Hard Lefschetz Conjecture and Hodge Standard Conjecture on blowing-up of Projective Spaces

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Contents

Acknowledgements

I wish to take this opportunity to express my sincere thanks to Prof. (Dr.) Pascal Boyer for giving me his valuable time to discuss the subject matter and suggesting the framework of this thesis. But for his encouragement and consideration, this work would not have been possible.

I am thankful to Prof. (Dr.) Fontanne of the University of Paris Sud, who was instrumental in inculcating a sense of confidence in me to carry on my academic endeavor in the University.

I am also thankful to the authorities, who preside over the Erasmus Mundus ALGANT program and the European Commission for affording me an opportunity to study Mathematics in two celebrated Universities in Europe.

I shall be failing in my duty if I do not place on record my sincere gratitude to all my Professors in the University of Padova, Italy and the University of Paris Sud, France, who have helped me to peep into the world of Mathematics.

And finally, I wish to thank my father, my mother and my brother for their unflinching support throughout the time of preparation of this thesis and for making my life so beautiful and meaningful with their love and affection.

Chapter 1

Blowing-up of Projective Spaces.

1.1 Preliminaries

Let X be a noetherian scheme and $\mathfrak{B}$ a quasi-coherent sheaf of $\mathcal{O}_X$-modules which has a structure of a sheaf of graded $\mathcal{O}_X$-algebras. i.e $\mathfrak{B} = \bigoplus_{d \geq 0} \mathfrak{B}_d$, where $\mathfrak{B}_d$ is the homogenous part of degree d with $\mathfrak{B}_0 = \mathcal{O}_X$, $\mathfrak{B}_1$ a coherent $\mathcal{O}_X$-module and $\mathfrak{B}$ generated by $\mathfrak{B}_1$ as an $\mathcal{O}_X$-algebra.

For an open affine subset $U = \text{Spec } A$ of X, $\mathfrak{B}(U)$ is a graded A-Algebra. $\pi_U : \text{Proj } \mathfrak{B}(U) \longrightarrow U$ is a projective A-scheme. Then $\mathfrak{B}$ quasi-coherent implies that $\text{Proj } \mathfrak{B}(U_f) \cong \pi_U^{-1}(U_f)$. Hence for two affine open subests U and V, we have a natural isomorphism $\pi_U^{-1}(U \cap V) = \pi_V^{-1}(U \cap V)$. Thus we can glue these schemes to obtain a scheme which we denote by **Proj** $\mathfrak{B}$. Also, there is a natural morphism $\pi : \text{Proj } \mathfrak{B} \longrightarrow X$ with the property that for every affine open subset $U = \text{Spec } A$, $\pi^{-1}(U) \cong \text{Proj } \mathfrak{B}(U)$.

Since a closed subscheme corresponds to a coherent sheaf of ideals, we can speak of blowing up along a coherent sheaf of ideals $\mathfrak{I}$ on X.

Definition 1.1.1. Consider a sheaf of graded algebras $\mathfrak{B} = \bigoplus_{d \geq 0} \mathfrak{I}^d$, $\mathfrak{I}^d$ being the d^{th} power of $\mathfrak{I}$ with $\mathfrak{I}^0 = \mathcal{O}_X$. We define $\widetilde{X} = \text{Proj } \mathfrak{B}$ to be the ***Blowing-up*** of X with respect to $\mathfrak{I}$. If Y is the closed subscheme corresponding to $\mathfrak{I}$, then we say $\widetilde{X}$ is the Blowing-up of X along Y.

Definition 1.1.2. Let $f : X \longrightarrow Y$ be a morphism of schemes, $\mathfrak{I} \subseteq \mathcal{O}_Y$ be a sheaf of ideals on Y. Let $f^{-1}\mathfrak{I}$ be the inverse image of the sheaf $\mathfrak{I}$. $f^{-1}\mathfrak{I}$ is a sheaf of ideals in $f^{-1}\mathcal{O}_Y$. Using the natural map $f^{-1}\mathcal{O}_Y \longrightarrow \mathcal{O}_X$, we define the *Inverse Image Ideal Sheaf of* $\mathfrak{I}' \subseteq \mathcal{O}_X$ to be the sheaf of ideals generated by $f^{-1}\mathfrak{I}$.

Let A be a ring. On $\mathbb{P}^n_A$ we have the invertible sheaf $\mathcal{O}(1)$ and the homogenous co-ordinates $(x_0, x_1, ..., x_n)$ as global sections in $\Gamma(\mathbb{P}^n_A, \mathcal{O}(1))$ generate $\mathcal{O}(1)$. Let X be a scheme over A and $\phi : X \longrightarrow \mathbb{P}^n_A$ be an A-morphism. Then $\mathcal{L} = \phi^*(\mathcal{O}(1))$ is an invertible sheaf on X and the global sections $s_i = \phi^*(x_i) \in \Gamma(X, \mathcal{L})$ generate the sheaf $\mathcal{L}$. Conversely

Theorem 1.1.3. *If $\mathcal{L}$ is an invertible sheaf on X and if $(s_0, s_1, ..., s_n) \in \Gamma(X, \mathcal{L})$ generate $\mathcal{L}$, the there exists a unique A-morphism $\phi : X \longrightarrow \mathbb{P}^n_A$ such that $\mathcal{L} \cong \phi^*(\mathcal{O}(1))$ and $s_i = \phi^*(x_i)$.*

Proof. Let us define the open subset $X_i = \{P \in X | (s_i)_P \notin \mathfrak{m}_P\mathcal{L}_P\}$ for $i = 0, 1, ..., n$. If $P \notin \cup_i X_i$, then $(s_i)_P \in \mathfrak{m}_P\mathcal{L}_P$ $\forall i$ which implies that $\{s_i\}_{i=0,...,n}$ do not generate $\mathcal{L}_P$, a contradiction. Hence, the $\{X_i\}_i$ cover X. We prove the theorem by providing a map from X_i to U_i where U_i is the open subset $\{x_i \neq 0\}$ of $\mathbb{P}^n_A$ and then glueing them to obtain a map $\phi : X \longrightarrow \mathbb{P}^n_A$.

We have that U_i is isomorphic to $\mathbb{A}^n_A \cong$ spec $A[T_0, ..., T_n]$ with $T_i = x_j/x_i$ where x_i as above. Now consider the ring homomorphism $A[T_1, ..., T_n] \longrightarrow \Gamma(\mathcal{O}_X, X_i)$ given by $T_j \longrightarrow s_j/s_i$. Now $s_i \notin \mathfrak{m}_P\mathcal{L}_P$ $\forall P \in X_i$ and because $\mathcal{L}$ is locally free of rank 1, s_j/s_i makes sense as an element of $\Gamma(\mathcal{O}_X, X_i)$. Also this map is well defined. Hence we have a map $X_i \longrightarrow U_i$. By glueing these maps, we get a A-morphism $\phi : X \longrightarrow \mathbb{P}^n_A$ such that $s_i = \phi^*(x_i)$. Any map ϕ satisfying the given properties is unique by construction above. Also, $\mathcal{L} \cong \phi^*(\mathcal{O}(1))$.

■

Theorem 1.1.4 (Universal Property of blowing-up)**.** *Let X be a noetherian scheme, $\mathcal{I}$ a coherent sheaf of ideals and $\pi : \widetilde{X} \longrightarrow X$ the blowing-up with* *respect to $\mathcal{I}$. If $f : Z \longrightarrow X$ is any morphism such that $f^{-1}\mathcal{I}.\mathcal{O}_Z$ is an invertible sheaf of ideals on Z, then there exists a unique morphism $g : Z \longrightarrow \widetilde{X}$ factoring through f.*

$$\begin{tikzcd} Z \arrow[r, dashed, "g"] \arrow[dr, "f"'] & \widetilde{X} \arrow[d, "\pi"] \\ & X \end{tikzcd}$$

Proof. We may assume without loss of generality that $X = \text{spec } A$ is affine with A noetherian. Further we have that the sheaf of ideals $\mathcal{I}$ is given by the ideal I whose generators are $\{a_1, ...a_n\}$. We have the natural morphism $A[x_1, ..., x_n] \longrightarrow \mathcal{S} = \oplus_{d \geq 0} I^d$ sending x_i to a_i with kernel $\{h | h(a_1, ..., a_n) = 0$. This induces a closed immersion $\widetilde{X} \longrightarrow \mathbb{P}^{n-1}_A$. Let $\mathcal{L} = f^{-1}\mathcal{I}.\mathcal{O}_Z$ be the invertible sheaf. Then the inverse images of $\{a_1, ..., a_n\}$, as global sections, generate $\mathcal{L}$.

By the previous theorem there is a unique morphism $g : Z \longrightarrow \mathbb{P}^{n-1}_A$ such that $\mathcal{L} = g^*(\mathcal{O}(1))$ and $s_i = g^* X_i$. Then the map g factors through the closed subscheme $\widetilde{X}$ of $\mathbb{P}^{n-1}_A$. If h is such that $h(a_1, ...a_n) = 0$, then $f(s_1, ...s_n) = 0$ in $\Gamma(Z, \mathcal{L}^d)$ where h is homogenous of deg d. Thus g is a map factoring f. This forces $f^{-1}\mathcal{I}.\mathcal{O}_Z = g^{-1}(\pi^{-1}\mathcal{I}.\mathcal{O}_{\widetilde{X}}).\mathcal{O}_Z = g^{-1}\mathcal{O}_{\widetilde{X}(1)}.\mathcal{O}_Z$. Thus there is a surjective map $g^*\mathcal{O}_{\widetilde{X}}(1) \cong \mathcal{L}$.

Since a surjective map of invertible sheaves on a locally ringed spaces is an isomorphism, we have $g^*\mathcal{O}_{\widetilde{X}(1)} \cong \mathcal{L}$. Further we have that the sections s_i of $\mathcal{L}$ pull-backs sections x_i of $\mathcal{O}(1)$. Now by Theorem 1.1.3 we have that g is unique.

■

Corollary 1.1.5. Let $f : X \longrightarrow Y$ be a morphism of noethreian schemes and let $\mathfrak{I}$ be a coherent sheaf of ideals on X. Let $\widetilde{X}$ be the blowing-up of $\mathfrak{I}$ and let $\widetilde{Y}$ be the blowing-up of Y with respect to the inverse image sheaf

$\mathfrak{J} = f^{-1}\mathfrak{I}.\mathfrak{O}_Y$. Then there exists a unique morphism $\widetilde{f} : \widetilde{Y} \longrightarrow \widetilde{X}$ such that the diagram

$$\begin{tikzcd} \widetilde{Y} \arrow[r, dashed, "\widetilde{f}"] \arrow[d] & \widetilde{X} \arrow[d] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

is commutative. Also, if f is a closed immersion, then so is $\widetilde{f}$.

Proof. The existence and uniqueness follows from the theorem above. The only thing that is to be checked is that $\widetilde{f}$ is a closed immersion if f is. $\widetilde{X} = \mathrm{Proj}\mathfrak{B}$ where $\mathfrak{B} = \bigoplus_{d\geq 0} \mathfrak{I}^d$ and $\widetilde{Y} = \mathrm{Proj}\ \mathfrak{B}^{'}$ where $\mathfrak{B}' = \bigoplus_{d\geq 0} \mathfrak{J}^d$. Y a closed subscheme implies that $\mathfrak{B}^{'}$ is a sheaf of graded algebras on X. Hence there is a natural surjectove homomorphism of graded rings $\mathfrak{B} \longrightarrow \mathfrak{B}^{'}$ which gives a closed immersion.

■

Definition 1.1.6. Let Y be a closed subscheme. We call the closed subscheme $\widetilde{Y}$ of $\widetilde{X}$ as above to be the *strict transform* of Y under the blowing-up $\pi : \widetilde{X} \longrightarrow X$.

1.2 Blowing-up of Projective Spaces

Definition 1.2.1. Let us denote by $\mathbb{P}^n$ for $\mathbb{P}^n_{\overline{\mathbb{F}}_q}$. We define $V \subset \mathbb{P}^n$ to be a *linear subvariety* if $\phi^{-1}(V \cup \{0\}) \subset \mathbb{A}^{n+1}$ is a linear subspace, where $\phi : \mathbb{A}^{n+1}\backslash\{0\} \longrightarrow \mathbb{P}^n$ is the natural map.

Let $Gr_d(\mathbb{P}^n)$ be the grassmann variety of linear subvarieties of dimension d and $Gr_*(\mathbb{P}^n) = \coprod_d Gr_d(\mathbb{P}^n)$. Let $Gr_d(\mathbb{P}^n)(\mathbb{F}_q)$ be the linear subvarieties in $\mathbb{P}^n$ of dimesion d defined over $\mathbb{F}_q$. We construct a projective smooth variety $\mathbb{B}^n$ of dimension n over $\overline{\mathbb{F}}_q$ by successive blowing-up of $\mathbb{P}^n$ along linear subvarieties with a birational map $f : \mathbb{B}^n \longrightarrow \mathbb{P}^n$.

1.2.1 Construction of $\mathbb{B}^n$

First we put $Y_0 := \mathbb{P}^n$. Let Z_0 be the disjoint union of linear subvarieties of dimension 0 in $\mathbb{P}^n$ defined over $\mathbb{F}_q$ i.e all the $\mathbb{F}_q$-rational points on $\mathbb{P}^n$. Let Y_1 be the blow-up of Y_0 along Z_0. Let Z_1 be the strict transform of lines in $\mathbb{P}^n$ over $\mathbb{F}_q$ and Y_2 the blow-up of Y_1 along Z_1. Construct Y_{k+1} inductively as follows. Assume Y_k was already constructed. Let Z_k be the union of strict transforms of linear subvarieties of dimension k in $\mathbb{P}^n$ defined over $\mathbb{F}_q$. Let $f_k : Y_{k+1} \longrightarrow Y_k$ be the blow-up of Y_k along Z_k. We set $\mathbb{B}^n = Y_{n-1}$ and $f = f_0 \circ ..... \circ f_{n-2} : \mathbb{B}^n \longrightarrow \mathbb{P}^n$

Hence $\mathbb{B}^n$ can be described by the following sequence of blow-ups.

$$\mathbb{B}^n = Y_{n-1} \longrightarrow Y_{n-1} \longrightarrow ..... \longrightarrow Y_1 \longrightarrow Y_0 = \mathbb{P}^n$$

The construction is equivariant under the action of $PGL_{n+1}(\mathbb{F}^q)$ which is obtained from the natural action of $PGL_{n+1}(\mathbb{F}^q)$ on $\mathbb{P}^n$.

1.2.2 Divisors on $\mathbb{B}^n$

Definition 1.2.2. Let $V \in Gr_k(\mathbb{P}^n)(\mathbb{F}_q)$. We define a smooth irreducible divisor for $k = n-1$ to be $D_V \subset \mathbb{B}^n$, the strict transform of V. For $k < n-1$, consider the strict transform $\widetilde{V} \subset Y_k$ of V which is a connected component of Z_k. Define the divisor $D_V \subset \mathbb{B}^n$ to be the stict transform of the $\mathbb{P}^{n-d-1}$-bundle, $f_k^{-1}(\widetilde{V}) \subset Y_{k+1}$, over $\widetilde{V}$.

Proposition 1.2.3. Let $V, W \in Gr_*(\mathbb{P}^n)(\mathbb{F}_q)$, $D_V \cap D_W \neq \phi$ if and only if $V \subset W$ or $W \subset V$.

Proof. If $V \subset W$ or $W \subset V$, then clearly $D_V \cap D_W \neq \phi$. To prove the other way, let us assume $V \nsubseteq W$ and $W \nsubseteq V$. Then blowing-up along the strict transform of $V \cap W$ in $\mathbb{B}^n$, the strict transforms of V and W become disjoint.

■

Proposition 1.2.4. Let $V \in Gr_d(\mathbb{P}^n)(\mathbb{F}_q)$ we have a non-canonical isomorphism,

$$D_V \cong \mathbb{B}^d \times \mathbb{B}^{n-d-1}$$

where $\mathbb{B}^d$(resp. $\mathbb{B}^{n-d-1}$) is obtained from $\mathbb{P}^d$(resp. $\mathbb{P}^{n-d-1}$) in the same way as $\mathbb{B}^n$.

Proof. Consider the normal bundle $N_{V/\mathbb{P}^n}$ of V. It is isomorphic to $\mathcal{O}_V(1)^{\oplus n-d}$. Now $N_{V/\mathbb{P}^n} \otimes \mathcal{L}$ is a trivial vector bundle over V for some line bundle $\mathcal{L}$ over V. Then, so is $N_{\widetilde{V}/Y_d}$ over Y_d where $\widetilde{V}$ is the strict transform of V. Hence, $g_d^{-1}(\widetilde{V})$ is a trivial $\mathbb{P}^{n-d-1}$-bundle over $\widetilde{V}$. $\widetilde{V}$ is isomorphic to $\mathbb{B}^d$. Consider a sequence of regular embeddings $Z \longrightarrow Y \longrightarrow X$, the strict transform of Y in the blow up of X along Z being isomorphic to the blow-up of Y along Z.

Let $x \in V$ be a point which does not lie in any linear subvariety strictly contained in V defined over $\mathbb{F}_q$. Let $g_d^{-1}(\widetilde{V}) \cong \widetilde{V} \times \mathbb{P}(\breve{N}_{V/\mathbb{P}^n,x})$ is a trivialization. $\breve{N}_{V/\mathbb{P}^n,x}$ is the fiber at x of the conormal bundle $\breve{N}_{V/\mathbb{P}^n}$. For $W \in Gr_*(\mathbb{P}^n)(\mathbb{F}_q)$ with $V \subset W \subset \mathbb{P}^n$, we have $N_{V/W} \subset N_{V/\mathbb{P}^n}$ and hence $\mathbb{P}(\breve{N}_{V/W}) \subset \mathbb{P}(\breve{N}_{V/\mathbb{P}^n})$. Thus $\mathbb{P}(\breve{N}_{V/W,x})$ is a linear subvariety of $\breve{N}_{V/\mathbb{P}^n,x}$. Hence the blowing-up of $g_d^{-1}(\widetilde{V})$ along the strict transform of such $W \supsetneq V$ coincides with the blowing-up of $g_d^{-1}(\widetilde{V}) \cong \widetilde{V} \times \mathbb{P}(\breve{N}_{V/\mathbb{P}^n,x})$ along $\mathbb{P}(\breve{N}_{V/\mathbb{P}^n,x})$. From this we get the isomorphism $D_V \cong \mathbb{B}^d \times \mathbb{B}^{n-d-1}$.

■

Corollary 1.2.5. For $W \in Gr_*(\mathbb{P}^n)(\mathbb{F}_q)$, $W \subsetneq V$, D_W intersects transversally with D_V and the intersection $D_W \cap D_V$ is of the form $\widetilde{D}_W \times \mathbb{B}^{n-d-1}$ on D_V where $\widetilde{D}_W$ is a divisor on $\mathbb{B}^d$ correspoding to the inclusion $W \hookrightarrow V \cong \mathbb{P}^d$.

Proposition 1.2.6. For $W' \in Gr_*(\mathbb{P}^n)(\mathbb{F}^q)$ strictly containing $V, D_{W'}$ intersects transversally with D_V and the intersection $D_{W'} \cap D_V$ is of the form

$\mathbb{B}^d \times \tilde{\tilde{D}}_{W'}$ on D_V where $\tilde{\tilde{D}}_{W'}$ is a divisor on $\mathbb{B}^{n-d-1}$ correspoding to the inclusion $\mathbb{P}(\check{N}_{V/W',x}) \hookrightarrow \mathbb{P}(\check{N}_{V/\mathbb{P}^n,x}) \cong \mathbb{P}^{n-d-1}$.

Corollary 1.2.7. Let H be an $\mathbb{R}$-divisor on $\mathbb{P}^n, V \in Gr_d(\mathbb{P}^n)(\mathbb{F}_q)$, and $f : \mathbb{B}^n \longrightarrow \mathbb{P}^n$ the nartural map. Also, $D_V \cong \mathbb{B}^d \times \mathbb{B}^{n-d-1}$. Let $p_1 : D_V \longrightarrow \mathbb{B}^d$ be the morphism to the first factor. Let $f^{'} : \mathbb{B}^d \longrightarrow \mathbb{P}^d$ be the map similar map to f. Then,

$$(f^*\mathcal{O}(H))|_{D_V} = p_1^*(f^{'*}(\mathcal{O}(H)|_{D_V})) \text{ in } H^2(D_V)$$

Proposition 1.2.8. For $V \in Gr_{n-1}(\mathbb{P}^n)(\mathbb{F}_q)$, we have

$$f^*\mathcal{O}(V) = \textstyle\sum_{W \in Gr_*(\mathbb{P}^n)(\mathbb{F}_q), W \subset V} D_W \text{ in } H^2(\mathbb{B}^n)$$

Proof. When we have $W \subset V$, the multiplicity of V along $W \in Gr_*(\mathbb{P}^n)(\mathbb{F}_q)$ is equal to 1. Hence in the expansion $f^*\mathcal{O}(V) = \sum_{W \in Gr_*(\mathbb{P}^n)(\mathbb{F}_q), W \subset V} a_W D_W$ we get $a_W = 1$. Hence we have the result.

■

Proposition 1.2.9. For an $\mathbb{R}$-divisor H on $\mathbb{P}^n$, $f^*\mathcal{O}(H)$ is $PGL_{n+1}(\mathbb{F}_q)$-invariant.

Proof. We have already seen that the construction of $\mathbb{B}^n$ is equivariant under the action of $PGL_{n+1}(\mathbb{F}_q)$. Hence f^* is also $PGL_{n+1}(\mathbb{F}_q)$-equivariant. Also $PGL_{n+1}(\mathbb{F}_q)$ acts trivially on $H^2(\mathbb{P}^n)$. From this we can conclude that $f^*\mathcal{O}(H)$ is $PGL_{n+1}(\mathbb{F}_q)$-invariant.

Now $H^2(\mathbb{P}^n)$ is generated by the class of $\mathcal{O}_{\mathbb{P}^n}(1)$. Let $H = \alpha\mathcal{O}_{\mathbb{P}^n}(1)$ in $H^2_{\mathbb{P}^n}$. The number of n-1 dimensional linear subvarieties over $\mathbb{F}_q = |Gr_{n-1}(\mathbb{P}^n)(\mathbb{F}_q)| = (q^{n+1}-1)/(q-1) = |\mathbb{P}^n(\mathbb{F}_q)|$ and for each $W \in Gr_d(\mathbb{P}^n)(\mathbb{F}_q)$, the number of elements(linear subvarieties) in $Gr_{n-1}(\mathbb{P}^n)(\mathbb{F}_q)$ containing $W = |\mathbb{P}^{n-d-1}(\mathbb{F}_q)|$. Thus we have

$$f^*\mathcal{O}(H) = \frac{\alpha}{|\mathbb{P}^n(\mathbb{F}_q)|} \sum_{V \in Gr_{n-1}(\mathbb{P}^n)(\mathbb{F}_q)} \Big(\sum_{W \in Gr_*(\mathbb{P}^n)(\mathbb{F}_q), W \subset V} D_W \Big)$$

$$= \frac{\alpha}{|\mathbb{P}^n(\mathbb{F}_q)|}\sum_{d=0}^{n-1}|\mathbb{P}^{n-d-1}(\mathbb{F}_q)|D_d$$

which is $PGL_{n+1}(\mathbb{F}_q)$-invariant.

■

Proposition 1.2.10. D be a $PGL_{n+1}(\mathbb{F}_q)$-invariant $\mathbb{R}$-divisor on $\mathbb{B}^n$ which is written as

$$D = \alpha f^*\mathcal{O}_{\mathbb{P}^n}(1) + \textstyle\sum_{d=0}^{n-1} a_d D_d \qquad (\alpha, a_d \in \mathbb{R}) \qquad \text{in } H^2(\mathbb{B}^n)$$

Then D is positive iff $\alpha > 0$ and

$$a_d + \alpha\tfrac{|\mathbb{P}^{n-d-1}(\mathbb{F}_q)|}{|\mathbb{P}^n(\mathbb{F}_q)|} > 0 \text{ for all } 0 \le d \le n-1.$$

Proof. We are given that

$$D = \alpha f^*\mathcal{O}_{\mathbb{P}^n}(1) + \sum_{d=0}^{n-1} a_d D_d$$

From the above calculation in the previous proposition, we can rewrite this as

$$D = \sum_{d=0}^{n-1}(\alpha\frac{|\mathbb{P}^{n-d-1}(\mathbb{F}_q)|}{|\mathbb{P}^n(\mathbb{F}_q)|} + a_d)D_d$$

Hence from this expression we can conclude that D is positive if and only if $\alpha > 0$

$$a_d + \alpha\tfrac{|\mathbb{P}^{n-d-1}(\mathbb{F}_q)|}{|\mathbb{P}^n(\mathbb{F}_q)|} > 0 \text{ for all } 0 \le d \le n-1.$$

■

Lemma 1.2.11. Let D be a $PGL_{n+1}(\mathbb{F}_q)$-invariant $\mathbb{R}$-divisor on $\mathbb{B}^n$. Let $V \in Gr_d(\mathbb{P}^n)(\mathbb{F}_q)$. We have the ismomorphism $D_V \cong \mathbb{B}^d \times \mathbb{B}^{n-d-1}$. Let $p_1 : D_V \longrightarrow \mathbb{B}^d$ and $p_2 : D_V \longrightarrow \mathbb{B}^{n-d-1}$ be the projections. Then there is a $PGL_{d+1}(\mathbb{F}_q)$-invariant $\mathbb{R}$-divisor $D^{'}$ on $\mathbb{B}^d$ and a $PGL_{n-d}(\mathbb{F}_q)$-invariant $\mathbb{R}$-divisor $D^{''}$ on $\mathbb{B}^{n-d-1}$ such that

$$\mathcal{O}(D)|_{D_V} = p_1^*\mathcal{O}(D^{'}) + p_2^*\mathcal{O}(D^{''}) \text{ in } H^2(D_V)$$

Proof. To prove this result let us write

$$D = \alpha f^*\mathcal{O}_{\mathbb{P}^n}(1) + \textstyle\sum_{0\le k\le n-1, k\ne d} a_k D_k \qquad (\alpha, a_k \in \mathbb{R}) \qquad \text{in } H^2(\mathbb{B}^n)$$

Then there is no self-intersection and we can consider the case when $D = \alpha f^*\mathcal{O}_{\mathbb{P}^n}(1)$ and $D = D_k$ cases separately.

The case $D = \alpha f^*\mathcal{O}_{\mathbb{P}^n}(1)$ is a direct consequence of the corollary 1.2.9 and proposition 1.2.11. The case $D = D_k$ is a consequence of proposition 1.2.6, 1.2.7 and 1.2.8.

■

Proposition 1.2.12. Let D be an ample $PGL_{n+1}(\mathbb{F}_q)$-invariant $\mathbb{R}$-divisor on $\mathbb{B}^n$. Then D is positive.

Proof. The proof is by induction on n. The case $n = 1$ is obvious. Assume the result proved for upto $n-1$. An $\mathbb{R}$-divisor D on $\mathbb{B}^n$ is written uniquely as

$$D = f^*\mathcal{O}(H) + \sum_{V \in Gr_*(\mathbb{P}^n)(\mathbb{F}^q), 0 \leq k \leq n-2} a_V D_V$$

where H is an $\mathbb{R}$-divisor on $\mathbb{P}^n$. Rewriting this gives

$$D = \alpha f^*\mathcal{O}_{\mathbb{P}^n}(1) + \sum_{d=0}^{n-2} a_d D_d$$

Using the above proposition 1.2.12 we have D is positive if and only if $\alpha > 0$ and

$$a_d + \alpha \frac{|\mathbb{P}^{n-d-1}(\mathbb{F}_q)|}{|\mathbb{P}^n(\mathbb{F}_q)|} > 0$$

for all $0 \leq k \leq n-2$. Fix $V \in Gr_{n-1}(\mathbb{P}^n)(\mathbb{F}^q)$. Then we have $D_V \cong \mathbb{B}^{n-1}$.

Let us denote by $f^{'}$ the map $\mathbb{B}^{n-1} \longrightarrow \mathbb{P}^{n-1}$ obtained as in the construction of $\mathbb{B}^n$. Let $'$ be the distinction in the case when we are talking in the $n-1$ case. Consider the restriction of $\mathcal{O}(D)$ to D_V. We have

$$\mathcal{O}(D)|_{D_V} = \alpha f^*\mathcal{O}_{\mathbb{P}^{n-1}}(1) + \sum_{d=0}^{n-2} a_d D_d^{'}$$

By induction $\mathcal{O}(D)|_{D_V}$ is positive. Thus we have $\alpha > 0$. We have

$$a_d + \alpha \frac{|\mathbb{P}^{n-d-2}(\mathbb{F}_q)|}{|\mathbb{P}^{n-1}(\mathbb{F}_q)|} > 0$$

for all $0 \leq d \leq n-2$. Hence we are through if we can show that $\frac{|\mathbb{P}^{n-d-2}(\mathbb{F}_q)|}{|\mathbb{P}^{n-1}(\mathbb{F}_q)|} < \frac{|\mathbb{P}^{n-d-1}(\mathbb{F}_q)|}{|\mathbb{P}^n(\mathbb{F}_q)|}$ since then we satisfy the above mentioned condition.

We have the above inequality if and only if

$$\frac{(q^{n-d-1}-1)/(q-1)}{(q^n-1)/(q-1)} < \frac{(q^{n-d}-1)/(q-1)}{(q^{n+1}-1)/(q-1)}$$

if and only if

$$(q^{n-d-1}-1)(q^{n+1}-1) < (q^{n-d}-1)(q^n-1)$$

if and only if

$$q^{n-d} < q^n$$

which is true. Hence we have the proposition.

■

1.2.3 The Drinfeld Upper Half spaces $\widehat{\Omega}^d_K$

The Drinfeld upper half spaces $\widehat{\Omega}^d_K$ of dimension d over K is the rigid analytic space obtained by removing all K-rational hyperplanes from $\mathbb{P}^d_K$.

Consider the projective space $\mathbb{P}^d_{\mathcal{O}_K}$ over $\mathcal{O}_K$. We blow up $\mathbb{P}^d_{\mathcal{O}_K}$ along the linear subvarieties in the special fibre of $\mathbb{P}^d_{\mathbb{F}_q}$ successively, as in the construction of $\mathbb{B}^n$. We continue this process along all exceptional divisors occuring in the blow-up. In this way we obtain a formal scheme $\widehat{\Omega}^d_{\mathcal{O}_K}$ over the formal spectrum Spf $\mathcal{O}_K$. The rigid analytic space associated to $\widehat{\Omega}^d_{\mathcal{O}_K}$ is isomorphic to the space $\widehat{\Omega}^d_K$.

Let $\Gamma \subset PGL_{d+1}(K)$ be a torsion free discrete subgroup. Then there is a natural action of Γ on $\widehat{\Omega}^d_{\mathcal{O}_K}$. Then we take the quotient $\widehat{\mathfrak{X}}_\Gamma := \Gamma\backslash\widehat{\Omega}^d_{\mathcal{O}_K}$ as a formal scheme.

Theorem 1.2.13. *The relative dualizing sheaf $\omega_{\widehat{\mathfrak{X}}_\Gamma/\mathcal{O}_K}$ is ample and invertible.*

For the proof of this theorem refer to the paper by Mustafin and Kurihara.

From this we conclude that $\widehat{\mathfrak{X}}_\Gamma$ can be algebraized to a projective scheme $\mathfrak{X}_\Gamma$ over $\mathcal{O}_K$. The generic fiber $X_\Gamma := \mathfrak{X}_\Gamma \otimes K$ is a projectie smooth variety over K. The rigid analytic space associated to this space is the rigid analytic quotient $\widehat{\mathfrak{X}}_\Gamma := \Gamma\backslash\widehat{\Omega}^d_K$

Chapter 2

Hard Lefschetz and Hodge Standard Conjecture

2.1 The Hard Leftschetz Conjecture

Let F be an algebraically closed field of any charasteric and l a prime number different from characteristic of F. X be a projective scheme over F of dimension n. Let $cl_k^X : C^k(X) \longrightarrow H_{ét}^{2k}(X, \mathbb{Q}_l(k))$ be the cycle map for the l-adic cohomology where $C^k(X)$ is the group of algebraic cycles of codimension k. Let us denote by $C_{num}^k(X) \subset C^k(X)$ the subgroup of algebraic cycles numerically equivalent to zero and $N^k(X) := C^k(X)/C_{num}^k(X)$ a finitely generated $\mathbb{Z}$-module.

[Assumption] cl^k ***induces an isomorphism***

$$\mathbf{N^k}(\mathbf{X}) \otimes_{\mathbb{Z}} \mathbb{Q}_\mathbf{l} \cong \mathbf{H}_{\mathbf{ét}}^{\mathbf{2k}}(\mathbf{X}, \mathbb{Q}_\mathbf{l}(\mathbf{k}))$$

We define $H^k(X)$ to be $N^{k/2}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ when k is even and 0 when k is odd. We denote by $H^*(X) = \bigoplus_k H^k(X)$ which can be considered as a cohomology with coefficients in $\mathbb{R}$.

Definition 2.1.1. A $\mathbb{Q}$-divisor is a formal sum of divisors with co-efficients in $\mathbb{Q}$. A $\mathbb{Q}$-divisor is called an ample if it is of the form $L = a_1L_1 + ... + a_rL_r$ with $r \geq 1, a_1, ..., a_r \in \mathbb{Q}_{>0}$ and $L_1, ..., L_r$ ample. An $\mathbb{R}$-divisor is defined similarly.

There is a natural map from the group of $\mathbb{R}$-divisors on X to $H^2(X)$. Hence we can identify an $\mathbb{R}$-divisor, a formal sum of $\mathbb{R}$-line bundles and its class in $H^2(X)$.

Definition 2.1.2. By taking the cup product with L, we get an $\mathbb{R}$-linear map

$$L : H^k(X) \longrightarrow H^{k+2}(X)$$

We call L the Leftschetz operator.

Conjecture 1 : (Hard Leftschetz Conjecture)
For all k, L^k induces an isomorphism

$$L^k : H^{n-k}(X) \longrightarrow H^{n+k}(X)$$

2.2 Hodge Standard Conjecture

Definition 2.2.1. Assume that the conjecture holds for (X, L) We define the primitive part $P^k(X)$ by

$$P^k(X) = ker(L^{n-k+1} : H^k(X) \longrightarrow H^{2n-k+2}(X))$$

for $0 \leq k \leq n$ and in the case of $k < 0$ and $k > 0$, we define $P^k(X) = 0$.

Then we have a decomposition called the primitive decomposition

$$H^k(X) = \bigoplus_{i \geq 0} L^i P^{k-2i}(X) \cong \bigoplus_{i \geq 0} P^{k-2i}(X)$$

for all k.

Definition 2.2.2. We define a pairing $\langle , \rangle_{H^k(X)}$ by the composite of the following two maps.

$$\begin{array}{cccccc} \langle , \rangle_{H^k(X)} : H^k(X) \times H^k(X) & \longrightarrow & H^{2n}(X) & \xrightarrow{\sigma} & R \\ (x, y) & \longrightarrow & (-1)^{k/2} L^{n-k} x \cup y & & \end{array}$$

where the map

$$\sigma : H^{2n}(X) = \bigoplus_{i=1}^{m} H^{2n}(X_i) = \bigoplus_{i=1}^{m} \mathbb{R} \longrightarrow \mathbb{R}$$

sends $(a_1, ..., a_m) \longrightarrow \sum a_i$ for $a_i \in \mathbb{R}$ for $0 \leq k \leq n$. Otherwise we define the pairing to be the zero pairing.

The pairing $\langle , \rangle_{H^k(X)}$ are non-degenerate by Poincare duality and the Hard Leftschetz conjecture. Let us denote the restriction of $\langle , \rangle_{H^k(X)}$ to $P^k(X)$ by $\langle , \rangle_{P^k(X)}$. For all k with $0 \leq k \leq n$, $\langle , \rangle_{H^k(X)}$ is isomorphic to the alternating sum $\langle , \rangle_{P^{k-2i}(X)}$ for $i \geq 0$.

$$\langle , \rangle_{H^k(X)} = \sum_{i \geq 0} (-1)^i \langle , \rangle_{P^{k-2i}(X)} \ on \ H^k(X) \cong \bigoplus_{i \geq 0} P^{k-2i}(X)$$

Also the primitive decompostion above is an orthogonal decomposition with respect to $\langle , \rangle_{H^k(X)}$. Hence the pairing $\langle , \rangle_{P^k(X)}$ is non-degenerate. The primitive decomposition may depend on the choice of L. However the dimension of $P^k(X)$ is independant of the choice of L.

Assume that the the hard leftschetz conjecture holds for (X, L) along with with assumption. Then

Conjecture 2 : (Hodge Standard Conjecture)
For all k, the pairing $\langle , \rangle_{P^k(X)}$ is positive definite.

Proposition 2.2.3. Let $L, L^{'}$ be ample $\mathbb{R}$-divisors on X. If the Hard Leftschetz conjecture holds for $(X, tL + (1-t)L^{'})$ for all $0 \leq t \leq 1$, then the Hodge standard conjecture for (X, L) and $(X, L^{'})$ are equivalent to each other. In particular if the Hard Leftschetz conjecture holds for (X, L) for all ample $\mathbb{R}$-divisors, then the Hodge standard conjecture for (X, L) for all ample $\mathbb{R}$-divisors are equivalent.

Proof. For a given pairing $\langle , \rangle_{H^k(X)}$, let us denote the difference between number of positive eigenvalues and negative values by $sign(H^k(X), \langle , \rangle_{H^k(X)})$.

This is independant of the basis chosen to count the positive/negative eigenvalues.

If we show that the Hodge standard conjecture holds for (X, L) if and only if

$$sign(H^k(X), \langle,\rangle_{H^k(X)}) = \sum_{i\geq 0}(-1)^i dim_{\mathbb{R}} P^{k-2i}(X) \qquad (*)$$

for all $0 \leq k \leq n$, since $sign$ is locally constant on the set of non-degenerate quadratic forms over $\mathbb{R}$, the proposition follows.

Hence it is enough to show that the above condition is an equivalence for the Hodge standard conjecture on (X, L). We have

$$dim_{\mathbb{R}} P^k(X) = dim_{\mathbb{R}} H^k(X) - dim_{\mathbb{R}} H^{k-2}(X)$$

which implies that the right hand side of $(*)$ is independant of choice of L. Since

$$\langle,\rangle_{H^k(X)} = \sum_{i\geq 0}(-1)^i \langle,\rangle_{P^{k-2i}(X)} \text{ } on \text{ } H^k(X) \cong \bigoplus_{i\geq 0} P^{k-2i}(X)$$

we have that for all $0 \leq k \leq n$

$$sign(H^k(X), \langle,\rangle_{H^k(X)}) = \sum_{i\geq 0}(-1)^i sign(P^{k-2i}(X), \langle,\rangle_{P^{k-2i}(X)}) \qquad (**)$$

Now the Hodge standard conjecture is equivalent to

$$dim_R P^k(X) = sign(P^k(X), \langle,\rangle_{P^k(X)}) \qquad (***)$$

for all k such that $0 \leq k \leq n$. Hence the equality $(**)$ implies $(*)$.

Now let us assume $(*)$. We shall prove $(***)$. The proof is by induction on k. For the case $k = 0$, $P^0(X) \cong \mathbb{R}$ and since L^n is positive, the result follows. Assume $(***)$ proved for $k < l \leq n$. Then $(*)$ implies

$$sign(H^l(X), \langle,\rangle_{H^l(X)}) = \sum_{i\geq 0}(-1)^i dim_{\mathbb{R}} P^{l-2i}(X)$$

$$= dim_{\mathbb{R}} P^l(X) + \sum_{i\geq 1}(-1)^i sign(P^{l-2i}(X), \langle,\rangle_{P^{l-2i}(X)})$$

now with the equality $\langle,\rangle_{H^k(X)} = \sum_{i\geq 0}(-1)^i\langle,\rangle_{P^{k-2i}(X)}$, we get the $(***)$ for l. Hence we get the result.

■

Theorem 2.2.4. *Let D be an ample $PGL_{n+1}(\mathbb{F}_q)$-invariant $\mathbb{R}$-divisor on $\mathbb{B}^n$ where $\mathbb{B}^n$ be the space obtained by successive blowing up of Projective Space along linear varieties. Then the hard Leftschetz conjecture and the standard Hodge conjecture holds for $(\mathbb{B}^n, D)$.*

Monodromy filtration

Let X be a proper smooth variety over K. Let $V := H^w_{et}(X_{\overline{K}}, \mathbb{Q}_l)$ where l is a prime numer. Let I_K be the inertia group of K. Then $Gal(\overline{\mathbb{F}}_Q/\mathbb{F}_Q)$ acts of I_K by conjugation. For $\tau \in Gal(\overline{\mathbb{F}}_Q/\mathbb{F}_Q)$ and $\sigma \in I_K$,

$$\tau : \sigma \longrightarrow \tau\sigma\tau^{-1}$$

The pro l-part of I_K is isomorphic to $\mathbb{Z}_l(1)$ as a $Gal(\overline{\mathbb{F}_q}/\mathbb{F}_q)$-module. By Grothendieck's monodromy theorem, there exists $r, s \geq 1$ such that $(\rho(\sigma)^r - 1)^s = 0$ for all $\sigma \in I_K$. Therefore I_K acts unipotently on V. Then there is a unique nilpotent map of $Gal(\overline{K}/K$-representations called monodromy operator $N : V(1) \longrightarrow V$ such that $\rho(\sigma) = \exp\ (t_l(\sigma)N)\ \ \forall\ \sigma \in I_K$

Definition 2.2.5. There exists a unique filtration $M_\bullet$ on V called the monodromy filtration characterized by the following characteristics.

1. $M_\bullet$ is an increasing filtration. i.e ...... $\subset M_{i-1}V \subset M_iV \subset M_{i+1}V \subset$ ..... of $Gal(\overline{K}/K)$-representations such that $M_iV = 0$ for sufficiently small i and $M_iV = V$ for sufficiently large i .

2. $N(M_i(V(1)) \subset M_{i-2}V$

3. Let us define $Gr^M_i V := M_iV/M_{i-1}V$. Then by above, we can define $N : Gr^M_i V(1) \longrightarrow Gr^M_{i-2}V$ satisfying $N^r : Gr^M_r V(1) \longrightarrow Gr^M_{-r}V$ is an isomorphism.

Weight filtration

Let $Fr_q \in Gal(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ be the inverse of the q-th power map on $\overline{\mathbb{F}}_q$. A $Gal(\overline{\mathbb{F}}_q/\mathbb{F}_q)$-representation is sadi to have a weight k if all eigenvalues of the action of $Fr_q \in Gal(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ are algebraic integers whose conjugates have complex absolute value $q^{k/2}$.

Definition 2.2.6. There exists a unique filtration $W_\bullet$ on V called the weight filtratoin characterized by the following properties.

1. $W_\bullet$ is an increasing filtration. i.e ..... $\subset W_{i-1}V \subset W_iV \subset W_{i+1}V \subset$ .... of $Gal(\overline{K}/K)$-representations such that$W_iV = 0$ for sufficiently small i and $W_iV = V$ for sufficiently large i .

2. For a lift $\widetilde{Fr}_q$ of $Fr_q \in Gal(\overline{K}/K)$, all eigenvalues of the action of $\widetilde{Fr}_q$ on each $Gr_i^W V := W_iV/W_{i-1}V$ are algebraic integers whose all complex conjugates have complex absolute value $q^{i/2}$.

Theorem 2.2.7. *Let X be a proper smooth variety over K which has a proper strictly semistable model $\mathfrak{X}$ over $\mathcal{O}_K$. Let $X_1, ..., X_m$ be irreducible components of the special fiber of $\mathfrak{X}$. Let $\mathfrak{X}$ be projective over $\mathcal{O}_K$ with an ample line bundle $\mathcal{L}$. If for $1 \leq i_1 \leq ..... \leq i_k \leq m$ every irreducible component Y of $X_{i_1} \cap ... \cap X_{i_k}$ satisfies the assumption above, and the Hodge standard conjecture holds for $(Y, \mathcal{L}|_Y)$. Then*

$$M_iV = W_{i+w}V \quad \forall i$$

holds for X

Let us accept the notations used in section 1.3. We have that $\mathfrak{X}_\Gamma$ is a strictly semistable model of the generic fiber X_Γ over $\mathcal{O}_K$.

Theorem 2.2.8. *Let $\Gamma \subset PGL_{d+1}(\mathbb{F}_q)$ be a cocompact torsion free discrete subgroup. Then we have*

$$M_iV = W_{i+w}V \quad \forall i$$

for X_Γ

Chapter 3

Cohomology of Varieties obtained by Blowing-up

3.1 Some Fundamental Thoerems

Let X be a base scheme. let C/X denote the full subcategory of Sch/X satisfying the following two conditions.

(a) C/X is closed under fiber products.

(b) For any $Y \longrightarrow X$ in C/X and an E-morphism $U \longrightarrow Y$, the morphism $U \longrightarrow Y \longrightarrow X$ is in C/X.

Definition 3.1.1. An E**-covering** of $Y \in C/X$ is a family of E-morphims $(U_i \xrightarrow{f_i} Y)_{i \in I}$ such that $Y = \cup_{i \in I} f_i(U_i)$. The class of all such coverings on all such objects is called the E**-topology** on C/X. The category C/X with the E-topology is called the E**-Site** over X, written as $(C/X)_E$ or as X_E.

Remark. The category C/X together with the family of E-coverings is a Grothendieck topoplogy.

The E-site $(E/X)_E$ is called the *small E-site* on X when $E = (Zar)$ or $E = (ét)$. In the case when all E-morphism is locally of finite type, the *big E-site* is the full subcategory $(LFT/X)_E$ of Sch/X consisting of objects with locally finite type structure morphisms.

Definition 3.1.2 (Morphism of sites)**.** Let $(C^{'}/X^{'})_{E^{'}}$ and $(C/X)_E$ be sites. A morphism $\pi : X^{'} \longrightarrow X$ of schemes defines a morphism of sites if the following conditions are satisfied :

(a) for $Y \in C/X$, $Y_{X'}$ is in C'/X';

(b) for any E-morphism $U \longrightarrow Y$ in C/X, $U_{(X')} \longrightarrow Y_{(X')}$ is an E'-morphism.

π is referred to as a continuous morphism $\pi : X'_{E'} \longrightarrow X_E$.

Since the base change of a surjective family of morphisms is again surjective, π gives a functor $\pi^{\bullet} : C/X \longrightarrow C'/X'$ sending $(Y \longrightarrow Y_{(X')})$ taking coverings to coverings.

Definition 3.1.3. $\pi : X'_{E'} \longrightarrow X_E$ be continuous. $\mathcal{P}'$ be a presheaf on X'. We associate the presheaf $\pi_p(\mathcal{P}') = \mathcal{P}' \circ \pi$. The presheaf $\pi_p(\mathcal{P}')$ is called the direct image of $\mathcal{P}'$. π_p defines a functor $\mathbb{P}(X'_{E'}) \longrightarrow \mathbb{P}(X_E)$. We define the inverse image functor $\pi^p : \mathbb{P}(X_E) \longrightarrow \mathbb{P}(X'_{E'})$ to be the left adjoint of π_p. Hence

$$Hom_{\mathbb{P}(X'_{E'})}(\pi^p(\mathcal{P}), \mathcal{P}') = Hom_{\mathbb{P}(X_E)}(\mathcal{P}, \pi_p(\mathcal{P}'))$$

.

Definition 3.1.4. Let $\pi : X' \longrightarrow X$ be a morphism of sites $(C'/X')_{E'} \longrightarrow (C/X)_E$. The direct image of a sheaf $\mathcal{F}'$ on $X'_{E'}$ is defined to be $\pi_*\mathcal{F}' = \pi_p\mathcal{F}'$. The inverse image of a sheaf $\mathcal{F}$ on X_E is defined to be $\pi^*\mathcal{F} = a(\pi^p\mathcal{F})$, the sheaf associated to the presheaf $\pi^p\mathcal{F}$. There are canonical ismorphisms

$$Hom_{\mathbb{S}(X_E)}(\mathcal{F}, \pi_*\mathcal{F}') = Hom_{\mathbb{P}(X'_{E'})}(\pi^p\mathcal{F}, \mathcal{F}') = Hom_{\mathbb{S}(X'_{E'})}(\pi^*\mathcal{F}, \mathcal{F}')$$

Hence π_* and π^* are adjoint functors $\mathbb{S}(X'_{E'}) \rightleftarrows \mathbb{S}(X_E)$.

Now let X be a scheme and U an open subscheme of X. Z be the subscheme with underlying space $X \backslash U$. Let

$$Z \xrightarrow{i} X \xleftarrow{j} U$$

be the inclusions. $\mathcal{F}$ is a sheaf on $X_{\acute{e}t}$. Let us denote $\mathcal{F}_1 = i^*\mathcal{F}$ and $\mathcal{F}_2 = j^*\mathcal{F}$ sheaves on Z and U respectively. Since $Hom(\mathcal{F}, j_*j^*\mathcal{F}) \cong Hom(j^*\mathcal{F}, j^*\mathcal{F})$, there is a canonical isomorphism $F \longrightarrow j_*j^*\mathcal{F}$ corresponding to the identity.

By applying i^* we get a canonical morphism $\phi_{\mathcal{F}} : \mathcal{F}_1 \longrightarrow i^*j_*\mathcal{F}$. Define $T(X)$ to be the category whose objects are triples $(\mathcal{F}_1, \mathcal{F}_2, \phi)$ with $\mathcal{F}_1 \in \mathbb{S}(Z_{\acute{e}t})$ and $\mathcal{F}_2 \in \mathbb{S}(U_{\acute{e}t})$ and $\phi : \mathcal{F}_1 \longrightarrow i^*j_*\mathcal{F}_2$. A morphism of triples $(\mathcal{F}_1, \mathcal{F}_2, \phi) \longrightarrow (\mathcal{F}_1^{'}, \mathcal{F}_2^{'}, \phi^{'})$ is a pair (ψ_1, ψ_2) where ψ_1 is a morphism $\mathcal{F}_1 \longrightarrow \mathcal{F}_1^{'}$ and $\psi_2 : \mathcal{F}_2 \longrightarrow \mathcal{F}_2^{'}$ and they are compatible with ϕ and $\phi^{'}$.

$$\begin{tikzcd}
F_1 \arrow[r, "\phi"] \arrow[d, "\psi_1"] & i^*j_*F_2 \arrow[d, "i^*j_*(\psi_2)"] \\
F_1^{'} \arrow[r, "\phi^{'}"] & i^*j_*F_2^{'}
\end{tikzcd}$$

This makes it into a category $\mathbb{T}(X)$ which is equivalent to the category of sheaves $\mathbb{S}(X_{\acute{e}t})$.

In this context we can define the following functors which are adjoint to the one below it. We identify the category $\mathbb{S}(X_E)$ with $\mathbb{T}(X)$.

Definition 3.1.5. $i^* : \mathbb{T}(X) \longrightarrow \mathbb{S}(Z)$ sending the triple $(F_1, F_2, \phi) \longmapsto F_1$

$i_* : \mathbb{S}(Z) \longrightarrow \mathbb{T}(X)$ sending $F_1 \longmapsto (F_1, 0, 0)$

$i^! : \mathbb{T}(X) \longrightarrow \mathbb{S}(Z)$ sending the triple $(F_1, F_2, \phi) \longmapsto \ker \phi$

Definition 3.1.6. $j_! : \mathbb{S}(U) \longrightarrow \mathbb{T}(X)$ sending the triple $F_2 \longmapsto (0, F_2, 0)$

$j^* : \mathbb{T}(X) \longrightarrow \mathbb{S}(U)$ sending $(F_1, F_2, \phi) \longmapsto F_2$

$j_* : \mathbb{S}(U) \longrightarrow \mathbb{T}(X)$ sending the triple $F_2 \longmapsto (i^*j_*F_2, F_2, 1)$

$j_!$ is called the "extension by zero" and $i^!$ is "form subsheaf of sections with support on Z".

Recall that an object I of an abelian category $\mathfrak{A}$ is called an injective object if $Hom_{\mathfrak{A}}(\centerdot\,, I) : \mathfrak{A} \longrightarrow \mathfrak{Ab}$ is an exact functor. $\mathfrak{A}$ is said to have enough injectives if for every $M \in \mathfrak{A}$ there is a monomorphism from M into an injective object.

Definition 3.1.7. Let $\mathfrak{A}$ be an abelian category with enough injectives and $f : \mathfrak{A} \longrightarrow \mathfrak{B}$ be a left exact functor where $\mathfrak{B}$ is another abelian catogory. We define the right derived functors $R^i f : \mathfrak{A} \longrightarrow \mathfrak{B}$, $i > 0$, the unique sequence of functors satisfying the following conditions :

(a) $R^0 f = f$

(b) $R^i f(I) = 0$ if I is injective and $i > 0$

(c) for any exact sequence $0 \longrightarrow M^{'} \longrightarrow M \longrightarrow M^{''} \longrightarrow 0$ in $\mathfrak{A}$, there are morphisms $\delta_i : R^i f(M^{''}) \longrightarrow R^{i+1}(M^{'})$, $i \geq 0$ such that

$$....... \longrightarrow R^i f(M) \longrightarrow R^i f(M^{''}) \longrightarrow R^{i+1} f(M^{'}) \longrightarrow R^{i+1} f(M) \longrightarrow ....$$

is exact. The associaltion of this long exact sequence to a short exact sequence being functorial.

Definition 3.1.8. An object M in $\mathfrak{A}$ is called f-acyclic if $R^i f(M) = 0$ for all $i > 0$. If

$$0 \longrightarrow M \longrightarrow N^0 \longrightarrow N^1 \longrightarrow N^2 \longrightarrow .....$$

is a resolution of M by f-acyclic objects N^i, then the objects $R^i f(M)$ are canonically isomorphic to the cohomology objects of the complex

$$0 \longrightarrow fN^0 \longrightarrow fN^1 \longrightarrow fN^2 \longrightarrow ....$$

Theorem 3.1.9. *The category $\mathbb{S}(X_E)$ has enough injectives.*

Hence we can define the right derived functors of any left exact functor from $\mathbb{S}(X_E)$ into an abelian category.

Definition 3.1.10. The functor $\Gamma(X, -) : \mathbb{S}(X_E) \longrightarrow \mathfrak{Ab}$ is left exact. Its right derived functors are written

$$R^i\Gamma(X, -) = H^i(X, -) = H^i(X_E, -)$$

The group $H^i(X_E, F)$ is called the i^{th}-cohomology group of X_E with values in F.

Let $F^{\boldsymbol{\cdot}}$ and $G^{\boldsymbol{\cdot}}$ be complexes of sheaves over a scheme X. We write $F^{\boldsymbol{\cdot}} \otimes G^{\boldsymbol{\cdot}}$ for the total complex of the double complex $(F^r \times G^s)$. i.e

$$(F^{\boldsymbol{\cdot}} \otimes G^{\boldsymbol{\cdot}})^m = \sum_{r+s=m} F^r \otimes G^s$$

and

$$d^m = \sum_{r+s=m} d_F^r \otimes 1 + (-1)^r 1 \otimes d_G^s$$

Definition 3.1.11. A map $A_1^\cdot \longrightarrow A_2^\cdot$ is said to be a quasi-isomorphim if the induced maps on cohomology $H^r(A_1^\cdot) \cong H^r(A_2^\cdot)$ are isomorphic for all r. It is written as $A_1^\cdot \simeq A_2^\cdot$

Lemma 3.1.12. Let $g : G_1^\cdot \longrightarrow G_2^\cdot$ be a quasi-isomorphism and let $F^\cdot$ be a bounded above sequence of flat sheaves. Then $1 \otimes g : F^\cdot \otimes G_1^\cdot \longrightarrow F^\cdot \otimes G_2^\cdot$ is a quasi-isomorphism if one of the following two conditions is satisfied :

(a) $G_1^\cdot$ and $G_2^\cdot$ are bounded above.

(b) $F^\cdot$ is bounded above.

Proposition 3.1.13. If $F^\cdot$ is a complex of sheaves such that $H^r(F^\cdot) = 0$ for $r \gg 0$, then there exists a quasi-isomorphism $P^\cdot \longrightarrow F^\cdot$ with $P^\cdot$ a bounded above complex of flat sheaves.

Theorem 3.1.14. *If $f : X \longrightarrow S$ be a proper morphism with S quasi-compact. For any complex $F^\cdot$ of sheaves on X, there is a quasi-isomorphism $F^\cdot \simeq A^\cdot(F^\cdot)$ with $A^\cdot(F^\cdot)$ a complex of f_*-acyclic sheaves. If $H^r(F^\cdot) = 0$ for $r \ll 0$, then we can take $A^\cdot(F^\cdot)$ to be a bounded below complex of injectives. If $F^\cdot \simeq A_1^\cdot$ and $F^\cdot \simeq A_2^\cdot$ with $A_1^\cdot$ and $A_2^\cdot$ complexes of f_*-acyclics, then $f_*A_1^\cdot \cong f_*A_2^\cdot$. Now let $\alpha : F_1^\cdot \longrightarrow F_2^\cdot$ be a map of complexes, then we can choose $A^\cdot(F_1^\cdot)$ and $A^\cdot(F_2^\cdot)$ making the following diagram commutative.*

$$\begin{tikzcd} F_1^\cdot \arrow[r, "\alpha"] \arrow[d, "\sim"] & F_2^\cdot \arrow[d, "\sim"] \\ A^\cdot(F_1^\cdot) \arrow[r, "\beta"] & A^\cdot(F_2^\cdot) \end{tikzcd}$$

If α is a quasi-isomorphism, then so is $f_*\beta$.

Let S be as above and let $f : X \longrightarrow S$ be a compactifiable morphism with compactification $X \overset{j}{\hookrightarrow} \overline{X} \xrightarrow{\overline{f}} S.$ For any complex of sheaves $F^\cdot$ on

X we write $\mathbf{R}_c f_* F^{\cdot}$ for $\overline{f}_* A^{\cdot}(j_! F^{\cdot})$ where $A^{\cdot}(j_! F^{\cdot})$ is a complex of $\overline{f}_*$-acyclic sheaves as above. By the above theorem $\mathbf{R}_c f_* F^{\cdot}$ is well-defined upto quasi-isomorphism. When $F^{\cdot}$ consists of a single sheaf F, $A^{\cdot}(j_! F^{\cdot})$ may be taken to be the injective resolution of F and $H^r(\mathbf{R}_c f_* F^{\cdot}) = R_c f_* F$. If $H^r(F^{\cdot}) = 0$ for $r \gg 0$, then $H^r(\mathbf{R}_c f_* F^{\cdot}) = 0$ for $r \gg 0$ and by the above proposition there is a bounded above complex of flat sheaves $\overline{\mathbf{R}}_c f_* F^{\cdot}$ and a quasi-isomorphism $\overline{\mathbf{R}}_c f_* F^{\cdot} \longrightarrow \mathbf{R}_c f_* F^{\cdot}$. In this case we always choose $A^{\cdot}(j_! F^{\cdot})$ and hence $\mathbf{R}_c f_* F^{\cdot}$ to be bounded below.

Theorem 3.1.15 (Künneth formula)**.** *Let F and G be sheaves on X and Y respectively, with F flat. Now consider the following diagram :*

$$\begin{tikzcd}
 & X \times_S Y \arrow[dl, "p"'] \arrow[dd, "h"] \arrow[dr, "q"] & \\
X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\
 & S &
\end{tikzcd}$$

with S quasi-compact. Let f and g be compactifiable. Denote by $F \boxtimes G$ the sheaf $p^*F \otimes q^*G$ on $X \times_S Y$. Then with the above notations, we have the following canonical quasi isomorphism

$$\overline{\boldsymbol{R}}_c f_* F \otimes \boldsymbol{R}_c g_* G \simeq \boldsymbol{R}_c h_*(F \boxtimes G)$$

Let F be a sheaf on $X_{\acute{e}t}$. Let F be a sheaf on $X_{\acute{e}t}$. Then $i_* i^! F$, for a closed immersion $i : Z \hookrightarrow X$ of smooth S-schemes, is the largest sheaf on X that is zero outside Z. Let $U = X - Z$. The functor $F \longrightarrow \Gamma(X, i_* i^! F) = \Gamma(Z, i^! F) = \ker(F(X) \longrightarrow F(U))$ is a left exact functor, the right derived functors of which are denoted $H^p_Z(X, F)$. A smooth S-pair (Z, X) is a closed immersion $i : Z \hookrightarrow X$ of smooth S-schemes. Then we have the commutative diagram

$$\begin{tikzcd}
Z \arrow[r, "i"'] \arrow[dr] & X \arrow[d] & U \arrow[l, "j"] \arrow[dl] \\
 & S &
\end{tikzcd}$$

We say (Z, X) has codimension c if Z_s has pure codimension c in X_s, for all $s \in S$. Locally for the *é*tale topology any smooth pair is isomorphic to the standard pair $(\mathbb{A}_S^{m-c}, \mathbb{A}_S^m)$. Let $z \in Z$ and $X^{'}$ be a neighbourhood of $i(z)$ in X with an *é*tale morphism $X^{'} \longrightarrow \mathbb{A}_s^m = \text{spec}\ \mathcal{O}_S[T_1, ..., T_m]$ with $Z \cap X^{'}$ the inverse image of closed subscheme $V(I)$ where $I = < T_{m-c+1}, ..., T_m >$. We denote $\underline{H}_Z^p(X, F)$ for the p^{th} derived functors of $i^!$. $R^p i^! F$ is a sheaf on Z. We have canonical isomorphisms $H^i(Z, \underline{H}_Z^{2c}(X, F)) \cong H_Z^{2c+i}(X, F)$ for locally free sheaves of finite rank on X. Hence

$$\Gamma(Z, \underline{H}_Z^{2c}(X, F)) \cong H_Z^{2c}(X, F)$$

Let k be a separably closed field. Let n be an integer coprime to the char(k). Let us denote Λ for $\mathbb{Z}/n\mathbb{Z}$ and let $\Lambda(r)$ be the product of r copies of the subsheaf defined by the n^{th} roots of unity of the structure sheaf of X. i.e $\Lambda(r) = \omega_n \otimes ..... \otimes \omega_n (r \text{ times})$ where $\omega_n(U) = n^{th}$ roots of unity in $\Gamma(U, \mathcal{O}_U)$.

Definition 3.1.16. We define the fundamental class to be the canonical class $s_{Z/X} \in H_Z^{2c}(X, \Lambda(c))$ that generates $\underline{H}_Z^{2c}(X, \Lambda(c))$. (i.e the map from $\Lambda \longrightarrow \underline{H}_Z^{2c}(X, \Lambda(c))$ sending 1 to restriction of $s_{Z/X}$ is an isomorphism).

In what follows, Λ denotes the constant sheaf $\mathbb{Z}/n\mathbb{Z}$ with n prime to char(k). A prime r-cycle is a closed integral subscheme of codimension r. An algebraic r-cycle is an element of the free abelian group $C^r(X)$ generated by the set of prime r-cycles. Let $C^*(X)$ be the graded algebra $\sum C^r(X)$. An algebraic cycle is an element of $C^*(X)$. A prime r-cycle W and a prime s-cycle Z is said to intersect properly if every irreducible componenet of the intersection $Z \cap W$ has codimension $r + s$. Two algebraic cycles intersect properly and $W.Z$ is defined if every prime cycle of W intersects every prime cycle of Z properly.

Let $H^*(X) = \sum_r H^r(X, \Lambda[r/2])$. $H^*(X)$ is an anti-commutative graded ring. Let Z be a smooth prime r-cycle and hence a smooth subvariety of X. We define the cycle map $cl_X : C^*(X) \longrightarrow H^*(X)$ as follows : $cl_X(Z)$ the image of the fundamental class $s_{Z/X}$ of Z in X under the map $H_Z^{2r}(X, \Lambda(r)) \longrightarrow H^{2r}(X, \Lambda(r))$.

Theorem 3.1.17. *Let E be a vector bundle of rank m over X and let $p : P = \mathbb{P}(E) \longrightarrow X$ be the associated projective bundle. Consider the map $Pic(P) \longrightarrow H^2(P, \Lambda(1))$ obtained from the kummer sequence. Let ξ be the image of the canonical line bundle $\mathcal{O}_P(1)$ on P. Then the morphim of graded algebras $H^*(X)[T]/(T^m) \longrightarrow H^*(P)$ which is p^* on $H^*(X)$ and sending T to ξ is an $H^*(X)$-isomorphism.*

$H^*(P)$ is a free $H^*(X)$-module with basis $\{1, \xi, \xi^2, ..., \xi^{m-1}\}$. Hence there are unique elements $c_r(E) \in H^{2r}(X)$ such that

$$\sum_{r=0}^{m} c_r(E)\xi^{m-r} = 0$$

with $c_0(E) = 1$ and $c_r(E) = 0$ for $r > m$.

Definition 3.1.18. The $c_r(E)$ is called the r-th chern class of E, $c(E) = \sum c_r(E)$ the total chern class of E and $c_t(E) = 1 + c_1(E)t + c_2(E)t^2 + ... + c_m(E)t^m$ the total chern polynomial of E.

3.2 Proof of Results

3.2.1 Cohomology of Blowing-ups

Let X be a projective smooth irreducible variety over F of dimension n. Let $Y_1, ..., Y_r \subset X$ be mutually disjoint closed irreducible subvarieties of X of dimension $d \geq 2$. Let $i : Y = \amalg_{k=1}^{r} Y_k \hookrightarrow X$ be the closed immersion and $f : X' \longrightarrow X$ be the blowing up of X along Y with $Y' \subset X'$ the strict transform of Y. Let $g : Y' \longrightarrow Y$ and $j : Y' \longrightarrow X$ be the canonical maps that arise during blowing-up.

Theorem 3.2.1. *We have the following explicit description of cohomology groups of X'.*

$$H^k(X') \cong H^{k-2}(Y) \oplus ..... \oplus H^{k-2-2(d-2)}(Y)\xi^{d-2} \oplus H^k(X)$$

For proof of this theorem refer to SGA5, section VII.

If $X, Y_1, ..., Y_k$ satisfy assumption in section 2.1, then so do the blow-up and the strict transform.

Proposition 3.2.2. Let $x^{'} \in H^k(X^{'})$ be a cohomology class on $X^{'}$ whose restriction to $Y^{'}$ is zero. Then $x^{'}$ is of the form $x^{'} = f^*(x)$ for a unique $x \in H^k(X)$

Proof. Using the above theorem 3.2.1, we can write

$$x^{'} = x_0 + x_1\xi + ... + x_{d-2}\xi^{d-2} + x$$

This relation is obtained by the above isomorphism. Then there is a relation

$$x^{'} = j_*(x_0 + x_1\xi + ... + x_{d-2}\xi^{d-2}) + f^*(x)$$

If we show that $x_0 = ... = x_{d-2} = 0$, we prove the result. For that we consider the restriction $\mathcal{O}_{X'}$ to $Y^{'}$. It is isomorphic to $\mathcal{O}_{Y'}(-1)$. Hence we have $j^*(j_*(1_{Y'})) = -\xi$. Hence taking the pull back,

$$j^*x^{'} = j^*j_*(x_0 + x_1\xi + ... + x_{d-2}\xi^{d-2}) + j^*f^*(x)$$

Rewriting this we get

$$j^*x^{'} = -x_0\xi - x_1\xi^2 - ...... - x_{d-2}\xi^{d-1} + g^*i^*(x)$$

However, since restriction of $x^{'}$ to $Y^{'}$ is zero and since $H^*(Y^{'})$ is a free $H^*(Y)$-module with basis $\{1, \xi, \xi^2, ..., \xi^{d-1}\}$, we have $x_0 = ... = x_{d-2} = i^*(x) = 0$

The uniqueness follows from the fact $f_*(x^{'}) = f_*(f^*(x)) = x$.

■

Proposition 3.2.3. If the hard Lefschetz conjecture holds for (X, L_X) and Y, L_Y, it also holds for $(X \times Y, p_1^*\mathcal{O}(L_X) + p_2^*\mathcal{O}(L_Y))$ where p_1 and p_2 are projection morphisms.

Theorem 3.2.4. *Let L be an ample $\mathbb{R}$-divisor on X. If the hard Lefshcetz conjecture and Hodge standard conjecture hold for (X, L) and $(Y, \mathcal{O}(L)|_Y)$,* *then there exists $\beta \in \mathbb{R}_{>0}$ such that for all $0 < \varepsilon < \beta$, the $\mathbb{R}$-divisor $L^{'}$ on $X^{'}$ of the form*

$$L^{'} = f^{*}\mathcal{O}(L) - \varepsilon Y^{'}$$

is an ample $\mathbb{R}$-divisor on $X^{'}$ for which both the conjectures hold.

Proof. First we note that

$$H^{*}(X^{'}) \cong H^{*}(Y \times \mathbb{P}^{d-2}) \otimes H^{*}(X)$$

as $\mathbb{R}$-vector spaces, which is a consequence of Theorem 3.2.1 .

Let $\{x_i\}$ and $\{y_j\}$ be basis of $H^{*}(X)$ and $H^{*}(Y)$. Then $\{\varepsilon^{\frac{-(d-2)}{2}+k} y_j \xi^k\}$ is a basis of $H^{*}(Y)\xi^k$ for $0 \leq k \leq d-2$. Again by Theorem 3.2.1 , we have the isomorphism

$$H^{*}(X^{'}) \cong H^{*}(Y) \oplus H^{*}(Y)\xi \oplus .... \oplus H^{*}(Y)\xi^{d-2} \oplus H^{*}(X)$$

Using the above, we have

$$\{\varepsilon^{\frac{-(d-2)}{2}} y_j\} \cup \{\varepsilon^{\frac{-(d-2)}{2}+1} y_j \xi\} \cup ..... \cup \{\varepsilon^{\frac{(d-2)}{2}} y_j \xi^{d-2}\} \cup \{x_i\}$$

is a basis of $H^{*}(X^{'})$. Using this basis we have a $\mathbb{R}$-Vector Space isomorphism

$$H^{k}(X^{'}) \cong H^{k-2}(Y \times \mathbb{P}^{d-2}) \oplus H^{k}(X)$$

sending $\varepsilon^{\frac{-(d-2)}{2}+m} y_j \xi^m$ to $c_1(\mathcal{O}_{\mathbb{P}^{d-2}}(1))^m$

Let us denote by $\widetilde{H}^{k}(X^{'}) := H^{k-2}(Y \times \mathbb{P}^{d-2}) \oplus H^{k}(X)$ and define the map

$$\widetilde{L} : \widetilde{H}^{k}(X^{'}) \longrightarrow \widetilde{H}^{k+2}(X^{'})$$

as follows : Let $p_1 : Y \times \mathbb{P}^{d-2} \longrightarrow Y$ and $p_2 : Y \times \mathbb{P}^{d-2} \longrightarrow \mathbb{P}^{d-2}$ be the projection maps. Then

$$L_{Y \times \mathbb{P}^{d-2}} = p_1^{*}(\mathcal{O}(L)|_{Y}) + p_2^{*}(\mathcal{O}_{\mathbb{P}^{d-2}}(1))$$

is an ample $\mathbb{R}$-divisor on $Y \times \mathbb{P}^{d-2}$. So we define our operator $\widetilde{L}$ as direct sum of the lefschetz operators $L_{Y \times \mathbb{P}^{d-2}}$ and L. Hence by Proposition 3.2.3, we have that both the conjectures hold for $(Y \times \mathbb{P}^{d-2}, L_{Y \times \mathbb{P}^{d-2}})$. Thus $\widetilde{L}$ is an isomorphism. Hence, $\widetilde{L}^{k}$ induces the isomorphism

$$\widetilde{L}^{k} : \widetilde{H}^{n-k}(X^{'}) \longrightarrow \widetilde{H}^{n+k}(X^{'})$$

This cleary is analogous to the hard lefschetz conjecture. Let $\widetilde{P}^k(X^{'})$ be the kernel of $\widetilde{L}^{n-k+1}$ in $\widetilde{H}^k(X^{'})$. $\widetilde{P}^k(X^{'})$ is isomorphic to $P^{k-2}(Y\times\mathbb{P}^{d-2})\oplus P^k(X)$. Now we define the pairing

$$\widetilde{P}^k(X^{'}) \times \widetilde{P}^k(X^{'}) \longrightarrow \widetilde{H}^{2n}(X^{'})$$

given by

$$(a,b) \longrightarrow (-1)^{k/2}\widetilde{L}^{n-k}a\widetilde{\cup}b$$

Here $\widetilde{\cup}$ is the difference between the cup products on $H^*(Y\times\mathbb{P}^{d-2})$ and $H^*(X)$ given as follows

$$\widetilde{\cup} : ((x_0,y_0),(x_1,y_1)) \longrightarrow -y_0\cup y_1 + x_0\cup x_1$$

where $((x_0,y_0),(x_1,y_1)) \in \widetilde{H}^*(X^{'}) \times \widetilde{H}^*(X^{'})$ Hence an analogue of Hodge conjecture holds for $\widetilde{L}$

So to prove the theorem, it is enough to show that as $\varepsilon \longrightarrow 0$ the matrix representing $L^{'}$ with respect to the given choice of basis, converges to the matrices of $\widetilde{L}$ via the isomorphism $H^k(X^{'}) \cong H^{k-2}(Y\times\mathbb{P}^{d-2})\oplus H^k(X)$ described above. Similarily the matrices of the cup product pairing between $H^k(X^{'})$ and $H^{2n-k}(X^{'})$ converges to the matrices respresenting $\widetilde{\cup}$ between $\widetilde{H}^k(X^{'})$ and $\widetilde{H}^{2n-k}(X^{'})$

We have $L^{'} = f^*\mathcal{O}(L) - \varepsilon Y^{'}$ and $[Y] = j_*(1_{Y^{'}})$ in $H^2(X^{'})$. Computing the cup product

$$(f^*\mathcal{O}(L) - \varepsilon j_*(1_{Y^{'}})\cup(f^*(x)+j_*(y))$$

for $x\in H^k(X), y\in H^{k-2}(Y^{'})$ with $g^*(y)=0$. Expanding the above expression we get

$$f^*\mathcal{O}(L)\cup f^*(x) + f^*\mathcal{O}(L)\cup j_*(y) - \varepsilon j_*(1_{Y^{'}})\cup f^*(x) - \varepsilon j_*(1_{Y^{'}})\cup j_*(y)$$

The pull back preserves the cup product. Hence $f^*\mathcal{O}(L)\cup f^*(x) = f^*(\mathcal{O}(L)\cup x) = f^*(L\cup x)$. Next consider $f^*\mathcal{O}(L)\cup j_*(y) = j_*(j^*(f^*\mathcal{O}(L))\cup y) = j_*(g^*(i^*\mathcal{O}(L))\cup y) = j_*(g^*(\mathcal{O}(L)|_Y)\cup y)$. Third we have $\varepsilon j_*(1_{Y^{'}})\cup f^*(x) =$

$\varepsilon j_*(1_{Y'} \cup j^* f^*(x)) = \varepsilon j_*(g^*(x|_Y))$. Finally we have $j^* j_*(1_{Y'}) = -\xi$. Hence $\varepsilon j_*(1_{Y'}) \cup j_*(y) = j_*(\varepsilon j^* j_*(1_{Y'}) \cup y) = -j_*(\varepsilon \xi \cup y)$. Piecing together all this we have,

$$f^*(L \cup x) + j_*((g^*(\mathcal{O}(L))|_Y + \xi\varepsilon) \cup y) - \varepsilon j_*(g^*(x|_Y))$$

The first term corresponds to the action of L on $H^*(X)$ and the second term is analogous to the action of $L_{Y \times \mathbb{P}^{d-2}}$ via the isomorphism

$$H^k(X^{'}) \cong H^{k-2}(Y \times \mathbb{P}^{d-2}) \oplus H^k(X)$$

when ε converges to zero. The third term vanishes trivially when $\varepsilon \longrightarrow 0$. This proves that matrix of $L^{'}$ with respect to the given basis converges to the matix representing $\widetilde{L}$. So if we have proved that the matrices of the cup product pairing between $H^k(X^{'})$ and $H^{2n-k}(X^{'})$ converges to the matrices respresenting $\widetilde{\cup}$ between $\widetilde{H}^k(X^{'})$ and $\widetilde{H}^{2n-k}(X^{'})$ we finish the proof of the given proposition.

We shall compute the cup product pairing on $H^*(X^{'})$.

$$(f^*(x_0) + j_*(y_0)) \cup (f^*(x_1) + j_*(y_1))$$

$x_0 \in H^k(X), y_0 \in H^{k-2}(Y^{'}), x_1 \in H^{2n-k}(X), y_1 \in H^{2n-k-2}(Y^{'})$ with $g_*(y_0) = 0, g_*(y_1) = 0$. It is enough to compute

$$f_*((f^*(x_0) + j_*(y_0)) \cup (f^*(x_1) + j_*(y_1)))$$

since f_* is an isomorphism on $H^{2n}(X^{'}) = \mathbb{R}$. This gives

$$= f_*(f^*(x_0) + j_*(y_0)) \cup f_*(f^*(x_1) + j_*(y_1))$$

$$= f_*(f^*(x_0) \cup f^*(x_1)) + f_*(j_*(y_0) \cup f^*(x_1)) + f_*(f^*(x_0) \cup j_*(y_1)) + f_*(j_*(y_0) \cup j_*(y_1))$$

The first term we have $f_*(f^*(x_0) \cup f^*(x_1)) = f_*(f^*(x_0 \cup x_1))$. The pullback preserves the cup product. Hence this is equal to $x_0 \cup x_1$. The second term is equal to $f_*(j_*(y_0)) \cup x_1 = i_*(g_*(y_0)) \cup x_1 = 0$ since $g_*(y_0) = 0$. The third term is zero due to similar reasons $g_*(y_1) = 0$. Finally we have $j^*(j_*(y_0)) = -\xi \cup y_0$.

Hence we have, $f_*(j_*(y_0)\cup j_*(y_1)) = f_*(j_*(j^*j_*(y_0)\cup y_1)) = i_*(g_*(-\xi\cup y_0\cup y_1))$. Now $g_*(\xi^{d-1}) = 1$ and hence $g_*(\xi^m) = 0$ for $0 \leq m \leq d-2$. So taking limite we have the matrix of the pairing

$$(y_0, y_1) \longrightarrow i_*(g_*(-\xi \cup y_0 \cup y_1)$$

converging to the negative of the matrix representing the cup product pairing on $Y \times \mathbb{P}^{d-2}$ via the isomorphism in the beginning. Hence we prove the result.

■

3.2.2 The Main result

Proposition 3.2.5. For $0 \leq k \leq n-1$ and $a \in H^k(\mathbb{B}^n)$, if the restriction of a to D_V is zero for all $V \in Gr_*(\mathbb{P}^n(\mathbb{F}_q))$, then $a = 0$.

Proof. By applying the above proposition 3.2.2 to the Y_k's in the construction of $\mathbb{B}^n$ successively, we get that $a = f^*a^{'}$ for $a^{'} \in H^k(\mathbb{P}^n)$. Now the restriction of a to D_V is zero. Hence the restriction of $a^{'}$ to V is zero. The restriction map

$$H^k(\mathbb{P}^n) \longrightarrow H^k(V)$$

is an isomorphism. Hence we conclude that $a = 0$.

■

Theroem 2.2.4 *Let D be an ample $PGL_{n+1}(\mathbb{F}_q)$-invariant $\mathbb{R}$-divisor on $\mathbb{B}^n$ where $\mathbb{B}^n$ be the space obtained by successive blowing up of Projective Space along linear varieties. Then the hard Leftschetz conjecture and the standard Hodge conjecture holds for $(\mathbb{B}^n, D)$.*

Proof. We prove the theorem by induction on n. The case $n = 1$ is obvious. Let us assume that the theorem is proved for all dimensions less than n. By theorem 3.2.4 , there exists at least one $PGL_{n+1}(\mathbb{F}_q)$-invariant $\mathbb{R}$-divisor on $\mathbb{B}^n$ for which the Hard Lefschetz conjecture and the Standard Hodge conjecture holds.

Now consider two ample $PGL_{n+1}(\mathbb{F}_q)$-invariant $\mathbb{R}$-divisors D and $D^{'}$ on $\mathbb{B}^n$. Then $tD + (1-t)D^{'}$ is also an ample $PGL_{n+1}(\mathbb{F}_q)$-invariant $\mathbb{R}$-divisor on $\mathbb{B}^n$ for $0 \leq t \leq 1$. Hence, if we prove that the hard Lefschetz conjecture holds for ample $PGL_{n+1}(\mathbb{F}_q)$-invariant $\mathbb{R}$-divisors D, then we prove that the standard Hodge conjecture also holds for these divisors.

Let us assume that L is an ample $PGL_{n+1}(\mathbb{F}_q)$-invariant $\mathbb{R}$-divisor on $\mathbb{B}^n$ and $a \in H^k(\mathbb{B}^n)$ for $0 \leq k \leq n-1$, a non-zero cohomology class such that

$$L^{n-k} \cup a = 0$$

For $0 \leq d \leq n-1$, take $V \in Gr_d(\mathbb{P}^n)(\mathbb{F}_q)$ and $D_V \cong \mathbb{B}^d \times \mathbb{B}^{n-d-1}$. Let $p_1 : D_V \longrightarrow \mathbb{B}^d$ and $p_2 : D_V \longrightarrow \mathbb{B}^{n-d-1}$ be projections. Then by proposition 1.2.11 ,

$$\mathcal{O}(L)|_{D_V} = p_1^*(\mathcal{O}(D^{'})) + p_2^*(\mathcal{O}(D^{''}))$$

where $D^{'}$ and $D^{''}$ are ample $PGL_{n+1}(\mathbb{F}_q)$-invariant $\mathbb{R}$-divisor on $\mathbb{B}^d$ and $\mathbb{B}^{n-d-1}$ respectively. By using induction hypothesis and above proposition, the hard Lefschetz conjecture holds for $(D_V, \mathcal{O}(L)|_{D_V}$.

By restricting $L^{n-k} \cup a$ to D_V, we get

$$\mathcal{O}(L)|_{D_V} \cup a|_{D_V} = 0$$

But dimension of $D_V = n-1$. Hence $a|_{D_V} \in H^k(D_V)$ is in the primitive part $P^k(D_V)$. So by induction hypothesis we can apply the hodge conjecture for $(D_V, \mathcal{O}(L)|_{D_V})$. Hence we have the pairing $\langle , \rangle_{P^k(D_V)}$ is positive definite. Hence we have

$$(-1)^{k/2}.(\mathcal{O}(L)|_{D_V})^{n-k-1} \cup (a|_{D_V}) \cup a|_{D_V} \geq 0$$

and again by induction hypothesis, the hard lefschetz conjecture tells us that the equality holds if and only if $a|_{D_V} = 0$. Since $L \in H^2(\mathbb{B}^n)$, we have by by proposition 1.2.12 , L is positive. Hence,

$$L = \sum_{V \in Gr_*(\mathbb{P}^n)(\mathbb{F}^q)} a_V D_V$$

such that $a_V > 0$.

Thus we have

$$(-1)^{k/2}.L^{n-k} \cup a \cup a = \sum_{V \in Gr_*(\mathbb{P}^n)(\mathbb{F}^q)} a_V D_V \cup ((-1)^{k/2}.L^{n-k-1} \cup a \cup a)$$

From the above we have

$$(-1)^{k/2}.L^{n-k} \cup a \cup a = \sum_{V \in Gr_*(\mathbb{P}^n)(\mathbb{F}^q)} a_V (\mathcal{O}(L)|_{D_V} \cup a|_{D_V} \cup a|_{D_V})$$

However we have assumed that $L^{n-k} \cup a = 0$ which implies that

$$(-1)^{k/2}.L^{n-k} \cup a \cup a = 0$$

and also we have that $a_V > 0$ as L is positive. Thus we have that $a|_{D_V} = 0$ for all $V \in Gr_*(\mathbb{P}^n)(\mathbb{F}^q)$. The proposition 3.2.5 says that $a = 0$. Hence we have contradiction since we assumed that a is a non zero cohomology class. Hence we have that the leftschetz operator is an isomorphism which proves the hard lefschetz conjecture in our case and hence the Standard Hodge conjecture as well.

■

Theorem 2.2.8 *Let $\Gamma \subset PGL_{d+1}(\mathbb{F}_q)$ be a cocompact torsion free discrete subgroup. Then we have*

$$M_i V = W_{i+w} V \quad \forall i$$

for X_Γ

Proof . We have that all irreducible components, $X_1, ..., X_m$ of the special fiber of $\mathfrak{X}_\Gamma$ are isomorphic to the variety $\mathbb{B}^d$. Now from Proposition 1.2.5 and Corollary 1.2.6, we have that the irreducible components of $X_i \cap X_j$ is isomorphic to a divisor of the form D_V on $\mathbb{B}^d$. Hence by induction we prove that the irreducible components Y of $X_{i_1} \cap X_{i_2} \cap X_{i_k}$ is isomorphic to the product

$$Y \cong \mathbb{B}^{n_1} \times .... \times \mathbb{B}^{n_k}$$

with $n_1 + ... + n_k = d - k + 1$.

Let L be the relative dualizing sheaf $\omega_{\mathfrak{X}_\Gamma/\mathcal{O}_K}$. Then by Theorem 1.2.13, we have that L is invertible and ample. For $1 \leq i \leq m$, fix an isomorphism $X_i \cong \mathbb{B}^d$. The restriction of L to X_i will look like

$$-(n+1)f^*\mathcal{O}_{\mathbb{P}^d}(1) + \sum_{d=0}^{n-1}(n-d)D_d$$

This is an ample $PGL_{n+1}(\mathbb{F}_q)$-invariant divisor on X_i. Now by the theorem 2.2.4, we have that the Hodge standard conjecture holds for $(X_i, L|_{X_i})$. Also, by Kunneth formula, Y satisfies the assumption above. Let $p_j : Y \longrightarrow \mathbb{B}^{n_j}$ be the projection to the $i-th$ factor for $j = 1, ..., k$. Then again by the same arguement as above,

$$L|_Y = p_1^*\mathcal{O}(L_1) + ... + p_k^*\mathcal{O}(L_k)$$

where L_j is an ample $PGL_{n+1}(\mathbb{F}_q)$-invariant divisor on $\mathbb{B}^{n_j}$ for $j = 1, ..., k$. Again by Theorem 2.2.4, we have that the Hodge conjecture holds for $(\mathbb{B}^{n_j}, L_j)$ for $j = 1, ...k$.

So now by Proposition 3.2.3 and induction the Hodge conjecture holds for $(Y, L|_Y)$. In other words $\mathfrak{X}_\Gamma$ satisfies all conditions of Theorem 2.2.7. Hence we have the theorem.

■

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