Higher genus counterexamples to relative Manin–Mumford

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CHAPTER 1

Introduction

The goal of this work is to provide a higher genus analog of Edixhoven's construction [**2**, Appendix] of Bertrand's [**2**] counter-example to Pink's relative Manin–Mumford conjecture (cf. Section 4.1 for the statement of the conjecture). This is done in Chapter 4, where further discussion of the conjecture and past work can be found. To give our construction, we must first develop the theory of pinchings a la Ferrand [**8**] in flat families and understand the behavior of the relative Picard functor for such pinchings, and this is the contents of Chapter 3. In Chapter 2 we recall some of the results and definitions we will need from algebraic geometry; the reader already familiar with these results should have no problem beginning with Chapter 3 and referring back only for references and to fix definitions. Similarly, the reader uninterested in the technical details of pinching should have no problem beginning with Chapter 4 and referring back to Chapter 3 only for the statements of theorems.

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CHAPTER 2

Preliminaries

2.1. Conventions and notation

We make several notational remarks: if X and Y are both schemes over a base S, we will often denote by X_Y the fibered product $X \times_S Y$, especially when we are considering it as a base extension from S to Y. We will denote the structure sheaf of a ringed space X by $\mathcal{O}_X$, or just $\mathcal{O}$ if the ringed space in question is clear. The symbol $\mathcal{L}$ will almost always denote an invertible sheaf, except in part of Section 3.2 where it will also be used for quasi-coherent sheaves and general $\mathcal{O}_X$–modules. If D is a Cartier divisor then by $\mathcal{O}(D)$ we mean the invertible subsheaf of the sheaf of rational functions generated locally by a local equation for $-D$. By a variety we will mean a reduced scheme of finite type over a field.

2.2. Algebraic curves

Definition 2.1. Let S be a scheme. A *relative curve* X/S is a morphism $X \to S$ separated and locally of finite presentation whose geometric fibers are reduced, connected, and 1–dimensional. If the base scheme S is the spectrum of a field, we will call X a *curve*. A relative curve X/S is *semistable* if it is flat, proper, and the geometric fibers of X have only ordinary double points.

Note that by this definition a curve over a field is always geometrically connected. By a standard result, a smooth proper curve over a field is projective (one can prove this quickly using Riemann–Roch).

2.2.1. Divisors on curves. Let k be an algebraically closed field and let X/k and Y/k be smooth proper curves. If $\alpha\colon X \to Y$ is a non-constant morphism, then it is a finite surjective morphism and it induces two homomorphisms of the corresponding divisor groups: the pullback α^* and the pushforward α_*.

If D is a divisor on Y, then the pullback of D by α is defined as

$$\alpha^* D = \sum_{P \in X(k)} v_P(f_{\alpha(P)} \circ \alpha) P$$

where f_Q for $Q \in Y$ is any rational function defining D in a neighborhood of Q. If f is a rational function on Y then $\alpha^* \mathrm{Div} f = \mathrm{Div}(f \circ \alpha)$, and thus α^* induces a morphism between the divisor class groups (which agrees with the morphism α^* defined on the Picard group when the usual identification is made between the divisor class group and Picard group).

If D is a divisor on X, then the pushforward of D by α is defined as

$$\alpha_* D = \sum_{P \in X(k)} v_P(D) \alpha(P).$$

(where $v_P D$ is just the coefficient of P in D). If f is a rational function on X, then $\alpha_* \mathrm{Div} f = \mathrm{Div}\mathrm{Norm}_\alpha f$ where Norm_α is the norm map of the field extension $K(X)/K(Y)$, and thus α_* induces a morphism between the divisor class groups. It follows from the definitions that $\alpha_* \alpha^*$ as a map from $\mathrm{Div} X \to \mathrm{Div} X$ is multiplication by $\deg \alpha$.

More generally, if $\alpha : X \to Y$ is any finite surjective map of regular integral varieties then we obtain in this way maps α^* and α_* on their Weil divisors.

For a divisor D on a curve X we denote by $\mathrm{Supp} D$ the support of D, that is, the set of $P \in X(k)$ such that $v_P(D) \neq 0$. If f is a rational function on X and D is a divisor with support disjoint from $\mathrm{Div} f$ we define

$$f(D) = \prod_{P \in X(k)} f(P)^{v_P(D)}.$$

The classical Weil reciprocity theorem states the following:

Theorem 2.2 (Weil Reciprocity — see, e.g.,[**1**, Section 2 of Appendix B])**.** *Let X/k be a smooth proper curve over an algebraically closed field. If f and g are two rational functions on X such that* $\mathrm{SuppDiv} f \cap \mathrm{SuppDiv} g = \emptyset$*, then*

$$f(\mathrm{Div} g) = g(\mathrm{Div} f).$$

Remark. There exists a more general version of Weil reciprocity written with local symbols that holds without the hypothesis of disjoint support — see [**22**, Proposition III.7].

2.3. Group schemes

2.3.1. Group schemes. Our principal references for group schemes are the books of Demazure and Gabriel [**6**] and Oort [**19**]. For abelian schemes we refer to Milne [**17**] and Mumford *et al.* [**18**].

Definition 2.3. A *sheaf of (abelian) groups* over a scheme S is an fppf sheaf on Sch/S with values in (abelian) groups. A *group scheme* over S is a representable sheaf of groups over S. A group scheme is called *commutative* if it is a sheaf of abelian groups. An *action* of a sheaf of groups $\mathcal{G}$ on a fppf sheaf $\mathcal{S}$ is a morphism $\mathcal{G} \times \mathcal{S} \to \mathcal{S}$ that is a group action on T–points for every $T \to S$.

By Yoneda and the fact that a representable functor is an fppf sheaf (see, e.g., [**24**, Theorem 2.55]), an equivalent definition of a group scheme over S is as a scheme X/S with morphisms $m : X \times X \to X$ and $e : S \to X$ such that for any $T \in \mathrm{Sch}/S$, $X(T)$ is a group with multiplication induced by m and identity element e_T.

Again by Yoneda, (commutative) group schemes over S form a full sub-category of the category of sheaves of (abelian) groups over S. The category of commutative group schemes over S is not abelian, however, the category of sheaves of abelian groups over S is. Thus, when working with commutative group schemes we will always consider them as embedded inside the category of sheaves of abelian groups, even when we are only interested in commutative group schemes. In particular, by an exact sequence of commutative group schemes we mean a sequence of commutative group schemes that is exact in the category of sheaves of abelian groups.

Definition 2.4. Let S be a scheme. An *abelian scheme X/S* is a smooth and proper group scheme over S with connected geometric fibers.

An abelian scheme is commutative (see [**18**, Section 6.1, Corollary 6.5] for the Noetherian case and [**7**, Remark I.1.2] for the general case).

Definition 2.5. Let S be a scheme. A *torus X/S* is a group scheme which that is fppf–locally over S isomorphic to a finite product of copies of $\mathbb{G}_m$. A *semiabelian scheme G/S* of an abelian variety A_s by a torus T_s (i.e. there is an exact sequence is a smooth separated commutative group scheme with geometrically connected fibers such that each fiber G_s is an extension $0 \to T_s \to G_s \to A_s \to 0$ in the fppf topology).

We will say more about extensions of group schemes in the next section.

2.3.2. Torsors. Our principal references for torsors are Demazure and Gabriel [**6**, Section III.4], Milne [**16**, Section III.4], and the Stacks Project [**23**]. Note that in the definition below we use the fppf topology, differing from the Stacks Project where torsors are defined as locally trivial in fpqc. Because we will be working almost exclusively with $\mathbb{G}_m$–torsors this choice will not make a difference, however working over fppf allows us to make accurate citations to theorems stated for more general group schemes in Demazure and Gabriel who also work with the fppf topology.

Definition 2.6 (cf. [**23**, Definitions 0498 and 049A])**.** Let G be a group scheme over S and X a scheme over S. A *G--torsor over X* is an fppf sheaf $\mathcal{S}$ on Sch/X with an action of G such that there is an fppf covering $(U_i \to X)$ and for each i, $\mathcal{S}_{U_i}$ with its G action is isomorphic (as a sheaf with G–action) to G with the G–action of left multiplication.

The following is a standard result:

Proposition 2.7 (see, e.g., [**6**, Proposition III.4.1.9])**.** *Let G be an affine group scheme over S and X a scheme over S . If P is a G–torsor over X, then P is representable by a scheme that is affine over X. If G/S is flat, so is P/X.*

There is a natural way to associate to any G–torsor a class in the Čech cohomology group $\check{H}^1(X_{fppf}, G)$. If we denote by $PHS(X, G)$ the set of isomorphism classes of G torsors, then this gives a bijection $PHS(X, G) \leftrightarrow \check{H}^1(X_{fppf}, G)$ [**16**, Corollary III.4.7] (the name PHS is an abbreviation for principal homogenous spaces, which is another name for torsors).

When G is commutative, the group structure on $\check{H}^1(X_{fppf}, G)$ induces a group structure on $PHS(X, G)$, which can be described as follows ([**16**, Remark III.4.8b]): if Y_1 and Y_2 are two G–torsors then their sum $Y_1 \vee Y_2$ is the G–torsor obtained by taking the product $Y_1 \times_X Y_2$ and quotienting by the action of G given by $(g, (y_1, y_2)) \to (gy_1, g^{-1}y_2)$. The action of G on $Y_1 \vee Y_2$ is that defined by the action $(g, (y_1, y_2)) \mapsto (gy_1, y_2) = (y_1, gy_2)$ on the corresponding presheaf.

Of particular interest for us are $\mathbb{G}_m$–torsors. By Hilbert's Theorem 90 [**23**, Theorem tag 03P8], $\check{H}^1(X_{fppf}, \mathbb{G}_m) \cong \check{H}^1(X_{et}, \mathbb{G}_m) \cong \check{H}^1(X_{zar}, \mathbb{G}_m)$. In particular, any $\mathbb{G}_m$–torsor can be trivialized locally in the Zariski topology, and $PHS(X, \mathbb{G}_m) \cong \mathrm{Pic}(X)$. This isomorphism maps the class of a torsor to the class of the invertible sheaf defined by the same cocycle.

If we have an exact sequence of group schemes over S (exact in the fppf–topology)

$$0 \to \mathbb{G}_m \to Y \to X \to 0$$

then Y has a natural structure as a $\mathbb{G}_m$–torsor over X: indeed, surjectivity of the arrow $Y \to X$ implies that there exists an fppf cover X'/X such that $\mathrm{id} : X' \to X'$ lifts to a map $X' \to Y_{X'} = Y \times_X X'$ and thus the exact sequence of group schemes over X'

$$0 \to \mathbb{G}_m \to Y_{X'} \to X' \to 0$$

splits and $Y_{X'} \cong X' \times \mathbb{G}_m$. We call an exact sequence of commmutative group schemes over S

$$0 \to \mathbb{G}_m \to Y \to X \to 0$$

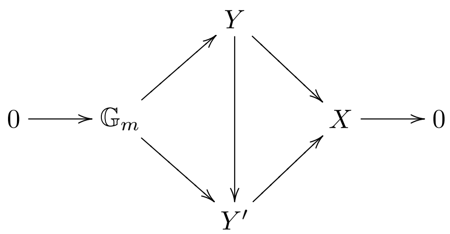
an *extension* of X by $\mathbb{G}_{m,S}$ and we say that two extensions

$$0 \to \mathbb{G}_m \to Y \to X \to 0$$

and

$$0 \to \mathbb{G}_m \to Y' \to X \to 0$$

are isomorphic if there is an isomorphism $Y \to Y'$ (over S) making the following diagram commute



We denote by $\mathrm{Ext}(X, \mathbb{G}_m)$ the set of isomorphism classes of extensions of X by $\mathbb{G}_m$. The map sending an extension to the class of the torsor it defines is an injection from $\mathrm{Ext}(X, \mathbb{G}_m)$ to $\mathrm{Pic}(X)$. The Barsotti–Weil theorem, which we will state in the next section as Theorem 2.20 after introducing the relative Picard functor, refines this statement.

2.4. The relative Picard functor

In this section we recall some facts about the relative Picard functor. Our primary references are Bosch *et al.* [**4**, Chapters 8 and 9] and Kleiman [**14**]. Note that the definitions of the functor $\mathrm{Pic}_{X/S}$ given in these two sources are not equivalent — we adopt the definition of Bosch *et al.* [**4**, Definition 8.1.2]:

Definition 2.8. Let X/S be a scheme. We define the relative Picard functor of X over S:

$$\mathrm{Pic}_{X/S} : \text{Schemes}/S \to \mathrm{Ab}$$

to be the fppf–sheafification of the functor

$$T \to \mathrm{Pic}(X \times T)$$

If $S = \mathrm{Spec}A$ for A a ring we will often write $\mathrm{Pic}_{X/A}$ for $\mathrm{Pic}_{X/S}$.

Under certain conditions, the relative Picard functor admits a particularly amenable description:

Proposition 2.9 (see, e.g., [**14**, Theorem 9.2.5] or [**4**, Proposition 8.1.4])**.** *Let X be an S–scheme with structural morphism f such that $f_*(\mathcal{O}_X) = \mathcal{O}_S$ holds universally (i.e. under any change of base $S' \to S$) and admitting a section $S \to X$. Then the relative Picard functor* $\mathrm{Pic}_{X/S}$ *is given by*

$$T \to \mathrm{Pic}(X \times T)/\mathrm{Pic}(T)$$

where $\mathrm{Pic}(T)$ *is mapped to* $\mathrm{Pic}(X \times T)$ *by pullback via the projection map $X \times T \to T$. Furthermore* $\mathrm{Pic}_{X/S}$ *is an fpqc sheaf on* Sch/S.

Remark. The description of $\mathrm{Pic}_{X/S}$ in the conclusion of 2.9 is taken as the definition of $\mathrm{Pic}_{X/S}$ in [**14**], and then different names are attached to its sheafifications in various topologies.

Definition 2.10. Let X/S be a scheme. Given a section $\epsilon : S \to X$ and an invertible sheaf $\mathcal{L}$ on $X \times_S T$, a *rigidification* of $\mathcal{L}$ along ϵ is an isomorphism $\alpha\colon \mathcal{O}_T \to \epsilon_T^*\mathcal{L}$ where $\epsilon_T = \epsilon \times \mathrm{id}_T$. We define the rigidified Picard functor along ϵ:

$$P(X, \epsilon)\colon\ \mathrm{Sch}/S \to \mathrm{Ab}$$

by

$$P(X, \epsilon)(T) = \{(\mathcal{L}, \alpha) \mid \mathcal{L} \text{ an invertible sheaf on } X_T,\ \alpha \text{ a rigidifcation along } \epsilon\}/\sim$$

Where $\sim$ means up to isomorphism, where two pairs $(\mathcal{L}, \alpha)$ and $(\mathcal{L}', \alpha')$ are isomorphic if there is an isomorphism $\mathcal{L} \to \mathcal{L}'$ carrying α to α'.

Proposition 2.11 (see, e.g., [**14**, Lemma 9.2.9])**.** *If X/S admits a section ϵ, the map $P(X, \epsilon)(T) \to$ $\mathrm{Pic}(X \times T)/\mathrm{Pic}(T)$, $(\mathcal{L}, \alpha) \mapsto \mathcal{L}$ is an isomorphism.*

Remark. If $f_*(\mathcal{O}_X) = \mathcal{O}_S$ holds universally then a rigidified line bundle $(\mathcal{L}, \alpha)$ on X_T does not admit any non-trivial automorphisms (see, e.g., [**14**, Lemma 9.2.10]), and this can be used together with Proposition 2.11 to show that $P(X, \epsilon)$ is an fpqc sheaf in order to prove Proposition 2.9.

We will need some results on the representability of $\mathrm{Pic}_{X/S}$ by a scheme. If $\mathrm{Pic}_{X/S}$ is representable we will say the Picard scheme of X/S exists and denote such a representing scheme by $\mathbf{Pic}_{X/S}$. We first discuss some properties of such a representing scheme, if it exists.

Suppose that $\mathrm{Pic}_{X/S}$ is described on T points as $\mathrm{Pic}_{X/S}(T) = \mathrm{Pic}(X_T)/\mathrm{Pic}(T)$ (as is the case, for example, if the hypotheses of Proposition 2.9 are satisfied) and is representable by a scheme $\mathbf{Pic}_{X/S}$. Then, for a fixed section ϵ, $\mathbf{Pic}_{X/S}$ also represents $P(X, \epsilon)$, and corresponding to the identity element in $\mathrm{Hom}(\mathbf{Pic}_{X/S}, \mathbf{Pic}_{X/S})$ there is a unique (up to unique isomorphism) "universal" line bundle $\mathcal{U}$ on $X \times \mathbf{Pic}_{X/S}$ rigidified along $\epsilon \times \mathrm{id}_{\mathbf{Pic}_{X/S}}$ such that for any T and any line bundle $\mathcal{L}$ on $X \times T$ rigidified over $\epsilon \times \mathrm{id}_T$, there exists a unique morphism $\phi : T \to \mathbf{Pic}_{X/S}$ such that $\mathcal{L}$ is uniquely isomorphic to $(id_X \times \phi)^*(\mathcal{U})$.

We make a tautological observation about the behavior of $\mathrm{Pic}_{X/S}$ under base change.

Proposition 2.12. *Let X be a scheme over S, $f\colon S' \to S$ a morphism, and let $X' = X_{S'}$. Then the map ${\pi_X}^*\colon \mathrm{Pic}_{X/S} \times_S S' \to \mathrm{Pic}_{X'/S'}$ is an isomorphism of functors on Sch/S'. Furthermore, if $\mathrm{Pic}_{X/S}$ is represented by a scheme $\mathbf{Pic}_{X/S}$ with universal line bundle $\mathcal{U}$ then $\mathrm{Pic}_{X'/S'}$ is represented by $\mathbf{Pic}_{X'/S'} = \mathbf{Pic}_{X/S} \times_S S'$ with universal bundle $\mathcal{U}' = (\pi_X \times \pi_P)^*\mathcal{U}$ (where $\pi_X : X' \times_{S'} \mathbf{Pic}_{X'/S'} \to X' \to X$ and $\pi_P\colon X' \times_{S'} \mathbf{Pic}_{X'/S'} \to \mathbf{Pic}_{X'/S'} \to \mathbf{Pic}_{X/S}$ are the projection morphisms). If $\mathcal{U}$ is rigidified along $\epsilon\colon S \to X$ then $\mathcal{U}'$ is rigidified along the canonical extension $\epsilon'\colon S' \to X'$.*

Proof. For a scheme T/S' consider the commutative diagram with cartesian squares

$$\begin{array}{ccccc} T \times_{S'} X' & \xrightarrow{\pi_{X'}} & X' & \xrightarrow{\pi_X} & X \\ \downarrow{\scriptstyle \pi_T} & & \downarrow & & \downarrow \\ T & \longrightarrow & S' & \xrightarrow{f} & S \end{array}$$

There is a natural $S-$isomorphism $T \times_{S'} X' \to T \times_S X$ given by the map $id_T \times \pi_X$, and pullback of bundles by this map induces an isomorphism of the functors $\mathrm{Pic}_{X/S} \times_S S' \to \mathrm{Pic}_{X'/S'}$.

To see the statement about the universal bundles, let $\mathcal{L} \in \mathrm{Pic}_{X'/S'}(T)$. Considering it as an element $\tilde{\mathcal{L}}$ of $\mathrm{Pic}_{X/S}(T)$ we see there is a unique morphism $\phi : T \to \mathbf{Pic}_{X/S}$ such that $(\phi \times id_X)^*\mathcal{U} = \tilde{\mathcal{L}}$. This induces a unique morphism $\phi' : T \to \mathbf{Pic}_{X'/S'}$ such that $(\pi_P \circ \phi' \times id_X)^*\mathcal{U} = \tilde{\mathcal{L}}$. Then, pulling back by $id_T \times \pi_X$, we see

$$(\pi_P \circ \phi' \times \pi_X)^*\mathcal{U} = \mathcal{L}$$

and thus

$$(\phi' \times id_{X'})^*(\pi_P \times \pi_X)^*\mathcal{U} = (\phi' \times id_{X'})^*\mathcal{U}'.$$

□

We now state some results on the existence of the Picard scheme.

Theorem 2.13 ([**4**, Theorem 8.2.1]). *Let $f : X \to S$ be projective, flat, and finitely presented, with reduced and irreducible geometric fibers. Then the Picard scheme $\mathbf{Pic}_{X/S}$ exists and is a separated S–scheme, locally of finite presentation.*

We note that $\mathrm{Pic}_{X/S}$ has a natural group law and thus if it is representable by a scheme then it is equipped with a natural structure as a group scheme. In the case where the base is a field we can relax the conditions of Theorem 2.13:

Theorem 2.14 ([**4**, Theorem 8.2.3]). *Let X be a proper scheme over a field k. The Picard scheme $\mathbf{Pic}_{X/k}$ exists and is locally of finite type over k.*

When it exists, $\mathbf{Pic}_{X/k}$ is a group scheme and we can consider its identity component $\mathbf{Pic}^0_{X/k}$ (i.e. the maximal connected subgroup scheme).

Theorem 2.15 ([**4**, Theorem 8.4.3]). *Let X/k be a smooth, proper, and geometrically integral scheme. Then the identity component $\mathbf{Pic}^0_{X/k}$ is a proper scheme over k.*

Over a general base S, we define $\mathrm{Pic}^0_{X/S}$ for X/S proper to be the subfunctor of $\mathrm{Pic}_{X/S}$ consisting of all elements whose restriction to each fiber X_s, $s \in S$ belongs to $\mathbf{Pic}^0_{X_s/k(s)}$. If k is an algebraically closed field then, for a smooth proper curve X/k, $\mathrm{Pic}^0_{X/k}(k)$ consists of line bundles of degree 0; for an abelian variety A/k, $\mathrm{Pic}^0_{A/k}(k)$ consists of translation invariant line bundles.

We are primarily interested in the functor $\mathrm{Pic}^0_{X/S}$ in two cases: when X is a relative curve and when X is an abelian scheme.

In the case of relative curves the theory of $\mathrm{Pic}^0_{X/S}$ is better known as the theory of relative Jacobians. In the case of a smooth proper connected curve X/k, $\mathbf{Pic}^0_{X/k}$ is an abelian variety which we call the *Jacobian variety* Jac$X$. In the case of proper curve with a single ordinary double point, $\mathrm{Pic}^0_{X/k}$ is represented by a $\mathbb{G}_m$–extension of the Jacobian of its normalization, one of the generalized Jacobians of Rosenlicht; we will explore and generalize this result in Section 3.3.

We will need some of the following theorems on the representability of $\mathrm{Pic}^0_{X/S}$.

Theorem 2.16 ([**4**, Theorem 9.4.1]). *Let $X \to S$ be a semistable relative curve. Then $\mathrm{Pic}^0_{X/S}$ is represented by a smooth separated S–scheme.*

In the case that it exists for a relative curve, we will sometimes call the scheme representing $\mathrm{Pic}^0_{X/S}$ the *relative Jacobian* J of X/S.

We will need a result on the structure of Jacobians over fields.

Theorem 2.17 ([**4**, Proposition 9.2.3]). *Let X be a smooth proper geometrically connected curve over a field k. Then the Jacobian J (i.e. $\mathbf{Pic}^0_{X/k}$) of X/k exists and is an abelian variety.*

Theorem 2.18 ([**4**, Proposition 9.4.4]). *Let $X \to S$ be a smooth relative curve. Then $\mathrm{Pic}^0_{X/S}$ is an abelian S–scheme.*

Theorem 2.19 ([**7**, Theorem I.1.9]). *Let A/S be an abelian scheme. Then $\mathrm{Pic}^0_{A/S}$ is represented by an abelian scheme $A^\vee/S$ called the dual abelian scheme.*

We conclude this section with the Barsotti–Weil theorem, which ties together our discussion of torsors and extensions and our discussion of the relative Picard functor. We motivate this with the following (see [**17**, Proposition I.9.2]): if A/k is an abelian variety and $\mathcal{L}$ is an invertible sheaf whose class is in $\mathrm{Pic}^0(A/k)$ then,

$$m^*\mathcal{L} \cong p^*\mathcal{L} \otimes q^*\mathcal{L}.$$

where $m\colon A \times A \to A$ is the multiplication map and p_1 and p_2 are the two projections $A \times A \to A$. An identity sectin can be chosen such that this gives a group structure on the associated $\mathbb{G}_m$–torsor $Y \to A$ making Y an extension of A by $\mathbb{G}_m$. Over a more general base, the following theorem shows that the only obstruction to this group structure existing is that $\mathcal{L}$ must also be rigidified in order to provide an identity section.

Theorem 2.20 (Barsotti–Weil). *Let S be a scheme and let A/S be an abelian scheme with dual $A^\vee$. The map $\mathrm{Ext}(A, \mathbb{G}_m) \to P^0(A, e_A) \cong \mathrm{Pic}^0(A)/\mathrm{Pic}(S) = A^\vee(S)$ given by considering an extension as a $\mathbb{G}_m$–torsor is an isomorphism of groups.*

Remark. It is difficult to find a reference for Barsotti–Weil where S is allowed to be an arbitrary scheme. One proof is contained in Oort [**19**, III.18.1], that, as indicated in a footnote in Jossen's thesis [**13**, Theorem 1.2.2 and footnote], extends to the general case once more recent existence results for the dual abelian scheme are taken into account. For more details on these existence results, see e.g. Faltings and Chai [**7**, pages 2-5]. Also in Faltings and Chai [**7**, page 9] one can find a single sentence asserting that the extensions of an abelian scheme A by a torus T are classified by $\mathrm{Hom}(\mathcal{X}(T), A^\vee)$ where $\mathcal{X}(T)$ is the character group of T, which is a more general statement that reduces to the version of Barsotti–Weil above when $T = \mathbb{G}_m$.

2.5. Jacobian varieties

In this section we recall some results from the theory of Jacobians over algebraically closed fields.

2.5.1. The Rosati Involution. Let k be an algebraically closed field, X/k a smooth proper curve, and J/k its Jacobian variety, which, by Theorem 2.17 is an abelian variety. For any closed point $P_0 \in X$ there exists a unique map $f_{P_0}\colon X \to J$ sending a k–point P to the divisor class of $(P) - (P_0)$. By pullback of invertible sheaves we obtain a map $f_{P_0}^* : J^\vee \to J$, and a classic result says that this map is independent of the base point P_0 and is an isomorphism $J^\vee \to J$ (see, e.g., [**17**, Lemma III.6.9]). We will denote the inverse of $-f_{P_0}^*$ by λ (or λ_X if there is ambiguity as to the curve we are working with) so that $-f_{P_0}^*$ is equal to λ^{-1} for any P_0. We will sometimes refer to λ as the canonical principal polarization.

Using λ we define an involution on $\mathrm{End}(J)$,

$$\psi \mapsto \psi^\dagger\colon = \lambda^{-1}\psi^\vee\lambda$$

where

$$\psi^\vee\colon J^\vee \to J^\vee$$

is the dual map to ψ given by pullback of invertible sheaves by ψ. This is called the Rosati involution and $\psi^\dagger$ is called the Rosati dual of ψ. The Rosati involution is linear, reverses the order of composition, and for any $\psi \in \mathrm{End}(J)$, $\psi^{\dagger\dagger} = \psi$ (see, e.g., [**17**, Section I.14]). We will call ψ symmetric if $\psi^\dagger = \psi$ and antisymmetric if $\psi^\dagger = -\psi$. For any ψ we will denote by $\bar{\psi} = \psi - \psi^\dagger$ its antisymmetrization. antisymmetric endomorphisms will play an important role in Chapter 4.

2.5.2. The Weil Pairing. We now define the Weil pairing, following Milne [**17**, Section I.13] where more details can be found. Fix k an algebraically closed field. For any abelian variety over k and any n coprime to the characteristic of k, there is a natural pairing $e_n\colon A[n] \times A^\vee[n] \to \mu_n$ (where here we mean the sets of n torsion in the k–points) defined as follows: if $y \in A^\vee[n]$ is represented by a divisor D, then n_A^*D (where n_A is multiplication by n on A) is the divisor of a function g on A, and we define $e_n(x,y) = g/(g \circ \tau_x)$ where τ_x is translation by x (one shows this to be a constant value contained in μ_n). The Weil pairing is skew-symmetric and non-degenerate (in the sense that if $e_n(x,y) = 1$ for all $y \in A^\vee[n]$ then x is the identity of A).

On the Jacobian J of a smooth proper connected curve X/k, we obtain the Weil–pairing on $J[n]\times J[n]$ from the Weil pairing on $J[n]\times J^\vee[n]$ by composition in the right component with the canonical principal polarization λ, and by abuse of notation we will continue to write this as $e_n : J[n] \times J[n] \to \mu_n$.

Lemma 2.21. *If $\psi \in \mathrm{End}(J)$ then $e_n(x, \psi^\dagger(y)) = e_n(\psi(x), y)$.*

Proof. Let $y \in J[n]$. Then $\lambda(\psi^\dagger(y)) = \psi^*\lambda(y)$ is represented by ψ^*D' where D' represents $\lambda(y)$. Then $n^*\psi^*D' = (\psi \circ n)^*D' = (n \circ \psi)^*D' = \psi^*n^*D'$, so if g is a function with $\mathrm{Div} g = n^*D'$ then $\mathrm{Div} g \circ \psi = n^*\psi^*D'$. Then for $x \in J[n]$, we see

$$\begin{aligned} e_n(x, \psi^\dagger(y)) =& g \circ \psi / g \circ \psi \circ \tau_x \\ =& g/g \circ \tau_{\psi(x)} \\ =& e_n(\psi(x), y) \end{aligned}$$

the equality on the second line following from the fact that these functions are constant. □

In other words, † is an adjoint operator for e_n.

Finally, we note that one can also give a description of the Weil pairing on $J[n] \times J[n]$ in terms of divisors on X. Namely, if $x = [D_x]$ and $y = [D_y]$ where D_x and D_y are divisors on X with disjoint support, and if $nD_x = \mathrm{Div} f_x$ and $nD_y = \mathrm{Div} f_y$ then

$$e_n(x,y) = \frac{f_x(D_y)}{f_y(D_x)}.$$

For a proof of this fact, see Theorem 1 and the remarks afterwards in [**12**].

2.5.3. Endomorphisms of the Jacobian. Let C and C' be smooth proper curves over an algebraically closed field k with Jacobians J and J', respectively.

Definition 2.22. A *divisorial correspondence* on $C \times C'$ is an element of

$$\mathrm{Corr}(C \times C') := \mathrm{Pic}(C \times C')/(p_C^* \mathrm{Pic}(C) \cdot p_{C'}^* \mathrm{Pic}(C'))$$

There is a natural bijection between divisorial correspondences on $C \times C'$ (that is, invertible sheaves $C \times C'$ considered up to equivalence by pullbacks of invertible sheaves on C and C') and the group $\mathrm{Hom}(J, J')$.

To any invertible sheaf $\mathcal{L}$ on $C \times C'$ we associate the morphism $\Phi_{\mathcal{L}}$ that sends a degree 0 divisor on C

$$\sum_{i=1}^{k} n_i P_i$$

to the bundle on C'

$$(P_1 \times id)^* \mathcal{L}^{n_k} \otimes \cdots \otimes (P_k \times id)^* \mathcal{L}^{n_k}$$

which is of degree 0 by virtue of the fact that the degree of these pullbacks onto C' is constant.

Proposition 2.23. *The map* $\mathcal{L} \mapsto \Phi_{\mathcal{L}}$ *from* $\mathrm{Corr}(C \times C')$ *to* $\mathrm{Hom}(J, J')$ *is a bijection.*

Proof. For details, see e.g. [**17**, Corollary III.6.3]. □

Proposition 2.24. *Let k be an algebraically closed field and let C/k and C'/k be smooth projective curves with Jacobian J and J' respectively. Any morphism $\psi \colon J \to J'$ can be written as a sum*

$$\psi = \sum_i \alpha_{i*} \gamma_i^*$$

where α_i and γ_i are non constant morphisms of smooth proper curves $Y_i \to C'$ and $Y_i \to C$, respectively. Furthermore if $C = C'$ then given any such representation, we have the representation

$$\psi^\dagger = \sum_i \gamma_{i*} \alpha_i^*$$

and thus the antisymmetrization of ψ can be represented as

$$\bar{\psi} = \psi - \psi^\dagger = \sum_i (\alpha_{i*} \gamma_i^* - \gamma_{i*} \alpha_i^*)$$

Proof. Consider the divisorial correspondence given by the sheaf $\mathcal{O}(D)$ associated to the divisor $D = A$ where A is an irreducible curve in $C \times C'$ not equal to a fiber of the form $\{P\} \times C'$ or $C \times \{P'\}$. If $Y \xrightarrow{\pi} A$ is the normalization of A and α is the composition $\pi_{C'} \circ \pi \colon Y \to C'$ and γ is the composition $\pi_C \circ \pi \colon Y \to C$ then the map $J \to J'$ defined by D is given on divisors by $\alpha_* \gamma^*$. Note that α and γ are both non-constant morphisms of smooth proper curves.

We note that any correspondence can be given by a sheaf $\mathcal{O}(D)$ where D is an effective divisor whose support does not contain any fibers of the form $\{P\} \times C'$ or $C \times \{P'\}$ for $P \in C$ or $P' \in C'$ closed points. Indeed, it suffices to show that any correspondence can be given by D effective since removing the fibers of this form do not change the class of the correspondence. To see that any correspondence can be given by an effective divisor, observe that for any $P \in C$, $P' \in C'$ the sheaf $\mathcal{O}(\{P\} \times C + \{P'\} \times C')$ is ample (as the tensor product of the pullback of ample sheaves on C and C' — see, e.g., [**10**, Exercise II.5.12]), and thus we can tensor a sheaf $\mathcal{L}$ with a sufficient power of $\mathcal{O}(\{P\} \times C + \{P'\} \times C')$ to obtain a sheaf giving the same correspondence as $\mathcal{L}$ and admitting a non-zero global section, which gives the desired effective divisor D.

Now, given any morphism $\psi \colon J \to J'$, we can describe it as the morphism associated to the correspondence $\mathcal{O}(D_0)$ for some effective divisor $D_0 = \sum n_i A_i$ on $C \times C'$ as above. Then ψ can be described on divisors as the map

$$\sum n_i \alpha_{i*} \gamma_i^*$$

where $\gamma_i \colon Y_i \to C$ and $\alpha_i \colon Y_i \to C'$ are non-constant morphisms of smooth proper curves associated to A_i as before.

Consider now a single curve C with Jacobian J and the ring $\mathrm{End}(J)$. We can describe the action of the Rosati dual in terms of correspondences. Indeed, since an endomorphism is given by a divisorial correspondence on $C \times C$, it is clear that we can obtain a duality on $\mathrm{End}(J)$ by swapping the roles of the two copies of C. This is, in fact, the same as the Rosati dual (cf. Birkenhake and Lange [**3**, Proposition 11.5.3] for a proof over $\mathbb{C}$). In particular, if $\psi\colon J \to J$ is an endomorphism that can be written in the form $\alpha_*\gamma^*$ where α and γ are both maps $Y \to C$ for Y another smooth proper curve over k, then $\psi^\dagger = \gamma_*\alpha^*$. This proves the final statement of the proposition. □

Remark. The key point here is that we can produce a lift of ψ to a morphism that is defined already at the level of divisors and such that the Rosati dual can be lifted in a compatible way.

CHAPTER 3

Pinching, line bundles, and $\mathbb{G}_m$–extensions

In this chapter we use pinching to describe some families of $\mathbb{G}_m$–extensions of the Jacobian of a smooth proper curve over an algebraically closed field.

3.1. Amalgamated sums and pinching

In this section we recall some results of Ferrand [**8**] on amalgamated sums and pinchings of closed schemes, and then develop some complements on pinchings of flat families.

In addition to Ferrand [**8**], other useful references include Schwede [**21**] and Demazure and Gabriel [**6**, III.2.3].

3.1.1. Definitions and basic results.

Suppose

$$\begin{array}{ccc} Z & \xrightarrow{f_Y} & Y \\ \big\downarrow{\scriptstyle f_X} & & \\ X & & \end{array}$$

is a diagram of morphisms of ringed spaces. The amalgamated sum of this diagram is the ringed space

$$(X \sqcup_Z Y, \mathcal{O}_{X\sqcup_Y Z})$$

described as follows: $X \sqcup_Z Y$ is the almagamated sum of X and Y as topological spaces over Z, formed by taking the disjoint union $X \sqcup Y$ and quotienting by the equivalence relation generated by the relations $f_Y(z) \sim f_X(z)$ for all $z \in Z$ (we remark that the definition stated by Ferrand in [**8**, Scolie 4.3.a.i] is incorrect because it claims that these are the only relations, however, this does not seem to pose any problems in his other results). There are natural maps of sets $g_X : X \to X \sqcup_Z Y$ and $g_Y : Y \to X \sqcup_Z Y$ and the topology on $X \sqcup_Z Y$ is the quotient topology — that is, a set U in $X \sqcup_Z Y$ is open if and only if both $g_X^{-1}(U)$ and $g_Y^{-1}(U)$ are open (i.e. it is the strongest topology such that g_X and g_Y are both continuous). Denote by $f_{X\sqcup_Z Y} : Z \to X \sqcup_Z Y$ the map $g_X \circ f_X = g_Y \circ f_Y$. The structure sheaf $\mathcal{O}_{X\sqcup_Z Y}$ and maps $g_X^{\#}$ and $g_Y^{\#}$ are defined for an open set $U \subset X \sqcup_Z Y$ as the fibered product making the following diagram of rings cartesian,

$$\begin{array}{ccc} \mathcal{O}_{X\sqcup_Y Z}(U) & \xrightarrow{g_Y^{\#}} & \mathcal{O}_Y(g_Y^{-1}(U)) \\ \big\downarrow{\scriptstyle g_X^{\#}} & & \big\downarrow{\scriptstyle f_Y^{\#}} \\ \mathcal{O}_X(g_X^{-1}(U)) & \xrightarrow{f_X^{\#}} & \mathcal{O}_Z(f_{X\sqcup_Z Y}^{-1}(U)) \end{array}$$

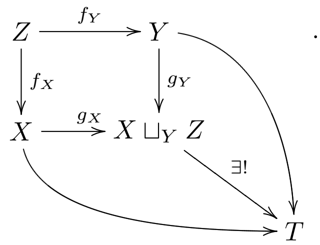
i.e. $\mathcal{O}_{X\sqcup_Y Z}$ is the sheaf fibered product $g_{X*}\mathcal{O}_X \times_{f_{X\sqcup_Z Y*}\mathcal{O}_Z} g_{Y_*}\mathcal{O}_Y$. Equivalently,

$$\mathcal{O}_{X\sqcup_Y Z}(U) = \{(x, y) \in g_{X*}\mathcal{O}_X(U)) \times g_{Y*}\mathcal{O}_Y(U) \mid f_X^{\#}(x) = f_Y^{\#}(y)\}$$

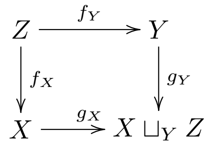
The amalgamated sum is a *fibered co-product* (or *push-out*) in the category of ringed spaces: one can check that it satisfies the universal property that for any commutative diagram of ringed spaces

$$\begin{array}{ccc} Z & \xrightarrow{f_Y} & Y \\ \big\downarrow{\scriptstyle f_X} & & \big\downarrow \\ X & \longrightarrow & T \end{array}$$

there exists a unique morphism $X \sqcup_Z Y \to T$ making the following diagram commute



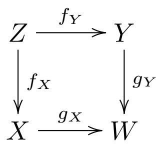
We also say that the commutative diagram



is *co-cartesian* in the category of ringed spaces.

Suppose now that X, Y, and Z are locally ringed spaces and the morphisms are of locally ringed spaces. If the amalgamated sum $X \sqcup_Z Y$ is also a locally ringed space, then it is a fibered co-product in the category of locally ringed spaces (one verifies that if morphisms $X \to T$ and $Y \to T$ are local then the induced morphism $X \sqcup_Z Y \to T$ must also be local). In fact, this is always the case, as the following theorem shows.

Theorem 3.1. *Let $f_X : Z \to X$ and $f_Y : Z \to Y$ be morphisms of locally ringed spaces and let $W = X \sqcup_Z Y$ be the amalgamated sum in the category of ringed spaces making the following diagram co-cartesian*



Then W is locally ringed and the morphisms g_X and g_Y are morphisms of locally ringed spaces.

Proof. The proof depends on the following lemma.

Lemma 3.2. *Let $(a,b) \in \mathcal{O}_W(W) = \mathcal{O}_X(X) \times_{\mathcal{O}_Z(Z)} \mathcal{O}_Y(Y)$ and let $w \in W$. The following are equivalent:*

1. *There exists $x \in g_X^{-1}(w)$ such that a_x is invertible in $\mathcal{O}_{X,x}$ or there exists $y \in g_Y^{-1}(w)$ such that b_y is invertible in $\mathcal{O}_{Y,y}$.*
2. *For all $x \in g_X^{-1}(w)$, a_x is invertible in $\mathcal{O}_{X,x}$ and for all $y \in g_Y^{-1}(w)$, a_y is invertible in $\mathcal{O}_{Y,y}$.*

Proof. (of lemma). The direction (2) $\implies$ (1) is trivial after noting that at least one of the sets $g_X^{-1}(w)$ and $g_Y^{-1}(w)$ is nonempty, and so we prove (1) $\implies$ (2).

We claim that for any $z \in Z$, $a_{f_X(z)}$ is invertible in $\mathcal{O}_{X,f_X(z)}$ if and only if $b_{f_Y(z)}$ is invertible in $\mathcal{O}_{Y,f_Y(z)}$. Indeed, if $a_{f_X(z)}$ is invertible then $f_X^{\#}(a)_z$ is invertible in $\mathcal{O}_{Z,z}$. But $f_X^{\#}(a) = f_Y^{\#}(b)$ and since the morphism $f_Y^{\#}$ is local, $f_Y^{\#}(b)_z$ invertible implies $b_{f_Y(z)}$ is invertible. The other direction follows by symmetry. Now, since the equivalence relation giving W from $X \sqcup Y$ is generated by the relations of the form $f_X(z) \sim f_Y(z)$ for $z \in Z$, we see that if $x \in g_X^{-1}(w)$ and $y \in g_Y^{-1}(w)$ then x and y are connected by a finite chain of such relations and thus a_x is invertible if and only if b_y is invertible, and similarly for x and x' both in $g_X^{-1}(w)$ and y and y' both in $g_Y^{-1}(w)$. The result follows. □

We now prove the theorem. Let $w_0 \in W$. We want to show that the ring $\mathcal{O}_{W,w_0}$ is local. So, let I be the ideal consisting of all elements of $\mathcal{O}_{W,w_0}$ that can be represented by (U,a,b) for $U \subset W$ open and $(a,b) \in \mathcal{O}_W(U)$ such that for all $x \in g_X^{-1}(w_0)$, $a_x \in \mathfrak{m}_x$ and for all $y \in g_Y^{-1}(w_0)$, $b_y \in \mathfrak{m}_y$. We will show this ideal is maximal by showing anything outside of I is invertible. So, let $(U,a,b) \notin I$. Then, applying the lemma (note that U is the amalgamated sum of its preimages so we can apply the lemma to it), we see that for all $x \in g_X^{-1}(w_0)$, a_x is invertible and for all $y \in g_Y^{-1}(w_0)$, b_y is invertible. Let

$$V = \{w \in U \mid \forall x \in g_X^{-1}(w),\, a_x \notin \mathfrak{m}_x \text{ and } \forall y \in g_Y^{-1}(w),\, b_y \notin \mathfrak{m}_y\}$$

Then by the above $w_0 \in V$, and we claim that V is open. Indeed, suppose $x \in f_X^{-1}(V)$. Then $a_x \notin \mathfrak{m}_x$, and thus there is an open neighborhood $N_x \subset f_X^{-1}(U)$ of x such that $a_{x'}$ is invertible for all $x' \in N_x$. But then by the lemma, $f_X(x') \in V$, and we see $N_x \subset f_X^{-1}(V)$ and thus $f_X^{-1}(V)$ is open. A symmetric argument shows $f_Y^{-1}(V)$ is also open, and thus by definition V is an open subset of W. It remains to see that $(a,b)|_V$ is invertible. But indeed, $a|_{f_X^{-1}(V)}$ is invertible and $b|_{f_Y^{-1}(V)}$ is invertible (by definition of V they are both everywhere locally invertible), and thus we obtain an inverse for (U,a,b) in $\mathcal{O}_{W,w}$ represented by $(V,(a|_{f_X^{-1}(V)})^{-1},(b|_{f_Y^{-1}(V)})^{-1})$. This shows $\mathcal{O}_{W,w_0}$ is local with maximal ideal I. That the morphisms f_X and f_Y are local morphisms follows immediately from the definition of I. □

Furthermore, if X, Y, and Z are schemes and the amalgamated sum $X \sqcup_Z Y$ is also a scheme then $X \sqcup_Z Y$ is a fibered co-product in the category of schemes. However, the amalgamated sum of schemes in the category of ringed spaces is not necessarily a scheme, as the following example shows.

Example 3.3. Let k be a field and let U be $\mathbb{A}_k^1 \backslash \{0\}$. If j is the inclusion $U \to \mathbb{A}_k^1$ and f is the structure morphism $U \to \mathrm{Spec} k$ then we obtain an amalgamated sum

$$\begin{array}{ccc} U & \xrightarrow{f} & \mathrm{Spec} k \\ \downarrow{\scriptstyle j} & & \downarrow \\ \mathbb{A}_k^1 & \longrightarrow & \mathbb{A}_k^1 \sqcup_U \mathrm{Spec} k \end{array} \quad .$$

This amalgamated sum as a topological space is a two point set with one closed point which is not open (the image of 0) and one open point which is not closed (the image of U). The stalk at either point is equal to k, and the global sections are also equal to k. In particular, it cannot be a scheme since the only open containing the closed point is the entire space but since $\mathbb{A}_k^1 \sqcup_U \mathrm{Spec} k$ has two points, it is not equal to $\mathrm{Spec} k$, and thus the closed point is not contained in an open affine.

If a fibered co-product exists in a category, then the universal property implies that it is unique up to unique isomorphism. Thus if we restrict ourselves to schemes and the amalgamated sum $X \sqcup_Z Y$ is also a scheme then it is the unique co-product in the category of schemes. However, if $X \sqcup_Z Y$ is not a scheme then it is still possible for there to be a fibered co-product in the category of schemes, but it will not be equal to $X \sqcup_Z Y$ and thus will not be a fibered co-product in the category of ringed spaces, as the following example illustrates.

Example 3.4. Taking up again the notations of Example 3.3, we note that the commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{f \circ j} & \mathrm{Spec} k \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle id} \\ \mathbb{A}_k^1 & \xrightarrow{f} & \mathrm{Spec} k \end{array}$$

is co-cartesian in the category of schemes over k. Indeed, suppose we have a morphism from $\mathbb{A}_k^1$ to a scheme T/k whose restriction to U factors through $\mathrm{Spec} k$. The image of $\mathrm{Spec} k$ in T is a closed point with residue field k, and since the inverse image of a closed set in $\mathbb{A}_k^1$ must be closed and the inverse image of this closed point already contains U, it must be all of $\mathbb{A}_k^1$. Thus the map factors through a closed subscheme of T with a single point and residue field k. But since $\mathbb{A}_k^1$ is reduced this closed subscheme is $\mathrm{Spec} k$, as desired.

In fact, when the amalgamated sum of schemes is a scheme, then it is a fibered co-product in the category of schemes that satisfies a particularly nice property: namely, if U is an open subset of $X \sqcup_Z Y$ then $(U, \mathcal{O}_{X \sqcup_Z Y}|_U)$ is, by construction, the amalgamated sum of $f_X^{-1}(U)$ and $f_Y^{-1}(U)$ over $f_{X \sqcup_Z Y}^{-1}(U)$, and thus if $X \sqcup_Z Y$ is a scheme it gives a co-product such that each open subset is the co-product of its inverse images.

We are particularly interested in the amalgamated sum (in the category of ringed spaces) of schemes in the case of a diagram

$$\begin{array}{ccc} Z & \xrightarrow{q} & Z' \\ \downarrow{\scriptstyle \iota} & & \\ X & & \end{array}$$

where $\iota : Z \to X$ is a closed immersion and $q : Z \to Z'$ is affine and dominant. In this case, if $X' = X \sqcup_Z Z'$ is a scheme and the induced morphism $Z' \to X'$ is a closed immersion, we will call X' the *pinching* of X along Z by q and we will say that we can *pinch* X along Z by q or that the pinching of X along Z by q exists.

Remark 3.5. The terminology "pinching" is taken from [**8**], however, the definition given in [**8**] does not require the morphism $q : Z \to Z'$ to be dominant. Not only is this at odds with the geometric intuition of what a pinching should be, but it is also less convenient to work with because it confuses the roles of X and Z' in the construction — we do not want new points (or any more than necessary) to be appearing from Z', which we view as a scheme that the closed subscheme Z of X is pinched down to, but if q is not dominant then $X \sqcup_Z Z'$ has a non-empty open set outside of the image of X coming from $Z' - \overline{q(Z)}$. Furthermore, we do not lose any generality by requiring q to be dominant — if q is not dominant then we can replace X with $X \sqcup Z'$, Z with $Z \sqcup Z'$, and q with $q \sqcup \mathrm{id}$ to obtain the same ringed space now with a dominant morphism.

We will make extensive use of the following theorem taken from a combination of results of Ferrand [**8**] on the existence and properties of pinchings.

Theorem 3.6 (Ferrand [**8**, Théorèmes 5.6 and 7.1])**.** *Let $\iota : Z \to X$ be a closed immersion of schemes and let $q : Z \to Z'$ be a finite surjective morphism of schemes such that for every point $z' \in Z'$, the set $q^{-1}(z')$ is contained in an open affine of X. The pinching X' of X along Z by q exists. Denote by $\pi : X \to X'$ and $\iota' : Z' \to X'$ the natural maps so that we have a commutative diagram*

$$\begin{array}{ccc} Z & \xrightarrow{q} & Z' \\ \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \iota'} \\ X & \xrightarrow{\pi} & X' \end{array}$$

The morphism ι' is a closed immersion, and π is finite surjective and induces an isomorphism $X - Z \to X' - Z'$. The commutative diagram above is both cartesian and co-cartesian.

Furthermore, if Z, Z', and X are schemes over S and q and ι are morphisms over S then X' has a unique structure of a scheme over S making π and ι' morphisms over S. If X/S is separated (resp. proper) then X'/S is separated (resp. proper).

Proof. Except for the final remark, this statement is derived directly from Ferrand [**8**, Théorèmes 5.6 and 7.1]. The universal property of the fibered co-product gives the desired morphism $X' \to S$. If $Z \to Z'$ is surjective, then $X \to X'$ is surjective. If X/S is separated then since $X \times_S X \to X' \times_S X'$ is proper (it is the product of two proper maps) and since $Z \to Z'$ surjective implies $X \to X'$ is surjective, we see that the diagonal is a closed set and thus X' is separated. If $X \to S$ is proper then since for any base extension $T \to S$ the map $X'_T \to X_T$ is both finite and surjective, and since $X_T \to T$ is proper and thus a closed map we conclude that $X'_T \to T$ is as well and thus $X' \to S$ is universally closed and X' is proper (since we have already shown it to be separated). ☐

Example 3.7. Here are some examples of different types of pinchings:

(1) *Glueing two schemes along a closed subscheme.* If $Z \xrightarrow{\iota_X} X$ and $Z \xrightarrow{\iota_Y} Y$ are closed immersions then we can glue together X and Y along Z. This corresponds to the pinching

$$\begin{array}{ccc} Z \sqcup Z & \xrightarrow{\mathrm{id} \sqcup \mathrm{id}} & Z \\ \downarrow{\scriptstyle \iota_X \sqcup \iota_Y} & & \downarrow \\ X \sqcup Y & \longrightarrow & X' \end{array}$$

(2) *A nodal cubic.* We obtain a nodal cubic by glueing two distinct points in the affine line over a field k. To make the calculation work out nicely we take $\operatorname{char} k \neq 2$. In this case we can take the pinching X' of X along Z over q where $X = \operatorname{Spec} k[x]$, $Z = \operatorname{Spec} k[x]/I$ with $I = (x+1)(x-1)$, $Z' = \operatorname{Spec} k$, $\iota \colon Z \to X$ is the natural closed immersion, and $q : Z \to Z'$ is the structure morphism, which collapses the two points $x = -1$ and $x = 1$ to a single point. Then, as in the description of the affine case following this example, X' is the spectrum of the subring A of $k[x]$ of f such that $f(-1) = f(1)$, and the map $k[u, v] \to A$ sending u to $(x + 1)(x - 1)$ and v to $x(x + 1)(x - 1)$ is a surjection with kernel $v^2 = u^3 - u^2$, and thus X' is the nodal cubic defined by this equation.

(3) *A cuspidal cubic.* We obtain a cuspidal cubic by pinching a point with nilpotents in the affine line over a field. Explicitly, take the pinching X' of X along Z over q where $X = \operatorname{Spec} k[x]$, $Z = \operatorname{Spec} k[x]/I$ with $I = (x^2)$, $Z' = \operatorname{Spec} k$, $\iota \colon Z \to X$ is the natural closed immersion and $q : Z \to Z'$ is the structure morphism. Then X' is the spectrum of the subring of A of $k[x]$ consisting of f such that $f'(0) = 0$ and the map $k[u, v] \to A$ sending u to x^2 and v to x^3 is an isomorphism with kernel $v^2 = u^3$, and thus X' is the cuspidal cubic defined by this equation.

The cubics of examples 2 and 3 will resurface again in Example 3.23 where we discuss pinchings in flat families and their relative Picard functors.

We conclude this section with a more detailed discussion of the amalgamated sum in the affine case. Suppose that

$$\begin{array}{ccc} \mathrm{Spec}B & \longrightarrow & \mathrm{Spec}B' \\ \downarrow & & \downarrow \\ \mathrm{Spec}A & \longrightarrow & \mathrm{Spec}A' \end{array}$$

is co-cartesian in the category of ringed spaces. Then it is co-cartesian in the category of affine schemes, and thus we obtain that in the opposite category

$$\begin{array}{ccc} A' & \longrightarrow & A \\ \downarrow & & \downarrow \\ B' & \longrightarrow & B \end{array}$$

is a cartesian diagram. Conversely, suppose we are given such a cartesian diagram of rings and furthermore that $A \to B$ is surjective (i.e. $\mathrm{Spec}B \to \mathrm{Spec}A$ is a closed immersion). Then by Ferrand [**8**, Théorème 5.1] (which is used to prove the more general case stated above as Theorem 3.6), the corresponding diagram of spectra is co-cartesian in the category of locally ringed spaces. Furthermore, in Lemme 1.3 and the subsequent discussion of [**8**] it is shown that a commutative diagram of rings

$$\begin{array}{ccc} A' & \xrightarrow{f} & A \\ \downarrow & & \downarrow \\ B' & \longrightarrow & B \end{array}$$

with $A \to B$ surjective is cartesian if and only if $A' \to B'$ is surjective with kernel I and f induces a bijection between I and $\ker A \to B$. Intuitively, we obtain such a diagram by taking a quotient B of A and letting A' be the pre-image in A of a sub-ring B' of B (at least when $B' \to B$ is injective). Putting this together we see

Theorem 3.8 (Ferrand [**8**, Lemme 1.3, Théorème 5.1])**.** *Let*

$$\begin{array}{ccc} \mathrm{Spec}B & \xrightarrow{q} & \mathrm{Spec}B' \\ \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \iota'} \\ \mathrm{Spec}A & \xrightarrow{\pi} & \mathrm{Spec}A' \end{array}$$

be a commutative diagram of affine schemes with ι a closed immersion. It is co-cartesian in the category of ringed spaces if and only if ι' is a closed immersion and $\pi^{\#}$ induces a bijection between $\ker(A' \to B')$ *and* $\ker(A \to B)$*.*

3.1.2. Amalgamated sums in flat families. In general the amalgamated sum does not commute with base change, as the following example shows:

Example 3.9. Let k be a field and let $A = k[\epsilon]$ ($\epsilon^2 = 0$). Consider the diagram of rings

$$\begin{array}{ccc} A[\epsilon']/\epsilon\epsilon' & \longleftarrow & A \\ \uparrow & & \uparrow{\scriptstyle x\mapsto 0} \\ A[\epsilon'] & \underset{x\mapsto\epsilon'\epsilon}{\longleftarrow} & A[x]/(x^2, x\epsilon) \end{array}$$

Where ϵ' is another nilpotent with $\epsilon'^2 = 0$. By Theorem 3.8, the corresponding diagram of affine schemes is co-cartesian in the category of locally ringed spaces. Consider now the map $A \to k$ given by $\epsilon \mapsto 0$. Tensoring with this map, we obtain the diagram

$$
\begin{array}{ccc}
k[\epsilon'] & & k[\epsilon] \\
\uparrow & & \uparrow{\scriptstyle x\mapsto 0} \\
k[\epsilon'] & \xleftarrow[x\mapsto 0]{} & k[x]/(x^2)
\end{array}
$$

and since the corresponding diagram of schemes has closed immersions for the vertical maps but the bottom arrow does not induce a bijection between their kernels, we see by Theorem 3.8 that it is not a co-product in the category of affine schemes.

We will show now, however, that if the schemes involved are flat over a base then amalgamated sums are preserved by base extension.

Lemma 3.10. *Let*

$$
\begin{array}{ccc}
\mathrm{Spec}B & \xrightarrow{q} & \mathrm{Spec}B' \\
\downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \iota'} \\
\mathrm{Spec}A & \xrightarrow{\pi} & \mathrm{Spec}A'
\end{array}
$$

be a commutative diagram of morphisms of schemes over Spec$R$ *with* ι *a closed immersion. Suppose furthermore that the diagram is co-cartesian in the category of ringed spaces. If* A*,* B*, and* B' *are flat over* R *then:*

(1) A' *is flat over* R
(2) *For any morphism of rings* $R \to \tilde{R}$*, the diagram obtained through base extension by* Spec$\tilde{R}$ *is co-cartesian in the category of ringed spaces.*

Proof. Let

$$
\begin{array}{ccc}
A' & \xrightarrow{f} & A \\
\downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\
B' & \xrightarrow{g} & B
\end{array}
$$

be the corresponding commutative diagram of morphisms of R–algebras.

By Theorem 3.8, p' is also surjective and f induces an isomorphism of R–modules between kerp' and ker$p$. By a standard result, if $0 \to M_1 \to M_2 \to M_3 \to 0$ is an exact sequence of R–modules, then all three are flat as soon as either M_2 and M_3 are flat or M_1 and M_3 are flat. One application of this shows that since A and B are flat, $\ker p$ is also flat. Since $\ker p$ and $\ker p'$ are isomorphic, and since B' is also flat, another application shows that A' is flat.

The diagram obtained by change of base can be written

$$
\begin{array}{ccc}
A' \otimes \tilde{R} & \xrightarrow{\tilde{f}} & A \otimes \tilde{R} \\
\downarrow{\scriptstyle \tilde{p}'} & & \downarrow{\scriptstyle \tilde{p}} \\
B' \otimes \tilde{R} & \xrightarrow{\tilde{g}} & B \otimes \tilde{R}
\end{array} .
$$

The vertical arrows remain surjective and, by flatness of B' and B, $\ker \tilde{p} = \ker p \otimes \tilde{R}$ and $\ker \tilde{p}' = \ker p' \otimes \tilde{R}$. Since f was an isomorphism $\ker p \to \ker p'$, $\tilde{f}$ is an isomorphism between these kernels. Thus, by Theorem 3.8, the diagram obtained by base extension is co-cartesian in the category of ringed spaces. □

Theorem 3.11. *Let*

$$
\begin{array}{ccc}
Z & \xrightarrow{q} & Z' \\
\downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \iota'} \\
X & \xrightarrow{\pi} & X'
\end{array}
$$

be a commutative diagram of schemes over S *with* ι *and* ι' *closed immersions and both* q *and* π *affine. Suppose furthermore that the diagram is co-cartesian in the category of ringed spaces. If* Z*,* Z'*, and* X *are flat over* S *then:*

(1) X' *is flat over* S

(2) *For any $T \to S$, the diagram obtained by base extension*

$$\begin{array}{ccc} Z_T & \longrightarrow & Z'_T \\ \downarrow & & \downarrow \\ X_T & \longrightarrow & X'_T \end{array}$$

is co-cartesian in the category of ringed spaces.

Proof. We recall that an open subscheme of X'_T is the fibered co-product (in the category of ringed spaces) of its inverse images in X_T and Z'_T over its inverse image in Z_T. Using this, we can show the universal property of a fibered co-product holds. Indeed, to construct a morphism out of X'_T we can construct it on an appropriate affine cover using Lemma 3.10 and the fact that q, ι, π, and ι' are all affine, and an application of the universal property shows that these will glue. Flatness of X' also follows directly from Lemma 3.10. □

3.2. Pinching and locally free sheaves

In this section we develop the theory of locally free sheaves of finite rank on pinchings. Our ultimate goal is to use these results to describe the relative Picard functor of pinchings in flat families, however, it is natural to work in the more general setting of locally free sheaves of finite rank.

The main result of this section is Theorem 3.13, which describes the category of locally free sheaves of finite rank on pinchings as a fiber product of the categories of locally free sheaves of finite rank on the schemes involved in the pinching. Ferrand [**8**, Section 2] gives a similar description of modules over fiber products of rings, also noting that analogous statements hold for quasi-coherent modules over appropriate pinchings, although he leaves it to the reader to make and verify such statements [**8**, Section 7.4]. Our approach is inspired from that of Ferrand, however, because it is not too difficult we give a full proof in the case of locally free sheaves of finite rank without citing any of Ferrand's results on quasi-coherent modules. Our proof has the advantage that parts of it take place over arbitrary locally ringed spaces, and it is in a certain sense more geometric, whereas Ferrand's proof, couched already in the language of rings, is forcibly algebraic in nature. In Section 3.2.1 we reformulate Ferrand's result on modules over fiber products of rings in the case of quasi-coherent modules on schemes, which recovers a stronger version of Theorem 3.13 as well as similar results for flat quasi-coherent modules.

Although we develop the theory in a much more general setting, the motivating example to keep in mind is that of line bundles on curves, where we would like line bundles on the pinching of a curve to be given by pinchings of line bundles on the original curve (in some sense to be made precise later).

Definition 3.12. Let $\mathcal{C}_0$, $\mathcal{C}_1$, and $\mathcal{C}_2$ be categories and let F_i, $1 \le i \le 2$ be a functor from $\mathcal{C}_i$ to $\mathcal{C}_0$. We denote by $\mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_2$ the *fiber product* of the categories $\mathcal{C}_1$ and $\mathcal{C}_2$ over $\mathcal{C}_0$ with respect to the functors F_i, defined as follows: The objects of $\mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_2$ are triplets (A_1, A_2, σ) where A_i is an object of $\mathcal{C}_i$ and $\sigma\colon F_1(A_1) \to F_2(A_2)$ is an isomorphism in $\mathcal{C}_0$. The morphisms in $\mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_2$ from one triplet (A_1, A_2, σ) to another triplet (A'_1, A'_2, σ') are the pairs (f_1, f_2) where $f_i : A_i \to A'_i$ are morphisms in $\mathcal{C}_i$ such that $\sigma' \circ F(f_1) = F(f_2) \circ \sigma$.

For a ringed space X, let $\mathcal{D}_X$ denote the category of $\mathcal{O}_X$–modules, let $\mathcal{D}_W^{\mathrm{lf}}$ denote the category of locally free of finite rank $\mathcal{O}_W$–modules, and let $\mathcal{D}_W^{\mathrm{lf};n}$ denote the category of locally free $\mathcal{O}_W$–modules of rank n. If we have morphisms of ringed spaces

$$\begin{array}{ccc} Z & \xrightarrow{f_Y} & Y \\ \downarrow{\scriptstyle f_X} & & \\ X & & \end{array}$$

then we can form the fiber product category $\mathcal{D}_X \times_{\mathcal{D}_Z} \mathcal{D}_Y$ over the functors $f_Y^* : \mathcal{D}_Y \to \mathcal{D}_Z$ and $f_X^* : \mathcal{D}_X \to \mathcal{D}_Z$. Restating the definition above in this special case, an object of $\mathcal{D}_X \times_{\mathcal{D}_Z} \mathcal{D}_Y$ is a triplet $(\mathcal{L}_X, \mathcal{L}_Y, \sigma)$ where $\mathcal{L}_X$ is an object of $\mathcal{D}_X$, $\mathcal{L}_Y$ is an object of $\mathcal{D}_Y$, and σ is an isomorphism $f_X^*\mathcal{L}_X \to f_Y^*\mathcal{L}_Y$. A morphism from $(\mathcal{L}_X, \mathcal{L}_Y, \sigma)$ to $(\mathcal{L}'_X, \mathcal{L}'_Y, \sigma')$ is a pair of morphisms (α_X, α_Y), $\alpha_X : \mathcal{L}_X \to \mathcal{L}'_X$, $\alpha_Y : \mathcal{L}_Y \to \mathcal{L}'_Y$ such that $\sigma' \circ (f_X^*\alpha_X) = (f_Y^*\alpha_Y) \circ \sigma$.

Let Z, X, and Y be ringed spaces and $f_X : Z \to X$ and $f_Y : Z \to Y$ morphisms. Let $T = X \sqcup_Z Y$ be the amalgamated sum making the following diagram co-cartesian in the category of ringed spaces

$$\begin{array}{ccc} Z & \xrightarrow{f_Y} & Y \\ {\scriptstyle f_X}\downarrow & \searrow^{f_T} & \downarrow{\scriptstyle g_Y} \\ X & \xrightarrow{g_X} & T \end{array}$$

Denote by ξ the standard isomorphism of functors $\xi\colon f_X^* g_X^* \to f_Y^* g_Y^*$ obtained by adjunction and the identity of functors $f_{X*}g_{X*} = f_{T*} = f_{Y*}g_{Y*}$. Using ξ we define the functor $F\colon \mathcal{D}_T \to \mathcal{D}_X \times_{\mathcal{D}_Z} \mathcal{D}_Y$ by $\mathcal{L}_T \mapsto (g_X^*\mathcal{L}_T, g_Y^*\mathcal{L}_T, \xi_{\mathcal{L}_T})$.

We also define a functor $G\colon \mathcal{D}_X \times_{\mathcal{D}_Z} \mathcal{D}_Y \to \mathcal{D}_{X'}$ that under certain circumstances will serve as an inverse to F (as shown below in Theorem 3.13, the main result of this section). We define G as follows: Given $(\mathcal{L}_X, \mathcal{L}_Y, \sigma)$ an object of $\mathcal{D}_X \times_{\mathcal{D}_Z} \mathcal{D}_Y$, let $\mathcal{B} \overset{\text{def}}{=} f_{T*}f_Y^*\mathcal{L}_Y$. From the identity of functors

$$f_{T*} = g_{X*}f_{X*} = g_{Y*}f_{Y*}$$

we obtain a natural map $\alpha_Y : g_{Y*}\mathcal{L}_Y \to \mathcal{B}$ which is the pushforward by g_Y of the map $\mathcal{L}_Y \to f_{Y*}f_Y^*\mathcal{L}_Y$ obtained by adjunction from $\mathrm{id}\colon f_Y^*\mathcal{L}_Y \to f_Y^*\mathcal{L}_Y$. Similarly we obtain a map $\alpha_X : g_{X*}\mathcal{L}_X \to \mathcal{B}$ which is the pushforward by g_X of the map $\mathcal{L}_X \to f_{X*}f_Y^*\mathcal{L}_Y$ obtained by adjunction from the isomorphism $\sigma\colon f_X^*\mathcal{L}_X \to f_Y^*\mathcal{L}_Y$. Finally,

$$G(\mathcal{L}_X, \mathcal{L}_Y, \sigma) \overset{\text{def}}{=} g_{X*}\mathcal{L}_X \times_{\mathcal{B}} g_{Y*}\mathcal{L}_Y.$$

We note that $G(\mathcal{O}_X, \mathcal{O}_Y, c)$ is equal, by definition, to $\mathcal{O}_T$ where c is the canonical isomorphism $f_X^*\mathcal{O}_X \to \mathcal{O}_Z \to f_Y^*\mathcal{O}_Y$, and thus we obtain for any $(\mathcal{L}_X, \mathcal{L}_Y, \sigma)$ an action of $\mathcal{O}_T$ on $G(\mathcal{L}_X, \mathcal{L}_Y, \sigma)$, making it an $\mathcal{O}_T$–module. The following is the main result of this section:

Theorem 3.13. *Let X, Z, and Z' be schemes, let $\iota\colon Z \to X$ be a closed immersion and let $q\colon Z \to Z'$ be a finite surjective morphism such that for any $z' \in Z'$, $q^{-1}(z')$ is contained in an open affine of X. Let $\pi\colon X \to X'$ be the pinching of X in Z by q and $\iota'\colon Z' \to X'$ the associated closed immersion*

$$\begin{array}{ccc} Z & \xrightarrow{q} & Z' \\ {\scriptstyle \iota}\downarrow & & \downarrow{\scriptstyle \iota'} \\ X & \xrightarrow{\pi} & X' \end{array}$$

The functors F and G defined above are mutually quasi-inverse equivalences of categories between $\mathcal{D}^{\mathrm{lf}}_{X'}$ and $\mathcal{D}^{\mathrm{lf}}_X \times_{\mathcal{D}^{\mathrm{lf}}_Z} \mathcal{D}^{\mathrm{lf}}_{Z'}$, and between $\mathcal{D}^{\mathrm{lf},n}_{X'}$ and $\mathcal{D}^{\mathrm{lf},n}_X \times_{\mathcal{D}^{\mathrm{lf},n}_Z} \mathcal{D}^{\mathrm{lf},n}_{Z'}$ for any positive integer n.

Before proving Theorem 3.13, we will develop some preliminary results relating the functors F and G in a more general setting.

Lemma 3.14. *If*

$$\begin{array}{ccc} Z & \xrightarrow{f_Y} & Y \\ {\scriptstyle f_X}\downarrow & \searrow^{f_T} & \downarrow{\scriptstyle g_Y} \\ X & \xrightarrow{g_X} & T \end{array}$$

is a co-cartesian diagram of locally ringed spaces then the functor $G\colon \mathcal{D}_X \times_{\mathcal{D}_Z} \mathcal{D}_Y \to \mathcal{D}_T$ is the right adjoint of $F\colon \mathcal{D}_T \to \mathcal{D}_X \times_{\mathcal{D}_Z} \mathcal{D}_Y$.

Proof. (Of lemma). Suppose we have a morphism $\phi \in \mathrm{Hom}(\mathcal{L}', G(\mathcal{L}_X, \mathcal{L}_Y, \sigma))$. Since $G(\mathcal{L}_X, \mathcal{L}_Y, \sigma)$ is a cartesian product, this is equivalent to a commutative diagram

$$\begin{array}{ccc} \mathcal{L}' & \longrightarrow & g_{Y*}\mathcal{L}_Y \\ \downarrow & & \downarrow{\scriptstyle \alpha_Y} \\ g_{X*}\mathcal{L}_X & \xrightarrow{\alpha_X} & \mathcal{B} \end{array}$$

By adjunction, to give morphisms $\mathcal{L}' \to g_{Y*}\mathcal{L}_Y$ and $\mathcal{L}' \to g_{X*}\mathcal{L}_X$ is the same as to give a pair of morphisms $g_Y^*\mathcal{L}' \to \mathcal{L}_Y$ and $g_X^*\mathcal{L}' \to \mathcal{L}_X$. Thus it remains only to verify that the original morphisms after composition to $\mathcal{B}$ agree if and only if the diagram obtained by pullback of these morphisms by f_X^* and f_Y^*

$$\begin{array}{ccc} f_X^* g_X^* \mathcal{L}' & \xrightarrow{\xi_{\mathcal{L}'}} & f_Y^* g_Y^* \mathcal{L}' \\ \downarrow & & \downarrow \\ f_X^* \mathcal{L}_X & \xrightarrow{\sigma} & f_Y^* \mathcal{L}_Y \end{array}$$

commutes. But this diagram commutes if and only if the composed morphism $f_X^* g_X^* \mathcal{L}' \to f_X^* \mathcal{L}_X \to f_Y^* \mathcal{L}_Y$ gives the same map $\mathcal{L}' \to \mathcal{B}$ by adjunction (applied first to f_X then to g_X) as the map $\mathcal{L}' \to \mathcal{B}$ given by adjunction from $f_Y^* g_Y^* \mathcal{L}' \to f_Y^* \mathcal{L}_Y$ (adjunction applied first to f_Y then to g_Y). From the definitions of $\mathcal{B}$, α_X, and α_Y, this occurs if and only if the original diagram was commutative, as desired. □

We observe that both G and F commute with restriction in the following sense: if U is an open subset of T then U with the restricted structure sheaf is the amalgamated sum of its inverse images with their restricted structure sheaves. This defines functors from sheaves over T to sheaves over U and from sheaves on $X, Y,$ and Z to sheaves on the inverse images of X, Y, and Z, and the latter extend to a functor from the fibered product of categories on the full space to the fiber product of categories on the inverse images of U. By abuse of notation we denote all of these restriction functors by $|_U$. Then there are natural isomorphisms $F \circ |_U \cong |_U \circ F$ and $G \circ |_U \cong |_U \circ G$ where on one side F and G are defined for sheaves over T and on the other for sheaves over U. Furthermore, these isomorphisms preserve the adjunction in the natural sense.

Lemma 3.15. *If*

$$\begin{array}{ccc} Z & \xrightarrow{f_Y} & Y \\ {\scriptstyle f_X}\downarrow & \searrow^{f_T} & \downarrow{\scriptstyle g_Y} \\ X & \xrightarrow{g_X} & T \end{array}$$

is a co-cartesian diagram of locally ringed spaces then the morphism of adjunction id $\to GF$ *is an isomorphism of functors* $\mathcal{D}_T^{\mathrm{lf}} \to \mathcal{D}_T^{\mathrm{lf}}$.

PROOF. We claim first that $F(\mathcal{O}_T) = (g_X^* \mathcal{O}_T, g_Y^* \mathcal{O}_T, \xi_{\mathcal{O}_T})$ is isomorphic to the triplet $(\mathcal{O}_X, \mathcal{O}_Y, c)$ where c is the canonical isomorphism $f_X^* \mathcal{O}_X \to \mathcal{O}_Z \to f_Y^* \mathcal{O}_Y$. Indeed, there are canonical isomorphisms $c_X : g_X^* \mathcal{O}_T \to \mathcal{O}_X$ and $c_Y : g_Y^* \mathcal{O}_T \to \mathcal{O}_Y$, and we claim the diagram

$$\begin{array}{ccc} f_X^* g_X^* \mathcal{O}_T & \xrightarrow{\xi_{\mathcal{O}_T}} & f_Y^* g_Y^* \mathcal{O}_T \\ \downarrow{\scriptstyle f_X^* c_X} & & \downarrow{\scriptstyle f_Y^* c_Y} \\ f_X^* \mathcal{O}_X & \xrightarrow{c} & f_Y^* \mathcal{O}_Y \end{array}$$

commutes. This diagram commutes if and only if the diagram

$$\begin{array}{ccc} f_X^* g_X^* \mathcal{O}_T & \xrightarrow{\xi_{\mathcal{O}_T}} & f_Y^* g_Y^* \mathcal{O}_T \\ \downarrow{\scriptstyle f_X^* c_X} & & \downarrow{\scriptstyle f_Y^* c_Y} \\ f_X^* \mathcal{O}_X & & f_Y^* \mathcal{O}_Y \\ \downarrow & & \downarrow \\ \mathcal{O}_Z & \xrightarrow{\mathrm{id}} & \mathcal{O}_Z \end{array}$$

commutes where the bottom vertical arrows are the canonical isomorphisms. This commutes if and only if the two adjoint morphisms $\mathcal{O}_T \to f_{T*} \mathcal{O}_Z$ corresponding to the two columns are the same. But both of these adjoints are the arrow $f_T^{\#} : \mathcal{O}_T \to f_{T*} \mathcal{O}_Z$.

Now, by definition $\mathcal{O}_T = G(\mathcal{O}_X, \mathcal{O}_Y, c)$, and after the isomorphism above, the corresponding morphism $\mathcal{O}_T \to \mathcal{O}_T$ is the identity. Indeed, it is the morphism corresponding by adjunction to the map $(g_X^* \mathcal{O}_T, g_T^* \mathcal{O}_Y, \xi_{\mathcal{O}_T}) \to (\mathcal{O}_X, \mathcal{O}_Y, c)$, which is by definition the morphism $\mathcal{O}_T \to \mathcal{O}_T$ corresponding by the universal property of the fiber product to the two projection morphisms $\mathcal{O}_T \to g_{X*} \mathcal{O}_X$ and $\mathcal{O}_T \to g_{Y*} \mathcal{O}_Y$, which is the identity. Thus, the adjunction morphism $\mathcal{O}_T \to GF(\mathcal{O}_T)$ is an isomorphism.

Similarly, the adjunction morphism $\mathcal{O}_T^n \to GF(\mathcal{O}_T^n)$ is an isomorphism and thus the adjunction morphism is an isomorphism on free sheaves of finite rank. For the general case of a locally free sheaf $\mathcal{L}$

on $\mathcal{O}_T$, we observe that the adjunction morphism commutes with restriction to an open subset of T as described immediately before the statement of the lemma, and the result follows. □

Lemma 3.16. *If*

$$\begin{array}{ccc} Z & \xrightarrow{f_Y} & Y \\ {\scriptstyle f_X}\downarrow & \searrow^{f_T} & \downarrow{\scriptstyle g_Y} \\ X & \xrightarrow{g_X} & T \end{array}$$

is a co-cartesian diagram of locally ringed spaces then the adjoint morphism $FG(\mathcal{O}_X, \mathcal{O}_Y, c) \to (\mathcal{O}_X, \mathcal{O}_Y, c)$ is an isomorphism (where c is the canonical isomorphism $f_X^*\mathcal{O}_X \to f_Y^*\mathcal{O}_Y$).

Proof. We remark that $G(\mathcal{O}_X, \mathcal{O}_Y, c) = \mathcal{O}_T$ and the map $FG(\mathcal{O}_X, \mathcal{O}_Y, c) \to (\mathcal{O}_X, \mathcal{O}_Y, c)$ is given by the maps adjoint to $\mathcal{O}_T \to g_{X*}\mathcal{O}_X$ and $\mathcal{O}_T \to g_{Y*}\mathcal{O}_Y$. These are the canonical isomorphisms $g_X^*\mathcal{O}_T \to \mathcal{O}_X$ and $g_Y^*\mathcal{O}_T \to \mathcal{O}_Y$, and thus the map is an isomorphism. □

We are now ready to prove Theorem 3.13.

Proof. (Of Theorem 3.13). We need to show that F and G are mutually inverse equivalences of categories on locally free sheaves. On the one hand, it is clear that F maps $\mathcal{D}^{\text{lf}}_{X'}$ to $\mathcal{D}^{\text{lf}}_X \times_{\mathcal{D}^{\text{lf}}_Z} \mathcal{D}^{\text{lf}}_{Z'}$, and we have established that the adjunction gives an isomorphism of functors $GF \to \text{id}$ when restricted to locally free sheaves. To do so we first showed that it preserved free objects and then were able to conclude the result because we could always restrict locally to open subsets of the amalgamated sum X' where the object was free.

In Lemma 3.16 we established that FG is equivalent to the identity on free objects. Thus, in order to apply the same argument as before with GF to show that FG is equivalent to the identity on locally free objects, by the compatibility of F and G with restriction it suffices to show that for any object $(\mathcal{L}_X, \mathcal{L}_{Z'}, \sigma)$ of $\mathcal{D}^{\text{lf}}_X \times_{\mathcal{D}^{\text{lf}}_Z} \mathcal{D}^{\text{lf}}_{Z'}$ and any $x' \in X'$ that there exist an open $V \subset X'$ containing x' such that

$$(\mathcal{L}_X|_{g_X^{-1}(V)}, \mathcal{L}_{Z'}|_{g_{Z'}^{-1}(V)}, \sigma)$$

is isomorphic to

$$(\mathcal{O}^n_X|_{g_X^{-1}(V)}, \mathcal{O}^n_{Z'}|_{g_{Z'}^{-1}(V)}, c).$$

where here c is the nth tensor power of the canonical isomorphism. Note that this will also show that G maps $\mathcal{D}^{\text{lf}}_X \times_{\mathcal{D}^{\text{lf}}_Z} \mathcal{D}^{\text{lf}}_{Z'}$ to $\mathcal{D}^{\text{lf}}_{X'}$, and thus show that G and F are mutually quasi-inverse equivalences of categories.

We remark that since π is an isomorphism from $X - Z$ to $X' - Z'$, it remains only to consider neighborhoods of points $z' \in Z'$. We are given that for any $z' \in Z'$, all of the preimages $z_1, ..., z_m$ of z' in Z are contained in an open affine of X. Furthermore, at each of these pre-images $\mathcal{L}_X$ must be locally free of rank n equal to the rank of $\mathcal{L}_{Z'}$ at z' since $\iota^*\mathcal{L}_X \cong q^*\mathcal{L}_{Z'}$. The following lemma shows then that there exists an open $U \subset X$ containing $z_1, ..., z_m$ such that $\mathcal{L}_X|_U \cong \mathcal{O}_X|_U^n$.

Lemma 3.17. *Let $X = \text{Spec}A$ be an affine scheme and let $\mathfrak{p}_1, ..., \mathfrak{p}_k$ be prime ideals of A. Then for any locally free sheaf $\mathcal{L} = \tilde{M}$ on X that is free of rank n in neighborhoods of each of the $\mathfrak{p}_i$, there exists an open $U \subset \text{Spec}A$ containing $\mathfrak{p}_1, ..., \mathfrak{p}_k$ such that $\mathcal{L}|_U \cong \mathcal{O}_U^n$.*

Proof. (Of lemma) It suffices to prove the assertion when the $\mathfrak{p}_i$ are disjoint maximal ideals, since in the general case we can replace each $\mathfrak{p}_i$ with a maximal ideal containing it and throw out any duplicates (note that if it is rank n at $\mathfrak{p}_i$ it is rank n at any maximal ideal containing $\mathfrak{p}_i$). By the Chinese remainder theorem, $A \twoheadrightarrow \prod_{i=1}^k A/\mathfrak{p}_i$ and tensoring with M we see $M \twoheadrightarrow \prod_{i=1}^k M/\mathfrak{p}_i M$. Since each $M/\mathfrak{p}_i M$ is an n dimensional vector space over $A/\mathfrak{p}_i$, we can take $m_1, ..., m_n \in M$ whose image in each $M/\mathfrak{p}_i M$ is a basis. We claim that these m_j form a basis for $\mathcal{L}$ in a neigborhood of each $\mathfrak{p}_i$. Indeed, we can pick some affine neighborhood $\text{Spec}B$ of $\mathfrak{p}_i$ where $\mathcal{L}$ is free of rank n and take a basis there $l_1, ..., l_n$. Then there is a matrix $D \in M_{n\times n}(B)$ such that D maps $(l_1, ..., l_n)$ to $(m_1, ..., m_n)$ and its determinant must be invertible in a neighborhood of $\mathfrak{p}_i$ since both $l_1, ..., l_n$ and $m_1, .., m_n$ reduce to bases of $M/\mathfrak{p}_i M$, and thus in that neigborhood $m_1, ..., m_n$ is a basis. Taking U to be the union of such neighborhoods over all $\mathfrak{p}_i$ we obtain an isomorphism $\mathcal{O}_U^n \tilde{\to} \mathcal{L}|_U$ via $(x_1, ..., x_n) \mapsto \sum x_i m_i$. □

Now, for such a U, $U' = X' - \pi(X \backslash U)$ is an open subset of X' containing $\iota'(z')$ (open because π is finite and thus a closed map) such that $\mathcal{L}_X|_{\pi^{-1}(U')} \cong \mathcal{O}_X|^n_{\pi^{-1}(U')}$. Since ι' is a closed immersion, by replacing U' with a smaller open set we also obtain that $\mathcal{L}_{Z'}|_{\iota'^{-1}(U')} \cong \mathcal{O}_{Z'}|^n_{\iota'^{-1}(U')}$, and furthermore we can also take U' to be affine. Replacing X, Z', and Z by the respective preimages of U', we reduce to the case where $\mathcal{L}_X = \mathcal{O}_X^n$, $\mathcal{L}_{Z'} = \mathcal{O}_{Z'}^n$, and all of the schemes involved are affine (recall that an open subset of the amalgamated sum in the category of ringed spaces is the amalgamated sum of its preimages). We now claim that in this case there is a neighborhood of z' and an automorphism of $\mathcal{O}_X^n$ that sends σ to c. Indeed, σ can be identified with an automorphism of $\mathcal{O}_Z^n$, and thus it suffices to see that this comes locally from an automorphism of $\mathcal{O}_X^n$. Since $Z \to X$ is a closed immersion of affine schemes and σ is given by a matrix in $\mathrm{Mat}_{n\times n}(\Gamma(\mathcal{O}_Z))$, there exists a matrix σ' in $\mathrm{Mat}_{n\times n}(\Gamma(\mathcal{O}_X))$ mapping to σ. The matrix σ' is not necessarily an automorphism of $\mathcal{O}_X^n$, but if we restrict to the open subset $V = D(\det \sigma')$ which contains Z (since σ is invertible), then it is, and on V we obtain the desired isomorphism between

$$(\mathcal{L}_X|_{g_X^{-1}(V)}, \mathcal{L}_{Z'}|_{g_{Z'}^{-1}(V)}, \sigma)$$

and

$$(\mathcal{O}_X^n|_{g_X^{-1}(V)}, \mathcal{O}_{Z'}^n|_{g_{Z'}^{-1}(V)}, c).$$

□

3.2.1. Quasi-coherent sheaves. Ferrand [**8**, Section 2] developed the theory of modules on fiber products of rings by defining functors S and T which, for affine schemes under the equivalence of categories between quasi-coherent sheaves and modules, correspond to our G and F (respectively). His Théorème 2.2 describes certain properties of these functors and, in certain cases, gives an equivalence of categories between modules on the fiber product of two rings and the fiber product of the categories of modules on the component rings. Ferrand [**8**, Section 7.4] also mentions that similar statements hold for quasi-coherent modules over schemes, however, he has chosen to "laisser au lecteur le soin d'énoncer et vérifier" such statements. We do so now.

For any scheme W we denote by $\mathcal{D}_W^{\mathrm{qc}}$ the category of quasi-coherent $\mathcal{O}_W$–modules.

Proposition 3.18. *Let*

$$\begin{array}{ccc} Z & \xrightarrow{f_Y} & Y \\ {\scriptstyle f_X}\downarrow & \searrow^{f_T} & \downarrow{\scriptstyle g_Y} \\ X & \xrightarrow{g_X} & T \end{array}$$

be a commutative diagram of morphisms of schemes, co-cartesian in the category of ringed spaces. If the maps f_X and g_Y are closed immersions and the maps g_X and f_Y are affine, then

1. *G and F induce functors between $\mathcal{D}_X^{\mathrm{qc}} \times_{\mathcal{D}_Z^{\mathrm{qc}}} \mathcal{D}_Y^{\mathrm{qc}}$ and $\mathcal{D}_T^{\mathrm{qc}}$ and the morphism of adjunction $FG \to \mathrm{id}$ is an isomorphism of functors (when restricted to these categories).*
2. *A quasi-coherent sheaf $\mathcal{L}$ on T is 0 if and only if $F(\mathcal{L})$ is 0.*
3. *The morphism of adjunction $\mathcal{L} \to GF(\mathcal{L})$ is surjective for any quasi-coherent sheaf $\mathcal{L}$ on T, and an isomorphism if $\mathcal{L}$ is also flat.*
4. *The functor G maps pairs of locally finitely generated modules in $\mathcal{D}_X^{\mathrm{qc}} \times_{\mathcal{D}_Z^{\mathrm{qc}}} \mathcal{D}_Y^{\mathrm{qc}}$ to locally finitely generated modules in $\mathcal{D}_T^{\mathrm{qc}}$. Furthermore, if $\mathcal{C}_\square$ (for $\square$ one of X, Y, Z or T) denotes the category of flat quasi-coherent modules (resp. flat locally finitely generated, resp. locally free of finite rank) then G and F are quasi-inverse equivalences of categories between $\mathcal{C}_X \times_{\mathcal{C}_Z} \mathcal{C}_Y$ and $\mathcal{C}_T$.*

Proof. In fact, F always maps a quasi-coherent sheaf to a pair of quasi-coherent sheaves because they are formed by pullback. Consider now a triplet $(\mathcal{L}_X, \mathcal{L}_Y, \sigma)$ where $\mathcal{L}_X$ and $\mathcal{L}_Y$ are quasi-coherent and let $\mathcal{L}_T \stackrel{\mathrm{def}}{=} G(\mathcal{L}_X, \mathcal{L}_Y, \sigma)$. Since the morphisms g_Y, g_X, and f_T are all affine, the pushforwards $g_{Y*}\mathcal{L}_Y$, $g_{X*}\mathcal{L}_X$, and $f_{T*}f_Y^*\mathcal{L}_Y = \mathcal{B}$ are all quasi-coherent, and thus the fiber product $\mathcal{L}_T = g_{Y*}\mathcal{L}_Y \times_{\mathcal{B}} g_{X*}\mathcal{L}_X$ is as well.

The remaining claims follow from the corresponding claims of Ferrand [**8**, Théorème 2.2] after observing that F and G commute with restriction and are identified with Ferrand's functors "T" and "S" by the equivalence of categories between quasi-coherent sheaves and modules. In order to obtain the result for modules that are locally free of finite rank, one must observe that this is equivalent to locally projective finitely generated (or flat locally of finite presentation). We make one further note: number (3) is stated differently here than in [**8**, Théorème 2.2] — in particular, our statement that the map is an isomorphism when $\mathcal{L}$ is flat is not contained in Ferrand's statement, however, it is apparent in the proof where the kernel of the map is identified with the image of a Tor functor that vanishes when the module is flat, and it is also used implicitly in the proof of his statement corresponding to (4). □

Remark. The locally free of finite rank case of proposition 3.18-(4) gives a more general version of Theorem 3.13.

3.2.2. Pinching and the Picard group. We now place ourselves in the situation of Theorem 3.13 and use this equivalence to examine $\mathrm{Pic}(X')$ and the map π^* from $\mathrm{Pic}(X')$ to $\mathrm{Pic}(X)$. The equivalence of categories gives an isomorphism of groups between $\mathrm{Pic}(X')$ and the set of isomorphism classes of triplets $(\mathcal{L}_X, \mathcal{L}_{Z'}, \sigma)$ (where the latter is a group under the natural multiplication $(\mathcal{L}_X, \mathcal{L}_{Z'}, \sigma) \cdot (\mathcal{L}'_X, \mathcal{L}'_{Z'}, \sigma') \to (\mathcal{L}_X \otimes \mathcal{L}'_X, \mathcal{L}_{Z'} \otimes \mathcal{L}'_{Z'}, \sigma \otimes \sigma')$), and the map π^* maps a triplet $(\mathcal{L}_X, \mathcal{L}_{Z'}, \sigma)$ to the isomorphism class of $\mathcal{L}_X$. Thus, the class of an invertible sheaf $\mathcal{L}_X$ is contained in the image of π^* if and only if the class of $\iota^*\mathcal{L}_X$ is in the image of q^*(viewed as a map from $\mathrm{Pic}(Z') \to \mathrm{Pic}(Z)$). Furthermore, the fiber of π^* over the class of an invertible sheaf $\mathcal{L}_X$ can be identified with the set of isomorphism classes of pairs $(\mathcal{L}_{Z'}, \sigma)$ where $\mathcal{L}_{Z'}$ is an invertible sheaf on Z' and σ is an isomorphism $\sigma : \iota^*\mathcal{L}_X \to q^*\mathcal{L}_{Z'}$, modulo the actions of $\mathcal{O}_X(X)^*$ and $\mathcal{O}_{Z'}(Z')^*$ (the automorphisms of $\mathcal{L}_X$ and $\mathcal{L}_{Z'}$).

3.2.3. Pinching in flat families and the relative Picard functor. We now want to put ourselves in a situation where we can apply the above description of the Picard group given by the equivalence of categories of Theorem 3.13 in order to describe the relative Picard functor of a pinching. So, let X, Z, and Z' be schemes over S satisfying the hypotheses of Theorem 3.6 with the added condition that the morphisms be morphisms over S. Applying Theorem 3.6, let $\pi \colon X \to X'$ be the pinching of X along Z by q and let $\iota' : Z' \to X'$ be the corresponding closed immersion, and note that X' is equipped with a canonical structure morphism $X' \to S$ such that π and ι' are both morphisms over S. We want to describe the relative Picard functor $\mathrm{Pic}_{X'/S}$ by describing its T–points. Thus, we also require that X, Z, and Z' be flat over S so that, by Theorem 3.11, X' is also flat and a base extension of X' is the pinching of the corresponding base extensions of X, Z, and Z'. This allows us to describe $\mathrm{Pic}(X' \times_S T)$, which will give us a nice description of $\mathrm{Pic}_{X'/S}(T)$ if we are in the situation where $\mathrm{Pic}_{X'/S}(T) = \mathrm{Pic}(X' \times_S T)/\mathrm{Pic}(T)$. We now introduce some further conditions to put ourselves in this situation. Suppose $f_{X*}\mathcal{O}_X = \mathcal{O}_S$ (where f_X is the structure morphism) holds universally and X/S admits a section. Proposition 2.9 implies that $\mathrm{Pic}_{X/S}(T) = \mathrm{Pic}(X \times_S T)/\mathrm{Pic}(T)$, and by composition X'/S also admits a section so that if we can show that also $f_{X'*}\mathcal{O}_{X'} = \mathcal{O}_S$ holds universally then again from Proposition 2.9 we will obtain the desired description of $\mathrm{Pic}_{X'/S}$.

Lemma 3.19. *If q is also flat then $f_{X'*}\mathcal{O}_{X'} = \mathcal{O}_S$ holds universally.*

Proof. Since $q : Z \to Z'$ is faithfully flat (q was already assumed surjective) we see that $\mathcal{O}_{Z'} \to q_*\mathcal{O}_Z$ is injective. By definition of the structure sheaf on $\mathcal{O}_{X'}$ and since $\mathcal{O}_{Z'} \to q_*\mathcal{O}_Z$ is injective, we obtain that $\mathcal{O}_{X'} \to \pi_*\mathcal{O}_X$ is injective. The morphism $O_S \to f_{X*}\mathcal{O}_X$ factors as

$$\mathcal{O}_S \to f_{X'*}\mathcal{O}_{X'} \to (f_{X'} \circ \pi)_*\mathcal{O}_X = f_{X*}\mathcal{O}_X$$

and since the second arrow is injective and the composition is the identity we see that $f_{X'*}\mathcal{O}_{X'} = \mathcal{O}_S$. Furthermore, since q will remain faithfully flat under base extension over S, the same argument holds under any base extension and thus $f_{X'*}\mathcal{O}_X = \mathcal{O}_S$ holds universally. □

Theorem 3.20. *In the situation of Theorem 3.13, if X, Z, and Z' are flat over S, $q : Z \to Z'$ is flat, both q and ι are morphisms over S, $f_{X*}\mathcal{O}_X = \mathcal{O}_S$ holds universally (where $f_X : X \to S$ is the structure morphism), and X/S admits a section, then*

$$\mathrm{Pic}_{X/S}(T) = \mathrm{Pic}(X_T)/\mathrm{Pic}(T)$$

and

$$\begin{aligned}\mathrm{Pic}_{X'/S}(T) =& \mathrm{Pic}(X'_T)/\mathrm{Pic}(T)\\ =& \{(\mathcal{L}_{X_T}, \mathcal{L}_{Z'_T}, \sigma) \,|\, \sigma \colon \iota^*\mathcal{L}_{X_T} \tilde{\to} q^*\mathcal{L}_{Z'_T}\}/\sim\end{aligned}$$

where $\mathcal{L}_{X_T}$ is an invertible sheaf of X_T, $\mathcal{L}_{Z'_T}$ is an invertible sheaf on Z'_T, and σ is an isomorphism. The relation $\sim$ is equivalence by isomorphism of triplets and by the subgroup of elements of the form $(f^*_{X_T}\mathcal{L}, f^*_{Z'_T}\mathcal{L}, \sigma)$ where $\mathcal{L}$ is an invertible sheaf on T and σ is the canonical isomorphism $\iota^* f^*_{X_T}\mathcal{L} \to q^* f^*_{Z'_T}\mathcal{L}$.

Furthermore, under this description the map π^* sends the class of a triplet $(\mathcal{L}_{X_T}, \mathcal{L}_{Z'_T}, \sigma)$ to the class of $\mathcal{L}_X$, and

$$\ker \pi^*(T) = \{(\mathcal{L}, \sigma) \,|\, \mathcal{L} \text{ an invertible line bundle on } Z'_T \text{ and } \sigma \colon \mathcal{O}_{Z_T} \tilde{\to} q^*\mathcal{L}\}/\sim$$

where $\sim$ is the relation $(\mathcal{L}, \sigma) \sim (\mathcal{L}', \sigma')$ if there exists an isomorphism $\mathcal{L} \to \mathcal{L}'$ sending σ to σ'. Under this description the embedding $\ker \pi^* \to \mathrm{Pic}_{X'/S}$ is given by $(\mathcal{L}, \sigma) \mapsto (\mathcal{O}_{X_T}, \mathcal{L}, \sigma \circ c)$ where c is the canonical isomorphism $c \colon \iota^*\mathcal{O}_{X_T} \tilde{\to} \mathcal{O}_{Z_T}$.

Proof. Following the discussion at the start of this section, the hypotheses and Lemma 3.19 imply that $\mathrm{Pic}_{X'/S}(T) = \mathrm{Pic}(X' \times_S T)/\mathrm{Pic}(T)$ and similarly for $\mathrm{Pic}_{X/S}$. Combining this with Theorem 3.13, we obtain the desired description. Since $X_T = X' \times_S T$ is, by Theorem 3.11, the pinching of X_T in Z_T by $q_T \colon Z_T \to Z'_T$, the map π^* is the map $\pi^* \colon \mathrm{Pic}(X'_T) \to \mathrm{Pic}(X_T)$ taken to the quotient by $\mathrm{Pic}(T)$. To obtain the description of the kernel, we observe that since X_T and X'_T both admit sections, the pullback maps $\mathrm{Pic}(T) \to \mathrm{Pic}(X_T)$ and $\mathrm{Pic}(T) \to \mathrm{Pic}(X'_T)$ are injective, and thus the kernel of π^* can now be identified with isomorphism classes of pairs $(\mathcal{L}_{Z'}, \sigma)$ with $\sigma : \mathcal{O}_{Z_T} \to q^*\mathcal{L}_{Z'}$ an isomorphism, since by hypothesis $\mathcal{O}_{X_T}(X_T)^*$ is $\mathcal{O}_T(T)^*$ and thus this action is already included in the isomorphisms of $\mathcal{L}_{Z'}$. □

Example 3.21. *Glueing of two flat families along a section.* In Example 3.7, we saw how to glue together two sub-schemes along a closed sub-scheme. We can do the same in the relative case by glueing along any flat closed subscheme. In particular, we can glue along sections. So, let X and Y be flat and separated over S and let $S \xrightarrow{\epsilon_X} X$ and $S \xrightarrow{\epsilon_Y} Y$ be two sections. Let W be the pinching of the diagram

$$\begin{array}{ccc} S \sqcup S & \xrightarrow{\mathrm{id} \sqcup \mathrm{id}} & S \\ \downarrow{\scriptstyle \epsilon_X \sqcup \epsilon_Y} & & \downarrow \\ X \sqcup Y & \longrightarrow & W \end{array} .$$

If $f_{X*}\mathcal{O}_X = \mathcal{O}_X$ and $f_{Y*}\mathcal{O}_Y = \mathcal{O}_Y$ hold universally then it is clear that $f_{W*}\mathcal{O}_W = \mathcal{O}_S$ holds universally and since W also admits a section, by Proposition 2.9 we obtain $\mathrm{Pic}_{W/S}(T) = \mathrm{Pic}(W_T)/\mathrm{Pic}(T)$. In this case $\mathrm{Pic}(W_T)$ is given by isomorphism classes of pairs $(\mathcal{L}_{X_T}, \mathcal{L}_{Y_T})$ such that $\epsilon^*_{X_T}\mathcal{L}_{X_T} \cong \epsilon^*_{Y_T}\mathcal{L}_{Y_T}$ (we can forget about the specific isomorphism because our conditions guarantee these always come from automorphisms in $X \sqcup Y$). In fact, the map $\mathrm{Pic}_{W/S} \to \mathrm{Pic}_{X/S} \times \mathrm{Pic}_{Y/S}$ defined on T points by sending $(\mathcal{L}_{X_T}, \mathcal{L}_{Y_T})$ to $\{\mathcal{L}_{X_T}\} \times \{\mathcal{L}_{Y_T}\}$ is an isomorphism: any element of $\mathrm{Pic}_{X/S}(T) \times_S \mathrm{Pic}_{Y/S}(T)$ can be represented by bundles with trivial pullback to S and thus it is surjective, and any pair $(\mathcal{L}_{X_T}, \mathcal{L}_{Y_T})$ mapping to the identity maps to $\mathrm{Pic}(T)$ in both components and thus since $\epsilon^*_{X_T}\mathcal{L}_{X_T} \cong \epsilon^*_{Y_T}\mathcal{L}_{Y_T}$ it is contained in the pullback of $\mathrm{Pic}(T)$ to $\mathrm{Pic}(W_T)$. Note that in this example we are not in the situation of Theorem 3.20 because $f_{X \sqcup Y*}\mathcal{O}_{X \sqcup Y} = \mathcal{O}_S \times \mathcal{O}_S$.

3.2.4. Pinching sections of relative curves. Let X be a smooth projective curve over an algebraically closed field k and suppose we have two disjoint closed points $P, Q \in X$. Then we can pinch the points P and Q together to obtain a new curve X' as in Theorem 3.6. Concretely, this gives the following diagram of schemes, co-cartesian in the category of ringed spaces:

$$\begin{array}{ccc} \mathrm{Spec} k \sqcup \mathrm{Spec} k & \longrightarrow & \mathrm{Spec} k \\ \downarrow{\scriptstyle P \sqcup Q} & & \downarrow \\ X & \xrightarrow{\pi} & X' \end{array}$$

and following Serre [**22**, Chapter 4, Proposition 4.1.2], we see that X' is a singular curve over k with one singular point, a node at the image of P and Q, and the map $X \xrightarrow{\pi} Y$ is the normalization of X' (cf. also Examples 3.7-2 and 3.7-3). More generally, we can perform a similar construction by pinching more than two points together or pinching together multiple groups of points. If X is a relative curve over S then we can also pinch disjoint sections over S, which in light of the compatibility of pinching and base change in flat families (Theorem 3.11) can be viewed as a family of pinchings parameterized by S. The rest of this section is dedicated to describing the behavior of the relative Picard functor in this situation.

Remark. Line bundles on pinchings of curves were first studied by Rosenlicht, who developed the theory of generalized Jacobians that is presented by Serre in [**22**]. Deligne [**5**, Section 10.3] has also developed the theory of 1–motives attached to singular and quasi-projective curves and studied their Picard groups in this context. We do not know of another work besides the present one that develops the theory explicitly in the relative case.

If X/S is a relative curve and $P_{i,j}$, $1 \le i \le n$ and $0 \le j \le m_i$ are pairwise non-intersecting sections $P_{i,j} \colon S \to X$ of X/S, then by the pinching of X/S in $\{P_{i,j}\}$ we mean the pinching obtained by for each $1 \le i_0 \le n$ collapsing all of the sections $\{P_{i_0,j}\}$ to a single section which we will call P_{i_0}. This is the pinching X' corresponding to the diagram

$$\begin{array}{ccc} \sqcup_i(\sqcup_j S_{ij}) & \xrightarrow{q} & \sqcup_i S_i \\ \downarrow \iota & & \downarrow \iota' \\ X & \xrightarrow{\pi} & X' \end{array}$$

where S_{ij} and S_i are indexed copies of S, the morphism ι maps S_{ij} by P_{ij} to X, the morphism q maps each S_{ij} to S_i by the identity, and the morphisms π and ι' are the induced maps so that, in particular, ι' maps S_i to X' by the new section P_i obtained by collapsing all of the P_{ij} with the same i. This pinching exists by Theorem 3.6.

Applying our results on pinching to this situation we obtain the following:

Theorem 3.22. *Let S be a scheme and let X/S be a smooth projective curve. Let $\{P_{i,j}\}$ $1 \leq i \leq n$ and $0 \leq j \leq m_i$ be pairwise non-intersecting sections of X. The pinching $\pi\colon X \to X'$ in the $\{P_{i,j}\}$ exists and is stable under base extension. In particular, the geometric fibers of X' are proper curves with exactly n singularities and normalization $\pi_s\colon X_s \to X'_s$. Furthermore, $\mathrm{Pic}^0_{X/S}$ is an abelian scheme, $\mathrm{Pic}^0_{X'/S}$ is a commutative group scheme, and there is an exact sequence of Zariski sheaves*

$$0 \longrightarrow G \longrightarrow \mathrm{Pic}^0_{X'/S} \xrightarrow{\pi^*} \mathrm{Pic}^0_{X/S} \longrightarrow 0$$

where G is a split torus over S of rank $N = \sum m_i$.

Remark. We consider only disjoint sections because in general if the sections are not disjoint then in Theorem 3.22 the kernel will not be a torus — cf. Example 3.23.

Proof. The first two parts of the statement follow from Theorems 3.6 and 3.11 after noting that by projectivity any finite number of points lying in a single fiber X_s are contained in an open affine of X.

We observe that projectivity gives us that $f_{X*}\mathcal{O}_X = \mathcal{O}_S$ holds universally, and thus we can apply Theorem 3.20. The kernel of $\pi^* : \mathrm{Pic}_{X'/S} \to \mathrm{Pic}_{X/S}$ is, on T–points, given by isomorphism classes of pairs $(\mathcal{L}, \sigma)$ where $\mathcal{L}$ is a line bundle on $\sqcup_i T_i$ and σ is an isomorphism $q^*\mathcal{L} \to \mathcal{O}_{\sqcup_{i,j}T_{i,j}}$. The existence of such a σ implies $\mathcal{L}$ is isomorphic to $\mathcal{O}_{\sqcup_i T_i}$, and so the kernel of π^* is given by pairs $(\mathcal{O}_{\sqcup_i T_i}, \sigma)$, $\sigma : q^*\mathcal{O}_{\sqcup_i T_i} \to \mathcal{O}_{\sqcup_{i,j}T_{i,j}}$ up to automorphism of $\mathcal{O}_{\sqcup_i T_i}$. Let c be the canonical such isomorphism — then any other isomorphism can be written as $u \cdot c$ where $u \in \Gamma(\mathcal{O}_{\sqcup_{i,j}T_{i,j}})^*$. In this way we obtain a canonical surjective morphism $f^*_{\sqcup_{i,j}S_{i,j}}\mathbb{G}_m \cong \mathbb{G}_m^{N+n} = \Pi_{i,j}\mathbb{G}_m \to \ker \pi^*$ (where $f_{\sqcup_{i,j}S_{i,j}}$ is the structure morphism $\sqcup_{i,j}S_{i,j} \to S$). A point $t = (t_{i,j}) \in \mathbb{G}_m^{N+n}(T)$ is in the kernel of this map if and only if for each i and each pair j and j', $t_{i,j} = t_{i,j'}$ (so that it comes from an automorphism of $\mathcal{O}_{\sqcup_i T_i}$). Thus, the kernel of this map is a torus of rank n and we see that $\ker \pi^*$ is a split torus of rank N. We must also show that $\ker \pi^*$ is contained in $\mathrm{Pic}^0_{X'/S}$. But at every geometric point $\bar{s}$ of S, $\ker\, \pi^*$ is also a torus, which is connected and contains the identity, and thus is contained in the identity component of $\mathrm{Pic}^0_{X'_{\bar{s}}/\bar{s}}$, and thus by definition $\ker \pi^* \subset \mathrm{Pic}^0_{X'/S}$.

We now observe that $\pi^* : \mathrm{Pic}_{X'/S} \to \mathrm{Pic}_{X/S}$ is surjective in the Zariski topology. Indeed, if $\mathcal{L}$ is a bundle on X_T, then for a fixed $t \in T$ and for each i and j there is an open neighborhood of t in $T_{i,j}$ where $P^*_{i,j}(\mathcal{L})$ is trivial. If U is the intersection of these neighborhoods over all i and j we find $\iota^*(\mathcal{L})$ is trivial on U. Since $q^*(\mathcal{O}_{\sqcup_i T_i})$ is also trivial on U, there is an isomorphism $\sigma : \iota^*(\mathcal{L}) \to q^*(\mathcal{O}_{\sqcup T_{i,j}})$ on U, and thus we see there is a line bundle on X'_U (coming from this isomorphism) whose pullback to X_U is $\mathcal{L}$. It remains to see that this surjection is preserved on the Pic^0. But indeed, at a geometric point the map is a surjective map of group schemes with connected kernel, and so is surjective between the identity components. Thus if $\mathcal{L}'$ on X_T maps to something in $\mathrm{Pic}^0_{X/S}(T)$ then it must be in $\mathrm{Pic}^0_{X'/S}(T)$.

Thus, we have shown that the sequence is exact. By Theorem 2.18, $\mathrm{Pic}^0_{X/S}$ is an abelian scheme, and since the sequence is exact $\mathrm{Pic}^0_{X'/S}$ is a torsor for a split torus over $\mathrm{Pic}^0_{X/S}$. In particular, since a split torus is affine, by Proposition 2.7 (a standard result), $\mathrm{Pic}^0_{X'/S}$ is representable and thus a commutative group scheme. □

Remark. In the proof of Theorem 3.22 we could also have shown that the exact sequence of Picard functors restricts correctly to the identity components by considering degrees as in [**4**, Section 9.1].

Remark. Theorem 3.22 generalizes [**4**, Example 9.2.8].

Continuing with the notation of Theorem 3.22, we now describe an explicit splitting for the torus $G = \ker \pi^*$ which we will use frequently in the rest of this work. We define a morphism $\phi_{X'} : \mathbb{G}_m^N \to \mathrm{Pic}^0_{X'/S}$ that on T–points sends a point $(t_{i,j})$ $1 \leq i \leq n$, $1 \leq j \leq m_i$ (notice we have omitted the index 0 for j) to the triple $(\mathcal{O}_{X_T}, \mathcal{O}_{\sqcup_i T_i}, \sigma)$, where $\sigma|_{T_{i,j}}$ acts by $t_{i,j}$ times the canonical morphism $c : j^*\mathcal{O}_{X_T} \to \mathcal{O}_{\sqcup_{i,j} T_{i,j}} \to q^*\mathcal{O}_{\sqcup_i T_i}$. It follows from our description of G in the proof of 3.22 that this is an isomorphism onto G. By fixing this map we obtain a $\mathbb{G}_m^N$–extension of $\mathrm{Pic}^0_{X/S}$

$$0 \longrightarrow \mathbb{G}_m^N \xrightarrow{\phi_{X'}} \mathrm{Pic}^0_{X'/S} \xrightarrow{\pi^*} \mathrm{Pic}^0_{X/S} \longrightarrow 0$$

There is a natural product morphism $\mathrm{prod} : \mathbb{G}_m^N \to \mathbb{G}_m$ and pushing out along this morphism we obtain a $\mathbb{G}_m$–extension E fitting in the following commutative diagram with exact rows:

(3.2.1)
$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{G}_m^N & \xrightarrow{\phi_{X'}} & \mathrm{Pic}^0_{X'/S} & \xrightarrow{\pi^*} & \mathrm{Pic}^0_{X/S} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{prod}} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{G}_m & \xrightarrow{\bar{\phi}_{X'}} & E & \xrightarrow{\pi^*} & \mathrm{Pic}^0_{X/S} & \longrightarrow & 0
\end{array}$$

In the rest of this work, this extension is the extension we refer to when we say the $\mathbb{G}_m$–extension of $\mathrm{Pic}^0_{X/S}$ associated to X' or to the pinching of X in $(P_{i,j})$. In the next section we determine the associated extension in terms of divisors and the auto-duality of the Jacobian when S is an algebraically closed field.

Example 3.23. *Intersecting sections.* One might ask why we work only with non-intersecting sections. Indeed, the glueing theorems of the previous section can still be applied, however, if the sections are not disjoint then generally the extensions obtained will not be semiabelian: Consider, for example, $\mathbb{P}^1_k \times \mathbb{A}^1_k$ as a family of projective lines parameterized by $\mathbb{A}^1_k$ with coordinate x, and consider a copy of $\mathbb{A}^2_k \subset \mathbb{P}^1_k \times \mathbb{A}^1_k$ inside by taking an $\mathbb{A}^1_k \subset \mathbb{P}^1_k$ with coordinate y. We will pinch the familiy $\mathbb{P}^1_k \times \mathbb{A}^1_k$ in the two crossing diagonal lines defined by $(x-y)$ and $(x+y)$ in $\mathbb{A}^2_k$ by identifying points with the same x–coordinate. This is the pinching corresponding to the diagram

$$\begin{array}{ccc}
\mathrm{Spec}k[x,y]/(x+y)(x-y) & \longrightarrow & \mathbb{A}^1_k = \mathrm{Spec}k[x] \\
\downarrow & & \downarrow \\
\mathbb{P}^1_k \times \mathbb{A}^1_k & \longrightarrow & X'
\end{array}$$

Its fiber over the point $x = 0$ in $k[x]$ is given by the pinching diagram

$$\begin{array}{ccc}
\mathrm{Spec}k[y]/(y^2) & \longrightarrow & \mathrm{Spec}k\ . \\
\downarrow & & \downarrow \\
\mathbb{P}^1_k & \longrightarrow & X'_0
\end{array}$$

We claim that $\mathrm{Pic}^0_{X'_0/k}$ is the group scheme $\mathbb{G}_a$. Indeed, $\mathrm{Pic}^0_{\mathbb{P}^1_k/k} = \mathrm{Spec}k$, so that for any T/k there is only a single element in $\mathrm{Pic}^0_{\mathbb{P}^1_k}(T)$, and thus everying in $\mathrm{Pic}^0_{X'_0/k}(T)$ is in the kernel of the pullback map $\pi^* : \mathrm{Pic}_{X'_0/k}(T) \to \mathrm{Pic}_{\mathbb{P}^1_k}(T)$. Conversely, the kernel of the map map $\pi^* : \mathrm{Pic}_{X'_0/k} \to \mathrm{Pic}_{X'_0/k}$ is given by fiber over $\mathcal{O}_{\mathbb{P}^1_T}$ which can be identified with $\mathbb{G}_a(T) = \Gamma(T)$. Indeed, by Theorem 3.20 this fiber is given by invertible elements of $\Gamma(T)[y]/y^2$ mod invertible elements of $\Gamma(T)$, and since the invertible elements of $\Gamma(T)[y]/y^2$ are the elements $a + by$ with a invertible, the map $\mathbb{G}_a(T) \to \mathrm{Pic}_{X'_0/k}(T)$ given by $b \mapsto 1 + by$ is an isomorphism onto $\ker \pi^*$. In particular, since $\ker \pi^* = \mathbb{G}_a$ is connected and $\mathrm{Pic}^0_{X'_0/k} \subset \ker \pi^*$ we see that $\ker \pi^* = \mathrm{Pic}^0_{X'_0/k}$. Thus, even though in this family outside of the fiber at 0 we obtain $\mathbb{G}_m$, at 0 we obtain an additive group $\mathbb{G}_a$.

We note that there is another way to understand this example. The family of curves obtained in this pinching is easily seen to be a family of nodal cubics degenerating to a cuspidal cubic at $x = 0$: restricting to the $\mathbb{A}^2_k$ where all of the pinching occurs, X' becomes the spectrum of the sub-$k[x]$–algebra of $k[x,y]$ generated by $(x+y)(x-y)$ and $y(x+y)(x-y)$ which is isomorphic as a $k[x]$–algebra to $k[x,v,w]/(w^2 = -v^3 + x^2v^2)$ by $v \mapsto (x+y)(x-y)$ and $w \mapsto y(x+y)(x-y)$ (cf. also Examples 3.7-2 and 3.7-3). It is a standard result that the Pic^0 of a projective nodal cubic is $\mathbb{G}_m$ and of a projective cuspidal cubic is $\mathbb{G}_a$.

3.3. Pinching and $\mathbb{G}_m$–extensions of Jacobians over fields

Let k be an algebraically closed field and let C/k be a smooth proper curve of genus $g \geq 1$. Let $J = \mathrm{Pic}^0_{C/k}$ be its Jacobian variety. Let λ be the canonical principal polarization $J \to J^\vee$ as described in Section 2.5.

By the Barsotti–Weil theorem (stated earlier as Theorem 2.20), $\mathbb{G}_m$–extensions of J are in one to one correspondence with k–points of $J^\vee$ by associating to any $\mathbb{G}_m$–extension the line bundle with the same cocycle, and thus by composition with λ^{-1} we get an isomorphism $\mathrm{Ext}(J, \mathbb{G}_m) \to J(k)$. If we then identify a line bundle on C with the divisor class that gives rise to the same cocycle (under the identification that sends a divisor defined on an open cover $\{U_i\}$ by rational functions f_i to the cocycle defined on the open cover U_i by f_i/f_j on $U_{ij} = U_i \cap U_j$), then we get an isomorphism from the group of $\mathbb{G}_m$–extensions of J to the divisor class group of C. We now calculate the image under this map of the $\mathbb{G}_m$–extension associated to a pinching in terms of divisor classes under this isomorphism.

Theorem 3.24. *Let $\pi : C \to C'$ be the pinching associated to a disjoint set of points $\{P_{i,j}\}$ $1 \leq i \leq n$, $0 \leq j \leq m_i$ on a smooth curve C over an algebraically closed field k. The image of the associated $\mathbb{G}_m$–extension (cf. 3.2.1) under the isomorphism* $\mathrm{Ext}(J, \mathbb{G}_m) \to J(k)$ *described above is the divisor class of*

$$\sum_{i=1}^{n}(\sum_{j=1}^{m_i} P_{ij} - m_i \cdot P_{i0})$$

Proof. Let J' be the relative Jacobian of C'. The associated $\mathbb{G}_m$–extension can be placed in the bottom row of the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{G}_m^N & \xrightarrow{\phi_{C'}} & J' & \xrightarrow{\pi^*} & J & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{prod}} & & \downarrow & & \downarrow{\scriptstyle id} & & \\ 0 & \longrightarrow & \mathbb{G}_m & \xrightarrow{\bar{\phi}_{C'}} & E & \xrightarrow{\pi^*} & J & \longrightarrow & 0 \end{array}$$

We will calculate the image of the extension under λ^{-1} by giving sections of the associated torsor E over a cover by two open sets and calculating the corresponding cocycle. We let U_1 be the set $C\backslash\{P_{i,j}\}$. To find the other set, we first let f_{ij} for each $1 \leq i \leq n$, $1 \leq j \leq m_i$ be a function whose divisor is

$$D_{ij} = P_{ij} - P_{i0} + \tilde{D}_{ij}$$

where $\tilde{D}_{ij}$ is a divisor with support disjoint from $\{P_{i,j}\}$ (such functions exist by either a simple argument involving Riemann–Roch or by working in a suitable open affine), and let $f_{ij} = 1$ when $j = 0$. We then let U_2 be the set $C\backslash \cup \mathrm{Supp}\tilde{D}_{ij}$.

Fix a point $P_0 \in C$. For notational simplicity, we will calculate the image of the extension under $-\lambda^{-1}$ instead of λ^{-1}. The map $-\lambda^{-1}$ is the pullback $f^*_{P_0}$, and the morphism $f_{P_0} : C \to J$ is given by the line bundle $\mathcal{L}_{\Delta - C\times\{P_0\}}$ on $C \times C$ where Δ is the diagonal. Thus, a section of $f^*_{P_0}J'$ over an open set $U \subset C$, which is a point of $J'(U)$ that is mapped by π^* to $f_{P_0}|_U$, is a line bundle on $U \times C'$ whose pull back to $U \times C$ is $\mathcal{L}_{\Delta - C\times\{P_0\}}$. Since $U \times C'$ is the pinching of $U \times C$ in the $\{U \times P_{i,j}\}$, we can construct such a section by using our description of line bundles in pinchings:

Denote by P_i the image of $P_{i,j}$ in C'. The pullback of $\mathcal{L}_{\Delta - C\times\{P_0\}}$ to $Z_1 = \sqcup_{i,j} U_1 \times P_{i,j}$ is canonically isomorphic to $\mathcal{O}_{Z_1}$ since $\Delta - C \times \{P_0\}$ and $\cup_{i,j} U_1 \times P_{i,j}$ are disjoint. Thus we obtain a section s_1 over U_1 with the bundle on $U_1 \times C'$ given by the triple $(\mathcal{L}_{\Delta - C\times\{P_0\}}|_{U_1\times C}, \mathcal{O}_{Z'_1}, \sigma_1)$ where $Z'_1 = \sqcup_i U_1 \times P_i$ and σ_1 is the canonical isomorphism.

The pullback of $\mathcal{L}_{\Delta - C\times\{P_0\}}$ to $Z_2 = \sqcup_{i,j} U_2 \times P_{i,j}$ is canonically isomorphic to $\mathcal{L}_D$ where D is the divisor

$$D = \sum_{i,j} P_{i,j} \times P_{i,j}$$

Let D' be the divisor $\sqcup_{i,j} P_{i,0} \times P_i$ on $Z'_2 = \sqcup_i U_1 \times P_i$. The pull back of $\mathcal{L}_{D'}$ is canonically isomorphic to $\mathcal{L}_{\tilde{D}}$ where

$$\tilde{D} = \sum_{i,j} P_{0,j} \times P_{i,j}.$$

If we let f be the function on Z_2 equal to $f_{i,j}$ on $U_2 \times P_{i,j}$ then $D' = \tilde{D} + f$ and thus multiplication by $1/f$ is an isomorphism from $\mathcal{L}_D \to \mathcal{L}_{\tilde{D}}$. So we obtain a section s_2 over U_2 with the bundle on $U_2 \times C'$ given by the triple $(\mathcal{L}_{\Delta - C\times\{P_0\}}|_{U_2}, \mathcal{L}_{D'}, \sigma_2)$ where σ_2 is composition of multiplication by $1/f$ with the canonical isomorphisms.

On the intersection $U_1 \cap U_2$, the section s_2 is $1/f$ times the section s_1 (where $1/f$ is interepreted as the action of the image of $1/f$ in $\mathbb{G}_m^N$). Thus the composed sections to E differ on $U_1 \cap U_2$ by the action of the image of $1/f$ in $\mathbb{G}_m$, which is $\Pi_{i,j} 1/f_{ij}$. So, $(\{U_1, U_2\}, \Pi_{i,j} 1/f_{ij}|_{U_{12}})$ provides a cocycle for $f_{P_0}^* E$. Since $\lambda^{-1} = -f_{P_0}^*$, the image of the extension under the map we consider is given by the cocycle $(\{U_1, U_2\}, \Pi_{ij} f_{ij}|_{U_{12}})$, which corresponds to the Cartier divisor given on U_1 by 1 and by U_2 by $\Pi_{ij} f_{ij}$, which is equivalent to the Weil divisor

$$\sum_{i=1}^{n} (\sum_{j=1}^{m_i} P_{ij} - m_i \cdot P_{i0})$$

as desired. □

3.3.1. The universal extension over C^g. We now interpret this in light of the "universal" extension: let A be an abelian variety over an algebraically closed field and $A^\vee$ its dual. By Barsotti–Weil (Theorem 2.20), the map $\mathrm{id} : A^\vee \to A^\vee$ corresponds to a universal $\mathbb{G}_m$–extension E of $A_{A^\vee}$

$$0 \longrightarrow \mathbb{G}_m \longrightarrow E \longrightarrow A_{A^\vee} \longrightarrow 0$$

which is defined by the property that the fiber over a point $x \in A^\vee(k)$ is the $\mathbb{G}_m$–extension of A defined by x. In the case of the Jacobian J of a smooth curve C/k, if $\lambda : J \to J^\vee$ is the canonical polarization then this is the extension

$$0 \longrightarrow \mathbb{G}_m \longrightarrow E \longrightarrow J_J \longrightarrow 0$$

whose fiber over a point $p \in J$ is the extension defined by $\lambda(p)$.

In practice this extension is difficult to work with. Instead, let us consider the product of curves C^g where g is the genus of C. If we fix a set $Q_1, ..., Q_g$ of g disjoint closed points of C we obtain a canonical morphism $\phi_{(Q_1,...Q_g)} : C^g \to J$ sending a closed point $(P_1, ..., P_g)$ to the divisor class of $\sum_i (P_i) - (Q_i)$. Again by Barsotti–Weil this defines an extension

$$0 \longrightarrow \mathbb{G}_m \longrightarrow E \longrightarrow J_{C^g} \longrightarrow 0$$

whose fiber over a closed point $p = (P_1, ..., P_g) \in C^g$ is the extension defined by $\lambda(\sum (P_i) - (Q_i))$, and so we can view this as a sort of "de-symmetrization" of the universal extension. One advantage of using this extension with base C^g instead of the universal extension with base J is that we can construct it over an open subset of C^g as the extension associated to a pinching.

Let U be the open subset of C^g given by $U = C^g \backslash W$ where W is the closed set consisting of the union of the inverse image of the diagonal in $C \times C$ under each projection $C^g \to C \times C$ and the inverse image of $\{Q_i\} \subset C$ under each projections $C^g \to C$; that is U is C^g minus points with repeated coordinates and points with a coordinate equal to Q_i. On U we have $2g$ disjoint sections p_i, q_i of $U \times C$, where p_i sends a point $(P_1, ..., P_g)$ to $(P_1, ..., P_g) \times (P_i)$ and q_i sends a point $(P_1, ..., P_g)$ to $(P_1, ..., P_g) \times (Q_i)$. Pinching these sections in pairs (p_i, q_i) as in Theorem 3.22, we obtain the associated extension (cf. 3.2.1)

$$0 \longrightarrow \mathbb{G}_m \longrightarrow E' \longrightarrow J_U \longrightarrow 0$$

that, over a closed point $u = (P_1, ..., P_g) \in U(k)$, corresponds by Theorem 3.24 to $\lambda(\sum_i (P_i) - (Q_i))$, and thus we see that this is the extension E over C^g restricted to U.

CHAPTER 4

Higher genus counterexamples to relative Manin–Mumford

In the appendix of an article of Bertrand [**2**], Edixhoven gives a geometric construction of a counterexample to Pink's relative Manin–Mumford conjecture (see Section 4.1 for the statement of this conjecture) using pinching on an elliptic curve with complex multiplication. In this chapter our goal is to generalize this construction to curves of any genus such that the Jacobian admits an antisymmetric (with respect to the Rosati dual) isogeny to itself. We now describe the setting of our construction.

Let k be an algebraically closed field and C/k a smooth projective curve of genus g. Let J be the Jacobian variety of C and let $Q_1, ..., Q_g$ be g distinct closed points of C. As in Section 3.3.1, let $\phi = \phi_{(Q_1,...,Q_g)} : C^g \to J$ be the map sending a closed point $(P_1, ..., P_g) \in C^g$ to the divisor class of $\sum_{i=1}^{g}(P_i) - (Q_i)$. Letting λ denote the canonical principal polarization $J \to J^\vee$ and applying Barsotti–Weil to $\lambda \circ \phi$, we obtain an extension

$$0 \longrightarrow \mathbb{G}_m \longrightarrow E \xrightarrow{\rho} J_{C^g} \longrightarrow 0$$

that, over a closed point $(P_1, ..., P_g)$ is the extension associated to $\lambda(\sum(P_i) - (Q_i))$. For any endomorphism $\psi : J \to J$, we obtain a section $\beta_\psi = \psi \circ \phi \times \mathrm{id}$ of J_{C^g} over C^g. For an endomorphism $\psi : J \to J$ we will denote by $\bar{\psi} = \psi - \psi^\dagger$ its antisymmetrization with respect to the Rosati dual $\dagger$ defined by the canonical polarization. We will prove the following lifting theorem, analogous to Theorem 1 of Bertrand [**2**].

Theorem 4.1. *For any endomorphism ψ of J, there exists an open dense subset $U \subset C^g$ and a section $\eta_{\bar{\psi}} : U \to E$ such that*

(1) $\rho \circ \eta_{\bar{\psi}} = \beta_{\bar{\psi}}|_U$
(2) *If $P = (P_1, ..., P_g) \in U(k)$ is such that $\phi(P)$ is torsion of order n, then $\eta_{\bar{\psi}}(P)$ is torsion of order dividing n^2.*

In the case that $k = \mathbb{C}$ and $\bar{\psi}$ is an isogeny, this theorem can be used to provide counterexamples to relative Manin–Mumford, and moreover a counterexample can be given using this technique for a given curve if and only if there exists an antisymmetric (with respect to $\dagger$) isogeny $J \to J$. In Section 4.1 we recall Pink's relative Manin–Mumford conjecture, discuss the counterexamples of Bertrand and Edixhoven and situate our work in relation to theirs, and show how Theorem 4.1 can be used to give a counterexample. In Section 4.2 we prove Theorem 4.1 by explicitly constructing the section $\eta_{\bar{\psi}}$. In Section 4.3 we show that if J is simple, $\bar{\psi}$ is an isogeny, and n is coprime to $2 \deg \bar{\psi}$, then there is a point $P \in U(k)$ such that $\phi(P)$ is torsion of order n and $\eta_{\bar{\psi}}(p)$ is torsion of order n^2, and thus the bound on the order of the torsion points in Theorem 4.1 cannot be improved (at least for a section that is produced by our construction). Finally, in Section 4.4 we classify antisymmetric isogenies coming from automorphisms of curves and use this classification to exhibit an explicit hyperelliptic curve for every genus $g \geq 1$ whose Jacobian admits an antisymmetric isogeny and thus where the above construction provides a counterexample to relative Manin–Mumford.

4.1. Relative Manin–Mumford

In this section we recall Pink's relative Manin–Mumford conjecture and the counterexamples given by Bertrand and Edixhoven [**2**] and then situate our work in respect to theirs. We also show how Theorem 4.1 can be used to give a counterexample. We first restate the conjecture.

Conjecture 4.2 (Pink's relative Manin–Mumford [**20**, Conjecture 6.2])**.** *Consider an algebraic family of semiabelian varieties B/X over an irreducible variety $X/\mathbb{C}$ and a closed subvariety $Y \subset B$. Assume that Y is not contained in any proper closed subgroup scheme of $B \to X$, and that it contains a Zariski–dense subset of torsion points. Then* $\dim Y \geq \dim B/X$ *(where* $\dim B/X$ *is the dimension of the geometric fibers of B/X).*

The first counterexample was given by Bertrand [**2**, Section 1] using Ribet sections on an elliptic curve with complex multiplication, a technique that generalizes to any higher dimensional abelian variety A admitting an antisymmetric isogeny $A^\vee \to A$ (cf. [**2**, Remark 1.ii]). Edixhoven [**2**, Appendix] then used pinching to give a concrete geometric construction of a similar counterexample in the case of an elliptic curve with complex multiplication. Edixhoven's approach also has the advantage that it gives precise control over the order of the torsion points that appear. Our contribution is a generalization of Edixhoven's pinching construction to the case where the abelian variety A admitting an antisymmetric isogeny is the Jacobian of a curve of any genus.

Remark 4.3. We note that the counterexample of Bertrand is *not* a counterexample to Pink's general conjecture on special subvarieties of mixed Shimura varieties [**20**, Conjectures 1.1–1.3], from which the relative Manin–Mumford conjecture was deduced. Indeed, Bertrand [**2**, Section 2] shows that the image of the section he constructs is contained in a proper special subvariety, and thus argues that in this light it is actually an example in support of Pink's general conjecture. The problem with the relative Manin Mumford conjecture as stated is that the deduction by Pink of the special subvarieties of a family of semiabelian varieties in [**20**, Theorems 5.7 and 6.3] is incorrect, as remarked by Bertrand [**2**, Remark 2.2(i)]. Similarly, although we have not carried out the calculation, we expect that our counterexample will also be contained in a proper special subvariety different than a closed subgroup scheme, and thus will also not be a counterexample to Pink's general conjecture. We plan to carry out this calculation in a future work.

Remark 4.4. The relative Manin–Mumford conjecture, and more generally the Zilber–Pink conjecture for a family of *abelian* varieties, has been proven in some specific cases involving products of elliptic curves — see, e.g., Masser and Zannier [**15**] and Habegger [**9**].

We describe now how Theorem 4.1 can be used to give a counterexample to Conjecture 4.2 (cf. also [**2**, Section 1]).

Proposition 4.5. *Let $C/\mathbb{C}$ be a smooth connected projective curve with Jacobian J. If $\psi \in \mathrm{End}(J)$ is such that $\bar\psi$ is an isogeny, then, in the notation of Theorem 4.1, $\eta_{\bar\psi}(U) \subset E_U$ is a counterexample to Conjecture 4.2 (Pink's relative Manin–Mumford).*

Remark. Such a ψ exists if and only if J admits an antisymmetric isogeny. Indeed, $\bar\psi$ is always antisymmetric, and if ψ is an antisymmetric isogeny then $\bar\psi = 2\psi$ is also an isogeny.

Proof. The extension E/C^g has relative dimension $g+1$, and since $\bar\psi$ is an isogeny the image of $\eta_{\bar\psi}$ has dimension g, and thus it remains only to show that torsion points are dense and that the image is not contained in any proper closed sub-group variety. That torsion points are dense follows from the density of torsion points in J and the property that for a point $P \in U(\mathbb{C})$, $\eta_{\bar\psi}(P)$ is torsion whenever $\phi(P)$ is. Finally, the following lemma shows that $\eta_{\bar\psi}(U)$ is not contained in a proper closed sub-group scheme:

Lemma 4.6. *If $\bar\psi$ is an isogeny then $\overline{\cup_{n\in\mathbb{Z}} n\cdot\eta_{\bar\psi}(U)} = E_U$.*

Proof. (Of lemma) Suppose $u \in U(\mathbb{C})$ is a point such that the multiples of $\bar\psi \circ \phi(u)$ are dense in $J(\mathbb{C})$. Note that this implies that $\phi(u)$ has infinite order in $J(\mathbb{C})$. We claim that at such a u the multiples of $\eta_{\bar\psi}(u)$ are also dense in E_u. Indeed, let X denote the closure of the multiples of $\eta_{\bar\psi}(u)$ in E_u. The image of X in $J(\mathbb{C})$ contains the multiples of $\bar\psi \circ \phi(u)$ and thus is dense. Since the image of X in J is constructible and dense, it must contain an open set, and then since it is closed under the group law it must be equal to all of J. Thus, X surjects onto J. Since X is a group variety, the fiber of X over any point in $J(\mathbb{C})$ is isomorphic to the fiber over the identity element $0 \in J(\mathbb{C})$. But the fiber X_0 over the identity is a closed subvariety of $\mathbb{G}_m$, and thus either finite or all of $\mathbb{G}_m$. If it were finite, then $X \to J$ would be a finite étale map, and as $E_u \times_J X \to X$ admits a section, this contradicts the fact that $\phi(u)$ has infinite order, since the extension E_u is the extension associated to the point $\phi(u)$. Thus the fiber X_0 is all of $\mathbb{G}_m$, and we conclude that X contains the entire fiber over every point of J, showing that $X = E_u$.

Thus, $\overline{\cup_{n\in\mathbb{Z}} n\eta_{\bar\psi}(U)}$ contains the fiber E_u over every point $u \in U(\mathbb{C})$ such that the multiples of $\bar\psi \circ \phi(u)$ are dense in $J(\mathbb{C})$. We claim that this is a dense subset of U, and the conclusion will follow. Indeed, the set

$$W = \{x \in J(\mathbb{C}) \,|\, \text{the multiples of } x \text{ are dense in } J\}$$

has dense intersection with every open subset of J because J is an abelian variety over $\mathbb{C}$. Now, the image of U under $\bar\psi \circ \phi$ is dense in J since $\bar\psi$ is an isogeny and the image of ϕ is dense in J, and since the map $\bar{\psi} \circ \phi$ is generically finite, there exists a non-empty open subset V of J such that $\bar{\psi} \circ \phi$, restricted to a map $(\bar{\psi} \circ \phi)^{-1}(V) \to V$, is a finite morphism (see [**10**, Exercise II.3.7] — note also that in the case we are in, one could also prove directly the existence of such a V). But then $(\bar{\psi} \circ \phi)^{-1}(V \cap W)$ must be dense in $(\bar{\psi} \circ \phi)^{-1}(V)$, because otherwise $V \cap W$ cannot be dense in V (the image under $\bar{\psi} \circ \phi$ of $\overline{(\bar{\psi} \circ \phi)^{-1}(V \cap W)}$ would otherwise be a proper closed subset containing $V \cap W$). Thus, the set of $u \in U(\mathbb{C})$ such that the multiples of $\bar{\psi} \circ \phi(u)$ are dense in $J(\mathbb{C})$ is dense in U, and we are done. □

This concludes the proof of the proposition. □

4.2. Constructing the lift

The following lemma will be crucial in our construction:

Lemma 4.7. *Let k be an algebraically closed field and let X and Y be smooth proper curves over k. Let α and γ be non-constant morphisms from Y to X and let D be a divisor on X such that $nD = (f)$ for a rational function f on X (i.e. $[D]$ is torsion of order dividing n in $\mathrm{Pic}^0(X)$). If*

$$g = \frac{N_\alpha(f \circ \gamma)}{N_\gamma(f \circ \alpha)}$$

so that $\mathrm{Div}g = n \cdot (\alpha_*\gamma^*D - \gamma_*\alpha^*D)$, and if $\mathrm{Div}f$ and $\mathrm{Div}g$ have disjoint support and $\mathrm{Div}(f \circ \alpha)$ and $\mathrm{Div}(f \circ \gamma)$ have disjoint support, then $g(D)^n = 1$.

Proof. The result follows from multiple applications of Weil reciprocity (stated in Theorem 2.2) as in [**2**, Appendix], where an analogous result is shown in the case of divisors of the form $P - Q$ on elliptic curves when $Y = X$ and α is the identity. Indeed,

$$\begin{aligned} g(D)^n &= g(nD) \\ &= g(\mathrm{Div}f) \end{aligned}$$

and, by Weil reciprocity,

$$\begin{aligned} g(D)^n &= f(\mathrm{Div}g) \\ &= \frac{f(\alpha_*\gamma^*\mathrm{Div}f)}{f(\gamma_*\alpha^*\mathrm{Div}f)} \\ &= \frac{f \circ \alpha(\gamma^*\mathrm{Div}f)}{f \circ \gamma(\alpha^*\mathrm{Div}f)} \end{aligned}$$

and, applying Weil reciprocity to the numerator,

$$\begin{aligned} g(D)^n &= \frac{f \circ \gamma(\alpha^*\mathrm{Div}f)}{f \circ \gamma(\alpha^*\mathrm{Div}f)} \\ &= 1 \end{aligned}$$

as desired. □

We now want to construct a lift $\eta_{\bar{\psi}}$ of $\beta_{\bar{\psi}}$ over an open subset $U \subset C^g$ to prove Theorem 4.1.

We will use the construction of the extension E over an open set explained in Section 3.3.1. Namely, let U be the open subset of C^g consisting of points not containing any of the coordinates Q_i and not containing any points with repeated coordinate. On U we have $2g$ disjoint sections p_i, q_i of $U \times C$, where p_i sends a point $(P_1, ..., P_g)$ to $(P_1, ..., P_g) \times (P_i)$ and q_i sends a point $(P_1, ..., P_g)$ to $(P_1, ..., P_g) \times (Q_i)$. Pinching these sections in pairs (p_i, q_i) as in Theorem 3.22, we obtain the associated extension (cf. 3.2.1), which is determined by the property that its fiber over each point $P = (P_1, ..., P_g) \in C^g(k)$ is the extension associated to $\phi(P) \in J(k)$, and is thus equal to E_U.

We write the pinching of the (p_i, q_i) as

$$\begin{array}{ccc} Z & \xrightarrow{q} & Z' \\ \big\downarrow{\scriptstyle \iota} & & \big\downarrow{\scriptstyle \iota'} \\ X & \xrightarrow{\pi} & X' \end{array}$$

where $X = U \times C$, $Z = \sqcup(U_{i,1} \sqcup U_{i,2})$ and $Z' = \sqcup U_i$ where U_i, $U_{i,1}$ and $U_{i,2}$ are all copies of U, q is the map sending both $U_{i,1}$ and $U_{i,2}$ to U_i by the identity map, ι is the closed immersion given on $U_{i,1}$ by p_i and on $U_{i,2}$ by q_i, and X' is the corresponding pinching.

Over U the extension fits into the following commutative diagram with exact rows (cf. Theorem 3.22 and the discussion following it):

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{G}_m^g & \xrightarrow{\phi_{C'}} & J' & \xrightarrow{\pi^*} & J_U & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \text{prod}} & & \downarrow & & \downarrow{\scriptstyle \text{id}} & & \\ 0 & \longrightarrow & \mathbb{G}_m & \xrightarrow{\bar{\phi}_{C'}} & E_U & \xrightarrow{\pi^*} & J_U & \longrightarrow & 0 \end{array}$$

where J' is the relative Jacobian of X'/U. We will produce $\eta_{\bar{\psi}}$ by giving a section of J' and then composing with the quotient map $J' \to E_U$.

Fix a representation $\psi = \sum_i \alpha_{i*}\gamma_i^*$ as in Proposition 2.24 so that $\bar{\psi} = \sum_i(\alpha_{i*}\gamma_i^* - \gamma_{i*}\alpha_i^*)$. We can assume that $\gamma_i \neq \alpha_i$ for all i since in that case we may remove them from our representation of $\bar{\psi}$. We will need a quick lemma:

Lemma 4.8. *If $\alpha \neq \gamma$ are non-constant morphisms $Y \to X$ of smooth proper curves over an algebraically closed field k then for any $n \in \mathbb{Z}^+$ there exists a non-empty open subset $U_{\alpha,\gamma} \subset X^n$ such that for all closed points $(P_1, ..., P_n) \in U_{\alpha,\gamma}$, $\alpha^{-1}(\{P_1, ..., P_n\}) \cap \gamma^{-1}(\{P_1, ..., P_n\}) = \emptyset$.*

Proof. We rewrite $\alpha^{-1}(\{P_1, ..., P_n\}) \cap \gamma^{-1}(\{P_1, ..., P_n\}) = \bigcup_{l,m} \alpha^{-1}(P_l) \cap \gamma^{-1}(P_m)$. Thus, it suffices to show for each l, m that the set such that $\alpha^{-1}(P_l) \cap \gamma^{-1}(P_m) = \emptyset$ is open. But $\alpha^{-1}(P_l) \cap \gamma^{-1}(P_m) \neq \emptyset$ if and only if there is a $Q \in Y$ s.t. $\alpha(Q) = P_l$ and $\gamma(Q) = P_m$. If $l \neq m$ then this happens only on a proper closed set corresponding to the image of X in X^2 under the map (α, γ) (that is, on the inverse image of this closed set under the projection $X^n \to X^2$ corresponding to the lth and mth places). If $l = m$ then since $\alpha \neq \gamma$ they agree only on a closed finite set of X and so we are finished. □

Consider the Weil divisor D on X (recall $X = U \times C$) given by $D = \sum_{i=1}^g p_i(U) - q_i(U)$ and let $D' = \sum_i(\alpha_{i*}\gamma_i^* - \gamma_{i*}\alpha_i^*)D$, where γ_i and α_i are here considered as maps $U \times Y_i \to X$, and remove from the base U the projections of $\text{Supp}D \cap \text{Supp}D'$ and $\text{Supp}\alpha_i^*D \cap \text{Supp}\gamma_i^*D$ for all i. We want our base to remain a non-empty open set, and so we claim that we can make choices that ensure that these are proper subsets of U (they are automatically closed since projection to U is a closed map): indeed, it suffices to show that for each of these there is at least one point of U where they do not intersect the fibers. After potentially changing the Q_i, Lemma 4.8 guarantees that this is possible.

To give a section of J' over U is to give an element of $\text{Pic}^0_{X'/S}(U)$. Consider the triplet $(\mathcal{L}_{D'}, \mathcal{O}_{Z'}, \sigma)$ where σ is the composition of the canonical isomorphisms $\iota^*\mathcal{L}_{D'} \to \mathcal{O}_Z \to q^*\mathcal{O}_{Z'}$ (the first one coming from the fact that $Z \cap \text{Supp}D'$ is trivial). This defines a line bundle on X', and we let $\eta_{J'}$ be the corresponding section of J' and $\eta_{\bar{\psi}}$ be the section of E given by composition with the quotient $J' \to E$. Clearly $\eta_{\bar{\psi}}$ is a lift of $\beta_{\bar{\psi}}$, and the following lemma shows that it behaves as desired with respect to torsion points.

Lemma 4.9. *Let $P = (P_1, ..., P_g) \in U(k)$ be such that the class of the divisor $D = \sum_{i=1}^g P_i - Q_i$ is torsion of order n in $\text{Pic}^0(C)$. Then $\eta_{\bar{\psi}}(P)$ is torsion of order dividing n^2 in $E_P(k)$.*

Proof. By the compatibility of the relative Picard functor with base extension (Proposition 2.12) and by the compatibility of pinching in flat families with base extension (Theorem 3.11), J'_P is the relative Jacobian of the curve C'_P obtained by pinching along the $\{P_i, Q_i\}$, and the point $\eta_J(P)$ corresponds to the triplet $(\mathcal{L}_{D'_P}, \mathcal{O}_{Z'_P}, \sigma_P)$ where we set $D_P = \sum_{i=1}^g P_i - Q_i$, $D'_P = \sum_i(\alpha_{i*}\gamma_i^* - \gamma_{i*}\alpha_i^*)D_P$ and σ_P is the composition of the canonical isomorphisms $\iota_P^*\mathcal{L}_{D'_P} \to \mathcal{O}_{Z_P} \to q_P^*\mathcal{O}_{Z'_P}$. Multiplying by n^2, we obtain the triplet $(\mathcal{L}_{n^2D'_P}, \mathcal{O}_{Z'_P}, c_P)$ where c_P is the canonical isomorphism $\iota_P^*\mathcal{L}_{n^2D'_P} \to \mathcal{O}_{Z_P} \to q_P^*\mathcal{O}_{Z'_P}$. By Lemma 4.7, $n^2D'_P = \text{Div}g$ where g is a function such that $g(D_P) = 1$ (take g equal to the nth power of the function given by the lemma). Multiplication by g gives an isomorphism $\mathcal{L}_{n^2D'_P} \to \mathcal{O}_C$, and thus identifies $(\mathcal{L}_{n^2D'_P}, \mathcal{O}_{Z'_P}, \alpha_P)$ with $(\mathcal{O}_C, \mathcal{O}_{Z'_P}, (1/g) \cdot c_P)$ where c_P is the canonical isomorphism $\iota_P^*\mathcal{O}_C \to \mathcal{O}_{Z_P} \to q_P^*\mathcal{O}_{Z'_P}$ (this is really the same c_P as above since $\mathcal{L}_{n^2D'_p} = \mathcal{O}_C$ in an open set viewed as sub-sheafs of the sheaf of rational functions). Under our identification of the kernel of π^* with $\mathbb{G}_m^g$, $(\mathcal{O}_C, \mathcal{O}_{Z'_P}, (1/g) \cdot c_P)$ corresponds to the point $(g(P_i)/g(Q_i))_i$. In particular, under the map prod this goes to $g(D_P)$, which is equal to 1, and thus $n^2\eta_{\bar{\psi}}(P) = 1$, as desired. □

4.3. Controlling the order of lifts

In this section we assume that the characteristic of k is 0. As with Edixhoven's construction on elliptic curves (cf. [**2**, Appendix, Remark 3.i]), we can use the Weil pairing to show that if $\bar{\psi}$ is an isogeny and J is simple then for n coprime to $2 \cdot \deg \bar{\psi}$ there exists a point $P \in U(\mathbb{C})$ such that $\eta_{\bar{\psi}}(P)$ has order equal to n^2. To show this, we begin with two lemmas connecting the order of $\eta_{\bar{\psi}}(P)$ and the Weil pairing:

Lemma 4.10. *Let D be a divisor such that $[D]$ is of order n in $J(k)$ and let $D' = \sum_i (\alpha_{i*}\gamma_i^* - \gamma_{i*}\alpha_i^*)D$ so that D' represents $\bar{\psi}([D])$. Let f be a function with* $\mathrm{Div} f = nD$ *and let*

$$g = \prod_i \frac{N_{\alpha_i}(f \circ \gamma_i)}{N_{\gamma_i}(f \circ \alpha_i)}$$

so that $\mathrm{Div} g = n \cdot D'$*. If e_n is the Weil pairing on $J(k)[n]$ then*

$$e_n([D], \bar{\psi}([D])) = 1/g(D)^2.$$

Proof. By the characterization of the Weil pairing given at the end of Section 2.5.2,

$$e_n([D], \bar{\psi}([D])) = f(D')/g(D)$$

and thus it suffices to show that $f(D') = 1/g(D)$. Indeed,

$$\begin{aligned} f(D') &= \prod_i f(\alpha_{i*}\gamma_i^* D - \gamma_{i*}\alpha_i^* D) \\ &= \prod_i \left[\frac{(N_{\gamma_i}(f \circ \alpha_i))(D)}{(N_{\alpha_i}(f \circ \gamma_i))(D)} \right] \\ &= 1/g(D) \end{aligned}$$

□

Remark. When n is not divisible by 2, Lemma 4.10 gives another proof of Lemma 4.7.

Lemma 4.11. *The pairing $\langle , \rangle \colon (x, y) \mapsto e_n(x, \bar{\psi}y)$ on $J[n] \times J[n]$ is symmetric. If $\bar{\psi}$ is an isogeny of degree prime to n then $\langle , \rangle$ is non-degenerate (in the sense that if $\langle x, y \rangle = 1 \; \forall y \in J[n]$ then $x = 0$).*

Proof. By Lemma 2.21 (which says that † is the adjoint operator for e_n), we have

$$\langle x, y \rangle = e_n(x, \bar{\psi}y) = e_n(\bar{\psi}^\dagger x, y),$$

and since $\bar{\psi}$ is antisymmetric and e_n is skew-symmetric,

$$e_n(\bar{\psi}^\dagger x, y) = e_n(-\bar{\psi}x, y) = e_n(y, \bar{\psi}x) = \langle y, x \rangle,$$

and thus the pairing is symmetric. If $\bar{\psi}$ is an isogeny of degree prime to n then it is an automorphism on the n–torsion and non-degeneracy follows from non-degeneracy of e_n. □

Now, for $P = (P_1, ..., P_g) \in U(\mathbb{C})$ such that $D = \sum P_i - Q_i$ has order n, $\eta_{\bar{\psi}}(P)$ has order n^2 if and only if $g(D)$ is a primitive nth root of unity (cf. the proof of Lemma 4.9). Thus, by Lemma 4.10, if n is coprime to 2 then $\eta_{\bar{\psi}}(P)$ has order n^2 if and only if $e_n([D], \bar{\psi}[D])$ is a primitive nth root of unity. If n is also coprime to the degree of $\bar{\psi}$ then the pairing $\langle , \rangle$ of Lemma 4.11 is symmetric non-degenerate (assuming still that $\bar{\psi}$ is an isogeny). Since n is coprime to 2, this implies that there exists an element $x \in J[n]$ such that $\langle x, x \rangle$ is a primitive nth root of unity: suppose not, then, since $\langle x, y \rangle = [\langle x + y, x + y \rangle / (\langle x, x \rangle \cdot \langle y, y \rangle)]^{1/2}$ (the square root is well defined since we are coprime to 2), we see that the image of $\langle , \rangle$ is contained in μ_m for some m that is a proper divisor of n. But then m times any element of order n is degenerate for the pairing, a contradiction. Thus we obtain a point $P \in C^g(\mathbb{C})$ such that $\phi(P)$ has order n and such that, if it is in $U(\mathbb{C})$, its image under $\eta_{\bar{\psi}}$ has order n^2. The complement of the image of U in J is contained in a closed and a proper subset of J, and thus by a general form of Mordell–Lang there is some finite set of translates of proper abelian subvarieties of J containing all of the torsion points that are outside of the image of U (see the statement "Lang's conjecture (absolute form, characteristic 0)" on page 10 of [**11**]). If J is simple then this implies that U must contain all but finitely many torsion points, and thus for sufficiently large n we can find a point of order n as above in $U(\mathbb{C})$.

Remark. I do not know if U will or will not contain a point of order n of the form desired if J is not simple.

4.4. Explicit examples

Proposition 4.5 gives a counterexample to relative Manin–Mumford on the Jacobian of any curve admitting an antisymmetric isogeny. In this section, we characterize the automorphisms α of a smooth projective curve $C/\mathbb{C}$ such that

$$\overline{\alpha_*} = \alpha_* - \alpha^* = \alpha_* - (\alpha^{-1})_*$$

is an isogeny from Jac$C$ to itself, and use this characterization in order to give explicit examples. In Theorem 4.12 we characterize such C and α, and in Example 4.13 we use this characterization to give examples of curves of every genus $g \geq 1$ admitting antisymmetric isogenies, and thus to which we can apply Proposition 4.5 to give explicit counterexamples to relative Manin–Mumford.

Theorem 4.12. *Let $C/\mathbb{C}$ be a smooth projective curve of genus $g \geq 1$ and let α be an automorphism of C of order n. Then $\overline{\alpha_*} = \alpha_* - \alpha^*$ is an isogeny if and only if either*

- *n is odd and the quotient C' of C by the action of $\mathbb{Z}/n\mathbb{Z}$ on C induced by α is isomorphic to $\mathbb{P}^1$.*
- *n is even and the quotient C' of C by the action of $\mathbb{Z}/\frac{n}{2}\mathbb{Z}$ on C induced by α^2 is isomorphic to $\mathbb{P}^1$.*

Proof. In the case $n = 1$, we have $\alpha = \text{id}$, $\overline{\alpha_*} = 0$, and $C' \cong C$, and thus C' is not isomorphic to $\mathbb{P}^1$ since $g \geq 1$. Thus the result holds in this case. So, we can assume $n > 1$.

Let f_α be the minimal polynomial of α_* considered as an element of the finite dimensional $\mathbb{Q}$–algebra $\text{End}^0(\text{Jac}C) = \mathbb{Q} \otimes \text{End}(\text{Jac}C)$. We obtain an injection $\phi\colon \mathbb{Q}[x]/f_\alpha \hookrightarrow \text{End}^0(\text{Jac}C)$ by $x \mapsto \alpha_*$. We observe that

$$(\alpha_*)^n - \text{id}_{\text{Jac}C} = (\alpha^n)_* - \text{id}_{\text{Jac}C} = 0$$

since α has order n, and thus $f_\alpha | x^n - 1$.

We observe that x is invertible in $\mathbb{Q}[X]/f_\alpha$ and x^{-1} maps to $(\alpha_*)^{-1} = (\alpha^{-1})_* = \alpha^*$. Note that $\overline{\alpha_*}$ is invertible in $\text{End}^0(\text{Jac}C)$ if and only if $\overline{\alpha_*}$ is an isogeny, and furthermore, since α_* is invertible, $\overline{\alpha_*}$ is an isogeny if and only if $(\alpha_*)(\overline{\alpha_*}) = (\alpha^2)_* - \text{id}_{\text{Jac}C}$ is an isogeny. Note that $\phi(x^2 - 1) = (\alpha^2)_* - \text{id}_{\text{Jac}C}$. Thus, to show that $\overline{\alpha_*}$ is an isogeny it suffices to show that $x^2 - 1$ is invertible in $\mathbb{Q}[X]/f_\alpha$. Decomposing f_α as a product of cyclotomic polynomials and applying the Chinese remainder theorem, we see this is the case if and only if f_α is not divisible by $(x - 1)$ or $(x + 1)$. On the other hand, to show that $\overline{\alpha_*}$ is *not* an isogeny, it suffices to show that there exists $g \in \mathbb{Q}[x]/f_\alpha$ such that $\phi(g) \neq 0$ and $(x^2 - 1) \cdot g = 0$ in $\mathbb{Q}[x]/f_\alpha$, since then $\overline{\alpha_*}$ is either zero or a zero divisor and in particular is not invertible. We now break into cases.

Suppose n is odd. We first show that if the quotient is $\mathbb{P}^1$ then $\overline{\alpha_*}$ is invertible. By the remarks above, since $(x + 1) \nmid (x^n - 1)$ (n is odd), it suffices to show that f_α divides $(x^n - 1)/(x - 1) = \sum_{j=0}^{n-1} x^j$, or equivalently that $\sum_{j=0}^{n-1} (\alpha_*)^j = 0$ in $\text{End}^0(\text{Jac}C)$. But for $P \in C(\mathbb{C})$,

$$\sum_{j=0}^{n-1} (\alpha_*)^j(P) = \sum_{j=0}^{n-1} (\alpha^j(P)) = \pi^*(\pi(P))$$

where π is the quotient morphism $\pi\colon C \to C'$. Thus, by linearity, for all degree 0 divisors D on C,

$$\sum_{j=0}^{n-1} (\alpha_*)^j[D] = \pi^*\pi_*[D] = 0$$

since $[\pi_*D] = 0$ as it is the class of a degree zero divisor on $C' \cong \mathbb{P}^1$.

We now show the other direction (still assuming n to be odd). To do so, we will show that if the quotient C' is not $\mathbb{P}^1$ then $\overline{\alpha_*}$ is not invertible. Observe that

$$(x^2 - 1) \cdot \sum_{j=0}^{n-1} x^j = 0$$

in $\mathbb{Q}[x]/f_\alpha$ since it is divisible by $x^n - 1$, and so by the remarks above it suffices to show that

$$\phi(\sum_{j=0}^{n-1} x^j) = \sum_{j=0}^{n-1} (\alpha_*)^j \neq 0$$

in $\text{End}^0(\text{Jac}C)$. So, suppose D is a degree 0 divisor on C such that $\sum_{j=0}^{n-1} (\alpha_*)^j([D]) = 0$. As above, this implies

$$\pi^*\pi_*[D] = 0.$$

Since $\pi_*\pi^*$ is multiplication by $\deg \pi$ on $\mathrm{Jac}C'$, this implies $n \cdot \pi_*[D] = 0$. Since π_* is surjective and the dimension of $\mathrm{Jac}C'$ is greater than 0 (C' has genus greater than 0), there exists a $[D] \in \mathrm{Jac}C(\mathbb{C})$ such that $n \cdot \pi_*[D]$ is non-zero, and thus $\sum_{j=0}^{n-1}(\alpha_*)^j$ is non-zero in $\mathrm{End}^0(\mathrm{Jac}C)$, as desired.

Thus we have completed the odd case. The case of n even is proven identically, except in both directions we must use the polynomial

$$(x^n - 1)/(x^2 - 1) = \sum_{j=0}^{n/2-1} x^{2j}$$

instead of

$$(x^n - 1)/(x - 1) = \sum_{j=0}^{n-1} x^j$$

because when n is even $x + 1$ also divides $x^n - 1$. In particular, the fact that only even powers appear in the sum is why in the even case we must consider the quotient by the action of α^2 instead of that by the action of α. $\square$

Example 4.13. We give some applications of Theorem 4.12:

(1) Let K be a cyclic extension of $\mathbb{C}(t)$ of odd degree and let $C \to \mathbb{P}^1$ be the corresponding ramified covering of smooth curves. Suppose that the genus of C is greater than or equal to 1. Then for α an automorphism of C generating the Galois group of $K/\mathbb{C}(t)$, the quotient of C by α is $\mathbb{P}^1$, and thus by Theorem 4.12 $\overline{\alpha_*}$ is an isogeny and Proposition 4.5 applies, giving a counterexample to relative Manin–Mumford.

(2) Let $n \geq 3$ and let $C/\mathbb{C}$ be the hyperelliptic curve defined by the equation $y^2 = x^n - 1$, and let $J = \mathrm{Jac}C$. Let α be the automorphism of C given by $(x_0, y_0) \mapsto (\zeta_n x_0, y_0)$ where ζ_n is a primitive n–th root of unity. Then Theorem 4.12 shows $\alpha_* - \alpha^* = \overline{\alpha_*}$ is an antisymmetric isogeny of J. Indeed, for n odd the quotient C' of C by α is given by the curve $y^2 = x - 1$, which is isomorphic to $\mathbb{P}^1$, with the quotient map $C \to C'$ given by $(x_0, y_0) \mapsto (x_0^n, y_0)$. For n even the quotient C' of C by α^2 is given by the curve $y^2 = x^2 - 1$, again isomorphic to $\mathbb{P}^1$, with the quotient map given by $(x_0, y_0) \mapsto (x_0^{n/2}, y_0)$. In particular, since the curve $y^2 = x^n - 1$ has genus $\lfloor \frac{n-1}{2} \rfloor$, we obtain for every $g \geq 1$ an explicit curve of genus g whose Jacobian admits an explicit antisymmetric isogeny, and thus to which Proposition 4.5 applies. This gives an explicit counterexample to relative Manin–Mumford for every genus $g \geq 1$.

(3) For any smooth projective curve $C'/\mathbb{C}$ of genus $g \geq 1$, let $C \to C'$ be the curve corresponding to a cyclic extension of $k(C')$. Then C admits an automorphism α generating the cyclic group of the extension such that $C \to C'$ is the quotient by α. Then whether α has even or odd order, the quotient appearing in Theorem 4.12 lies between C and C' and in particular is not isomorphic to $\mathbb{P}^1$, implying that $\overline{\alpha_*}$ is not an isogeny.

Remark 4.14. For p an odd prime, the proof of Theorem 4.12 also shows that if $C/\mathbb{C}$ is a smooth projective curve of genus $g = \frac{p-1}{2}$ admitting an automorphism α of order p such that the quotient of C by $\mathbb{Z}/p\mathbb{Z} \cong \langle \alpha \rangle$ is $\mathbb{P}^1$, then $\mathrm{Jac}C$ has complex multiplication. Indeed, the proof shows that $\mathrm{End}^0(\mathrm{Jac}C)$ contains the CM-field $\mathbb{Q}[x]/(x^{p-1} + ... + 1)$ of dimension $2g = 2 \cdot \dim \mathrm{Jac}C$ over $\mathbb{Q}$ with the Rosati involution extending the complex conjugation. This remark applies, for example, to the hyperelliptic curves $y^2 = x^p - 1$ of Example 4.13-(2).

Bibliography

[1] E. Arbarello, M. Cornalba, P. A. Griffiths, and J. Harris. *Geometry of algebraic curves. Vol. I*, volume 267 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, 1985.

[2] Daniel Bertrand. Special points and Poincaré bi-extensions, with an Appendix by Bas Edixhoven. arXiv:1104.5178v1, April 2011.

[3] Christina Birkenhake and Herbert Lange. *Complex abelian varieties*, volume 302 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 2004.

[4] Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud. *Néron models*, volume 21 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1990.

[5] Pierre Deligne. Théorie de Hodge : III. *Publications mathématiques de l'I.H.É.S.*, 44:5–77, 1974.

[6] Michel Demazure and Peter Gabriel. *Introduction to algebraic geometry and algebraic groups*. North-Holland Pub. Co. ; sole distributors for the U.S.A. and Canada, Elsevier North-Holland, Amsterdam ; New York : New York :, 1980.

[7] Gerd Faltings and Ching-Li Chai. *Degeneration of abelian varieties*, volume 22 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1990. With an appendix by David Mumford.

[8] Daniel Ferrand. Conducteur, descente et pincement. *Bull. Soc. Math. France*, 131(4):553–585, 2003.

[9] Philipp Habegger. Torsion points on elliptic curves in Weierstrass form. 2011.

[10] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[11] Marc Hindry. Introduction to abelian varieties and the Mordell-Lang conjecture. In *Model theory and algebraic geometry*, volume 1696 of *Lecture Notes in Math.*, pages 85–100. Springer, Berlin, 1998.

[12] Everett W. Howe. The Weil pairing and the Hilbert symbol. *Math. Ann.*, 305(2):387–392, 1996.

[13] Peter Jossen. *On the arithmetic of 1-motives*. PhD thesis.

[14] Steven L. Kleiman. The Picard scheme. In *Fundamental algebraic geometry*, volume 123 of *Math. Surveys Monogr.*, pages 235–321. Amer. Math. Soc., Providence, RI, 2005.

[15] David Masser and Uwe Zannier. Torsion anomalous points and families of elliptic curves. *C. R. Acad. Sci. Paris Ser. I*, pages 491–494.

[16] James S. Milne. *Étale cohomology*, volume 33 of *Princeton Mathematical Series*. Princeton University Press, Princeton, N.J., 1980.

[17] James S. Milne. Abelian varieties (v2.00), 2008. Available at www.jmilne.org/math/.

[18] D. Mumford, J. Fogarty, and F. Kirwan. *Geometric invariant theory*, volume 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)]*. Springer-Verlag, Berlin, third edition, 1994.

[19] F. Oort. *Commutative group schemes*. Springer-Verlag, Berlin :, 1966.

[20] Richard Pink. A common generalization of the conjectures of André-Oort, Manin-Mumford, and Mordell-Lang. Preprint (13p.), April 2005.

[21] Karl Schwede. Gluing schemes and a scheme without closed points. In *Recent progress in arithmetic and algebraic geometry*, volume 386 of *Contemp. Math.*, pages 157–172. Amer. Math. Soc., Providence, RI, 2005.

[22] Jean-Pierre Serre. *Algebraic groups and class fields*, volume 117 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1988. Translated from the French.

[23] The Stacks Project Authors. *Stacks Project*. `http://math.columbia.edu/algebraic_geometry/stacks-git`.

[24] Angelo Vistoli. Grothendieck topologies, fibered categories and descent theory. In *Fundamental algebraic geometry*, volume 123 of *Math. Surveys Monogr.*, pages 1–104. Amer. Math. Soc., Providence, RI, 2005.