background, we state the cobordism hypothesis in Section 3.3.

In the final chapter, we detail two computations involving the cobordism hypothesis, both sketched in [Lur09c]. The first example is in an explicit computation of the topological field theories where $\mathcal{C}$ is a category of E_n-algebras in a symmetric monoidal category, for example E_n-algebras in chain complexes of k-modules. The second example is a characterisation of fully dualisable objects in symmetric monoidal $(\infty, 2)$-categories.

Chapter 2

Symmetric monoidal (∞, n)-categories

In this chapter, we will present some categorical preliminaries needed in our subsequent definition of the cobordism hypothesis. We will construct a model of (∞, n) categories, namely, the complete Segal n-spaces and its variants. Complete Segal 1-spaces were originally defined by Rezk [Rez01], and the theory was extended to complete Segal n-spaces by Barwick in his unpublished PhD thesis (and later published in [BSP12]). We will give a variant of that construction.

We will then show how we can endow certain Segal n-spaces with a (symmetric) monoidal structure, that is, a functor

$$\otimes : X \times X \to X$$

satisfying the associativity (and commutative) axioms. We can then perform algebraic operations on such a Segal n-space X. We also define symmetric monoidal functors between symmetric monoidal categories and conclude with the key result that the infinity category of symmetric monoidal functors $\mathbf{Fun}^{\otimes}(X, Y)$ is an ∞-groupoid.

We will begin our discussion with $(\infty, 1)$-categories. They form the central part of all our constructions, as (∞, n)-categories for $n > 1$ are built from the $n = 1$ case by a simple induction.

2.1 Simplicial spaces

We will first define the basic objects from which we build our $(\infty, 1)$-categories. Segal spaces are presented as simplicial spaces, that is, simplicial objects in the category of spaces, $\mathbf{Sp}$.

We will be deliberately vague over the notion of space. Depending on the context, we will allow $\mathbf{Sp} = \mathbf{Top}$ the category of topological spaces, $\mathbf{CGHaus}$ the category of compactly generated Hausdorff topological spaces, $\mathbf{Sp} = \mathbf{KComp} \cong \mathbf{CW}$ the category of Kan complexes or CW-complexes (they are equivalent under their standard model category structures) or $\mathbf{Sp} = \mathbf{sSet}$ the category of simplicial sets. We have the inclusions

$$\mathbf{sSet} \supset \mathbf{KComp} \cong \mathbf{CW} \subset \mathbf{CGHaus} \subset \mathbf{Top}.$$

$\mathbf{Top}$ is the most general context over which we can construct the complete Segal space. However, it is often useful to restrict our attention to nicer topological spaces. The categories $\mathbf{CGHaus}$ is a suitable candidate as it is Cartesian closed, that is, it has an internal hom-object that is right adjoint to the product.

Sometimes, it is useful to move from a topological space construction to a simplicial set construction, which is notationally simpler and allows for simpler proofs. This is when we take $\mathbf{Sp} = \mathbf{KComp}$. Indeed, this is the context under which Rezk defined Segal spaces in [Rez01] (he defined them on the category of simplicial sets, but the Reedy-fibrant condition imposed implies that the simplicial sets in question are Kan complexes). However, for many constructions, we have to start with simplicial spaces where we take $\mathbf{Sp} = \mathbf{sSet}$. This is not a problem as the Kan complexes are precisely the fibrant-cofibrant objects in $\mathbf{sSet}$.

Recall, from the theory of simplicial sets, that we have a category Δ which is a full subcategory of the category of categories, whose objects are the finite categories $[r]$ with $n + 1$ objects $\{0, \ldots, r\}$ and a composable chain of morphisms $\{0 \to 1 \cdots \to r\}$. Δ is equivalent to the category whose objects are finite sets $[r] = \{0, \dots, r\}$ and whose morphisms are order-preserving maps (not strict, i.e., $\phi(k) \le \phi(l)$ if $k \le l$). Indeed, all the morphisms in Δ can be generated from a subset of morphisms

$$d^i : [r] \to [r+1] : j \mapsto \begin{cases} j & j \le i \\ j+1 & j > i \end{cases} \qquad s^i : [r] \to [r-1] : j \mapsto \begin{cases} j & j \le i \\ j-1 & j > i \end{cases}.$$

The maps d^i and s^i are called the face and degeneracy maps respectively. They satisfy the **cosimplicial identities**

$$\begin{cases} d^j d^i = d^i d^{j-1} & i < j \\ s^j d^i = d^i s^{j-1} & i < j \\ s^j d^j = 1 = s^j d^{j+1} & \\ s^j d^i = d^{i-1} s^j & i > j+1 \\ s^j s^i = s^i s^{j+1} & i \le j \end{cases} \tag{2.1.1}$$

Definition 2.1.1. A **simplicial space** (or simplicial 1-space or 1-fold simplicial space) is a simplicial object in the category **Sp** of spaces, that is, a functor $X : \Delta^{\mathrm{op}} \to \mathbf{Sp}$. Let X_n be the topological space $X([n])$ and let d_i and s_i be the images of d^i and s^i under X. Let $1\text{-}\mathbf{sSp} = \mathbf{Fun}(\Delta^{\mathrm{op}}, \mathbf{Sp})$ denote the category of simplicial spaces. A map of simplicial spaces $f : X \to Y$ is thus a collection of continuous maps $f_n : X_n \to Y_n$ that commute with d_i and s_i. d_i and s_i satisfy the **simplicial identities**

$$\begin{cases} d_i d_j = d_{j-1} d_i & i < j \\ d_i s_j = s_{j-1} d_i & i < j \\ d_j s_j = 1 = d_{j+1} s_j & \\ d_i s_j = s_j d_{i-1} & i > j+1 \\ s_i s_j = s_{j+1} s_i & i \le j \end{cases} \tag{2.1.2}$$

There is an embedding $\mathbf{Sp} \hookrightarrow 1\text{-}\mathbf{sSp}$ sending X to the **constant simplicial space**, also denoted by X, where all $X_n = X$ and the face and degeneracy maps are the identity maps.

A **standard k-simplex** Δ^k is a simplicial set defined by $[n] \mapsto \mathrm{Hom}_\Delta([n], [k])$ with the face and degeneracy maps d_i and s_i given by precomposition with d^i and s^i respectively. Let $|\Delta^k| \in \mathbf{Top}$ be the geometric n-simplex, that is, the geometric realisation of Δ^k [GJ99].

Suppose **Sp** is a subcategory of **Top**. To every pair of objects $X, Y \in 1\text{-}\mathbf{sSp}$, we can associate a simplicial set $\mathcal{M}(X, Y)$, defined by

$$\mathcal{M}(X, Y)_k = \mathrm{Hom}_{1\text{-}\mathbf{sSp}}(X \times |\Delta^k|, Y)$$

where $|\Delta^k|$ is taken as a constant simplicial space. This is called the **function complex** from X to Y. We define the **mapping space** to be the geometric realisation of the simplicial set

$$\mathrm{Map}_{1\text{-}\mathbf{sSp}}(X, Y) = |\mathcal{M}(X, Y)|.$$

For $\mathbf{Sp} = \mathbf{KComp}$ or in general $\mathbf{Sp} = \mathbf{sSet}$, the mapping space is equal to the function complex given by the simplicial set

$$\mathrm{Map}_{1\text{-}\mathbf{sSp}}(X, Y)_n = \mathrm{Hom}_{1\text{-}\mathbf{sSp}}(X \times \Delta^n, Y)$$

where Δ^n is taken as a constant simplicial space.

We will now define a model category structure on $1\text{-}\mathbf{sSp}$.

Recall the definitions of a Cartesian closed, proper and cofibrantly generated model category structure.

Definition 2.1.2 ([Rez01])**.** We say that a category C is **Cartesian closed** if it has a final object, and for any pair of objects X, Y, there exists a product $X \times Y$ and an internal hom-object Y^X satisfying for all triplets of objects $X, Y, Z \in C$, we have a bijection

$$\mathrm{Hom}(X \times Y, Z) \cong \mathrm{Hom}(X, Z^Y).$$

Let M be a Cartesian closed model category. The model category structure is said to be **compatible with Cartesian closure** if for any cofibrations $i : A \to B$ and $j : C \to D$ and any fibration $p : X \to Y$,

(i) the induced map $A \times D \sqcup_{A \times C} B \times C \to B \times D$ is a cofibration, which is trivial if either i or j is; or

(ii) the induced map $X^B \to X^A \times_{Y^A} Y^B$ is a fibration, which is trivial if either i or p is.

Definition 2.1.3 ([Hir03, Def. 11.1.1])**.** A model category is said to be **proper** if

(i) the pushout of a weak equivalence along a cofibration is a weak equivalence; and

(ii) the pullback of a weak equivalence along a fibration is a weak equivalence.

Definition 2.1.4 ([Hir03, Def. 13.2.1])**.** A model category is said to be **cofibrantly generated** if there exist sets I and J of cofibrations and trivial cofibrations respectively such that a map f is fibration (resp., trivial fibration) if and only if it has the right lifting property with respect to J (resp., I).

There is a Quillen model category structure on **Top** and **CGHaus**.

Theorem 2.1.5. *There exists a model category structure on* **Top** *and* **CGHaus***, where $f : X \to Y$ is a*

(i) weak equivalence if it is a **weak homotopy equivalence***, that is, $\pi_0 f : \pi_0 X \to \pi_0 Y$ is a bijection and $\pi_i f : \pi_i X \to \pi_i Y$ is an isomorphism for all $i \geq 1$;*

(ii) cofibration if it satisfies the homotopy extension property, that is, for all continuous maps $g : Y \to Z$ and homotopies $H : X \times I \to Z$ such that $g \circ f = H \circ (\mathrm{id} \times \{0\})$, there exists $G : Y \times I \to Z$ such that $G|_{Y \times \{0\}} = g$ and $G \circ (f \times \mathrm{id}_I) = H$;

(iii) fibration if it is a **Serre fibration***, that is, it satisfies the right lifting property (RLP) with respect to all inclusions $i_0 : D^n \hookrightarrow D^n \times I$ where $i_0(D^n) = D^n \times \{0\}$.*

CGHaus *is Cartesian closed and the Quillen model category structure on* **CGHaus** *is compatible with Cartesian closure. Both* **Top** *and* **CGHaus** *are proper.* **Top** *and* **CGHaus** *are cofibrantly generated by generating cofibrations and generating trivial cofibrations*

$$I = \{|\partial\Delta^n| \to |\Delta^n|\} \qquad \textit{and} \qquad J = \{|\Lambda^n_k| \to |\Delta^n|\}$$

respectively

Remark 2.1.6. **Top** is not Cartesian closed as it does not have internal hom-objects for all pairs $X, Y \in$ **Top**. We say that X is **exponentiable** if there is an internal hom-object Y^X for all Y. X is exponentiable if and only if it is core-compact, that is, for any $x \in X$ and $U \ni x$ an open neighbourhood, there exists an open set V such that $x \in V \subset U$ and every open cover of U admits a finite subcover of V [EH01]. If X is Hausdorff, then it is core-compact if and only if it is locally compact (every point has a compact neighbourhood), and in this case, if Y is Hausdorff as well, $Y^X = \mathrm{Hom}_{\mathbf{Top}}(X, Y)$ is given the compact-open topology.

The product and the internal hom-objects in **CGHaus** is not the same as that in **Top**, since in general, **CGHaus** is not closed under products and internal-homs. For any Hausdorff space X, the associated compactly generated space $k(X)$ is the set X endowed with the topology such that $U \subset k(X)$ is open if and only if $U \cap C$ is open for all compact subsets $C \subset X$ [Ste67]. The topology on $k(X)$ is finer than that on X. We thus define $X \times_{\mathbf{CGHaus}} Y = k(X \times Y)$ and $(Y^X)_{\mathbf{CGHaus}} = k(Y^X)$. $k(X \times Y) = X \times Y$ and $k(Y^X) = Y^X$ if X is locally compact.

In all the discussions below, all the spaces we consider as the exponential in internal-homs will be Hausdorff and locally compact. In addition, all the spaces in the bordism categories will lie in **CGHaus** and are locally compact. So, we can safely work in either category and we will not make the distinction.

Remark 2.1.7. For Hausdorff spaces, cofibrations can be characterised in another way. An inclusion $A \to X$ is a **neighbourhood deformation retract** (NDR) if there exists a map $u : X \to I$ such that $u^{-1}(0) = A$ and a homotopy $H : X \times I \to X$ such that $H_0 = \mathrm{id}_X$ and $H(a, t) = a$ for all $a \in A$ and $t \in I$ and $h(x, 1) \in A$ if $u(x) < 1$. That is, there is a neighbourhood $U = u^{-1}[0, 1) \subset X$ that deformation retracts onto A. A theorem states that an inclusion $A \to X$ is a cofibration if and only if it is a NDR [May99, Ch. 6.4].

We have a similar Quillen model category structure on **sSet**:

Theorem 2.1.8. *There exists a model category structure on* **sSet**, *where* $f : X \to Y$ *is a*

(i) weak equivalence if the induced map on the geometric realisation $|f| : |X| \to |Y|$ *is a weak homotopy equivalence;*

(ii) cofibration if it is a monomorphism;

(iii) fibration if it is a **Kan fibration**, *that is, it satisfies the right lifting property (RLP) with respect to all inclusions* $i_0 : \Lambda^n_k \hookrightarrow \Delta^n$.

sSet *is Cartesian closed and the Quillen model category structure on* **sSet** *is compatible with Cartesian closure and proper.* **sSet** *is cofibrantly generated by generating cofibrations and generating trivial cofibrations*

$$I = \{\partial\Delta^n \to \Delta^n\} \qquad \text{and} \qquad J = \{\Lambda^n_k \to \Delta^n\}$$

The full subcategory of fibrant-cofibrant objects is **KComp**.

The relationship between the Quillen model category structures on topological spaces and simplicial sets is given by the following Quillen equivalences.

Proposition 2.1.9. *The geometrical realisation functor* $|-| : \mathbf{Top} \to \mathbf{sSet}$ *is left adjoint to the singular complex functor* $\mathrm{Sing} : \mathbf{sSet} \to \mathbf{CGHaus} \subset \mathbf{Top}$, *inducing Quillen equivalences*

$$|-| : \mathbf{CGHaus} \rightleftarrows \mathbf{sSet} : \mathrm{Sing} \qquad \text{and} \qquad |-| : \mathbf{Top} \rightleftarrows \mathbf{sSet} : \mathrm{Sing}.$$

Just as the standard simplices Δ^k for $k \geq 0$ form an indexing set for simplicial sets, there is also a set of indexing simplicial spaces $\{F(k)\}$. For every k, let the **standard k-th simplicial space** $F(k)$ be the simplicial space with the n-space given by the discrete space associated to the set

$$F(k)_n = \mathrm{Hom}_\Delta([n], [k])$$

and the face and degeneracy maps d_i and s_i given by the canonical maps induced by d^i and s^i.

If we regard a simplicial space as a bisimplicial set, then $F(k)$ and the constant simplicial space given by Δ^k correspond to the bisimplicial sets $\Delta^k \times \{*\}$ and $\{*\} \times \Delta^k$ respectively.

Let $\mathbf{F}$ be the **simplex category** whose objects are $F(k)$ and morphisms are generated by the canonical face and degeneracy maps d^i_* and s^i_*.

Note that any map of simplicial spaces $F(k) \to X$ is determined by the image of $\mathrm{id}_{[k]} \in F(k)_k$ since all other points in $F(k)$ can be obtained from $\mathrm{id}_{[k]}$ through a sequence of face and degeneracy maps. Hence, for any simplicial (topological) space X, we have the weak homotopy equivalence

$$\begin{aligned} \mathrm{Map}_{1\text{-}\mathbf{sSp}}(F(k), X) &= |(\mathrm{Hom}_{1\text{-}\mathbf{sSp}}(F(k) \times |\Delta^n|, X))_n| \\ &\cong |(\mathrm{Hom}_{\mathbf{Sp}}(\{\mathrm{id}_{[k]} \times |\Delta^n|, X_k))_n| = |\mathrm{Sing}(X_k)| \xrightarrow{\sim} X_k. \end{aligned}$$

For $\mathbf{Sp}$ being a category of simplicial sets, we have the same weak equivalence.

We let $\partial F(k)$ be the largest simplicial subspace of $F(k)$ not containing $\mathrm{id}_{[k]}$ and for any $X \in 1\text{-}\mathbf{sSp}$, we define

$$\partial X_k = \mathrm{Map}_{1\text{-}\mathbf{sSp}}(\partial F(k), X).$$

We can define a model category structure on 1-**sSp** using the Reedy model category construction (see [Hir03]). The model structure can be given as follows:

Theorem 2.1.10. *There exists a model category structure on* 1-**sSp**, *called the* **Reedy model structure**, *where* $f : X \to Y$ *is a*

(i) weak equivalences if $f_k : X_k \to Y_k$ *are degree-wise weak equivalences;*

(ii) cofibrations if the induced maps

$$X_k \coprod_{\bigcup s_i X_{k-1}} (\bigcup s_i Y_{k-1}) \to Y_k$$

are cofibrations in **Sp***;*

(iii) fibrations if the induced maps

$$X_k \to Y_k \times_{\partial Y_k} \partial X_k \tag{2.1.3}$$

are fibrations in **Sp**.

1-**sSp** *is Cartesian closed and the Reedy model structure is compatible with Cartesian closure and proper. It is cofibrantly generated by generating cofibrations*

$$\partial F(k) \times \Delta^l \sqcup_{\partial F(k) \times \partial \Delta^l} F(k) \times \partial \Delta^l \to F(k) \times \Delta^l, \qquad k, l \geq 0$$

and generating trivial cofibrations

$$\partial F(k) \times \Delta^l \sqcup_{\partial F(k) \times \Lambda^l_t} F(k) \times \Lambda^l_t \to F(k) \times \Delta^l, \qquad k \geq 0, 0 \leq t \leq l.$$

Remark 2.1.11. Our examples of **Sp** are actually cellular model categories, that is, cofibrantly generated model categories satisfying some small conditions on the sets of generating cofibrations and generating trivial cofibrations (see [Hir03] for the precise definition). Hirschhorn showed that the Reedy model category structure on $\mathbf{Sp}^{\Delta^{\mathrm{op}}}$ is also a cellular model structure. We need this fact in the construction of the localisation of this model structure later.

To end of this section, we want to introduce a homotopy version of the mapping space. For any simplicial set X, let $(\mathbf{F} \times \Delta) \downarrow X$ denote the category whose objects are the maps $\sigma : F(k) \times \Delta^l \to X$ and the arrows are commutative diagrams of simplicial spaces

$$\begin{tikzcd}
F(k) \times \Delta^l \arrow[r, "\sigma"] \arrow[d, "\theta"'] & X \\
F(k') \times \Delta^{l'} \arrow[ur, "\tau"'] &
\end{tikzcd}$$

Proposition 2.1.12. *Let* **Sp** = **KComp** *or* **CW** *and X be a simplicial space. There is an isomorphism*

$$X \cong \operatorname*{colim}_{(F(k)\times\Delta^l \to X) \in (\mathbf{F}\times\Delta)\downarrow X} F(k) \times \Delta^l.$$

Proof. This is because every functor $D^{\mathrm{op}} \to \mathbf{Set}$ from a small category D to the category of sets is the colimit of representable functors (see [Mac71, Ch. III.7]). □

Corollary 2.1.13. *Let X be a discrete simplicial space. There is an isomorphism*

$$X \cong \operatorname*{colim}_{(F(k)\to X)\in \mathbf{F}\downarrow X} F(k).$$

Proof. This is because every discrete simplicial set X can be written as the colimit of $F(k) \times \Delta^0$. □

We thus have for any discrete simplicial space X and simplicial space Y,

$$\mathrm{Map}_{1\text{-}\mathbf{sSp}}(X, Y) \cong \mathrm{Map}_{1\text{-}\mathbf{sSp}}\big(\operatorname*{colim}_{(F(k)\to X)} F(k), Y\big) \cong \lim_{(F(k)\to X)} \mathrm{Map}_{1\text{-}\mathbf{sSp}}(F(k), Y).$$

If Y is Reedy-fibrant, then $\mathrm{Map}_{1\text{-}\mathbf{sSp}}(F(k), Y) \xrightarrow{\sim} Y_k$ are trivial fibrations for all k, so the limit is the same as homotopy limit, and we have

$$\lim_{(F(k)\to X)} \mathrm{Map}_{1\text{-}\mathbf{sSp}}(F(k), Y) \xrightarrow{\sim} \lim_{(F(k)\to X)} Y_k.$$

However, in general, limits do not preserve homotopy. To solve this problem, we define the following:

Definition 2.1.14. Let X be a simplicial space ($\mathbf{Sp} = \mathbf{KComp}$ or $\mathbf{CW}$). We define the **homotopy mapping space** of X to Y to be

$$\mathrm{HoMap}_{1\text{-}\mathbf{sSp}}(X,Y) = \operatorname*{holim}_{(F(k)\times\Delta^l \to X)} \mathrm{Map}_{1\text{-}\mathbf{sSp}}(F(k)\times\Delta^l, Y).$$

If X is discrete,

$$\mathrm{HoMap}_{1\text{-}\mathbf{sSp}}(X,Y) = \operatorname*{holim}_{(F(k)\to X)} \mathrm{Map}_{1\text{-}\mathbf{sSp}}(F(k), Y) \xrightarrow{\sim} \operatorname*{holim}_{(F(k)\to X)} Y_k.$$

In fact, we will only use the discrete case.

The homotopy mapping space in our discussion of non-Reedy-fibrant Segal spaces is the generalisation of the mapping space in Rezk's discussion of Reedy-fibrant Segal spaces.

Example 2.1.15. It is clear that $\mathrm{HoMap}_{1\text{-}\mathbf{sSp}}(F(k),X) \cong \mathrm{Map}_{1\text{-}\mathbf{sSp}}(F(k),X)$ for all k and simplicial space X.

Since we can write $\partial F(k) = \cup_{i=0}^{k} d^i F(k-1)$, we have $\mathrm{HoMap}_{1\text{-}\mathbf{sSp}}(\partial F(k),X)$ is equal to the homotopy limit taken over the diagram with $k+1$ copies of X_{k-1}, indexed as $X_{k-1}^0,\dots,X_{k-1}^k$ with the arrows

$$\begin{tikzcd} & X_{k-1}^j \arrow[d, "d_i"] \\ X_{k-1}^i \arrow[r, "d_{j-1}"] & X_{k-2}^{ij} \end{tikzcd}$$

for all $i<j$.

Let $R : 1\text{-}\mathbf{sSp} \to 1\text{-}\mathbf{sSp}$ be a fibrant replacement functor. Then, the weak equivalence $Y \xrightarrow{\sim} R(Y)$ induces a weak equivalence

$$\mathrm{HoMap}_{1\text{-}\mathbf{sSp}}(X,Y) \xrightarrow{\sim} \mathrm{HoMap}_{1\text{-}\mathbf{sSp}}(X,R(Y)) = \mathrm{Map}_{1\text{-}\mathbf{sSp}}(X,R(Y)).$$

Thus, for any map of simplicial spaces $f : A \to B$ and simplicial space X, $\mathrm{HoMap}_{1\text{-}\mathbf{sSp}}(f,X)$ is a weak equivalence if and only if $\mathrm{Map}_{1\text{-}\mathbf{sSp}}(f,R(X))$ is.

2.2 Segal spaces

We are now ready to define Segal spaces.

Definition 2.2.1. Let $\mathbf{Sp} = \mathbf{Top}$, $\mathbf{CGHaus}$ or $\mathbf{KComp}$. A **Segal space** (or Segal 1-space) W is a Reedy-cofibrant simplicial space satisfying: for all $m,n\in\mathbb{N}$, the square

$$\begin{tikzcd} W_{m+n} \arrow[r] \arrow[d] & W_m \arrow[d, "t"] \\ W_n \arrow[r, "s"] & W_0 \end{tikzcd} \tag{2.2.4}$$

is a homotopy pullback (in the model structure for $\mathbf{Sp}$), where s and t are induced by $[0] \ni 0 \mapsto 0 \in [n]$ and $[0] \ni 0 \mapsto m \in [m]$ respectively. That (2.2.4) is a homotopy pullback is called the **Segal condition**.

Let $1\text{-}\mathbf{SeSp} \subset 1\text{-}\mathbf{sSp}$ denote the full subcategory of Segal spaces in the category of simplicial spaces.

Remark 2.2.2. We impose the condition that a Segal space is Reedy-cofibrant in order to define a completion functor later (which is a pushout). The completion functor can be constructed without the Reedy-cofibrant hypothesis, but it is only well-defined up to homotopy. Reedy-cofibrancy is a very weak condition. For $\mathbf{Sp} = \mathbf{KComp}$, cofibrations are monomorphisms, so all simplicial spaces are Reedy-cofibrant. For $\mathbf{Sp} = \mathbf{Top}$ or $\mathbf{CGHaus}$, a simplicial space X is Reedy-cofibrant if (but not only if) the spaces X_k are Hausdorff and the inclusion

$$\left(\bigcup s_i X_{k-1}\right) \subset X_k$$

is an embedding. All the simplicial spaces we will encounter are Reedy-cofibrant.

For $0 \le i < k$, let $\alpha^i : [1] \to [k]$ be the map sending $(0,1) \mapsto (i, i+1)$. Let $G(k) \subset F(k)$ be the simplicial subspace generated by $\alpha^i \in F(k)_1$. Equivalently, α^i induces a map $F(1) \to F(k)$ and we define $G(k)$ to be

$$G(k) = \cup_{i=0}^{k-1} \alpha^i F(1) \subset F(k). \tag{2.2.5}$$

The inclusion $\phi^n : G(n) \hookrightarrow F(n)$ induces a map

$$\mathrm{HoMap}_{\textbf{1-sSp}}(\phi^n, W) : \mathrm{HoMap}_{\textbf{1-sSp}}(F(n), W) \cong W_n \to \mathrm{HoMap}_{\textbf{1-sSp}}(G(n), W). \tag{2.2.6}$$

Proposition 2.2.3. *Let W be a Reedy-cofibrant simplicial space. The following are equivalent:*

(i) W is a Segal space;

(ii) For each $n \ge 2$,

$$\phi_n : W_n \to \mathrm{holim}(W_1 \xrightarrow{d_0} W_0 \xleftarrow{d_1} W_1 \xrightarrow{d_0} W_0 \xleftarrow{d_1} \cdots \xrightarrow{d_0} W_0 \xleftarrow{d_1} W_1) \tag{2.2.7}$$

is a weak equivalence;

(iii) For each $n \ge 2$, $\mathrm{HoMap}_{\textbf{1-sSp}}(\phi^n, W)$ given by (2.2.6) is a weak equivalence.

Proof. (i)⇒(ii): Suppose W is a Segal space. By (2.2.4), $W_2 \to \mathrm{holim}(W_1 \to W_0 \leftarrow W_1)$ is a weak equivalence. Suppose we have proven (2.2.7) for $n-1$, then by (2.2.4) and the induction hypothesis,

$$W_n \xrightarrow{\sim} \mathrm{holim}(W_{n-1} \to W_0 \leftarrow W_1) \xrightarrow{\sim} \mathrm{holim}(W_1 \to W_0 \leftarrow \cdots \to W_0 \leftarrow W_1).$$

(ii)⇒(i): Suppose (2.2.7) is a weak equivalence. Then, for $m = n = 1$, we have the homotopy pullback square in (2.2.4). By induction, suppose we have shown the Segal condition for all $m + n < p$, then we have the diagram

$$\begin{tikzcd}
X_p \arrow[rr] \arrow[dr, "\sim"'] & & \mathrm{holim}(X_m \xrightarrow{s} X_0 \xleftarrow{t} X_{p-m}) \arrow[dl, "\sim"] \\
& \mathrm{holim}(X_1 \to X_0 \leftarrow \cdots \to X_0 \leftarrow X_1) &
\end{tikzcd}$$

so the horizontal map is also a weak equivalence.

(ii)⇔(iii): Using (2.2.5), we easily see that $\mathrm{HoMap}_{\textbf{1-sSp}}(\phi^n, W) = \phi_n$. □

Recall that in **Top**, a homotopy pullback can be explicitly constructed as a homotopy fibre product

$$X \times_Z^{\mathrm{ho}} Y = X \times_Z Z^{[0,1]} \times_Z Y.$$

Remark 2.2.4. At this point, we should contrast our definition with that given by Rezk in [Rez01]. Rezk defined a Segal space as a Reedy-fibrant simplicial space ($\mathbf{Sp} = \mathbf{sSet}$) satisfying (2.2.4). The Reedy-fibrant condition immediately implies that the simplicial sets in question are Kan complexes. Our definition when $\mathbf{Sp} = \mathbf{KComp}$ is weaker as we do not require Reedy-fibrancy. Indeed, most of the Segal spaces we will construct are not Reedy fibrant. However, Reedy fibrancy is just a formal condition guaranteeing that the simplicial space is a fibrant object in the model structure for Segal spaces. We can obtain a Reedy fibrant Segal space by taking the fibrant replacement without any loss of information.

If we assume W is Reedy fibrant, the limits are automatically homotopy limits, so W is a Segal space if and only if $W_{m+n} \to W_m \times_{W_0} W_n$ are weak equivalences if and only if for all n, there exist weak equivalences

$$\phi_n : W_n \to \lim(W_1 \xrightarrow{d_0} W_0 \xleftarrow{d_1} W_1 \xrightarrow{d_0} W_0 \xleftarrow{d_1} \cdots \xrightarrow{d_0} W_0 \xleftarrow{d_1} W_1)$$

if and only if for all n, there exist weak equivalences

$$\mathrm{Map}_{\textbf{1-sSp}}(\phi^n, W) : \mathrm{Map}_{\textbf{1-sSp}}(F(n), W) \cong W_n \to \mathrm{Map}_{\textbf{1-sSp}}(G(n), W).$$

The following examples are due to Rezk [Rez01].

Example 2.2.5 (Discrete nerve). Let C be a category. The discrete nerve $\operatorname{discnerve} C$ of C is the simplicial space whose n-space is given by the set

$$\{c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} c_n | c_i \in \operatorname{Ob} C, f_i \in \operatorname{Mor} C\}$$

of chains of morphisms in C of length n. The face and degeneracy maps are defined by

$$\begin{aligned}
&d_0(c_0 \xrightarrow{f_1} \cdots \xrightarrow{f_n} c_n) = (c_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} c_n), \qquad d_n(c_0 \xrightarrow{f_1} \cdots \xrightarrow{f_n} c_n) = (c_0 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} c_{n-1})\\
&d_i(c_0 \xrightarrow{f_1} \cdots \xrightarrow{f_n} c_n) = (c_0 \xrightarrow{f_1} \cdots \xrightarrow{f_{i-1}} c_{i-1} \xrightarrow{f_{i+1}\circ f_i} c_{i+1} \xrightarrow{f_{i+2}} \cdots \xrightarrow{f_n} c_n) \quad \text{for } 1 \leq i < n\\
&s_i(c_0 \xrightarrow{f_1} \cdots \xrightarrow{f_n} c_n) = (c_0 \xrightarrow{f_1} \cdots \xrightarrow{f_i} c_i \xrightarrow{\mathrm{id}_{c_i}} c_i \xrightarrow{f_{i+1}} \cdots \xrightarrow{f_n} c_n) \quad \text{for } 0 \leq i \leq n.
\end{aligned}$$

It is clear that $\operatorname{discnerve} C$ is a Segal space. Indeed it is a Reedy fibrant Segal space since all discrete simplicial spaces are Reedy fibrant.

For example, we have $F(k) = \operatorname{discnerve}[k]$. Note that discrete nerves do not preserve categorical equivalences: two equivalent categories may have non-Reedy equivalent discrete nerves.

Example 2.2.6 (Classifying diagrams of categories). This is another construction of a Segal space from a category. Let C be a category and W be a subcategory such that $\operatorname{Ob} W = \operatorname{Ob} C$ and $\operatorname{Hom}_W(x,y) \subset \operatorname{Hom}_C(x,y)$ for all $x, y \in \operatorname{Ob}(C)$. We call the morphisms in W weak equivalences. For any category D, we define the category $\operatorname{we}(C^D)$ to be the subcategory of the category C^D consisting of functors $D \to C$ with natural transformations $\alpha : F \to G$ such that $\alpha_d \in W$ for all $d \in \operatorname{Ob} D$.

Given a pair (C, W), we define a simplicial space $N(C,W)$ ($\mathbf{Sp} = \mathbf{sSet}$) where

$$N(C,W)_m = \operatorname{nerve} \operatorname{we}(C^{[m]})$$

and the face and degeneracy maps are determined by precomposition with d^i or s^i applied on $[m]$. nerve : $\mathbf{Cat} \to \mathbf{sSet}$ is the simplicial nerve functor [GJ99]. It is convenient to view an n-simplex of $N(C,W)_m$ as a commutative diagram

$$\begin{array}{ccccccc}
c_{00} & \to & c_{01} & \to & \cdots & \to & c_{0n}\\
\downarrow & & \downarrow & & & & \downarrow\\
c_{10} & \to & c_{11} & \to & \cdots & \to & c_{1n}\\
\downarrow & & \downarrow & & & & \downarrow\\
\vdots & & \vdots & & \ddots & & \vdots\\
\downarrow & & \downarrow & & & & \downarrow\\
c_{m0} & \to & c_{m1} & \to & \cdots & \to & c_{mn}
\end{array} \tag{2.2.8}$$

where the horizontal arrows are in W.

The discrete nerve $\operatorname{discnerve} C$ can be seen as $N(C, C_0)$ where C_0 is the subcategory where $\operatorname{Ob} C_0 = \operatorname{Ob} C$ and only contains the identity morphisms.

Let $W = \operatorname{iso} C$ be the subcategory consisting of all isomorphisms in C. We write $N(C)$ for $N(C, \operatorname{iso} C)$, the **classifying diagram** of the category C.

There is a canonical inclusion $\operatorname{discnerve} C \to N(C)$ induced by the inclusion $C_0 \to \operatorname{iso} C$.

Proposition 2.2.7. *$N(C)$ is a Reedy-fibrant Segal space.*

Proof. First, we have to prove that $N(C)_m \in \mathbf{KComp} \subset \mathbf{sSet}$. This is true since $\operatorname{we}(C^{[m]})$ is a groupoid as all the natural transformations are indeed natural isomorphism, so its simplicial nerve is a Kan complex.

The proofs that $N(C)$ is Reedy-fibrant and satisfies the Segal condition are simply checks on the extension of the diagram (2.2.8). ☐

Rezk also proved that a functor $F : C \to D$ is an equivalence of categories if and only if the map of simplicial spaces $N(F) : N(C) \to N(D)$ is a weak equivalence.

Example 2.2.8 (Classifying diagrams of model categories). This is one of the most important examples of Segal spaces. Let $C = \mathcal{M}$ be a closed model category and W be its subcategory of weak equivalences. We write $N(\mathcal{M}) = N(\mathcal{M}, W)$ for its classifying diagram. We remark that this notation will not risk confusion with that in the previous example. Any category C with finite limits and colimits can be regarded as a model category with weak equivalences being the isomorphisms and all maps being fibrations and cofibrations. Then, the classifying diagram with this model structure is precisely the classifying diagram of the category.

In general, however, $N(\mathcal{M})$ is not a Segal space, since the m-spaces $N(\mathcal{M})_m$ are not Kan complexes as not all weak equivalences are invertible (they are weak Kan complexes). To each model category, we have the associated category $\pi\mathcal{M}_{cf}$ of fibrant-cofibrant objects and homotopy classes of morphisms. This is obtained from $\mathcal{M}$ by applying the fibrant and cofibrant replacement functors R and Q. They induce a Reedy weak equivalence $N(\mathcal{M}) \to N(\pi\mathcal{M}_{cf})$.

Thus, $N\mathcal{M})$ satisfies the Segal condition (2.2.7). The Reedy-fibrant replacement $N^f(\mathcal{M})$ is thus a Reedy-fibrant Segal space. This is the **classifying diagram** of the model category $\mathcal{M}$.

Segal spaces can be seen as a model for $(\infty, 1)$-categories.

Definition 2.2.9. Let W be a Segal space. Let $\mathrm{Ob}\, W$ be the points of W_0 (or the 0-simplices of W_0 in the case where $\mathbf{Sp} = \mathbf{KComp}$). Given any $x, y \in \mathrm{Ob}\, W$, let $\mathrm{map}_W(x, y)$ be the homotopy fibre of (x, y) in the map

$$W_1 \xrightarrow{(d_1,d_0)} W_0 \times W_0.$$

For any $x \in \mathrm{Ob}\, W$, let $\mathrm{id}_x = s_0 x \in \mathrm{map}_W(x, x)$.

Remark 2.2.10. If W is Reedy-fibrant, then $W_1 \to W_0 \times W_0$ is a fibration, so $\mathrm{map}_W(x, y)$ is simply the fibre of (x, y). For every Segal space W, there is a functorial Reedy-fibrant replacement $W \mapsto R(W)$ which is a Reedy weak equivalence. Hence, $\mathrm{map}_W(x, y) \xrightarrow{\sim} \mathrm{map}_{R(W)}(R(x), R(y))$.

This gives a natural realisation of a Segal space as an $(\infty, 1)$-category. The 1-morphisms are the points of $\mathrm{map}_W(x, y)$, the 2-morphisms are the paths (or 1-simplices) in $\mathrm{map}_W(x, y)$, etc. Since the objects of $\mathbf{Sp}$ are topological spaces or Kan complexes, the n-morphisms for $n > 1$ are invertible. We will often write 1-morphisms as $f : x \to y$ for $f \in \mathrm{map}_W(x, y)$.

For any $(n+1)$-uple of objects $(x_0, \ldots, x_n)$ in $\mathrm{Ob}\, W$, we can define $\mathrm{map}_W(x_0, \ldots, x_n)$ to be the homotopy fibre of the map

$$X_n \to (X_0)^n$$

induced by $\coprod_{i=0}^n [0] \mapsto (0, \ldots, n)$. Using the Segal condition (2.2.7) and the commutative diagram

$$\begin{tikzcd} X_n \arrow[rr, "\sim"] \arrow[dr] & & X_1 \times^{\mathrm{ho}}_{X_0} \cdots \times^{\mathrm{ho}}_{X_0} X_1 \arrow[dl] \\ & X_0 \times \cdots \times X_0 & \end{tikzcd},$$

we have a weak equivalence of homotopy fibres

$$\mathrm{map}_W(x_0, \ldots, x_n) \xrightarrow{\sim} \mathrm{map}_W(x_0, x_1) \times \cdots \times \mathrm{map}_W(x_{n-1}, x_n).$$

If there is no risk of confusion, we will drop the subscript W.

Let us now define the homotopy category of a Segal space W.

Two 1-morphisms (i.e., points) $f, g : x \to y$ are **homotopic** (written $f \sim g$) if f and g lie in the same path component of $\mathrm{map}(x, y)$ (or there exists a 1-simplex $h \in \mathrm{map}(x, y)$ such that $d_1(h) = f$ and $d_0(h) = g$). Given two 1-morphisms $f : x \to y$ and $g : y \to z$, we can define the composition $g \circ f : x \to z$ as follows: by the Segal condition, we can lift (f, g) to $h \in \mathrm{map}(x, y, z)$, and let $g \circ f = d_1(h)$. The sequence of maps is given in the diagram below, where the horizontal map is a weak equivalence:

$$\begin{tikzcd} \mathrm{map}(x,y,z) \arrow[r, "(d_2,d_0)", "\sim"'] \arrow[d, "d_1"'] & \mathrm{map}(x,y) \times \mathrm{map}(y,z) \\ \mathrm{map}(x,z) & \end{tikzcd} \tag{2.2.9}$$

Proposition 2.2.11. *Composition of 1-morphisms as described above is well-defined up to homotopy. The composition is associative and respects the identity up to homotopy, that is, given $f : w \to x$, $g : x \to y$ and $h : y \to z$, we have $(h \circ g) \circ f \sim h \circ (g \circ f)$ and $f \circ \mathrm{id}_w \sim f \sim \mathrm{id}_x \circ f$.*

Proof. Let h and h' be two choices of the lifting of (f, g) described above. (2.2.9) gives an isomorphism $\pi_0(d_2, d_0) : \pi_0\mathrm{map}(w, x, y) \to \pi_0(\mathrm{map}(w, x) \times \mathrm{map}(x, y))$. So, there exists a path (or 1-simplex) k joining h and h'. Then, $d_1(k)$ is a path (1-simplex) joining $d_1(h)$ and $d_1(h')$, so $d_1(h) \sim d_1(h')$.

To prove the second part of the proposition, we show that there exist choices of the composition that give equal, and not just homotopic results. Suppose we are given $f : w \to x$, $g : x \to y$ and $h : y \to z$. The construction of $h \circ (g \circ f)$ can be given by the diagram

$$\begin{tikzcd}
\mathrm{map}(w,x,y,z) \arrow[r, "(d_3, d_0 d_0)", "\sim"'] \arrow[d, "d_1"'] & \mathrm{map}(w,x,y) \times \mathrm{map}(y,z) \arrow[r, "(d_2,d_0)\times\mathrm{id}", "\sim"'] \arrow[d, "d_1\times\mathrm{id}"'] & \mathrm{map}(w,x) \times \mathrm{map}(x,y) \times \mathrm{map}(y,z) \\
\mathrm{map}(w,y,z) \arrow[r, "(d_2,d_0)", "\sim"'] \arrow[d, "d_1"'] & \mathrm{map}(w,y) \times \mathrm{map}(y,z) & \\
\mathrm{map}(w,z) & &
\end{tikzcd}.$$

$h \circ (g \circ f)$ is determined by a lift $k \in \mathrm{map}(w, x, y, z)$ of (f, g, h). In the analogous diagram for $(h \circ g) \circ f$, k is also a lift of (f, g, h) by the commutative diagram (a consequence of the simplicial identities (2.1.2))

$$\begin{tikzcd}
\mathrm{map}(w,x,y,z) \arrow[r, "(d_3, d_0 d_0)", "\sim"'] \arrow[d, "(d_2 d_3, d_0)"', "\sim"] & \mathrm{map}(w,x,y) \times \mathrm{map}(y,z) \arrow[d, "\sim"', "(d_2,d_0)\times\mathrm{id}"] \\
\mathrm{map}(w,x) \times \mathrm{map}(x,y,z) \arrow[r, "\mathrm{id}\times(d_2,d_0)", "\sim"'] & \mathrm{map}(w,x) \times \mathrm{map}(x,y) \times \mathrm{map}(y,z)
\end{tikzcd}$$

Hence, $(h \circ g) \circ f = d_1 d_2 k = d_1 d_1 k = h \circ (g \circ f)$.

To show that $f \circ \mathrm{id}_w \sim f$, we note that $k = s_0 f \in \mathrm{map}(w, w, x)$ satisfies $(d_2, d_0)k = (\mathrm{id}_w, f)$ and $d_1 k = f$, so we can choose $f \circ \mathrm{id}_w = d_1 k = f$. Similarly, $\mathrm{id}_x \circ f \sim f$. $\square$

By the above proposition, we have a well-defined category:

Definition 2.2.12. Let W be a Segal space. We define the **homotopy category** $\mathrm{Ho}\, W$ to be the category with $\mathrm{Ob}\,\mathrm{Ho}\, W = \mathrm{Ob}\, W$ and for all $x, y \in \mathrm{Ob}\, W$, $\mathrm{Hom}_{\mathrm{Ho}\, W}(x, y) = \pi_0\mathrm{map}_W(x, y)$.

Example 2.2.13. For any category C, $\mathrm{Ho}\, NC \cong \mathrm{Ho}\,\mathrm{discnerve}\, C \cong C$.

We want a notion of equivalences of $(\infty, 1)$-categories. A good notion of equivalence must necessarily imply an equivalence of the homotopy categories. However, it is not sufficient, as we also want an equivalence in the higher homotopy structures. An equivalence of homotopy categories only tells us that there is a bijection between the path components of the mapping space, but nothing about the higher homotopy groups. An appropriate notion of equivalence was introduced by Dwyer and Kan in the context of simplicial categories [DK80]. Rezk adapted it for Segal spaces.

Definition 2.2.14. A map $f : U \to V$ of Segal spaces is a **Dwyer-Kan equivalence** if

(i) the induced map $\mathrm{Ho}\, f : \mathrm{Ho}\, U \to \mathrm{Ho}\, V$ is an equivalence of categories; and

(ii) for each pair of objects $x, x' \in U$, the induced function $\mathrm{map}_U(x, x') \to \mathrm{map}_V(fx, fx')$ is a weak equivalence.

Let $\mathrm{Ob}\, U/\sim$ denote the equivalence classes of objects in U under homotopy equivalence. We can reformulate condition (i) as

(i') the induced map $\mathrm{Ob}\, U/\sim \to \mathrm{Ob}\, V/\sim$ is a bijection on the equivalence classes of objects.

The pair of conditions (i') and (ii) is equivalent to the pair (i) and (ii).

Proposition 2.2.15. *In the diagram of maps of Segal spaces*

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[rr, bend right, "h"'] & V \arrow[r, "g"] & W \end{tikzcd},$$

if two of the maps are Dwyer-Kan equivalences, so is the third.

Proof. The result is clear on the homotopy categories. On the mapping spaces, it is a consequence of the similar diagram for weak equivalences in model categories. □

As an important example, we relate discrete nerves to classifying diagrams.

Proposition 2.2.16. *The canonical inclusion* $\operatorname{discnerve} C \to N(C)$ *is a Dwyer-Kan equivalence.*

Proof. We have $\operatorname{Ho}\operatorname{discnerve} C \cong \operatorname{Ho} N(C) \cong C$. For any $x, y \in \operatorname{Ob} C$, $\operatorname{map}_{\operatorname{discnerve} C}(x, y)$ is the discrete simplicial set $\operatorname{Hom}_C(x, y)$ while $\operatorname{map}_{N(C)}(x, y)$ is homotopy equivalent to $\operatorname{Hom}_C(x, y)$ (we can construct a homotopy similar to that in Example 2.3.4). □

The following construction is a technical tool we need in defining the cobordism category. We wish to consider non-unital categories, that is, categories that do not contain the identity maps on objects.

Definition 2.2.17. Let $\Delta_0 \subset \Delta$ be the sub-non-unital-category of Δ with the same objects but only allow morphisms $[m] \to [n]$ which are strictly increasing. The morphisms in Δ_0 are generated by the face maps d^i.

Definition 2.2.18. A **semisimplicial space** is a functor $W : \Delta_0^{\mathrm{op}} \to \mathbf{Sp}$, with the induced face maps d_i. The category of semisimplicial spaces is denoted by 1-**semisSp** $= \mathbf{Sp}^{\Delta_0^{\mathrm{op}}}$.

A **semiSegal space** W is a semisimplicial space satisfying the Segal condition (2.2.4) for all $m, n \in \mathbb{N}$. The category of semiSegal spaces 1-**semiSeSp** is a full subcategory of 1-**semisSp**.

For any semiSegal space W, we can similarly define its homotopy category $\operatorname{Ho} W$, which will be a non-unital category.

There is a forgetful functor For : 1-**sSp** $\to$ 1-**semisSp** that sends $W \mapsto W|_{\Delta_0^{\mathrm{op}}}$. Note that the forgetful functor induces the identity map on the n-spaces: $\operatorname{For}(W)_n = W_n$. So, $\operatorname{For}(W)$ satisfies the Segal condition (2.2.4), which is a homotopy pullback square on **Sp**, if and only if W does. We summarise this in the following proposition:

Proposition 2.2.19. *Let W be a simplicial space, then W is a Segal space if and only if* $\operatorname{For}(W)$ *is a semiSegal space.*

2.3 Complete Segal spaces

While a $(\infty, 1)$-category can be modelled by a Segal space, such a model is not completely satisfactory. There are too many Segal spaces in the following sense. We can impose a model category structure 1-$\mathcal{SS}$ on 1-**sSp** related to Segal spaces (Thm. 2.4.6). However, a Dwyer-Kan equivalence of Segal spaces may not be a weak equivalence in that model structure. This is not a problem with the choice of the model structure. There are non-equivalent Segal spaces that give rise to equivalent $(\infty, 1)$-categories (Dwyer-Kan equivalence). For example, we have already noted in Example 2.2.5 that discrete nerves of equivalent categories may not be equivalent.

To solve this problem, we will construct a refinement of Segal spaces, called the complete Segal spaces. In the next section, we will show that there is an appropriate model structure 1-$\mathcal{CSS}$ on 1-**sSp** (Thm. 2.4.7) such that weak equivalence of complete Segal spaces in that model are precisely Dwyer-Kan equivalences.

In fact, we have more: a map of Segal spaces in 1-$\mathcal{CSS}$ is weak equivalence if and only if it is a Dwyer-Kan equivalence.

We begin the construction. Let W be a Segal space and $x, y \in \operatorname{Ob} W$. We say that a 1-morphism $g : x \to y$ is a **homotopy equivalence** if there exist $f, h : y \to x$ such that $g \circ f \sim \mathrm{id}_y$ and $h \circ g \sim \mathrm{id}_x$. By associativity up to homotopy, we have $h \sim h \circ g \circ f \sim f$.

Definition 2.3.1. Let W be a Segal space. The **space of homotopy equivalence** is the subspace $W_{\mathrm{hoequiv}} \subset W_1$ of homotopy equivalences.

Remark 2.3.2. Suppose W is Reedy fibrant, then Rezk showed that W_{hoequiv} is a union of path components of W_1 [Rez01]. This is not true in general.

Note that the map $s_0 : W_0 \to W_1$ factors through W_{hoequiv} since $s_0 x = \mathrm{id}_x \in W_{\mathrm{hoequiv}}$ for all $x \in W_0$.

Definition 2.3.3. A complete Segal space is a Segal space W where the map $s_0 : W_0 \to W_{\mathrm{hoequiv}}$ is a weak equivalence.

Example 2.3.4. The classifying diagram $N(C)$ of a category C is a complete Segal space. It is easy to check that $N(C)_{\mathrm{hoequiv}} \subset N(C)_1$ consist of commutative diagrams of the form (2.2.8) where all maps are isomorphisms. Consider the maps

$$s_0 : N(C)_0 \to N(C)_{\mathrm{hoequiv}} \quad : \quad (c_0 \to \cdots \to c_n) \mapsto \begin{pmatrix} c_0 \to \cdots \to c_n \\ \downarrow \mathrm{id} \qquad \mathrm{id} \downarrow \\ c_0 \to \cdots \to c_n \end{pmatrix}$$

$$d_1 : N(C)_{\mathrm{hoequiv}} \to N(C)_0 \quad : \quad \begin{pmatrix} c_{00} \to \cdots \to c_{0n} \\ \downarrow \qquad\qquad \downarrow \\ c_{10} \to \cdots \to c_{1n} \end{pmatrix} \mapsto (c_{00} \to \cdots \to c_{0n}).$$

It is clear that $d_1 s_0 = \mathrm{id}_{N(C)_0}$ and we have a homotopy

$$H \quad : \quad N(C)_{\mathrm{hoequiv}} \times \Delta^1 \longrightarrow N(C)_{\mathrm{hoequiv}}$$

$$\left(\begin{pmatrix} c_{00} \to \cdots \to c_{0n} \\ \downarrow \qquad\qquad \downarrow \\ c_{10} \to \cdots \to c_{1n} \end{pmatrix}, [0, \dots, 0, 1, \dots, 1] \right) \mapsto \begin{pmatrix} c_{00} \to \cdots \to c_{0m} \to c_{0,m+1} \to \cdots \to c_{0n} \\ \downarrow \qquad\qquad \downarrow \qquad \downarrow \qquad\qquad \downarrow \\ c_{00} \to \cdots \to c_{0m} \to c_{1,m+1} \to \cdots \to c_{1n} \end{pmatrix}$$

between $\mathrm{id}_{N(C)_{\mathrm{hoequiv}}}$ and $s_0 d_1$. Hence, $N(C)_0 \to N(C)_{\mathrm{hoequiv}}$ is a weak equivalence.

Example 2.3.5. The classifying diagram $N^f(\mathcal{M})$ of a model category $\mathcal{M}$ is a complete Segal space since it is Reedy-weak equivalent to $N(\pi \mathcal{M}_{cf})$ which is a complete Segal space.

As in 2.2.6, we will like to be able to represent the condition of Segal completeness. Let $E = \operatorname{discnerve} I[1]$ where $I[1]$ is the category with objects $0, 1$ and non-identity morphisms $0 \to 1$ and $1 \to 0$ which are inverse to each other. We have the natural inclusion $i : F(1) \hookrightarrow E$. We can decompose i into a chain of inclusions

$$F(1) = E^{(1)} \subset E^{(2)} \subset \cdots \subset E$$

where $E^{(j)}$ is the smallest sub-simplicial space containing the chain of maps $0 \to 1 \to 0 \to \cdots$ of length j. We have $E = \operatorname{colim} E^{(j)}$.

The following theorem gives a representation of the inclusion $W_{\mathrm{hoequiv}} \to W_1$ up to weak equivalence:

Theorem 2.3.6 ([Rez01, Thm. 6.2, Prop. 11.1])**.** *Suppose W is a Segal space. Then the map*

$$\mathrm{HoMap}_{1\text{-}\mathbf{sSp}}(E, W) \to W_1$$

induced by the inclusion $i : F(1) \hookrightarrow E$ factors through $W_{\mathrm{hoequiv}} \subset W_1$ and induces a weak equivalence

$$\mathrm{HoMap}_{1\text{-}\mathbf{sSp}}(E, W) \xrightarrow{\sim} W_{\mathrm{hoequiv}}.$$

Proof. The proof is technical and is presented in [Rez01, Sec. 11] for Reedy-fibrant Segal spaces. The general proof is the same, replacing mapping spaces with homotopy mapping spaces. ☐

We can thus reformulate the completeness condition as follows

Proposition 2.3.7. *Let W be a Segal space. The following are equivalent:*

(i) *W is a complete Segal space;*

(ii) *the map $W_0 \to \mathrm{HoMap}_{\mathbf{1\text{-}sSp}}(E, W)$ induced by the unique map $E \to F(0)$ is a weak equivalence;*

(iii) *for all $n \geq 1$ and every map $f : \partial\Delta^n \times F(1) \to W$ such that $f(\partial\Delta^n \times \{\mathrm{id}_{[1]}\}) \subset W_{\mathrm{hoequiv}}$, there exists a homotopy pushout square*

$$\begin{tikzcd}
\partial\Delta^n \times F(1) \arrow[r, "f"] \arrow[d, "\mathrm{id}\times s^0"] & W \arrow[d, "\mathrm{id}"] \\
\partial\Delta^n \times F(0) \arrow[r] & W
\end{tikzcd} .$$

Proof. (i)⇔(ii) The map $s^0 : [1] \to [0]$ extends to $I[1] \to [0]$, so we have a map of discrete nerves $F(1) \hookrightarrow E \to F(0)$. Applying $\mathrm{HoMap}_{\mathbf{1\text{-}sSp}}(-, W)$ and Thm. 2.3.6, we have the commutative diagram

$$\begin{tikzcd}
W_0 \arrow[r] \arrow[dr] & \mathrm{HoMap}_{\mathbf{1\text{-}sSp}}(E, W) \arrow[d] \arrow[drr] & & \\
& W_{\mathrm{hoequiv}} \arrow[rr] & & W_1
\end{tikzcd}$$

The vertical map is a weak equivalence, so $W_0 \to W_{\mathrm{hoequiv}}$ is a weak equivalence if and only if $W_0 \to \mathrm{HoMap}_{\mathbf{1\text{-}sSp}}(E, W)$ is.

(i)⇔(iii) This is a purely formal result. $\pi_{n-1}W_0 \to \pi_{n-1}W_{\mathrm{hoequiv}}$ is an isomorphism (bijection for $n = 1$) if and only if the pushout square in (iii) exists for all maps f such that $f(\partial\Delta^n \times \{\mathrm{id}_{[1]}\}) \subset W_{\mathrm{hoequiv}}$. □

The 0-space of a Reedy-fibrant complete Segal space can be characterised as follows:

Proposition 2.3.8. *Let W be a complete Segal space. Then there is a bijection $\pi_0 W_0 \cong \mathrm{Ob}\, W/\sim$ where $\mathrm{Ob}\, W/\sim$ is the set of homotopy equivalence classes ($x \sim y$ if they are homotopy equivalent).*

Proof. Since $W_0 \to W_{\mathrm{hoequiv}}$ is a weak equivalence, we have a bijection $\pi_0 W_0 \to \pi_0 W_{\mathrm{hoequiv}}$. Hence, there is a path from x to y in W_0 if and only if there is a path in W_{hoequiv} from id_x to id_y.

Consider the homotopy pullback diagram

$$\begin{tikzcd}
\mathrm{hoequiv}(x, y) \arrow[r, hook] \arrow[d] & \mathrm{map}(x, y) \arrow[r] \arrow[d] & \{(x, y)\} \arrow[d] \\
W_{\mathrm{hoequiv}} \arrow[r, hook] & W_1 \arrow[r] & W_0 \times W_0
\end{tikzcd} .$$

Thus, $x \sim y$ if and only if $\mathrm{hoequiv}(x, y)$ is non-empty if and only if there is a path in W_{hoequiv} from id_x to id_y. □

We will like to construct a functor to obtain a complete Segal space from a Segal space in a universal way.

Definition 2.3.9. Let W be a Segal space, a **Segal completion** (or simply completion) of W is a complete Segal space $\widetilde{W}$ with a map $W \to \widetilde{W}$ which is universal among all maps from W to complete Segal spaces.

Proposition 2.3.10. *There exists a functorial Segal completion $W \to \widetilde{W}$.*

Proof. We construct the completion using the small object argument. We can decompose the map $s^0 : F(1) \to F(0)$ into a cofibration followed by a trivial fibration:

$$F(1) \hookrightarrow N(I[1]) \to F(0) = N([0]).$$

The second map is a trivial fibration since $I[1]$ is equivalent to $[0]$, and so their classifying diagrams are Reedy-weakly equivalent.

We inductively define Segal spaces W^i. Let $W^0 = W$. Assume that W^i has been defined. Consider the set D^i of all maps

$$f : \partial\Delta^n \times F(1) \to W^i$$

ranging over all n and all maps such that $f(\partial\Delta^n \times \{\mathrm{id}_{[1]}\}) \subset W^i_{\mathrm{hoequiv}}$. We define W^{i+1} to be the pushout

$$\begin{tikzcd} \coprod_{f\in D^i}(\partial\Delta^n \times F(1)) \arrow[r, "\coprod f"] \arrow[d, "\coprod(\mathrm{id}\times s^0)"] & W^i \arrow[d] \\ \coprod_{f\in D^i}(\partial\Delta^n \times N(I[1])) \arrow[r] & W^{i+1} \end{tikzcd} . \tag{2.3.10}$$

Note that since the left hand map is a cofibration and W^i is cofibrant, the square is a homotopy pushout.

Let $\widetilde{W} = \operatorname{colim} W^i$. Since $\partial\Delta^n \times F(1)$ is compact, any map $f : \partial\Delta^n \times F(1) \to \widetilde{W}$ factors through some W^i, so we have a commutative diagram

$$\begin{tikzcd} \partial\Delta^n \times F(1) \arrow[r, "f"] \arrow[d, "\mathrm{id}\times s^0"] & W^i \arrow[rr] \arrow[d] \arrow[dr] & & \widetilde{W} \arrow[d, "\mathrm{id}"] \\ \partial\Delta^n \times N(I[1]) \arrow[r] & W^i_f \arrow[r] & W^{i+1} \arrow[r] & \widetilde{W} \end{tikzcd}$$

where W^i_f is the pushout of the first square. The larger rectangle is in fact also a pushout: by composition $\partial\Delta^n \times F(1) \xrightarrow{f} W^i \to W^{i+1} \to \cdots \to W^j$, we obtain a series of pushouts W^j_f for $j \geq i$. The commutative diagram

$$\begin{tikzcd} W^i \arrow[r] \arrow[d] & W^{i+1} \arrow[r] \arrow[d] & \cdots \arrow[r] & \widetilde{W} \\ W^i_f \arrow[r] \arrow[ur] & W^{i+1}_f \arrow[r] \arrow[ur] & \cdots \end{tikzcd}$$

shows that $\operatorname{colim} W^j_f = \operatorname{colim} W^j = \widetilde{W}$. So , we have

$$\operatorname{colim} \left(\begin{tikzcd} \partial\Delta^n \times F(1) \arrow[r, "f"] \arrow[d, "\mathrm{id}\times s^0"] & \widetilde{W} \\ \partial\Delta^n \times N(I[1]) \end{tikzcd} \right) = \operatorname*{colim}_{j\geq i} \operatorname{colim} \left(\begin{tikzcd} \partial\Delta^n \times F(1) \arrow[r, "f"] \arrow[d, "\mathrm{id}\times s^0"] & \widetilde{W}^j \\ \partial\Delta^n \times N(I[1]) \end{tikzcd} \right) = \operatorname*{colim}_{j\geq i} W^j_f = \widetilde{W}.$$

Thus, $\widetilde{W}$ is a complete Segal space by Prop. 2.3.7(iii).

We need to prove that $j : W \to \widetilde{W}$ is initial among all maps from W to a complete Segal space. Given any map $G^0 : W^0 \to Y$ where Y is complete, we note that $G^0(W_{\mathrm{hoequiv}}) \subset Y_{\mathrm{hoequiv}}$. The pushout squares (2.3.10) (which are also homotopy pushouts) and Prop. 2.3.7(iii) thus give us the homotopy squares

$$\begin{tikzcd} \coprod_{f\in D^i}(\partial\Delta^n \times F(1)) \arrow[r, "\coprod f"] \arrow[d, "\coprod(\mathrm{id}\times s^0)"] & W^i \arrow[r, "G^i"] \arrow[d] & Y \arrow[d, "\mathrm{id}"] \\ \coprod_{f\in D^i}(\partial\Delta^n \times N(I[1])) \arrow[r] & W^{i+1} \arrow[r, "G^{i+1}"] & Y \end{tikzcd} .$$

We hence obtain a unique (up to isomorphism) $G^\infty = \operatorname{colim} G^i : \widetilde{W} = \operatorname{colim} W^i \to Y$ satisfying $G^\infty \circ j = G^0$.

The construction can easily be seen to be functorial (this is true of all constructions using the small object argument). □

To conclude this section, we will construct a variant of the completion functor for Reedy-fibrant Segal spaces.

Definition 2.3.11. Let W be a Segal space, a **Reedy-fibrant Segal completion** of W is a Reedy-fibrant complete Segal space $\widehat{W}$ with a map $W \to \widehat{W}$ which is initial among all maps from W to Reedy-fibrant complete Segal spaces.

The Reedy-fibrant Segal completion $\widehat{W}$ is a Reedy-fibrant replacement of the Segal completion $\widetilde{W}$, and in Sec. 2.4, we show that it is Dwyer-Kan equivalent to W, and hence so is $\widetilde{W}$.

Proposition 2.3.12. *There exists a functorial Reedy-fibrant Segal completion $W \to \widehat{W}$.*

Proof. First, let $R(W)$ be a Reedy-fibrant replacement of W. Prop. 2.3.7(ii) applied to a Reedy-fibrant Segal space X implies that X is complete if and only if $X_0 \xrightarrow{\sim} \mathrm{Map}_{\mathbf{1\text{-}sSp}}(E, X)$. This implies that X is complete if and only if

$$\begin{tikzcd}
\partial\Delta^n \times E \arrow[r, "f"] \arrow[d, "\mathrm{id}\times s^0"] & X \arrow[d, "\mathrm{id}"] \\
\partial\Delta^n \times F(0) \arrow[r] & X
\end{tikzcd}$$

is a homotopy pushout for all maps $f : \Delta^n \times E \to X$.

The map $E \to F(0)$ can be decomposed into a cofibration followed by a trivial fibration

$$E \hookrightarrow N(I(1)) \to F(0). \tag{2.3.11}$$

Using the small object argument as in the proof of Prop. 2.3.10, we obtain $j : R(W) \to \widehat{W}$ which is initial among all maps W to Reedy-fibrant complete Segal spaces. $\square$

Note that the Segal completion and Reedy-fibrant Segal completion functors are unique by the universal properties.

2.4 Segal space model structures and equivalences

We will now define the Segal space and complete Segal space model structures. We construct them as a localisation of the Reedy model category structure.

First, we present some background on localisation.

Definition 2.4.1. Let $\mathcal{M}$ be a model category, and S a set of morphisms in $\mathcal{M}$. An object W is **S-local** if it is fibrant and for every map $f : A \to B$ in S, the induced map of function complexes

$$\mathrm{Map}_{\mathcal{M}}(f, W) : \mathrm{Map}_{\mathcal{M}}(B, W) \to \mathrm{Map}_{\mathcal{M}}(A, W)$$

is a weak equivalence of simplicial sets.

A morphism $g : X \to Y$ in $\mathcal{M}$ is a **S-local equivalence** if for every S-local object W, the induced map of simplicial sets

$$\mathrm{Map}_{\mathcal{M}}(g, W) : \mathrm{Map}_{\mathcal{M}}(Y, W) \to \mathrm{Map}_{\mathcal{M}}(X, W)$$

is a weak equivalence of simplicial sets.

Definition 2.4.2. The **left Bousfield localisation** of $\mathcal{M}$ with respect to S is a set of data $L_S\mathcal{M}$ consisting of the underlying category $\mathcal{M}$ and the following classes of morphisms:

(i) the class of weak equivalences is the class of S-local equivalences of $\mathcal{M}$;

(ii) the class of cofibrations is the class of cofibrations of $\mathcal{M}$;

(iii) the class of fibrations is the class of maps with the RLP with respect to all cofibrations which are also S-local equivalences.

$L_S\mathcal{M}$ as defined above, in general, does not satisfy the model category axioms. However, when it is, it corresponds to the usual notion of "localisation", in the following sense.

Theorem 2.4.3 ([Hir03, Thm. 3.3.19])**.** *Let $\mathcal{M}$ be a model category and S a set of morphisms in $\mathcal{M}$. Suppose $L_S\mathcal{M}$ is a model category. Then, the identity map $j : \mathcal{M} \to L_S\mathcal{M}$ is a* **left localisation** *of $\mathcal{M}$ with respect to S, that is, for any model category $\mathcal{N}$ and left Quillen functor $F : \mathcal{M} \to \mathcal{N}$ such that*

$$LF : \mathrm{Ho}\,(\mathcal{M}) \to \mathrm{Ho}\,(\mathcal{N})$$

sends S to invertible morphisms in $\mathrm{Ho}\,(\mathcal{N})$, F factors through j.

The main existence theorem for left Bousfield localisation can be stated as follows:

Theorem 2.4.4. *Let $\mathcal{M}$ be a left proper cellular model category (e.g. $\mathcal{M} = 1\text{-}\mathbf{sSp}$). Then, for any set of morphisms S in $\mathcal{M}$, $L_S\mathcal{M}$ is a left proper cellular model category. The fibrant objects of $L_S\mathcal{M}$ are precisely the S-local objects of M. A weak equivalence in $\mathcal{M}$ is an S-local equivalence. Conversely, an S-local equivalence between S-local objects is a weak equivalence in $\mathcal{M}$.*

Proof. See [Hir03]. □

Note that left Bousfield localisation does not preserve the property of compatibility with Cartesian closure. For simplicial spaces, Rezk gave a simple criterion to check if the localisation is compatible.

Proposition 2.4.5 ([Rez01, Prop. 9.2])**.** *Let S be a set of morphisms in $1\text{-}\mathbf{sSp}$. Suppose for each S-local object W, $W^{F(1)}$ is also an S-local object. Then, $L_S(1\text{-}\mathbf{sSp})$ is compatible with Cartesian closure.*

Theorem 2.4.6. *There exists a model category structure on $1\text{-}\mathbf{sSp}$ with the following properties:*

(i) The cofibrations are the Reedy cofibrations.

(ii) The weak equivalences are maps f such that $\mathrm{Map}_{1\text{-}\mathbf{sSp}}(f, W)$ is a weak equivalence for all Reedy-fibrant Segal spaces W.

(iii) The fibrations are the maps that satisfy the right lifting property with respect to all trivial cofibrations.

This is called the **Segal space model category structure** *on $1\text{-}\mathbf{sSp}$, and is denoted as $1\text{-}\mathcal{SS}$. The fibrant objects are precisely the Reedy-fibrant Segal spaces. A Reedy weak equivalence between two objects X, Y is a weak equivalence in $1\text{-}\mathcal{SS}$ and the converse is true if X, Y are Reedy-fibrant Segal spaces.*

For $\mathbf{Sp} = \mathbf{CGHaus}$ or $\mathbf{sSet}$, this model structure is compatible with the Cartesian closure.

Proof. $1\text{-}\mathcal{SS}$ is obtained as the left Bousfield localisation of the Reedy model structure on $1\text{-}\mathbf{sSp}$ with respect to the set of maps

$$S = \{G(k) \to F(k) | k \in \mathbb{N}\}.$$

By (2.2.6), we see that the S-local objects are precisely the Reedy-fibrant Segal spaces. Apply Theorem 2.4.4.

For the proof of compatibility with Cartesian closure, refer to [Rez01, Sec. 10]. □

Theorem 2.4.7. *There exists a model category structure on $1\text{-}\mathbf{sSp}$ with the following properties:*

(i) The cofibrations are the Reedy cofibrations.

(ii) The weak equivalences are maps f such that $\mathrm{Map}_{1\text{-}\mathbf{sSp}}(f, W)$ is a weak equivalence for all Reedy-fibrant complete Segal spaces W.

(iii) The fibrations are the maps that satisfy the right lifting property with respect to all trivial cofibrations.

This is called the **complete Segal space model category structure** *on* 1-**sSp**, *and is denoted as* 1-$\mathcal{CSS}$. *The fibrant objects are precisely the Reedy-fibrant complete Segal spaces. A Reedy weak equivalence between two objects X, Y is a weak equivalence in* 1-$\mathcal{CSS}$ *and the converse is true if X, Y are Reedy-fibrant complete Segal spaces.*

For **Sp** = **CGHaus** *or* **sSet**, *this model structure is compatible with the Cartesian closure.*

Proof. 1-$\mathcal{CSS}$ is obtained as the left Bousfield localisation of the Reedy model structure on 1-$\mathcal{SS}$ with respect to the map $f : E \to F(0)$. By Cor. 2.3.7, we see that the f-local objects are precisely the Reedy-fibrant complete Segal spaces. Apply Theorem 2.4.4.

For the proof of compatibility with Cartesian closure, refer to [Rez01, Sec. 12]. □

Definition 2.4.8. We will denote by 1-**SeSp** and 1-**CSeSp** the full subcategories of Segal spaces and complete Segal spaces respectively, endowed with the *complete* Segal space model structure.

Corollary 2.4.9. *Let W be a (complete) Segal space that is exponential (i.e. the internal hom-object W^X exists for all X), then for any simplicial space X, W^X is a (complete) Segal space.*

Proof. First, suppose W is Reedy-fibrant and X is Reedy-cofibrant, so W and X are fibrant and cofibrant objects in 1-$\mathcal{SS}$ (resp., 1-$\mathcal{CSS}$) respectively. By the compatibility with Cartesian closure, W^X is also a fibrant object in 1-$\mathcal{SS}$ (1-$\mathcal{CSS}$), hence a complete Segal space.

In general, let $W \to R(W)$ be a Reedy-fibrant replacement of W and $Q(X) \to X$ be a Reedy-cofibrant replacement of X, so they are Reedy weak equivalences. Hence, $W^X \to R(W)^{Q(X)}$ is a Reedy weak equivalence, hence W^X is a (complete) Segal space. □

We will now present some results on equivalences of Segal spaces. Recall that aside from the weak equivalences in the various model category structures, we have also introduced Dwyer-Kan equivalence. We summarise the results that Rezk proved in [Rez01] regarding the relationships between these equivalences, while providing slightly simpler proofs and generalisations to non-Reedy fibrant Segal spaces.

Lemma 2.4.10. *A Reedy weak equivalence of Segal spaces (as objects in* 1-**sSp**) *is a Dwyer-Kan equivalence.*

Proof. Let $f : U \to V$ be a Reedy weak equivalence of Segal spaces and $\operatorname{Ho} f : \operatorname{Ho} U \to \operatorname{Ho} V$ the induced functor on the homotopy categories. We want to show that $\operatorname{Ho} f$ is essentially surjective. For any $v \in \operatorname{Ob} \operatorname{Ho} V = \operatorname{Ob} V$, since $\pi_0 f_0 : \pi_0 U_0 \to \pi_0 V_0$ is a bijection, there exists $u \in \operatorname{Ob} U$ such that $f(u)$ and v are in the same path component of V_0. Hence, $\operatorname{map}_V(f(u), v) \sim \operatorname{map}_V(v, v) \sim \operatorname{map}_V(v, f(u))$ are weakly homotopic in V_1. Let $f \in \operatorname{map}_V(f(u), v)$ and $g \in \operatorname{map}_V(v, f(u))$ be lifts of id_v and F and G be the homotopies from f to id_v and g to id_v respectively. Using the weak equivalence $V_2 \xrightarrow{\sim} V_1 \times V_1$, we get the composition $F \circ G = d_1(\text{lift of } (G, F))$ which is a homotopy in V_1 from $f \circ g$ to id_v. Hence, $f \circ g = \mathrm{id}_v$ in Ho_V. Similarly, $g \circ f = \mathrm{id}_{f(u)}$, $f(u) \cong v$ in $\operatorname{Ho} V$.

For any $x, y \in U$, we have

$$\operatorname{map}_U(x, y) = \operatorname{holim}((x, y) \to U_0 \times U_0 \leftarrow U_1) \xrightarrow{\sim} \operatorname{holim}((f(x), f(y)) \to V_0 \times V_0 \leftarrow V_1) = \operatorname{map}_V(f(x), f(y))$$

since $f : U \to V$ is a degree-wise weak equivalence. Hence, f is a Dwyer-Kan equivalence. □

Proposition 2.4.11. *Let $f : U \to V$ be a morphism between two Reedy-fibrant complete Segal spaces. Then, the following are equivalent:*

(i) f is a Reedy weak equivalence;

(ii) f is a weak equivalence in 1-$\mathcal{SS}$;

(iii) f is a weak equivalence in 1-$\mathcal{CSS}$;

(iv) f is a Dwyer-Kan equivalence;

Proof. (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii) is a consequence of the Bousfield localisation (Thm. 2.4.4) and that U and V are fibrant objects in 1-$\mathcal{CSS}$.

(i)$\Rightarrow$(iv) is proven in Lemma 2.4.10.

(iv)$\Rightarrow$(i): Suppose f is a Dwyer-Kan equivalence between complete Segal spaces. Condition (i') of Def. 2.2.14 and Prop. 2.3.8, we have a bijection

$$\pi_0 U_0 \cong \mathrm{Ob}\, U/\sim \to \pi_0 V_0 \cong \mathrm{Ob}\, V/\sim .$$

Since for any base point $x \in \mathrm{Ob}\, x$, $\mathrm{map}(x,x) \to \mathrm{map}(fx, fx)$ is a weak equivalence, we have $U_0 \to V_0$ is a weak equivalence. Consider the homotopy diagram

$$\begin{tikzcd}
\mathrm{map}(x,y) \arrow[rr] \arrow[dr] \arrow[dd] & & \{(x,y)\} \arrow[dr] \arrow[dd] & \\
& U_1 \arrow[rr] \arrow[dd] & & U_0 \times U_0 \arrow[dd] \\
\mathrm{map}(fx,fy) \arrow[rr] \arrow[dr] & & \{(fx,fy)\} \arrow[dr] & \\
& V_1 \arrow[rr] & & V_0 \times V_0
\end{tikzcd}$$

The back square is a homotopy pullback since $\mathrm{map}(x,y) \to \mathrm{map}(fx, fy)$ is a weak equivalence. Since the top and bottom squares are also homotopy pullbacks, so is the front square. Hence, $U_0 \times U_0 \to V_0 \times V_0$ being a weak equivalence implies that $U_1 \to V_1$ is a weak equivalence. Weak equivalences in the higher degrees follow from the Segal condition. $\square$

Corollary 2.4.12. *Let $f : U \to V$ be a morphism between two complete Segal spaces. All the equivalences in Prop. 2.4.11 are the same:*

$$\textit{Dwyer-Kan equivalence} \Longleftrightarrow \textit{Reedy weak equivalence} \Longleftrightarrow$$
$$\textit{Weak equivalence in } 1\text{-}\mathcal{SS} \Longleftrightarrow \textit{Weak equivalence in } 1\text{-}\mathcal{CSS}$$

Proof. Recall that there is a Reedy-fibrant replacement functor $R : 1\text{-}\mathbf{sSp} \to 1\text{-}\mathbf{sSp}$ which preserves Segal spaces and complete Segal spaces and such that $X \to R(X)$ is a Reedy weak equivalence for all simplicial spaces X. Since Reedy weak equivalences are weak equivalences in 1-$\mathcal{SS}$ and 1-$\mathcal{CSS}$, the commutative diagram

$$\begin{tikzcd}
U \arrow[r, "f"] \arrow[d] & V \arrow[d] \\
R(U) \arrow[r, "R(f)"] & R(V)
\end{tikzcd}$$

implies that a map $f : U \to V$ of Segal spaces is a Reedy weak equivalence (or weak equivalence in 1-$\mathcal{SS}$ or 1-$\mathcal{CSS}$) if and only if $R(f)$ is a Reedy weak equivalence (or weak equivalence in 1-$\mathcal{SS}$ or 1-$\mathcal{CSS}$, respectively).

Similarly, by Lemma 2.4.10, Reedy weak equivalences are Dwyer-Kan equivalences, so the vertical arrows are Dwyer-Kan equivalences. Hence, $f : U \to V$ is a Dwyer-Kan equivalence if and only if $R(f)$ is. $\square$

Some of these equivalences can be generalised to Segal spaces.

Theorem 2.4.13. *Let $\mathbf{Sp} = \mathbf{sSet}$ or $\mathbf{CGHaus}$ and $f : U \to V$ be a morphism between two Segal spaces. f is a Dwyer-Kan equivalence if and only if f is a weak equivalence in 1-$\mathcal{CSS}$.*

Proof. The functorial Segal completion gives us a diagram

$$\begin{tikzcd}
U \arrow[r, "f"] \arrow[d] & V \arrow[d] \\
\widetilde{U} \arrow[r, "\tilde{f}"] & \widetilde{V}
\end{tikzcd}.$$

The vertical arrows are weak equivalences in 1-$\mathcal{CSS}$ and Dwyer-Kan equivalences by Cor. 2.4.16 below. Hence, f is a weak equivalence in 1-$\mathcal{CSS}$ (Dwyer-Kan equivalence, respectively) if and only if $\tilde{f}$ is. The result then follows from Cor. 2.4.12. ☐

We now show that the completion functor is a weak equivalence in 1-$\mathcal{CSS}$ and a Dwyer-Kan equivalence.

Lemma 2.4.14. *The inclusion $E \to N(I[1])$ is a weak equivalence in* 1-$\mathcal{CSS}$ *and a Dwyer-Kan equivalence.*

Proof. Recall that $N(I[1]) \to F(0) = N([0])$ is a Reedy weak equivalence since $[0] \to I[1]$ is an equivalence of categories. So, it suffices to show that $E \to F(0)$ is a weak equivalence in 1-$\mathcal{CSS}$, but this is true since 1-$\mathcal{CSS}$ is defined to be the Bousfield localisation of 1-$\mathcal{SS}$ with respect to this map.

That it is a Dwyer-Kan equivalence is a consequence of Prop. 2.2.16 since $E = \operatorname{discnerve} I[1]$. ☐

Proposition 2.4.15. *Let* $\mathbf{Sp} = \mathbf{sSet}$ *or* $\mathbf{CGHaus}$. *The Reedy fibrant Segal completion $W \to \widehat{W}$ is a weak equivalence in* 1-$\mathcal{CSS}$ *and a Dwyer-Kan equivalence.*

Proof. First, let us assume that pushout squares which are also homotopy pushouts send Dwyer-Kan equivalences to Dwyer-Kan equivalences, and that the filtrant colimit of a diagram of Dwyer-Kan equivalences is a Dwyer-Kan equivalence.

We will show that the construction by small object argument given in Prop. 2.3.12 is a weak equivalence in 1-$\mathcal{CSS}$ (Dwyer-Kan equivalence, respectively). By Lemma 2.2.16, $E = \operatorname{discnerve} I[1] \to N(I[1])$ is a weak equivalence in 1-$\mathcal{CSS}$ and Dwyer-Kan equivalence. Since the push out squares are also homotopy pushouts, they preserve weak equivalences (Dwyer-Kan equivalences). $W^i \to W^{i+1}$ is a weak equivalence (Dwyer-Kan equivalences) for all i. Hence, $W \to \widehat{W} = \operatorname{colim} W^i$ is a weak equivalence (Dwyer-Kan equivalences).

It now remains to prove the two claims. In both cases, the corresponding result is clear with homotopy categories and equivalences of categories. For $\mathbf{Sp} = \mathbf{sSet}$ or $\mathbf{CGHaus}$, they are locally finitely presentable categories, and so is 1-$\mathbf{sSp} = \mathbf{Fun}(\Delta^{\mathrm{op}}, \mathbf{Sp})$. Hence, filtrant colimits commute with finite limits (in particular, pullbacks). This implies that given a filtrant colimit $f = \operatorname{colim} f_i$, the filtrant colimit of $\operatorname{map}(f_i x, f_i y)$ is $\operatorname{map}(fx, fy)$. The result now follows since weak equivalences are preserved by homotopy pushouts and filtrant colimits. ☐

Corollary 2.4.16. *Let* $\mathbf{Sp} = \mathbf{sSet}$ *or* $\mathbf{CGHaus}$. *The Segal completion $W \to \widetilde{W}$ is a weak equivalence in* 1-$\mathcal{CSS}$ *and a Dwyer-Kan equivalence.*

Proof. By the universal properties, there exists a unique map $\widetilde{W} \to \widehat{W}$ and $\widehat{\widetilde{W}} = \widehat{W}$. In the diagram

$$\begin{tikzcd} W \arrow[r] \arrow[rr, bend right, "\sim"'] & \widetilde{W} \arrow[r, "\sim"] & \widehat{W} \end{tikzcd},$$

since two of the maps are weak equivalences (Dwyer-Kan equivalences, resp.), so is the remaining. ☐

In light of the relation between the complete Segal space model structure and Dwyer-Kan equivalences, it is reasonable to take complete Segal spaces as our notion of $(\infty, 1)$-categories.

Definition 2.4.17. An $(\infty, 1)$**-category** is a complete Segal space.

Remark 2.4.18. As previously mentioned, this notion is not entirely standard. Most authors (e.g. [Rez01]) take Reedy-fibrant complete Segal spaces for $\mathbf{Sp} = \mathbf{KComp}$ as a model of $(\infty, 1)$-categories.

Finally, we will like to understand what are the ∞-groupoids in the category of $(\infty, 1)$-categories. Intuitively, we can make the following definition.

Definition 2.4.19. We say that a Segal space W is an ∞-groupoid if $\operatorname{Ho} W$ is a groupoid. W is an ∞-groupoid if and only if all 1-morphisms have homotopy inverses.

We will like to compare this definition of ∞-groupoids with other known models of ∞-groupoids, for example, the Kan complexes. To do so, we need to quickly review the notions of quasicategories (for a more detailed discussion, one can refer to, for example, [Joy08]).

A **quasicategory** is a simplicial set X satisfying the internal Kan condition, that is, for all $k > 0$ and $0 < i < k$, any map $\Lambda_i^k \to X$ can be extended to $\Delta^k \to X$. This can be given by the diagram

$$\begin{tikzcd} \Lambda_i^K \arrow[r] \arrow[d] & X \\ \Delta^k \arrow[ur, dashed, "\exists"'] & \end{tikzcd} .$$

Joyal showed that quasicategories give a good model of $(\infty, 1)$-categories. There is another model structure on **sSet**, known as the Joyal model structure, which we will denote by $\mathcal{QC}$, in which the cofibrations are monomorphisms and the weak equivalence are categorical equivalences (see [Joy08]). The fibrant-cofibrant objects are precisely the quasicategories. Indeed the standard model structure on **sSet** is the left Bousfield localisation of $\mathcal{QC}$ with respect to the outer horn inclusions. The two model category structures coincide on the full subcategory **KComp** of Kan complexes. Joyal and Tierney also showed that the quasicategory model is equivalent to the complete Segal space model (where $\mathbf{Sp} = \mathbf{sSet}$):

Theorem 2.4.20 ([JT07, Thm. 4.11]]). *There is an adjoint pair of functors*

$$p_1^* : \mathcal{QC} \to 1\text{-}\mathcal{CSS} : i_1^*$$

which is a Quillen equivalence.

Proof. See [JT07]. □

The functor p_1^* sends a simplicial set X to the discrete simplicial space given by $[k] \mapsto X_k$, while the functor i_1* sends a simplicial space Y to the simplicial set $[k] \mapsto (Y_k)_0$. The notations i_1 and p_1 refer to the fact that they are induced by the inclusion $i_1 : \Delta \to \Delta \times \Delta$ into the first component and the projection $p_1 : \Delta \times \Delta \to \Delta$ of the first component if we view simplicial spaces as bisimplicial sets (with the vertical Reedy model structure).

If X is a Kan complex, then $p_1^* X$ is a simplicial space in which all 1-morphisms have homotopy inverses by the outer horn extensions. Taking its fibrant replacement in $1\text{-}\mathcal{CSS}$, we thus get an ∞-groupoid. Conversely, if Y is an ∞-groupoid in $1\text{-}\mathcal{CSS}$, $i_1^* Y$ is a Kan complex since every 1-morphism has a homotopy inverse. Hence, we get an equivalence of categories:

Corollary 2.4.21. *The Quillen equivalence of Thm. 2.4.20 induces an equivalence of homotopy categories*

$$\mathbb{L}(R \circ p_1^*) : \mathrm{Ho}\,\mathcal{QC} \supset \mathrm{Ho}\,\mathbf{KComp} \to \mathrm{Ho}\,1\text{-}\mathbf{SeSp}_0 \subset \mathrm{Ho}\,1\text{-}\mathcal{CSS} : \mathbb{R}i_1^*$$

where R is a fibrant replacement functor in $1\text{-}\mathcal{CSS}$ and $1\text{-}\mathbf{SeSp}_0$ is the full subcategory of ∞-groupoids in $1\text{-}\mathcal{CSS}$.

2.5 Segal n-spaces and complete Segal n-spaces

In this section, we will generalise the notions defined in the previous sections to give a suitable construction for (∞, n) categories. Most of the work will proceed by induction on the $(\infty, 1)$ category case.

Definition 2.5.1. A **simplicial n-space** (or n-fold simplicial space) is a simplicial object in the category $(n-1)$-**sSp** of simplicial $(n-1)$-spaces, that is, a functor $X : \Delta^{\mathrm{op}} \to (n-1)\text{-}\mathbf{sSp}$. Let $X_k = X([k])$ denote the k-th simplicial $(n-1)$-space, with the usual face and degeneracy maps d_i and s_i. The simplicial n-space can also be viewed as a functor $X : (\Delta^{\mathrm{op}})^n \to \mathbf{Sp}$. Let $X_{k_1,\dots,k_n} = X([k_1], \dots, [k_n])$, with the order of the indices such that $X_k = X([k]) = X(\{[k]\} \times \Delta^{\mathrm{op}} \times \cdots \times \Delta^{\mathrm{op}}) = X_{k,\bullet,\dots,\bullet}$.

The category of simplicial n-spaces is the functor category

$$n\text{-}\mathbf{sSp} = \mathbf{Fun}\,(\Delta^{\mathrm{op}}, (n-1)\text{-}\mathbf{sSp}) \cong \mathbf{Fun}\,((\Delta^{\mathrm{op}})^n, \mathbf{Sp}).$$

There is an embedding $\mathbf{Sp} \hookrightarrow n\text{-}\mathbf{sSp}$ that takes a space X to the **constant simplicial n-space** X where $X_{k_1,\ldots,k_n} = X$ for all $k_1,\ldots,k_n \geq 0$. For $r < n$, we can also embed $r\text{-}\mathbf{sSp} \hookrightarrow n\text{-}\mathbf{sSp}$ as a **r-constant simplicial n-space** by mapping a simplicial r-space X to the simplicial n-space, also called X, where $X_{k_1,\ldots,k_{n-r},\bullet,\ldots,\bullet} = X$.

We say that X is a **discrete** simplicial n-space if X_k is a constant simplicial $(n-1)$-space for all k. We say that X is **r-discrete** if X_k is an $(r-1)$-discrete simplicial $(n-1)$-space, that is, $X_{k_1,\ldots,k_r,\bullet,\ldots,\bullet}$ is a constant simplicial $(n-r)$-space.

The **function complex** from X to Y is the simplicial set $\mathcal{M}(X,Y)$ where

$$\mathcal{M}(X,Y)_k = \mathrm{Hom}_{n\text{-}\mathbf{sSp}}(X \times \Delta^k, Y) \text{ or } \mathrm{Hom}_{n\text{-}\mathbf{sSp}}(X \times |\Delta^k|, Y)$$

where Δ^k ($|\Delta^k|$ for $\mathbf{Sp} = \mathbf{Top}$ or $\mathbf{CGHaus}$) is the (geometric) k-simplex taken as a constant simplicial n-space.

For each $0 \leq r \leq n$, we can define a notion of **r-mapping space**, which is a simplicial r-space (simplicial 0-space = space).

For $r = 0$, as in the $n = 1$ case, we define $\mathrm{Map}^0_{n\text{-}\mathbf{sSp}}(X,Y) = |\mathcal{M}(X,Y)|$ for $\mathbf{Sp} = \mathbf{Top}$ or $\mathbf{CGHaus}$ and $\mathrm{Map}^0_{n\text{-}\mathbf{sSp}}(X,Y) = \mathcal{M}(X,Y)$ if $\mathbf{Sp} = \mathbf{KComp}$ or $\mathbf{sSet}$.

For $r \geq 1$, let $F_r(k)$ be the discrete simplicial r-space where

$$F_r(k)_l = \mathrm{Hom}_\Delta([l],[k]).$$

We can further regard $F_r(k)$ as a r-constant simplicial n-space for any $n \geq r$. Explicitly,

$$F_r(k)_{l_1,\ldots,l_n} = \mathrm{Hom}_\Delta([l_{n-r+1}],[k]).$$

We have $F_1(k) = F(k)$ as defined in Section 2.1. We can also set $F_0(k) = \Delta^k$ or $|\Delta^k|$. Note that $F_r(k)$ is generated by $\mathrm{id}_{[k]} \in F_r(k)_k$.

We can construct the r-mapping space by induction on r: the k-simplicial $(r-1)$-space of $\mathrm{Map}^r_{n\text{-}\mathbf{sSp}}(X,Y)$ is defined to be

$$\mathrm{Map}^r_{n\text{-}\mathbf{sSp}}(X,Y)_k = \mathrm{Map}^{r-1}_{n\text{-}\mathbf{sSp}}(X \times F_r(k), Y)$$

For $r = n$, we obtain the internal hom-object $Y^X = \mathrm{Map}^n_{n\text{-}\mathbf{sSp}}(X,Y)$.

Alternatively, we see that for $r < n$,

$$\mathrm{Map}^r_{n\text{-}\mathbf{sSp}}(X,Y)_{k_1,\ldots,k_r} = \mathrm{Map}^0_{n\text{-}\mathbf{sSp}}(X \times F_r(k_1) \times \cdots \times F_1(k_r), Y).$$

For simplicity, write $F^r_n(k_1,\ldots,k_r)$ for $F_n(k_1) \times \cdots \times F_{n-r+1}(k_r)$. Let $\mathbf{F}^r_n$ be the category consisting of objects $F^r_n(k_1,\ldots,k_r)$.

We have

$$\mathrm{Map}^0_{n\text{-}\mathbf{sSp}}(F^n_n(k_1,\ldots,k_n), Y) = \mathrm{Hom}_{n\text{-}\mathbf{sSp}}(\{\mathrm{id}_{[k_1]}\} \times \cdots \times \{\mathrm{id}_{[k_n]}\}, Y_{k_1,\ldots,k_n}) \cong Y_{k_1,\ldots,k_n}.$$

Hence,

$$\begin{aligned} \mathrm{Map}^{n-1}_{n\text{-}\mathbf{sSp}}(F_n(k),Y) &= (\mathrm{Map}^0_{n\text{-}\mathbf{sSp}}(F_n(k) \times F^{n-1}_n(k_1,\ldots k_{n-1}), Y))_{k_1,\ldots,k_{n-1}} \\ &\cong (Y_{k,k_1,\ldots,k_{n-1}})_{k_1,\ldots,k_{n-1}} = Y_k. \end{aligned}$$

Let $\partial F_n(k)$ be the largest simplicial subspace of $F_n(k)$ not containing $\mathrm{id}_{[k]}$.

As for $n = 1$, there is a Reedy model category structure on $n\text{-}\mathbf{sSp}$.

Theorem 2.5.2. *There exists a model category structure on $n\text{-}\mathbf{sSp} = \mathbf{Fun}(\Delta^{\mathrm{op}}, (n-1)\text{-}\mathbf{sSp})$, called the* **Reedy model structure***, where $f : X \to Y$ is a*

(i) weak equivalences if $f_k : X_k \to Y_k$ are degree-wise weak equivalences (in $(n-1)$-$\mathbf{sSp}$);

(ii) *cofibrations if the induced maps*

$$X_k \coprod_{\bigcup s_i X_{k-1}} (\bigcup s_i Y_{k-1}) \to Y_k$$

are cofibrations in $(n-1)$-**sSp***;*

(iii) *fibrations if the induced maps*

$$X_k \to Y_k \times_{\partial Y_k} \partial X_k$$

are fibrations in $(n-1)$-**sSp***.*

n-**sSp** *is Cartesian closed and the Reedy model structure is compatible with Cartesian closure and proper. It is cofibrantly generated and cellular.*

Note that this is equivalent to the Reedy model structure obtained regarding n-**sSp** as $\mathbf{Fun}\left((\Delta^{\mathrm{op}})^n, \mathbf{Sp}\right)$ since they have the same weak equivalences and fibrations.

As for $n = 1$, any simplicial space can be written as the colimit of some "standard simplices".

Proposition 2.5.3. *Let* $\mathbf{Sp} = \mathbf{KComp}$ *or* $\mathbf{CW}$*, then for any simplicial* n*-space* X*, we have*

$$X \cong \operatorname*{colim}_{(F_n^n(k_1,\ldots,k_n)\times\Delta^l \to X)\in \mathbf{F}_n^n \downarrow X} F_n^n(k_1, \ldots, k_n) \times \Delta^l.$$

If X *is* r*-discrete, then*

$$X \cong \operatorname*{colim}_{(F_n^r(k)\to X)\in \mathbf{F}_n^r \downarrow X} F_n^r(k).$$

The notion of mapping spaces $\mathrm{Map}^r_{n\text{-}\mathbf{sSp}}(X, Y)$ is not homotopy invariant if Y is not Reedy-fibrant. We can use the decomposition of a simplicial space to define a homotopy invariant version of the mapping space.

Definition 2.5.4. Let X be a simplicial space with $\mathbf{Sp} = \mathbf{KComp}$ or $\mathbf{CW}$ and Y be any simplicial space, then the **homotopy** s**-mapping space** is defined to be

$$\mathrm{HoMap}^s_{n\text{-}\mathbf{sSp}}(X, Y) = \operatorname*{holim}_{(F_n^n(k_1,\ldots,k_n)\times\Delta^l \to X)} \mathrm{Map}^s_{n\text{-}\mathbf{sSp}}(F_n^n(k_1, \ldots, k_n) \times \Delta^l, Y).$$

If X is r-discrete, the homotopy $(n-r)$-mapping space is

$$\mathrm{HoMap}^{n-r}_{n\text{-}\mathbf{sSp}}(X, Y) = \operatorname*{holim}_{(F_n^r(k_1,\ldots,k_r)\to X)} \mathrm{Map}^{n-r}_{n\text{-}\mathbf{sSp}}(F_n^r(k_1, \ldots, k_r), Y) \xrightarrow{\sim} \operatorname*{holim}_{F_n^r(k_1,\ldots,k_r)\to X} Y_{k_1,\ldots,k_r,\bullet,\ldots,\bullet}.$$

We are now ready to define Segal n-spaces.

Definition 2.5.5. A simplicial n-space X is **essentially constant** if there exists a weak equivalence $X \to X'$ where X' is a constant simplicial n-space.

Definition 2.5.6. A **Segal** n**-space** W is a Reedy-cofibrant simplicial n-space satisfying

(i) W_k are Segal $(n-1)$-spaces;

(ii) for all $k, l \in \mathbb{N}$, the square

$$\begin{tikzcd} W_{k+l} \arrow[r] \arrow[d] & W_k \arrow[d, "t"] \\ W_l \arrow[r, "s"] & W_0 \end{tikzcd}$$

is a homotopy pullback in the Reedy model category structure on $(n-1)$-**sSp**; and

(iii) W_0 is essentially constant.

The category of Segal n-spaces n-**SeSp** is a full subcategory of n-**sSp**.

We will sometimes call a Reedy-cofibrant simplicial n-space satisfying (i) and (ii) but not (iii) a **pre-Segal n-space**.

As in the $n = 1$ case, the Segal condition can be reformulated as

$$W_k \xrightarrow{\sim} \mathrm{holim}(W_1 \xrightarrow{d_0} W_0 \xleftarrow{d_1} W_1 \xrightarrow{d_0} \cdots \xrightarrow{d_0} W_0 \xleftarrow{d_1} W_1)$$

is a weak equivalence or equivalently

$$\mathrm{HoMap}^{n-1}_{n\text{-}\mathbf{sSp}}(F_n(k), W) \xrightarrow{\sim} \mathrm{HoMap}^{n-1}_{n\text{-}\mathbf{sSp}}(G_n(k), W)$$

is a weak equivalence, where

$$G_n(k) = \cup_{i=0}^{k-1} \alpha^i F_n(1) \subset F_n(k).$$

The essentially constant condition can also be reformulated as a weak equivalence

$$\mathrm{HoMap}^{0}_{n\text{-}\mathbf{sSp}}(F^n_n(0, k_2, \ldots, k_n), W) \xrightarrow{\sim} \mathrm{HoMap}^{0}_{n\text{-}\mathbf{sSp}}(F^n_n(0, \ldots, 0), W).$$

for each $(k_2, \ldots, k_n)$.

We can unfold Def. 2.5.6 to give a precise description of a Segal n-space in terms of Segal spaces:

Definition 2.5.6A. A **Segal n-space** is a Reedy-cofibrant simplicial n-space $W : (\Delta^{\mathrm{op}})^n \to \mathbf{Sp}$ satisfying

(i) for all $1 \leq i \leq n$ and $k_1, \ldots, k_{i-1}, k_{i+1}, \ldots, k_n$, $W_{k_1,\ldots,k_{i-1},\bullet,k_{i+1},\ldots,k_n}$ is a Segal space; and

(ii) for all $1 \leq i \leq n$ and $k_1, \ldots, k_{i-1}$, $W_{k_1,\ldots,k_{i-1},0,\bullet,\ldots,\bullet}$ is essentially constant.

As before, Def. 2.5.6A can be rephrased using (2.2.6)

(i) for all $1 \leq i \leq n$ and n-uple $(k_1, \ldots, k_n)$, there is a weak equivalence

$$\begin{aligned}&\mathrm{HoMap}^{0}_{n\text{-}\mathbf{sSp}}(F^n_n(k_1, \ldots, k_n), W) \to \\ &\qquad \mathrm{HoMap}^{0}_{n\text{-}\mathbf{sSp}}(F^{i-1}_n(k_1, \ldots, k_{i-1}) \times G_{n-i}(k_i) \times F^{n-i}_{n-i}(k_{i+1}, \ldots, k_n), W);\end{aligned}$$

(ii) for all $1 \leq i \leq n$ and $(n-1)$-uple $(k_1, \ldots, k_{i-1}, k_{i+1}, \ldots, k_n)$, there is a weak equivalence

$$\mathrm{HoMap}^{0}_{n\text{-}\mathbf{sSp}}(F^n_n(k_1, \ldots, k_{i-1}, 0, k_i, \ldots, k_n), W) \to \mathrm{HoMap}^{0}_{n\text{-}\mathbf{sSp}}(F^n_n(k_1, \ldots, k_{i-1}, 0, \ldots, 0), W).$$

We will now define the homotopy category associated to a Segal n-space.

Definition 2.5.7. Let W be a Segal n-space. Let $\mathrm{Ob}\, W$ be the points in $W_{0,\ldots,0}$. For any $x_0, \ldots, x_k \in \mathrm{Ob}_W$, let $\mathrm{map}^1_W(x_0, \ldots, x_k)$ be the homotopy fibre of the map of simplicial $(n-1)$-spaces $W_k \to (W_0)^{k+1}$ at $(x_0, \ldots, x_k)$. $\mathrm{map}^1_W(x_0, \ldots, x_k)$ is a Segal $(n-1)$-space since, using Def. 2.5.6A, we can easily check that the weak equivalences are preserved under homotopy pullbacks. By the Segal condition, we have the weak equivalence

$$\mathrm{map}^1_W(x_0, \ldots, x_k) \xrightarrow{\sim} \mathrm{map}^1_W(x_0, x_1) \times \ldots \times \mathrm{map}^1_W(x_{k-1}, x_k).$$

Inductively, for $1 \leq r \leq n$ and $f_0, \ldots, f_l \in \mathrm{map}^{r-1}_W(x_0, \ldots, x_k)$, let $\mathrm{map}^r_W(f_0, \ldots, f_l)$ be the homotopy fibre of the map of simplicial $(n-r)$-spaces

$$\mathrm{map}^{r-1}_W(x_1, \ldots, x_k)_l \to (\mathrm{map}^{r-1}_W(x_1, \ldots, x_k)_0)^{l+1}.$$

A 1-morphism is an object $f : x \to y$ in $\mathrm{map}^1_W(x, y)$ for objects $x, y \in \mathrm{Ob}\, W$. An r-morphism $f : x \to y$ is an object in $\mathrm{map}^r_W(x, y)$ where x and y are $(r-1)$-morphisms. Two r-morphisms $f, g : x \to y$ are **homotopic** if they lie in the same component of $\mathrm{map}^r(x, y)_{0,\ldots,0}$.

Given two r-morphisms $f : x \to y$ ans $g : y \to z$, let k be a lift of (f, g) under the weak equivalence

$$\mathrm{map}^r_W(x, y, z) \xrightarrow{\sim} \mathrm{map}^r_W(x, y) \times \mathrm{map}^r_W(y, z). \tag{2.5.12}$$

The composition $g \circ f$ is defined to be $d_1 k \in \mathrm{map}^r_W(x, z)$.

Proposition 2.5.8. *Composition of r-morphisms is well-defined up to homotopy in* $\mathrm{map}^r_W(x,z)$*. Composition is associative and has identity up to homotopy.*

Proof. The proof is identical to that of Prop. 2.2.11. $\square$

Definition 2.5.9. Let W be a Segal n-space. The **homotopy 1-category** (or simply homotopy category) $\mathrm{Ho}\,W = \mathrm{Ho}_1 W$ is the category with objects $\mathrm{Ob}\,W$ and for each pair $x, y \in \mathrm{Ob}\,W$, $\mathrm{Hom}^1_W(x,y) = \pi_0 \mathrm{map}^1_W(x,y)_{0,\ldots,0}$ is the set of path components of the space $\mathrm{map}^1_W(x,y)_{0,\ldots,0}$.

We can also define a higher categorical version of homotopy categories. However, since the homotopy r-category is an example of a weak r-category and is difficult to construct, we will only define the homotopy 2-category which we will need later in studying the adjoints in a Segal n-space.

Definition 2.5.10. A **bicategory** C is the following collection of data (see [Lei98] for the complete definition):

- a class of objects $\mathrm{Ob}\,\mathcal{C}$, called **0-cells**;
- for each pair of objects A, B, a category $\mathcal{C}(A,B)$ whose objects are call **1-cells** and morphisms **2-cells**;
- for objects A, B, C, an identity functor $i_A : * \to \mathcal{C}(A,A)$ and a composition functor $\mu_{A,B,C} : \mathcal{C}(A,B) \times \mathcal{C}(B,C) \to \mathcal{C}(A,C)$;
- natural isomorphisms $\mu_{A,C,D} \circ (\mu_{A,B,C} \times \mathrm{id}_{\mathcal{C}(C,D)}) \xrightarrow{\sim} \mu_{A,B,D} \circ (\mathrm{id}_{\mathcal{C}(A,B)} \times \mu_{B,C,D})$ (associativity), $\mu_{A,B,B} \circ (\mathrm{id}_{\mathcal{C}(A,B)} \times i_B) \xrightarrow{\sim} \mathrm{id}_{\mathcal{C}(A,B)}$ and $\mu_{A,A,B} \circ (\mathrm{id}_{\mathcal{C}(A,B)} \times i_B) \xrightarrow{\sim} \mathrm{id}_{\mathcal{C}(A,B)}$ (identity) satisfying the pentagon law.

Definition 2.5.11. The **homotopy 2-category** $\mathrm{Ho}_2 W$ of a Segal n-space W is the bicategory where

- $\mathrm{Ob}\,\mathrm{Ho}_2 W = \mathrm{Ob}\,W$;
- for each pair $x, y \in \mathrm{Ob}\,W$, $\mathrm{Ho}_2 W(x,y) = \mathrm{Ho}\,\mathrm{map}^1_W(x,y)$;
- the identity functor is defined by $* \mapsto s_0 x \in \mathrm{Ho}\,\mathrm{map}^1_W(x,x)$ and the weak equivalence (2.5.12) gives an equivalence of homotopy categories and hences a functor

$$\mathrm{Ho}\,\mathrm{map}_W(x,y) \times \mathrm{Ho}\,\mathrm{map}_W(y,z) \to \mathrm{Ho}\,\mathrm{map}_W(x,y,z) \to \mathrm{Ho}\,\mathrm{map}_W(x,z);$$

- Prop. 2.5.8 gives the natural isomorphisms which we can check to satisfy the pentagon law.

We will extend the definition of Dwyer-Kan equivalences (Def. 2.2.14) inductively to Segal n-spaces.

Definition 2.5.12. A map $f : U \to V$ of Segal n-spaces is a **Dwyer-Kan equivalence** if

(i) the induced map $\mathrm{Ho}\,f : \mathrm{Ho}\,U \to \mathrm{Ho}\,V$ is an equivalence of categories; and

(ii) for each pair of objects $x, x' \in U$, the induced function $\mathrm{map}_U(x,x') \to \mathrm{map}_V(fx, fx')$ is a Dwyer-Kan equivalence of Segal $(n-1)$-spaces.

Let $\mathrm{Ob}\,U/\sim$ denote the equivalence classes of objects in U under homotopy equivalence. We can reformulate condition (i) as

(i') the induced map $\mathrm{Ob}\,U/\sim \to \mathrm{Ob}\,V/\sim$ is a bijection on the equivalence classes of objects.

The pair of conditions (i') and (ii) is equivalent to the pair (i) and (ii).

Using Δ_0, we obtain a series of definitions and results similar to before.

Definition 2.5.13. A **semisimplicial n-space** is a functor $X : (\Delta_0^{\mathrm{op}})^n \to \mathbf{Sp}$. Equivalently, for $n \geq 2$, it is a functor $X : \Delta_0^{\mathrm{op}} \to (n-1)$-**semisSp** where n-**semisSp** is the category of semisimplicial n-spaces.

A **semiSegal n-space** is a Reedy-cofibrant semisimplicial n-space $X : \Delta_0^{\mathrm{op}} \to (n-1)$-**semisSp** satisfying the conditions in Def. 2.5.6 (replacing Segal $(n-1)$-space with semiSegal $(n-1)$-space).

Proposition 2.5.14. *Let W be a simplicial n-space. Then W is a Segal n-space if and only if $W|_{(\Delta_0^{\mathrm{op}})^n}$ is a semiSegal n-space.*

As before, Segal n-spaces do not give a correct notion of (∞, n)-categories. We will like to define complete Segal n-spaces.

Definition 2.5.15. A 1-morphism $g : x \to y$ is a **homotopy equivalence** if there exist $f, h : y \to x$ such that $g \circ f \sim \mathrm{id}_y$ and $h \circ g \sim \mathrm{id}_x$. Let $W_{\mathrm{hoequiv}} \subset W_1$ be the maximal subsimplicial $(n-1)$-space with objects being homotopy equivalences between 1-morphisms. The embedding $W_0 \xrightarrow{s_0} W_1$ factors through W_{hoequiv}.

Since all homotopy equivalences g lie in $W_{1,0,\dots,0}$, we get that W_{hoequiv} is a constant simplicial $(n-1)$-space. Indeed, g is a homotopy equivalence if and only if it is a homotopy equivalence in the Segal 1-space $(W_{k,0,\dots,0})_k$. Hence, W_{hoequiv} is the constant simplicial $(n-1)$-space determined by the space $(W_{\bullet,0,\dots,0})_{\mathrm{hoequiv}}$. Thus, $W_0 \to W_{\mathrm{hoequiv}}$ is a weak equivalence if and only if $W_{0,\dots,0} \to (W_{\bullet,0,\dots,0})_{\mathrm{hoequiv}}$ is.

Definition 2.5.16. A **complete Segal n-space** W is a Segal n-space satisfying:

(i) W_k is a complete Segal $(n-1)$-space for each k; and

(ii) $W_0 \to W_{\mathrm{hoequiv}}$ is a Reedy weak equivalence of Segal $(n-1)$-spaces.

By the argument above, we can rewrite condition (ii) as

(ii') $W_{\bullet,0,\dots,0}$ is a complete Segal space.

We will sometimes call a pre-Segal n-space satisfying (i) and (ii) a **pre-complete Segal n-space**.

Unfolding the definition, we can explicitly describe a Segal n-space as follows:

Definition 2.5.16A. A **complete Segal n-space** is a Segal n-space W satisfying: for all $1 \leq i \leq n$ and $k_1, \dots, k_{i-1}$, $W_{k_1,\dots,k_{i-1},\bullet,0,\dots,0}$ is a complete Segal space.

Let E_r be the r-discrete r-constant simplicial n-space generated by the space $E = \operatorname{discnerve} I[1]$. We can define the Segal space and complete Segal space model structures as localisations of the Reedy model structure on n-**sSp**. By Prop. 2.3.7, the condition in Def. 2.5.16A can be represented by the weak equivalence

$$\begin{aligned}&\mathrm{HoMap}^0_{n\text{-}\mathbf{sSp}}(F_n^n(k_1,\dots,k_{i-1},0\dots,0), W) \xrightarrow{\sim} \\ &\qquad \mathrm{HoMap}^0_{n\text{-}\mathbf{sSp}}(F_n^{i-1}(k_1,\dots,k_{i-1}) \times E_i \times F_{n-i}^{n-i}(k_{i+1},\dots,k_n), W).\end{aligned}$$

As in Section 2.3, we can construct completion functors.

Definition 2.5.17. Let W be a Segal n-space. A **Segal completion** (or simply completion) of W is a complete Segal n-space $\widetilde{W}$ with a map $W \to \widetilde{W}$ which is universal among all maps from W to complete n-Segal spaces. A **Reedy-fibrant Segal completion** of W is a Reedy-fibrant complete Segal n-space $\widehat{W}$ with a map $W \to \widehat{W}$ which is universal among all maps from W to Reedy-fibrant complete n-Segal spaces.

Proposition 2.5.18. *Let W be a Segal n-space. There exists a functorial Segal completion $W \to \widetilde{W}$ and a functorial Reedy-fibrant Segal completion $W \to \widehat{W}$.*

Proof. The proof is by the small object argument and is similar to that for Props. 2.3.10 and 2.3.12.

The Segal completion $\widetilde{W}$ is constructed as the colimit of pushouts of the diagrams

$$\begin{tikzcd}
\partial\Delta^l \times F_n^n(k_1,\dots,k_{i-1},1,0,\dots,0) \arrow[r, "f"] \arrow[d] & W \\
\partial\Delta^l \times F_n^{i-1}(k_1,\dots,k_{i-1}) \times N(I[1])_i \times F_{n-i}^{n-i}(0,\dots,0) &
\end{tikzcd}$$

where $N(I[1])_i$ is the i-discrete i-constant simplicial n-space and

$$f(\partial\Delta^l \times \{\mathrm{id}_{[k_1]} \times \cdots \mathrm{id}_{[k_{i-1}]} \times \mathrm{id}_{[1]} \times \mathrm{id}_{[0]} \times \cdots \times \mathrm{id}_{[0]}) \subset (W_{k_1,\dots,k_{i-1},\bullet,0,\dots,0})_{\mathrm{hoequiv}}.$$

The Reedy-fibrant Segal completion $\widehat{W}$ is constructed using the diagrams

$$\begin{tikzcd}
\partial\Delta^l \times F_n^{i-1}(k_1,\dots,k_{i-1}) \times E_i \times F_{n-i}^{n-i}(0,\dots,0) \arrow[r, "f"] \arrow[d] & W \\
\partial\Delta^l \times F_n^{i-1}(k_1,\dots,k_{i-1}) \times N(I[1])_i \times F_{n-i}^{n-i}(0,\dots,0) &
\end{tikzcd}.$$

□

Theorem 2.5.19. *There exists a model category structure on n-$\mathbf{sSp} = \mathbf{Fun}((\Delta^{\mathrm{op}})^n, \mathbf{Sp})$ with the following properties:*

(i) The cofibrations are the Reedy cofibrations.

(ii) The weak equivalences are maps f such that $\mathrm{Map}^0_{n\text{-}\mathbf{sSp}}(f, W)$ is a weak equivalence for all Reedy-fibrant Segal n-spaces W.

(iii) The fibrations are the maps that satisfy the right lifting property with respect to all trivial cofibrations.

This is called the **Segal n-space model category structure** *on n-$\mathbf{sSp}$, and is denoted as n-$\mathcal{SS}$. The fibrant objects are precisely the Reedy-fibrant Segal n-spaces. A Reedy weak equivalence between two objects X, Y is a weak equivalence in n-$\mathcal{SS}$ and the converse is true if X, Y are Reedy-fibrant Segal n-spaces.*

For $\mathbf{Sp} = \mathbf{CGHaus}$ or $\mathbf{sSet}$, this model structure is compatible with the Cartesian closure.

Proof. n-$\mathcal{SS}$ is obtained as the left Bousfield localisation of the Reedy model structure on n-$\mathbf{sSp}$ with respect to the set of maps

$$\begin{aligned} S \quad = \quad & \{F_n^{i-1}(k_1,\dots,k_{i-1}) \times G_{n-i}(k_i) \times F_{n-i}^{n-i}(k_{i+1},\dots,k_n) \to F_n^n(k_1,\dots,k_n)\}, \\ & \cup \{F_n^n(k_1,\dots,k_{i-1},0,\dots,0) \to F_n^n(k_1,\dots,k_{i-1},0,k_i,\dots,k_n)\}. \end{aligned}$$

Theorem 2.4.4 implies the rest of the statements.

The proof of compatibility with Cartesian closure is similar to that given in [Rez01, Sec. 10]. □

Theorem 2.5.20. *There exists a model category structure on n-$\mathbf{sSp} = \mathbf{Fun}((\Delta^{\mathrm{op}})^n, \mathbf{Sp})$ with the following properties:*

(i) The cofibrations are the Reedy cofibrations.

(ii) The weak equivalences are maps f such that $\mathrm{Map}^0_{n\text{-}\mathbf{sSp}}(f, W)$ is a weak equivalence for all Reedy-fibrant complete Segal n-spaces W.

(iii) The fibrations are the maps that satisfy the right lifting property with respect to all trivial cofibrations.

This is called the **complete Segal n-space model category structure** *on n-$\mathbf{sSp}$, and is denoted as n-$\mathcal{CSS}$. The fibrant objects are precisely the Reedy-fibrant complete Segal n-spaces. A Reedy weak equivalence between two objects X, Y is a weak equivalence in n-$\mathcal{CSS}$ and the converse is true if X, Y are Reedy-fibrant complete Segal n-spaces.*

For $\mathbf{Sp} = \mathbf{CGHaus}$ or $\mathbf{sSet}$, this model structure is compatible with the Cartesian closure.

Proof. n-$\mathcal{SS}$ is obtained as the left Bousfield localisation of the Reedy model structure on n-$\mathcal{SS}$ with respect to the set of maps

$$S = \{F_n^{i-1}(k_1, \dots, k_{i-1}) \times E_i \times F_{n-i}^{n-i}(k_{i+1}, \dots, k_n) \to F_n^n(k_1, \dots, k_{i-1}, 0 \dots, 0)\}.$$

Theorem 2.4.4 implies the rest of the statements.

The proof of compatibility with Cartesian closure is similar to that given in [Rez01, Sec. 12]. $\square$

Definition 2.5.21. We denote by n-**SeSp** and n-**CSeSp** the full subcategories of Segal n-spaces and complete Segal n-spaces endowed with the model structure of n-$\mathcal{CSS}$.

We have same results regarding equivalences of Segal n-spaces as for Segal 1-spaces.

Theorem 2.5.22. *Let $f : U \to V$ be a morphism between two complete Segal n-spaces. Then, the following notions are all equivalent:*

$$\text{Dwyer-Kan equivalence} \iff \text{Reedy weak equivalence} \iff$$
$$\text{Weak equivalence in } n\text{-}\mathcal{SS} \iff \text{Weak equivalence in } n\text{-}\mathcal{CSS}$$

Let $f : U \to V$ be a morphism between two Segal n-spaces. f is a Dwyer-Kan equivalence if and only if f is a weak equivalence in n-$\mathcal{CSS}$.

The Segal completion map $W \to \widetilde{W}$ and the Reedy-fibrant Segal completion map $W \to \widehat{W}$ are both weak equivalences in n-$\mathcal{CSS}$ and Dwyer-Kan equivalences.

Proof. The proof follows by induction on n. Assuming that the identification between the different equivalences has already been shown for Segal $(n-1)$-spaces and using the fact that

$$F_n^{i-1}(k_1, \dots, k_{i-1}) \times E_i \times F_{n-i}^{n-i}(k_{i+1}, \dots, k_n) \to F_n^n(k_1, \dots, k_{i-1}, 0 \dots, 0)$$

is both a weak equivalence in n-$\mathcal{CSS}$ and a Dwyer-Kan equivalence, the remainder of the proof for Segal n-spaces is exactly the same as in the $n = 1$ case. $\square$

As in the previous section, it is reasonable to take complete Segal n-spaces as our notion of (∞, n)-categories.

Definition 2.5.23. An (∞, n)**-category** is a complete Segal n-space.

We can also embed r-**SeSp** into n-**SeSp** for $r < n$. Let **Sp** = **sSet**.

Definition 2.5.24. A **Segal** $(n-1)$**-space in** n-$\mathcal{CSS}$ is a Segal n-space such that all n-morphisms are invertible up to homotopy, that is, for all $(n-2)$-morphisms x, y, the category $\mathrm{Ho}\,\mathrm{map}^{n-1}(x, y)$ is a groupoid.

Claim 2.5.25. *There is an equivalence of homotopy categories*

$$\mathbb{L}(R \circ p_1^*) : \mathrm{Ho}\,(n-1)\text{-}\mathbf{SeSp} \to \mathrm{Ho}\,n\text{-}\mathbf{SeSp}_{n-1} : \mathbb{R}i_1^*$$

*where R is the fibrant replacement functor in n-$\mathcal{CSS}$ and n-$\mathbf{SeSp}_{n-1}$ is the full subcategory of Segal $(n-1)$-spaces in n-**SeSp**.*

We give an idea of how to prove the above claim.

Sketch of proof. First, note that we can define a Reedy model structure on n-**sSp** over the Joyal model structure $\mathcal{QC}$ on **sSet**. Applying the left Bousfield localisation as in Thm. 2.5.20, we obtain a model structure which we will call n-$\mathcal{CSS}_{\mathcal{QC}}$. The fibrant-cofibrant objects are Joyal-Reedy-fibrant simplicial n-spaces satisfying the Segal, completeness and essentially constant conditions. In particular, for any fibrant-cofibrant object X, $X_{k_1,\dots,k_n}$ are quasicategories.

By induction on n and Thm. 2.4.20, we obtain a Quillen equivalence

$$p_1^* : (n-1)\text{-}\mathcal{CSS}_{\mathcal{QC}} \to n\text{-}\mathcal{CSS} : i_1^*$$

where p_1^* takes a simplicial $(n-1)$-space X to a simplicial n-space $[k_1,\ldots,k_n]\mapsto X_{k_1,\ldots,k_{n-1}}$ and i_1^* takes a simplicial n-space Y to a simplicial $(n-1)$-space i_1^*Y where $(i_1^*Y)_{k_1,\ldots,k_{n-1}}$ are simplicial sets given by $[k]\mapsto (Y_{k_1,\ldots,k_{n-1},k})_0$.

The identity functor $(n-1)$-$\mathcal{CSS}\to(n-1)$-$\mathcal{CSS}_{\mathcal{QC}}$ is compatible with the model category structure and the two model category structures are the same on the subcategory $(n-1)$-**SeSp** (where we require each underlying simplicial set to be a Kan complex).

As in the proof of Cor. 2.4.21, given any Segal $(n-1)$-space X, all n-morphisms in p_1^*X are invertible up to homotopy, so its fibrant replacement in n-$\mathcal{CSS}$ is a Segal $(n-1)$-space. Conversely, for any Segal $(n-1)$-space Y in n-**SeSp**, i_1^*Y is a Segal $(n-1)$-space. This completes the proof of the theorem. □

Definition 2.5.26. For $1\le r<n-1$, a **Segal r-space X in n-$\mathcal{CSS}$** is a Segal r-space i_1^*X in $(n-1)$-$\mathcal{CSS}$. An **∞-groupoid in n-$\mathcal{CSS}$** is an ∞-groupoid in $(n-1)$-$\mathcal{CSS}$.

2.6 Symmetric monoidal Segal n-spaces

We have thus far defined a notion of (∞,n)-categories, specifically the complete Segal n-spaces. We would like to endow the category with an additional "algebraic" structure. Informally, given an (∞,n) category C, we want to define a functor $\otimes : C\times C\to C$ that is associative, symmetric and has an identity.

First, we recall the definition of a symmetric monoidal (1-)category.

Definition 2.6.1. A **(symmetric) monoidal category** is a category $\mathcal{C}$ equipped with a bifunctor $\otimes :$ $\mathcal{C}\times\mathcal{C}\to\mathcal{C}$, a unit $\mathbf{1}\in\mathcal{C}$ and the following natural isomorphisms:

$$\begin{aligned}&a\otimes(b\otimes c)\xrightarrow{\sim}(a\otimes b)\otimes c\\&a\otimes\mathbf{1}\xrightarrow{\sim}a\xrightarrow{\sim}\mathbf{1}\otimes a\\&(a\otimes b\xrightarrow{\sim}b\otimes a)\end{aligned}$$

for all $a,b,c\in\mathcal{C}$, satisfying the some coherence properties (see [Mac71]). A **monoidal functor** $F:\mathcal{C}\to\mathcal{D}$ between two (symmetric) monoidal categories is a functor with the following natural isomorphisms in $\mathcal{D}$:

$$F(c)\otimes F(c')\xrightarrow{\sim}F(c\otimes c')\qquad\text{and}\qquad F(\mathbf{1}_{\mathcal{C}})\xrightarrow{\sim}\mathbf{1}_{\mathcal{D}}$$

satisfying some coherence properties (see [Mac71])

The product of (symmetric) monoidal categories is (symmetric) monoidal, by taking $\otimes$ to be componentwise.

For infinity categories, the logical extension of the definition will be to replace natural isomorphisms with natural weak equivalences and to ask that they are coherent up to homotopy. Checking the coherence properties, while already difficult for 1-categories, will be forbidding in higher category settings.

To generalise the construction to higher categories, we will use an alternative internal definition of symmetric monoidal categories. The ideas are derived from Segal's Γ-space construction.

Let Γ_∞ be the skeleton of the category of pointed finite sets. We write the objects of Γ_∞ as $\langle r\rangle=\{0,\ldots,r\}$ pointed at 0. The morphisms are pointed maps of sets. Define Γ_1 to be the category with the same objects as Γ_∞, and such that each morphism is the data of a pointed map of sets $\phi:\langle k\rangle\to\langle l\rangle$ together with an ordering within each set $\phi^{-1}(i)$ for $1\le i\le l$. There is an obvious forgetful functor $\iota:\Gamma_1\to\Gamma_\infty$.

In particular, we have the maps

$$\rho_i:\langle r\rangle\to\langle 1\rangle : j\mapsto\delta_{ij}$$

in Γ_1 and Γ_∞ for $1\le i\le r$.

Definition 2.6.1A. A **pre-monoidal category** is a functor $X:\Gamma_1\to\mathbf{Cat}$. Let $X_r=X(\langle r\rangle])$.

A **monoidal category** is a a pre-monoidal category X such that the induced maps

$$\prod_{i=1}^{r}\rho_{i*} : X_r\to\prod_{i=1}^{r}X_1$$

are equivalences of categories for all $r \geq 0$. For $r = 0$, this implies that $X_0 = \{*\}$ is a single point.

A **pre-symmetric monoidal category** is a functor $X : \Gamma_\infty \to \mathbf{Cat}$. Let $X_r = X(\langle r \rangle)$.

A **symmetric monoidal category** is a pre-symmetric monoidal category X such that the induced maps

$$\prod_{i=1}^{r} \rho_{i*} : X_r \to \prod_{i=1}^{r} X_1$$

are equivalences of categories for all $r \geq 0$.

The 2-category of (symmetric) monoidal categories is thus a full subcategory of the 2-category of categories. There is a forgetful functor from the 2-category of symmetric monoidal categories to the 2-category of monoidal categories given by pre-composition with $\iota : \Gamma_1 \to \Gamma_\infty$.

The two definitions are equivalent by the following identification:

Proposition 2.6.2. *Let X be a monoidal category as in Def. 2.6.1A. The unique inclusion $s_0 : \langle 0 \rangle \to \langle 1 \rangle$ defines an object $\mathbf{1} = s_{0*}(*) \in X_1$. The diagram*

$$\begin{tikzcd} X_2 \arrow[r, "d^{1*}"] \arrow[d, "d^{2*} \times d^{0*}"', "\sim"] & X_1 \\ X_1 \times X_1 \arrow[ur, dashed, "\otimes"'] & \end{tikzcd}.$$

defines a bifunctor $\otimes : X_1 \times X_1 \to X_1$. $(X_1, \otimes, \mathbf{1})$ is a symmetric monoidal category in the sense of Def. 2.6.1.

Conversely, given any monoidal category $(\mathcal{C}, \otimes, \mathbf{1})$, there is a well-defined functor $X : \Gamma_1 \to \mathbf{Cat}$ such that $X_r = \mathcal{C}^{\otimes r}$ is the category of r-uples of objects in $\mathcal{C}$. This gives a monoidal category as in Def. 2.6.1A.

There is a similar identification for symmetric monoidal categories.

Proof. We omit the proof. □

We can extend the ideas of Def. 2.6.1A to infinity categories. Toën and Vezzosi [TV11] gave a construction for symmetric monoidal $(\infty, 1)$-categories. We give a simple generalisation to (∞, n)-categories. For infinity categories, the right notion of symmetric monoidal structure should allow for associativity and commutativity up to homotopy. Therefore, it is natural to replace Γ_1 and Γ_∞ with their infinity counterparts. As Γ_1 and Γ_∞ are 1-categories, all higher morphisms are invertible, so it suffices to replace them with $(\infty, 1)$-categories.

We recall the notions of the classifying diagrams for categories and model categories in Examples 2.2.6 and 2.2.8. For $\eta = 1, \infty$, $N(\Gamma_\eta) = N(\Gamma_\eta, \mathrm{iso}\Gamma_\eta)$ is the classifying diagram of the ordinary category Γ_η. $N(n\text{-}\mathcal{CSS}) = N(n\text{-}\mathcal{CSS}, \mathrm{we}\, n\text{-}\mathcal{CSS})$ is the classifying diagram of the model category n-$\mathcal{CSS}$ with respect to the weak equivalences.

Definition 2.6.3. A **pre-monoidal simplicial n-space** is a map of simplicial spaces $X : N(\Gamma_1) \to N(n\text{-}\mathcal{CSS})$. Let $X_r = X(\langle r \rangle) \in n\text{-}\mathcal{CSS}$.

A **monoidal simplicial n-space** is a a pre-monoidal simplicial n-space X such that the induced maps

$$\prod_{i=1}^{r} \rho^{i*} : X_r \to \prod_{i=1}^{r} X_1$$

are Dwyer-Kan equivalences of Segal n-spaces for all $r \geq 0$. For $r = 0$, this implies that X_0 is weakly equivalent to a point (seen as a constant simplicial $(n-1)$-space).

A **pre-symmetric simplicial n-space** is a map of simplicial spaces $X : N(\Gamma_\infty) \to N(n\text{-}\mathcal{CSS})$. Let $X_r = X(\langle r \rangle) \in n\text{-}\mathcal{CSS}$.

A **symmetric simplicial n-space** is a pre-symmetric monoidal category X such that the induced maps

$$\prod_{i=1}^{r} \rho_{i*} : X_r \to \prod_{i=1}^{r} X_1$$

are Dwyer-Kan equivalences of Segal n-spaces for all $r \geq 0$.

A **(pre)-(symmetric) monoidal Segal n-space** (**complete Segal n-space**, respectively) is a (pre)-(symmetric) monoidal simplicial n-space such that X_1 is a Segal n-space (resp., complete Segal n-space).

The underlying Segal n-space of a (symmetric) monoidal Segal n-space X as defined above is X_1. The identity object is given by $s_{0*}(*) \in \mathrm{Ob}\, X_1$. The diagram

$$\begin{tikzcd} X_2 \arrow[r, "\rho_{12*}"] \arrow[d, "\rho_{1*}\times\rho_{2*}"', "\sim"] & X_1 \\ X_1 \times X_1 \arrow[ur, dashed, "\otimes"'] & \end{tikzcd}$$

where $\rho_{12} : [2] \to [1]$ sends $1, 2 \mapsto 1$ (we choose any ordering in Γ_1), gives a non-functorial map (i.e. not a map of simplicial n-spaces) $\otimes : X_1 \times X_1 \to X_1$: we can lift each element in $X_1 \times X_1$ to X_2 but not necessarily in a functorial way. Nevertheless, the induced map

$$\otimes : \mathrm{Ho}\, X_1 \times \mathrm{Ho}\, X_1 \to \mathrm{Ho}\, X_1$$

is a functor and exhibits $\mathrm{Ho}\, X_1$ as a (symmetric) monoidal category. By the equivalence $\mathrm{Ho}\, X_r \to \mathrm{Ho}\,(X_1)^r$, $\mathrm{Ho}\, X_r$ is a (symmetric) monoidal category as well.

If X_2 and X_1 are Reedy-fibrant complete Segal spaces, then the Dwyer-Kan equivalence $X_2 \to X_1 \times X_1$ is a homotopical equivalence, so it has a homotopy inverse. The composition of this homotopy inverse with $\rho_{12*} : X_2 \to X_1$ gives a functorial map (i.e. a map of simplicial n-spaces) $\otimes : X_1 \times X_1 \to X_1$.

Conversely, given a complete Segal n-space W and a map of simplicial n-spaces $\otimes : W \times W \to W$, it has a (symmetric) monoidal structure if there is a map of simplicial spaces $X : N(\Gamma_\eta) \to N(n\text{-}\mathcal{CSS})$ ($\eta = 1, \infty$) such that $X_r \to W^r$ is a Dwyer-Kan equivalence for all r.

Let X and Y be two (symmetric) monoidal (∞, n)-categories. We will like to consider (symmetric) monoidal functors between these two categories. Intuitively, a symmetric monoidal functor is a natural transformation between the two functors $X, Y : N(\Gamma_\eta) \to N(n\text{-}\mathcal{CSS})$. We can view the collection of (symmetric) monoidal functors $\mathbf{Fun}^{\otimes}(X, Y)$ as a sub-(∞, n)-category of $\mathbf{Fun}\,(X, Y)$, the category of all functors from $X \to Y$ obtained by forgetting the monoidal structure. Formally, we define $\mathbf{Fun}^{\otimes}(X, Y)$ as follows:

Definition 2.6.4. The collection of (symmetric) monoidal functors between two (symmetric) monoidal simplicial n-spaces X and Y forms a simplicial n-space $\mathrm{Fun}^{\otimes}(X, Y)$ given by

$$\mathrm{Fun}^{\otimes}(X, Y)_{k_1,\ldots,k_n} = \mathrm{map}(X \times F(k_1, \ldots, k_n), Y)$$

where $\mathrm{map}(-,-)$ are the mapping spaces in the $(\infty, 1)$-category $N^f(n\text{-}\mathcal{CSS})^{N(\Gamma_\eta)}$ and $F(k_1, \ldots, k_n)$ are constant functors $N(\Gamma_\eta) \to N^f(n\text{-}\mathcal{CSS})$. The face and degeneracy maps are induced by those on $F(k_1, \ldots, k_n)$.

A **(symmetric) monoidal functor** from X to Y is an object in $\mathrm{Fun}^{\otimes}(X, Y)$.

Proposition 2.6.5. $\mathrm{Fun}^{\otimes}(X, Y)$ *is Reedy weakly equivalent to the homotopy limit of the diagram*

$$\begin{tikzcd} Y_l^{X_l} \arrow[dr, "\phi_*"] & \\ & Y_{l'}^{X_l} \\ Y_{l'}^{X_{l'}} \arrow[ur, "\phi^*"'] & \end{tikzcd} \qquad \forall\, \phi : [l] \to [l'] \text{ in } \Gamma_\eta.$$

Hence, by Cor. 2.4.9, $\mathrm{Fun}^{\otimes}(X, Y)$ *is a (complete) Segal n-space if Y is a pre-(symmetric) monoidal (complete) Segal n-space.*

Proof. We will give the proof for monoidal simplicial n-spaces. It suffices to check the assertion on each space, that is that $\mathrm{Fun}^{\otimes}(X,Y)_{k_1,\dots,k_n}$ is the homotopy limit of the diagram

$$\begin{tikzcd}[column sep=small] \mathrm{map}(X_l \times F(k_1,\dots,k_n), Y_l) \arrow[dr, "\phi_*"] & & \\ & \mathrm{map}(X_l \times F(k_1,\dots,k_n), Y_{l'}) & \forall\, \phi : [l] \to [l'] \\ \mathrm{map}(X_{l'} \times F(k_1,\dots,k_n), Y_{l'}) \arrow[ur, "\phi^*"'] & & \end{tikzcd}$$

where the mapping spaces are in $N^f(n\text{-}\mathcal{CSS})$. The inclusion $\tau_l : F(0) \to N(\Gamma_\eta)$ sending $[0] \mapsto [l]$ induces the commutative square

$$\begin{tikzcd} \mathrm{Map}(N(\Gamma_\eta) \times F(1), N^f(n\text{-}\mathcal{CSS})) \arrow[r, "\tau_l^*"] \arrow[d] & \mathrm{Map}(F(0) \times F(1), N^f(n\text{-}\mathcal{CSS})) \arrow[d] \\ (\mathrm{Map}(N(\Gamma_\eta), N^f(n\text{-}\mathcal{CSS})))^2 \arrow[r] & (\mathrm{Map}(F(0), N^f(n\text{-}\mathcal{CSS})))^2. \end{tikzcd}$$

Taking the fibre at the point $(X \times F(k_1,\dots,k_n), Y) \in (\mathrm{Map}(N(\Delta^{\mathrm{op}}), N^f(n\text{-}\mathcal{CSS})))^2$, we get the commutative diagram

$$\begin{tikzcd} \mathrm{map}(X \times F(k_1,\dots,k_n), Y) \arrow[r, "\tau_l^*"] \arrow[d] & \mathrm{map}(X_l \times F(k_1,\dots,k_n), Y_l) \arrow[d] \\ (X \times F(k_1,\dots,k_n), Y) \arrow[r] & (X_l \times F(k_1,\dots,k_n), Y_l). \end{tikzcd}$$

Any $F \in \mathrm{Ob}\,(\mathrm{map}(X \times F(k_1,\dots,k_n), Y))$ is uniquely determined by a collection $\{\tau_l^* F\}_{l \geq 0}$ satisfying, for any $\phi : [l] \to [l']$, the homotopy commutative diagram

$$\begin{tikzcd} X_l \times F(k_1,\dots,k_n) \arrow[r, "\tau_l^* F"] \arrow[d, "\phi_*"] & Y_l \arrow[d, "\phi_*"] \\ X_{l'} \times F(k_1,\dots,k_n) \arrow[r, "\tau_{l'}^* F"] & Y_{l'}. \end{tikzcd}$$

Hence, $\mathrm{Fun}^{\otimes}(X,Y)_{k_1,\dots,k_n}$ is the required homotopy limit. □

Remark 2.6.6. The above proposition also establishes a morphism from the (∞,n)-category of (symmetric) monoidal functors $\mathrm{Fun}^{\otimes}(X,Y)$ to the (∞,n)-category of functors $\mathrm{Fun}(X,Y) = Y_1^{X_1}$.

In general, $\mathrm{Fun}^{\otimes}(X,Y)$ does not have a monoidal structure. This can be seen for monoidal ordinary categories: given two monoidal functors $F, G : X \to Y$ and $x, x' \in X$,

$$(F \otimes G)(x \otimes x') \cong (F \otimes G)(x) \otimes (F \otimes G)(x') \cong F(x) \otimes G(x) \otimes F(x') \otimes G(x') \ncong F(x \otimes x') \otimes G(x \otimes x'),$$

so $F \otimes G$ is not a monoidal functor unless Y is symmetric monoidal. We have the same result for (∞,n)-categories.

Proposition 2.6.7. *Let X be a pre-monoidal simplicial n-space and Y be a symmetric monoidal (complete) Segal n-space. Then, there exists a symmetric monoidal (complete) Segal n-space $\mathbf{Fun}^{\otimes}(X,Y)$ such that $\mathbf{Fun}^{\otimes}(X,Y)_1 = \mathrm{Fun}^{\otimes}(X,Y)$.*

Proof. We define two functors $\alpha_l, \beta_k : \Gamma_\infty \to \Gamma_\infty$. On objects, they send $\langle k \rangle \mapsto \langle kl \rangle$ and $\langle l \rangle \mapsto \langle kl \rangle$ respectively. Let $0 < x < kl$, then there exists a unique representation $x = ak + b$ where $0 < b \leq k$. For any $\psi : \langle k \rangle \to \langle k' \rangle$, define $\alpha_l(\psi)(x) = ak' + \psi(b)$. For any $\phi : \langle l \rangle \to \langle l' \rangle$, define $\beta_k(\phi)(x) = \phi(a)k + b$.

Define $Y^{\otimes l}$ to be the composition of functors $Y \circ N(\alpha_l)$, so $Y_k^{\otimes l} = Y_{kl}$. It is easy to check that $Y^{\otimes l}$ is a symmetric monoidal (complete) Segal n-space.

For any $\phi : \langle l\rangle \to \langle l'\rangle$, we can define a functor $\phi_\beta : Y^{\otimes l} \to Y^{\otimes l'}$ by $Y_{kl} \xrightarrow{\beta_k(\phi)_*} Y_{kl'}$ for each $k \geq 0$. This is a symmetric monoidal functor since the squares

$$\begin{tikzcd} Y_{kl} \arrow[r, "\beta_k(\phi)_*"] \arrow[d, "\alpha_l(\psi)_*"'] & Y_{kl'} \arrow[d, "\alpha_{l'}(\psi)_*"] \\ Y_{k'l} \arrow[r, "\beta_{k'}(\phi)_*"] & Y_{k'l'} \end{tikzcd}$$

commute for all $\psi : \langle k\rangle \to \langle k'\rangle$.

Define the functor $\mathbf{Fun}^{\otimes}(X,Y)$ by $\mathbf{Fun}^{\otimes}(X,Y)_l = \mathrm{Fun}^{\otimes}(X, Y^{\otimes l})$ and the maps induced by ϕ_β. It is now immediate that $\mathbf{Fun}^{\otimes}(X,Y)$ is a symmetric monoidal (complete) Segal n-space. $\square$

Note that we can construct the Segal completion of a (symmetric) monoidal Segal n-space by completing it degree-wise.

Proposition 2.6.8. *Let X be a (symmetric) monoidal Segal n-space. We define its* **Segal completion** *to be the (symmetric) monoidal complete Segal space $\widetilde{X}$ where $(\widetilde{X})_r = \widetilde{X_r}$. The Segal completion is initial among all (symmetric) monoidal functors $X \to Y$ where Y is a (symmetric) monoidal complete Segal n-space.*

Proof. That $\widetilde{X}$ is a pre-(symmetric) monoidal complete Segal n-space is clear by the universality of the Segal completions $X_r \to \widetilde{X}_r$. It is symmetric monoidal since the Segal completion map is a Dwyer-Kan equivalence, so it induces a diagram of Dwyer-Kan equivalences

$$\begin{tikzcd} X_r \arrow[r, "\sim"] \arrow[d, "\sim"] & (X_1)^r \arrow[d, "\sim"] \\ \widetilde{X}_r \arrow[r, "\sim"] & (\widetilde{X}_1)^r \end{tikzcd}.$$

The universal property of the Segal completion map for symmetric monoidal Segal n-spaces follows immediately from the universal property for the Segal completion on each degree. $\square$

The constructions of (symmetric) monoidal (∞,n)-categories given above can also be derived from a topological setting. A monoidal (∞,n)-category as defined above is an example of an E_1-algebra, that is an associative algebra.A symmetric monoidal (∞,n) category has the structure of an E_∞-algebra, which roughly translates to saying that the symmetric monoidal operation $\otimes$ is commutative up to a contractible space of homotopies. More generally, we will construct the little m-cube algebras $\mathbf{E}_m$, which are standard examples of an E_m-algebras, and show that $N(\Gamma_\eta)$ is equivalent to $\mathbf{E}_\eta$ as $(\infty,1)$-categories for $\eta = 1, \infty$.

Definition 2.6.9. The **little m-cube algebra $\mathbf{E}_m$** is a topological category whose objects are finite disjoint union of m-dimensional cubes $\mathbf{I}^m_n = \coprod_{i=1}^n I^m_i$ where $I^m_i = I^m = [0,1]^m$, and whose morphisms are

$$\mathrm{Map}_{\mathbf{E}_m}(\mathbf{I}^m_k, \mathbf{I}^m_l) = \coprod_{S \subset [k]} \mathrm{Rect}\big(\mathbf{I}^m_{|S|}, \mathbf{I}^m_l\big) = \coprod_{\phi \in \mathrm{Hom}_{\Gamma_\infty}(\langle k\rangle,\langle l\rangle)} \prod_{i=1}^{l} \mathrm{Rect}\big(\mathbf{I}^m_{|\phi^{-1}(i)|}, I^m_i\big)$$

where $\mathrm{Rect}\left(\mathbf{I}^m_k, I^m\right)$ is the space of rectilinear embeddings. Composition is given by composition of rectilinear maps.

The **little ∞-cube algebra** is the colimit $\mathbf{E}_\infty = \mathrm{colim}_m \mathbf{E}_m$. Explicitly, it is a category whose objects are finite disjoint unions of ∞-dimensional cubes and

$$\mathrm{Map}_{\mathbf{E}_\infty}(\langle k\rangle, \langle l\rangle) = \coprod_{\phi \in \mathrm{Hom}_{\Gamma_\infty}(\langle k\rangle,\langle l\rangle)} \prod_{i=1}^{l} \mathrm{Rect}\big(\mathbf{I}^\infty_{|\phi^{-1}(i)|}, I^\infty_i\big).$$

By the equivalences between the different models for $(\infty,1)$-categories (see, for example, [Ber10, Lur09b]), we can associate to each topological category $\mathcal{C}$ a complete Segal space $N(\mathcal{C})$. Explicitly, $N(\mathcal{C})$ can be given as the Segal completion of a Segal space discnerve $\mathcal{C}$ where discnerve C_0 is the discrete space of objects in $\mathcal{C}$ and for each $k > 0$,

$$(\operatorname{discnerve}\mathcal{C}_k)_l = \Big\{(h_1, \dots, h_k) \in \operatorname{Map}_{\mathbf{Top}}\big(|\Delta^l|, \operatorname{map}(x_0, x_1) \times \cdots \operatorname{map}(x_{k-1}, x_k)\big) \Big| \; x_i \in \operatorname{Ob}\mathcal{C}\Big\}.$$

Note that $f : x \to y$ in $N(C)$ is a homotopy equivalence if there exists $g : y \to x$ such that $g \circ f \sim \mathrm{id}_x$ and $f \circ g \sim \mathrm{id}_y$. $N(\mathcal{C})$ is a complete Segal space satisfying $\operatorname{Ho}\mathcal{C} \cong \operatorname{Ho} N(\mathcal{C})$ and for any $x, y \in \operatorname{Ob}\mathcal{C}$, $\operatorname{map}_{\mathcal{C}}(x, y) \cong \operatorname{map}_{N(\mathcal{C})}(x, y)$. The functor $C \mapsto N(C)$ from the category of topological categories to 1-$\mathcal{CSS}$ is a Quillen equivalence.

Proposition 2.6.10. *For $\eta = 1, \infty$, $N(\Gamma_\eta)$ and $\mathbf{E}_\eta$ are equivalent as $(\infty, 1)$-categories, that is, Dwyer-Kan equivalent.*

Proof. We will make free use of the fact that topological categories and simplicial categories are Quillen equivalent to complete Segal spaces as models of $(\infty, 1)$-categories.

There is an obvious functor $\mathbf{E}_\eta \to \Gamma_\eta$, which induces a map of simplicial spaces $N(\mathbf{E}_\eta) \to N(\Gamma_\eta)$. There is a clear bijection between the objects of $N(\Gamma_\eta)$ and $N(\mathbf{E}_\eta)$, so it suffices to show that for any $k, l \geq 0$, the induced map $\operatorname{map}_{N(\mathbf{E}_\eta)}(\langle k\rangle, \langle l\rangle) \to \operatorname{map}_{\Gamma_\eta}(\langle k\rangle, \langle l\rangle)$ is a weak equivalence.

For $\eta = 1$, we note that each map $\phi \in \operatorname{Rect}(I, I)$ is uniquely determined by the pair $(a, b) = (\phi(0), \phi(1) - \phi(0))$. Hence,

$$\operatorname{Rect}(\mathbf{I}_k, I) \cong \{(a_i, b_i)_{1\leq i\leq k}\} \subset I^{2k}.$$

Since the coordinates b_i can be retracted to 0, it follows that $\operatorname{Rect}(\mathbf{I}_k, I)$ is homotopic to the space of k-uples $\{(a_i)_{1\leq i\leq k} | a_i \neq a_j \;\forall\; i \neq j\}$. The latter space is homotopic to the discrete set of permutations Sym_k of k objects. Hence,

$$\begin{aligned}\operatorname{Map}_{\mathbf{E}_1}(\langle k\rangle, \langle l\rangle) &= \coprod_{\phi\in\operatorname{Hom}_{\Gamma_\infty}(\langle k\rangle,\langle l\rangle)} \prod_{i=1}^{l} \operatorname{Rect}\big(\mathbf{I}_{|\phi^{-1}(i)}, I_i\big) \\ &\xrightarrow{\sim} \coprod_{\phi\in\operatorname{Hom}_{\Gamma_\infty}(\langle k\rangle,\langle l\rangle)} \prod_{i=1}^{l} \operatorname{Sym}_{|\phi^{-1}(i)|} \cong \operatorname{Hom}_{\Gamma_1}(\langle k\rangle, \langle l\rangle).\end{aligned}$$

For $\eta = \infty$, we apply Prop. 3.1.3 proven in the next chapter to show that $\operatorname{Rect}(\mathbf{I}_k^\infty, I^\infty)$ is contractible, so $\operatorname{Hom}_{\mathbf{E}_\infty}(\langle k\rangle, \langle l\rangle)$ is homotopic to the discrete space

$$\coprod_{\phi\in\operatorname{Hom}_{\Gamma_\infty}(\langle k\rangle,\langle l\rangle)} \prod_{i=1}^{s} \{*\} = \operatorname{Hom}_{\Gamma_\infty}(\langle k\rangle, \langle l\rangle).$$

□

By the above proposition, we can equivalently define a (symmetric) monoidal Segal n-space to be a map $N(\mathbf{E}_1) \to N(n\text{-}\mathcal{CSS})$ ($N(\mathbf{E}_\infty) \to N(n\text{-}\mathcal{CSS})$ respectively). We will use this definition in the proving that the bordism category is symmetric monoidal. In general, we can define an **E_m-monoidal simplicial n-space** to be a map $N(\mathbf{E}_m) \to N(n\text{-}\mathcal{CSS})$.

Remark 2.6.11. There are several other definitions of monoidal infinity categories given by other authors. Another common definition is using the characterisation of a monoidal category as a functor $\Delta^{\mathrm{op}} \to$ **Cat**. This characterisation gives us a monoidal simplicial n-space as a functor $N(\Delta^{\mathrm{op}}) \to N(n\text{-}\mathcal{CSS})$ (for example, see [Lur09a], the exact definition defers slightly depending on the model of $(\infty, 1)$-categories chosen).

There is a morphism $N(\Delta^{\mathrm{op}}) \to N(\Gamma_1)$, but it is not an equivalence of $(\infty, 1)$-categories. Nevertheless, it induces a weak equivalence $N(n\text{-}\mathcal{CSS})^{N(\Gamma_1)} \to N(n\text{-}\mathcal{CSS})^{N(\Delta^{\mathrm{op}})}$ in 1-$\mathcal{CSS}$. Hence, the two definitions of monoidal simplicial n-spaces are equivalent.

This definition can also be extended to E_m monoidal categories for $n > 1$. A E_m simplicial n-space is a map of simplicial spaces $N(\Delta^{\mathrm{op}}) \to N(n\text{-}\mathcal{CSS})^{N(\Delta^{\mathrm{op}})^{m-1}}$. A symmetric monoidal simplicial n-space is thus a collection $X = (X_m)_{m\geq 0}$ of E_m simplicial n-spaces X_m such that $X_m([1]) = X_{m-1}$.

2.7 Duals

Let $\mathcal{C}$ be a symmetric monoidal Segal n-space. The cobordism hypothesis states that every symmetric monoidal functor $Z : \mathbf{Bord}_n^{\mathrm{fr}} \to \mathcal{C}$ is determined up to canonical isomorphism by $Z(*)$. However, not all objects in $\mathcal{C}$ can be written in the form $Z(*)$ for some Z. The class of such objects is precisely the fully dualisable subcategory of $\mathcal{C}$. We will explicitly construct it in this section.

First, we consider the notion of a dual in an ordinary monoidal category:

Definition 2.7.1. Let C be a monoidal category (as in Def. 2.6.1). An object $x^\vee \in \mathrm{Ob}\, C$ is a **right dual** of $x \in \mathrm{Ob}\, C$ if there exist **evaluation** (or **counit**) and **coevaluation** (or **unit**) **maps**

$$\mathrm{ev}_x : x \otimes x^\vee \to \mathbf{1} \qquad \text{and} \qquad \mathrm{coev}_x : \mathbf{1} \to x^\vee \otimes x,$$

such that the compositions

$$x \xrightarrow{\mathrm{id}_x \otimes \mathrm{coev}_x} x \otimes x^\vee \otimes x \xrightarrow{\mathrm{ev}_x \otimes \mathrm{id}_x} x$$
$$x^\vee \xrightarrow{\mathrm{coev}_x \otimes \mathrm{id}_{x^\vee}} x^\vee \otimes x \otimes x^\vee \xrightarrow{\mathrm{id}_{x^\vee} \otimes \mathrm{ev}_x} x$$

are equal to id_x and $\mathrm{id}_{x^\vee}$ respectively. The right dual $x^\vee$ is unique up to unique isomorphism.

We can analogously define a **left dual**. An object $x \in C$ is **dualisable** if it has both a left and a right dual.

If C is a symmetric monoidal category, then there is a unique isomorphism between the left and right duals, so we will call the unique object $x^\vee$ the **dual** of x.

We say that C **has duals (for objects)** if every object $x \in \mathrm{Ob}\, C$ has both left and right duals.

Example 2.7.2. Let $\mathbf{Vect}_k$ be the category of vectorspaces over a field k. It is a symmetric monoidal category under the usual tensor product $\otimes$. The dual of any finite dimensional vectorspace V is the usual dual vectorspace $V^\vee = \mathrm{Hom}_k(V, k)$. If V is infinite dimensional, the dual vectorspace $V^\vee$ is not a categorical dual of V: there exists a evaluation map $\mathrm{ev}_V : V \otimes V^\vee \to k$ but there is no coevaluation map satisfying the identities. Indeed, a vectorspace is dualisable if and only if it is finite dimensional.

Let C be a symmetric monoidal category, let C^{fd} be the full subcategory of objects with duals. Then, C^{fd} has duals. Indeed, the inclusion $C^{\mathrm{fd}} \hookrightarrow C$ is final in the category of functors from symmetric monoidal categories with duals to C since objects with duals map to objects with duals.

Readers may notice that the definition of a dual object is similar to the definition of adjunction between functors. In fact, we can generalise adjunction to 1-morphisms in any 2-category.

Definition 2.7.3. Let C be a 2-category. A 1-morphism $g : y \to x$ is **right adjoint** to a 1-morphism $f : x \to y$ if there exist **unit** and **counit maps**

$$u : \mathrm{id}_x \to g \circ f \qquad \text{and} \qquad v : f \circ g \to \mathrm{id}_y,$$

such that the compositions

$$f \cong f \circ \mathrm{id}_x \xrightarrow{\mathrm{id} \times u} f \circ g \circ f \xrightarrow{v \times \mathrm{id}} \mathrm{id}_y \circ f \cong f$$
$$g \cong \mathrm{id}_x \circ g \xrightarrow{u \times \mathrm{id}} g \circ f \circ g \xrightarrow{\mathrm{id} \times v} g \circ \mathrm{id}_y \cong g$$

are equal to id_f and id_g respectively.

A 2-category C **has adjoints (for 1-morphisms)** if every 1-morphism f has both left and right adjoints.

Example 2.7.4. Let **Cat** be the 2-category of (small) categories. Then a functor (1-morphism) $F : C \to D$ has a left/right adjoint if and only if it has a left/right adjoint in the classical sense.

Example 2.7.5. Let C be a monoidal 1-category, and BC be the two category which has a single object $*$ and such that $\mathrm{Hom}_{BC}(*, *) = C$.

Then $x \in C$ has a left/right dual $x^\vee$ if and only if $x \in \mathrm{Hom}_{BC}(*, *)$ has a left/right adjoint $x^\vee$. Hence, C has duals for objects if and only if BC has adjoints for 1-morphisms.

We can generalise the concept of having adjoints to higher categories. To avoid complications with notions of homotopy equivalences, we shall go directly to infinity categories using the formalism of Segal n-spaces.

Definition 2.7.6. Let W be a Segal n-space. W **has adjoints for 1-morphisms** if the homotopy 2-category $\mathrm{Ho}_2 W$ has adjoints for 1-morphisms. For $1 < r < n$, W **has adjoints for r-morphisms** if for all $x, y \in \mathrm{Ob}\, W$, $\mathrm{map}^1_W(x, y)$ has adjoints for $(r-1)$-morphisms. W **has adjoints** if W has adjoints for r-morphisms for all $1 \leq r < n$.

Intuitively, we want to say that a monoidal (∞, n)-category $\mathcal{C}$ has duals if it has dual for objects ($\mathrm{Ho}\, \mathcal{C}$ has duals) and it has adjoints for r-morphisms for all $1 \leq r < n$. Recall that with our construction, the underlying Segal n-space of a monoidal Segal n-space X is $\mathrm{map}^1_X(*, *)$.

Definition 2.7.7. Let X be a monoidal Segal n-space. X has **duals for objects** if $\mathrm{Ho}\, X_1$ has duals for objects. X **has duals** if $\mathrm{Ho}\, X_1$ has duals for objects and X_1 has adjoints.

Remark 2.7.8. If X is a monoidal Segal n-space defined by a strict functor $X : \Delta^{\mathrm{op}} \to n\text{-}\mathcal{CSS}$, then X can be regarded as a Segal $(n+1)$-space, so X has duals if and only if X has adjoints as a Segal $(n+1)$-space.

To every symmetric monoidal (∞, n)-category, we can associate a sub-symmetric monoidal (∞, n)-category with duals.

Theorem 2.7.9. *Let X be a (symmetric) monoidal (complete) Segal n-space. There exists a (symmetric) monoidal (complete) Segal n-space X^{fd} with duals and a (symmetric) monoidal functor $X^{\mathrm{fd}} \to X$ that is final in the category of (symmetric) monoidal functors $Y \to X$ where Y is a (symmetric) monoidal (complete) Segal n-space with duals.*

Proof. For each $k \geq 1$, $\mathrm{Ho}\, X_k$ is a (symmetric) monoidal category, so we have the full subcategory $(\mathrm{Ho}\, X_k)^{\mathrm{fd}}$. Let X'_k be the largest subsimplicial n-space whose objects are $\mathrm{Ob}\,(\mathrm{Ho}\, X_k)^{\mathrm{fd}}$. More precisely, X'_k is defined by the homotopy pullbacks

$$\begin{tikzcd} (X'_k)_0 \arrow[r] \arrow[d] & (X_k)_0 \arrow[d] \\ \mathrm{Ob}\,(\mathrm{Ho}\, X_k)^{\mathrm{fd}} \arrow[r] & \mathrm{Ob}\,(\mathrm{Ho}\, X_k) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} (X'_k)_l \arrow[r] \arrow[d] & (X_k)_l \arrow[d] \\ ((X'_k)_0)^l \arrow[r] & ((X_k)_0)^l \end{tikzcd} \quad \text{for } l > 0.$$

Thus, X'_k is a (complete) Segal n-space with duals for objects.

The inclusion $X'_k \to X_k$ is final in the category of functors from the underlying space of a (symmetric) monoidal Segal n-spaces with duals for objects to X_k. So, for any morphism $\phi : \langle k \rangle \to \langle l \rangle$, we have a unique factorisation

$$\begin{tikzcd} X'_k \arrow[r] \arrow[d, dashed, "\exists!\, X'(\phi)"'] & X_k \arrow[d, "X(\phi)"] \\ X'_l \arrow[r] & X_l \end{tikzcd}$$

which commutes with composition. This exhibits $X' = (X'_k)_{k \geq 0}$ as a (symmetric) monoidal full (complete) subSegal n-space of X with duals for objects, and $X' \to X$ is final in the category of functors from a (symmetric) monoidal (complete) Segal n-space with duals for objects to X.

To construct a (complete) subSegal n-space with adjoints for r-morphisms, we proceed by induction on r.

Let $r = 1$ and $n > 1$. Let $\mathrm{Ho}_2 X_1^{\mathrm{ad}}$ be the subcategory of $\mathrm{Ho}_2 X_1$ consisting of all objects and 1-morphisms which have both left and right adjoints. Then $\mathrm{Ho}_2 X_1^{\mathrm{ad}}$ is a 2-category with adjoints. Let X'_1 be the largest subsimplicial n-space such that $\mathrm{Ho}_2 X' = \mathrm{Ho}_2 X_1^{\mathrm{ad}}$. As before, this can be expressed in terms of homotopy pullbacks

$$\begin{tikzcd} (X_1')_1 \arrow[r] \arrow[d] & (X_1)_1 \arrow[d] \\ \mathrm{Mor}_1(\mathrm{Ho}\, X_1)^{\mathrm{ad}} \arrow[r] & \mathrm{Mor}_1(\mathrm{Ho}\, X_1) \end{tikzcd} \quad \text{and}$$

$$\begin{tikzcd} (X_1')_l \arrow[r] \arrow[d, "\sim"] & (X_1)_l \arrow[d, "\sim"] \\ (X_1')_1 \times_{X_1'{}_0} \cdots \times_{X_1'{}_0} (X_1')_1 \arrow[r] & (X_1)_1 \times_{X_{1\,0}} \cdots \times_{X_{1\,0}} (X_1)_1 \end{tikzcd} \quad \text{for } l > 0.$$

It is clear that X_1' is a Segal n-space. If X is complete, then so is X' since all homotopy eqiuivalences have left and right adjoints, so $(X_1')_{\mathrm{hoequiv}} \cong (X_1)_{\mathrm{hoequiv}}$.

X_1' has adjoints for 1-morphisms and $X_1' \to X_1$ is final in the category of functors from the underlying space of a (symmetric) monoidal complete Segal n-space with adjoints for 1-morphisms to X_1. Similar to before, using the identity $X_k \sim (X_1)^k$, we can construct the (complete) Segal n-space X_k' with adjoints for 1-morphisms for all $k \geq 1$ and show that $X' = (X_k')_{k \geq 0}$ is a (symmetric) monoidal (complete) Segal n-space with adjoints for 1-morphisms to X.

Now, suppose $r > 1$ and $n > r$. $([l] \mapsto (X_l)_1)$ is a (symmetric) monoidal (complete) Segal $(n-1)$-space. By the induction hypothesis, there exists a (symmetric) monoidal (complete) Segal $(n-1)$-space $((X_l)_1')_{l \geq 0}$ with adjoints for $(r-1)$-morphisms. We can define the (complete) Segal n-space X_1' with adjoints for r-morphisms as above, and hence $X' \to X$ the (symmetric) monoidal (complete) subSegal n-space with adjoints for r-morphisms.

Finally, given a symmetric monoidal Segal n-space X, we can make a chain of replacements

$$X^0 \to X^1 \to \cdots X^{n-2} \to X^{n-1} \to X$$

where X^0 is the replacement of X^1 with duals for objects and X^{r-1} is the replacement of X^r with adjoints for r-morphisms. Since each replacement preserves adjoints for i-morphisms for $i > r$, X^r has adjoints for i-morphisms for all $i > r$. So, $X^{\mathrm{fd}} = X^0$ has duals. By the universality of each replacement, $X^{\mathrm{fd}} \to X$ is final in the category of all functors from a (symmetric) monoidal (complete) Segal n-space with duals to X. $\square$

It follows from the unicity part of the theorem that a symmetric monoidal complete Segal n-space X has duals if and only if $X \cong X^{\mathrm{fd}}$.

Remark 2.7.10. Let n-$\mathbf{SeSp}^{\otimes}$ be a category whose objects are symmetric monoidal Segal n-spaces and for any two objects X, Y, $\mathrm{Hom}(X, Y)$ is the set of symmetric monoidal functors from X to Y. Then, Thm. 2.7.9 implies that $X \mapsto X^{\mathrm{fd}}$ defines a functor from n-$\mathbf{SeSp}^{\otimes}$ to the full subcategory of symmetric monoidal Segal n-spaces with duals, and this functor is right adjoint to the inclusion of the full sub-2-category into n-$\mathbf{SeSp}^{\otimes}$. That is, for any symmetric monoidal Segal n-space X with duals and any symmetric monoidal Segal n-space Y, there is a Dwyer-Kan equivalence

$$\mathrm{Hom}_{n\text{-}\mathbf{SeSp}^{\otimes}}(X, Y) \cong \mathrm{Hom}_{n\text{-}\mathbf{SeSp}^{\otimes}}(X, Y^{\mathrm{fd}}).$$

Corollary 2.7.11. *Let X be a symmetric monoidal Segal n-space with duals and Y be a symmetric monoidal Segal n-space. Then there is a Dwyer-Kan equivalence*

$$\mathbf{Fun}^{\otimes}(X, Y) \cong \mathbf{Fun}^{\otimes}(X, Y^{\mathrm{fd}}).$$

Proof. If X has duals, so does $X \times F(k_1, \ldots, k_n) \times \Delta^l$ for all $k_1, \ldots, k_n, l \in \mathbb{Z}_{\geq 0}$. Hence, Thm. 2.7.9 gives a degree-wise bijection of sets

$$(\mathrm{Fun}^{\otimes}(X, Y)_{k_1, \ldots, k_n})_l \cong (\mathrm{Fun}^{\otimes}(X, Y^{\mathrm{fd}})_{k_1, \ldots, k_n})_l.$$

The equivalence is functorial by Remark 2.7.10, so it commutes with face and degeneracy maps of X. This gives us a Reedy weak equivalence of Segal n-spaces

$$\mathrm{Fun}^{\otimes}(X, Y) \cong \mathrm{Fun}^{\otimes}(X, Y^{\mathrm{fd}}).$$

Finally, it remains to show that the equivalence is compatible with the symmetric monoidal structure. By the universal property, since $(Y^{\mathrm{fd}})^{\otimes k}$ has duals, there is a map $(Y^{\mathrm{fd}})^{\otimes k} \to (Y^{\otimes k})^{\mathrm{fd}} \to Y^{\otimes k}$. This induces a functor $\mathbf{Fun}^{\otimes}(X, Y^{\mathrm{fd}}) \to \mathbf{Fun}^{\otimes}(X, Y)$ which is a Reedy weak equivalence on degree 1. The Dwyer-Kan equivalences

$$\mathbf{Fun}^{\otimes}(X, Y)_k \cong (\mathbf{Fun}^{\otimes}(X, Y)_1)^k \cong (\mathbf{Fun}^{\otimes}(X, Y^{\mathrm{fd}})_1)^k \cong \mathbf{Fun}^{\otimes}(X, Y^{\mathrm{fd}})_k$$

give us the equivalence of symmetric monoidal Segal n-spaces. □

Definition 2.7.12. An object $x \in \operatorname{Ob} X$ is **fully dualisable** if it is contained in the essential image of $X^{\mathrm{fd}} \to X$, that is, if it is homotopically equivalent to some x' in the image of the map.

Corollary 2.7.13. *Suppose X is a symmetric monoidal Segal n-space with duals, then its Segal completion $\widetilde{X}$ has duals.*

Proof. The completion map $X \to \widetilde{X}$ factors through $\widetilde{X}^{\mathrm{fd}}$, so by the universality of the Segal completion, $\widetilde{X}^{\mathrm{fd}} \cong \widetilde{X}$. □

Finally, we prove a couple of results to relate duals and adjoints to ∞-groupoids.

Lemma 2.7.14. *Let C be a 2-category and $f : x \to y$ a 1-morphism in C. Then, f admits a left and a right adjoint if f is an isomorphism. The converse holds if all 2-morphisms in C are invertible.*

Proof. If f is an isomorphism, then there exists $g : y \to x$ such that $g \circ f \cong \mathrm{id}_x$ and $f \circ g \cong \mathrm{id}_y$. So, g is both a left and a right adjoint to f.

Conversely, suppose all 2-morphisms in C are invertible, and let g be a right adjoint to f. Then, there exist 2-isomorphisms

$$u : \mathrm{id}_x \to g \circ f, \qquad v : f \circ g \to \mathrm{id}_y.$$

Hence, g is isomorphic to f. Similarly for f admitting a left adjoint. □

Definition 2.7.15. An r-morphism $f : x \to y$ in a Segal n-space is **homotopy invertible** if it is invertible in $\operatorname{Ho}\operatorname{map}^r_W(x, y)$.

Proposition 2.7.16. *Let W be a Segal n-space. Suppose every r-morphism is homotopy invertible, then W has adjoints for r-morphisms. The converse holds if all $(r+1)$-morphisms are also homotopy invertible.*

Proof. This is a direct consequence of the above lemma. □

We immediately get the following corollary:

Corollary 2.7.17. *Let W be a Segal n-space. If W has adjoints for r-morphisms for all $r \leq n$, then W is an ∞-groupoid.*

Remark 2.7.18. Note that the above corollary implies that a Segal n-space W cannot have duals when viewed as a Segal $(n+1)$-space unless it is an ∞-groupoid.

For any (∞, n)-category, this gives us a construction of a maximal sub-∞-groupoid.

Proposition 2.7.19. *Let X be an (∞, n)-category. For each $0 \leq r < n$, there exists a* **maximal sub-(∞, r)-category** *$X^r \to X$ in n-$\mathcal{CSS}$ such that all maps from (∞, r)-categories to X_1 factors through X^r_1.*

The map $X \mapsto X_r$ is right adjoint to the inclusion r-$\mathbf{SeSp}^{\otimes} \to n$-$\mathbf{SeSp}^{\otimes}$. For any (∞, r)-category X in n-$\mathcal{CSS}^{\otimes}$ and (∞, n)-category Y, there is a Dwyer-Kan equivalence of (∞, n)-categories

$$\mathbf{Fun}(X, Y) \cong \mathbf{Fun}(X, Y^r).$$

Proof. The construction is similar to that given in the proof of Thm. 2.7.9. We can construct a chain

$$X^r \to X^{r+1} \to \cdots \to X^{n-1} \to X^n = X$$

where each map $X^{k-1} \to X^k$ is the universal replacement of X^k with a sub-(∞, n)-category where all k-morphisms have adjoints. Thus, X^r has adjoints for k-morphisms for all $r < k \leq n$ and is a (∞, r)-category by Cor. 2.7.17. The universality of each replacement and the fact that an (∞, r)-category has adjoints for k-morphisms for all $r < k \leq n$ gives the universality of the construction.

The last two statements follow from Remark 2.7.10 and Cor. 2.7.11. □

Chapter 3

The bordism category

In this chapter, we will use the formal machineries from the last chapter to define the bordism category in the form that Lurie gave in [Lur09c]. We will start off by proving some topological preliminaries. The main part of the chapter will be dedicated to proving that the bordism category is a symmetric monoidal Segal n-space with duals. The key steps in the construction and the proof are summarised in [Lur09c, Chap. 2.2], with some elaborations given in [Mal]. We will conclude the chapter with the formal statement of the cobordism hypothesis.

3.1 Moduli spaces of bordisms

In the construction of the bordism categories, we need to understand the topology on the set of cobordisms. We shall present a collection of relevant results in this section. Most of the constructions were originally presented in [GMTW09, Gal11].

Let M be a smooth compact abstract real manifold (with or without boundary) of dimension m and V a finite dimensional real vector space. Let $\mathrm{Emb}(M,V)$ be the set of all smooth embeddings of M into V. We can endow $\mathrm{Emb}(M,V)$ with the Whitney C^∞-topology. A basis of open neighbourhoods of $f \in \mathrm{Emb}(M,V)$ can be given by

$$\mathcal{V}(f,\overline{\varepsilon}) = \{g \in \mathrm{Emb}(M,V) | \|D^k(g-f)\| \leq \varepsilon_k \ \forall\ k\}$$

where $\overline{\varepsilon} = (\varepsilon_0, \varepsilon_1, \ldots)$ is a bounded sequence in $\mathbb{R}_{>0}$.

The set of all compact smooth submanifolds of V diffeomorphic to M can be identified with

$$B_M^V = \mathrm{Emb}(M,V)/\mathrm{Diff}(M).$$

Let $\mathbb{R}^\infty = \varinjlim_n \mathbb{R}^n$ given by standard inclusions $\mathbb{R}^n \subset \mathbb{R}^{n+1}$. We can then define the inductive limits

$$\mathrm{Emb}(M,\mathbb{R}^\infty) = \varinjlim_{\substack{V \subset \mathbb{R}^\infty \\ \dim_\mathbb{R} V < \infty}} \mathrm{Emb}(M,V) \qquad \text{and} \qquad B_M = \mathrm{Emb}(M,\mathbb{R}^\infty)/\mathrm{Diff}(M) \cong \varinjlim_{\substack{V \subset \mathbb{R}^\infty \\ \dim_\mathbb{R} V < \infty}} B_M^V$$

induced by inclusions of vectorspaces. We have the induced final topology on $\mathrm{Emb}(M,\mathbb{R}^\infty)$ and B_M.

The collection of all compact submanifolds of dimension $m-1$ in V is denoted by $B_0^V(m) = \coprod_{[M]} B_M^V$ where the disjoint union is taken over all diffeomorphism classes of smooth compact manifolds of dimension $m-1$. Let $B_0(m) = \coprod_{[M]} B_M \cong \varinjlim_V B_0^V(m)$.

The following results allow us to identify the space of subvarieties B_M with the abstract diffeomorphism group $\mathrm{Diff}(M)$.

Definition 3.1.1. Let G be a topological group. A principal bundle is a fibre bundle $\pi : E \to M$ with a right action $E \times G \to E$ that preserves the fibre and is free and transitive.

Proposition 3.1.2. *$\pi : \mathrm{Emb}(M,\mathbb{R}^\infty) \to B_M$ is a principal $\mathrm{Diff}(M)$-bundle.*

Proof. Set theoretically, there is a bijection $\mathrm{Emb}(M, \mathbb{R}^\infty) \to B_M \times \mathrm{Diff}(M)$. Hence, any $X \in B_M$ has a contractible neighbourhood U such that $\pi^{-1}(U) \cong U \times \mathrm{Diff}(M)$ is a trivial bundle. $\mathrm{Emb}(M, \mathbb{R}^\infty) \to B_M$ is principal by the definition of B_M. $\square$

Proposition 3.1.3 ([Sta])**.** $\mathrm{Emb}(M, \mathbb{R}^\infty)$ *is contractible.*

Proof. Consider the embedding $S_+ : \mathbb{R}^\infty \to \mathbb{R}^\infty$ sending $(x_i)_{i=1}^\infty \to (0, x_1, 0, x_2, 0, \ldots)$. This induces a split exact sequence

$$0 \to \mathbb{R}^\infty \xrightarrow{S_+} \mathbb{R}^\infty \xrightarrow{P_+} \mathbb{R}^\infty \to 0$$

where $P_+(x_i) = (x_1, x_3, x_5, \ldots)$. We thus have an isomorphism $\mathbb{R}^\infty \oplus \mathbb{R}^\infty \xrightarrow{S_+ + S_-} \mathbb{R}^\infty$ where $S_-(x_i) = (x_1, 0, x_2, 0, \ldots)$.

Consider the homotopy

$$H : \mathbb{R}^\infty \times I \to \mathbb{R}^\infty : (v, t) \mapsto (1-t)v + tS_+v.$$

For each $t \in [0, 1]$, H_t is injective, otherwise there exists $v \neq v'$ such that $S_+(v - v') = -t^{-1}(1-t)(v - v')$ but S_+ has no eigenvectors. Hence, for each t, H_t defines a smooth embedding.

H induces a smooth homotopy

$$H_* : \mathrm{Emb}(M, \mathbb{R}^\infty) \times I \to \mathrm{Emb}(M, \mathbb{R}^\infty) : (\phi, t) \mapsto H_t \circ \phi,$$

thus $\mathrm{id}_{\mathrm{Emb}(M,\mathbb{R}^\infty)} \sim S_{+*}$.

Pick $\tilde{\phi}_0 \in \mathrm{Emb}(M, \mathbb{R}^\infty)$ and let $\phi_0 = S_- \tilde{\phi}_0$. Consider the smooth homotopy

$$G : \mathrm{Emb}(M, \mathbb{R}^\infty) \times I \to \mathrm{Emb}(M, \mathbb{R}^\infty) : (\phi, t) \mapsto (1-t)S_+\phi + t\phi_0.$$

It is well-defined since for all $t \neq 0$, $S_- P_+((1-t)S_+\phi + t\phi_0) = t\phi_0$ is an embedding, so $(1-t)S_+\phi + t\phi_0$ is as well. G gives a homotopy of S_{+*} to the constant map $\mathrm{Emb}(M, \mathbb{R}^\infty) \mapsto \phi_0$, so composing with the homotopy H_* gives a deformation retraction of $\mathrm{Emb}(M, \mathbb{R}^\infty)$ onto $\{\phi_0\}$. Hence, $\mathrm{Emb}(M, \mathbb{R}^\infty)$ is contractible. $\square$

Corollary 3.1.4. *There is a weak equivalence from the classifying space* $B\mathrm{Diff}(M)$ *(the geometric realization of the nerve of* $\mathrm{Diff}(M)$*) to* B_M*.*

Proof. This follows immediately from the fact that $\mathrm{Emb}(M, V) \to B_M$ is a principal $\mathrm{Diff}(M)$-bundle and $\mathrm{Emb}(M, V)$ is contractible. $\square$

However, when considering bordisms, we will like to consider embeddings which are well-behaved on the boundaries. Given a smooth compact abstract real manifold M of dimension m with boundary $\partial M = \partial_0 M \sqcup \partial_1 M$ and V a finite dimensional vectorspace, we let $\mathrm{Emb}((M, \partial_\nu M), V \times [0, 1])$ be the set of all smooth embeddings ϕ of M into $V \times [0, 1]$ such that $\phi(M) \cap (V \times \{\nu\}) = \phi(\partial_\nu M)$ for $\nu = 0, 1$ and such that $\phi(M)$ intersects $V \times \{\nu\}$ transversely (that is, for any $x \in \phi(\partial_\nu M)$, $T_x(\phi(M)) + T_x(V \times \{\nu\}) = T_x V$). $\mathrm{Emb}((M, \partial_\nu M), V \times [0, 1]) \subset \mathrm{Emb}(M, V \times \mathbb{R})$ is endowed with the subspace topology.

The set of all compact smooth submanifolds $(X, \partial_0 X, \partial_1 X)$ of $V \times [0, 1]$ diffeomorphic to $(M, \partial_0 M, \partial_1 M)$, such that $X \cap V \times \{\nu\} = X_\nu$ for $\nu = 0, 1$ and such that X intersects $V \times \{0, 1\}$ transversely, can be identified with

$$B^V_{M,\partial_\nu M} = \mathrm{Emb}((M, \partial_\nu M), V \times [0, 1])/\mathrm{Diff}(M, \partial_\nu M)$$

where $\mathrm{Diff}(M, \partial_\nu M)$ is the space of diffeomorphisms of M that restrict to diffeomorphisms on $\partial_0 M$ and $\partial_1 M$. We similarly define

$$\begin{aligned}
\mathrm{Emb}((M, \partial_\nu M), \mathbb{R}^\infty \times [0, 1]) &= \varinjlim_{\substack{V \subset \mathbb{R}^\infty \\ \dim_{\mathbb{R}} V < \infty}} \mathrm{Emb}((M, \partial_\nu M), V \times [0, 1]), \\
B_{M,\partial_\nu M} &= \varinjlim_{\substack{V \subset \mathbb{R}^\infty \\ \dim_{\mathbb{R}} V < \infty}} B^V_{M,\partial_\nu M},
\end{aligned}$$

$$B^V_1(m) = \coprod_{[M,\partial_\nu M]} B^V_{M,\partial_\nu M} \qquad \text{and} \qquad B_1(m) = \coprod_{[M,\partial_\nu M]} B_{M,\partial_\nu M}$$

where the disjoint unions are taken on all diffeomorphism classes of smooth compact manifolds of dimension m with boundary (we allow all diffeomorphisms of manifolds, not imposing any conditions on the restrictions to partitions of the boundaries).

We can take it one step further. Let $k \geq 1$ be an integer. Let M be a smooth compact abstract manifold of dimension m. Let $M_i \subset M$ $(1 \leq i \leq k)$ be smooth closed submanifolds of dimension m with boundaries $\partial M_i = \partial_0 M_i \sqcup \partial_1 M_i$ such that $M = \cup_{i=1}^k M_i$, $M_i \cap M_{i+1} = \partial_1 M_i = \partial_0 M_{i+1}$ for $1 \leq i < k$, $\partial M = \partial_0 M_1 \sqcup \partial_1 M_k$ and $M_i \cap M_j = \emptyset$ if $j \neq i-1, i, i+1$. Let the set of such manifolds be denoted as $\mathcal{M}^1$.

Let $\mathrm{Emb}((M, \{\partial_\nu M_i\}), V \times [0,1])$ be the set of all smooth embeddings ϕ of M into $V \times [0,1]$ such that there exists a set $\{0 = a_0 < a_1 < \ldots < a_k = 1\}$ satisfying

(i) $\pi^{-1}(a_0) = \partial_0 M_0$ and $\pi^{-1}(a_i) = \partial_1 M_i$ for $i \geq 1$ where π is the composition $M \xrightarrow{\phi} V \times [0,1] \to [0,1]$; and

(ii) $\phi(M)$ intersects $V \times \{a_i, \ldots, a_k\}$ transversely.

$\mathrm{Emb}((M, \{\partial_\nu M_i\}), V \times [0,1]) \subset \mathrm{Emb}(M, V \times \mathbb{R})$ is given the subspace topology. We similarly define $\mathrm{Diff}(M, \{\partial_\nu M_i\})$, $B^V_{M,\{\partial_\nu M_i\}}$, $\mathrm{Emb}((M, \{\partial_\nu M_i\}), \mathbb{R}^\infty \times [0,1])$, $B_{M,\{\partial_\nu M_i\}}$, $B^V_n(m)$ and $B_n(m)$.

As with B_M, we can relate $B_{M,\{\partial_\nu M_i\}}$ and $\mathrm{Diff}(M, \{\partial_\nu M_i\})$.

Proposition 3.1.5. $\mathrm{Emb}((M, \{\partial_\nu M_i\}), \mathbb{R}^\infty \times [0,1]) \to B_{M,\{\partial_\nu M_i\}}$ *is a principal* $\mathrm{Diff}(M, \{\partial_\nu M_i\})$*-bundle.*

Proof. $\mathrm{Emb}((M, \{\partial_\nu M_i\}), \mathbb{R}^\infty \times [0,1]) \to B_{M,\{\partial_\nu M_i\}}$ is a fibre bundle since it is a restriction of the fibre bundle $\mathrm{Emb}(M, \mathbb{R}^\infty \times \mathbb{R}) \to B_M$ to $B_{M,\{\partial_\nu M_i\}}$. That it is principal follows immediately from the definition of $B_{M,\{\partial_\nu M_i\}}$. □

Lemma 3.1.6. $\mathrm{Emb}(M, [0,1) \times \mathbb{R}^\infty)$ *and* $\mathrm{Emb}(M, [0,1] \times \mathbb{R}^\infty)$ *are contractible.*

Proof. We can construct S_+ and P_+ as in Prop. 3.1.3, but the sequence

$$0 \to J \times \mathbb{R}^\infty \xrightarrow{S_+} J \times \mathbb{R}^\infty \xrightarrow{P_+} J \times \mathbb{R}^\infty \to 0$$

where $J = [0,1)$ or $[0,1]$ is no longer exact. However, we can still construct the section S_- and have an inclusion $J \times \mathbb{R}^\infty \oplus J \times \mathbb{R}^\infty \xrightarrow{S_+ + S_-} J \times \mathbb{R}^\infty$. The rest of the proof follows as above. □

Proposition 3.1.7. $\mathrm{Emb}((M, \{\partial_\nu M_i\}), \mathbb{R}^\infty \times [0,1])$ *is contractible.*

Proof. We first consider $\mathrm{Emb}((M, \partial_\nu M), \mathbb{R}^\infty \times [0,1])$. It is isomorphic to the fibred product

$$\mathrm{Emb}(\partial_0 M, \mathbb{R}^\infty \times \{0\}) \times_{\mathrm{Emb}(\partial_0 M, \mathbb{R}^\infty \times [0,1])} \mathrm{Emb}(M, \mathbb{R}^\infty \times [0,1]) \times_{\mathrm{Emb}(\partial_1 M, \mathbb{R}^\infty \times [0,1])} \mathrm{Emb}(\partial_1 M, \mathbb{R}^\infty \times \{1\})$$

where the maps are given by restrictions.

Since each of the spaces in the fibred product can be strongly deformation retracted to a single point in a compatible way, so can the fibred product. Hence, $\mathrm{Emb}((M, \partial_\nu M), \mathbb{R}^\infty \times [0,1])$ is contractible.

In general, $\mathrm{Emb}((M, \{\partial_\nu M_i\}), \mathbb{R}^\infty \times [0,1])$ is isomorphic to the fibred product

$$\left(\prod_{i=1}^{k-1} \mathrm{Emb}(\partial_1 M_i, \mathbb{R}^\infty) \times J \right) \times_{\mathrm{Emb}(\coprod_{i=1}^{k-1} \partial_1 M, \mathbb{R}^\infty \times [0,1])} \mathrm{Emb}((M, \{\partial_0 M_1, \partial_1 M_k\}), \mathbb{R}^\infty \times [0,1])$$

where $J = \{(x_1, \ldots, x_{k-1}) \in (0,1)^{k-1} | x_1 < x_2 < \cdots < x_{k-1}\}$. J and all other factors in the fibred product are strongly deformation retractable to a single point in a compatible way, so $\mathrm{Emb}((M, \{\partial_\nu M_i\}), \mathbb{R}^\infty \times [0,1])$ is contractible. □

Prop. 3.1.5 and 3.1.7 immediately gives us the weak equivalence as before:

Corollary 3.1.8. *There exists a weak equivalence* $B\mathrm{Diff}(M, \{\partial_\nu M_i\}) \to B_{M,\{\partial_\nu M_i\}}$.

One of our primary purposes in introducing the weak equivalence above is to establish a weak equivalence between the bordism moduli spaces defined above and a collared version that we will define below. The collared version is useful in allowing us to construct smooth composition of bordisms, which is not possible in the bordism moduli spaces defined above (Given $X \in B_{M,\partial_\nu M}$ and $Y \in B_{M',\partial_\nu M'}$ with a $\partial_1 X = \sigma_1 \partial_0 Y$, the union $X \cup \sigma_1 Y$ where $\sigma_1 Y$ is a shift of Y from $\mathbb{R}^\infty \times [0,1]$ to $\mathbb{R}^\infty \times [1,2]$ may not be smooth).

Let $\partial_0 M \subset \partial M$ be a component of the boundary of M. The tubular neighbourhood theorem [Hir94, IV, Thm 5.1] states that there exists an embedding $\partial_0 M \times [0,1) \to M$. We call the image of $N \times [0,1)$ a collar of N in M.

Recall the manifold with $n+1$-distinguished submanifolds $(M, \{\partial_\nu M_i\})$ defined above. For each $\partial_\nu M_i$, we can choose a collar $c_{\nu,i} : \partial_\nu M_i \times [0,1) \xrightarrow{\sim} C_{\nu,i} \subset M_i$ such that $C_{0,i} \cap C_{1,i} = \emptyset$.

Let $\mathrm{Emb}_c((M, \{\partial_\nu M_i\}), V \times [0,1]) \subset \mathrm{Emb}((M, \{\partial_\nu M_i\}), V \times [0,1])$ be the set of all smooth embeddings that maps each collar $c_{\nu,i}$ trivially into a cylinder $\partial_\nu M_i \times [a,b)$, that is, the set of smooth embeddings $\phi \in \mathrm{Emb}((M, \{\partial_\nu M_i\}), V \times [0,1])$ such that

$$\phi|_{C_{\nu,i}} = (\phi|_{\partial_\nu M_i} \times h_{\nu,i}) \circ c_{\nu,i}^{-1}$$

for some monotonous map $h_{\nu,i} : [0,1) \to [0,1]$.

We define $\mathrm{Diff}_c(M, \{\partial_\nu M_i\})$ to be set of diffeomorphisms whose action on the collars is the trivial extension of its action on the boundaries, that is, diffeomorphisms $f \in \mathrm{Diff}(M, \{\partial_\nu M_i\})$ such that

$$f|_{C_{\nu,i}} = c_{\nu,i} \circ (f|_{\partial_\nu M_i} \times \mathrm{id}_{[0,1)}) \circ c_{\nu,i}^{-1}.$$

Let $B^V_{c,M,\{\partial_\nu M_i\}} = \mathrm{Emb}_c((M, \{\partial_\nu M_i\}), V \times [0,1]) / \mathrm{Diff}_c(M, \{\partial_\nu M_i\})$.

Similarly define $\mathrm{Emb}_c((M, \{\partial_\nu M_i\}), \mathbb{R}^\infty \times [0,1])$, $B_{c,M,\{\partial_\nu M_i\}}$ and $B_{c,k}(m)$.

As before, we have the following results:

Proposition 3.1.9. $\mathrm{Emb}_c((M, \{\partial_\nu M_i\}), \mathbb{R}^\infty \times [0,1]) \to B_{c,M,\{\partial_\nu M_i\}}$ *is a principal* $\mathrm{Diff}_c(M, \{\partial_\nu M_i\})$*-bundle,* $\mathrm{Emb}_c((M, \{\partial_\nu M_i\}), \mathbb{R}^\infty \times [0,1])$ *is contractible and there is a weak equivalence*

$$B\mathrm{Diff}_c(M, \{\partial_\nu M_i\}) \xrightarrow{\sim} B_{c,M,\{\partial_\nu M_i\}}.$$

Proof. The only point that needs to be proven is that $\mathrm{Emb}_c((M, \{\partial_\nu M_i\}), \mathbb{R}^\infty \times [0,1])$ is contractible. Note that the collared embedding space is isomorphic to the fibred product

$$\mathrm{Emb}((M, \{\partial_\nu M_i\}), \mathbb{R}^\infty \times [0,1]) \times_{\mathrm{Emb}(\cup C_{\nu,i}, \mathbb{R}^\infty \times [0,1])} \left(\mathrm{Emb}(\partial_0 M_1, \mathbb{R}^\infty) \times \prod_{i=1}^{k} \mathrm{Emb}(\partial_1 M_i, \mathbb{R}^\infty) \right)$$

where the map $\mathrm{Emb}(\partial_0 M_1, \mathbb{R}^\infty) \times \prod_{i=1}^k \mathrm{Emb}(\partial_1 M_i, \mathbb{R}^\infty) \to \mathrm{Emb}(\cup C_{\nu,i}, \mathbb{R}^\infty \times [0,1])$ is given by

$$(\phi_0, \ldots, \phi_k) \mapsto (\phi_0 \times h_{0,1}) \circ c_{0,1}^{-1} \cup (\phi_1 \times h_{1,0}) \circ c_{1,0}^{-1} \cup \cdots \cup (\phi_k \times h_{1,k}) \circ c_{1,k}^{-1}.$$

As in the proof of Prop. 3.1.7, the spaces in the fibred product are all strongly deformation retractable in compatible ways to points, so they give a strong deformation retraction of the fibred product, so $\mathrm{Emb}_c((M, \{\partial_\nu M_i\}), \mathbb{R}^\infty \times [0,1])$ is contractible. □

We now want to show that the inclusion $B_{c,M,\{\partial_\nu M_i\}} \to B_{M,\{\partial_\nu M_i\}}$ is a weak homotopy equivalence.

Let $N \subset M$ be a submanifold and $\partial_0 M$ be a component of the boundary of M such that $\partial_0 M \subset N$. Define $\mathrm{Emb}(N, M, \partial_0 M)$ to be the set of all smooth embeddings ϕ of N into M such that $\phi|_{\partial_0 M}$ is a diffeomorphism on $\partial_0 M$.

Proposition 3.1.10. *The inclusion* $\mathrm{Diff}_c(M, \{\partial_\nu M_i\}) \to \mathrm{Diff}(M, \{\partial_\nu M_i\})$ *is a weak equivalence.*

Proof. Recall that

$$\mathrm{Diff}_c(M, \{\partial_\nu M_i\}) = \{f \in \mathrm{Diff}(M, \{\partial_\nu M_i\}) | \, f|_{C_{\nu,i}} = c_{\nu,i} \circ (f|_{\partial_\nu M_i} \times \mathrm{id}_{[0,1)}) \circ c_{\nu,i}^{-1} \; \forall \, \nu, i\}$$

We can thus construct $\mathrm{Diff}_c(M, \{\partial_\nu M_i\})$ via a sequence of pullbacks:

$$\mathrm{Diff}_c(M, \{\partial_\nu M_i\}) = D_{0,1} \to D_{1,1} \to D_{0,2} \to \ldots \to D_{1,k} \to D_{0,k+1} = \mathrm{Diff}(M, \{\partial_\nu M_i\})$$

where $D_{\nu,i}$ is the pullback of the following term $D_{\nu',i'}$ given by the square

$$\begin{tikzcd} D_{\nu,i} \arrow[r] \arrow[d] & D_{\nu',i'} \arrow[d, "\Phi_{\nu,i}"] & f \arrow[d, mapsto] \\ \mathrm{Diff}(\partial_\nu M_i) \arrow[r] & \mathrm{Emb}(C_{\nu,i}, M_i, \partial_\nu M_i) & f|_{C_{\nu,i}} \\ g \arrow[r, mapsto] & c_{\nu,i} \circ (g \times \mathrm{id}_{[0,1)}) \circ c_{\nu,i}^{-1} & \end{tikzcd}$$

where $\mathrm{Emb}(C_{\nu,i}, M_i, \partial_\nu M_i)$ is the space of embeddings ϕ of $C_{\nu,i}$ into M_i such that $\phi|_{\partial_\nu M_i}$ is a diffeomorphism of $\partial_\nu M_i$ and such that $ol\phi(C_{\nu,i}) - \partial_\nu M_i$ is contained in the interior of M_i.

The vertical arrow $D_{\nu',i'} \to \mathrm{Emb}(C_{\nu,i}, M_i, \partial_\nu M_i)$ given by the restriction map is a surjection and hence a fibration by [Cer61, Chap. II, 2.2.2, Cor. 2].

Let $\mathrm{Emb}_c(C_{\nu,i}, M_i, \partial_\nu M_i)$ be the image of the inclusion $\mathrm{Diff}(\partial_\nu M_i) \to \mathrm{Emb}(C_{\nu,i}, M_i, \partial_\nu M_i)$ given by the lower horizontal arrow. Then, $\mathrm{Emb}_c(C_{\nu,i}, M_i, \partial_\nu M_i) \to \mathrm{Emb}(C_{\nu,i}, M_i, \partial_\nu M_i)$ is a weak homotopy equivalence by [Cer61, Chap. II, 4.2.3, Cor. 3].

Hence, since **Top** is a proper model category, $D_{\nu,i} \to D_{\nu',i'}$ is also a weak homotopy equivalence. ☐

Proposition 3.1.11. *The inclusion $B_{c,M,\{\partial_\nu M_i\}} \to B_{M,\{\partial_\nu M_i\}}$, and hence $B_{c,k}(m) \to B_k(m)$, are weak homotopy equivalences.*

Proof. Consider the diagram of fibre bundles

$$\begin{tikzcd} \mathrm{Diff}_c(M, \{\partial_\nu M_i\}) \arrow[r] \arrow[d, "\sim"] & \mathrm{Emb}_c((M, \{\partial_\nu M_i\}), \mathbb{R}^\infty \times [0,1]) \arrow[r] \arrow[d, "\sim"] & B_{c,M,\{\partial_\nu M_i\}} \arrow[d] \\ \mathrm{Diff}(M, \{\partial_\nu M_i\}) \arrow[r] & \mathrm{Emb}((M, \{\partial_\nu M_i\}), \mathbb{R}^\infty \times [0,1]) \arrow[r] & B_{M,\{\partial_\nu M_i\}} \end{tikzcd}$$

The left and middle vertical arrows are weak homotopy equivalences, so by the long exact sequence of homotopy for fibrations, the right arrow also induces a weak equivalence. ☐

Remark 3.1.12. Indeed, we can generalise Prop. 3.1.11. Let $Z \subset B_{M,\{\partial_\nu M_i\}}$ be any subspace such that the preimage of Z in $\mathrm{Emb}((M, \{\partial_\nu M_i\}), \mathbb{R}^\infty \times [0,1])$ is contractible, then diagram of fibre bundles implies that $Z \cap B_{c,M,\{\partial_\nu M_i\}} \to Z$ is a weak equivalence.

We can inductively generalise $\mathrm{Emb}((M, \{\partial_\nu M_i\}), V \times [0,1])$ and $B_{M,\{\partial_\nu M_i\}}$ to higher dimensions.

Recall we defined $\mathcal{M}^1$ to be the set of pairs $(M, \{\partial_\nu M_i\})$. Now, suppose we have defined $\mathcal{M}^r$ for all $r < n$ and let $n \le m = \dim M$. We describe an element of $\mathcal{M}^n$. Let M be a smooth abstract compact manifold (with boundary) of dimension m. Fix positive integers $k_1, \ldots, k_n$ and partition M into $k_1 \cdots k_n$ closed submanifolds $M_{j_1,\ldots,j_n}$ with boundaries such that

$$\begin{aligned} \partial M_{j_1,\ldots,j_n} &= \bigcup_{i=1}^n \Big(\partial_{i,0} M_{j_1,\ldots,j_n} \cup \partial_{i,1} M_{j_1,\ldots,j_n} \Big), \\ \partial M &= \bigcup_{i=1}^n \Big(\Big(\bigcup \partial_{i,0} M_{\ldots,1,\ldots} \Big) \cup \Big(\bigcup \partial_{i,1} M_{\ldots,k_i,\ldots} \Big) \Big) \qquad \text{and} \\ M_{\ldots,j_i,\ldots} \cap M_{\ldots,j_i+1,\ldots} &= \partial_{i,1} M_{\ldots,j_i,\ldots} = \partial_{i,0} M_{\ldots,j_i+1,\ldots} \end{aligned}$$

and such that, for each fixed j_i, $\bigcup \partial_{i,\nu} M_{\ldots,j_i,\ldots}$ with this given partition is an element of $\mathcal{M}^{n-1}$. Such $(M, \{\partial_{\nu_i} M_{\{j_i\}}\})$ defines an element of $\mathcal{M}^n$.

We define $\mathrm{Emb}((M, \{\partial_{i,\nu} M_{\{j_i\}}\}), V \times [0,1]^n)$ to be the set of smooth embeddings of $M \in \mathcal{M}^n$ of dimension m into $V \times [0,1]^n$ such that there exist sets $\{0 = a_0^i < a_1^i < \cdots < a_{k_i}^i = 1\}$ for each $1 \leq i \leq n$ satisfying

(i) for each $1 \leq i \leq n$,

$$\pi_i^{-1}(a_j^i) = \begin{cases} \bigcup_{j_1,\ldots,j_{i-1},j_{i+1},\ldots,j_n} \partial_{i,0} M_{j_1,\ldots,j_{i-1},1,j_{i+1},\ldots,j_n} & j = 0 \\ \bigcup_{j_1,\ldots,j_{i-1},j_{i+1},\ldots,j_n} \partial_{i,1} M_{j_1,\ldots,j_{i-1},j,j_{i+1},\ldots,j_n} & j \geq 1 \end{cases}$$

where π_i is the projection $M \to V \times [0,1]^n \to [0,1]^{\{i\}}$ to the i-th component of $[0,1]^n$;

(ii) for any $S \subset \{1, \ldots, n\}$ and any collection of indices $\{j_s\}_{s \in S}$ where $0 \leq j_s \leq k_s$, X intersects $V \times [0,1]^{\{1,\ldots,n\}-S} \times \prod_{s \in S} \{a_{j_s}\}$ traversely; and

(iii) for any $1 \leq i < n$, the projection $M \to V \times [0,1]^n \to V \times [0,1]^{\{i+1,\ldots,n\}}$ is a submersion for all points $x \in M$ that project to the set $\{a_1^i, \ldots, a_{k_i}^i\}$ in $\mathbb{R}^{\{i\}}$.

As before, we can define $B^{n,V}_{M,\{\partial_{i,\nu} M_{\{j_i\}}\}} = \mathrm{Emb}((M, \{\partial_{i,\nu} M_{\{j_i\}}\}), V \times [0,1]^r)/\mathrm{Diff}(M, \{\partial_{i,\nu} M_{\{j_i\}}\})$. We similarly obtain the colimit $B^{n}_{M,\{\partial_{i,\nu} M_{\{j_i\}}\}}$ taken over all finite dimensional vectorspaces V.

For any n-uple $\mathbf{k} = (k_1, \ldots, k_n)$ of non-negative integers, let $k_{i_1}, \ldots, k_{i_r}$ be positive and the remaining indices be 0. We define $B^{n,V}_{\mathbf{k}}(m)$ and $B^{n}_{\mathbf{k}}(m)$ to be the disjoint union of $B^{n,V}_{M,\{\partial_{i_d,\nu} M_{\{j_{i_d}\}}\}}$ and $B^{n}_{M,\{\partial_{i_d,\nu} M_{\{j_{i_d}\}}\}}$ respectively over all isomorphism classes of $(M, \{\partial_{i_d,\nu} M_{\{j_{i_d}\}}\}_{1 \leq d \leq r})$ where $M \in \mathcal{M}^r$ is a manifold of dimension $m - n + r$.

We can also define a collared version $B^n_{c,\mathbf{k}}(m)$ and show that $B^n_{c,\mathbf{k}}(m) \to B^n_{\mathbf{k}}(m)$ is a weak equivalence.

3.2 Bordism category as a symmetric monoidal Segal space

In this section, we will give a precise construction of the bordism category as a symmetric monoidal Segal n-space. We proceed with the construction as follows: we first define $\mathbf{SemiBord}_n$ as a semisimplicial n-space. This construction is geometrically intuitive and it is easy to show that $\mathbf{SemiBord}_n$ is a semiSegal n-space. We then construct the simplicial space $\mathbf{PBord}_n$ and show that it is weakly equivalent to $\mathbf{SemiBord}_n$ as semiSegal spaces, so $\mathbf{PBord}_n$ is a Segal n-space. Finally, we show that there exists a well-defined symmetric monoidal structure on $\mathbf{PBord}_n$.

We start with a Segal 1-space. We define the following semisimplicial space.

Definition 3.2.1. We define a simplicial space $\mathbf{SemiCob}(m)^V$ as follows:

Let $\mathbf{SemiCob}(m)_0^V$ be the set of pairs (X, a_0) where X is a smooth compact submanifolds of V without boundary of dimension $m-1$ and $a_0 \in \mathbb{R}$.

For $k > 0$, let $\mathbf{SemiCob}(m)_k^V$ be the set of pairs $(X, (a_0 < \cdots < a_k))$ where $a_i \in \mathbb{R}$ and X is a smooth compact submanifold of $V \times [a_0, a_k]$ of dimension m such that

(i) $\partial X = X \cap (V \times \{a_0, a_k\})$,

(ii) X intersect $V \times \{a_i\}$ transversely for each $0 \leq i \leq k$ (i.e., for all $x \in X \cap (V \times \{a_i\})$, $T_x V = T_x X + T_x(V \times \{a_i\})$).

$\mathbf{SemiCob}(m)_k^V \subset B_m^V \times \mathbb{R}^2$ is given the subspace topology.

For any strictly increasing map $f : [k] \to [l]$, we define the induced map $f^* : \mathbf{SemiCob}(m)_l^V \to \mathbf{SemiCob}(m)_k^V$ by

$$f^*(X, (a_0 < \cdots < a_l))(X \cap (V \times [a_{f(0)}, a_{f(k)}]), (a_{f(0)} < \cdots < a_{f(k)})).$$

The spaces $\mathbf{SemiCob}(m)_k^V$ together with the above maps give a semisimplicial space $\mathbf{SemiCob}(m)^V$. We define

$$\mathbf{SemiCob}(m) = \varinjlim_{\substack{V \subset \mathbb{R}^\infty \\ \dim_{\mathbb{R}} V < \infty}} \mathbf{SemiCob}(m)^V.$$

We have $\mathbf{SemiCob}(m)_0 = B_0(m)$ and $\mathbf{SemiCob}(m)_k = B_k(m) \times J \subset B_k(m) \times \mathbb{R}^2$ for $k > 0$ where $J = \{(a_0, a_k) \in \mathbb{R}^2 | a_0 < a_k\}$.

We will sometimes write an element of $\mathbf{SemiCob}(m)_k$ as $\underline{X} = (X, (a_0 < \cdots < a_k))$, and if there is no ambiguity, we may simply denote it as X.

We can also define a collared version:

Definition 3.2.2. We define $\mathbf{SemiCob}_c(m)^V$ as follows: let $\mathbf{SemiCob}_c(m)^V_0 = B^V_0(m)$ and for $k > 0$, we define $\mathbf{SemiCob}_c(m)^V_k$ as above, with an additional condition

(iii) for every i, there exists a neighbourhood $W \subset [a_0, a_k]$ of a_i such that $X \cap (V \times W) = (X \cap (V \times \{a_i\})) \times W$.

By Prop. 3.1.11, we get that the inclusion map

$$\mathbf{SemiCob}_c(m) \to \mathbf{SemiCob}(m)$$

is a Reedy-weak equivalence.

Proposition 3.2.3. $\mathbf{SemiCob}_c(m)$ *is Reedy fibrant.*

Proof. We need to check that

$$\begin{tikzcd} \mathbf{SemiCob}_c(m)_n \arrow[r, "(d_0,\ldots,d_n)"] & \partial(\mathbf{SemiCob}_c(m)_n) \arrow[d, equal] \\ & \{(X_0, \ldots, X_n) \in (\mathbf{SemiCob}_c(m)_{n-1})^{n+1} | d_i X_j = d_{j-1} X_i \ \forall\, i < j\} \end{tikzcd}$$

is a Serre fibration for all n.

For $n = 0$, this holds since all objects in **Top** are fibrant.

For $n \geq 2$, the above map is an isomorphism since the elements of $\partial(\mathbf{SemiCob}(m)_n)$ are precisely of the form

$$\begin{aligned} &((X \cap (\mathbb{R}^\infty \times [a_1, a_k]), a_1 < \cdots < a_k), (X, a_0 < a_2 < \cdots < a_k), \ldots \\ &\quad \ldots, (X, a_0 < \cdots < a_{k-1} < a_k), (X \cap (\mathbb{R}^\infty \times [a_0, a_{k-1}]), a_0 < \cdots < a_{k-1})) \end{aligned}$$

and the collar allows us to glue $X \cap (\mathbb{R}^\infty \times [a_0, a_1])$ and $X \cap (\mathbb{R}^\infty \times [a_1, a_2])$ for $n = 2$.

The main part of the proof is for $n = 1$. We need to show that the map $\mathbf{SemiCob}_c(m)_1 \to \mathbf{SemiCob}(m)_0 \times \mathbf{SemiCob}(m)_0$ given by restriction to the boundaries is a fibration. Since $B_1(m) \times J \to B_1(m)$ is a fibration, it suffices to show that $B_1(m) \to B_0(m) \times B_0(m)$ is a fibration. Consider the commutative diagram

$$\begin{tikzcd} \coprod_{\dim[M]=m} \mathrm{Emb}_c((M, \partial_\nu M), \mathbb{R}^\infty \times [0,1]) \arrow[r] \arrow[d] & \Big(\coprod_{\dim[N]=m-1} \mathrm{Emb}(N, \mathbb{R}^\infty)\Big) \times \Big(\coprod_{\dim[N]=m-1} \mathrm{Emb}(N, \mathbb{R}^\infty)\Big) \arrow[d] \\ B_{c,1}(m) \arrow[r] & B_0(m) \times B_0(m). \end{tikzcd}$$

Since the vertical arrows are fibrations, we can lift any diagram

$$\begin{tikzcd} I^{k-1} \arrow[r] \arrow[d, "i_k"] & B_{c,1}(m) \arrow[d] \\ I^k \arrow[r] & B_0(m) \times B_0(m) \end{tikzcd}$$

to

$$\begin{tikzcd} I^{k-1} \arrow[r] \arrow[d, "i_k"] & \coprod_{\dim[M]=m} \mathrm{Emb}_c((M, \partial_\nu M), \mathbb{R}^\infty \times [0,1]) \arrow[d] \\ I^k \arrow[r] & \Big(\coprod_{\dim[N]=m-1} \mathrm{Emb}(N, \mathbb{R}^\infty)\Big) \times \Big(\coprod_{\dim[N]=m-1} \mathrm{Emb}(N, \mathbb{R}^\infty)\Big). \end{tikzcd}$$

Since I^k and I^{k-1} are connected, we can restrict to a single component. Note that a map $I^{k-1} \to \mathrm{Emb}_c((M, \partial_\nu M), \mathbb{R}^\infty \times [0,1])$ can be viewed as a continuous embedding

$$(M, \partial_\nu M) \times I^{k-1} \to \mathbb{R}^\infty \times I^{k-1} \times [0,1].$$

Hence, we just need to show that given continuous embeddings

$$(M, \partial_\nu M) \times I^{k-1} \xrightarrow{f} \mathbb{R}^\infty \times I^{k-1} \times [0,1], \qquad \partial_\nu M \times I^{k-1} \times I \xrightarrow{g_\nu} \mathbb{R}^\infty \times I^{k-1} \times I \quad \text{for } \nu = 0, 1$$

which are smooth and collared when restricted to any fixed point in I^{k-1} or I^k and such that $g_\nu(x, 0) = f|_{\partial_\nu M \times I^{k-1}}(x)$, we can extend them to a continuous embedding

$$(M, \partial_\nu M) \times I^{k-1} \times I \to \mathbb{R}^\infty \times I^{k-1} \times I \times [0,1]$$

which is smooth and collared when restricted to any fixed point in I^k.

Let

$$\tilde{M} = (\partial_0 M \times [-2, 0]) \coprod_{\partial_0 M} M \coprod_{\partial_1 M} (\partial_1 M \times [1, 3]),$$

so there is a homeomorphism $\phi : M \to M_t$ that sends the collars C_0 and C_1 into $\partial_0 M \times [-2, -1]$ and $\partial_1 M \times [2, 3]$ respectively.

Define $\tilde{f}_t : \tilde{M} \times I^{k-1} \xrightarrow{f} \mathbb{R}^\infty \times I^{k-1} \times [0,1]$ by

$$\tilde{f}_t(x) = \begin{cases} f(x) & x \in M \times I^{k-1} \\ g_0(z, -l) & x = (z, l) \in \partial_0 M \times I^{k-1} \times [-t, 0] \\ g_0(z, -t) & x = (z, l) \in \partial_0 M \times I^{k-1} \times [-2, -t] \\ g_1(z, l-1) & x = (z, l) \in \partial_1 M \times I^{k-1} \times [1, 1+t] \\ g_1(z, t) & x = (z, l) \in \partial_1 M \times I^{k-1} \times [1+t, 3]. \end{cases}$$

Let $\sigma : [-2, 3] \to [0, 1]$ be linear maps and let $f_t = (\mathrm{id}_{M \times I^{k-1}} \times \sigma) \circ \tilde{f}_t \circ \phi$. Thus,

$$f_t : (M, \partial_\nu M) \times I^{k-1} \times \{t\} \to \mathbb{R}^\infty \times I^{k-1} \times \{t\} \times [0,1]$$

is a continuous family of continuous embeddings. By Whitney's approximation theorem [Lee03], we can approximate the family f_t continuously with continuous embeddings that are smooth and collared when restricted to any fixed point in I^{k-1}. Furthermore, this approximation can be chosen to fix f_t where it is already smooth, that is, fixing f_0 and a collared neighbourhood around $\phi^{-1}(\partial_0 M \times [-2, -1] \cup \partial_1 M \times [2, 3]$. This gives a continuous embedding

$$\tilde{h} : (M, \partial_\nu M) \times I^{k-1} \times I \to \mathbb{R}^\infty \times I^{k-1} \times I \times [0,1]$$

which is smooth and collared when restricted to any fixed point in I^k and correspond to g_ν on the restrictions. Finally, since there exists a smooth transformation from f to f_0, we can extend it to a smooth transformation on $\tilde{h}$ that fixes the boundaries. Composing the transformation with $\tilde{h}$ gives us h that corresponds to f on $(M, \partial_\nu M) \times I^{k-1} \times \{0\}$. $\square$

Proposition 3.2.4. $\mathbf{SemiCob}(m)$ *is a semiSegal space.*

Proof. Since $\mathbf{SemiCob}_c(m) \to \mathbf{SemiCob}(m)$ is a weak equivalence, it suffices to prove the Segal condition for the former. $\mathbf{SemiCob}_c(m)$ is Reedy fibrant, so by Remark 2.2.4, we just need to show that the maps

$$\mathbf{SemiCob}_c(m)_k \xrightarrow{\phi} \mathbf{SemiCob}_c(m)_1 \times_{\mathbf{SemiCob}(m)_0} \cdots \times_{\mathbf{SemiCob}(m)_0} \mathbf{SemiCob}_c(m)_1$$

are weak homotopy equivalences.

By the strong deformation retractibility of J and $\mathbb{R}$, we have the diagram

$$\begin{tikzcd} B_{c,k}(m) \times J \arrow[r] & (B_{c,1}(m) \times J) \times_{B_0(m)} \cdots \times_{B_0(m)} (B_{c,1}(m) \times J) \\ B_{c,k}(m) \arrow[u, "\sim"] \arrow[r, "(\tau_1, \ldots, \tau_k)"] & B_{c,1}(m) \times_{B_0(m)} \cdots \times_{B_0(m)} B_{c,1}(m) \arrow[u, "\sim"] \end{tikzcd}$$

where $\tau_i : B_{c,k}(m) \to B_{c,1}(m)$ takes $X = \cup_{i=1}^k X_k \in B_k(m)$ to $(\mathrm{id}_{\mathbb{R}^\infty} \times \sigma)(X_i)$ where σ is a linear function taking $[a_{i-1}, a_i]$ to $[0,1]$.

It suffices to show that the bottom arrow is a weak equivalence. Indeed it is a homotopy equivalence. We can define a homotopy inverse

$$\psi : B_{c,1}(m) \times_{B_0(m)} \cdots \times_{B_0(m)} B_{c,1}(m) \to B_{c,k}(m)$$

by sending $(X_1, \ldots, X_k)$ to $X = \cup_{i=1}^k (\mathrm{id}_{\mathbb{R}^\infty} \times \sigma_i) X_i$ where σ_i is the linear map sending $[0,1]$ to $[\frac{i-1}{k}, \frac{i}{k}]$.

X is a smooth compact submanifold of $\mathbb{R}^\infty \times [0,1]$ since for each X_i, there exists $\epsilon > 0$ such that $X_i \cap \mathbb{R}^\infty \times [\frac{i-1}{k}, \frac{i-1}{k} + \epsilon) = \partial_0 X_i \times [\frac{i-1}{k}, \frac{i-1}{k} + \epsilon)$ and $X_i \cap \mathbb{R}^\infty \times (\frac{i}{k} - \epsilon, \frac{i}{k}] = \partial_1 X_i \times (\frac{i}{k} - \epsilon, \frac{i}{k}]$. Hence, ψ is well-defined.

We have $(\tau_1, \ldots, \tau_k) \circ \psi = \mathrm{id}$. For any sequence $0 = a_0 < \ldots < a_k = 1$, there exists a k-piecewise linear function $\sigma : [0,1] \to [0,1]$ such that $\sigma(a_i) = \frac{i}{k}$. We have a piecewise linear homotopy $h : [0,1] \times [0,1] \to [0,1]$ that takes $\mathrm{id}_{[0,1]}$ to σ. This induces a homotopy $H : B_{c,k}(m) \times [0,1] \to B_{c,k}(m)$ given by

$$H_t(X) = (\mathrm{id}_{\mathbb{R}^\infty} \times h_t) X.$$

$H_t(X)$ is smooth since X has a trivial collar at each of the non-smooth points of h_t. Hence, $\psi \circ (\tau_1, \ldots, \tau_k) \sim \mathrm{id}$. □

We can extend our definitions to higher Segal spaces.

Definition 3.2.5. Let $\mathbf{SemiBord}_{m,n}^V$ be a simplicial n-space in which each space $(\mathbf{SemiBord}_{m,n}^V)_{k_0,\ldots,k_n}$ is defined as follows:

Suppose $k_{i_1}, \ldots, k_{i_r}$ are positive and the remaining k_i are 0. An point in $(\mathbf{SemiBord}_{m,n}^V)_{k_0,\ldots,k_n}$ is a tuple $(X, (a_0^1 < \cdots < a_{k_1}^1), \ldots, (a_0^n < \cdots < a_{k_n}^n))$ where $a_{j_i}^i \in \mathbb{R}$ and X is a smooth compact submanifold of $V \times \prod_{i=1}^n [a_0^i, a_{k_i}^i]$ of dimension $m - n + r$ such that

(i) $\partial X = X \cap (V \times \prod_{d=1}^r \{a_0^{i_d}, a_{k_{i_d}}^{i_d}\})$;

(ii) for any $S \subset \{i_1, \ldots, i_r\}$ and r-uple $(a_{j_s}^s)_{s \in S}$ with $0 \le j_s \le k_s$ for each $s \in S$, X intersects $V \times \big(\prod_{t \notin S} [a_0^t, a_{k_t}^t]\big) \times \big(\prod_{s \in S} \{a_{j_s}^s\}\big)$ transversely;

(iii) for any $1 \le i < n$, the projection

$$X \hookrightarrow V \times \prod_{j=1}^n [a_0^j, a_{k_j}^j] \to \prod_{j=i+1}^r [a_0^j, a_{k_j}^j]$$

is a submersion at all points $x \in V$ whose projection into $\mathbb{R}^{\{i\}}$ lies in the set $\{a_0^i, \ldots, a_{k_i}^i\}$.

$(\mathbf{SemiBord}_{m,n}^V)_{k_0,\ldots,k_n} \subset B_{k_{i_0},\ldots,k_{i_r}}^r (m - n + r) \times \mathbb{R}^{n+r}$ is given the subspace topology.

For any strictly increasing map $f = (f_i) : \prod_{i=1}^n [k_i] \to \prod_{i=1}^n [l_i]$, we define the induced map $f^* : (\mathbf{SemiBord}_{m,n}^V)_{l_0,\ldots,l_n} \to (\mathbf{SemiBord}_{m,n}^V)_{k_0,\ldots,k_n}$ by

$$f^*(X, (a_0^j < \cdots < a_{l_j}^j)) = \Big(X \cap \big(V \times \prod_{e=1}^s [a_{f_j(0)}^j, a_{f_j(k_j)}^j]\big), (a_{f_i(0)}^i < \cdots < a_{f_i(k_i)}^i)\Big).$$

The spaces $(\mathbf{SemiBord}_{m,n}^V)_k$ together with the above maps give a semisimplicial n-space $\mathbf{SemiBord}_{m,n}^V$. We define

$$\mathbf{SemiBord}_{m,n} = \varinjlim_{\substack{V \subset \mathbb{R}^\infty \\ \dim_{\mathbb{R}} V < \infty}} \mathbf{SemiBord}_{m,n}^V.$$

We have $\mathbf{SemiBord}_{m,1} = \mathbf{SemiCob}(m)$ and we denote $\mathbf{SemiBord}_n = \mathbf{SemiBord}_{n,n}$.

Replacing with collared embeddings, we get $\mathbf{SemiBord}_{c,m,n}$.

Proposition 3.2.6. $\mathbf{SemiBord}_{c,m,n}$ *is Reedy-fibrant and* $\mathbf{SemiBord}_{m,n}$ *is a semiSegal* n*-space.*

Proof. The proof that $\mathbf{SemiBord}_{c,m,n}$ is Reedy-fibrant and that $\mathbf{SemiBord}_{c,m,n}$ satisfies the Segal condition is similar to that for $\mathbf{SemiCob}(m)$.

It remains to show that the Segal $(n-i)$-spaces $(\mathbf{SemiBord}_{m,n})_{k_1,\ldots,k_{i-1},0,\bullet,\ldots,\bullet}$ are essentially constant. As before, showing that $(\mathbf{SemiBord}_{m,n})_{k_1,\ldots,k_{i-1},0,k_{i+1},\ldots,k_n} \to (\mathbf{SemiBord}_{m,n})_{k_1,\ldots,k_{i-1},0,\ldots,0}$ is a weak equivalence is equivalent to showing that the map

$$\begin{aligned}\Phi \quad : \quad & B^n_{k_1,\ldots,k_{i-1},0,k_{i+1},\ldots,k_n} \to B^n_{k_1,\ldots,k_{i-1},0,\ldots,0}\\ & X \mapsto X \cap \left(\mathbb{R}^\infty \times \Big(\prod_{j=1}^{i}[a_0^j, a_{k_j}^j]\Big) \times \Big(\prod_{j=i+1}^{n}\{a_0^j\}\Big)\right) = \pi_X^{-1}(a_0^{i+1},\ldots,a_0^n)\end{aligned}$$

is a weak equivalence where π_X is the projection $X \to \prod_{j=i+1}^r [a_0^j, a_{k_j}^j]$.

Given any $X \in B^n_{k_1,\ldots,k_{i-1},0,k_{i+1},\ldots,k_n}$, π_X is a proper submersion. So, by Ehresmann's fibration theorem, π_X is a locally trivial fibration onto its image. By the conditions on the boundaries, we see that π_X is surjective, and since $\prod_{j=i+1}^r [a_0^j, a_{k_j}^j]$ is contractible, π_X is a trivial fibration. Hence, we have a homeomorphism

$$X \cong \pi_X^{-1}(a_0^{i+1},\ldots,a_0^n) \times \Big(\prod_{j=i+1}^{r}[a_0^j, a_{k_j}^j]\Big). \tag{3.2.1}$$

Recall that

$$B^n_{k_1,\ldots,k_{i-1},0,\ldots,0} = \coprod_{[M,\{\partial_{i,\nu}M_{\{j_i\}}\}]} B^n_{M,\{\partial_{i,\nu}M_{\{j_i\}}\}}.$$

For each M, define $\widetilde{M} = M \times \big(\prod_{i_d>i}[0,1]\big)$ with partitioning

$$\widetilde{M}_{\{j_i\},\{j_{i_d}\}} = M_{\{j_i\}} \times \Big(\prod_{i_d>i}\Big[\frac{j_{i_d}-1}{k_{i_d}}, \frac{j_{i_d}}{k_{i_d}}\Big]\Big)$$

for $1 \le j_{i_d} \le k_{i_d}$ for each $i_d > i$. Then, the homeomorphism (3.2.1) implies that

$$B^n_{k_1,\ldots,k_{i-1},0,k_{i+1},\ldots,k_n} = \coprod_{[M,\{\partial_{i,\nu}M_{\{j_i\}}\}]} B^n_{\widetilde{M},\{\partial_{i,\nu}\widetilde{M}_{\{j_i\},\{j_{i_d}\}}\}}$$

and Φ factors as a disjoint union of

$$\Phi_M : B^n_{M,\{\partial_{i,\nu}M_{\{j_i\}}\}} \to B^n_{\widetilde{M},\{\partial_{i,\nu}\widetilde{M}_{\{j_i\},\{j_{i_d}\}}\}}.$$

It suffices to show that Φ_M is a weak homotopy equivalence. This is true using the long exact sequence for fibre bundles as in the proof of Prop. 3.1.11 and the facts that $\mathrm{Emb}(M,\{\partial_{i,\nu}M_{\{j_i\}}\})$ and $\mathrm{Emb}(\widetilde{M},\{\partial_{i,\nu}\widetilde{M}_{\{j_i\},\{j_{i_d}\}}\})$ are contractible, and that

$$\mathrm{Diff}(M,\{\partial_{i,\nu}M_{\{j_i\}}\}) \to \mathrm{Diff}(\widetilde{M},\{\partial_{i,\nu}\widetilde{M}_{\{j_i\},\{j_{i_d}\}}\})$$

is a weak homotopy equivalence (cf. [Cer61, Chap. II, 4.2.3, Cor. 3]). □

We are now ready to define the Segal space $\mathbf{PBord}_{m,n}$ which will be our main object of study.

Let $\mathrm{Sub}(V\times\mathbb{R}^n)$ be the set of smooth closed (but not necessarily compact) submanifolds of $V\times\mathbb{R}^n$ of dimension m and without boundary. We can construct a topology on $\mathrm{Sub}(V\times\mathbb{R}^n)$ by the basis

$$\mathcal{N}(K,W) = \{Y \in B_m^V | Y\cap K = f(M)\cap K \text{ for some } f\in W\}$$

for all $K \subset V\times\mathbb{R}^n$ compact and open subsets $W \subset \mathrm{Emb}(M, V\times\mathbb{R}^n)$ where M is an abstract smooth manifold of dimension m. Let $\mathrm{Sub}(\mathbb{R}^\infty\times\mathbb{R}^n) = \varinjlim_V \mathrm{Sub}(V\times\mathbb{R}^n)$.

Definition 3.2.7. Suppose $m \ge n$. Let $(\mathbf{PBord}^V_{m,n})_{k_1,\ldots,k_n}$ be the set of tuples $(X,(a_0^1 \le \cdots \le a_{k_1}^1),\ldots,(a_0^n \le \cdots \le a_{k_n}^n))$ where $a_i \in \mathbb{R}$ and $X \subset V\times\mathbb{R}$ is a smooth (not necessarily compact) manifold of dimension m satisfying

(i) the composition $X \hookrightarrow V \times \mathbb{R}^n \xrightarrow{\mathrm{pr}_2} \mathbb{R}^n$ is proper (i.e. the preimages of compact sets are compact); and

(ii) for all $S \subset [n]$ and $(i_s)_{s\in S}$ where $0 \le i_s \le k_s$, X intersects $V \times \mathbb{R}^{[n]-S} \times \{a_{i_s} | s \in S\}$ transversely, that is, $(a_{i_s})_{s\in S}$ is not a critical value of the map $X \to V \times \mathbb{R}^n \to \mathbb{R}^S$.

(iii) for all $1 \le i \le n$, the projection map $X \to \mathbb{R}^{\{i+1,\dots,n\}}$ is a submersion at all $x \in X$ whose image under the projection to $\mathbb{R}^{\{i\}}$ lies in $\{a_0^i, \dots, a_{k_i}^i\}$.

$(\mathbf{PBord}^V_{m,n})_{k_1,\dots,k_n} \subset \mathrm{Sub}(V \times \mathbb{R}^n) \times \mathbb{R}^{n+k_1+\dots+k_n}$ is endowed with the subspace topology. For any morphism $f = (f_1, \dots, f_n) : [k_1] \times \dots \times [k_n] \to [l_1] \times \dots \times [l_n]$, we define the induced map $f^* : (\mathbf{PBord}^V_{m,n})_{l_1,\dots,l_n} \to (\mathbf{PBord}^V_{m,n})_{k_1,\dots,k_n}$ by

$$f^*(X, (a_0^1 \le \dots \le a_{l_1}^1), \dots, (a_0^n \le \dots \le a_{l_n}^n)) = (X, (a^1_{f_1(0)} \le \dots \le a^1_{f_1(l_1)}), \dots, (a^n_{f_n(0)} \le \dots \le a^n_{f_n(l_n)})).$$

The spaces $(\mathbf{PBord}^V_{m,n})_{k_1,\dots,k_n}$ together with the above maps give a simplicial n-space $\mathbf{PBord}^V_{m,n}$. We define

$$\mathbf{PBord}_{m,n} = \varinjlim_{\substack{V\subset\mathbb{R}^\infty \\ \dim_{\mathbb{R}} V<\infty}} \mathbf{PBord}^V_{m,n}.$$

Similarly, we let $\mathbf{PreCob}(m) = \mathbf{PBord}_{m,1}$ and $\mathbf{PBord}_n = \mathbf{PBord}_{n,n}$

Let $\mathbf{PBord}^0_{m,n}$ be the subsemisimplicial n-spaces consisting of all objects of the form $(X, (a_0^1 < \dots < a_{k_1}^1), \dots, (a_0^n < \dots < a_{k_n}^n))$.

There is a canonical morphism of semisimplicial spaces $\mathbf{PBord}^0_{m,n} \to \mathbf{SemiBord}_{m,n}$ given by sending

$$(X, (a_0^i < \dots < a_{k_i}^i)) \mapsto (X \cap (\mathbb{R}^\infty \times [a_0, a_k]), (a_0^{i_d} < \dots < a^{i_d}_{k_{i_d}}))$$

where $k_{i_1}, \dots, k_{i_r}$ are the indices greater than 0.

Proposition 3.2.8. *The arrows in the diagram*

$$\begin{tikzcd}
\mathbf{PBord}^0_{m,n} \arrow[r] \arrow[d] & \mathbf{PBord}_{m,n} \\
\mathbf{SemiBord}_{m,n} &
\end{tikzcd}$$

are all Reedy weak equivalences. Hence, since $\mathbf{SemiBord}_{m,n}$ *is a semiSegal n-space,* $\mathbf{PBord}_{m,n}$ *is a Segal n-space.*

Proof. $\mathbf{PBord}^0_{m,n}$ can be given as the product space $B \times J$ where B is the image of the projection $\mathbf{PBord}_{m,n} \to \mathrm{Sub}(V \times \mathbb{R}^n)$ and J is the contractible open subspace $\prod_{i=1}^n \{a_0^i < \dots < a_{k_i}^i\} \subset \mathbb{R}^{n+k_1+\dots+k_n}$. We have $\mathbf{PBord}_{m,n} = B \times \overline{J}$ and $\overline{J}$ is also contractible, so the horizontal arrow is a weak equivalence.

For the vertical arrow, we shall give the proof for the case $n = 1$ to avoid cumbersome notation. The proof for $n > 1$ is similar, and we will only describe the construction for one of the steps below.

We will show that for all $k \ge 0$, the map $\pi : \mathbf{PreCob}^0(m)_k \to \mathbf{SemiCob}(m)_k$ is a trivial fibration.

To show that it is a fibration, consider, for $K = I^r$, any commutative diagram

$$\begin{tikzcd}
K \arrow[r, "\Phi"] \arrow[d, "\mathrm{id}\times\{0\}"'] & \mathbf{PreCob}^0(m)_k \arrow[d, "\pi"] \\
K \times I \arrow[r, "\Psi"] & \mathbf{SemiCob}(m)_k.
\end{tikzcd}$$

Let $\Phi(k) = (X, (a_0 < \dots < a_k))$ and $\Psi(k, t) = (Y, (a_0' < \cdot < a_k'))$ where $k \in K$ and $t \in I$. Define

$$\begin{aligned}
\widetilde{Y} \;\; = \;\; & \sigma_{a_0'-a_0-t}\big(X \cap (\mathbb{R}^\infty \times (-\infty, a_0])\big) \cup Z(\sigma_{a_0'-a_0-t} X|_{a_0}, Y|_{a_0'}) \cup Y \\
& \cup Z(Y|_{a_k'}, \sigma_{a_k'-a_k+t} X|_{a_k}) \cup \sigma_{a_k'-a_k+t}\big(X \cap (\mathbb{R}^\infty \times [a_k, \infty))\big)
\end{aligned}$$

where σ_t is a shift on the last coordinate, $X|_a = X \cap (\mathbb{R}^\infty \times \{a\})$ and for $Z_1 \cong Z_2$, $Z(Z_1, Z_2) \cong Z_1 \times I$ is a smoothly embedded cylinder joining Z_1 and Z_2. Hence, $\widetilde{Y} \in \mathbf{PreCob}^0(m)_k$ and $\pi(\widetilde{Y}) = Y$. In other words, $\widetilde{Y}$ is a a smooth extension of Y by $X \cap (\mathbb{R}^\infty \times (-\infty, a_0])$ and $X \cap (\mathbb{R}^\infty \times [a_k, \infty))$.

Using Whitney's approximation theorem as in the proof of Prop. 3.2.3, we can construct

$$K \times I \to \mathbf{PreCob}^0(m)_k : (k,t) \mapsto (Y, (a_0 < \cdots < a_k)) \mapsto (\widetilde{Y}, (a_0 < \cdots < a_k))$$

continuous in $K \times I$, thus giving us the required lifting.

For the remainder of the proof, we will work with the collared version. Define $\mathbf{PreCob}^0_c(m)_k$ to be the pullback

$$\begin{tikzcd} \mathbf{PreCob}^0_c(m)_k \arrow[r] \arrow[d, "\pi_c"] & \mathbf{PreCob}^0(m)_k \arrow[d, "\pi"] \\ \mathbf{SemiCob}_c(m)_k \arrow[r] & \mathbf{SemiCob}(m)_k. \end{tikzcd}$$

Since π is a fibration, so is π_c. Since **Top** is proper, the two horizontal maps are weak equivalences. So π is trivial if and only if π_c is trivial. Indeed, we will show that π_c is a deformation retraction.

We can define an inclusion $\iota_c : \mathbf{SemiCob}_c(m)_k \to \mathbf{PreCob}^0_c(m)_k$ by extending any $Y \in \mathbf{SemiCob}_c(m)_k$ by cylinders. More precisely, let $\iota_c(Y, (a_0 < \cdots < a_k)) = (\widetilde{Y}, (a_0 < \cdots < a_k))$ where

$$\widetilde{Y} = (Y|_{a_0} \times (-\infty, a_0]) \cup Y \cup (Y|_{a_k} \times [a_k, \infty))).$$

It is clear that $\pi_c \circ \iota_c = \mathrm{id}_{\mathbf{SemiCob}_c(m)_k}$. It remains to show that $\iota_c \circ \pi_c \sim \mathrm{id}_{\mathbf{PreCob}^0_c(m)_k}$.

For any $X \in \mathbf{PreCob}^0_c(m)_k$ and $0 \le t < 1$, define

$$\begin{aligned} X_t \;=\; & \sigma_{-\tan(\pi t/2)}\big(X \cap (\mathbb{R}^\infty \times (-\infty, a_0])\big) \cup \big(X|_{a_0} \times [a_0 - \tan(\pi t/2), a_0]\big) \cup \big(X \cap (\mathbb{R}^\infty \times [a_0, a_k])\big) \\ & \cup \big(X|_{a_k} \times [a_k, a_k + \tan(\pi t/2)]\big) \cup \sigma_{\tan(\pi t/2)}\big(X \cap (\mathbb{R}^\infty \times [a_k, \infty))\big). \end{aligned}$$

X_t is a smooth embedded manifold because of the embedded collared neighbourhood at a_0 and a_k (see Fig. 3.1). For $t = 1$, let

$$X_1 = \big(X|_{a_0} \times (-\infty, a_0]\big) \cup \big(X \cap (\mathbb{R}^\infty \times [a_0, a_k])\big) \cup \big(X|_{a_k} \times [a_k, \infty)\big).$$

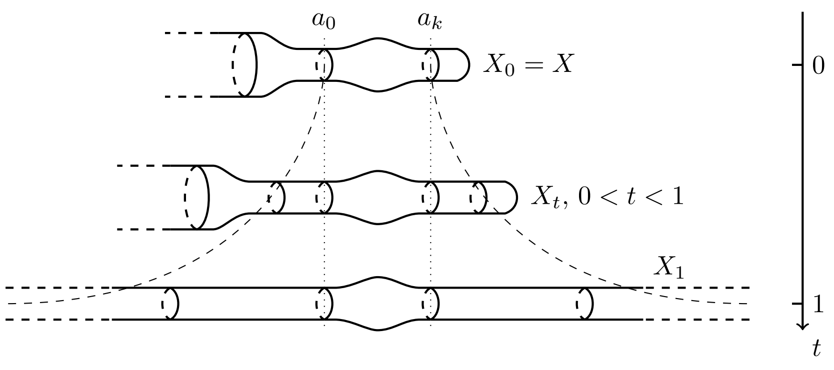


Figure 3.1: Schematic representation of the homotopy $t \mapsto X_t$.

In the topology of $\mathrm{Sub}(\mathbb{R}^\infty \times \mathbb{R})$, a basis of neighbourhoods of X_0 is given by $\mathcal{N}(K, W)$ where K is compact and $W \subset \mathrm{Emb}(X, \mathbb{R}^\infty \times \mathbb{R})$ such that there exists $f \in W$ with $f(X) = X_0 \cap K$. For any compact K, there exists t_K sufficiently close to 1 such that $K \subset \mathbb{R}^\infty \times [a_0 - \tan(\pi t/2), a_k + \tan(\pi t/2)]$, so $X_{t_K} \in \mathcal{N}(K, W)$ for all W. Hence, $X_1 = \lim_{t \to 1} X_t$.

This gives us a homotopy

$$H : \mathbf{PreCob}^0_c(m)_k \times I \to \mathbf{PreCob}^0_c(m)_k : (X, (a_0 < \cdots < a_k), t) \mapsto (X_t, (a_0 < \cdots < a_k))$$

where $H(-,0) = \mathrm{id}_{\mathbf{PreCob}_c^0(m)_k}$ and $H(-,1) = \iota_c \circ \pi_c$.

For $\mathbf{PBord}^0_{m,n} \to \mathbf{SemiBord}_{m,n}$ where $n > 1$, we describe the construction of ι_c. Given $X \in \mathbb{R}^\infty \times (\prod_{i=1}^n [a_0^i, a_{k_i}^i])$ a collared manifold, let $X_0 = X$ and define inductively, for $0 < r \leq n$,

$$\begin{aligned} X_r &= \big(X|_{a_0^r} \times \mathbb{R}^\infty \times \mathbb{R}^{r-1} \times (-\infty, a_0] \times (\prod_{i=r+1}^n [a_0^i, a_{k_i}^i])\big) \cup X_{r-1} \\ &\quad \cup \big(X|_{a_{k_r}^r} \times \mathbb{R}^\infty \times \mathbb{R}^{r-1} \times [a_{k_r}^r, \infty) \times (\prod_{i=r+1}^n [a_0^i, a_{k_i}^i])\big). \end{aligned}$$

Let $\iota_c(X, (a_j^i)) = (X_n, (a_j^i)) \in (\mathbf{PBord}^0_{m,n})_{k_1,\ldots,k_n}$. The rest of the proof is similar to the case $n = 1$. $\square$

This leads us to the definition

Definition 3.2.9. The **bordism (∞, n)-category of manifolds of dimension** m $\mathbf{Bord}_{m,n}$ is a Segal completion of $\mathbf{PBord}_{m,n}$. We also write $\mathbf{Cob}(m)$ for $\mathbf{Bord}_{m,1}$.

Remark 3.2.10. Prop. 2.3.10 gives a construction of $\mathbf{Bord}_{m,n}$ as an infinite sequence of pushouts, but this does not give an explicit definition of the complete Segal n-space. However, in this paper, a direct definition of $\mathbf{Bord}_{m,n}$ is unnecessary. As we are only interested in topological field theories, which are functors from $\mathbf{Bord}_{m,n}$ to a complete Segal n-space $\mathcal{C}$, it suffices to work with the Segal n-space $\mathbf{PBord}_{m,n}$, since all functors $\mathbf{PBord}_{m,n} \to \mathcal{C}$ factors through the completion.

We shall show that we can endow $\mathbf{PBord}_{m,n}$ with a symmetric monoidal structure and that $\mathbf{PBord}_{m,n}$ is a symmetric monoidal Segal n-space with duals. Informally, the symmetric monoidal functor $\otimes$ takes two bordisms M and N to their disjoint union. However, since $\mathbf{PBord}_{m,n}$ is the space of embedded manifolds, we need to specify a choice of embedding. Similarly, the dual of a bordism M with boundaries $\partial_0 M$ and $\partial_1 M$ can be regarded as the same bordism M taken with an opposite orientation. We will formalise these statements in the following results.

Theorem 3.2.11. *There exists a symmetric monoidal Segal n-space $\mathbf{PBord}^\otimes_{m,n} : N(\Gamma) \to N(n\text{-}\mathcal{CSS})$ such that $(\mathbf{PBord}^\otimes_{m,n})_1 \cong \mathbf{PBord}_{m,n}$.*

Proof. We will write $A = \mathbf{PBord}^\otimes_{m,n}$ for short. By Prop. 2.6.10, it suffices to construct an infinity functor from $\mathbf{E}_\infty$ to $N(n\text{-}\mathcal{CSS})$. In fact, we can do better, we will construct a strict functor $A : \mathbf{E}_\infty \to n\text{-}\mathcal{CSS}$.

For each r, we define A_r as in Def. 3.2.7, except that we take $X \subset (\coprod_{i=1}^r \mathbb{R}_i^\infty) \times \mathbb{R}^n$. Hence, we have precisely $A_0 = \{\emptyset\}$ and $A_1 = \mathbf{PBord}_{m,n}$.

There exists a diffeomorphism $\Phi : \mathrm{Int}(I^\infty) \to \mathbb{R}^\infty$ where $\mathrm{Int}(X)$ is the interior of a topological space X. Then, any morphism $(S, f) : \langle k \rangle \to \langle l \rangle$ in $\mathbf{E}_\infty$ where $S \subset [k]$ and $f \in \mathrm{Rect}(\mathbf{I}^\infty_{|S|}, \mathbf{I}^\infty_l)$ induces a corresponding morphism

$$\begin{aligned} (S, f)_* \quad &: \quad A_k \longrightarrow A_l \\ (X, (a_j^i)) &\mapsto \left(\Big(\coprod_{i=1}^l \Phi \times \mathrm{id}_{\mathbb{R}^n}\Big) \circ f \circ \Big(\coprod_S \Phi^{-1} \times \mathrm{id}_{\mathbb{R}^n}\Big)\Big(X \cap \Big(\coprod_{i \in S} \mathbb{R}_i^\infty\Big) \times \mathbb{R}^n\Big), (a_j^i)\right). \end{aligned}$$

It is clear that composition of maps is well-defined, hence this gives us the required functor.

Finally, we need to show that the pre-symmetric monoidal Segal n-space so constructed is symmetric monoidal, that is, $\prod \rho_{d*} : A_r \to \prod_{d=1}^r A_1$ is a Dwyer-Kan equivalence for all r.

Let

$$\tilde{\rho}_d = \mathrm{id}_{I^\infty} \in \mathrm{Rect}(I_d^\infty, I_1^\infty) \subset \mathrm{Map}_{\mathbf{E}_\infty}([r], [1])$$

be the lift of ρ_d in $\mathbf{E}_\infty$. By the contractibility of $\mathrm{Map}_{\mathbf{E}_\infty}([r], [1])$, it suffices to show that $\prod \tilde{\rho}_{d*} : A_r \to \prod_{d=1}^r A_1$ is a weak equivalence.

As in Prop. 3.2.8, we have a diagram of weak equivalences

$$\begin{tikzcd}
A_r^0 = \{(X, (a_j^i, <))\} \arrow[r] \arrow[d] & A_r = \{(X, (a_j^i, \leq))\} \\
\overline{A}_r^0 = \big\{\big(X \cap \big((\coprod \mathbb{R}^\infty) \times [a_0^1, a_{k_1}^1] \times \cdots \times [a_0^n, a_{k_n}^n]\big)\big), (a_j^i, <)\big)\big\} &
\end{tikzcd}$$

So, it suffices to check that $\Phi : \overline{A}_r^0 \to (\overline{A}_1^0)^r$ is a homotopy equivalence. By Remark 3.1.12, we can work with the collared version $\overline{A}_{c,r}^0$.

We construct a homotopy inverse as follows. First, note that for any choice of $(a_0^i < \cdots < a_{k_i}^i)_i$ and $(a_0'^i < \cdots < a_{k_i}'^i)_i$, there exists a piecewise linear function $\sigma_{(a_j^i),(a_j'^i)} : \prod_{i=1}^n [a_0^i, a_{k_i}^i] \to \prod_{i=1}^n [a_0'^i, a_{k_i}'^i]$ whose graph is obtained by linear interpolations among the points $\{((a_{j_0}^0, \ldots, a_{j_n}^n), (a_{j_0}'^0, \ldots, a_{j_n}'^n))\}$.

We can then define $\Psi : (A_{c,1}^0)^r \to A_{c,r}^0$ by

$$((X_d, (a_{d,j}^i)))_{d=1}^r \mapsto \left(\coprod_{d=1}^r (\mathrm{id}_{\mathbb{R}^\infty} \times \sigma_{(a_{d,j}^i),(a_{1,j}^i)}) X_j, (a_{1,j}^i) \right).$$

Fig. 3.2 shows an example of Ψ applied in the case $r = 2$.

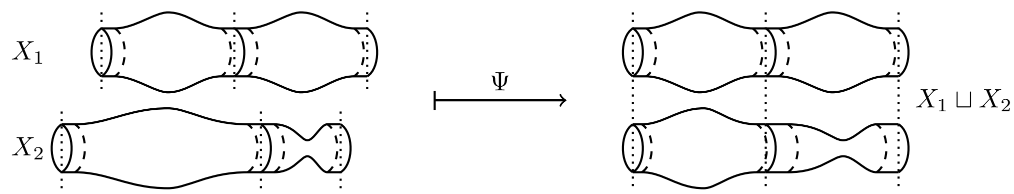


Figure 3.2: Schematic representation of the map Ψ.

Then, $\Psi \circ \Phi = \mathrm{id}$ and as in the proof of Prop. 3.2.4, we have a homotopy $\Phi \circ \Psi \sim \mathrm{id}$. $\square$

Remark 3.2.12. From the above construction, we see that in the symmetric monoidal structure of $\mathbf{PBord}_{m,n}$, the identity is given by the empty set $\emptyset$ and given any two object X and Y, the symmetric monoidal operation $X \coprod Y$ is given by the embedding of $\Psi(X, Y) \in A_2$ into A_1 via some embedding of $\mathbb{R}^\infty \sqcup \mathbb{R}^\infty \to \mathbb{R}^\infty$.

Theorem 3.2.13. $\mathbf{PBord}_{m,n}^{\otimes}$ *has duals.*

Proof. We need to show the following:

(i) for any $\underline{X} = (X, a_0^1, \ldots, a_0^n) \in (\mathbf{PBord}_{m,n})_{0,\ldots,0}$, there exists $\underline{Y} \in (\mathbf{PBord}_{m,n})_{0,\ldots,0}$ and spaces $\underline{M}, \underline{N} \in (\mathbf{PBord}_{m,n})_{1,0,\ldots,0}$ such that $d_1\underline{M} \cong d_0\underline{N} \cong \underline{X} \otimes \underline{Y}$ and $d_0\underline{M} = d_1\underline{N} = \underline{\emptyset}$, satisfying the adjoint identities.

(ii) for any $1 \leq r < n$ and r-morphism

$$\underline{X} = (X, (a_0^1 \leq a_1^1), \ldots, (a_0^r \leq a_1^r), a_0^{r+1}, \ldots, a_0^n) \in (\mathbf{PBord}_{m,n})_{1,\ldots,1,0,0,\ldots,0},$$

there exists a left and a right adjoint $\underline{Y}^L, \underline{Y}^R \in \mathrm{map}(\underline{X}|_{a_1^r}, \underline{X}|_{a_0^r})$ and counit and unit maps

$$\underline{Y}^L \circ \underline{X} \xrightarrow{\underline{M}^L} s_0(\underline{X}|_{a_0^r}),\ s_0(\underline{X}|_{a_1^r}) \xrightarrow{\underline{N}^L} \underline{X} \circ \underline{Y}^L,$$
$$\underline{X} \circ \underline{Y}^R \xrightarrow{\underline{M}^R} s_0(\underline{X}|_{a_1^r}),\ s_0(\underline{X}|_{a_0^r}) \xrightarrow{\underline{N}^R} \underline{Y}^R \circ \underline{X}$$

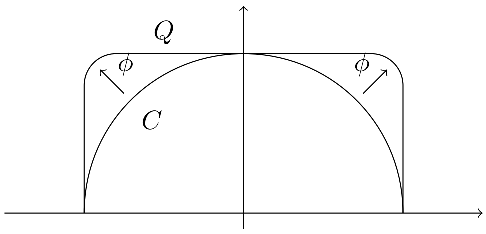
in $(\mathbf{PBord}_{m,n})_{1,\ldots,1,1,0,\ldots,0}$, satisfying the adjoint identities.

Indeed, it suffices to show the above in $\mathbf{SemiBord}_{m,n}$ since counit and unit maps in $\mathbf{SemiBord}_{m,n}$ satisfying the adjoint identities (with $s_0(X)$ replaced by $X \times [a_0, a_1]$ for some $a_0 < a_1$) can be extended to counit and unit maps in $\mathbf{PBord}^0_{m,n}$, and hence are equivalent to counit and unit maps in $\mathbf{PBord}_{m,n}$ by Prop. 3.2.8. Hence, by the same proposition, $\mathbf{PBord}_{m,n}$ has duals or adjoints if and only if $\mathbf{SemiBord}_{m,n}$ has.

First, we describe a "semi-circular" rotation construction that we will use in the proof later. Let $Z \subset V \times [0,\infty) \times \{0\}$ be a smooth embedded manifold. Define

$$\widetilde{C}_+(Z) = \{(x, t\cos\theta, t\sin\theta) \in V \times \mathbb{R} \times [0,\infty) |\, x \in Z, t \in [0,\infty), \theta \in [0,\pi]\}.$$

Let $C_+(Z) = (\mathrm{id}_V \times \phi)(\widetilde{C}_+(Z))$ where $\phi : \mathbb{R} \times [0,\infty) \to \mathbb{R} \times [0,\infty)$ is a diffeomorphism of the half plane that fixes the boundary $\mathbb{R} \times \{0\}$ and sends the semicircle $C = \{(x,y)|x^2+y^2 = 1\}$ into an open rectangular boundary Q with rounded corners (see figure below).



Similarly define $\widetilde{C}_-(Z)$ to be the points $(x, t\cos\theta, t\sin\theta)$ for $\theta \in [-pi, 0]$ and $C_-(Z) = (\mathrm{id}_V \times -\phi) \circ \widetilde{C}_-(Z)$.

(i) Let $\underline{X} = (X, a_0^1, \ldots, a_0^n) \in (\mathbf{SemiBord}_{m,n})_{0,\ldots,0}$. Without loss of generality, assume $a_0^1 = 0$ and $X \subset H_1 \times \{0\} \times \mathbb{R}^{n-1}$ where $H_1 = \{(x_i)_{i\geq 1} \in \mathbb{R}^\infty | 0 < x_1 < 1\}$. This is possible since X is compact.

Let $R(X)$ be the reflection of X across the hyperplane $\mathbb{R}^\infty_{\geq 2} = \{(x_i)_{i\geq 1} \in \mathbb{R}^\infty | x_1 = 0\}$ and $\underline{Y} = (R(X), a_0^1, \ldots, a_0^n)$. We shall show that $\underline{Y}$ is the dual of $\underline{X}$. $\underline{X} \otimes \underline{Y}$ can be represented by the manifold $X \coprod R(X) \subset \mathbb{R}^\infty \times \{0\} \times \mathbb{R}^{n-1}$, so we can take it to be

$$\underline{X} \otimes \underline{Y} = \big(X \sqcup R(X), a_0^1, \ldots, a_0^n\big).$$

We define the counit and unit to be

$$\underline{M} = \big(C_+(X), (0<1), a_0^2, \ldots, a_0^n\big) \in \mathrm{map}^1(\underline{X} \otimes \underline{Y}, \emptyset)$$
$$\underline{N} = \big(C_-(X), (-1<0), a_0^2, \ldots, a_0^n\big) \in \mathrm{map}^1(\emptyset, \underline{X} \otimes \underline{Y}).$$

(In this construction of $C_\pm(X|_0)$, we take $V \times [0,1] \times \{0\} = [0,1] \times \mathbb{R}^\infty_{\geq 2} \times \{0\} \times \mathbb{R}^{n-1}$.) The composition

$$\underline{X} \xrightarrow{(\underline{X}\times[0,1])\sqcup \underline{N}} \underline{X} \otimes \underline{Y} \otimes \underline{X} \xrightarrow{\underline{M}\sqcup(\underline{X}\times[-1,0])} \underline{X},$$

where $\underline{X} \times [a,b] = (X \times [a,b], (a<b), a_0^2, \ldots, a_0^n)$, can be represented by a manifold that is diffeomorphic to $X \times [0,1]$ (see Fig. 3.3), hence it is equivalent to $\underline{X} \times [0,1]$. The other composition is similar.

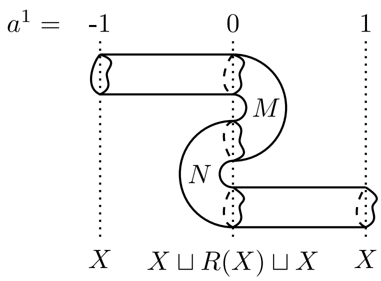


Figure 3.3: Schematic representation of the counit M, the unit N and the composition $(M \coprod (X \times [0,1])) \circ ((X \times [-1,0]) \coprod N)$.

(ii) By Prop. 3.1.11, it suffices to work in $\mathbf{SemiBord}_{c,m,n}$. Let

$$\underline{X} = (X, (a_0^1 \leq a_1^1), \ldots, (a_0^r \leq a_1^r)) \in (\mathbf{SemiBord}_{c,m,n})_{1,\ldots,1,0,0,\ldots,0}.$$

Without loss of generality, assume that X has a collared neighbourhood around a_0^r and a_1^r (so that we can smoothly join manifolds on the r-th coordinate) and assume that $0 = a_0^r < a_1^r = 1$.

Similarly to the first part, let $R(X)$ be the reflection of X across the hyperplane $\mathbb{R}^\infty \times \mathbb{R}^{r-1} \times \{0\}$ and let $\underline{Y} = (R(X), (a_0^1 \leq a_1^1), \ldots, (-1 < 0))$. Thus, $\underline{X} \circ \underline{Y}$ and $\underline{Y} \circ \underline{X}$ can be given by

$$\Big(R(X) \coprod_{X|_0} X, (a_0^1 \leq a_1^1), \ldots, (-1 < 0))\Big) \quad \text{and} \quad \Big(\sigma_1 R(X) \coprod_{X|_1} \sigma_{-1} X, (a_0^1 \leq a_1^1), \ldots, (-1 < 0))\Big)$$

where σ_a is a shift on the r-th coordinate by a.

$\underline{Y}$ is both the left and the right adjoint to $\underline{X}$: we can embed X and $R(X)$ into $\mathbb{R}^\infty \times \mathbb{R}^{r+1}$ using the embedding $\mathbb{R}^\infty \times \mathbb{R}^n \cong \mathbb{R}^\infty \times \mathbb{R}^n \times \{0\} \subset \mathbb{R}^\infty \times \mathbb{R}^{r+1}$. The counit and unit maps of $\underline{Y}$ as a left adjoint are given by

$$\begin{aligned}
\underline{M}^L &= \big(C_+(\sigma_1 R(X)), (a_0^1 \leq a_1^1), \ldots, (-1 < 1), (0 < 1)\big) \in \mathrm{map}^{r+1}(\underline{Y} \circ \underline{X}, \underline{X}|_0 \times [-a, a]) \\
\underline{N}^L &= \big(C_-(X), (a_0^1 \leq a_1^1), \ldots, (-1 < 1), (-1 < 0)\big) \in \mathrm{map}^{r+1}(\underline{X}|_1 \times [-a, a], \underline{X} \circ \underline{Y})
\end{aligned}$$

where $0 < a < 1$. As in the previous proof, we can check that the compositions $(\underline{N}^L \times \mathrm{id}) \circ (\mathrm{id} \times \underline{M}^L)$ and $(\mathrm{id} \times \underline{N}^L) \circ (\underline{M}^L \times \mathrm{id})$ are equivalent to the identities. The construction of the counit and unit maps for the right adjoint is similar. $\square$

Remark 3.2.14. Note that $\mathbf{PBord}_{m,n}^{\otimes}$ does not have adjoints for n-morphisms (otherwise, by Prop. 2.7.16, it will be an ∞-groupoid) since given any n-morphism X, although we can define $R(X)$ by taking the reflection on the last coordinate, we are not able to construct the counit and unit morphisms given by the "semi-circular" rotation, which requires an $n+1$-th coordinate.

Corollary 3.2.15. $\mathbf{Bord}_{m,n}^{\otimes}$ *is a symmetric monoidal complete Segal n-space with duals.*

Proof. This follows from Thms. 3.2.11 and 3.2.13, using Prop. 2.6.8 and Cor. 2.7.13. $\square$

3.3 The cobordism hypothesis

With the setup given in the previous sections, we are now ready to present the Baez-Dolan cobordism hypothesis as given by Lurie in [Lur09c].

Definition 3.3.1. Let M be a manifold of dimension m. A **framing** of M is a trivialisation of the tangent bundle of M, that is, an isomorphism $T_M \to \underline{\mathbb{R}}_M^m$ where $\underline{\mathbb{R}}_M^m$ is the trivial bundle of rank m on M. More generally, for $n \geq m$, a n-**framing** of M is a trivialisation of the stabilised tangent bundle $T_M \oplus \underline{\mathbb{R}}_M^{n-m}$.

An n-**framed manifold** of dimension $m \leq n$ is a manifold M with a specified n-framing $\phi : T_M \oplus \underline{\mathbb{R}}_M^{n-m} \to \underline{\mathbb{R}}^n$. An n-**framed morphism** of n-framed manifold is a smooth map $f : M \to M'$ and a map of vector bundles $F : T_M \oplus \underline{\mathbb{R}}_M^{n-m} \to T_{M'} \oplus \underline{\mathbb{R}}_{M'}^{n-m'}$ such that the induced maps on the stabilised tangent bundles and the trivial bundles commute with the trivialisation, that is, we have a commutative square

$$\begin{tikzcd}
T_M \oplus \underline{\mathbb{R}}_M^{n-m} \arrow[r, "F"] \arrow[d, "\phi"', "\sim"] & T_{M'} \oplus \underline{\mathbb{R}}_{M'}^{n-m'} \arrow[d, "\sim"', "\phi'"] \\
\underline{\mathbb{R}}_M^n \arrow[r, "f_*"'] & \underline{\mathbb{R}}_{M'}^n
\end{tikzcd}.$$

We can similarly construct $\mathbf{SemiBord}_{m,n}^{\mathrm{fr}}$ and $\mathbf{PBord}_{m,n}^{\mathrm{fr}}$ as sub-(semi)Segal n-spaces of $\mathbf{SemiBord}_{m,n}$ and $\mathbf{PBord}_{m,n}$ respectively, where we only consider framed embeddings of framed manifolds. As before, this gives us the symmetric monoidal complete Segal n-space $\mathbf{Bord}_{m,n}^{\otimes,\mathrm{fr}}$ with duals.

For any symmetric monoidal (∞, n)-category $\mathcal{C}$, there is an evaluation functor

$$\mathbf{Fun}^{\otimes}(\mathbf{Bord}_n^{\otimes,\mathrm{fr}}, \mathcal{C}) \to \mathcal{C}$$

determined by $Z \mapsto Z(*)$ where $*$ is a point in $(\mathbf{Bord}_n^{\mathrm{fr}})_{0,\ldots,0}$. The cobordism hypothesis can be stated as follows:

Theorem 3.3.2 (Cobordism Hypothesis)([Lur09c, Thm. 2.4.6]). *Let $\mathcal{C}$ be a symmetric monoidal (∞, n)-category with duals. The evaluation functor*

$$\mathbf{Fun}^{\otimes}(\mathbf{Bord}_n^{\otimes,\mathrm{fr}}, \mathcal{C}) \to \mathcal{C} : Z \mapsto Z(*)$$

factors through the sub-∞-groupoid $\mathcal{C}^0$ of $\mathcal{C}$ and the induced functor

$$\mathbf{Fun}^{\otimes}(\mathbf{Bord}_n^{\otimes,\mathrm{fr}}, \mathcal{C}) \to \mathcal{C}^0$$

is a Dwyer-Kan equivalence.

Remark 3.3.3. We can show that $\mathbf{Fun}^{\otimes}(\mathbf{Bord}_n^{\otimes,\mathrm{fr}}, \mathcal{C})$ is an ∞-groupoid, so the evaluation functor factors $Z \mapsto Z(*)$ factors through the sub-∞-groupoid.

Note that since $\mathcal{C}$ is complete Segal n-space, by the Segal completion (Prop. 2.5.18), we have a Dwyer-Kan equivalence

$$\mathbf{Fun}^{\otimes}(\mathbf{Bord}_n^{\otimes,\mathrm{fr}}, \mathcal{C}) \to \mathbf{Fun}^{\otimes}(\mathbf{PBord}_n^{\otimes,\mathrm{fr}}, \mathcal{C}),$$

so we can describe a topological field theory in terms of the Segal n-space $\mathbf{PBord}_n^{\otimes,\mathrm{fr}}$.

Since $\mathbf{Bord}_n^{\otimes,\mathrm{fr}}$ has duals, for any symmetric monoidal (∞, n)-category $\mathcal{C}$, Cor. 2.7.11 gives a weak equivalence

$$\mathbf{Fun}^{\otimes}(\mathbf{Bord}_n^{\otimes,\mathrm{fr}}, \mathcal{C}^{\mathrm{fd}}) \to \mathbf{Fun}^{\otimes}(\mathbf{Bord}_n^{\otimes,\mathrm{fr}}, \mathcal{C}).$$

Hence, the theorem can be reformulated for all symmetric monoidal (∞, n)-categories $\mathcal{C}$.

Corollary 3.3.4 (Cobordism hypothesis). *Let $\mathcal{C}$ be a symmetric monoidal (∞, n)-category. The evaluation functor*

$$\mathbf{Fun}^{\otimes}(\mathbf{Bord}_n^{\otimes,\mathrm{fr}}, \mathcal{C}) \to \mathcal{C} : Z \mapsto Z(*)$$

factors through the fully dualisable sub-∞-groupoid $(\mathcal{C}^{\mathrm{fd}})^0$ of $\mathcal{C}$ and the induced functor

$$\mathbf{Fun}^{\otimes}(\mathbf{Bord}_n^{\otimes,\mathrm{fr}}, \mathcal{C}) \to \left(\mathcal{C}^{\mathrm{fd}}\right)^0$$

is a Dwyer-Kan equivalence.

The cobordism hypothesis, in this form, states that any framed extended topological field theory is uniquely determined by its evaluation at a point. Furthermore, any object $X \in \mathcal{C}$ that is fully dualisable in $\mathcal{C}^0$ uniquely determines a framed extended topological field theory.

Chapter 4

Applications of the Cobordism Hypothesis

In this chapter, we will consider some more concrete examples of the cobordism hypothesis that are frequently studied. In the first example, we consider a special class of symmetric monoidal (∞, n)-categories $\mathcal{C}$ for which we can give an explicit construction of the topological field theories. In the second example, we study in detail the notion of fully dualisable for $n = 2$.

4.1 Algebras in a symmetric monoidal $(\infty, 1)$-category

First, we present some categorical preliminaries needed for the later discussions. Much of this section is based on the ideas in Chapter 4 of [Lur12].

Example 4.1.1 (Algebras in $\mathbf{Mod}_k$). Let k be a field. The abelian category of k-modules $\mathbf{Mod}_k$ has a natural symmetric monoidal structure given by the usual tensor product $\otimes_k$. A k-algebra A is a k-module, with an additional multiplication operation $A \times A \to A$ that is bilinear, has an identity and satisfies the associativity axiom. In other words, it is a k-module equipped with two maps

$$k \to A \qquad \text{and} \qquad A \otimes A \to A$$

satisfying a commutative diagram for associativity.

Given any k-algebra A, we can define a subcategory ${}_A\mathbf{Mod}$ (or $\mathbf{Mod}_A$, respectively) of left (resp., right) A-modules, whose objects are k-modules M equipped with a map $A \otimes M \to M$ (resp., $M \otimes A \to M$) satisfying some commutative diagrams. Given two k-algebras A and B, the subcategory ${}_A\mathbf{BiMod}_B$ of (A, B)-bimodules is the subcategory of right B-modules in the category of left A-modules. The objects can be given by k-modules M equipped with a map $A \otimes M \otimes B \to M$ satisfying some coherence properties.

Given two bimodules ${}_AM_B$ and ${}_BN_C$, there is a well-defined composition given by the relative tensor product ${}_AM_B \otimes_B {}_BN_C$. We can thus define the following 2-category $\mathbf{Alg}_1(k)$. Let the objects of $\mathbf{Alg}_1(k)$ be the k-algebras in $\mathbf{Mod}_k$. For any two k-algebras A and B, the category of morphisms $\mathrm{Hom}(A, B)$ is given by the category of (A, B)-bimodules ${}_A\mathbf{BiMod}_B$. Composition of 1-morphisms is given by the relative tensor product.

The purpose of this section is to generalise the construction in the example above to any symmetric monoidal $(\infty, 1)$-category. Recall the definitions of monoidal and symmetric monoidal $(\infty, 1)$-categories using Γ_1 and Γ_∞ (Def. 2.6.3).

Let $\mathbf{S}$ be a symmetric monoidal $(\infty, 1)$-category, that is, a map of simplicial spaces $N(\Gamma_\infty) \to N(1\text{-}\mathcal{CSS})$ with $\mathbf{S}_r$ being complete Segal spaces. We can define the algebra and module objects in $\mathbf{S}$. The following definitions are similar to those given in [Lur12, Chap. 4].

The topological category $\mathbf{E}_1$ can be given a pre-symmetric monoidal structure. Let $\mathbf{E}_1^{\otimes k}$ denote the category consisting of k-uples of objects of $\mathrm{Ob}\,\mathbf{E}_1$ and component-wise morphisms. There exists a functor

$E_1 : \Gamma_\infty \to \mathbf{TopCat}$ from Γ_∞ to the category of topological categories taking $\langle k\rangle \mapsto \mathbf{E}_1^{\otimes k}$. For any morphism $\phi : \langle k\rangle \to \langle l\rangle$ in Γ_∞, $E_1(\phi) : \mathbf{E}_1^{\otimes k} \to \mathbf{E}_1^{\otimes l}$ is a functor which takes

$$\Big(X_i = \coprod_{1\le j\le r_i} I\Big)_{1\le i\le k} \mapsto \Big(\coprod_{j\in\phi^{-1}(i)} X_j\Big)_{1\le i\le l}$$

and any morphism

$$(X_i \xrightarrow{\psi_i} Y_i)_{1\le i\le k} \mapsto \Big(\coprod_{j\in\phi^{-1}(i)} X_j \xrightarrow{\coprod \psi_j} \coprod_{j\in\phi^{-1}(i)} Y_j\Big)_{1\le i\le l}.$$

The pre-symmetric monoidal category E_1 is in fact symmetric monoidal. We are only interested in the underlying pre-monoidal structure though.

Definition 4.1.2. Let $\mathbf{Ass}^{\otimes} : N(\Gamma_1) \to N(1\text{-}\mathcal{CSS})$ be the monoidal complete Segal space obtained from the composition

$$Ass : \Gamma_1 \to \Gamma_\infty \xrightarrow{E_1} \mathbf{TopCat} \xrightarrow{N} 1\text{-}\mathcal{CSS}.$$

Definition 4.1.3. Let $\mathbf{S}$ be a symmetric monoidal $(\infty,1)$-category. The symmetric monoidal $(\infty,1)$-category of associative algebras in $\mathbf{S}$ is the symmetric monoidal $(\infty,1)$-category of monoidal functors

$$\mathbf{Alg}^{\otimes}(\mathbf{S}) = \mathbf{Fun}^{\otimes}(\mathbf{Ass}, \mathbf{S}).$$

Let $\mathbf{Alg}(\mathbf{S}) = \mathbf{Alg}^{\otimes}(\mathbf{S})_1$ be the underlying $(\infty,1)$-category. The full subcategory $\{I_1\} \subset \mathbf{E}_1$ induces an inclusion $* \to \mathbf{Ass}$. This induces a monoidal functor

$$\mathbf{Alg}^{\otimes}(\mathbf{S}) = \mathbf{Fun}^{\otimes}(\mathbf{Ass}, \mathbf{S}) \to \mathbf{Fun}^{\otimes}(*, \mathbf{S}) \cong \mathbf{S}.$$

An **associative algebra** in $\mathbf{S}$ is a monoidal functor $\mathbf{Ass} \to \mathbf{S}$, that is, an object in $\mathbf{Alg}(\mathbf{S})$.

Remark 4.1.4. We can justify this definition as follows. To each associative algebra $\mathbf{A} \in \mathbf{Alg}(\mathbf{S})$, let $A \in \mathbf{S}_1$ be the image of $I_1 \in \mathbf{Ass}_1 = \mathbf{E}_1$ and k-times product $A \otimes \cdots \otimes A$ be the image of $\langle k\rangle$. The unique map $\emptyset \to I_1\rangle$ in $\mathbf{E}_1$ and any embedding $I \sqcup I \to I$ determine maps

$$\mathbf{1} \to A \qquad \text{and} \qquad A \otimes A \to A.$$

The remaining morphisms in $\mathbf{E}_1$ give the associativity and identity axioms up to coherence. The monoidal structure of $\mathbf{Ass}$ is compatible with the monoidal structure of $\mathbf{S}$.

To each associative algebra $\mathbf{A}$, we can also define an opposite algebra, where the multiplication action is reversed. This is done as follows.

There exists a functor $\mathrm{op} : \mathbf{E}_1 \to \mathbf{E}_1$ that is the identity on objects and for any morphism $\phi : \coprod_S I \to \coprod_B I$, $\mathrm{op}(\phi)$ is given by the composition $(\coprod_B \sigma) \circ \phi \circ (\coprod_S \sigma)$ where $\sigma : I \to I$ is given by $\sigma(t) = 1 - t$.

op induces functor $\mathrm{op} : \mathbf{E}_1^{\otimes k} \to \mathbf{E}_1^{\otimes k}$ for all $k \ge 1$ and hence a natural transformation $\mathrm{op} : \mathbf{Ass} \to \mathbf{Ass}$.

Definition 4.1.5. The **opposite algebra $\mathbf{A}^{\mathrm{op}}$** of $\mathbf{A}$ is the associative algebra $A \circ \mathrm{op} : \mathbf{Ass} \to \mathbf{S}$. We immediately have $(\mathbf{A}^{\mathrm{op}})^{\mathrm{op}} = \mathbf{A}$.

Since $\mathbf{S}$ is symmetric monoidal, we get $\mathbf{1}^{\mathrm{op}} \cong \mathbf{1}$.

We can further proceed to define the modules of associative algebras.

Definition 4.1.6. Let LM be a topological category defined as follows:

- the objects of LM are

$$\mathrm{Ob}\,(\mathrm{LM}) = \Big\{\mathbf{I}_r = \coprod_{i=1}^{r} I_i \,|\, r \in \mathbb{Z}_{\ge 0}\Big\} \cup \big\{\mathbf{I}_r \sqcup I_* \,|\, r \in \mathbb{Z}_{\ge 0}\big\}$$

  where $I_i = I_* = [0,1]$;

- for any finite subsets of indices $A \subset \mathbb{Z}_{\geq}0 \cup \{*\}$, let $\mathbf{I}_A = \coprod_{a\in A} I_a$. For any $\mathbf{I}_A, \mathbf{I}_B \in \mathrm{Ob}\,(\mathrm{LM})$, the space of morphisms $\mathrm{Map}_{\mathrm{LM}}(\mathbf{I}_A, \mathbf{I}_B)$ is the space of all rectilinear embeddings $\phi : \mathbf{I}_S \to \mathbf{I}_B$ with $S \subset A$ (we allow $S = \emptyset$) satisfying

  (i) if $* \in A$, then $* \in S$ and $* \in B$;

  (ii) $\phi(I_*) \subset I_*$; and

  (iii) $\phi(I_*)$ is the rightmost component of the image in I_*, that is, for all $I_s \subset \phi^{-1}(I_*)$, $\phi(0_{I_s}) \leq \phi(0_{I_*})$ where 0_{I_s} is the point $0 \in I_s$;

  this construction gives $\mathrm{Map}_{\mathrm{LM}}(A, B)$ as a subspace of $\coprod_{S\subset A} \mathrm{Rect}\big(\coprod_{s\in S} I_s, \coprod_{b\in B} I_b\big)$;
- composition is given by composition of maps.

Note that if $* \in A$ and $* \notin B$, then $\mathrm{Map}_{\mathrm{LM}}(A, B) = \emptyset$.

$\mathbf{E}_1$ is a full subcategory of LM.

We can analogously define RM by requiring $\phi(I_*)$ to be the leftmost component in condition (iii).

Let $\mathrm{LM}^{\otimes k}$ denote the topological category of k-uples of the forms

$$\big(\mathbf{I}_{r_1}, \ldots, \mathbf{I}_{r_k}\big) \quad \text{or} \quad \big(\mathbf{I}_{r_1}, \ldots, \mathbf{I}_{r_{k-1}}, \mathbf{I}_{r_k} \sqcup I_*\big)$$

and component-wise morphisms.

We can similarly endow LM with a pre-monoidal structure $LM : \Gamma_1 \to \mathbf{TopCat}$ sending $\langle k\rangle \mapsto \mathrm{LM}^{\otimes k}$. The morphisms in LM are given by disjoint unions in a similar manner to that in the construction of E_1 above.

Definition 4.1.7. Let $\mathbf{LM} : N(\Gamma_1) \to N(1\text{-}\mathcal{CSS})$ be the pre-monoidal complete Segal space induced by the composition $\Gamma_1 \xrightarrow{LM} \mathbf{TopCat} \xrightarrow{N} 1\text{-}\mathcal{CSS}$.

Let $\mathbf{S}$ be a symmetric monoidal $(\infty, 1)$-category. The symmetric monoidal $(\infty, 1)$-category of left modules in $\mathbf{S}$ is the symmetric monoidal $(\infty, 1)$-category of monoidal functors

$$\mathbf{LMod}^{\otimes}(\mathbf{S}) = \mathbf{Fun}^{\otimes}(\mathbf{LM}, \mathbf{S}).$$

Let $\mathbf{LMod}(\mathbf{S}) = \mathbf{LMod}^{\otimes}(\mathbf{S})_1$ be the underlying $(\infty, 1)$-category. The inclusion $\mathbf{Ass} \to \mathbf{LM}$ induces a monoidal functor $\mathbf{LMod}^{\otimes}(\mathbf{S}) \to \mathbf{Alg}^{\otimes}(\mathbf{S})$. Let $A \in \mathbf{Alg}(\mathbf{S})_1$ be an associate algebra. The $(\infty, 1)$-category of left A-modules ${}_A\mathbf{LMod}(\mathbf{S})$ is given as the homotopy fibre of the functor $\mathbf{LMod}(\mathbf{S}) \to \mathbf{Alg}(\mathbf{S})$ of the underlying (∞, n)-categories at A, that is, it is the homotopy pullback in the diagram

$$\begin{tikzcd} {}_A\mathbf{LMod}(\mathbf{S}) \arrow[r] \arrow[d] & \{A\} \arrow[d] \\ \mathbf{LMod}(\mathbf{S}) \arrow[r] & \mathbf{Alg}(\mathbf{S}). \end{tikzcd}$$

Note that ${}_A\mathbf{LMod}(\mathbf{S})$ does not have a natural monoidal structure as the fibre does not preserve the unit of the monoidal operation.

A **left** A**-module** in $\mathbf{S}$ is an object in ${}_A\mathbf{LMod}(\mathbf{S})$.

We can similarly define the symmetric monoidal $(\infty, 1)$-category of right modules $\mathbf{RMod}^{\otimes}(\mathbf{S})$ and the $(\infty, 1)$-category of right A-modules $\mathbf{RMod}_A(\mathbf{S})$ using $\mathbf{RM}$.

Remark 4.1.8. An object $\mathbf{M} \in \mathbf{LMod}(\mathbf{S})$ gives an associative algebra $A = \mathbf{M}(I_1)$ and a left A-module $M = \mathbf{M}(*)$. Any embedding $I_1 \sqcup I_* \to I_*$ gives the action $A \otimes M \to M$ of A on M. The two embedding pathways gives a homotopy commutative diagram

$$\begin{tikzcd} I_1 \sqcup I_2 \sqcup I_* \arrow[r, "i_{12}\sqcup \mathrm{id}"] \arrow[d, "\mathrm{id}\sqcup i_{2*}"'] & I \sqcup I_* \arrow[d] \\ I \sqcup I_* \arrow[r] & I_* \end{tikzcd}$$

where $i_{12} : I_1 \sqcup I_2 \to I$ and $i_{2*} : I_2 \sqcup I_* \to I_*$ are some embeddings, demonstrate the associativity of the left action of A on M up to homotopy.

Note that there is also a functor LM $\to$ Ass which is the identity on the subcategory Ass and sends $\mathbf{I}_r \coprod I_*$ to $\mathbf{I}_{r+1}$, and sending a morphism in LM to the corresponding map. This induces a monoidal functor $\mathbf{Alg}^{\otimes}(\mathbf{S}) \to \mathbf{LMod}^{\otimes}(\mathbf{S})$ which sends an associative algebra A to a left A-module M where $M(I_1) = M(I_*) = A$ and the module structure $A \otimes A \to A$ given by the left multiplication in the algebra. This functor thus exhibits each associative algebra A as a left module over itself. When we write A as a left module, we will always refer to this module structure unless stated otherwise.

We can also define bimodules in $\mathbf{S}$.

Definition 4.1.9. Let BM be a topological category defined as follows:

- the objects of BM are

$$\mathrm{Ob}\,(\mathrm{BM}) = \Big\{{}_r\mathbf{I} = \coprod_{i=1}^{r} {}_iI \,|\, r \in \mathbb{Z}_{\geq 0}\Big\} \cup \Big\{\mathbf{I}_s = \coprod_{i=1}^{s} I_s \,|\, s \in \mathbb{Z}_{\geq 0}\Big\} \cup \big\{{}_r\mathbf{I} \sqcup I_* \sqcup \mathbf{I}_s \,|\, r, s \in \mathbb{Z}_{\geq 0}\big\}$$

  where ${}_iI = I_i = I_* = [0,1]$; we call ${}_rI$ and I_s the left and the right components respectively and I_* the module component;

- for any $\mathbf{I}_A, \mathbf{I}_B \in \mathrm{Ob}\,(\mathrm{BM})$, the space of morphisms $\mathrm{Map}_{\mathrm{BM}}(\mathbf{I}_A, \mathbf{I}_B)$ is the space of all rectilinear embeddings $\phi : \mathbf{I}_S \to \mathbf{I}_B$ with $S \subset A$ satisfying

  (i) if $* \in A$, then $* \in S$ and $* \in B$;

  (ii) ϕ takes left (resp., right, module) components of $\mathbf{I}_S$ into the left (resp., right, module) components of $\mathbf{I}_B$; and

  (iii) $\phi(I_*)$ lies to the right of the images of all left components of $\mathbf{I}_{\phi^{-1}(*)}$ and to the left of all right components, that is, for all ${}_rI \subset \phi^{-1}(I_*)$, $\phi(0_{{}_rI}) \leq \phi(0_{I_*})$ and for all $I_s \subset \phi^{-1}(I_*)$, $\phi(0_{I_s}) \geq \phi(0_{I_*})$;

  this construction gives $\mathrm{Map}_{\mathrm{BM}}(A, B)$ as a subspace of $\coprod_{S\subset A} \mathrm{Rect}\big(\coprod_{s\in S} I_s, \coprod_{b\in B} I_b\big)$;

- composition is given by composition of maps.

LM and RM are full subcategories of BM. Ass can be viewed as a full subcategory of BM in two different ways, by embedding into $\{{}_r\mathbf{I}\}$ or $\{\mathbf{I}_s\}$. We denote the two embeddings Ass $\to$ BM as ι_L and ι_S respectively.

Let $\mathrm{BM}^{\otimes k}$ denote the topological category of k-uples of the forms

$$\big({}_{r_1}\mathbf{I}, \ldots, {}_{r_k}\mathbf{I}\big) \quad \text{or} \quad \big(\mathbf{I}_{s_1}, \ldots, \mathbf{I}_{s_k}\big) \quad \text{or} \quad \big({}_{r_1}\mathbf{I}, \ldots, {}_{r_{j-1}}\mathbf{I}, {}_{r_j}\mathbf{I} \sqcup I_*\mathbf{I}_{s_j}, \mathbf{I}_{s_{j+1}}, \ldots, \mathbf{I}_{s_k}\big)$$

and component-wise morphisms. As before, we can endow BM with a pre-monoidal structure $BM : \Gamma_1 \to$ **TopCat** sending $\langle k\rangle \mapsto \mathrm{BM}^{\otimes k}$.

Definition 4.1.10. Let $\mathbf{BM} : N(\Gamma_1) \to N(1\text{-}\mathcal{CSS})$ be the pre-monoidal complete Segal space induced by the composition $\Gamma_1 \xrightarrow{BM} \mathbf{TopCat} \xrightarrow{N} 1\text{-}\mathcal{CSS}$.

Let $\mathbf{S}$ be a symmetric monoidal $(\infty,1)$-category. The symmetric monoidal $(\infty,1)$-category of bimodules in $\mathbf{S}$ is the $(\infty,1)$-category of monoidal functors

$$\mathbf{BiMod}^{\otimes}(\mathbf{S}) = \mathbf{Fun}^{\otimes}(\mathbf{BM}, \mathbf{S}).$$

Let $\mathbf{BiMod}(\mathbf{S}) = \mathbf{BiMod}^{\otimes}(\mathbf{S})_1$ be the underlying $(\infty,1)$-category. The two inclusion $\iota_L, \iota_R : \mathbf{Ass} \to \mathbf{BM}$ induce monoidal functors $\iota_L^*, \iota_R^* : \mathbf{BiMod}^{\otimes}(\mathbf{S}) \to \mathbf{Alg}^{\otimes}(\mathbf{S})$. Let $A, B \in \mathbf{Alg}(\mathbf{S})$ be associate algebras. The $(\infty,1)$-category of A,B-bimodules is given by the homotopy fibre

$$\begin{tikzcd}
{}_A\mathbf{BiMod}_B(\mathbf{S}) \arrow[r] \arrow[d] & \{(A,B)\} \arrow[d] \\
\mathbf{BiMod}(\mathbf{S}) \arrow[r, "(\iota_L,\iota_R)"] & \mathbf{Alg}(\mathbf{S}) \times \mathbf{Alg}(\mathbf{S}).
\end{tikzcd}$$

We also define ${}_A\mathbf{BiMod}(\mathbf{S})$ and $\mathbf{BiMod}_A(\mathbf{S})$ to be the homotopy fibre of ι_L and ι_R respectively at A.

An (A,B)**-bimodule** in $\mathbf{S}$ is an object in ${}_A\mathbf{BiMod}(\mathbf{S})_B$.

Remark 4.1.11. As before, an object $\mathbf{M} \in \mathbf{BiMod}(\mathbf{S})$ gives the data of two associative algebras $A = \mathbf{M}({}_1I)$ and $B = \mathbf{M}(I_1)$ and a module $M = \mathbf{M}(I_*)$, together with the left and right actions $A \otimes M \to M$ and $M \otimes B \to M$. The homotopy commutative diagram

$$\begin{tikzcd} A \otimes M \otimes B \arrow[r] \arrow[d] & A \otimes M \arrow[d] \\ M \otimes B \arrow[r] & M \end{tikzcd}$$

demonstrates the compatibility of the left and right module structures.

As for left modules, there is a monoidal functor $\mathbf{Alg}^{\otimes}(\mathbf{S}) \to \mathbf{BiMod}^{\otimes}(\mathbf{S})$ that exhibits each associative algebra A as a bimodule over itself with the left and right module structures given by left and right multiplication respectively.

Lemma 4.1.12. *There exists a monoidal functor* $\mathrm{op} : \mathbf{LMod}^{\otimes}(\mathbf{S}) \to \mathbf{RMod}^{\otimes}(\mathbf{S})$ *which sends a pair* (A, M) *to* (A^{op}, M)*, and which is a Dwyer-Kan equivalence. Hence, a left* A*-module is a right* A^{op}*-module. Similarly, for* $\mathbf{RMod}^{\otimes}(\mathbf{S}) \to \mathbf{LMod}^{\otimes}(\mathbf{S})$ *and* $\mathbf{BiMod}^{\otimes}(\mathbf{S}) \to \mathbf{BiMod}^{\otimes}(\mathbf{S})$.

Proof. We can extend the functor $\mathrm{op} : \mathbf{E}_1 \to \mathbf{E}_1$ to $\mathrm{op} : \mathrm{RM} \to \mathrm{LM}$ using the same operation on all objects and morphisms of RM. Note that the reflection $\sigma(t) = 1 - t$ sends the leftmost object of I_* to the rightmost object, hence op takes a morphism in RM to a morphism in LM.

The commutative diagram

$$\begin{tikzcd} \mathbf{E}_1 \arrow[r, "\mathrm{op}"] \arrow[d] & \mathbf{E}_1 \arrow[d] \\ \mathrm{RM} \arrow[r, "\mathrm{op}"] & \mathrm{LM} \end{tikzcd}$$

thus induce a homotopy commutative diagram

$$\begin{tikzcd} \mathbf{LMod}^{\otimes}(\mathbf{S}) \arrow[r, "\mathrm{op}"] \arrow[d] & \mathbf{RMod}^{\otimes}(\mathbf{S}) \arrow[d] \\ \mathbf{Alg}^{\otimes}(\mathbf{S}) \arrow[r, "\mathrm{op}"] & \mathbf{Alg}^{\otimes}(\mathbf{S}). \end{tikzcd}$$

where the lower horizontal arrow takes A to A^{op}. The upper horizontal arrow gives the required map.

To check that op is an Dwyer-Kan equivalence, we first note that it is essentially surjective on objects, since each pair $(A, M) \in \mathrm{Ob}\, \mathbf{RMod}^{\otimes}(\mathbf{S})$ has a preimage (A^{op}, M). The weak equivalences

$$\mathrm{map}_{\mathbf{LMod}^{\otimes}(\mathbf{S})}((A, M), (B, N)) \to \mathrm{map}_{\mathbf{RMod}^{\otimes}(\mathbf{S})}((A^{\mathrm{op}}, M), (B^{\mathrm{op}}, N))$$

are consequences of the fact that $\mathrm{op} : \mathrm{Ass} \to \mathrm{Ass}$ is an equivalence of categories, and hence, so is $\mathrm{op} : \mathrm{RM} \to \mathrm{LM}$. $\square$

Lemma 4.1.13. *Let* $\mathbf{1} = \mathbf{1}_{\mathbf{S}}$ *be the unit of the monoidal operation on* $\mathbf{S}$*. The composite functor*

$${}_{\mathbf{1}}\mathbf{LMod}(\mathbf{S}) \to \mathbf{LMod}(\mathbf{S}) \to \mathbf{S}_1$$

where the second map is induced by $* \mapsto I_* \in \mathrm{LM}$ *is a Dwyer-Kan equivalence. Hence,* ${}_{\mathbf{1}}\mathbf{BiMod}(\mathbf{S}) \to \mathbf{RMod}(\mathbf{S})$ *is a Dwyer-Kan equivalence.*

Proof. It is clear that the composite functor is essentially surjective since every $M \in \mathbf{S}$ can be regarded as a left $\mathbf{1}$-module.

$\mathrm{map}_{{}_{\mathbf{1}}\mathbf{LMod}(\mathbf{S})}(M, N)$ is contained in the space $\mathrm{map}(F(0) \times F(1), \mathbf{S}) \subset \mathrm{map}(\mathbf{LM} \times F(1), \mathbf{S})$ since we are restricting to morphisms that fix the associative algebra $\mathbf{1}$. Hence, $\mathrm{map}_{{}_{\mathbf{1}}\mathbf{LMod}(\mathbf{S})}(M, N) \to \mathrm{map}_{\mathbf{S}}(M, N)$ is a weak equivalence. $\square$

Corollary 4.1.14. *There exist monoidal functors* $\mathbf{BiMod}^{\otimes}(\mathbf{S}) \to \mathbf{LMod}^{\otimes}(\mathbf{S})$ *and* $\mathbf{BiMod}^{\otimes}(\mathbf{S}) \to \mathbf{RMod}^{\otimes}(\mathbf{S})$ *taking* (A, B, M) *to* $(A \otimes B^{\mathrm{op}}, M)$ *and* $(A^{\mathrm{op}} \otimes B, M)$ *respectively. Hence, an* (A, B)*-bimodule* M *can be seen as a left* $A \otimes B^{\mathrm{op}}$*-module or right* $A^{\mathrm{op}} \otimes B$*-module, and by Lemma 4.1.13, as a* $(A \otimes B^{\mathrm{op}}, \mathbf{1})$*-bimodule or a* $(\mathbf{1}, A^{\mathrm{op}} \otimes B)$*-bimodule.*

Proof. We will only construct the first functor, the other is similar. The inclusions $\mathbf{LM} \to \mathbf{BM}$ and $\mathbf{RM} \to \mathbf{BM}$ induce

$$\mathbf{BiMod}^{\otimes}(\mathbf{S}) \xrightarrow{L} \mathbf{LMod}^{\otimes}(\mathbf{S}) \qquad \text{and} \qquad \mathbf{BiMod}^{\otimes}(\mathbf{S}) \xrightarrow{R} \mathbf{RMod}^{\otimes}(\mathbf{S}).$$

Composing op to the second map gives us a functor

$$\mathbf{BiMod}^{\otimes}(\mathbf{S}) \xrightarrow{(L, \mathrm{op} \circ R)} \mathbf{LMod}^{\otimes}(\mathbf{S}) \times \mathbf{LMod}^{\otimes}(\mathbf{S}) \xrightarrow{\otimes} \mathbf{LMod}^{\otimes}(\mathbf{S})$$

where the second map uses the pre-monoidal structure on LM. The composition takes an object (A, B, M) to $(A \otimes B^{\mathrm{op}}, M)$. ☐

Let $\mathbf{BiMod}^{\otimes}(\mathbf{S})^{\otimes k}$ denote the homotopy limit of the diagram

$$\mathbf{BiMod}^{\otimes}(\mathbf{S}) \xrightarrow{\iota_L^*} \mathbf{Alg}^{\otimes}(\mathbf{S}) \xleftarrow{\iota_R^*} \mathbf{BiMod}^{\otimes}(\mathbf{S}) \xrightarrow{\iota_L^*} \cdots \xleftarrow{\iota_R^*} \mathbf{BiMod}^{\otimes}(\mathbf{S})$$

with k copies of $\mathbf{BiMod}^{\otimes}(\mathbf{S})$.

For any symmetric monoidal $(\infty, 1)$-category $\mathbf{S}$, Lurie demonstrated the existence of a symmetric monoidal functor

$$\mathbf{BiMod}^{\otimes}(\mathbf{S})^{\otimes 2} \to \mathbf{BiMod}^{\otimes}(\mathbf{S})$$

that takes a pair of bimodules $({}_A M_B, {}_B N_C)$ to an (A, C)-bimodule ${}_A M \otimes_B N_C$. This is done using the two-sided bar construction ([Lur12, Chap. 4.3.5]). Informally, on each pair of bimodules $({}_A M_B, {}_B N_C)$, define the simplicial object $\mathbf{Bar}(M, B, N) \in s\mathbf{BiMod}(\mathbf{S})$ by

$$\mathbf{Bar}({}_A M, B, N_C)_k = {}_A M \otimes B^{\otimes k} \otimes N_C$$

where $B^{\otimes k}$ is the n-fold tensor product of B. The outer face maps d_0 and d_k are induced by the module structure maps $M \otimes B \to M$ and $B \otimes N \to N$ respectively while the inner face maps d_i are induced by the algebra structure map $B \otimes B \to B$ between the i- and $(i+1)$-th copies of B. The degeneracy maps s_i are induced the algebra unit map $\mathbf{1} \to B$ for $i = 1, \ldots, k-1$. Then,

$${}_A M \otimes_B N_C = \operatorname{hocolim} \mathbf{Bar}({}_A M, B, N_C).$$

We call this construction the **relative tensor product** of bimodules in $\mathbf{S}$. It can be made precise as a monoidal functor of $(\infty, 1)$-categories using constructions similar to that given above. The relative tensor product satisfies all the properties of the usual relative tensor product of modules, including

$$A \otimes_A M \sim M, \qquad (M \otimes_A N) \otimes_B P \sim M \otimes_A (N \otimes_B P)$$

up to coherent homotopy.

Definition 4.1.15. Let $\mathbf{Alg}_{(1)}^{\otimes}(\mathbf{S})$ denote the symmetric monoidal pre-complete Segal 2-space whose 0-space is $(\mathbf{Alg}_{(1)}^{\otimes}(\mathbf{S})_\bullet)_0 = \mathbf{Alg}^{\otimes}(\mathbf{S})$ and whose k-space is $(\mathbf{Alg}_{(1)}^{\otimes}(\mathbf{S})_\bullet)_k = \mathbf{BiMod}^{\otimes}(\mathbf{S})^{\otimes k}$. The degeneracy maps are given by canonical inclusions. The outer face maps $d_0, d_k : (\mathbf{Alg}_{(1)}^{\otimes}(\mathbf{S})_\bullet)_k \to (\mathbf{Alg}_{(1)}^{\otimes}(\mathbf{S})_\bullet)_{k-1}$ are induced by the maps ι_L^* and ι_R^* respectively while the internal face maps d_i for $1 \leq i < k$ are induced by the relative tensor products. Let $\mathbf{Alg}_{(1)}(\mathbf{S}) = \mathbf{Alg}_{(1)}^{\otimes}(\mathbf{S})_1$ be its underlying $(\infty, 2)$-category.

$\mathbf{Alg}_{(1)}^{\otimes}(-)$ is a functor from the category of (small) symmetric monoidal $(\infty, 1)$-categories to the category of (small) symmetric monoidal pre-complete Segal 2-spaces since the construction is functorial on each k-space.

We can describe $\mathbf{Alg}_{(1)}(\mathbf{S})$ as follows: the objects of $\mathbf{Alg}_{(1)}(\mathbf{S})$ are associative algebras in $\mathbf{S}$ and a 1-morphism from A to B is an (A, B)-bimodule. 2-morphisms are given by maps of bimodules (A, B-bilinear maps), and higher morphisms are homotopies between such maps.

Proposition 4.1.16. $\mathbf{Alg}_{(1)}^{\otimes}(\mathbf{S})$ *has duals for objects.*

Proof. Let $A : \mathbf{Ass} \to \mathbf{S}$ be an associative algebra in $\mathbf{S}$. We claim that the opposite algebra A^{op} is a dual of A.

A is a (A, A)-bimodule, with the left and right module structures given by the algebra structure $A \otimes A \to A$. Indeed, it represents the identity morphism $A \to A$. We can also regard it as a $(\mathbf{1}, A \otimes A^{\mathrm{op}})$-bimodule or an $(A \otimes A^{\mathrm{op}}, \mathbf{1})$-bimodule. Similarly, A^{op} is both a $(A^{\mathrm{op}} \times A, \mathbf{1})$-bimodule and a $(\mathbf{1}, A^{\mathrm{op}} \times A)$-bimodule. So, to prove that A^{op} is the dual of A, it suffices to show that the compositions

$$ {}_A(A \otimes A^{\mathrm{op}}) \otimes_{A \otimes A^{\mathrm{op}} \otimes A} (A \otimes A)_A \quad \text{and} \quad {}_{A^{\mathrm{op}}}(A^{\mathrm{op}} \otimes A^{\mathrm{op}}) \otimes_{A^{\mathrm{op}} \otimes A \otimes A^{\mathrm{op}}} (A^{\mathrm{op}} \otimes A)_{A^{\mathrm{op}}} $$

are equivalent to ${}_A A_A$ and ${}_{A^{\mathrm{op}}} A^{\mathrm{op}}_{A^{\mathrm{op}}}$ respectively as $(\mathbf{1}, \mathbf{1})$-bimodules, that is, as objects in $\mathbf{S}$.

By the property that $A \otimes_A M \cong M \cong M \otimes_A A$ if the associative algebra A acts on the module A by right or left multiplication respectively, we have

$$ (A \otimes A^{\mathrm{op}}) \otimes_{A \otimes A^{\mathrm{op}} \otimes A} (A \otimes A) \cong A^{\mathrm{op}} \otimes_{A^{\mathrm{op}} \otimes A} (A \otimes A) \cong A^{\mathrm{op}} \otimes_{A^{\mathrm{op}}} A \cong A $$

as objects in $\mathbf{S}$, and similarly for the other map. $\square$

However, in general, $\mathbf{Alg}_{(1)}^{\otimes}(\mathbf{S})$ does not have adjoints for 1-morphisms. Nevertheless, we can consider its maximal $(\infty, 1)$-subcategory.

Definition 4.1.17. Let $\mathbf{Alg}_{(1)}^{0}(\mathbf{S})$ be the maximal symmetric monoidal sub-$(\infty, 1)$-category of $\mathbf{Alg}_{(1)}^{\otimes}(\mathbf{S})$ (see Prop. 2.7.19). We view it as an object in 1-$\mathcal{CSS}$ by applying the functor i_1^*.

We immediately get:

Corollary 4.1.18. $\mathbf{Alg}_{(1)}^{0}(\mathbf{S})$ *has duals as an* $(\infty, 1)$*-category.*

More generally, we can define an (∞, n)-category $\mathbf{Alg}_{(n)}^{\otimes}(\mathbf{S})$ of E_n-algebras [Lur09c].

Definition 4.1.19. Let $\mathbf{Alg}^{(1)\otimes}(\mathbf{S}) = \mathbf{Alg}^{\otimes}(\mathbf{S})$ and inductively define $\mathbf{Alg}^{(n)\otimes}(\mathbf{S}) = \mathbf{Alg}^{\otimes}(\mathbf{Alg}^{(n-1)\otimes}(\mathbf{S}))$. We call objects in this symmetric monoidal $(\infty, 1)$-category E_n**-algebras** of $\mathbf{S}$, so associative algebras are E_1-algebras. Let $\mathbf{BiMod}^{(n)\otimes}(\mathbf{S})$ denote the symmetric monoidal $(\infty, 1)$-category given by the homotopy pullback

$$ \begin{array}{ccc} \mathbf{BiMod}^{(n)\otimes}(\mathbf{S}) & \longrightarrow & \mathbf{Alg}^{(n)\otimes}(\mathbf{S}) \\ \downarrow & & \downarrow \\ \mathbf{BiMod}^{\otimes}(\mathbf{S}) & \longrightarrow & \mathbf{Alg}^{\otimes}(\mathbf{S}). \end{array} $$

Similarly define $\mathbf{BiMod}^{(n)\otimes}(\mathbf{S})^{\otimes k}$ to be the homotopy limit of the diagram

$$ \mathbf{BiMod}^{(n)\otimes}(\mathbf{S}) \xrightarrow{\iota_L^*} \mathbf{Alg}^{(n)\otimes}(\mathbf{S}) \xleftarrow{\iota_R^*} \mathbf{BiMod}^{(n)\otimes}(\mathbf{S}) \xrightarrow{\iota_L^*} \cdots \xleftarrow{\iota_R^*} \mathbf{BiMod}^{(n)\otimes}(\mathbf{S}) $$

with k copies of $\mathbf{BiMod}^{(n)\otimes}(\mathbf{S})$.

Remark 4.1.20. An E_n-algebra A can be regarded as a monoidal functor $\mathbf{Ass}^n \to \mathbf{S}$, with each copy of $\mathbf{Ass}$ defining a distinct "multiplicative" structure $A \otimes A \xrightarrow{m_i} A$ on the algebra, such that they are mutually compatible, that is, there are homotopy commutative diagrams

$$ \begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{m_i \otimes \mathrm{id}_A} & A \otimes A \\ {\scriptstyle \mathrm{id}_A \otimes m_j}\downarrow & & \downarrow{\scriptstyle m_j} \\ A \otimes A & \xrightarrow{m_i} & A. \end{array} $$

This gives A an E_n-monoidal structure. Alternatively, we can construct $\mathbf{Ass}^{(n)}$ as for $\mathbf{Ass}$ replacing $\mathbf{E}_1$ with $\mathbf{E}_n$ and show that $\mathbf{Alg}^{(n)\otimes}(\mathbf{S}) \cong \mathbf{Fun}^{\otimes}(\mathbf{Ass}^{(n)}, \mathbf{S})$.

The map $\mathbf{Alg}^{(n)\otimes}(\mathbf{S}) \to \mathbf{Alg}^{(n-1)\otimes}(\mathbf{S})$ takes an E_n-algebra and view it as an E_{n-1}-algebra by forgetting one of the algebra structures. Hence, such a map is not canonical as it depends on the choice of the structure to forget. However, by the Eckmann-Hilton argument, all compatible multiplicative structures on an E_n-algebra are equivalent, so the different maps are equivalent.

The objects of $\mathbf{BiMod}^{(n)}(\mathbf{S})$ are bimodules ${}_AM_B$ of E_n-algebras A, B, taken with respect to some global choice of one of the algebra structures on each E_n-algebra A, that is, by viewing the E_n-algebra as an E_1-algebra.

Definition 4.1.21. Let $\mathbf{Alg}^{\otimes}_{(n)}(\mathbf{S})$ be the symmetric monoidal pre-complete Segal $(n+1)$-space whose 0-space is $(\mathbf{Alg}^{\otimes}_{(n)}(\mathbf{S})_\bullet)_0 = \mathbf{Alg}_{(n-1)}(\mathbf{Alg}^{(n)\otimes}(\mathbf{S}))$ and whose k-space is

$$(\mathbf{Alg}^{\otimes}_{(n)}(\mathbf{S})_\bullet)_k = \mathbf{Alg}^{\otimes}_{(n-1)}(\mathbf{BiMod}^{(n)\otimes}(\mathbf{S})^{\otimes k}).$$

The degeneracy and face maps are given by applying the functor $\mathbf{Alg}^{\otimes}_{(n-1)}(-)$ to the degeneracy and face maps of $\mathbf{Alg}^{\otimes}_{(1)}(\mathbf{S})$.

The construction gives $\mathbf{Alg}^{\otimes}_{(n)}(-)$ as a functor from the category of symmetric monoidal $(\infty,1)$-categories to the category of symmetric monoidal pre-complete Segal $(n+1)$-spaces.

Claim 4.1.22. $\mathbf{Alg}^{\otimes}_{(n)}(\mathbf{S})$ *has duals for objects and adjoints for r-morphisms for all $r < n$.*

Sketch of proof. The proof that $\mathbf{Alg}^{\otimes}_{(n)}(\mathbf{S})$ has duals for objects is the same as in Prop. 4.1.16.

To show that $\mathbf{Alg}^{\otimes}_{(n)}(\mathbf{S})$ has adjoints for r-morphisms, we want to proceed by induction on n. The main idea of the proof is to endow $\mathbf{BiMod}^{(n)}(\mathbf{S})$ with a different monoidal structure that is compatible with composition in $\mathbf{Alg}^{\otimes}_{(n)}(\mathbf{S})$.

$\mathbf{Alg}_{(1)}(\mathbf{S})$ can be seen as a functor $\Delta^{\mathrm{op}} \to 1\text{-}\mathcal{CSS}$. By Remark 2.6.11, there is an equivalence $N(1\text{-}\mathcal{CSS})^{N(\Gamma_1)} \cong N(1\text{-}\mathcal{CSS})^{N(\Delta^{\mathrm{op}})}$, so $\mathbf{Alg}_{(1)}(\mathbf{S})$ can be regarded as a "non-unital" monoidal $(\infty,1)$-category of $\mathbf{BiMod}(\mathbf{S})$ where the monoidal operation is given by the relative tensor product. It is non-unital as there is no unique unit object $\mathbf{1}$, alternatively, we may regard every associative algebra as a unit object when viewed as a bimodule over itself.

Restricting to the sub-$(\infty,1)$-category $\mathbf{BiMod}^{(n)}(\mathbf{S})$ of bimodules over E_n-algebras, the monoidal operation can be shown to be E_n-monoidal, that is, the restricted functor $\mathbf{Alg}^{(n)}_{(1)}(\mathbf{S}) : \Delta^{\mathrm{op}} \to 1\text{-}\mathcal{CSS}$ can be shown to factor through $\mathbf{E}_n$.

For any non-unital E_n-monoidal $(\infty,1)$-category X, we can construct $\mathbf{Alg}^{\otimes}(X)$ and $\mathbf{BiMod}^{\otimes}(X)$ as before, but as non-unital E_{n-1}-monoidal $(\infty,1)$-categories. Hence, $\mathbf{Alg}^{\otimes}_{(n-1)}(X)$ is well-defined as non-unital E_{n-1}-monoidal pre-complete Segal n-space and $\mathbf{Alg}^{\otimes}_{(n-1)}(X)$ has duals for objects.

Since $(\mathbf{Alg}^{\otimes}_{(n)}(\mathbf{S})_\bullet)_1 = \mathbf{Alg}^{\otimes}_{(n-1)}(\mathbf{BiMod}^{(n)}(\mathbf{S}))$ and $\mathbf{Alg}^{\otimes}_{(n-1)}(\mathbf{Alg}_{(1)}(\mathbf{S}))$ have the same underlying pre-complete Segal n-space $\mathbf{Alg}(\mathbf{BiMod}^{(n)}(\mathbf{S}))$ and the monoidal structure in the latter is given by composition, $\mathbf{Alg}^{\otimes}_{(n)}(\mathbf{S})$ has adjoints for r-morphisms for $1 \leq r < n$ if and only if $\mathbf{Alg}^{\otimes}_{(n-1)}(\mathbf{Alg}_{(1)}(\mathbf{S}))$ has duals for objects and adjoints for r-morphisms for $1 \leq r < n-1$. The result then follows by induction. $\square$

Definition 4.1.23. Let $\mathbf{Alg}^{0}_{(n)}(\mathbf{S})$ be the maximal symmetric monoidal sub-(∞,n)-category of $\mathbf{Alg}^{\otimes}_{(n)}(\mathbf{S})$, seen as an object in $n\text{-}\mathcal{CSS}$.

Corollary 4.1.24. $\mathbf{Alg}^{0}_{(n)}(\mathbf{S})$ *has duals.*

Example: Differential graded category over a commutative ring k

The key example that motivates the construction of algebras comes from a generalisation of the 1-category of modules to higher categories.

Definition 4.1.25. A **differential graded category** (or **dg-category**) is a category enriched over chain complexes.

Definition 4.1.26. Let k be a commutative ring. Define dg-$\mathbf{Ch}(k)$ to be the dg-category whose objects are chain complexes of k-modules (unbounded or bounded below) and for any chain complexes $A_\bullet, B_\bullet$, the morphisms form a chain complex $\mathrm{map}_{\text{dg-}\mathbf{Ch}_k}(A_\bullet, B_\bullet)_\bullet$ where

$$\mathrm{map}_{\text{dg-}\mathbf{Ch}_k}(A_\bullet, B_\bullet)_i = \{f_\bullet : A_\bullet \to B_{\bullet+i}\}$$

and the differential maps $d : \mathrm{map}_{\text{dg-}\mathbf{Ch}_k}(A_\bullet, B_\bullet)_i \to \mathrm{map}_{\text{dg-}\mathbf{Ch}_k}(A_\bullet, B_\bullet)_{i-1}$ are given by

$$d(f) = d^B \circ f - (-1)^i f \circ d^A.$$

It is easy to check that $d^2 = 0$.

Tabuada showed a "zig-zag" series of Quillen adjunctions between the model category of dg-categories and the model category of simplicial categories [Tab10]. The construction for a functor from dg-categories to simplicial categories is given explicitly by Lurie in [Lur12, Chap. 1.3.1]. Applying the classifying diagram construction to the simplicial category then gives a complete Segal space. The main idea of the constructions is to truncate the chain complexes of morphisms and only consider the sub-chain complex that is bounded below at 0.

In the case of dg-$\mathbf{Ch}(k)$, we can construct the associated complete Segal space directly. Let $\mathbf{Ch}_0(k)$ denote the underlying ordinary category of dg-$\mathbf{Ch}(k)$, that is, the category with $\mathrm{Ob}\,\mathbf{Ch}_0(k) = \mathrm{Ob}\,\text{dg-}\mathbf{Ch}(k)$ and for each pair of chain complexes $A_\bullet, B_\bullet$,

$$\mathrm{Hom}_{\mathbf{Ch}_0(k)}(A_\bullet, B_\bullet) = \mathrm{map}_{\text{dg-}\mathbf{Ch}_k}(A_\bullet, B_\bullet)_0.$$

For any two chain maps $f, g : A_\bullet \to B_\bullet$, write

$$f \xrightarrow{h} g \qquad \text{or} \qquad \begin{tikzcd} A \arrow[d, "f"'] \arrow[r, phantom, "\overset{h}{\Longrightarrow}"] & A \arrow[d, "g"] \\ B & B \end{tikzcd}$$

for a chain homotopy between f and g, that is, a chain map $h_\bullet : A_\bullet \to B_{\bullet+1}$ satisfying

$$d^B_{n+1} \circ h_n + h_{n-1} \circ d^A_n = f_n - g_n \qquad \forall\ n.$$

We can compose chain homotopies: given $f_1, g_1 : A_\bullet \to B_\bullet$ and $f_2, g_2 : B_\bullet \to C_\bullet$ and chain homotopies $f_1 \xrightarrow{h_1} g_1$ and $f_2 \xrightarrow{h_2} g_2$, the composition is given by the chain homotopy

$$\left(f_2 \circ f_1 \xrightarrow{(h_2,h_1)} g_2 \circ g_1\right) = \left(f_2 \circ f_1 \xrightarrow{h_2 \circ f_1 + g_2 \circ h_1} g_2 \circ g_1\right).$$

It is easy to check that composition is associative. This gives a strict 2-category $\mathbf{Ch}_1(k)$ where $\mathrm{Ob}\,\mathbf{Ch}_1(k) = \mathrm{Ob}\,\text{dg-}\mathbf{Ch}(k)$ and for any two chain complexes $A_\bullet, B_\bullet$, $\mathrm{Hom}_{\mathbf{Ch}_1(k)}(A_\bullet, B_\bullet)$ is the category with objects being chain maps and morphisms being chain homotopies.

For $n > 0$, let $B[n]$ denote the 2-category with 2 distinct objects x, y, $\mathrm{Hom}(x,x) = \{\mathrm{id}_x\}$, $\mathrm{Hom}(y,y) = \{\mathrm{id}_y\}$, $\mathrm{Hom}(y,x) = \emptyset$ and $\mathrm{Hom}(x,y) = [n]$. So, $\mathrm{Hom}(B[n], \mathbf{Ch}_1(k))$ is the category of chains of homotopies

$$f^0 \xrightarrow{h^1} f^1 \xrightarrow{h^2} \cdots \xrightarrow{h^n} f^n.$$

A map $g : A_\bullet \to B_\bullet$ is a homotopy equivalence if there exists $f, f' : B_\bullet \to A_\bullet$ and homotopies $g \circ f \to \mathrm{id}_B$ and $f' \circ g \to \mathrm{id}_A$. Let $\mathbf{Ch}_{\text{hoequiv}}(k)$ be the sub-2-category of $\mathrm{Ob}\,\mathbf{Ch}_1(k)$ with $\mathrm{Ob}\,\mathbf{Ch}_{\text{hoequiv}}(k) = \mathrm{Ob}\,\mathbf{Ch}_1(k)$ and such that $\mathrm{Hom}_{\mathbf{Ch}_{\text{hoequiv}}(k)}(A_\bullet, B_\bullet)$ is the full subcategory of $\mathrm{Hom}_{\mathbf{Ch}_1(k)}(A_\bullet, B_\bullet)$ consisting of the homotopy equivalences.

Definition 4.1.27. Let $\mathbf{Ch}(k)$ denote the complete Segal space defined by $\mathbf{Ch}(k)_0 = \mathrm{nerve}\,\mathbf{Ch}_{\text{hoequiv}}(k)$ and, for $m > 0$, $\mathbf{Ch}(k)_m$ is given by

$$[n] \mapsto N(\mathrm{Hom}(B[n], \mathbf{Ch}_1(k)), \mathrm{Hom}(B[n], \mathbf{Ch}_{\text{hoequiv}}(k)))_{m,m}.$$

The proof that $\mathbf{Ch}(k)$ is a complete Segal space is easy and similar to that for the construction for Segal completion that Rezk gave in [Rez01, Sec. 14]. We instead give a visualisation: An n-simplex in $\mathbf{Ch}(k)_m$ can be viewed as a diagram of chain homotopies

$$\left(\begin{array}{ccc} A^{00} \xrightarrow{g^{110}} \cdots \xrightarrow{g^{m10}} A^{m0} \\ \downarrow f^{110} \qquad\qquad \downarrow f^{m10} \\ \vdots \qquad\qquad\qquad \vdots \\ \downarrow f^{1m0} \qquad\qquad \downarrow f^{1m0} \\ A^{0m} \xrightarrow{g^{1m0}} \cdots \xrightarrow{g^{mm0}} A^{mm} \end{array}\right) \xRightarrow{\begin{pmatrix} h^{111} & \cdots & h^{m11} \\ \vdots & & \vdots \\ h^{1m1} & \cdots & h^{mm1} \end{pmatrix}} \cdots \xRightarrow{\begin{pmatrix} h^{11n} & \cdots & h^{m1n} \\ \vdots & & \vdots \\ h^{1mn} & \cdots & h^{mmn} \end{pmatrix}} \left(\begin{array}{ccc} A^{00} \xrightarrow{g^{11n}} \cdots \xrightarrow{g^{m1n}} A^{m0} \\ \downarrow f^{11n} \qquad\qquad \downarrow f^{m1n} \\ \vdots \qquad\qquad\qquad \vdots \\ \downarrow f^{1mn} \qquad\qquad \downarrow f^{1mn} \\ A^{0m} \xrightarrow{g^{1mn}} \cdots \xrightarrow{g^{mmn}} A^{mm} \end{array}\right)$$

where g^{ijk} are homotopy equivalences.

Tensor products of dg-algebras are defined as follows: let $A_\bullet, B_\bullet$ be chain complexes of k-modules. Then their tensor product $(A \otimes B)_\bullet$ is given by

$$(A \otimes B)_n = \bigoplus_{p+q=n} A_p \otimes_k B_q$$

and the differential maps are given by

$$d(a \otimes b) = d^A a \otimes b + (-1)^p a \otimes d^B b \qquad \text{where } a \otimes b \in A_p \otimes B_q.$$

The tensor product defines a symmetric monoidal structure on $\mathbf{Ch(k)}$. The monoidal structure (associativity and identity) is strict and can be verified on the underlying category $\mathbf{Ch}_0(k)$. The symmetry map is given by

$$a \otimes b \mapsto (-1)^{pq} b \otimes a \qquad \text{where } a \otimes b \in A_p \otimes B_q.$$

We thus get a symmetric monoidal structure on $\mathbf{Ch}(k)$ induced from that of $\mathbf{Mod}(k)$. However, this symmetric monoidal structure is not homotopy invariant in $\mathbf{Ch}(k)$, unless k is a field, in which case all dg-k-modules are projective.

Instead, we can define the derived tensor product [Toë07]. $A \otimes_k -$ is a left Quillen functor (viewing $\mathrm{Ch}(k)$ as a dg-category), so we can define its left derived functor $A \otimes_k^{\mathbb{L}} -$. Explicitly, the derived tensor product is given by

$$A \otimes^{\mathbb{L}} B = Q(A) \otimes_k^{\mathbb{L}} B$$

where $Q(A)$ is a cofibrant replacement of A in the compatible model category structure of $\mathrm{Ch}_0(k)$ (that is, the model structure that makes $N(\mathrm{Ch}_0(k)) \cong \mathrm{Ch}(k)$. If we take $\mathrm{Ch}(k)$ to be the category of chain complexes bounded below, then $Q(A)$ is precisely a degree-wise projective replacement of A in $\mathbf{Mod}(k)$.

We take the symmetric monoidal structure on $\mathbf{Ch}(k)$ to be that induced by the derived tensor product $\otimes_k^{\mathbb{L}}$.

An object in $\mathbf{A} \in \mathbf{Alg}(\mathbf{Ch}(k))$ is a monoidal functor $\mathbf{Ass} \to \mathbf{Ch}(k)$. Let $A = \mathbf{A}(I_1)$, then by Remark 4.1.4, there are morphisms

$$k \to A \quad \text{and} \quad A \times A \to A \otimes A \to A$$

in $\mathbf{Ch}(k)$, which satisfy associativity and identity axioms, thus endowing A with a dg-k-algebra structure. 1-morphisms $\mathbf{A} \to \mathbf{B}$ in $\mathbf{Alg}(\mathbf{Ch}(k))$ are natural transformations between $\mathbf{A}, \mathbf{B} : \mathbf{Ass} \to \mathbf{Ch}(k)$, giving commutative diagrams

$$\begin{tikzcd} A \times A \arrow[r] \arrow[d] & B \times B \arrow[d] \\ A \arrow[r] & B \end{tikzcd},$$

hence they are precisely the dg-k-algebra maps. Higher morphisms give the chain homotopies between 1-morphisms. Thus, $\mathbf{Alg}(\mathbf{Ch}(k))$ is the symmetric monoidal $(\infty,1)$-category of dg-k-algebras.

By a similar argument, objects in $\mathbf{LMod}(\mathbf{Ch}(k))$ are pairs (A, M) of a dg-k-algebra A and a left dg-A-module M. A morphism $(A, M) \to (B, N)$ is a pair of maps: dg-k-algebra map $A \to B$ and a dg-A-module map $M \to N$ viewing N as a A-module. Hence, the $(\infty,1)$-category ${}_A\mathbf{LMod}(\mathbf{Ch}(k))$ is precisely the $(\infty,1)$-category of left dg-A-modules. Similar arguments hold for $\mathbf{RMod}(\mathbf{Ch}(k))$ and $\mathbf{BiMod}(\mathbf{Ch}(k))$.

The relative tensor product in $\mathbf{Ch}(k)$ is defined by the two-sided bar construction. Since associativity and identity are strict, it is easy to see that the underived relative tensor product is given by the colimit

$$M \otimes_B N = \operatorname{colim}_{\mathbf{BiMod}(\mathbf{Ch}_0(k))} \mathbf{Bar}(M, B, N) \cong \operatorname{colim}_{\mathbf{BiMod}(\mathbf{Ch}_0(k))} \left(M \otimes B \otimes N \underset{\mathrm{id}_M \otimes m_{B,N}}{\overset{m_{M,B} \otimes \mathrm{id}_N}{\rightrightarrows}} M \otimes N \right)$$

in the category $\mathbf{BiMod}(\mathbf{Ch}_0(k))$. So, for any dg-algebra B and right and left dg-B-modules M_B and ${}_BN$ respectively, we have

$$(M \otimes_B N)_n = \left(\bigoplus_{p+q=n} M_p \otimes N_q \right) / R$$

where R is the sub-module generated by $mb \otimes n - m \otimes bn$ where $m \in M_p, n \in N_q, b \in B_r$ and $p+q+r = k$.

We take the compositions in $\mathbf{Alg}_{(1)}(\mathbf{Ch}(k))$ to be the derived relative tensor products

$$M \otimes_B^{\mathbb{L}} N = Q(M) \otimes_{Q(B)} N.$$

$\mathbf{Alg}_{(1)}(\mathbf{Ch}(k))$ is the symmetric monoidal $(\infty, 2)$-category with objects being dg-k-algebras, 1-morphisms being bimodules, and 2-morphisms being morphisms of bimodules. Composition of 1-morphisms is given by the standard derived relative tensor product of dg-modules.

$\mathbf{Alg}^{(2)}(\mathbf{Ch}(k))$ is the category of strictly commutative dg-k-algebras. However, given a commutative dg-k-algebra A, the space of algebra structures on the underlying dg-k-module A may not be contractible. Indeed, A is an E_n-algebra if and only if the space of algebra structures, given by the 0-space of the pullback

$$\begin{tikzcd} \mathrm{AlgStr}(A) \arrow[r] \arrow[d] & \{A\} \arrow[d] \\ \mathbf{Alg}(\mathbf{Ch}(k)) \arrow[r] & \mathbf{Ch}(k)_1, \end{tikzcd}$$

is $(n-1)$-connected. By the Eckmann-Hilton argument, we know that if a dg-k-algebra A has two different compatible monoidal structures, then they are equivalent and all other monoidal structures on A are equivalent to them. Hence, if A is an associative dg-k-algebra that is not an E_2-algebra, then the 0-space of $\mathrm{AlgStr}(A)$ is discrete.

$\mathbf{Alg}_{(n)}(\mathbf{Ch}(k))$ is an $(\infty, n+1)$-category whose objects are E_n dg-k-algebras, r-morphisms are bimodules of E_{n-r+1} dg-k-algebras equipped with E_{n-r} dg-k-algebra structures for $1 \leq r < n$, n-morphisms are bimodules, $(n+1)$-morphisms are maps of bimodules, and higher morphisms are homotopies between such maps.

Application: Topological chiral homology

The cobordism hypothesis (Thm. 3.3.2) as stated in the last chapter gives a unique (up to homotopy) topological field theory $Z_C : \mathbf{Bord}_n^{\mathrm{fr}} \to \mathcal{C}$ for each fully dualisable object $C \in \mathrm{Ob}\,\mathcal{C}$ such that $Z_C(*) \cong C$. However, in general, given a bordism M in $\mathbf{Bord}_n^{\mathrm{fr}}$ (more formally, $\underline{M} \in \mathbf{SemiBord}_n^{\mathrm{fr}}$), it is difficult to compute $Z_C(M)$.

However, in the case of $\mathcal{C} = \mathbf{Alg}_{(n)}^0(\mathbf{S})$, it is possible to explicitly compute $Z_C(M)$ for all M. We give a sketch of the construction of topological chiral homology and its application to extended topological field theories, as given in [Fra12, Gin13, Lur12].

Let $\mathcal{A}$ be the topological category whose objects are finite disjoint unions of open discs $\coprod D^n$ and whose morphisms are open rectilinear embeddings. Thus, $\mathcal{A}$ can be seen as a subcategory of $\mathbf{E}_n$. $\mathcal{A}$ has a monoidal structure given by disjoint union on objects.

Let M be a framed manifold of dimension n. Let $\mathcal{A}_M$ be a topological category whose objects are pairs (X, j, h) where $X \in \mathrm{Ob}\,\mathcal{A}$, $j : X \to M$ is a framed open embedding and h is a homotopy between the canonical framing on $\coprod D^n$ and the pullback of the framing of M. The set of objects of $\mathcal{A}_M$ can be seen as a subspace of

$$\coprod_{i=1}^{\infty} \mathrm{Emb}^f\left(\coprod_i I, M \right)$$

where $\mathrm{Emb}^f(X, M) \subset \mathrm{Emb}(X, M)$ is the space of framed embeddings. For any $(X, j, h), (X', j', h') \in \mathrm{Ob}\,\mathcal{A}_M$, define $\mathrm{Map}_{\mathcal{A}_M}((X, j, h), (X', j', h'))$ to be the space of pairs (ϕ, f) where $\phi : X \to X'$ is an open rectilinear embedding, that is a morphism in $\mathcal{A}$ and $f : I \to \mathcal{A}_M$ is a continuous path in $\mathcal{A}_M$ such that $f(0) = j$ and $f(1) = j' \circ \phi$. It is given a topology as a subspace of $\mathrm{Map}_{\mathcal{A}}(X, X') \times \mathrm{Emb}^f(X, M)^I$. There is a canonical forgetful topological functor $\mathcal{A}_M \to \mathcal{A}$.

Let A be an E_n-algebra in $\mathbf{S}$. There is a monoidal functor $\mathcal{A} \to \mathbf{Alg}^0_{(n)}(S)$ which sends the disjoint union of k copies of D to the k-times tensor product $A \otimes \cdots \otimes A$. Define the **topological chiral homology** of M, denoted by $\int_M A$, to be the homotopy colimit of the composition

$$\mathcal{A}_M \to \mathcal{A} \to \mathbf{Alg}^0_{(n)}(\mathbf{S}).$$

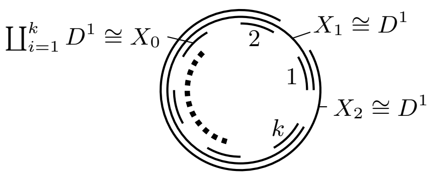
We need to impose some conditions on $\mathbf{S}$ to ensure that the homotopy colimit exists. A **good** monoidal (∞, n)-category $\mathbf{S}$ is one that admits small sifted colimits and such that the tensor product $\otimes : \mathbf{S} \times \mathbf{S} \to \mathbf{S}$ preserves small sifted colimits (see [Lur09b]).

If M is an n-framed manifold of dimension $m \le n$, define

$$\int_M A = \int_{M \times D^{n-m}} A.$$

If $i : D^n \to M$ is a homeomorphism, then all other embeddings $\coprod D^n \to M$ factors through i, and hence $\int_M A = A$.

Example 4.1.28. Let $M = S^1$ with the canonical framing. For any $k > 0$, we have diagrams of embeddings



$$\begin{tikzcd} A \otimes \cdots \otimes A \arrow[r, "a_1 \otimes \cdots \otimes a_k \mapsto a_1 \cdots a_k"] \arrow[d, "a_1 \otimes \cdots \otimes a_k \mapsto a_2 \cdots a_k a_1"'] & A \arrow[d] \\ A \arrow[r] & \int_{S^1} A \end{tikzcd}$$

which induces a cyclic permutation of the terms in the maps $A \otimes \cdots \otimes A \to A$. Hence, $\int_{S^1} A$ is the universal object that is equivariant under the action of any finite cyclic group, which can be computed to be the Hochschild homology

$$\int_{S^1} A = A \underset{A \otimes A^{\mathrm{op}}}{\otimes} A.$$

Since S is a good symmetric monoidal $(\infty, 1)$-category, $\int_{\_} A$ gives a monoidal covariant functor from the $(\infty, 1)$-category of n-framed manifolds to $\mathbf{Alg}_{(n)}(\mathbf{S})$, taking finite disjoint unions to the monoidal operation $\otimes$ in $\mathbf{S}$.

Let M be an framed manifold of dimension n with boundary ∂M. Let M^0 be the interior of M. By the collared neighbourhood theorem, we can write $M^0 \cong M^0 \sqcup ([0, 1) \times \partial M)$, so there is an open embedding $M^0 \sqcup (D^1 \times \partial M) \to M^0$ which induces a map

$$\Big(\int_{M^0} A\Big) \otimes \Big(\int_{\partial M} A\Big) \to \int_{M^0} A.$$

If $\int_{\partial M} A$ is an algebra, this gives $\int_{M^0} A$ a $\int_{\partial M} A$-module structure.

If $M = M_1 \cup M_2$ is a framed manifold such that $X = M_1 \cap M_2 \subset \partial M_1 \cap \partial M_2$, then $\int_{M_1} A$ and $\int_{M_2} A$ are right and left $(\int_X A)$-modules respectively. The inclusions

$$\begin{tikzcd} X \arrow[r] \arrow[d] & M_1 \\ M_2 & \end{tikzcd}$$

gives $\int_M A$ as the homotopy pushout in the diagram

$$\begin{tikzcd} \int_X A \arrow[r] \arrow[d] & \int_{M_1} A \arrow[d] \\ \int_{M_2} A \arrow[r] & \int_M A \end{tikzcd},$$

hence $\int_M A \cong (\int_{M_1} A) \otimes_{\int_X A} \int_{M_2} A$.

This gives another way to look at Example 4.1.28. We can decompose S^1 as the disjoint union of two closed intervals I glued together at their endpoints. Taking care of the orientations of the endpoints, we obtain

$$\int_{S^1} A = \int_I A \underset{\int_{*\sqcup *} A}{\otimes} \int_I A \cong A \underset{A\otimes A^{\mathrm{op}}}{A}.$$

Recall that a manifold in $M \in (\mathbf{SemiBord}_n^{\mathrm{fr}})_{1,\dots,1}$ has boundary $\partial M = \cup \partial_{i,\nu} M$ where $1 \leq i \leq n$ and $\nu = 0, 1$. the inclusions $(D^1 \times \partial_{i,0} M) \sqcup M^0 \sqcup (D^1 \times \partial_{i,1} M) \to M^0$ gives $\int_{M^0} A$ the structure of a $(\int_{\partial_{i,0} M} A, \int_{\partial_{i,1} M} A)$-bimodule. The interiors $(\partial_{i,\nu} M)^0$ of the boundary components are mutually disjoint, the intersection of the boundary of the boundary components establish relations between the different module structures on $\int_{M^0} A$. Inductively, this establishes $\int_{M^0} A$ as an n-morphism in $\mathbf{Alg}_{(n)}(\mathbf{S})$.

Any manifold $M \in (\mathbf{SemiBord}_n^{\mathrm{fr}})_{k_1,\dots,k_n}$ can be decomposed as a union of $k_1 \cdots k_n$ submanifolds $M_{i_1,\dots,i_n}$ in $(\mathbf{SemiBord}_n^{\mathrm{fr}})_{1,\dots,1}$, intersecting only on their boundaries (see Chap. 3.1), hence $\int_{M^0} A$ can be identified as the relative tensor products of the $\int_{M^0_{i_1,\dots,i_n}} A$ with respect to the shared boundaries. Hence, $\int_{M^0} A$ is an object in $\mathbf{Alg}^0_{(n)}(\mathbf{S})_{k_1,\dots,k_n}$.

Hence, $M \mapsto \int_A M$ defines a functor from $\mathbf{SemiBord}_n^{\mathrm{fr}}$ to $\mathbf{Alg}^0_{(n)}(\mathbf{S})$, which extends in an obvious way to a functor $\mathbf{PBord}_n^{\mathrm{fr}} \to \mathbf{Alg}^0_{(n)}(\mathbf{S})$ (by truncating the manifolds).

The above argument gives a sketch of the proof of the following theorem

Theorem 4.1.29 ([Lur09c, Thm. 4.1.24]). *Let $\mathbf{S}$ be any symmetric monoidal $(\infty, 1)$-category. Given any E_n-algebra $A \in \mathbf{Alg}^0_{(n)}(\mathbf{S})$, there exists a unique (up to homotopy in $\mathbf{Fun}^{\otimes}(\mathbf{Bord}_n^{\mathrm{fr}}, \mathbf{Alg}^0_{(n)}(\mathbf{S}))$) framed extended topological field theory $Z : \mathbf{Bord}_n^{\mathrm{fr}} \to \mathbf{Alg}^0_{(n)}(\mathbf{S})$ such that $Z(*) = A$. Z can be defined explicitly be*

$$\mathbf{PBord}_n^{\mathrm{fr}} \ni M \mapsto \int_M A.$$

Example 4.1.30. Let $n = 1$ and $A \in \mathbf{Alg}_{(1)}(\mathbf{S})$. Let $Z : \mathbf{SemiBord}_1^{\mathrm{fr}} \to \mathbf{Alg}^0_{(1)}(\mathbf{S})$ be a topological field theory with $Z(*) = A$. The objects of $\mathbf{SemiBord}_1^{\mathrm{fr}}$ are disjoint unions of points, each with an orientation. In Example 1.1.3, we showed that if Z takes the positively oriented points $*_+$ to A, then it takes the negatively oriented points $*_-$ to its dual A^{op} (see also the argument in [Lur09c, Prop. 1.1.8, Example 1.1.9]). Hence, the images of disjoint union of points are tensor products of copies of A and A^{op}.

As in Example 1.1.3, the connected 1-morphisms or framed bordisms are line segments and circles. Line segments I can determine framed bordisms in 4 ways: $d_1(I_1) = d_0(I_1) = *_+$; $d_1(I_2) = d_0(I_2) = *_-$, $d_1(I_3) = *_+ \sqcup *_-, d_0(I_3) = \emptyset$; or $d_1(I_4) = \emptyset, d_0(I_4) = *_- \sqcup *_+$. Their images under Z are the bimodules ${}_A A_A$, ${}_{A^{\mathrm{op}}} A^{\mathrm{op}}{}_{A^{\mathrm{op}}}$, ${}_{A\otimes A^{\mathrm{op}}} A_{\mathbf{1}}$ and ${}_{\mathbf{1}} A_{A^{\mathrm{op}}\otimes A}$ respectively. We have $S^1 = I_3 \sqcup_{*_+\sqcup *_-} I_4$, so

$$Z(S^1) \cong Z(I_3 \cup I_4) \cong Z(I_3) \circ Z(I_4) = {}_{\mathbf{1}}A \underset{A\otimes A^{\mathrm{op}}}{\otimes} A_{\mathbf{1}}$$

is the composition $\mathrm{ev}_A \circ \mathrm{coev}_A$.

However, in contrast to Example 1.1.3, Z encodes more higher homotopical information. For example, the mapping space $\mathrm{map}(\emptyset, \emptyset)$ has a connected component

$$B_{S^1} = \mathrm{Emb}^{\mathrm{fr}}(S^1, \mathbb{R}^\infty \times \mathbb{R})/\mathrm{Diff}^{\mathrm{fr}}(S^1) \cong \mathbb{CP}^\infty.$$

It carries the group action of $\mathrm{Diff}^{\mathrm{fr}}(S^1) \cong \mathrm{Diff}^{\mathrm{or}}(S^1) = SO(2) \cong S^1$. The $SO(2)$ action on an embedded manifold S^1 induces an $SO(2)$ action on $Z(S^1) \cong A \otimes_{A\otimes A^{\mathrm{op}}} A$. Since S^1 is invariant under the $SO(2)$ action and Z is a symmetric monoidal functor, $Z(S^1)$ is equivariant under the same action. We have already seen it for finite cyclic actions in Example 4.1.28.

Example 4.1.31. Let $n = 2$ and $A \in \mathbf{Alg}_{(2)}(\mathbf{S})$ be an E_2-algebra. S^1 is taken as a 2-framed manifold and $\int_{S^1} A$ is an object in $(\mathbf{Alg}^0_{(2)}(\mathbf{S}))_{1,0} = \mathbf{Alg}_{(1)}(\mathbf{BiMod}^{(1)\otimes}(\mathbf{S}))$. More specifically, since $S^1 \in \mathrm{Map}^1(\emptyset, \emptyset)$, $\int_{S^1} A$ is a $(\mathbf{1}, \mathbf{1})$-bimodule which is also an associative algebra. Recall that by definition of topological chiral homology for 2-framed 1-manifolds,

$$\int_{S^1} A = \int_{S^1 \times D^1} A = A \underset{A \otimes A^{\mathrm{op}}}{\otimes} A$$

and the multiplicative structure is induced by an inclusion $(S^1 \times D^1) \sqcup (S^1 \times D^1) \to (S^1 \times D^1)$.

Specifically, for $\mathbf{S} = \mathbf{Ch}(k)$, the multiplcation is given by the standard product of tensor algebras $(a \otimes b)(a' \otimes b') = (-1)^{\deg a \deg a'} aa' \otimes bb'$. It is well defined since A is a commutative dg-algebra.

4.2 The cobordism hypothesis in dimension 2

Recall from the previous section that the symmetric monoidal (∞, n)-category $\mathbf{Alg}^0_{(n)}(\mathbf{S})$ of E_n-algebras has duals, so every object is fully dualisable and determines a unique topological field theory by the cobordism hypothesis. However, in general, given an object C in a symmetric monoidal (∞, n)-category $\mathcal{C}$, it is difficult to determine if C is fully dualisable. In this section gives a simple condition to check this in the case $n = 2$.

Fully dualisable objects in a symmetric monoidal $(\infty, 2)$-category

Let $\mathcal{C}$ be a symmetric monoidal $(\infty, 2)$-category. For any object X to be fully dualisable, X must have a dual $X^\vee$. Furthermore, the counit and unit maps ev_X and coev_X must be contained in $\mathcal{C}^{\mathrm{fd}}$, that is to say, ev_X and coev_X have left and right adjoints, and furthermore, their adjoints themselves have left and right adjoints, and so on. However, the following lemma shows that it suffices to check that either ev_X or coev_X has left and right adjoints.

Lemma 4.2.1. *Let $\mathcal{C}$ be a symmetric monoidal $(\infty, 2)$-category. Suppose $X \in \mathrm{Ob}\,\mathcal{C}$ has a dual $X^\vee$ and that the counit map $\mathrm{ev}_X : X \otimes X^\vee \to \mathbf{1}$ has both a left adjoint ev^L_X and a right adjoint ev^R_X. Then, the unit map coev_X has both left and right adjoints as well, and there exists $S, T \in \mathrm{Ob}\,\mathrm{map}_{\mathbf{S}}(X, X)$ such that*

(i) $T = S^{-1}$, that is $T \circ S \cong S \circ T \cong \mathrm{id}_X$; and

(ii)

$$\begin{array}{ll} \tilde{\mathrm{ev}}^L_X = (\mathrm{id}_{X^\vee} \otimes S) \circ \mathrm{coev}_X & \tilde{\mathrm{ev}}^R_X = (\mathrm{id}_{X^\vee} \otimes T) \circ \mathrm{coev}_X \\ \tilde{\mathrm{coev}}^R_X = \mathrm{ev}_X \circ (S \otimes \mathrm{id}_{X^\vee}) & \tilde{\mathrm{coev}}^L_X = \mathrm{ev}_X \circ (T \otimes \mathrm{id}_{X^\vee}) \end{array}$$

where $\tilde{\mathrm{ev}}^{L/R}_X$ and $\tilde{\mathrm{coev}}^{L/R}_X$ are the adjoints post- and pre-composed with the symmetry map $X \otimes X^\vee \cong X^\vee \otimes X$ respectively.

Proof. The duality of X and $X^\vee$ gives the Dwyer-Kan equivalence

$$\begin{array}{rcl} \mathrm{map}_{\mathbf{S}}(X, X) & \xrightarrow{\sim} & \mathrm{map}_{\mathbf{S}}(\mathbf{1}, X^\vee \otimes X) \\ f & \mapsto & (\mathrm{id}_{X^\vee} \otimes f) \circ \mathrm{coev}_X \\ (\mathrm{ev}_X \otimes \mathrm{id}_X) \circ (\mathrm{id}_X \otimes g) & \leftarrow\!\shortmid & g \end{array}$$

which associates to $\tilde{\mathrm{ev}}^L_X$ and $\tilde{\mathrm{ev}}^R_X$ objects S and T in $\mathrm{map}(X, X)$ respectively. Hence, we can take

$$\tilde{\mathrm{ev}}^L_X = (\mathrm{id}_{X^\vee} \otimes S) \circ \mathrm{coev}_X \quad \text{and} \quad \tilde{\mathrm{ev}}^R_X = (\mathrm{id}_{X^\vee} \otimes T) \circ \mathrm{coev}_X.$$

Similarly, there is a Dwyer-Kan equivalence $\mathrm{map}_{\mathbf{S}}(X \otimes X^\vee, \mathbf{1}) \cong \mathrm{map}_{\mathbf{S}}(X, X)$ and the images of S and T under this equivalence are

$$\tilde{\mathrm{coev}}^R_X = \mathrm{ev}_X \circ (S \otimes \mathrm{id}_{X^\vee}) \quad \text{and} \quad \tilde{\mathrm{coev}}^L_X = \mathrm{ev}_X \circ (T \otimes \mathrm{id}_{X^\vee}).$$

Using the fact that ev_X^L and ev_X^R are adjoints of ev_X, it is immediate to verify that coev_X^R and coev_X^L are the right and left adjoints of coev_X respectively.

By duality, $(\mathrm{ev}_X \otimes \mathrm{id}_X) \circ (\mathrm{id}_X \otimes \mathrm{coev}_X) \cong \mathrm{id}_X$, so it is self-adjoint. We can compute its left adjoint explicitly to get

$$\begin{aligned} \mathrm{id}_X &\cong (\mathrm{id}_X \otimes \mathrm{coev}_X^L) \circ (\mathrm{ev}_X^L \otimes \mathrm{id}_X) \cong (\tilde{\mathrm{coev}}_X^L \otimes \mathrm{id}_X) \circ (\mathrm{id}_X \otimes \tilde{\mathrm{ev}}_X^L) \\ &\cong (\mathrm{ev}_X \otimes \mathrm{id}_X) \circ (T \otimes \mathrm{id}_{X^\vee \otimes X}) \circ (\mathrm{id}_{X \otimes X^\vee} \otimes S) \circ (\mathrm{id}_X \otimes \mathrm{coev}_X) \\ &\cong S \circ (\mathrm{ev}_X \otimes \mathrm{id}_X) \circ (T \otimes \mathrm{id}_{X^\vee \otimes X}) \circ (\mathrm{id}_X \otimes \mathrm{coev}_X) \cong S \circ T \end{aligned}$$

by the dualities given above. Similarly, using the right adjoint, we get $\mathrm{id}_X \cong T \circ S$. ☐

Proposition 4.2.2. *Let $\mathcal{C}$ be a symmetric monoidal $(\infty, 2)$-category. Then $X \in \mathrm{Ob}\,\mathcal{C}$ is fully dualisable if and only if X has a dual $X^\vee$ and the counit map $\mathrm{ev}_X : X \otimes X^\vee \to \mathbf{1}$ has both a left and a right adjoint.*

Proof. The necessity of the conditions is clear. Conversely, suppose X has a dual $X^\vee$ and $\mathrm{ev}_X : X \otimes X^\vee \to \mathbf{1}$ has a left adjoint and a right adjoint $\mathrm{ev}_X^L, \mathrm{ev}_X^R : \mathbf{1} \to X \otimes X^\vee$. By the lemma above coev_X have left and right adjoints as well.

The formulation of $\mathrm{ev}_X^{L/R}$ and $\mathrm{coev}_X^{L/R}$ in terms of S and T in the lemma above allows us to define further adjoints. For example, ev_X^L has a left adjoint given by

$$\tilde{\mathrm{ev}}_X^{L^{(2)}} = \mathrm{ev}_X \circ (S^{-2} \otimes \mathrm{id}_{X^\vee}) = \mathrm{ev}_X \circ (T^2 \otimes \mathrm{id}_{X^\vee}).$$

In general, for all integers n, $\mathrm{ev}_X \circ (S^n \otimes \mathrm{id}_{X^\vee})$ has a left adjoint $(\mathrm{id}_{X^\vee} \otimes S^{1-n}) \circ \mathrm{coev}_X$ and a right adjoint $(\mathrm{id}_{X^\vee} \otimes S^{-1-n}) \circ \mathrm{coev}_X$ while $(\mathrm{id}_{X^\vee} \otimes S^n) \circ \mathrm{coev}_X$ has a left adjoint $\mathrm{ev}_X \circ (S^{-1-n} \otimes \mathrm{id}_{X^\vee})$ and a right adjoint $\mathrm{ev}_X \circ (S^{1-n} \otimes \mathrm{id}_{X^\vee})$. Hence, ev_X and coev_X lie in $\mathcal{C}^{\mathrm{fd}}$ and X is fully dualisable. ☐

Recall from the previous section that the symmetric monoidal $(\infty, 2)$-category $\mathbf{Alg}^{\otimes}_{(1)}(\mathbf{S})$ has duals for objects but not adjoints for 1-morphisms. We can use this proposition to characterise the fully dualisable objects in $\mathbf{Alg}^{\otimes}_{(1)}(\mathbf{S})$.

First, note that for any associative algebra A, we can endow the $(\infty, 1)$-category of (A, A)-bimodules with a symmetric monoidal structure as follows: consider the underlying pre-complete Segal 2-space $\mathbf{Alg}_{(1)}(\mathbf{S})$ of $\mathbf{Alg}^{\otimes}_{(1)}(\mathbf{S})$. Let $\mathbf{Mod}(A)$ denote the maximal sub-complete Segal 2-space of $\mathbf{Alg}_{(1)}(\mathbf{S})$ consisting of a single object $\{A\}$. We can regard it as a functor $\Delta^{\mathrm{op}} \to 1\text{-}\mathcal{CSS}$. By Remark 2.6.11, there is a Dwyer-Kan equivalence $N(1\text{-}\mathcal{CSS})^{N(\Delta^{\mathrm{op}})} \cong N(1\text{-}\mathcal{CSS})^{N(\Gamma_1)}$, so $\mathbf{Mod}(A)$ can be regarded as a monoidal $(\infty, 1)$-category. The monoidal operation is given by the relative tensor product with respect to A. We will call objects of $\mathbf{Mod}(A)$ **A-modules**.

In fact, if A is an E_n-algebra, the maximal sub-complete Segal $(n+1)$-space $\mathbf{Mod}(A)$ of $\mathbf{Alg}_{(n)}(\mathbf{S})_1$ containing a single object $\{A\}$ can be regarded as a functor $(\Delta^{\mathrm{op}})^n \to 1\text{-}\mathcal{CSS}$. The Dwyer-Kan equivalence $N(1\text{-}\mathcal{CSS})^{N(\Delta^{\mathrm{op}})^n} \cong N(1\text{-}\mathcal{CSS})^{N}(\mathbf{E}_n)$ implies that $\mathbf{Mod}(A)$ is an E_n-monoidal $(\infty, 1)$-category. If A is an E_∞-algebra, that is, if A is an E_n-algebra for all n, then, by the strictification theorem [Rez01, TV02],

$$\begin{aligned} N(1\text{-}\mathcal{CSS}^{\mathrm{colim}\,(\Delta^{\mathrm{op}})^n}) &\cong N(1\text{-}\mathcal{CSS})^{\mathrm{colim}\,(N(\Delta^{\mathrm{op}}))^n} \cong \mathrm{colim}\, N(1\text{-}\mathcal{CSS})^{(N(\Delta^{\mathrm{op}}))^n} \\ &\cong \mathrm{colim}\, N(1\text{-}\mathcal{CSS})^{N(\mathbf{E}_n)} \cong N(1\text{-}\mathcal{CSS})^{\mathrm{colim}\, N(\mathbf{E}_n)} \cong N(1\text{-}\mathcal{CSS})^{N(\mathbf{E}_\infty)}, \end{aligned}$$

so $\mathbf{Mod}(A)$ can be regarded as a symmetric monoidal $(\infty, 1)$-category.

The informal discussion above allows us to give the following proposition/definition.

Proposition 4.2.3. *Let A be an E_n-algebra in $\mathbf{S}$. Then ${}_A\mathbf{BiMod}_A(\mathbf{S})$ can be given an E_n-monoidal structure $\mathbf{Mod}(A)$ with the monoidal operation given by relative tensor products with respect to A.*

Example 4.2.4. Let k be a commutative ring. If $\mathbf{S} = N(\mathbf{Mod}(k))$, then $\mathbf{Mod}(A) = N(\mathbf{Mod}(A))$ is the monoidal $(\infty, 1)$-category of A-modules (as defined classically) and it is symmetric if A is commutative.

More generally, if $\mathbf{S} = \mathbf{Ch}(k)$, then $\mathbf{Mod}(A)$ is the monoidal $(\infty, 1)$-category of chain complexes $M_\bullet$ such that M_i is an A_i-module for each $i \in \mathbb{Z}$, with the module structures compatible with the differential maps.

Let A be an associative algebra in $\mathbf{S}$ and let $A^e = A \otimes A^{\mathrm{op}} \in \mathbf{Alg}(\mathbf{S})$. We characterise the full dualisability of A in terms of its dualisability in $\mathbf{Mod}(\mathbf{1})$ and $\mathbf{Mod}(A^e)$.

Let B be is an E_2-algebra, that is, there is an equivalence $B^{\mathrm{op}} \cong B$. Let M be a right B-module, then by the E_2 structure, we can also view it as a right $B \otimes B$-module, with the right multiplication done twice. Note that the space of such right $B \otimes B$-module structures on M fixing a right B-module structure is connected, but not contractible unless B is an E_∞-algebra. However, they are equal in the homotopy 2-category of $\mathbf{Alg}_{(1)}(\mathbf{S})$, where we will be working when studying adjoints, so we shall not distinguish them. Hence, we can regard M as a (B, B)-bimodule.

It is easy to see that A^e is an E_2-algebra. Recall that an associative algebra A can be regarded as bimodules ${}_AA_A$, ${}_{A^e}A_{\mathbf{1}}$ or ${}_{\mathbf{1}}A_{(A^e)^{\mathrm{op}}} \cong {}_{\mathbf{1}}A_{A^e}$. By the above argument, we may also regard A as a bimodule ${}_{A^e}A_{A^e}$ or by forgetting the A-module structure, as ${}_{\mathbf{1}}A_{\mathbf{1}}$.

Definition 4.2.5. Let A be an associate algebra in $\mathbf{Alg}(\mathbf{S})$. We say that A is **proper** if A is dualisable as an object in $\mathbf{S} \cong \mathbf{Mod}(\mathbf{1})$. We say that A is **smooth** if A is dualisable as an object in $\mathbf{Mod}(A^e)$.

Theorem 4.2.6. *Let A be an associative algebra in $\mathbf{S}$. Then, A is fully dualisable in $\mathbf{Alg}^{\otimes}_{(1)}(\mathbf{S})$ if and only if it is smooth and proper.*

Proof. By Prop. 4.1.16, A has a dual A^{op}.

First suppose A is fully dualisable. Hence, the counit map ${}_{A^e}A_{\mathbf{1}}$ has both a left and a right adjoint, given by ${}_{\mathbf{1}}S_{A^e}$ and ${}_{\mathbf{1}}T_{A^e}$ respectively. We shall show that S and T are the duals of A in $\mathbf{Mod}(A^e)$ and $\mathbf{Mod}(\mathbf{1})$ respectively.

Given that S is the left adjoint of A in $\mathbf{Alg}_{(1)}(\mathbf{S})$, we have unit and counit maps

$$u : \mathbf{1} \to S \underset{A^e}{\otimes} A \qquad \text{and} \qquad A \underset{\mathbf{1}}{\otimes} S \to A^e.$$

By the arguments above, we can regard S as a (A^e, A^e)-bimodule as necessary. It is easy to see that the adjunctions in $\mathbf{Alg}_{(1)}(\mathbf{S})$ induce adjoint functors

$$\begin{gathered} - \underset{\mathbf{1}}{\otimes} {}_{\mathbf{1}}S_{A^e} : \mathbf{BiMod}_{\mathbf{1}}(\mathbf{S}) \to \mathbf{BiMod}_{A^e}(\mathbf{S}) : - \underset{A^e}{\otimes} {}_{A^e}A_{\mathbf{1}}, \\ {}_{A^e}A_{\mathbf{1}} \underset{\mathbf{1}}{\otimes} - : {}_{\mathbf{1}}\mathbf{BiMod}(\mathbf{S}) \to {}_{A^e}\mathbf{BiMod}(\mathbf{S}) : {}_{\mathbf{1}}S_{A^e} \underset{A^e}{\otimes} -, \end{gathered}$$

in particular, they induce Dwyer-Kan equivalences

$$\begin{array}{rcl} \mathrm{Map}_{\mathbf{BiMod}(\mathbf{S})_{A^e}}(M \underset{\mathbf{1}}{\otimes} S_{A^e}, N) & \cong & \mathrm{Map}_{\mathbf{BiMod}(\mathbf{S})_{\mathbf{1}}}(M, N \underset{A^e}{\otimes} A_{\mathbf{1}}) \\ (\mathrm{id}_N \otimes_{A^e} v) \circ (f \otimes_{\mathbf{1}} \mathrm{id}_S) & \leftarrow\!\shortmid & f \\ \mathrm{Map}_{{}_{A^e}\mathbf{BiMod}(\mathbf{S})}({}_{A^e}A \underset{\mathbf{1}}{\otimes} {}_{\mathbf{1}}P, Q) & \cong & \mathrm{Map}_{{}_{\mathbf{1}}\mathbf{BiMod}(\mathbf{S})}(P, {}_{\mathbf{1}}S \underset{A^e}{\otimes} Q) \\ g & \mapsto & (\mathrm{id}_S \otimes_{A^e} g) \circ (u \otimes_{\mathbf{1}} \mathrm{id}_{A^e}) \end{array}$$

for bimodules M, N, P, Q.

Set $f : {}_{A^e}A_{\mathbf{1}} \to {}_{A^e}A_{\mathbf{1}}$ the identity map and $g : {}_{A^e}A \otimes_{\mathbf{1}} A^e{}_{A^e} \to {}_{A^e}A_{A^e}$ the right A^e-module structure map in the adjunction equations above.

The first equation gives the A^e-bilinear map $v : A \otimes_{\mathbf{1}} S \to A^e$. By the universal property of the relative tensor product, v factors through $A \otimes_{\mathbf{1}} S \cong A \otimes_{\mathbf{1}} A^e \otimes_{A^e} S \xrightarrow{g \otimes \mathrm{id}} A \otimes_{A^e} S$ to give a homotopy commutative diagram

$$\begin{tikzcd} A \underset{\mathbf{1}}{\otimes} S \arrow[r, "v"] \arrow[d] & A^e \\ A \underset{A^e}{\otimes} S \arrow[ur, "\exists!", "\mathrm{ev}"'] & \end{tikzcd}$$

The second equation gives us an $(\mathbf{1}, A^e)$-bilinear map $\tilde{u} : A^e \to S \otimes_{A^e} A$. Since A^e is E_2, any left or right

A^e-linear map is homotopic to some A^e-bilinear map. This gives us a homotopy commutative diagram

$$\begin{tikzcd} {}_{\mathbf{1}}A^e{}_{A^e} \arrow[r, "\tilde{u}"] \arrow[d, "\mathrm{id}"'] & {}_{\mathbf{1}}S \underset{A^e}{\otimes} A_{A^e} \arrow[d, "\mathrm{id}"] \\ {}_{A^e}A^e{}_{A^e} \arrow[r, "\mathrm{coev}"] & {}_{A^e}S \underset{A^e}{\otimes} A_{A^e} \end{tikzcd}$$

We want to check that ev and coev are the evaluation and coevaluation maps for S as a right dual of A in $\mathbf{Mod}(A^e)$. We have a homotopy commutative diagram

$$\begin{tikzcd} {}_{A^e}A_{A^e} \arrow[r, "\mathrm{id}\otimes\mathrm{coev}"] & {}_{A^e}A \underset{A^e}{\otimes} S \underset{A^e}{\otimes} A_{A^e} \arrow[r, "\mathrm{ev}\otimes\mathrm{id}"] & {}_{A^e}A_{A^e} \\ {}_{A^e}A \underset{\mathbf{1}}{\otimes} A^e{}_{A^e} \arrow[u, "g"] \arrow[r, "\mathrm{id}\otimes\tilde{u}"] & {}_{A^e}A \underset{\mathbf{1}}{\otimes} S \underset{A^e}{\otimes} A_{A^e} \arrow[u, "g\otimes\mathrm{id}"'] \arrow[ur, "v\otimes\mathrm{id}"'] & \\ {}_{A^e}A_{\mathbf{1}} \arrow[u, "a\mapsto a\otimes 1"] \arrow[uu, bend left=60, "\mathrm{id}"] \arrow[r, "u\otimes\mathrm{id}"] \arrow[rr, bend right, "\mathrm{id}"'] & {}_{A^e}A \underset{\mathbf{1}}{\otimes} S \underset{A^e}{\otimes} A_{\mathbf{1}} \arrow[u] \arrow[r, "v\otimes\mathrm{id}"] & {}_{A^e}A_{\mathbf{1}} \arrow[uu, "\mathrm{id}"'] \end{tikzcd}$$

which shows that the composition of the top row is homotopic to the identity. Similarly, we can show that

$$ {}_{A^e}S_{A^e} \xrightarrow{\mathrm{coev}\otimes\mathrm{id}} {}_{A^e}S \underset{A^e}{\otimes} A \underset{A^e}{\otimes} S_{A^e} \xrightarrow{\mathrm{id}\otimes\mathrm{ev}} {}_{A^e}S_{A^e} $$

is homotopic to the identity.

For the converse direction, suppose A has a right dual S in $\mathbf{Mod}(A^e)$. Then, we can similarly check that ${}_{\mathbf{1}}S_{A^e}$ is the left adjoint of ${}_{A^e}A_{\mathbf{1}}$ with the unit and counit maps given by

$$\mathbf{1} \to A^e \xrightarrow{\mathrm{coev}} S \underset{A^e}{\otimes} A \qquad \text{and} \qquad A \underset{\mathbf{1}}{\otimes} S \to A \underset{A^e}{\otimes} S \to A^e.$$

The proof that T is right adjoint of A in $\mathbf{Alg}_{(1)}(\mathbf{S})$ if and only if it is the dual of A in $\mathbf{Mod}(\mathbf{1})$ is similar. $\square$

While the theorem gives a characterisation of fully dualisable objects in $\mathbf{Alg}_{(1)}(\mathbf{S})$, the definitions of smooth and proper algebras given in Def. 4.2.5 is rather obscure. The terms smooth and proper come from algebraic geometry and correspond to the geometric properties of smoothness and properness on schemes (properness gives a notion of compactness). Kontsevich and Sobeilman gave some examples of smooth and proper algebras in categories over smooth schemes, which illustrate the geometric nature of these definitions [KS09]. In line with the discussions above, we will give some more explicit definitions for module categories.

Example 4.2.7. Consider the ordinary abelian category $\mathbf{Mod}(k)$ where k is a commutative ring. For any commutative ring A, a module $M \in \mathbf{Mod}(A)$ is dualisable if and only if it is finitely generated and projective (see Prop. 4.2.9 below). Hence, a k-algebra A is fully dualisable in $\mathbf{Alg}_{(1)}(\mathbf{S})$ if and only if it is finitely generated and projective in $\mathbf{Mod}(k)$ and $\mathbf{Mod}(A \otimes A^{\mathrm{op}})$ (see, for example, [SP11, Appendix A]). In fact, this is also equivalent to saying that A is finite semisimple k-algebra.

Example 4.2.8. In the case of $\mathbf{S} = \mathbf{Ch}(k)$, we can show the following:

Proposition 4.2.9. *Let $A = A_\bullet \in \mathbf{Ch}(k)$ be an associative dg-k-algebra and $M \in \mathbf{Mod}(A_\bullet)$. The following are equivalent:*

(i) M is dualisable in $\mathbf{Mod}(A)$.

(ii) There exists $n < \infty$ and $k_1, \ldots, k_n \in \mathbb{Z}$ such that there is a commutative diagram of A-linear maps

$$\begin{tikzcd} M \arrow[r] \arrow[rr, bend right, "\mathrm{id}_M"'] & \bigoplus_{i=1}^n A[k_i] \arrow[r] & M. \end{tikzcd}$$

(iii) M is finitely generated and projective as an A-module.

Proof. (i)$\Rightarrow$(ii): Suppose M has a dual $M^\vee$ in $\mathbf{Mod}(A)$, so there are evaluation and coevaluation maps

$$M \otimes_A M^\vee \xrightarrow{\varepsilon} A \qquad \text{and} \qquad A \xrightarrow{\eta} M^\vee \otimes_A M$$

satisfying the adjoint identities. Let $\eta(1) = \sum_{i=1}^n x_i \otimes y_i$ where $x_i \in M^\vee[-k_i]$ and $y_i \in M[k_i]$ for each i ($M[k]$ is the shift of M by degree k). The adjoint identity $\mathrm{id}_M = (\mathrm{id}_M \otimes \eta) \circ (\varepsilon \otimes \mathrm{id}_M)$ gives

$$m = (\mathrm{id}_M \otimes \eta) \circ (\varepsilon \otimes \mathrm{id}_M)(m) = \sum_{i=1}^n \varepsilon(m \otimes x_i) y_i.$$

Hence, the composition of the maps

$$M \to \bigoplus_{i=1}^n A[k_i] : m \mapsto (\varepsilon(m \otimes x_i))_{i=1}^n \qquad \text{and} \qquad \bigoplus_{i=1}^n A[k_i] \to M : (a_i)_{i=1}^n \mapsto \sum_{i=1}^n a_i y_i$$

gives id_M.

(ii)$\Rightarrow$(iii): That M is finitely generated is immediate as $\oplus_{i=1}^n A[k_i] \to M$ is surjective. $A[k_i]$ are projective A-modules, and so is $\bigoplus_{i=1}^n A[k_i]$. Hence, given any surjection $N \to N'$ in $\mathbf{Mod}(A)$ and a map $M \to N'$, the commutative diagram

$$\begin{tikzcd}
 & & & N \arrow[d] \\
M \arrow[r] \arrow[rr, bend right, "\mathrm{id}_M"'] & \bigoplus_{i=1}^n A[k_i] \arrow[r] \arrow[urr, dashed, "\exists"] & M \arrow[r] & N' \arrow[d] \\
 & & & 0
\end{tikzcd}$$

gives a lift $M \to N$. Hence, M is projective as an A-module.

(iii)$\Rightarrow$(i): Suppose M is finitely generated and projective as an A-module. Let $M^\vee = \mathrm{Hom}_A(M, A)$ be the k-chain complex given by

$$\mathrm{Hom}_A(M, A)_l = \mathrm{Hom}_{\mathbf{Mod}(A)}(M, A[-l])$$

with the differential $(df)(m) = d(f(m))$. Since M is finitely generated, there exists a surjection $\phi : \bigoplus_{i=1}^n A[k_i] \to M$. Let $y_i = \phi(0, \ldots, 1, \ldots, 0) \in M$ be the image of the i-th coordinate, so $(y_1, \ldots, y_n)$ is a set of generators for M. By the projectivity of M, this induces a surjection

$$\psi : \mathrm{Hom}_A\Big(M, \bigoplus_{i=1}^n A[k_i]\Big) \cong \bigoplus_{i=1}^n \mathrm{Hom}_A(M, A[-k_i]) \cong \bigoplus_{i=1}^n M^\vee[-k_i] \to \mathrm{Hom}_A(M, M).$$

Choose $(x_1, \ldots, x_n) \in \bigoplus_{i=1}^n M^\vee[-k_i]$ such that $\psi(x_1, \ldots, x_n) = \mathrm{id}_M$. $(x_1, \ldots, x_n)$ is a set of generator for $M^\vee$. Define the A-linear maps

$$\varepsilon : M \otimes_A M^\vee \to A : y_i \otimes x_j \mapsto \delta_{ij} \qquad \text{and} \qquad \eta : A \to M^\vee \otimes_A M : 1 \mapsto \sum_{i=1}^n x_i \otimes y_i.$$

It is easy to check that they are the evaluation and coevaluation maps for $M^\vee$, so M is dualisable in $\mathbf{Mod}(A)$. $\square$

The third condition can be phrased in a more general setting of dg-categories (see [Kel07]). An A-module satisfying the third condition is usually called **perfect**. Keller proved that an object in a dg-category is perfect if and only if it is **compact**, that is, for any small indexing category I and $(N_i)_{i \in I} \in \mathbf{Mod}(A)^I$, the map

$$\operatorname*{colim}_{i \in I} \mathrm{Map}_{\mathbf{Mod}(A)}(M, N_i) \to \mathrm{Map}_{\mathbf{Mod}(A)}(M, \operatorname*{hocolim}_{i \in I} N_i)$$

is an isomorphism in $\mathrm{Ho}(\mathbf{sSet})$ [Kel07].

Hence, a dg-k-algebra A is proper if it is a perfect dg-k-module and smooth if it is a perfect $A \otimes A^{\mathrm{op}}$-module. Explicitly, A is a perfect dg-k-module if and only if $\sum_i \dim_k H_i(A) < \infty$.

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