Artin conductors of tori

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Abstract

This article is based on the paper "Congruences of Néron models for tori and the Artin conductor" by Ching-Li Chai and Jiu-Kang Yu, published in Annal of Mathematics **154** (2001).

Let K be a complete discrete valuation field with perfect residue field. Let T be a torus over K, with Néron model T^{NR} over the ring of integers $\mathcal{O}_K$ of K. The Néron model does not commutate with the base change in general. Choose a finite Galois extension L/K which spits T. One can measure the change of Néron models by comparing $(\mathrm{Lie}\, T^{NR}) \otimes \mathcal{O}_L$ with $\mathrm{Lie}((T \otimes L)^{NR})$. We define an invariant $c(T) \in \mathbb{Q}$ by

$$c(T) = \frac{1}{e_{L/K}} \mathrm{length}_{\mathcal{O}_L} \frac{\mathrm{Lie}(T \otimes L)^{NR}}{(\mathrm{Lie}\, T^{NR}) \otimes \mathcal{O}_L}$$

where $e_{L/K}$ is the ramification index of L/K and Lie() denotes the Lie algebra. Let $X_*(T)$ be the ocharacter group of T and let $a(X_*(T) \otimes \mathbb{Q})$ be the Artin conductor of the Galois representation $X_*(T) \otimes \mathbb{Q}$ of $\mathrm{Gal}(\bar{K}/K)$. The main theorem **10.2** states that $c(T)$ is invariant by isogeny and

$$c(T) = \frac{1}{2} a(X_*(T) \otimes \mathbb{Q}),$$

answering a question of B. Gross. Note that in the final step of the proof of theorem **10.2**, we restricted ourself to the special case when K has characteristic 0.

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Contents

1 Notation

- Let $\mathcal{O} = \mathcal{O}_K$ be a discrete valuation ring with residue field κ and let K be its field of fractions. Let $\pi = \pi_K$ be the a prime element of $\mathcal{O}$. The strict henselization and the completion of $\mathcal{O}$ are denoted by $\mathcal{O}^{sh}$ and $\widehat{\mathcal{O}}$ respectively. Their fields of fraction are denoted by K^{sh} and $\widehat{K}$ respectively. The residue fields of $\mathcal{O}^{sh}$ is the separable closure κ^{sep} of κ. Denote the algebraic closure of K by $\overline{K}$.

- Denote the multiplicative group scheme over a ring A by $\mathbb{G}_{m,A}$.

- Let T be a torus over K. Denote by Λ the *cocharacter group*

$$X_*(T) = \mathrm{Hom}(\mathbb{G}_{m,\overline{K}}, T \otimes \overline{K})$$

of T and by

$$X^*(T) = \mathrm{Hom}(T \otimes \overline{K}, \mathbb{G}_{m,\overline{K}})$$

the *character group* of T. We will often denote by L/K a Galois extension such that T is split over L and by Γ the Galois group $\mathrm{Gal}(L/K)$.

- we will also work with another discrete valuation ring $\mathcal{O}_0$. We will analogous constructs by the same notation with a subscript 0. And introduce a series of congruence notation:
  - $(\mathcal{O}, \mathcal{O}_L) \equiv_\alpha (\mathcal{O}_0, \mathcal{O}_{L_0})$ (level N): this means that α is an isomorphism from $\mathcal{O}_L/\pi^N\mathcal{O}_L$ to $\mathcal{O}_{L_0}/\pi_0^N O_{L_0}$ and induce an isomorphism $\mathcal{O}/\pi^N\mathcal{O} \to \mathcal{O}_0/\pi_0^N\mathcal{O}_{L_0}$.
  - $(\mathcal{O}, \mathcal{O}_L, \Gamma) \equiv_{\alpha,\beta} (\mathcal{O}_0, \mathcal{O}_{L_0}, \Gamma_0)$ (level N): this means $(\mathcal{O}, \mathcal{O}_L) \equiv_\alpha (\mathcal{O}_0, \mathcal{O}_{L_0})$ (level N), β is an isomorphism $\Gamma \to \Gamma_0$, and α is $\Gamma-$equivalent relative to β: $\alpha(\gamma.x) = \beta(\gamma).\alpha(x)$.
  - $(\mathcal{O}, \mathcal{O}_L, \Gamma, \Lambda) \equiv_{\alpha,\beta,\phi} (\mathcal{O}_0, \mathcal{O}_{L_0}, \Gamma_0, \Lambda_0)$ (level N): this means that $(\mathcal{O}, \mathcal{O}_L, \Gamma) \equiv_{\alpha,\beta} (\mathcal{O}_0, \mathcal{O}_{L_0}, \Gamma_0)$ (level N), and ϕ is isomorphism $\Lambda \to \Lambda_0$ which is Γ-equivalent relative to β.
  - If it is not necessary to name the isomorphisms($\alpha, \beta, etc.$), we omit them from the notation.
- In this paper, "X is determined by $(\mathcal{O}/\pi^N\mathcal{O}, \mathcal{O}_L/\pi^N\mathcal{O}_L, \Gamma, \Lambda)$ " means if $(\mathcal{O}, \mathcal{O}_L, \Gamma, \Lambda) \equiv_{\alpha,\beta,\phi} (\mathcal{O}_0, \mathcal{O}_{L_0}, \Gamma_0, \Lambda_0)$(level N), then there is a canonical isomorphism $X \to X_0$ determined by (α, β, ϕ).
- All rings in this paper are $\mathcal{O}-$algebras or $\mathcal{O}_0$-algebra. All maps between two group schemes are the homomorphisms of group schemes.
- If X is an $\mathcal{O}$-scheme, we sometimes denote $X \times \operatorname{Spec} \mathcal{O}/\pi^N$ by $X \otimes \mathcal{O}/\pi^N$. Similarly, we have the same meaning for $X \otimes L$, etc.
- For a group scheme X over base scheme S, we denote the module of translation invariant top differential forms on X by $\omega(X)$.

2 Basic properties of tori

Definition 2.1. Let K be a field, a *torus* T over K is an affine group scheme T over K such that $T_{\bar{K}} = T \otimes_K \bar{K} \simeq G^d_{m,\bar{K}}$, where d is the dimension of T. We say that T is *split* over some field extension L/K if $T \otimes L$ is isomorphic to $\mathbb{G}^d_L$, and that L is a *splitting field* of T.

Assume L/K is a Galois extension, and X, Y are $K-$schemes, then there exists a right $\mathrm{Gal}(L/K)$-action on $\mathrm{Hom}_L(X_L, Y_L)$. Let $\sigma \in \mathrm{Gal}(L/K)$, $\phi \in \mathrm{Hom}_L(X_L, Y_L)$, we have $id \otimes \sigma : X \otimes L \to X \otimes L$. Define the action of σ on ϕ to be $(id_Y \otimes \sigma) \circ \phi \circ (id_X \otimes \sigma)^{-1}$, denoted by ϕ^σ. Then ϕ^σ is also an L-morphism.

If $\phi^\sigma = \phi$ for every $\sigma \in \mathrm{Gal}(L/K)$, there exists $\psi \in \mathrm{Hom}_K(X, Y)$ such that $\phi = \psi \otimes id_L$. Hence $\mathrm{Hom}_K(X, Y) = \mathrm{Hom}_L(X_L, Y_L)^{\mathrm{Gal}(L/K)}$, where subscript $\mathrm{Gal}(L/K)$ means the $\mathrm{Gal}(L/K)$-fixed morphisms.

Let G be a group and let M, N be two $\mathbb{Z}[G]$-modules. Then $\mathrm{Hom}_{\mathbb{Z}}(M, N)$ has a G-action defined as follows. Let $f \in \mathrm{Hom}_{\mathbb{Z}}(M, N)$, $g \in G$. We define $f^g(m) = g(f(g^{-1}(m)))$, for $m \in M$. Then similarly, we have $\mathrm{Hom}_{\mathbb{Z}}(M, N)^G = \mathrm{Hom}_{\mathbb{Z}[G]}(M, N)$.

Notation. In this section the character group $X^*(T)$ of a torus T over K will be denoted by $\hat{T}$.

From the above, we have a $\mathrm{Gal}(\bar{K}/K)$-action on $\hat{T}$. Let A be the affine ring of T. Let $\phi \in \hat{T}$, then ϕ is determined by the image of X in A, where $\mathbb{G}_{m,\bar{K}} = \bar{K}[X, X^{-1}]$. Suppose $\phi^{\#}(X) = \sum_{\text{finite sum}} k_i \otimes a_i$, where $k_i \in \bar{K}$, $a_i \in A$, then $(\phi^\sigma)^{\#}(X) = \sum_{\text{finite sum}} \sigma(k_i) \otimes a_i \in K' \otimes A$, K' is a finite Galois extension containing all k_i, hence the $\mathrm{Gal}(\bar{K}/K)$-action on $\hat{T}$ is continuous.

Proposition 2.2. *The category of tori over K is anti-equivalent to the category of finitely generated, torsion-free abelian groups with continuous $\Gamma_K = \mathrm{Gal}(\bar{K}/K)$-action.*

Proof. We have defined a functor F between two categories by $T \longrightarrow \hat{T}$. First, we want to show that $\mathrm{Hom}(T_1, T_2) = \mathrm{Hom}(\hat{T}_1, \hat{T}_2)$.

$$\begin{aligned}
\mathrm{Hom}(T_1, T_2) &\simeq \mathrm{Hom}_{\bar{K}}(T_1 \times \bar{K}, T_2 \times \bar{K})^{\Gamma_K} \\
&\simeq \mathrm{Hom}_{\bar{K}}(G^{d_1}_{m,\bar{K}}, G^{d_2}_{m,\bar{K}})^{\Gamma_K} \\
&\simeq \mathrm{Hom}(\widehat{G^{d_2}_{m,\bar{K}}}, \widehat{G^{d_1}_{m,\bar{K}}})^{\Gamma_K} \\
&\simeq \mathrm{Hom}_{\mathbb{Z}}(\hat{T}_2, \hat{T}_1)^{\Gamma_K} \\
&\simeq \mathrm{Hom}_{\mathbb{Z}[\Gamma_K]}(\hat{T}_2, \hat{T}_1)
\end{aligned}$$

For any $\mathbb{Z}$-torsion-free and finitely generated $\mathbb{Z}[\Gamma_K]$-module M, we want to construct a torus such that $\hat{T} = M$. Let $d = rank_{\mathbb{Z}} M$. Consider the group algebra $\bar{K}[M]$, where the group operation on M is written as multiplication. Let $A = \{x \in \bar{K}[M] : \sigma(x) = x, \forall \sigma \in \Gamma_K\}$. Since Γ_K-action is continuous, and M is finitely generated, Γ_K-action factors through $\mathrm{Gal}(L/K)$-action for some finite Galois extension L/K. By descend theory, we have $A \otimes \bar{K} = \bar{K}[M]$. Let $T = \mathrm{Spec}\, A$, then T is a torus over K, and $\hat{T} = \mathrm{Hom}(\bar{K}[X, X^{-1}], \bar{K}[M]) = M$. □

Corollary 2.3. *For every torus T, there exists a minimal (for the inclusion) spliting field L/K. Moreover L/K is a finite Galois extension.*

Proof. Since the Γ_K-action is continuous and $\hat{T}$ is finitely generated, it is enough to take L to be the field fixed by the kernel of the representation $\Gamma_K \to \mathrm{Aut}(\hat{T})$. □

Example 2.4. Let L/K be a finite Galois extension, $G = \mathrm{Gal}(L/K)$. Let $T = \mathrm{Res}_{L/K}(\mathbb{G}_{m,L})$ be the Weil restriction of $\mathbb{G}_{m,L}$ to K, then $\hat{T} = \mathbb{Z}[G]$.

Proof. Let $T = \mathrm{Spec}\, A$ be the torus such that $\hat{T} = \mathbb{Z}[G]$. Then $T' \otimes L = \mathrm{Spec}\, L[G] = \mathrm{Spec}\, L[x_\sigma, x_\sigma^{-1}]_{\sigma\in G}$. For any K-algebra R, the L-homomorphism $f : A \otimes L = L[x_\sigma, x_\sigma^{-1}]_{\sigma\in G} \mapsto R \otimes L$ is determined by the image of x_σ in $R\otimes L$. If $\sigma \circ f = f \circ \sigma$, this means $\sigma f(x_e) = f(x_\sigma)$. Thus the homomorphism $A \to R$ is naturally corresponding to an invertible element $f(x_e)$ in $R \otimes L$, which is also corresponding to a homomorphism from $L[X, X^{-1}] \to R \otimes L$. Hence $T'(X) = \mathrm{Hom}_L(X \otimes L, \mathbb{G}_{m,L})$ for any K-scheme X. Then by definition T' just is $\mathrm{Res}_{L/K}(\mathbb{G}_{m,L})$. □

Definition 2.5. Let T, T' be tori over a field K. A homomorphism $\alpha : T \to T'$ is an *isogeny* if α is a surjection with finite kernel. The map $\hat{\alpha} : \hat{T}' \to \hat{T}$ is then injective with finite cokernel. Note that the *degree* of α is equal to be the cardinality of $\mathrm{Coker}\, \hat{\alpha}$.

We write $T \sim T'$ when T is isogenous to T'.

For any $n \in \mathbb{Z}$, let us denote by $[n]_G$ the multiplication by n map on a group scheme G.

Proposition 2.6. *Let T, T' be tori defined over K, let $\alpha : T \to T'$ be an isogeny. Then there exists an isogeny $\beta : T' \to T$, such that $\beta \circ \alpha = [\deg \alpha]_T$, and $\alpha \circ \beta = [\deg \alpha]_{T'}$.*

Proof. Since $\hat{\alpha} : \hat{T}' \to \hat{T}$ is injective with finite cokernel, then there exists $\hat{\beta} : \hat{T} \to \hat{T}'$, such that $\hat{\beta} \circ \hat{\alpha} = (\deg \alpha).\mathrm{id}_{\hat{T}'}$, $\hat{\alpha} \circ \hat{\beta} = (\deg \alpha).\mathrm{id}_{\hat{T}}$. Let $\beta : T' \to T$ be the isogeny corresponding to $\hat{\beta}$. Then $\beta \circ \alpha = [\deg \alpha]_T$, and $\alpha \circ \beta = [\deg \alpha]_{T'}$. □

Proposition 2.7. *Let T, T' be tori over K and L be a common splitting field of T and T'. Let $G = \mathrm{Gal}(L/K)$. Then $T \sim T'$ if and only if $\hat{T} \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \hat{T}' \otimes_{\mathbb{Z}} \mathbb{Q}$ as G-module.*

Proof. If $T \sim T'$, we have an exact sequence

$$0 \longrightarrow \widehat{T} \longrightarrow \widehat{T}' \longrightarrow M \longrightarrow 0\,,$$

where M is a finite abelian group. After tensor with $\mathbb{Q}$, we get an exact sequence

$$0 \longrightarrow \hat{T} \otimes \mathbb{Q} \longrightarrow \hat{T}' \otimes \mathbb{Q} \longrightarrow 0\,.$$

Conversely, if $\hat{T}\otimes_{\mathbb{Z}}\mathbb{Q} \simeq \hat{T}'\otimes_{\mathbb{Z}}\mathbb{Q}$, then $n\hat{T} \hookrightarrow \hat{T}'$ (as $\mathbb{Z}[G]$-modules) with finite cokernel for some integer n. Let $\widehat{\alpha}$ be the composition of $\widehat{T} \xrightarrow{\cdot n} n\widehat{T} \longrightarrow \widehat{T'}$, then $\hat{\alpha} : \hat{T} \to \hat{T}'$ is injective with finite cokernel. By Proposition **2.2** it corresponds a homomorphism $\alpha : T' \to T$ which is a surjection and with finite kernel. Hence $T \sim T'$. □

Let T be a torus over K, split over L. Let $G = \mathrm{Gal}(L/K)$, $g \in G$ and $K_g := L^g = \{x \in L | g(x) = x\}$. Let χ_T be the character of the representation $\widehat{T}\otimes\mathbb{Q}$ over $\mathbb{Q}$ and $T_g = \mathrm{Res}_{K_g/K}(\mathbb{G}_m)$, then $\widehat{T_g}$ is $\mathbb{Z}[< g >]$ where $< g >$ is the subgroup generated by g in G. The character of corresponding representation is denoted by χ_{T_g}.

By a theorem of Artin [Serre2, thm **9.2**], there exist positive integers n_h,$n_{h'}$ and subsets H, H' of G such that $H \bigcap H' = \emptyset$, and

$$n\chi_T + \sum_{h'\in H'} n_{h'}\chi_{T_{h'}} = \sum_{h'\in H'} n_h\chi_{T_h}.$$

Hence we get:

Proposition 2.8. *There exist positive integers n_h, $n_{h'}$ such that,*

$$T^n \times \prod \mathrm{Res}_{K_{h'}/K}(\mathbb{G}^{n_{h'}}_{m,K_{h'}}) \sim \prod \mathrm{Res}_{K_h/K}(\mathbb{G}^{n_h}_{m,K_h}).$$

3 Dilatation

Let K be a discrete valuation field with valuation ring $\mathcal{O}$.

Definition 3.1. Let X be a $\mathcal{O}$-scheme of finite type, whose generic fibre X_K is smooth over K. Let W be a closed subscheme of X_κ. The *dilatation of W on X* is a pair $(X', u : X' \to X)$, where X' is a flat $\mathcal{O}$-scheme of finite type and $u_\kappa : X'_\kappa \to X_\kappa$ factors through W, satisfying the following universal property:

if Z is a flat $\mathcal{O}$-scheme, and if $v : Z \to X$ is an $\mathcal{O}$-morphism such that its restriction v_κ to the special fibre factors through W, then v factors uniquely through u.

Construction of dilatation

Let $\mathcal{J}$ be the sheaf of ideals defining W in X. Let X' is an open subset of the blow-up $Bl(X, W)$ of X with center W, where $Bl(X, W) = \mathrm{Proj} \bigoplus_{t\geq 0} \mathcal{J}^t$ and $X' = \{x \in Bl(X, W) : (\mathcal{J} \cdot \mathcal{O}_{Bl(X,W)})_x$ is generated by $\pi\}$. Locally, if X is affine and A is the affine ring of X , and the ideal sheaf J of W is generated $g_1, ..., g_n$, then $X' = \operatorname{Spec} A'$ and let $u : X' \to X$ be the canonical map corresponding to $A \to A'$, where

$$A' = A[\frac{g_1}{\pi}, ..., \frac{g_n}{\pi}]/(\pi - \text{torsion})$$

and

$$A[\frac{g_1}{\pi}, ..., \frac{g_n}{\pi}] = A[X_1, ..., X_n]/(\pi X_1 - g_1, ..., \pi X_n - g_n).$$

Proposition 3.2. *Let (X', u) be constructed as above, then (X', u) is the dilatation of W on X.*

Proof. We just need to show that (X', u) satisfies the universal property of dilatation. Since the problem is local, we can assume $Z = \operatorname{Spec} B$ is affine. Keep the notation as before. The fact that v_κ factors through Y_κ implies that the ideal $J \cdot B$ is contained in πB. Hence there exist elements $h_i \in B$ with $v^*(g_i) = h_i$; the elements h_i are unique, for B has no π-torsion. Thus the A- morphism $A[X_1, ..., X_n] \to X$ sending T_i to h_i yields a morphism $w^* : A' \to B$ and hence a morphism $w : Z \to X'$ such that $v = u \circ w$. □

Corollary 3.3. *Let X be a closed subscheme of an $\mathcal{O}$-scheme Z, and let Y_κ be a closed subscheme of X_κ. Then the dilatation X' of Y_κ on X is a closed subcheme of the dilatation Z' of Y_κ in Z.*

Proof. This is clear from the construction of dilatation. □

Proposition 3.4. *Let X be a smooth scheme over $\mathcal{O}$, and W be a closed subscheme over $X \otimes \kappa$. Let X' be the dilatation of W on X. Then $X' \otimes \mathcal{O}/\pi^N$ depends only on $X \otimes \mathcal{O}/\pi^{N+1}\mathcal{O}$ in a canonical way.*

Remark. *Canonicity.* Assume X_1 and X_2 are $\mathcal{O}$-schemes, and ϕ is an isomorphism $X_1 \otimes \mathcal{O}/\pi^{N+1}\mathcal{O} \to X_2 \otimes \mathcal{O}/\pi^{N+1}\mathcal{O}$. Assume also that $W_1 \subseteq X_1 \otimes \kappa, W_2 \subseteq X_2 \otimes \kappa$ are closed smooth subschemes over κ, and ϕ induces an isomorphism from W_1 to W_2. Form the dilatation X_i' and $Y_i = Bl'(X_i, \mathcal{J}_i) = \operatorname{Proj} \bigoplus_{t \geq 0} \mathcal{J}_i^t$, $i = 1, 2$. The canonicity statement is that the natural isomorphism $Bl'(\phi) : Y_1 \otimes \mathcal{O}/\pi^N \to Y_2 \otimes \mathcal{O}/\pi^N$ induces an isomorphism from the subschemes $X_1' \otimes \mathcal{O}/\pi^N$ of $Y_1 \otimes \mathcal{O}/\pi^N$ to $X_2' \otimes \mathcal{O}/\pi^N$.

Proof of Proposition 3.4. Let $i = 1, 2$. Let x_i' be a point on $X_i' \otimes \kappa$ which projects to $x_i \in X_i \otimes \kappa$. Since X_i and W_i are smooth , we can choose a system of local coordinates $f_1^{(i)}, ..., f_r^{(i)}, g_{r+1}^i, ..., g_n^{(i)}$ at x_i on X_i such that W_i defined by $(\pi, g_{r+1}^i, ..., g_n^{(i)})$ near an affine neighborhood U_i of x_i and X_i' above U_i is $\operatorname{Spec}(B_i'/\pi^\infty{-}torsion)$, where $B_i' = \mathcal{O}_{X_i}(U_i)[Y_{r+1}^i, ..., Y_n^i]/(\pi Y_{r+1}^i -$

$g^i_{r+1}, ..., \pi Y^i_n - g^i_n)$. The $f^{(i)}_1, ..., f^{(i)}_r, Y^i_{r+1}, ..., Y^{(i)}_n$ form a system of local coordinates at x'_i in X_i. We can shrink U_i such that B'_i is free of π^∞-torsion.

If $\phi(x_1) = x_2$, we can assume $\phi^*(f^{(2)}_j \mod \pi^N) \equiv f^{(1)}_j \mod \pi^N$ and $\phi^*(g^{(2)}_k \mod \pi^N) \equiv g^{(1)}_k \mod \pi^N$, and ϕ induces an isomorphism $\widetilde{\phi}^* : \mathcal{O}_{X_2}(U_2) \otimes \mathcal{O}/\pi^N \to \mathcal{O}_{X_1}(U_1) \otimes \mathcal{O}/\pi^N$. Clearly, there is an isomorphism $(\phi')^* : B'_2 \otimes \mathcal{O}/\pi^N \to B'_1 \otimes \mathcal{O}/\pi^N$ which extends $(\widetilde{\phi}^*)$ and sends $Y^{(2)}_j$ to $Y^{(1)}_j$. It remains to show that $\phi' : X'_1 \otimes \mathcal{O}/\pi^N \to X'_2 \otimes \mathcal{O}/\pi^N$ is induced by $\mathrm{Bl}'(\phi)$. Above $U_i \otimes \mathcal{O}/\pi^N$, the affine ring of $Bl'(X_i, \mathcal{J}) \otimes \mathcal{O}/\pi^N$ is $B''_i = (\bigoplus_{t \geq 0} Sym^t_{B^N_i} \mathcal{J}^N_i)^{\deg 0}_{\pi_1}$, where $B^N_i = \mathcal{O}_{X_i} \otimes \mathcal{O}/\pi^N$, $\mathcal{J}^N_i = (\pi, g^{(i)}_{r+1}, ..., g^{(i)}_n) \otimes \mathcal{O}/\pi^N$, π is regards as a homogeneous element of degree 1. The element π_1 is an element of degree 1 in $\bigoplus_{t \geq 0} Sym^t_{B^N_i} \mathcal{J}^N_i$, and the subscrip indicates localization. The ring B''_i maps to $B'_i \otimes \mathcal{O}/\pi^N$ by sending $\pi_1^{-1} g^{(i)}_k$ to Y_k. Then it is clear that $(\phi')^*$ is induced by $Bl'(\phi)$. $\square$

4 Néron's measure for the defect of smoothness

Let X be a scheme of finite type over $\mathcal{O}$ such that $X \otimes K$ is smooth over K. Consider $x \in X(\mathcal{O}^{sh})$ as a morphism $\mathrm{Spec}\, \mathcal{O}^{sh} \to X$.

Definition 4.1. Define $\delta(x)$ = the length of the torsion part of $x^*\Omega^1_{X/\mathcal{O}}$ as *Néron's measure for the defeat of smooth at* x, sometimes we also denote it by $\delta(x, X)$.

The rank of free part is just the rank of $\Omega^1_{X/K}$ at x_K, which is the dimension of X_K at x_K, since X_K is smooth.

Lemma 4.2. *Let x be an $\mathcal{O}^{sh}$-value point of X. Then x factors through the smooth locus of X if and only if $\delta(x) = 0$.*

Proof. If x is contained in the smooth locus X_{smooth} of X, then $x^*\Omega^1_{X/\mathcal{O}} = x^*\Omega^1_{X_{smooth}/\mathcal{O}}$, where $\Omega^1_{X_{smooth}/\mathcal{O}}$ is locally free, so $\delta(x) = 0$. Conversely, if $\delta(x) = 0$, then $x^*\Omega^1_{X/\mathcal{O}}$ can be generated by d elements where d is the dimension of X_K at x_K. In particular, $x^*\Omega^1_{X_\kappa/\kappa}$ can be generated by $d-$elements at x_κ. Since the relative dimension at x_κ is at least d. So X_κ is smooth over κ at x_κ of relative dimension d. Then X is smooth over $\mathcal{O}$ at x. $\square$

Let U be a neighborhood of x in X which can be realized as a closed subscheme of an $\mathcal{O}$-scheme Z where Z is smooth over $\mathcal{O}$, and has constant relative dimension n. Assume that there exist functions $z_1, ..., z_n$ on Z such that $dz_1, ..., dz_n$ generate $\Omega^1_{Z/\mathcal{O}}$, and let $g_1, ..., g_m$ be functions which generate the sheaf of ideal of $\mathcal{O}_Z$ defining U in Z. Then we have $dg_u = \sum \frac{\partial g_u}{\partial z_v} dz_v$, and define Jacobian matrix J of $g_1, ..., g_m$ to be $(\frac{\partial g_u}{\partial z_v})_{m\times n}$. Let d be the relative dimension of X_K at x_K, and $v(a) = \pi$-order of a in $\mathcal{O}$.

Lemma 4.3. *$\delta(x) = min\{v(\Delta)|\Delta : (n-d)$-minors of $J\}$.*

Proof. By Jacobi criterion, there exist a $(n-d)$-minors Δ with $x^*\Delta \neq 0$, and any minor Δ of J with more than $n-d$ rows will satisfying $x^*\Delta = 0$. We know $x^*\Omega^1_{X/\mathcal{O}}$ is representable as a quotient F/M, where $F := x^*\Omega^1_{Z/\mathcal{O}}$ is a free $\mathcal{O}^{sh}$-module of rank n, and M is the submodule generated by $x^*dg_1, ..., x^*dg_m$. Since the rank of M is $n-d$ and $\mathcal{O}^{sh}$ is P.I.D, one can find a base $e_1, ..., e_n$ of $x^*\Omega^1_Z$ such that M is generated by $a_{d+1}e_{d+1}, ..., a_ne_n$, where $a_i \in \mathcal{O}$ and $a_i \neq 0$. Thus by the theory of elementary divisors, we have $\delta(x) = v(a_{d+1}) + ... + v(a_n)$.

Now consider the ideals in $\mathcal{O}^{sh}$ generated by all elements $x^*\Delta$, where Δ is $(n-d)$-minor, and this ideal is generated by $a_{d+1}...a_n$, and there is a minor Δ with $x^*(\Delta) = a_{d+1}...a_n$. □

Proposition 4.4. *Let Y be the Zariski closure of $\{x \mod \pi \in X(\kappa) : x \in X(\mathcal{O}^{sh})\}$ as a closed subscheme of $X \otimes \kappa$. Let $X' \to X$ be the dilatation of Y on X. For each $x \in X(\mathcal{O}^{sh})$ with $x_\kappa \in Y$, denote $x' \in X'(\mathcal{O}^{sh})$ be the unique lifting of x. Then $\delta(x') \leq max\{0, \delta(x) - 1\}$.*

Proof. The proof takes too many pages, see the details in [BLR, 3.3 Prop 5]. □

Lemma 4.5. 1). *Suppose X is a group scheme over $\mathcal{O}$, and $e \in X(\mathcal{O}^{sh})$ is the identity element. Then $\delta(e) = \delta(x)$, for any $x \in X(\mathcal{O}^{sh})$.*

2). Change of base field. *Let $x \in X(\mathcal{O}^{sh})$, consider x as a point of $X\otimes\mathcal{O}_L$, then $\delta(x; X\otimes\mathcal{O}_L) = e(L/K)\cdot\delta(x, X)$, where e(L/K) is the ramification index of L/K.*

3). Closed immersion. *Let $i : X \subseteq X'$ be a closed immersion of $\mathcal{O}$-scheme such that i induce an isomorphism $X \otimes K \to X' \otimes K$. Then we have a surjection $i^*\Omega^1_{X'/\mathcal{O}} \to \Omega^1_{X/\mathcal{O}}$. Therefor, for any $x \in X(\mathcal{O}^{sh})$, we have $\delta(x; X) \leq \delta(i \circ x; X')$.*

Proof. Let $r_x : X\otimes\mathcal{O}^{sh} \to X\otimes\mathcal{O}^{sh}$ be the isomorphism of right multiplication by x. Then $x = r_x \circ e$, hence $e^*\Omega^1_{X/\mathcal{O}} = x^*\Omega^1_{X/\mathcal{O}}$, so $\delta(e) = \delta(x)$. The other two are clear. □

5 The construction of the Néron model of a torus

Let K be a discrete valuation field.

Definition 5.1. Let T be a torus over K, the (finite type) *Néron model* of T is a smooth group scheme T^{NR} over $\operatorname{Spec}\mathcal{O}_K$ with generic fibre isomorphic to T, such that the image of $T^{NR}(\mathcal{O}^{sh})$ is in $T(K^{sh})$ is the maximal bounded subgroup of $T(K^{sh})$.

Remark. The usual definition of Néron model for a smooth and separated K-scheme X of finite type is the following: it is a smooth, separated $\mathcal{O}$-scheme $\mathcal{X}$, locally of finite type, satisfying the following universal property:

For each smooth $\operatorname{Spec}\mathcal{O}$-scheme Y and each K-morphism $u_K : Y_K \to X$, there is a unique $\operatorname{Spec}\mathcal{O}$-morphism $u : Y \to \mathcal{X}$ extending u_K. For more details, see [BLR].

For a torus T over K, the (finite type) Néron model T^{NR} is an open subscheme of $\mathcal{T}$. Its special fiber consists in the union of the connected components of $\mathcal{T}_\kappa$ which are of finite order in the group of components $\Phi(T)$. When T is anisotropic (i.e. T does not contain any factor $\mathbb{G}_{m,K}$), then $T^{NR} = \mathcal{T}$. In general, both models have the same neutral component.

Follow the construction of the Néron model of T as explained in [BLR].

- Step 1, construct a group scheme T^0 over $\mathcal{O}$ such that $T^0(\mathcal{O}^{sh}) = T^{NR}(\mathcal{O}^{sh)}$= the maximal bounded subgroup of $T(K^{sh})$.

  Let $R = \operatorname{Res}_{L/K}(T \otimes L)$, then there exits a canonical closed embedding $T \to R$, and choose T^0 to be the schematic closure of T in $R^{NR} \simeq X_*(T) \otimes (\operatorname{Res}_{\mathcal{O}_L/\mathcal{O}_K}(\mathbb{G}_{m,\mathcal{O}_L}))$, where $X_*(T)$ is the cocharacter group of T.

  Proposition 5.2. *$T^0_K = T$ and $T^0(\mathcal{O}^{sh}) = T^{NR}(\mathcal{O}^{sh})$.*

  Proof. Since all schemes are affine, the first equality is easy from algebraic facts. Let A, B, C, D be the affine rings of R^{NR}, R, T, T^0 respectively, and assume $f : A \to B, g : B \to C, h : A \to C$ are the corresponding morphisms and $h = g \circ f$. Then $D = A/\mathrm{Ker} h$ and h induce a mapping $h' : D \to C$. Now we want to show $D \otimes K \to C$ is isomorphic. It is surjective since $A \otimes K = B$ and g is surjective. The injectivity follows from K is flat $\mathcal{O}$-module. Thus $h' \otimes id : D \otimes K \to C \otimes K$ is injective and $C \otimes K = C$.

Let $u \in T^0(\mathcal{O}^{sh})$, then it is in the maximal bounded subgroup of $R(K^{sh})$ since it is in $R^{NR}(\mathcal{O}^{sh})$. So we have $T^0(\mathcal{O}^{sh}) \subseteq T^{NR}(\mathcal{O}^{sh})$. Conversely, let $t \in T^{NR}(\mathcal{O}^{sh})$, then it lifts t' in $R^{NR}(\mathcal{O}^{sh})$, we want to show it factor through T^0. And this is clear from the universal property of quotient of rings. □

- Step 2, apply the smoothening process to T^0, then we can get the Néron model T^{NR} of T.
  Let Z^i be the Zariski closure of $\{x \mod \pi \in T^i(\kappa^{sep}) : x \in T^i(\mathcal{O}^{sh})\}$ as a closed subscheme of $T^i \otimes \kappa$with the reduced induced structure. Let T^{i+1} is the dilatation of Z^i on T^i.
  Let $\delta = \max\{\delta(x) : x \in T^0(\mathcal{O}^{sh})\}$, where $\delta(x)$ is the N'eron measure for the defect of smoothness. Then $T^{NR} = T^i$ for $i \geq \delta$.

  Similarly, do the same process to $R^0 = R^{NR}$. For $i \geq 0$, let W^i be the Zariski closure of

  $$\{x \mod \pi \in R^i(\kappa^{sep}) : x \in T^0(\mathcal{O}^{sh}) \subset R^i(\mathcal{O}^{sh})\},$$

  as a subscheme of $R^i \otimes \kappa$ with the reduced induced structure. Then R^{i+1} is the dilatation of W^i on R^i. Clearly, we have $T^0(\mathcal{O}^{sh}) \subset R^i(\mathcal{O}^{sh}) \subset (R^{i+1}(\mathcal{O}^{sh}))$.

Lemma 5.3. *For $i \geq 0, N \geq 1$, $R^{i+1} \otimes \mathcal{O}/\pi^N$ depends only on $R^i \otimes \mathcal{O}/\pi^{N+1}\mathcal{O}$ in a canonical way.*

Proof. This is just a corollary of Proposition **3.4**. □

Lemma 5.4. *The schematic closure of T in R^i is T^i for all $i \geq 1$. In particular, it is T^{NR} for $i \gg 0$.*

Proof. Prove it by induction on i. T^{i-1} is a closed subgroup of R^{i-1}, and W^{i-1} is the image of Z^{i-1} in $T^{i-1} \to R^{i-1}$. Then R^i is a closed subscheme of subgroup of R^i by Corollary **3.3**. So the schemematic closure of T^i's generic fibre T in R^i is itself. □

Remark. When $i \geq \delta(e; T^0)$, T^i is smooth, hence $T^{NR} = T^i$. So we want to control $\delta(e; T^0)$. Let $T^0_L = T^0 \otimes \mathcal{O}_L$, the schematic closure of $T \otimes L$ in $R^{NR} \otimes \mathcal{O}_L$. Let R'=$R^{NR} \otimes \mathcal{O}_L$, $R^\dagger = X_*(R \otimes_K L) \otimes_{\mathbb{Z}} (\mathbb{G}_{m/\mathcal{O}_L})$, $T^\dagger = X_*(T \otimes_K L) \otimes_{\mathbb{Z}} (\mathbb{G}_{m/\mathcal{O}_L})$. There are canonical morphisms $T^\dagger \to R^\dagger$, and $\varphi : R' \to R^\dagger$. Let $T' = T^\dagger \times_{R^\dagger} R'$. Since $T^\dagger \to R^\dagger$ is a closed immersion, hence $T' \to R'$ is also a closed immersion by base change. Since T' has generic fibre $T \otimes L$, T^0_L is equal to the subscheme closure of $T \otimes L$ in T'. By the lemma **4.5**, we have $\delta(e, T^0) \leq \frac{\delta(e,T')}{e(L/K)}$. So it is enough to control $\delta(e, T')$.

We can write $T^\dagger$ and $R^\dagger$ explicitly. In fact, $T^\dagger \simeq \mathbb{G}^d_{m,\mathcal{O}_L}$ and $R^\dagger \simeq \mathbb{G}^{nd}_{m,\mathcal{O}_L}$, and $T^\dagger$ is cut out by $nd-d$ equations $f_1, ..., f_{nd-d}$ on $R^\dagger$, where $d = dimT, n = [L : K]$. By base change, T' is cut out by the equations $\varphi^* f_1, ..., \varphi^* f_{nd-d}$ on R'. Let $z_1, ..., z_{nd}$ be a system of local coordinates near e, and put $M = (\frac{\partial(\varphi^* f_i)}{\partial z_j})$, then by Lemma **4.3**, $\delta(e, T^0)$ is the minimum of the valuation of $e^*\Delta$, for all (nd-d)-minors Δ of M.

Lemma 5.5. *Suppose that* $(\mathcal{O}, \mathcal{O}_L, \Gamma, \Lambda) \equiv (\mathcal{O}_0, \mathcal{O}_{L_0}, \Gamma_0, \Lambda_0)(levelN)$ *with* $Ne(L/K) > \delta(e, T')$. *Form* T'_0 *in the same way that we form* T'. *Then* $\delta(e; T'_0) = \delta(e; T')$.

Proof. All following objects are determined only by $(\mathcal{O}/\pi^N, \mathcal{O}_L/\pi^N, \Gamma, \Lambda)$:

$R^\dagger \otimes \mathcal{O}_L/\pi^N\mathcal{O}_L$, $T^\dagger \otimes \mathcal{O}_L/\pi^N\mathcal{O}_L$, $R' \otimes \mathcal{O}_L/\pi^N\mathcal{O}_L$, $T' \otimes \mathcal{O}_L/\pi^N\mathcal{O}_L$, and the matrix $e^*(M \mod \pi^N)$. And if $Ne(L/K) > \delta(e, T')$, and by Lemma**4.3**, $\delta(e; T')$ is also determined by $(\mathcal{O}/\pi^N, \mathcal{O}_L/\pi^N, \Gamma, \Lambda)$. So the lemma is true. □

6 Singularities of commutative group schemes

Definition 6.1. Suppose A is a noetherian local ring. We say that A is a *complete intersection ring* if $\widehat{A}$ is isomorphic to a quotient of a complete local regular ring B by a regular ideal J. We say that a locally noetherian scheme X is *complete intersection at a point* $x \in X$, if $\mathcal{O}_{X,x}$ is a complete intersection ring.

Definition 6.2. Suppose $f : X \to S$ is a flat, locally of finite presentation morphism. We say that X is *relative complete intersection (r.c.i)* over S at the point x if the fibre $f^{-1}(f(x))$ is complete intersection at x. We say that f is an *r.c.i morphism* if X is r.c.i over S at all its points.

Proposition 6.3. *Suppose* B *is a noetherian regular local ring,* J *is an ideal of* B*. Then* $A = B/J$ *is a complete intersection ring if and only if* J *is a regular ideal of* B*.*

Proof. If J is a regular ideal, then $J\widehat{B}$ is also a regular ideal in $\widehat{B}$, hence A is a complete intersection ring.

Conversely, suppose that A is a complete intersection ring, we need to show J is a regular ideal. We can assume A and B are both complete since $\widehat{A} = \widehat{B}/J\widehat{B}$.

Choose a presentation $A = B'/J'$, where B' is a noetherian, complete, regular local ring and J' is its regular ideal. Denote $\pi_1 : B \to A, \pi_2 : B' \to A$ be the canonical projections. Consider $B'' = B \times_A B'$, where $B'' = \{(b, b') \in$

$B \times B'|\pi_1(b) = \pi_2(b')\}$, a subring of $B \times B'$. We claim that B'' is complete local noetherian ring. It is easy to seen that B'' is a local ring with unique maximal deal $m = \{(b, b') : \pi_1(b) = \pi_2(b') \in m_A\}$. And $(b, b') \in m$ if and only if $b \in m_B$ and $b' \in m_{B'}$, so B'' is complete. Let $\mathfrak{a}$ be an ideal of B'', and let $\mathfrak{b}$ be the kernel of $B'' \to B$. Then we have

$$0 \longrightarrow \mathfrak{a} \cap \mathfrak{b} \longrightarrow \mathfrak{a} \longrightarrow \mathfrak{a}/\mathfrak{a} \cap \mathfrak{b} \longrightarrow 0$$

and $\mathfrak{a}/\mathfrak{a} \cap \mathfrak{b} \simeq (\mathfrak{a} + \mathfrak{b})/\mathfrak{b}$. Since $(\mathfrak{a} + \mathfrak{b})/\mathfrak{b}$ is corresponding to an ideal of B, and $\mathfrak{a} \cap \mathfrak{b}$ is corresponding to an ideal of B'; they are both of finite type. Hence $\mathfrak{a}$ is also finitely generated.

By Cohen's theorem, there exits a noetherian, complete, regular local ring C such that B'' is a quotient of C with regular ideal. Let $I = \mathrm{Ker}(C \to A)$, then I is the preiamge of the regular ideal J', hence I is regular. And J is image of I in a regular ring, hence regular. □

Proposition 6.4. *Let $k \subset k'$ be a filed extension. Suppose X is a locally of finite type k-scheme and $X' = X \times_k k'$. Suppose $x' \in X'$ and x is its projection on X. Then X is complete intersection at x if and only if X' is complete intersection at x'.*

Proof. The problem is local, so we can assume $X = \mathrm{Spec}\, A$, where A is a quotient of polynomial ring $k[X_1, ..., X_n]$ with ideal I. "only if" part is trivial. Assume $\{f_1, ..., f_n\}$ be a minimal generators of I at x, then they also generate $I' = I \otimes k'$ at x'. If they are not regular sequence in $I'_{x'}$, then some f_i is generated by others in $I'_{x'}$. Hence f_i is also generated by others in I_x by the faithfully flatness of k' over k. This is contradiction with the choice of $f_i's$. □

Proposition 6.5. (1). *Suppose $f : X \to S$ is an r.c.i morphism. Let $f' = f_{S'} : X \times S' \to S'$ be the base change compatible with $g : S' \to S$. Then f' is also a r.c.i morphism. If g is fpqc (ie. faithfully flat, quasi compact), then vice versa.*

(2). *If $f : X \to Y, g : Y \to Z$ are both r.c.i morphism. Then so is $g \circ f : X \to Z$.*

Proof. Clearly from Proposition **6.4**. □

Lemma 6.6. *Let G be a commutative group scheme, flat and of finite type over a notherain base scheme S. Then $G \to S$ is an r.c.i morphism.*

Proof. We can assume $S = \mathrm{Spec}\, k$, where k is algebraically closed. Suppose that $0 \longrightarrow G' \longrightarrow G \longrightarrow G'' \longrightarrow 0$ is an exact sequence of commutative group scheme over k. Assume that G' and G'' are r.c.i over $\operatorname{Spec} k$, we claim that $G \to G''$ is also an r.c.i morphism, hence $G \to G'' \to \operatorname{Spec} k$ is an r.c.i morphism. By proposition **6.5**, it is enough to check after a fpqc base change $G \to G''$, that is, look at $G \times_{G''} G \to G$. This morphism is canonically isomorphic to $G \times_{\operatorname{Spec} k} G' \to G$, which is projection to the first factor, and it is an r.c.i morphism since $G' \to \operatorname{Spec} k$ is.

For any G over k, G admit a composition series in which the factor are smooth, isomorphic to μ_p, or α_p. And these factors are clearly r.c.i over k, hence by induction, $G \to S$ is an r.c.i morphism. □

Lemma 6.7. *Suppose that X is a noetherian scheme and $X \to \operatorname{Spec} \mathcal{O}$ is a flat r.c.i morphism. Then for any $N \geq 1$, the collection of points:*

$$\bigcup \{x \mod \pi^N \in X(C/\pi^N C) : x \in X(C)\},$$

as C ranges over local $\widehat{\mathcal{O}}$-algebra which are flat, and r.c.i over $\widehat{\mathcal{O}}$, is schematically dense in $X \otimes \mathcal{O}/\pi^N$.

Proof. Since $\mathcal{O} \to \widehat{\mathcal{O}}$ is faithfully flat and $\operatorname{Spec} \widehat{\mathcal{O}} \to \operatorname{Spec} \mathcal{O}$ is surjective, we can assume $X = \operatorname{Spec} A$, and A is a complete noetherian local ring such that $\pi \in m_A$.

Choose a presentation $A = B/I$, where $B = [[X_1, ...X_b]]$. Since X is r.c.i over $\mathcal{O}$, then I is generated by a regular sequence $(t_1, ..., t_a)$. Hence, $(t_1, ..., t_a) \otimes \kappa$ is a regular sequence on $B \otimes \kappa$. Extend $(t_1, ..., t_a) \otimes \kappa$ to a system of regular parameters, and lift the sequence to a sequence $(t_1, ..., t_b)$ in B. Put $J_n = (t_1^n, ..., t_b^n)$. Then $\bigcap_n J_n \subset \bigcap_n m^n = 0$. Let $C_n = B/(I + J_n)$ and $\operatorname{Spec} C_n \to X$ is induced by $B/I \to B/(I + J_n)$. Then $\{\operatorname{Spec} C_n \to X; n \geq 1\}$ is schematically closed in X. $I + J_n = (t_1, ..., t_a, t_{a+1}^n, ..., t_b^n)$ and $(t_1, ..., t_a, t_{a+1}^n, ..., t_b^n)$ is also a regular system in B, hence C_n is r.c.i of relative dimension 0, and then finite over $\widehat{\mathcal{O}}$. Clearly, π^k is not in $I + J_n$ for any integers k, so C_n is also flat.

From above, the points $\{\operatorname{Spec} C_n \otimes \mathcal{O}/\pi^N \to X\mathcal{O}/\pi^N : n \geq 1\}$ is schematically dense in $X \otimes \mathcal{O}/\pi^N$ □

Proposition 6.8. *Let G be a commutative noetherian group scheme over $\mathcal{O}$, not necessary flat. Let $\overline{G}$ be the schematic closure of $G \otimes K$ in G. Then $\overline{G} \otimes \mathcal{O}/\pi^N$ is the schematic closure in $G \otimes \mathcal{O}/\pi^N$ of the following collection of points*

$$\bigcup \{x \mod \pi^N \in G(C/\pi^N C) : x \in G(C)\}$$

as C ranges over local $\widehat{\mathcal{O}}$-algebras which are flat, finite, and r.c.i over $\widehat{\mathcal{O}}$.

Proof. $\overline{G}(C) = G(C)$ for any flat $\mathcal{O}$-algebra. Then it is clear from the two lemmas before. $\square$

Lemma 6.9. *The collection of $\mathcal{O}/\pi^N$-algebras $\{C/\pi^N C$: C is a local, flat,finite,r.c.i $\widehat{\mathcal{O}}$-algebras$\}$ is just the collection of all local $\mathcal{O}/\pi^N$-algebras which are flat, finite, and r.c.i over $\mathcal{O}/\pi^N$.*

Proof. Since the property of being r.c.i is stable under any base change. So we only need to show that any local flat, finite, r.c.i $\mathcal{O}$-algebra is of the form C/π^N for some C.

Choose a presentation $A = B/I$, $B = \mathcal{O}[X_1, ..., X_n]_m, m = (\pi, X_1, ...X_n)$, $\pi^N \in I$. Since B is regular and A is r.c.i, then I is generated by a regular sequence $(\pi^N, f_1, ..., f_m)$. Since A is of dimension 0, we have $m = n$.

Lift f_i to $\widetilde{f}_i \in \widehat{\mathcal{O}}[X_1, ..., X_n]_{\widetilde{m}}$, where $\widetilde{m} = (\pi, X_1, ..., X_n)$. Then $C = \widehat{\mathcal{O}}[X_1, ..., X_n]_{\widetilde{m}}/(\widetilde{f}_1, ..., \widetilde{f}_n)$ is flat, finite, and r.c.i $\widetilde{\mathcal{O}}$-algebra and $A = C/\pi^N$. $\square$

7 Elkik's theory

In this section, let R be a noetherian $\mathcal{O}$-algebra, complete with respect to the π-adic topology. Consider $R[X] = R[X_1, ..., X_N]$, the polynomial ring in N variables. Let I be an ideal of $R[X]$ and put $B = R[X]/I, Y = \operatorname{Spec} B$. We assume that $Y \otimes_{\mathcal{O}} K \to \operatorname{Spec}(R \otimes_{\mathcal{O}} K)$ is smooth of relative dimension s. The Jacobian ideal of I is defined to be the ideal of $R[X]$ generated by the $(N - s)$-minors of $(\frac{\partial f_i}{\partial X_j})_{s\times N}$ for all $f_1, ..., f_s$ in a generating set of I. By smoothness assumption and Jaccobi Criterion, $J + I \supseteq \pi^h R[X]$ for some $h \geq 0$. Fix such an h in the following.

Lemma 7.1 (Elkik)**.** *Suppose that I can be generated by $N - s$ elements. Then for any $n > 2h$, the image of $Y(R) \to Y(R/\pi^{n-h}R)$ is the same as the image of $Y(R/\pi^n R) \to Y(R/\pi^{n-h}R)$.*

Proof. We restatement the lemma as following: If $\mathbf{a} = (a_1, ...a_N) \in R^N$ such that $I(\mathbf{a}) = 0 \mod \pi^n$, where $I(\mathbf{a}) = \{f(\mathbf{a}) : \forall f \in I\}$, then there exists $\mathbf{a}' \in R^N$ such that $\mathbf{a} \equiv \mathbf{a}' \mod \pi^{n-h}$ and $I(\mathbf{a}) = 0$.

Since R is complete and by approximation, it is enough to find $\mathbf{y} = (y_1, ..., y_N) \in R^N$ such that $y_i \equiv 0 \mod \pi^{n-h}, \forall i$ and $I(\mathbf{a} - \mathbf{y}) \subset (\pi^{2n-2h})$.

Let $\mathbf{M}$ be the Jacobian matrix of I, and by Taylor's expansion,

$$\begin{pmatrix} f_1(\mathbf{a}-\mathbf{y}) \\ ... \\ f_{N-s}(\mathbf{a}-\mathbf{y}) \end{pmatrix} = \begin{pmatrix} f_1(\mathbf{a}) \\ ... \\ f_{N-s}(\mathbf{a}) \end{pmatrix} - \mathbf{M}(\mathbf{a}) \begin{pmatrix} y_1 \\ ... \\ y_N \end{pmatrix} + \sum y_i y_j Q_{ij}(\mathbf{a}-\mathbf{y}),$$

Where Q_{ij} is an $(N-s)$-column vector whose components are the polynomial in $\mathbf{a}$ and $\mathbf{y}$. Hence we just need to find a $\mathbf{y} = (y_1, ..., y_n)$, such that $y_i \equiv 0 \mod \pi^{n-h}$ and

$$\begin{pmatrix} f_1(\mathbf{a}-\mathbf{y}) \\ ... \\ f_{N-s}(\mathbf{a}-\mathbf{y}) \end{pmatrix} = \mathbf{M}(\mathbf{a}) \begin{pmatrix} y_1 \\ ... \\ y_N \end{pmatrix} \mod \pi^{2n-2h}$$

Let δ be a nonzero $(N-s)$-minor of M, then exits $N \times (N-s)$ matrix M_δ such that $\mathbf{M}M_\delta = \delta Id$, where Id means the identity matrix. By assumption, we have $\sum_\delta \delta P_\delta + Q = \pi^h$ in $R[X]$ for some $Q \in I$.

$$\pi^h \begin{pmatrix} f_1(\mathbf{a}) \\ ... \\ f_{N-s}(\mathbf{a}) \end{pmatrix} = (\Sigma\delta P_\delta + Q)(\mathbf{a}) \begin{pmatrix} f_1(\mathbf{a}) \\ ... \\ f_{N-s}(\mathbf{a}) \end{pmatrix}$$

$$= \sum \delta P_\delta(\mathbf{a}) \begin{pmatrix} f_1(\mathbf{a}) \\ ... \\ f_{N-s}(\mathbf{a}) \end{pmatrix} \mod \pi^{2n}$$

$$= \sum P_\delta \mathbf{M}(\mathbf{a}) M_\delta(\mathbf{a}) \begin{pmatrix} f_1(\mathbf{a}) \\ ... \\ f_{N-s}(\mathbf{a}) \end{pmatrix} \mod \pi^{2n}$$

$$= \mathbf{M}(\mathbf{a})[\sum P_\delta M_\delta(\mathbf{a}) \begin{pmatrix} f_1(\mathbf{a}) \\ ... \\ f_{N-s}(\mathbf{a}) \end{pmatrix}] \mod \pi^{2n}$$

Let $\mathbf{y} = (\sum P_\delta M_\delta(\mathbf{a}) \begin{pmatrix} f_1(\mathbf{a}) \\ ... \\ f_{N-s}(\mathbf{a}) \end{pmatrix})/\pi^h$, then $\mathbf{y}$ is what we need. □

Lemma 7.2. *Suppose that R is a local ring, and $Y \to \operatorname{Spec} R$ is a flat r.c.i morphism. Then for any $n \geq 2h$, the image of $Y(R) \to Y(R/\pi^{n-h}R)$ is the same as the image of $Y(R/\pi^n R) \to Y(R/\pi^{n-h}R)$.*

Proof. Let $y : \operatorname{Spec} R/\pi^n \to Y$ be a closed point of $Y(R/\pi^n)$. Let $\mathbf{m}$ be the unique maximal ideal in R/π^n, $q = y(m)$.

Since $Y \to \operatorname{Spec} R$ is r.c.i, and $\operatorname{Spec} R[X] \to \operatorname{Spec} R$ has regular fibre. Then there exists $f \in R[x]$, such that $q \in Y_f$ and Y_f is cut out by $(N-s)$ equations in $\operatorname{Spec} R[X]_f$, and regard $\operatorname{Spec} R[X]_f$ as a closed subscheme of $\operatorname{Spec} R[X][Z]$ cut out by $Zf-1$. Then Y_f is cut out by $(N+1-s)$ equations in $\mathbb{A}^{N+1}$. By Elkik's lemma, there exists $y' \in Y_f(R) \subset Y(R)$ such that $y \equiv y' \mod \pi^{n-h}$.

□

8 Congruences of Néron models

In this section, assume K is complete for simplicity. Notations are the same as Section **5**.

Since R^{NR} is the Néron model of an induced torus, we can realize R^{NR} as a closed subscheme of $\mathbb{A}^{d(n+1)}_{/\mathcal{O}}$, defined by n explicit equations. Recall that the closed subscheme T' of R' is cut out by $(\dim R^{NR} - \dim T)$ equations, and $R' = R^{NR} \otimes \mathcal{O}_L$. Hence, T' can be realizes as a closed subscheme of $\mathbb{A}^{d(n+1)}_{/\mathcal{O}_L}$ defined by an ideal I' generated by $(d(n+1) - \dim T)$ equations. Let $\mathcal{J}'$ be the Jacobian ideal for I'. Since the generic fibre of T' is smooth, $I' + \mathcal{J}'$ contains π^h for some $h > 0$. Let $h = h\ (\mathcal{O}, \mathcal{O}_L, \Gamma, \Lambda)$ be the smallest integer with this property.

Lemma 8.1. *Suppose* $(\mathcal{O}, \mathcal{O}_L, \Gamma, \Lambda) \equiv (\mathcal{O}_0, \mathcal{O}_{L_0}, \Gamma_0, \Lambda_0)(level N)$. *Form the Jacobian ideals* $\mathcal{J}'$ *and* $\mathcal{J}'_0$ *and define the integer* h *and* h_0 *for both data. If* $h < N$ *or* $h_0 < N$, *then* $h = h_0$.

Proof. Suppose $h < N$. Since T' just depends on $(\mathcal{O}, \mathcal{O}_L, \Gamma, \Lambda)$, hence I' and J' just depends on $(\mathcal{O}, \mathcal{O}_L, \Gamma, \Lambda)$. Then $J' \otimes \mathcal{O}/\pi^N$ just depends on $(\mathcal{O}/\pi^N, \mathcal{O}_L/\pi^N, \Gamma, \Lambda)$. So $I'_0 + J'_0 + \pi_0^N \mathcal{O}_{L_0}[X_1, ..., X_{d(n+1)}]$contains π_0^h. Then by Nakayama's Lemma , we have $I'_0 + J'_0 \supset \pi_0^h \mathcal{O}_{L_0}[X_1, ..., X_{d(n+1)}]$ Therefore $h_0 \leq h \leq N$. Similarly, $h \leq h_0$, hence $h = h_0$. □

Definition 8.2. If $h < n$, define $h\ (\mathcal{O}, \mathcal{O}_L, \Gamma, \Lambda)$ to be h; otherwise define $h\ (\mathcal{O}, \mathcal{O}_L, \Gamma, \Lambda) = N$. This is justified by the lemma.

Proposition 8.3. *The group scheme* $T_L^0 \otimes \mathcal{O}_L/\pi^{N-h}$ *is determined by* $(\mathcal{O}/\pi^N, \mathcal{O}_L/\pi^N, \Gamma, \Lambda)$ *if* $N > 2h$.

Proof. By lemma **6.8**, it is enough to show that the collection of points

$$\bigcup_C \text{image } (T'(C) \to T'(C/\pi^{N-h})),$$

where C ranges over all local finite flat $\widehat{\mathcal{O}}$-algebra, is determined by $(\mathcal{O}/\pi^N, \mathcal{O}_L/\pi^N, \Gamma, \Lambda)$. Since T' is complete intersection and by Lemma **7.2**, this collection is the same as the union of the image $T'(C/\pi^N) \to T'(C/\pi^{N-h})$ over all local, flat, r.c.i over $\mathcal{O}/\pi^N$ and this is clearly determined by $(\mathcal{O}/\pi^N, \mathcal{O}_L/\pi^N, \Gamma, \Lambda)$. □

Corollary 8.4. *The group scheme* $T^0 \otimes \mathcal{O}/\pi^{N-h}$ *is determined by* $(\mathcal{O}/\pi^N, \mathcal{O}_L/\pi^N, \Gamma, \Lambda)$ *for* $N > 2h$.

Proof. We have $T^0_L = T^0 \otimes \mathcal{O}_L$, and by Proposition **8.3**, the corollary is clearly derived from the following easy lemma: Suppose X, X' are closed S-subschemes of an S-scheme Y such that $X \times_S S' = X' \times_S S'$ in $Y \times_S S'$ for some $S' \to S$ faithfully flat. Then $X = X'$ □

In the following, we use the notations and procedure in Section **4** and Section **5**. T^0 is a closed subscheme of $\mathbb{A}^{d(n+1)}$, defined by an ideal I and let $J \subset \mathcal{O}[X_1, ..., X_{d(n+1)}]$ be the Jaccobian ideal of I. Since $I' \subset I$, we have $J' \subset J$ and $\pi^h \in (J' + I')$.

Proposition 8.5. 1), *$T^0 \otimes \mathcal{O}/\pi^N$ is determined by $(\mathcal{O}/\pi^m, \mathcal{O}_L/\pi^m, \Gamma, \Lambda)$ for all $N \geq 1, m \geq max(N+h, 2h+1)$.*
2), *$R^i \otimes \mathcal{O}/\pi^{m-i}$ depends only on $(\mathcal{O}/\pi^m, \mathcal{O}_L/\pi^m, \Gamma, \Lambda)$ for all $m \geq max(2h+i, 3h+1)$.*
3), *W^i depends only on $fourm$, for $m \geq max(2h+i+1, 3h+1)$.*

Proof. 1). $T^0 \otimes \mathcal{O}/\pi^N$ is determined by $T^0 \otimes \mathcal{O}/\pi^{max(n,h+1)}$, and then the proposition follows immediately from Corollary **8.4**.

2), By Lemma **5.3** and by induction, $R^i \otimes \mathcal{O}/\pi^{m-i}$ is determined by $R^0 \otimes \mathcal{O}^m$, and $R^0 = \Lambda_*(T) \otimes \mathrm{Res}_{\mathcal{O}_L/\mathcal{O}_K}(\mathbb{G}_m)$, then the statement is clear.

3), For i=0. From definition of W^0, W^0 is determined by the image of $T^0(\mathcal{O}^{sh}) \to T^0(\mathcal{O}^{sh}/\pi^N)$ for any $N \geq 1$, in particular $N = h+1$. Moreover, W^0 is group scheme, hence is r.c.i. By lemma **8.2**, this image is determined by $T^0(\mathcal{O}^{sh}/\pi^{2h+1})$, and the latter is determined by $T^0 \otimes \mathcal{O}^{sh}/\pi^{2h+1}$, which is determined by $(\mathcal{O}/\pi^m, \mathcal{O}_L/\pi^m, \Gamma, \Lambda)$ for $m \geq 3h+1$, according to Corollary **8.4**.

In general, let B^i be the affine ring of R^i, and recall the notations in Section **3** ,

$$B^i = B^{i-1}[Y_1, ..., Y_n]/(\pi Y_1 - g_1, ..., \pi Y_n - g_n) \mod \pi - torsion,$$

where write the image of Y_i as $\frac{g_i}{\pi}$, suggestively. A point y in R^i is determined by the projection of y on R^{i-1}, together with the additional "coordinates" $(\pi^{-1}g_1(y), ..., \pi^{-1}g_n(y))$.

For $x \in T^0(\mathcal{O}^{sh})$, by the universal property of dilatations, x is also in $R^i(\mathcal{O}^{sh})$, denoted by x_i. Then $x_i \mod \pi$ is determined by $x_{i-1} \mod \pi^2$. Inductively, the image of $T^0(\mathcal{O}^{sh}) \to T^0((\mathcal{O}^{sh})/\pi^{i+1})$ determined W^i. As in the case i=0, this image is determined by $(\mathcal{O}/\pi^m, \mathcal{O}_L/\pi^m, \Gamma, \Lambda)$ whenever $m \geq max(2h+i+1, 3h+1)$.

□

Let $\delta = \lfloor \frac{\delta(e,T')}{e(L/K)} \rfloor$, we have $\delta \leq h$ from Section **4**. If $\delta < N$, we define $\delta\,(\mathcal{O}/\pi^N, \mathcal{O}_L/\pi^N, \Gamma, \Lambda)$ to be δ; otherwise, we define $\delta\,(\mathcal{O}/\pi^N, \mathcal{O}_L/\pi^N, \Gamma, \Lambda) = N$. The definition is justified by lemma **5.5**.

Lemma 8.6. *Let X be a smooth group scheme over $\mathcal{O}$. Then the schematic closure of the points $\{x : x \in X(\mathcal{O}^{sh}/\pi^N)\} = \{x \mod \pi^N : x \in X(\mathcal{O}^{sh})\}$ in $X \otimes \mathcal{O}/\pi^N$ is the whole $X \otimes \mathcal{O}/\pi^N$.*

Proof. We first show $\{x : x \in X(\mathcal{O}^{sh}/\pi^N)\} = \{x \mod \pi^N : x \in X(\mathcal{O}^{sh})\}$. The notations are the same as Section **7**. By lemma **4.2**, we have $h = 0$, then the equality is clear by lemma **7.2**.

The statement is local, we can assume $X = \operatorname{Spec} A$ is smooth over $\mathcal{O}$. Suppose $f \in A$ satisfies $x^* f = 0$, $\forall x$. Then $f \mod \pi$ is zero on every closed points of $X \otimes \kappa^{sep}$, hence $f \in \pi A$. And by induction, we have $f = 0$. □

Theorem 8.7 (Main Theorem). *Suppose that $N \geq 1, m \geq max(N + \delta + 2h, 3h + 1)$, where $h = h\ (\mathcal{O}/\pi^m, \mathcal{O}_L/\pi^m, \Gamma, \Lambda)$ as defined at the beginning of this section, $\delta = \delta\ (\mathcal{O}/\pi^m, \mathcal{O}_L/\pi^m, \Gamma, \Lambda)$ as defined above. Then, $T^{NR} \otimes \mathcal{O}/\pi^N$ is determined by $(\mathcal{O}/\pi^m, \mathcal{O}_L/\pi^m, \Gamma, \Lambda)$.*

Proof. By lemma **3.3** and remark in Section **5**, T^{NR} is the schematic closure of T in R^δ.

Let Y be the image of $T^{NR}(\mathcal{O}^{sh}/\pi^N)$ in $R^\delta(\mathcal{O}^{sh}/\pi^N)$, then the schematic closure of Y in $R^\delta \otimes \mathcal{O}/\pi^N$ is simply $T^{NR} \otimes \mathcal{O}/\pi^N$ by the precious lemma. So we just need to show $(\mathcal{O}/\pi^m, \mathcal{O}_L/\pi^m, \Gamma, \Lambda)$ determine Y.

As explained in the proof of Proposition **8.5(3)**, Y is determined by the image of $T^0(\mathcal{O}^{sh}) \to T^0(\mathcal{O}^{sh}/\pi^{\delta+N})$, which is determined by the image of $T^0(\mathcal{O}^{sh}) \to T^0(\mathcal{O}^{sh}/\pi^{max(\delta+N,h+1)})$, which is the same as the image of $T^0(\mathcal{O}^{sh}/\pi^{max(N+\delta,h+1)+h}) \to T^0(\mathcal{O}^{sh}/\pi^{max(\delta+N,h+1)})$ by lemma **7.2**. By Corollary **8.4**, $T^0(\mathcal{O}^{sh}/\pi^{max(\delta+N+h,2h+1)})$ is determined by $(\mathcal{O}/\pi^m, \mathcal{O}_L/\pi^m, \Gamma, \Lambda)$. Hence, the proof is over. □

9 The invariant c(T) and Artin conductor

Let K be a complete discrete valuation field. We define an invariant of a torus T over K as following: by the universal property of the Néron model, there is a canonical morphism $\mathcal{T} \otimes \mathcal{O}_L$ to the (usual) Néron model of $T \otimes L$ extending the identity morphism on the generic fibres. This morphism induces a morphism

$$\Phi_{T,L} : T^{NR} \otimes \mathcal{O}_L \to (T \otimes L)^{NR},$$

Definition 9.1. Let L be a splitting field of T, and let $e(L/K)$ be the ramification index of L/K. Define

$$c(T) = \frac{1}{e(L/K)} \text{length}_{\mathcal{O}_L} \frac{\omega(T^{NR}) \otimes \mathcal{O}_L}{\Phi^*_{T,L}(\omega((T \otimes L)^{NR}))}$$

where $\omega(T^{NR})$ (resp. $\omega((T \otimes L)^{NR})$) denotes the module of the translation invariant top differential forms on T^{NR} (resp. $(T \otimes L)^{NR}$). It can easily seen that this rational number does not depend on the choice of a splitting extension L/K.

Note that $\omega(G)$ is the dual of $\bigwedge^{top} \operatorname{Lie}(G)$ for any smooth group scheme G over $\mathcal{O}_L$.

Artin conductors of representations
Let L/K be a finite Galois extension with Galois group G. Let v_L be the normalized valuation of L and π_L be a prime element of $\mathcal{O}_L$. Let f be the residue degree of L/K. Let $\sigma \in G$ and set

$$a_G(\sigma) = -f \cdot v_L(\sigma(\pi_L) - \pi_L) \quad \text{if } \sigma \neq 1$$

$$a_G(1) = f \sum_{\sigma \neq 1} v_L(\sigma(\pi_L) - \pi_L)$$

Then the function a_G is the character of a linear representation $\rho : G \to GL(V)$ by [Serre1, VI.2 Thm 1].

Definition 9.2. The *Artin conductor* $a(V)$ of the presentation $\rho : G \to GL(V)$ is defined to be the number

$$\frac{1}{\operatorname{Card}(G)} \sum_{\sigma \in G} a_G(\sigma) \chi(\sigma^{-1}),$$

where χ is the character of the presentation.

Let G_i be the i-th ramification group of L/K, of cardinality g_i. Then

$$a(V) = \sum_{i \geq 0} \frac{g_i}{g_0} \dim(V/V^{G_i}).$$

Example 9.3. Let $T = \operatorname{Res}_{L/K}(\mathbb{G}_m)$, then

$$c(T) = \frac{1}{2} a(X_*(T) \otimes \mathbb{Q}) = \frac{1}{2} v_K(\Delta)$$

where $a(-)$ is the Artin conductor of a module over $\mathbb{Q}[\operatorname{Gal}(K^{sep}/K)]$, Δ is the discriminant of L/K, and v_K is the normalized valuation of K .

Proof. In Section **2**, we saw that $X_*(T) = \mathbb{Z}[G]$, where $G = \operatorname{Gal}(L/K)$. Hence $a(\mathbb{Q}[G]) = f v_L(\mathfrak{D}) = v_K(\Delta)$, where $\mathfrak{D}$ is the different of L/K. The first equality is attained by [Serre1, IV. Prop 4] and the second one follows from $N_{L/K}(\mathfrak{D}) = \Delta$, where $N_{L/K}$ is the norm of L/K.

Let $n = [L : K]$. Assume $G = \{\sigma_1, ..., \sigma_n\}$ and $\{\alpha_i, i = 1, ..., n\}$ is a base of $\mathcal{O}_L/\mathcal{O}_K$, then the norm N of $\sum(x_i\alpha_i)$ is a polynomial on the $x_i's$. Let $A = \mathcal{O}_K[X_1, ..., X_n, 1/N]$ and let R be any $\mathcal{O}_K$-algebra. If $f \in \mathrm{Hom}(A, R)$, then $\sum f(X_i) \otimes \alpha_i$ is a unit in $R \otimes \mathcal{O}_L$, and vice versa. Hence $\mathrm{Hom}(A, R) \simeq (R \otimes \mathcal{O}_L)^\times$ for any $\mathcal{O}_K$-algebra R, and $\mathrm{Res}_{\mathcal{O}_L/\mathcal{O}_K}(\mathbb{G}_m) = \mathrm{Spec}\, A$. Similarly, $\mathrm{Res}_{L/K}(\mathbb{G}_{m,L})$=Spec $K[X_1, ..., X_n, 1/N]$ with the same polynomial N. And the identity map $A \to A$ induce a unit $\sum_j(X_j\alpha_j)$ in $A \otimes \mathcal{O}_L$. Fix the isomorphism $\Psi : T \otimes L \to \mathrm{G}^n_{m,L}$ which is associated to the ring homomorphism $\Psi^\# : L[X_{\sigma_i}, X^{-1}_{\sigma_i}] \to L[X_1, ..., X_n, 1/N]$ given by $X_{\sigma_i} \to \sum_j \sigma_i(\alpha_j)X_j$.

The map Ψ induces an isomorphism $(T \otimes L)^{NR} \to \mathrm{G}^n_{m,\mathcal{O}_L}$, and we define the composition Θ of $T^{NR} \otimes \mathcal{O}_L \to (T \otimes L)^{NR} \to \mathrm{G}^n_{m,\mathcal{O}_L}$ as following. Let $\Theta^\#$ be the ring homomorphism associated to Θ. The map $\Theta^\#$ is defined as following:

$$\Theta^\# : \mathcal{O}_L[X_{\sigma_i}, X^{-1}_{\sigma_i}] \to A \otimes \mathcal{O}_L, \quad X_{\sigma_i} \to \sum_j \sigma_i(\alpha_j)X_j.$$

Now, it is clear that $c(T) = v_K(\det(\sigma_i(\alpha_j))) = \frac{1}{2}v_K(\Delta)$. □

Proposition 9.4. *The following two statements are equivalent:*

(1) *$c(T_1) = c(T_2)$ for any tori T_1, T_2 over K such that T_1 is isogenous to T_2 over K.*

(2) *$c(T) = \frac{1}{2}a(X_*(T) \otimes \mathbb{Q})$ for any torus T over K, where $a(-)$ is the Artin conductor of a module over $\mathbb{Q}[\mathrm{Gal}(K^{sep}/K)]$.*

Proof. Clearly (2) implies (1) by the Proposition **2.7**.

Assume (1). We have seen (2) is true when T is an induced torus. Since $c(-)$ and $a(-)$ are both additive with respect to fibre product. And by Proposition **2.8**, we have (2). □

Let $\alpha : T_1 \to T_2$ be an isogeny over K. Let L be a common splitting field of T_1 and T_2, then $T_i \otimes L \simeq X_*(T_i) \otimes \mathbb{G}_{m,L}$ and $\Omega^1_{T_i/K} = X^*(T_i) \otimes \Omega^1_{\mathbb{G}_{m,K}/K}$. We have the commutative diagram

$$\begin{array}{ccc} \omega((T_2 \otimes L)^{NR}) & \xrightarrow{(\alpha\otimes L)^*} & \omega((T_1 \otimes L)^{NR}) \\ \downarrow{\scriptstyle \Phi^*_{T_2}} & & \downarrow{\scriptstyle \Phi^*_{T_1}} \\ \omega(T_2^{NR} \otimes \mathcal{O}_L) & \xrightarrow{\alpha^*\otimes\mathcal{O}_L} & \omega(T_2^{NR} \otimes \mathcal{O}_L) \end{array}$$

with injective vertical maps. When $\mathrm{char}(K) = 0$, then the horizontal maps are also injective.

For any homomorphism $g : M \to N$ of $\mathbb{Z}$-modules with finite cokernel, we define

$$c(g) = \text{length}(N/g(M)).$$

Clearly
$c(g \circ h) = c(g) + c(h)$. Hence $c(\Phi^*_{T_2}) = c(\Phi^*_{T_1})$ if and only if $c((\alpha \otimes L)^*) = c(\alpha^* \otimes \mathcal{O}_L)$. We have $c(\Phi_{T_i}) = e(L/K)c(T_i)$, and $c((\alpha \otimes L)^*) = v_L(\deg \alpha)$, where v_L is the normalized valuation of L. Hence,

Proposition 9.5. $c(T_1) = c(T_2)$ *if and only if* $c(\alpha^*) = v_K(\deg \alpha)$*, where* v_K *is the discrete valuation of* K *with* $v_K(\pi) = 1$*, and* $\alpha^* : \omega(T_2^{NR}) \to \omega(T_1^{NR})$*.*

Corollary 9.6. *If the residue field* κ *of* $\mathcal{O}$ *has characteristic 0, then* $c(T_1) = c(T_2)$ *for any two isogenous tori* T_1 *and* T_2*.*

Proof. Let $\alpha : T_1 \to T_2$ be an isogeny. By Proposition **2.6**, there exists an isogeny $\beta : T_2 \to T_1$, such that $\beta \circ \alpha = [\deg \alpha]_{T_1}$, and $\alpha \circ \beta = [\deg \alpha]_{T_2}$. Since $\text{char}(\kappa) = 0$, $\deg \alpha$ is invertible in $\mathcal{O}_K$, hence $(\alpha \otimes L)^*$ and $\alpha^* \otimes \mathcal{O}_L$ are both isomorphisms. Then $c(\alpha^*) = c((\alpha \otimes L)^*) = c(\alpha^* \otimes \mathcal{O}_L) = 0$, thus $c(T_1) = c(T_2)$ □

10 Isogeny invariance in characteristic 0

In this section, we will prove that $c(T)$ is invariant by isogeny when K has characteristic 0. As we have already proved this when the residue field κ of $\mathcal{O}_K$ has characteristic 0, we can assume that $\text{char}\,\kappa = p > 0$.

Lemma 10.1. *Let* K *be a field equipped with a discrete valuation and let* T *be a torus over* K*. Let* T_s *be the maximal split subtorus of* T*, and let* T_a *be the quotient torus* T/T_s*. Then the canonical sequence*

$$1 \longrightarrow T_s^{NR} \longrightarrow T^{NR} \longrightarrow T_a^{NR} \longrightarrow 1$$

is exact.

Proof. By [SGA 7 VIII. Cor. 6.6], we can extend the sequence

$$1 \longrightarrow T_s \longrightarrow T \longrightarrow T_a \longrightarrow 1$$

to an exact sequence of smooth group schemes

$$1 \longrightarrow T_s^{NR} \longrightarrow T^* \longrightarrow T_a^{NR} \longrightarrow 1\,.$$

Hence we have the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & T_s^{NR}(\mathcal{O}^{sh}) & \longrightarrow & T^*(\mathcal{O}^{sh}) & \longrightarrow & T_a^{NR}(\mathcal{O}^{sh}) & \longrightarrow & 1 \\ & & \downarrow \simeq & & \downarrow & & \downarrow \simeq & & \\ 1 & \longrightarrow & T_s(K^{sh}) & \longrightarrow & T(K^{sh}) & \longrightarrow & T_a(K^{sh}) & & \end{array}$$

Since $T^* \to T_a$ is smooth, and by [BLR. 2.2 Prop 14], the first low is exact. Thus $T^*(\mathcal{O}^{sh}) = T(K^{sh})$, and by [BLR. 7.1 Thm 1], we have $T^* = T^{NR}$. $\square$

Theorem 10.2. *Let K be a complete discrete valuation field with mixed characteristic $(0, p)$ and perfect residue field. Let T_1, T_2 be two tori over K, and let $\alpha : T_1 \to T_2$ be a K-isogeny. Then two tori have the same invariant:*

$$c(T_1) = c(T_2) = \frac{1}{2} a(X_*(T_1) \otimes \mathbb{Q}).$$

Remark. I will restrict myself to the case when K is a finite extension of $\mathbb{Q}_p$. For the general case, see the original paper of Ching-Li Chai and Jiu-Kang Yu.

Proposition 10.3. *Consider the pull-back map $\alpha^* : \omega(T_2^{NR}) \to \omega(T_1^{NR})$. There exists an element $a \in \mathcal{O}_K$, unique up to $\mathcal{O}_K^\times$, such that $\alpha^*(\omega(T_2^{NR})) = a \cdot \omega(T_1^{NR})$. Denote the rational number $p^{\mathrm{ord}_p(a)}$ by $\deg_{diff}(\alpha)$. Then*

$$\deg_{diff}(\alpha) \leq p^{ord_p(\deg \alpha)}.$$

In the above, ord_p *denotes the valuation on K with* $\mathrm{ord}_p(p) = 1$.

Proof. Suppose K is a finite extension of $\mathbb{Q}_p$.

By lemma **10.1**, we may assume that T_1 and T_2 are anisotropic over the maximal unramified extension of K (replacing K by a finite unrmified extension L/K if necessary). Then $T_i^{NR}(\mathcal{O}_L) = T_i(L)$ for any unramified extension L/K, $i = 1, 2$.

Let $T_i^{NR\circ}$ be the neutral component of the Néron model T_i^{NR}, $i = 1, 2$. Let ω_i be an $\mathcal{O}_K$-generator of $\omega(T_i)^{NR}$, $i = 1, 2$. Let ord_K be the valuation of K with $\mathrm{ord}_K(\pi) = 1$. Let $M = \mathrm{Ker}(\alpha)$, the kernel of isogeny α. Consider finite unramified extension L/K, and let q_L be the cardinality of the residue field κ_L of $\mathcal{O}_L$. Let $|\omega_i|$ be the Haar measure on T_i^{NR} attached to ω_i, $i = 1, 2$. Hence we have

$$|\alpha^* \omega_2|(T_1^{NR\circ}(\mathcal{O}_L)) = \mathrm{Card}(M(L) \bigcap T_1^{NR\circ}(\mathcal{O}_L)) \cdot |\omega_2|(\alpha(T_1^{NR}))$$

By definition, for $i = 1, 2$, $|\omega_i|(T_i^{NR\circ})(\mathcal{O}_L)$ is equal to the number of κ_L-rational points of the closed fibre of $T_i^{NR\circ}$, divided by $q_L^{\dim T_i}$. Since T_i

is anisotropic, its closed fibre is a unipotent group over κ_L, and has the same number of κ_L-rational points as $\mathbb{A}^{dim(T_i)}$. Hence $|\omega_1|(T_1^{NR\circ}(\mathcal{O}_L)) = |\omega_2|(T_2^{NR\circ}(\mathcal{O}_L))$, and

$$[T_2^{NR\circ}(\mathcal{O}_L) : \alpha(T_1^{NR\circ})] = \mathrm{Card}(M(L) \cap T_1^{NR\circ}(\mathcal{O}_L)) \cdot q_L^{ord_K(a)}$$

Let C_{T_i} be the group of geometric connected components of the closed fibre of $T_i^{NR}, i = 1, 2$. For sufficiently large finite unramified extension L of K, we have

$$[T_2^{NR\circ}(\mathcal{O}_L) : \alpha(T_1^{NR\circ}(\mathcal{O}_L))] = \frac{\mathrm{Card}(\mathrm{C_{T_1}})}{\mathrm{Card}(\mathrm{C_{T_2}})}[T_2^{NR}(\mathcal{O}_L) : \alpha(T_1^{NR}(\mathcal{O}_L))].$$

On the other hand, by Tate's formula for the Euler-Poincaré characteristic for the Galois cohomologies of local fields, we have

$$\mathrm{Card}(\mathrm{H}^1(L, M)) = q_L^{ord_K(\deg\alpha)} \cdot \mathrm{Card}(M(L)) \cdot \mathrm{Card}(\mathrm{H}^2(L, M)).$$

By the local duality for Galois cohomology of local fields ([Milne, I, Cor. 2.3]), $\mathrm{H}^2(L, M)$ is the dual of $M^D(L)$, where M^D is the Cartier dual of the finite group scheme M over K.

From the long exact sequence of Galois cohomologies attached to the isogeny α, we get an injection from $T_2(L)/\alpha(T_1(L))$ to $\mathrm{H}^1(L, M)$. Thus we have

$$\frac{\mathrm{Card}(C_{T_2})}{\mathrm{Card}(\mathrm{C_{T_1}})}\mathrm{Card}(M(L)\cap T_1^{NR\circ}(\mathcal{O}_L))\cdot q_L^{\mathrm{ord}_K(a)} \le q_L^{ord_K(\deg\alpha)}\cdot\mathrm{Card}(M(L))\cdot\mathrm{Card}(\mathrm{H}^2(L, M)).$$

As L tends to K^{sh}, we have $q_L \to +\infty$. Hence, we get $\mathrm{ord}_K(a) \le \mathrm{ord}_K(\deg\alpha)$. Since $\mathrm{ord}_K = \mathrm{ord}_K(p) \cdot \mathrm{ord}_p$, we have

$$\deg_{diff}(\alpha) \le p^{ord_p(\deg\alpha)}.$$

□

Proof of Theorem 10.2. Choose an isogeny $\beta : T_2 \to T_1$ such that $\beta \circ \alpha = [n]_{T_1}$. Let $d = \dim T_1 = \dim T_2$. Write $n = p^m u$, where $m = ord_p(n)$. We have

$$p^{md} = \deg_{diff}(\beta \circ \alpha) = \deg_{diff}(\beta)\deg_{diff}(\alpha) \le p^{\mathrm{ord}_p(\deg\alpha)}p^{\mathrm{ord}_p(\deg\beta)} = p^{md}.$$

So the equality holds throughout the above inequality. Hence by Proposition **9.5**, we have $c(T_1) = c(T_2)$. □

11 Isogeny invariance in characteristic p ——Application of Deligne's theory

Deligne's theory
Let K be a complete local field with a perfect residue field κ. Let $\mathcal{O}$ be the ring of integers of K, and let $e \geq 1$. A Galois extension L/K is at most $e-$ramified if $\mathrm{Gal}(L/K)^e = 1$, where e refers to the upper numbering filtration of the ramifications groups. In other words, $\mathrm{Gal}(L/K)$ is a quotient of $\mathrm{Gal}(K^{sep}/K)/\mathrm{Gal}(K^{sep}/K)^e$.

Deligne [Deligne] shows that $\mathrm{Gal}(K^{sep}/K)/\mathrm{Gal}(K^{sep}/K)^e$ is canonically determined by $Tr_eK = (\mathcal{O}/\mathfrak{p}^e, \mathfrak{p}/\mathfrak{p}^{e+1}, \epsilon)$, where $\mathfrak{p}$ is the prime ideal of $\mathcal{O}$, and ϵ is the canonical map from $\mathfrak{p}/\mathfrak{p}^{e+1}$ to $\mathcal{O}/\mathfrak{p}^e$. Denote $\mathrm{Gal}(K^{sep}/K)/\mathrm{Gal}(K^{sep}/K)^e$ by $\Gamma(Tr_eK)$.

Suppose Tr_eK is isomorphic to Tr_eK_0 and L/K is at most e-ramified. Then there exits a corresponding L_0/K_0 and $(\mathcal{O}, \mathcal{O}_L) \equiv (\mathcal{O}_0, \mathcal{O}_{L_0})$ (level e). We can construct L_0 as following:
Suppose $\phi : \mathcal{O}/\pi^e \to \mathcal{O}/\pi_0^e$ and $\eta : \mathfrak{p}/\mathfrak{p}^{e+1} \to \mathfrak{p}_0/\mathfrak{p}_0^{e+1}$ define the isomorphism $Tr_eK \to Tr_eK_0$. Let π_L be a prime element of $\mathcal{O}_L$ satisfying the Eisenstein equation

$$X^n + \sum_{i=0}^{n-1} a^{(i)} X^i = 0, \quad a^{(i)} \in \mathfrak{p}.$$

Let $a_0^{(i)} \in \mathcal{O}_0$ be the lifting of $\eta(a^{(i)} \mod \mathfrak{p}^{e+1})$. Then the equation $X^n + \sum_{i=0}^{n-1} a_0^{(i)} X^i = 0$ defines the extension L_0/K_0.

Proposition 11.1. *Let T be a torus over K, then the invariant $c(T)$ is determined by Tr_eK for $e \gg 0$.*

Proof. Let $e \gg N \gg 0$ and $\Lambda = X_*(T)$. Since $(Tr_e(K), \Gamma = \Gamma(Tr_eK), \Lambda)$ determines $(\mathcal{O}/\pi^e, \mathcal{O}_L/\pi^e, \Gamma, \Lambda)$, hence determines the following morphisms by Section **8**: $T_L^0 \otimes \mathcal{O}_L/\pi^N \to (T \otimes L)^{NR} \otimes \mathcal{O}_L$; $R^{i+1} \otimes \mathcal{O}/\pi^N \to R^i$; $T^{NR} \otimes \mathcal{O}/\pi^N \to R^\delta \otimes \mathcal{O}/\pi^N$. The last morphism factors through the closed immersion $T^{NR} \otimes \mathcal{O}/\pi^N \to T^0 \otimes \mathcal{O}/\pi^N$, hence the morphism $T^{NR} \otimes \mathcal{O}/\pi^N \to T^0 \otimes \mathcal{O}/\pi^N$ is determined by (Tr_eK, Λ). Finally, we conclude that the morphism $T^{NR} \otimes \mathcal{O}_L/\pi^N \to (T \otimes L)^{NR} \otimes \mathcal{O}_L/\pi^N$ is determined by $(Tr_e(K), \Lambda)$ for $e \gg N$. Hence $c(T)$ is determined by $(Tr_e(K), \Lambda)$ for $e \gg N \gg 0$. □

Theorem 11.2. *Assume that K is of equal-characteristic p and the residue field of $\mathcal{O}_K$ is perfect. Let T be a torus over K. Then $c(T) = \frac{1}{2}a(X_*(T) \otimes \mathbb{Q})$. In particular, it is invariant under isogeny.*

Proof. Since $T^{NR} \otimes \widehat{\mathcal{O}} \simeq (T \otimes \widehat{K})^{NR}$, we can assume K is complete.

By Deligne's theory, choose a local field K_0 of characteristic 0 such that $Tr_e K_0 \simeq Tr_e K$, then $c(T) = c(T_0) = \frac{1}{2}a(X_*(T_0) \otimes \mathbb{Q})$. Since $X_*(T_0)$ is isomorphic to $X_*(T)$ as $\Gamma(Tr_e K) \simeq \Gamma(Tr_e K_0)$-module, we have $a(X_*(T_0) \otimes \mathbb{Q}) = a(X_*(T) \otimes \mathbb{Q})$. Hence $c(T) = \frac{1}{2}a(X_*(T) \otimes \mathbb{Q})$. □

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