Margulis Space Time:

Crooked Planes and The Margulis Invariant

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Chapter 1

Introduction

In 1977, Milnor [2] asked whether a non-amenable group (e.g a free group of rank 2) could act properly by affine transformations. He observed that by applying Tits alternative [3], this question is equivalent to whether a non-abelian free group could act properly by affine transformations. Furthermore, he proposed the following construction of such a group: Start with a free discrete subgroup of $\mathbb{SO}^0(2,1)$ (for example a Schottky group acting on the hyperbolic plane) and add translation components to obtain a group of affine transformations which may act freely.
In 1983, Margulis introduced the Margulis invariant and used it to show that such properly discontinuous groups do exist, realizing Milnor's suggestion. In 1991, Drumm [5] gave a generalization of Schottky's construction introducing the Crooked Planes. Following this work Drumm and Goldman [6] [9] in a series of papers studied the Crooked Planes in much detail. This approach gave a slightly stronger positive result as to which groups act properly discontinuously on $\mathbb{R}^n$.
In the above mentioned 1983 paper Margulis [4] used the Margulis invariant to detect properness. He gave a necessary condition for properness using the sign of the invariant. Goldman conjectured this necessary condition to be sufficient. In 2006 Charette [12] came up with an example of a one-holed torus suggesting that the conjecture is generally false.
In 2001, following Margulis's work, Labourie [10] extended the original Margulis invariant to higher dimensions.
Recently, in 2009, Margulis, Goldman and Labourie [7] found an equivalent condition for properness using the extended Margulis invariant and gave a very conceptual proof of the Opposite Sign Lemma.
In this paper we study the works of Drumm and Goldman on Crooked Planes [5] [6] [9] and the recent works of Margulis, Goldman and Labourie [7] on giving an equivalent condition for properness.

Chapter 2

Geometry of $\mathbb{R}^{2,1}$

Let, $\mathbb{R}^{2,1}$ be the 3-dimensional real vector space with inner product,

$$\mathbb{B}(v, w) := v_1 w_1 + v_2 w_2 - v_3 w_3$$

and $\mathbf{G}_0 = \mathbb{O}^0(2,1)$ the identity component of its group of isometries. $\mathbf{G}_0$ consists of linear isometries of $\mathbb{R}^{2,1}$ which preserve both an orientation of $\mathbb{R}^{2,1}$ and a connected component of the open light cone:

$$\{\ v \in \mathbb{R}^{2,1} : \mathbb{B}(v, v) < 0\ \}.$$

Then, $\mathbf{G}_0 = \mathbb{O}^0(2,1) \cong \mathbb{PSL}(2,\mathbb{R}) \cong \mathbf{Isom}^0(\mathbb{H}^2)$, where $\mathbb{H}^2$ denotes the real hyperbolic plane.

2.1 The Hyperboloid Model

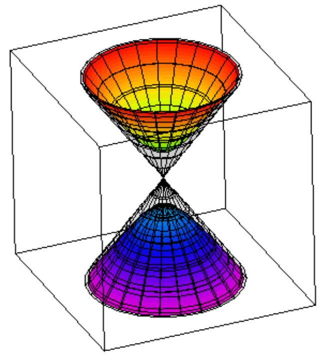
We define $\mathbb{H}^2$ as follows. We work in the irreducible representation $\mathbb{R}^{2,1}$ (isomorphic to the adjoint representation of $\mathbf{G}_0$). A non zero vector $v \in \mathbb{R}^{2,1}$ is called:

- Space like if $\mathbb{B}(v, v) > 0$
- Light like or Null if $\mathbb{B}(v, v) = 0$
- Time like if $\mathbb{B}(v, v) < 0$

The two sheeted hyperboloid:

$$\{\ v \in \mathbb{R}^{2,1} : \mathbb{B}(v, v) = -1\ \}$$

has two connected components. We fix a timelike vector e_3 and define $\mathbb{H}^2$ as the connected component with $\mathbb{B}(v, e_3) < 0$. The Lorentzian metric defined by $\mathbb{B}$ restricts to a Riemannian metric of constant curvature -1 on $\mathbb{H}^2$. The identity component $\mathbf{G}_0$ of the isometry group of $\mathbb{R}^{2,1}$ is the group of orientation preserving isometries of $\mathbb{H}^2$.



2.2 The Lorentz Product

On the space $\mathbb{R}^{2,1}$ defined as above we define a cross product called the Lorentzian cross product as follows:

$$\boxtimes : \mathbb{R}^{2,1} \times \mathbb{R}^{2,1} \longrightarrow \mathbb{R}^{2,1} \text{ such that,}$$
$$(u,v) \longmapsto \begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_2v_1 - u_1v_2 \end{bmatrix}$$

. Note that $\boxtimes$ is a skew-symmetric and bilinear form, satisfying the following properties:

- $\mathbb{B}(u, u \boxtimes v) = \mathbb{B}(v, u \boxtimes v) = 0$
- $u \boxtimes v = -v \boxtimes u$
- $\mathbb{B}(u \boxtimes v, u \boxtimes v) = \mathbb{B}(v,u)^2 - \mathbb{B}(u,u)\mathbb{B}(v,v)$

Let $\|.\|$ be the euclidean norm on the space $\mathbb{R}^3$. Denote by $< v_1, v_2, ..., v_n >$ the subspace generated by $v_1, v_2, ..., v_n$ and the sphere of unit euclidean length by $\mathbf{S}^2$. For any $v \in \mathbb{R}^{2,1}$ its Lorentz perpendicular plane is denoted by $\mathbf{P}(v)$. Let $\mathbf{C}$ be the light cone. It consists of all null vectors and its two components form the time orientations of $\mathbb{R}^{2,1}$. We define $\mathbf{C}^+$ and $\mathbf{C}^-$ respectively to be the positive and the negative time orientation. Similarly we denote by $\mathbf{U}$ the space of all time like vectors and its two components as $\mathbf{U}^+$ and $\mathbf{U}^-$. If $a, b \in \mathbf{C}^+ \cap \mathbf{S}^2$ are two non zero null vectors with positive time orientation, then $v = a \boxtimes b$ is space like and is a vector parallel to the intersection $\mathbf{P}(a) \cap \mathbf{P}(b)$. Let v be a null vector then the plane $\mathbf{P}(v)$ equals the plane tangent to the light cone containing the line $\mathbb{R}v$ and the null line $\mathbb{R}v$ divides the plane into two components. The mapping $x \longmapsto x \boxtimes v$ defines a diffeomorphism of $\mathbf{C}^+ \setminus \mathbb{R}_+ v$ onto one component. We denote the closure of this component by $\mathbf{P}^+(v)$, that is, $\mathbf{P}^+(v) := cl((\mathbf{C}^+ \setminus \mathbb{R}_+ v) \boxtimes v)$ and the other component by $\mathbf{P}^-(v)$, that is, $\mathbf{P}^-(v) := cl(v \boxtimes (\mathbf{C}^+ \setminus \mathbb{R}_+ v))$. If v is a space like vector then $\mathbf{P}(v) \cap \mathbf{C}$ is the union of two null lines. There exists a unique pair $x^-(v), x^+(v) \in \mathbf{P}(v) \cap \mathbf{C}^+$ such that $\|x^-(v)\| = \|x^+(v)\| = 1$ and $\mathbb{B}(v, x^-(v) \boxtimes x^+(v)) > 0$. If v is a space like then,

- $x^-(v) \boxtimes x^+(v) = 2\mathbb{B}(v,v)v/(\mathbb{B}(v,v) + \|v\|^2)$
- $v \boxtimes x^+(v) = -\mathbb{B}(v,v)^{1/2} x^+(v)$.

We define a conical neighbourhood $A \subset \mathbf{C}^+$ of $v \in \mathbf{C}^+$ to be an open connected subset of $\mathbf{C}^+$ containing v such that if $w \in A$ then $\mathbb{R}_+ w \subset A$. For disjoint conical open sets A and B we define,

$$\mathbf{T}(A, B) := \{v \in \mathbb{R}^{2,1} : \mathbb{B}(v, a \boxtimes b) > 0 \text{ for all } a \in cl(A) \text{ and } b \in cl(B)\}.$$

In particular, if A,B are connected then $\mathbf{T}(A, B)$ is an open infinite pyramid whose vertex is the origin and whose four edges are parallel to vectors in the boundary of A and B.

Chapter 3

Affine Geometry

In this chapter we collect general properties of affine spaces, affine transformations and affine deformations of linear group actions. We are primarily interested in affine deformations of linear actions factoring through the irreducible $2r+1$ dimensional real representation $\mathbb{V}_r$ of $\mathbf{G}_0$ where r is a positive integer. We note that for $r=1$ we get $\mathbb{V}_1 = \mathbb{R}^{2,1}$.

3.1 Affine Spaces and their automorphisms

Let $\mathbb{V}$ be a real vector space. An affine space $\mathbb{E}$ (modelled on $\mathbb{V}$) is a space equipped with a simply transitive action of $\mathbb{V}$. We call $\mathbb{V}$ the vector space underlying $\mathbb{E}$, and refer to its elements as translations. Translation τ_v by a vector $v \in \mathbb{V}$ is denoted by addition, that is,

$$\tau_v : \mathbb{E} \longrightarrow \mathbb{E} \text{ given by } x \longmapsto x + v.$$

Let $\mathbb{E}$ be an affine space with associated vector space $\mathbb{V}$. Choosing an arbitrary point $O \in \mathbb{E}$ (the origin) identifies $\mathbb{E}$ with $\mathbb{V}$ via the map,

$$f : \mathbb{V} \longrightarrow \mathbb{E} \text{ given by } v \longmapsto O + v.$$

An affine automorphism of $\mathbb{E}$ is the composition of a linear mapping (using the above identification of $\mathbb{E}$ and $\mathbb{V}$) and a translation, that is,

$$g : \mathbb{E} \longrightarrow \mathbb{E} \text{ given by } O + v \longmapsto O + \mathbb{L}(g)(v) + u(g)$$

where $\mathbb{L}(g) \in \mathbf{GL}(\mathbb{V})$ and $u(g) \in \mathbb{V}$. The affine automorphisms of $\mathbb{E}$ form a group $\mathbf{Aff}(\mathbb{E})$. The mapping,

$$(\mathbb{L}, u) : \mathbf{Aff}(\mathbb{E}) \longrightarrow \mathbf{GL}(\mathbb{V}) \ltimes \mathbb{V} \text{ given by } g \longmapsto (\mathbb{L}(g), u(g))$$

gives an isomorphism of groups. The linear mapping $\mathbb{L}(g) \in \mathbf{GL}(\mathbb{V})$ is called the linear part of the affine transformation g, and

$$\mathbb{L} : \mathbf{Aff}(\mathbb{E}) \longrightarrow \mathbf{GL}(\mathbb{V}) \text{ given by } g \longmapsto \mathbb{L}(g)$$

is a homomorphism. The vector $u(g) \in \mathbb{V}$ is called the translational part of g. The mapping,

$$u : \mathbf{Aff}(\mathbb{E}) \longrightarrow \mathbb{V} \text{ given by } g \longmapsto u(g)$$

satisfies the following identity (also known as the cocycle identity):

$$u(g_1 g_2) = u(g_1) + \mathbb{L}(g_1) u(g_2)$$

for $g_1, g_2 \in \mathbf{Aff}(\mathbb{E})$.

3.2 Affine deformations of linear actions

Let $\Gamma_0 \subset \mathbf{GL}(\mathbb{V})$ be a group of linear automorphisms of a vector space $\mathbb{V}$. Denote the corresponding Γ_0 module as $\mathbb{V}$ as well. An affine deformation of Γ_0 is a representation,

$$\rho : \Gamma_0 \longrightarrow \mathbf{Aff}(\mathbb{E})$$

such that $\mathbb{L} \circ \rho$ is the inclusion $\Gamma_0 \hookrightarrow \mathbf{GL}(\mathbb{V})$. We confuse ρ with its image $\Gamma := \rho(\Gamma_0)$, to which we aso refer to as an affine deformation of Γ_0. Note that ρ embedds Γ_0 as the subgroup Γ of $\mathbf{GL}(\mathbb{V})$. In terms of the semi-direct product decomposition $\mathbf{Aff}(\mathbb{E}) \cong \mathbf{GL}(\mathbb{V}) \ltimes \mathbb{V}$ an affine deformation is the graph $\rho = \rho_u$ (with image denoted by $\Gamma = \Gamma_u$) of a cocycle

$$u : \Gamma_0 \longrightarrow \mathbb{V},$$

that is, a map satisfying the aforementioned cocycle identity. We write $g = \rho(g_0) = (g_0, u(g_0)) \in \Gamma_0 \ltimes \mathbb{V}$ for the corresponding affine transformation $g(x) = g_0(x) + u(g_0)$. Cocycles form a vector space $\mathbf{Z}^1(\Gamma_0, \mathbb{V})$. Cocycles $u_1, u_2 \in \mathbf{Z}^1(\Gamma_0, \mathbb{V})$ are cohomologous if their difference $u_1 - u_2$ is a coboundary, a cocycle of the form,

$$\delta_{v_0} : \Gamma_0 \longrightarrow \mathbb{V} \text{ given by } g \longmapsto v_0 - g v_0$$

where $v_0 \in \mathbb{V}$. Moreover, cohomologous classes of cocycles form a vector space $\mathbf{H}^1(\Gamma_0, \mathbb{V})$. Affine deformations ρ_{u_1}, ρ_{u_2} are conjugate by translation by v_0 if and only if $u_1 - u_2 = \delta_{v_0}$. Thus $\mathbf{H}^1(\Gamma_0, \mathbb{V})$ parametrizes translational conjugacy classes of affine deformations of $\Gamma_0 \subset \mathbf{GL}(\mathbb{V})$. Note that when $u = 0$, the affine deformation Γ_u equals Γ_0 itself.

3.3 The Margulis Invariant

Consider the case that $\mathbf{G}_0 = \mathbf{PSL}(2, \mathbb{R})$ and $\mathbb{L}$ is an irreducible representation of $\mathbf{G}_0$. For every positive integer r, let $\mathbb{L}_r$ denote the irreducible representation of $\mathbf{G}_0$ on the $2r$-symmetric power $\mathbb{V}_r$ of the standard representation of $\mathbf{SL}(2, \mathbb{R})$ on $\mathbb{R}^2$. The dimension of $\mathbb{V}_r$ equals $2r + 1$. The central element $-\mathbb{I} \in \mathbf{SL}(2, \mathbb{R})$ acts by $(-1)^{2r} = 1$. So this representation of $\mathbf{SL}(2, \mathbb{R})$ defines a representation of

$$\mathbf{PSL}(2, \mathbb{R}) = \mathbf{SL}(2, \mathbb{R}) / \{\pm \mathbb{I}\}.$$

Note that the representation $\mathbb{R}^{2,1}$ introduced before is $\mathbb{V}_1$, the case when $r = 1$. Furthermore the $\mathbf{G}_0$ invariant non-degenerate skew-symmetric bilinear form on $\mathbb{R}^2$ induces a non-degenerate symmetric bilinear form $\mathbb{B}$ on $\mathbb{V}_r$, which we normalize in the following paragraph.

An element $g \in \mathbf{G}_0$ is hyperbolic if it corresponds to an element $\tilde{g}$ of $\mathbf{SL}(2,\mathbb{R})$ with distinct real eigenvalues. Necessarily these eigenvalues are reciprocals λ, λ^{-1} which we can uniquely specify by requiring $|\lambda| < 1$. Furthermore we choose eigenvectors $v_+, v_- \in \mathbb{R}^2$ such that:

- $\tilde{g}(v_+) = \lambda v_+$
- $\tilde{g}(v_-) = \lambda^{-1} v_-$
- The ordered basis $\{v_-, v_+\}$ is positively oriented.

Then the action $\mathbb{L}_r$ has eigenvalues λ^{2j}, for $j \in \{-r, 1-r, ..., 0, ..., r-1, r\}$ where the symmetric product $v_-^{r-j} v_+^{r+j} \in \mathbb{V}_r$ is an eigenvector with eigenvalue λ^{2j}. In particular g fixes the vector $x^0(g) := c v_-^r v_+^r$, where the scalar c is choosen so that $\mathbb{B}(x^0(g), x^0(g)) = 1$. Call $x^0(g)$ the neutral vector of g. The subspaces,

$$\mathbb{V}^-(g) := \sum_{j=1}^{r} \mathbb{R}(v_-^{r+j} v_+^{r-j}),$$
$$\mathbb{V}^+(g) := \sum_{j=1}^{r} \mathbb{R}(v_-^{r-j} v_+^{r+j})$$

are g invariant and $\mathbb{V}$ enjoys a g-invariant $\mathbb{B}$-orthogonal direct sum decomposition,

$$\mathbb{V} = \mathbb{V}^-(g) \oplus \mathbb{R}(x^0(g)) \oplus \mathbb{V}^+(g).$$

For any norm $\|.\|$ on $\mathbb{V}$, there exists $C, k > 0$ such that,

- $\|g^n(v)\| \leqslant Ce^{-kn}\|v\|$ for $v \in \mathbb{V}^+(g)$ and
- $\|g^{-n}(v)\| \leqslant Ce^{-kn}\|v\|$ for $v \in \mathbb{V}^-(g)$.

Furthermore, $x^0(g^n) = |n| x^0(g)$ if $n \in \mathbb{Z} \setminus \{0\}$ and $\mathbb{V}^\pm(g^n) = \begin{array}{l} \mathbb{V}^\pm(g) \text{ if } n > 0. \\ \mathbb{V}^\mp(g) \text{ if } n < 0 \end{array}$

Now, let us suppose that $g \in \mathbf{Aff}(\mathbb{E})$ is an affine transformation whose linear part $\mathbb{L}(g)$ is hyperbolic. Then there exists a unique affine line $l_g \subset \mathbb{E}$ which is g-invariant. The line l_g is parallel to $x^0(\mathbb{L}(g))$. The restriction of g to l_g is a translation by the vector,

$$\mathbb{B}(gx - x, x^0(\mathbb{L}(g)))x^0(\mathbb{L}(g))$$

where $x^0(\mathbb{L}(g))$ is as defined above with $\mathbb{B}(x^0(\mathbb{L}(g)), x^0(\mathbb{L}(g))) = 1$.
Suppose that $\Gamma_0 \subset \mathbf{G}_0$ be a Schottky group, that is, a non-abelian discrete subgroup containing only hyperbolic elements. Such a discrete subgroup is a free group of rank at least two. We define the Margulis invariant to be the function,

$$\begin{aligned} \alpha_u : \Gamma_0 &\longrightarrow \mathbb{R} \text{ given by} \\ g &\longmapsto \mathbb{B}(u(g), x^0(g)) \end{aligned}$$

where $u \in \mathbf{Z}^1(\Gamma_0, \mathbb{V})$. The Margulis invariant α_u associated to an affine deformation Γ_u is a well defined class function on Γ_0 satisfying the following properties:

- $\alpha_{[u]}(g^n) = |n|\alpha_{[u]}(g)$ for $g \in \Gamma_0$ and $[u] \in \mathbf{H}^1(\Gamma_0, \mathbb{V})$
- $\alpha_{[u]}(g) = 0 \Leftrightarrow g$ fixes a point in $\mathbb{V}$.
- The function $\alpha_{[u]}$ depends linearly on $[u]$.
- The map

$$\begin{gathered} \alpha : \mathbf{H}^1(\Gamma_0, \mathbb{V}) \longrightarrow \mathbb{R} \text{ given by} \\ [u] \longmapsto \alpha_{[u]} \end{gathered}$$

is injective.
In his 1983 paper Margulis used this invariant to detect properness. The significance of the Margulis invariant comes from the following result due to Margulis:

Theorem 3.3.1 (The Opposite Sign Lemma)**.** *Suppose* Γ *acts properly. Then either* $\alpha_{[u]}(g) > 0$ *for all* $g \in \Gamma_0$ *or* $\alpha_{[u]}(g) < 0$ *for all* $g \in \Gamma_0$.

We will give a proof of this result at the last chapter using tools developed jointly by Margulis, Goldman and Labourie.

Chapter 4

Margulis Space Times

Complete affinely flat manifolds correspond to subgroups $\Gamma \subset \mathbf{Aff}(\mathbb{R}^n)$ which act properly discontinuously on $\mathbb{R}^n$, and $\pi_1(\mathbb{M}) \cong \Gamma$ for $\mathbb{M} = \mathbb{R}^n/\Gamma$. Milnor showed that if Γ is virtually polycyclic then there exists some complete affinely flat manifold $\mathbb{M}$ such that $\pi^1(\mathbb{M}) \cong \Gamma$, and he asked if the converse was true.
Margulis demonstrated that there exist free subgroups $\Gamma \subset \mathbf{Aff}(\mathbb{R}^3)$ acting properly discontinuously on $\mathbb{R}^3$, thus answering Milnor's question negatively. By Fried and Goldman, the underlying linear group of Γ must be conjugate to a subgroup of $\mathbf{G}_0$. The corresponding quotient manifolds $\mathbb{M} \cong \mathbb{R}^3/\Gamma$ are called Margulis space-times. We note that Γ, a group of affine homeomorphisms of $\mathbb{R}^3$ acts properly discontinuously and freely on $\mathbb{R}^3$ if there exists a three dimensional submanifold $\mathcal{X}$ with boundary (a fundamental domain) such that no two elements of the interior of $\mathcal{X}$ are Γ-equivalent and every element of $\mathbb{R}^3$ is Γ-equivalent to an element of $\mathcal{X}$. In the following sections we try to construct a nice fundamental domain for a large class of Margulis space-times.

4.1 Fundamental Polyhedra for the linear part

Let $G =< g_1, g_2, ..., g_n >$ where $g_i \in \mathbf{G}_0$ for $i \in \{1, 2, ..., n\}$. Denote $G_i =< g_i >$. We note that, G "acts as a Schottky group on $\mathbf{C}^+$" [9] if there are conical neighbourhoods $A_i^{\pm}$ of $x^{\pm}(g_i)$ such that

- $cl(A_i^{\pm}) \cap cl(A_i^{\mp} \cup_{i \neq j} (A_j^+ \cup A_j^-) = \emptyset$ and
- $cl(g_i(A_i^-)) = \mathbf{C}^+ \setminus A_i^+$.

Denote the set $\mathbf{C} \setminus \{v \in \mathbf{C} : v = kx^{\pm}(g_i)$ for some $k \in \mathbb{R}\}$ by $\mathfrak{C}$ and the complement of the set $(A_i^+ \cup (-A_i^+) \cup A_i^- \cup (-A_i^-))$ by $\mathfrak{A}$.

Theorem 4.1.1. *$\mathfrak{A}$ is a fundamental domain for the action of G_i on $\mathfrak{C}$.*

Proof. We note that $cl(g_i(A_i^-)) = \mathbf{C}^+ \setminus A_i^+$. Using this we get that for $g \in G_i$ and $g \neq e$ we have $g_i(\mathfrak{A}) \cap \mathfrak{A} = \emptyset$.
Now, if we take $x \in \mathfrak{C}$ then $g_i^n(x) \to x^+(g_i)$. So for any $y \in \mathfrak{C}$ there exists a $x \in \mathfrak{A}$ such that $g_i^m(y) = x$ for some integer m. Hence $\mathfrak{A}$ is a fundamental domain for the action of G_i on $\mathfrak{C}$. $\square$

Define $v_{ij}^{\pm}$ for $j \in \{1,2\}$ to be of euclidean norm 1 and in the boundary of $A_i^{\pm}$ satisfying

- $v_{i1}^{\pm} \boxtimes v_{i2}^{\pm} \neq 0$ and
- $g_i(v_{ij}^{-})/\|g_i(v_{ij}^{-})\| = v_{ij}^{+}$.

The action under consideration is linear. So using the above theorem and linearity of the action we get that a fundamental domain for the action of G_i on $\mathbf{U}^{+}$ is the region bounded by $\mathfrak{A}$, $\mathfrak{F}_i^{+} := \{v \in < v_{i1}^{+}, v_{i2}^{+} >: \mathbb{B}(v,v) \leqslant 0\}$ and $\mathfrak{F}_i^{-} := \{v \in < v_{i1}^{-}, v_{i2}^{-} >: \mathbb{B}(v,v) \leqslant 0\}$.

Consider the half planes $\mathbf{P}^{\pm}(v_{ij}^{\pm})$. Note that

- $g_i(\mathbf{P}^{+}(v_{ij}^{-})) = \mathbf{P}^{+}(v_{ij}^{+})$ and
- $g_i(\mathbf{P}^{-}(v_{ij}^{-})) = \mathbf{P}^{-}(v_{ij}^{+})$,

since $g(x_v^{\pm})/\|g(x_v^{\pm})\| = x_{g(v)}^{\pm}$ for hyperbolic $g \in \mathbf{G}_0$ and $v \in \mathbb{R}^{2,1}$. Define the wedges $\mathfrak{W}_i^{\pm}$ to be the open region bounded by $\mathfrak{F}_i^{\pm} \cup \mathbf{P}^{+}(v_{i1}^{\pm}) \cup \mathbf{P}^{+}(v_{i2}^{\pm})$ and not containing $\mathfrak{F}_i^{\mp} \cup \mathbf{P}^{+}(v_{i1}^{\mp}) \cup \mathbf{P}^{+}(v_{i2}^{\mp})$ and the wedges $\mathfrak{M}_i^{\pm}$ to be the open region bounded by $\mathfrak{F}_i^{\pm} \cup \mathbf{P}^{-}(v_{i1}^{\pm}) \cup \mathbf{P}^{-}(v_{i2}^{\pm})$ and not containing $\mathfrak{F}_i^{\mp} \cup \mathbf{P}^{-}(v_{i1}^{\mp}) \cup \mathbf{P}^{-}(v_{i2}^{\mp})$.

Theorem 4.1.2. *A fundamental domain for the action of G_i*

- *on $\mathbb{R}^{2,1} \setminus (\mathbf{P}^{+}(x_{g_i}^{-}) \cup \mathbf{P}^{+}(x_{g_i}^{+}))$ is $\mathbb{R}^{2,1} \setminus (\mathfrak{W}_i^{+} \cup \mathfrak{W}_i^{-})$ and*
- *on $\mathbb{R}^{2,1} \setminus (\mathbf{P}^{-}(x_{g_i}^{-}) \cup \mathbf{P}^{-}(x_{g_i}^{+}))$ is $\mathbb{R}^{2,1} \setminus (\mathfrak{M}_i^{+} \cup \mathfrak{M}_i^{-})$.*

Proof. The result follows from the previous discussion and using the last theorem. □

4.2 Affine Fundamental Polyhedra

In this section we will discuss about the fundamental domains involving the $\mathbf{P}^{+}(v)$'s. The argument for the fundamental domains for $\mathbf{P}^{-}(v)$ is completely analogous. Let $H = < h_1, h_2, ..., h_n > \subset \mathbf{Aff}(\mathbb{R}^3)$ is such that $\mathbb{L}(h_i) = g_i$ and $u(h_i) = v_i$ where $v_i \in \mathbb{R}^3$. Denote $H_i =< h_i >$. The fundamental domain of a cyclic affine group is bounded by translates of components of the boundary of a fundamental domain for the corresponding cyclic linear group. We note that

$$cl(h_i(\mathfrak{W}_i^{-})) = cl(g_i(\mathfrak{W}_i^{-}) + v_i) = (\mathbb{R}^{2,1} \setminus \mathfrak{W}_i^{-}) + v_i$$

and a fundamental domain for the action of H_i on $\mathbb{R}^{2,1}$ is the complement of $\mathfrak{W}_i^{-}$ and $\mathfrak{W}_i^{+} + v_i$ if these two sets have disjoint closures.
Let ϱ denote the euclidean distance between two points in $\mathbb{R}^3$. If $\varrho(y, z + v_i) > 0$ for all choices of $y \in cl(\mathfrak{W}_i^{-})$ and $z \in cl(\mathfrak{W}_i^{+})$ then $cl(\mathfrak{W}_i^{-}) \cap cl(\mathfrak{W}_i^{+} + v_i) = \emptyset$. In particular, if for each pair of vectors $y \in cl(\mathfrak{W}_i^{-})$ and $z \in cl(\mathfrak{W}_i^{+})$ there is a vector $u \in \mathbb{R}^{2,1}$ such that $\mathbb{B}(y,u) \neq \mathbb{B}(z + v_i, u)$ then $cl(\mathfrak{W}_i^{-}) \cap cl(\mathfrak{W}_i^{+} + v_i) = \emptyset$.

To construct u given the vectors $y \in \mathfrak{W}_i^{-}$ and $z \in \mathfrak{W}_i^{+}$, first examine the case in which y and z are both space like. In this case, $x_y^{+} \in cl(A_i^{-})$, $x_z^{+} \in cl(A_i^{+})$ and

$$u = x_y^{+} \boxtimes x_z^{+}.$$

If y is not space like and z is space like, let

$$u = y/\|y\| \boxtimes x_z^+.$$

If z is not space like and y is space like, let

$$u = x_y^+ \boxtimes z/\|z\|.$$

If neither y nor z are space like, let

$$u = (y/\|y\|) \boxtimes (z/\|z\|).$$

Note that $\mathbb{B}(y, u) = 0$ if y is not space like. If y is space like then $x_y^+ = x_u^-$ and $\mathbb{B}(y, u) < 0$. Similarly, $\mathbb{B}(z, u) \geqslant 0$.
If $\mathbb{B}(v_i, u) > 0$ for all possible vectors u described above then $cl(\mathfrak{W}_i^-) \cap cl(\mathfrak{W}_i^+ + v_i) = \emptyset$ and a fundamental domain for H_i is the complement of $\mathfrak{W}_i^-$ and $\mathfrak{W}_i^+ + v_i$. For $y \in cl(\mathfrak{W}_i^-)$ and $z \in cl(\mathfrak{W}_i^+)$ one can construct a vector u as above such that $\mathbb{B}(z + v_i, u) = \mathbb{B}(z, u) + \mathbb{B}(v_i, u) > k$ and $\mathbb{B}(y, u) \leqslant 0$. The set of v_i's such that $\mathbb{B}(v_i, u) > 0$ for a fixed u is a half space bounded by $\mathbf{P}(u)$. The set of translations giving rise to a fundamental domain for the given $A_i^\pm$ in this construction is

$$\mathbf{T}(A_i^-, A_i^+),$$

the set of allowable translations. Another set of allowable translations is obtained by noting that $cl(h_i(\mathfrak{W}_i^- - g_i^{-1}(v_i))) = \mathbb{R}^{2,1} \setminus \mathfrak{W}_i^+$. In this case the wedges separate if $\mathbb{B}(-g_i^{-1}(v_i), u) < 0$ for all $u = z \boxtimes w$ where $w \in cl(A_i^+)$ and $z \in cl(A_i^-)$. Equivalently, $\mathbb{B}(v_i, g_i(u)) > 0$ for all $g_i(u) = z \boxtimes w$ where $w \in cl(g_i(A_i^+))$ and $z \in cl(g_i(A_i^-))$ and the set of allowable translation is

$$\mathbf{T}(g_i(A_i^-), g_i(A_i^+)).$$

These two sets of allowable translations can be combined together to make a larger third set of allowable translations. If $v_{i1} \in \mathbf{T}(A_i^-, A_i^+)$ and $v_{i2} \in \mathbf{T}(g_i(A_i^-), g_i(A_i^+))$ then $v_i = v_{i1} + v_{i2}$ is also an allowable translation. We see that the set
$\{v : v_i = v_{i1} + v_{i2}$ where $v_{i1} \in \mathbf{T}(A_i^-, A_i^+)$ and $v_{i2} \in \mathbf{T}(g_i(A_i^-), g_i(A_i^+))\}$ is same as the set

$$\mathbf{T}(A_i^-, g_i(A_i^+)).$$

Define $\mathcal{W}_i^- := \mathfrak{W}_i^- - g_i^{-1}(v_{i2})$ and $\mathcal{W}_i^+ := \mathfrak{W}_i^+ + v_{i1}$.

Theorem 4.2.1. *A fundamental domain for the action of H_i on $\mathbb{R}^{2,1}$ is $\mathbb{R}^{2,1} \setminus (\mathcal{W}_i^- \cup \mathcal{W}_i^+)$, if $v_i \in \mathbf{T}(A_i^-, g_i(A_i^+))$ where $v_{i1} \in \mathbf{T}(A_i^-, A_i^+)$ and $v_{i2} \in \mathbf{T}(g_i(A_i^-), g_i(A_i^+))$ are such that $v_i = v_{i1} + v_{i2}$.*

Proof. The result follows easily from the discussion above. □

The fundamental domain for the action of H on $\mathbb{R}^{2,1}$ is the intersection of the fundamental domains of the action of H_i's on $\mathbb{R}^{2,1}$ provided the fundamental domain for each H_i completely contains the complement of the fundamental domain for the other H_j's where $j \neq i$. In passing from the fundamental domain of H_i to the fundamental domain of H, it is useful to demand both wedges be translated away from the origin. We note

that, in order to guarantee that the closures of the translated wedges are distinct, it is useful to consider each wedge paired with the other wedges.
Let $\mathcal{A}_i^{\pm} = A_i^{\pm} \cup_{j\neq i} (A_j^+ \cup A_j^-)$. If $v_i \in \mathbf{T}(g_i(\mathcal{A}_i^+), \mathcal{A}_i^-)$ then $v_i = v_{i1} + v_{i2}$ for some $v_{i1} \in \mathbf{T}(A_i^+, \mathcal{A}_i^-)$ and $v_{i2} \in \mathbf{T}(g_i(\mathcal{A}_i^+), g_i(A_i^-)$, and if $-v_i \in \mathbf{T}(g_i(\mathcal{A}_i^+), \mathcal{A}_i^-)$ then $v_i = v_{i1}+v_{i2}$ for some $-v_{i1} \in \mathbf{T}(A_i^+, \mathcal{A}_i^-)$ and $-v_{i2} \in \mathbf{T}(g_i(\mathcal{A}_i^+), g_i(A_i^-)$. In this case, let $\mathcal{W}_i^+ = \mathfrak{W}_i^+ + v_{i1}$, $\mathcal{W}_i^- = \mathfrak{W}_i^- - g_i^{-1}(v_{i2})$, $\mathcal{M}_i^+ = \mathfrak{M}_i^+ + v_{i1}$ and $\mathcal{M}_i^- = \mathfrak{M}_i^- - g_i^{-1}(v_{i2})$.

Theorem 4.2.2. *Let $h_i(x) = g_i(x) + v_i$ for $i \in \{1, 2, ..., n\}$ and $H =< h_1, h_2, ..., h_n >$. If $\mathbb{L}(H)$ acts as a Schottky subgroup on $\mathbf{C}^+$ and:*

- *$v_i \in \mathbf{T}(g_i(\mathcal{A}_i^+), \mathcal{A}_i^-)$, then H acts properly discontinuously on $\mathbb{R}^{2,1}$. In this case, $\mathbb{R}^{2,1} \setminus (\cup_i(\mathcal{W}_i^- \cup \mathcal{W}_i^+))$ is a fundamental polyhedron for the action of H on $\mathbb{R}^{2,1}$.*

- *$-v_i \in \mathbf{T}(g_i(\mathcal{A}_i^+), \mathcal{A}_i^-)$, then H acts properly discontinuously on $\mathbb{R}^{2,1}$. In this case, $\mathbb{R}^{2,1} \setminus (\cup_i(\mathcal{M}_i^- \cup \mathcal{M}_i^+))$ is a fundamental polyhedron for the action of H on $\mathbb{R}^{2,1}$.*

Proof. It suffices to prove the theorem for the case $v_i \in \mathbf{T}(g_i(\mathcal{A}_i^+), \mathcal{A}_i^-)$. It is clear from the construction that no two elements of $\mathcal{X} := \mathbb{R}^{2,1} \setminus (\cup_i(\mathcal{W}_i^- \cup \mathcal{W}_i^+))$ are H-equivalent.

Assume that there exists a $p \in \mathbb{R}^{2,1}$ which is not H-equivalent to any point in $\mathcal{X}$. Thus, one can construct an infinite sequence of embedded images of the wedges all containing p in the following manner:

Let $\gamma_0 = e$ and $\mathcal{W}_{i_0}^{j_0} = \omega$. For integers $n > 1$ choose $\gamma_n \in H$, $i_n \in \{1, 2\}$ and $j_n \in \{-1, 1\}$, so that $p \in \gamma_n(\mathcal{W}_{i_n}^{j_n})$, $\gamma_{n+1}(\mathcal{W}_{i_{n+1}}^{j_{n+1}}) \subset \gamma_n(\mathcal{W}_{i_n}^{j_n})$ and $\gamma_{n+1} = \gamma_n h_{i_n}^{j_n}$.

The leading term of γ_n is $h_{i_0}^{j_0}$ and by an application of the Brouwer fixed point theorem [9] it can be shown that $x^+(\gamma_n) \in A_{i_0}^{j_0}$. Let $\gamma_{n+1/2}$ is defined to be $h_k\gamma_n$, where $k \in \{1, 2\}$ is chosen so that $k \neq i_0$. Using the same argument and applying the Brouwer fixed point theorem [9] one gets that $x^+(\gamma_{n+1/2}) \in g_k(A_{i_0}^{j_0})$.
Define the plane

$$\mathbf{S}_m :=< x^+(\gamma_m), x^0(\gamma_m) > \text{ for all } m \in \{0, 1/2, 1, 3/2, 2, ...\}.$$

Consider the intersection of the embedded images of the wedges and the plane

$$\mathbb{P} := \{x \in \mathbb{R}^{2,1} : x_3 = p_3\}.$$

p is H-equivalent to elements in all of the wedges $\mathcal{W}_i^{\pm}$. One can assume that p is contained in a "small" wedge ω, where the angle between every pair of rays contained in $\omega \cap \mathbb{P}$ is $\leqslant \pi/2$. In particular, $\mathbf{S}_n \cap \mathbb{P}$ contains a ray lying completely within $\omega \cap \mathbb{P}$ for all positive integers n.
Choose $L_0 \subset \mathbb{P}$ to be the line closest to p which bounds a half plane in $\mathbb{P}$ containing all of $\omega \cap \mathbb{P}$ and whose normal in $\mathbb{P}$ forms an angle of less than $\pi/4$ with all the rays contained in $\omega \cap \mathbb{P}$. Let $L_n \subset \mathbb{P}$ be the closest line to p parallel to L_0 and bounding a half plane in $\mathbb{P}$ containing $\gamma_n(\mathcal{W}_{i_n}^{j_n}) \cap \mathbb{P}$. The set $\{L_0, L_1, L_2, ...\}$ is an infinite sequence of parallel lines in $\mathbb{P}$ constructed so that $\varrho(p, L_{n+1}) \leqslant \varrho(p, L_n)$. To arrive at a contradiction it is enough

to show that $(\varrho(p, L_n) - \varrho(p, L_{n+1}))$ is bounded from below.

Now there exists an $\varepsilon > 0$ such that for any $x \in \mathcal{X}$ the ε-ball centered at x, $\mathbf{B}(x, \varepsilon)$, is contained in $\mathcal{X} \cap h_1(\mathcal{X}) \cap h_1^{-1}(\mathcal{X}) \cap h_2(\mathcal{X}) \cap h_2^{-1}(\mathcal{X})$. Therefore we get that $(\varrho(p, L_0) - \varrho(p, L_1)) > \varepsilon$.
Now we consider the case when γ_n is δ-hyperbolic for positive integer n. For every $y \in \gamma_n^{-1}(L_n)$, $\mathbf{B}(y, \varepsilon)$ is contained in the complement of $h_{i_n}^{j_n}(\mathcal{W}_{i_{n+1}}^{j_{n+1}})$. And since the angle between $\mathbb{P} \cap \mathbf{S}_n$ and the normal to L_n in $\mathbf{P}$ was constructed to be less than $\pi/4$, $\mathbf{B}(x, \varepsilon\delta/2^{3/2})$ for all $x \in L_n$ is contained in the complement of $\gamma_{n+1}(\mathcal{W}_{i_{n+1}}^{j_{n+1}})$ and

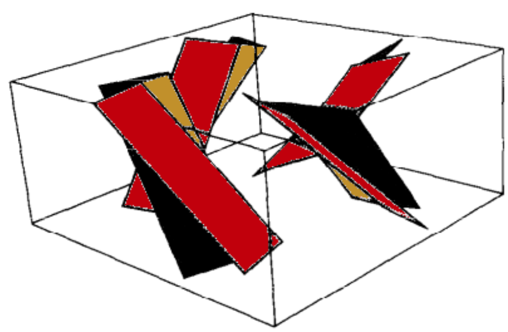
$$(\varrho(p, L_n) - \varrho(p, L_{n+1})) > \varepsilon\delta/2^{3/2}.$$

Now if γ_n is not δ-hyperbolic then by a theorem from another paper by Drumm and Goldman we get that $\gamma_{n+1/2}$ is δ-hyperbolic. We also notice that the action of g_k^{-1} does not contract any vector by more than a factor of $\lambda(g_k)$. Combining these facts we get that

$$(\varrho(p, L_n) - \varrho(p, L_{n+1})) > \lambda(g_k)\varepsilon\delta/2^{3/2}.$$

Therefore we have that $(\varrho(p, L_n) - \varrho(p, L_{n+1}))$ is bounded from below. So we get a contradiction. Thus, there is no $p \in \mathbb{R}^{2,1}$ which is not H-equivalent to an element of $\mathcal{X}$. Hence, $\mathcal{X}$ is a fundamental domain for the action of H on $\mathbb{R}^{2,1}$.

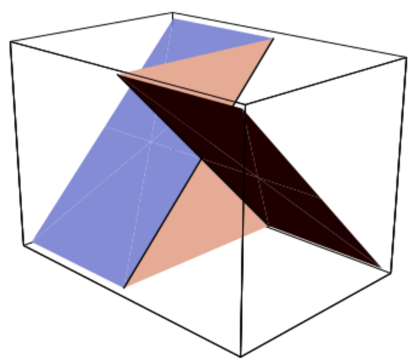
□



Chapter 5

The Geometry of Crooked Planes

We saw in the last chapter that the fundamental domains that we constructed for certain Margulis space times, are bounded by certain polyhedral hyper-surfaces in $\mathbb{R}^{2,1}$. These polyhedral hyper-surfaces are called the Crooked planes.
A crooked plane consists of three parts: two half planes, called wings and a pair of opposite planar sectors, called its stem. The wings lie in null planes and the stem (whose interior has two connected components) lies in a time like (indefinite) plane.



In this chapter we study the intersections of two crooked planes.

5.1 Anatomy of a Crooked Plane

Let $p \in \mathbb{R}^{2,1}$ be a point and $v \in \mathbb{R}^{2,1}$ a space like vector. Define the positively oriented crooked plane $\mathscr{C}(v,p) \subset \mathbb{R}^{2,1}$ with vertex p and direction vector v to be the union of two wings

$$\mathscr{W}^{+}(v,p) := p + \mathbf{P}^{+}(x^{+}(v)),$$
$$\mathscr{W}^{-}(v,p) := p + \mathbf{P}^{+}(x^{-}(v))$$

and a stem

$$\mathscr{S}(v,p) := p + \{x \in \mathbb{R}^{2,1} : \mathbb{B}(v,x) = 0, \mathbb{B}(x,x) \leqslant 0\}.$$

Each wing is a half plane and the stem is the union of two quadrants in a space like plane. The positively oriented crooked plane itself is a piecewise linear submanifold, which stratifies into four connected open subsets of planes, four null rays and a vertex. Note that for a crooked plane $\mathscr{C}(v,p)$ with $v \in \mathbb{R}^{2,1}$ a vector and $p \in \mathbb{R}^{2,1}$ a point we call the line $p + \mathbb{R}v$, the spine of the given crooked plane.
We also define the negatively oriented crooked plane $\mathscr{K}(v,p) \subset \mathbb{R}^{2,1}$ with vertex p and direction vector v to be the union of two wings

$$\mathscr{M}^+(v,p) := p + \mathbf{P}^-(x^+(v)),$$
$$\mathscr{M}^-(v,p) := p + \mathbf{P}^-(x^-(v))$$

and a stem

$$\mathscr{S}(v,p) := p + \{x \in \mathbb{R}^{2,1} : \mathbb{B}(v,x) = 0, \mathbb{B}(x,x) \leqslant 0\}.$$

In the following sections we will only consider positively oriented half planes. Furthermore, an unspecified orientation for a crooked plane will assumed to be the positive one. The case for the negatively oriented half planes is similar. The following sections describe intersections of two wings, a wing and a stem, and two stems. From these results follow necessary and sufficient conditions for the intersection of two crooked planes.

5.2 Intersection of Wings

A plane in $\mathbb{R}^{2,1}$ may be written as $p + \mathbf{P}(v)$, for $p \in \mathbb{R}^{2,1}$ and $v \in \mathbb{R}^{2,1}$. Suppose that $p_1, p_2 \in \mathbb{R}^{2,1}$ and that $v_1, v_2 \in \mathbb{R}^{2,1}$ are linearly independent. Then the intersection $\mathscr{P}_1 \cap \mathscr{P}_2$ of the two planes $\mathscr{P}_i := p_i + \mathbf{P}(v_i)$ is a line which can be parametrized as

$$p + \mathbb{R}(v_1 \boxtimes v_2)$$

for some $p \in \mathscr{P}_1 \cap \mathscr{P}_2$.

Lemma 5.2.1. *Let $x_1, x_2 \in \mathbf{C}^+$ be two linearly independent null vectors and $p_1, p_2 \in \mathbb{R}^{2,1}$ be two points. The corresponding null half planes $\mathscr{P}_i^+ := p_i + \mathbf{P}^+(x_i)$ for $i \in \{1,2\}$ are disjoint if and only if $\mathbb{B}(p_2 - p_1, x_1 \boxtimes x_2) > 0$. Otherwise, $\mathscr{P}_1^+ \cap \mathscr{P}_2^+$ is a point if and only if $\mathbb{B}(p_2 - p_1, x_1 \boxtimes x_2) = 0$ and $\mathscr{P}_1^+ \cap \mathscr{P}_2^+$ is a space like line segment if and only if $\mathbb{B}(p_2 - p_1, x_1 \boxtimes x_2) < 0$.*

Proof. Let l be the intersection of the planes that contain $\mathscr{P}_1^+$ and $\mathscr{P}_2^+$. Then $\mathscr{P}_1^+ \cap \mathscr{P}_2^+ \subset l$. Let $w := x_1 \boxtimes x_2$ and $p = l \cap (p_1 + \mathbf{P}(w))$ so that $l = p + \mathbb{R}w$.
The subsets $l \cap \mathscr{P}_i^+$ of l are characterized as the set of all $p + kw$ where $k \in \mathbb{R}$ satisfying the inequalities

$$\mathbb{B}((p + kw) - p_1, -w) \geqslant 0 \text{ and}$$
$$\mathbb{B}((p + kw) - p_2, w) \geqslant 0$$

respectively. Now p was chosen so that $\mathbb{B}(p - p_1, w) = 0$ and for the given parametrization of l

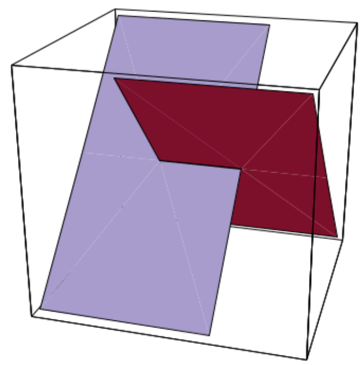
$$l \cap \mathscr{P}_1^+ \longleftrightarrow (-\infty, 0] \text{ and}$$
$$l \cap \mathscr{P}_2^+ \longleftrightarrow [\mathbb{B}(p_2 - p_1, w)/\mathbb{B}(w,w), \infty).$$

Then

- $\mathscr{P}_1^+ \cap \mathscr{P}_2^+$ is empty if and only if $\mathbb{B}(p_2 - p_1, w) > 0$
- $\mathscr{P}_1^+ \cap \mathscr{P}_2^+$ is a point if and only if $\mathbb{B}(p_2 - p_1, w) = 0$ and
- $\mathscr{P}_1^+ \cap \mathscr{P}_2^+$ is a space like line segment if and only if $\mathbb{B}(p_2 - p_1, w) < 0$.

and the proof is complete.

□



Here is the corresponding lemma for pairs of half planes with opposite orientations.

Lemma 5.2.2. *Let $x_1, x_2 \in \mathbf{C}^+$ be two linearly independent null vectors and $p_1, p_2 \in \mathbb{R}^{2,1}$ be two points. The intersection of the half planes $\mathscr{P}^+ := p_1 + \mathbf{P}^+(x_1)$ and $\mathscr{P}^- := p_2 + \mathbf{P}^-(x_2)$ is never empty.*

Proof. Using a similar method of argument as used in the previous theorem we get the result.

□

Now we consider the intersection of null half planes for the degenerate case when the null half planes are parallel:

Lemma 5.2.3. *Let $x \in \mathbf{C}^+$ be a null vectors and $p_1, p_2 \in \mathbb{R}^{2,1}$ be two points. The corresponding null half planes $\mathscr{P}_i^+ := p_i + \mathbf{P}^+(x)$ for $i \in \{1, 2\}$ are disjoint if and only if $\mathbb{B}(p_2 - p_1, x) \neq 0$. Otherwise, one of the half planes contains the similarly oriented half plane and their intersection is a null half plane.*

Proof. Using a similar method of argument as used in the first theorem of this section we get the result.

□

5.3 Intersection of Stem with Wing

In this section we describe when a positively oriented null half plane intersects a stem. From this section onwards we will just state the lemmas as the proof follows a similar line of reasoning as was used to prove the first theorem of the previous section.

Lemma 5.3.1. *Let x be a positively oriented null vector and v a unit space like vector such that $\mathbb{B}(x, v) < 0$. Let $p_1, p_2 \in \mathbb{R}^{2,1}$ be points. Then the stem $\mathscr{S}_1 := \mathscr{S}(v, p_1)$ and the positively oriented null half plane $\mathscr{P}_2^+ := p_2 + \mathbf{P}^+(x)$ are disjoint if and only if*

$$\mathbb{B}(p_2 - p_1, v \boxtimes x) > |\mathbb{B}(p_2 - p_1, x)|.$$

Otherwise $\mathscr{S}_1 \cap \mathscr{P}_2^+$ consists of,

- *a point (which lies on $\partial \mathscr{S}_1$ but not on $\partial \mathscr{P}_2^+$) if and only if*

$$\mathbb{B}(p_2 - p_1, v \boxtimes x) < 0 = \mathbb{B}(p_2 - p_1, x).$$

- *a point (which lies on $\partial \mathscr{S}_1$ and on $\partial \mathscr{P}_2^+$) if and only if*

$$\mathbb{B}(p_2 - p_1, v \boxtimes x) = |\mathbb{B}(p_2 - p_1, x)|.$$

- *a space like line segment (with atleast one vertex on $\partial \mathscr{S}_1$) if and only if*

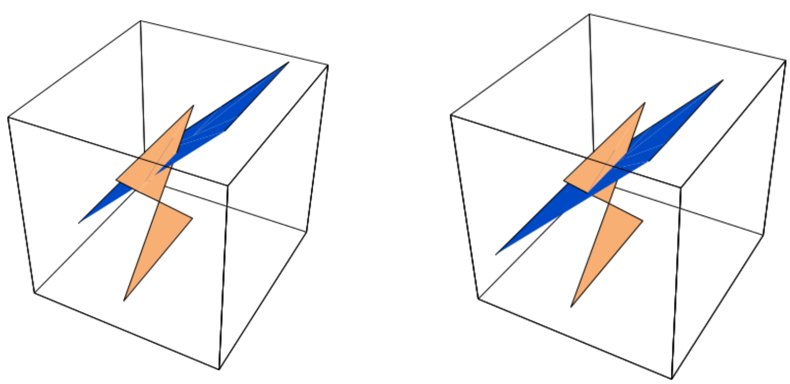
$$\mathbb{B}(p_2 - p_1, v \boxtimes x) < |\mathbb{B}(p_2 - p_1, x)|$$

where the other end point lies on $\partial \mathscr{P}_2^+$ if and only if

$$\mathbb{B}(p_2 - p_1, v \boxtimes x) \geqslant -|\mathbb{B}(p_2 - p_1, x)|$$

and on $\partial \mathscr{S}_1$ if and only if

$$\mathbb{B}(p_2 - p_1, v \boxtimes x) \leqslant -|\mathbb{B}(p_2 - p_1, x)|.$$



Now we consider the case when the null half plane and the stem are orthogonal:

Lemma 5.3.2. *Let x be a positively oriented null vector and v a unit space like vector such that $\mathbb{B}(x, v) = 0$. Let $p_1, p_2 \in \mathbb{R}^{2,1}$ be points. Then the stem $\mathscr{S}_1 := \mathscr{S}(v, p_1)$ and the positively oriented null half plane $\mathscr{P}_2^+ := p_2 + \mathbf{P}^+(x)$ are disjoint if and only if*

- $\mathbb{B}(p_2 - p_1, v) < 0$ *when x is a multiple of $x^-(v)$*
- $\mathbb{B}(p_2 - p_1, v) > 0$ *when x is a multiple of $x^+(v)$*

Otherwise $\mathscr{S}_1 \cap \mathscr{P}_2^+$ is a ray with end point on $\partial \mathscr{P}_2^+$ (except if $\mathbb{B}(p_2 - p_1, x) = 0$, in which case it is an entire line).

5.4 Intersection of Stems

Let $\mathscr{C}(v_1, p_1)$ and $\mathscr{C}(v_2, p_2)$ be two crooked planes where $p_1, p_2 \in \mathbb{R}^{2,1}$ are points and $v_1, v_2 \in \mathbb{R}^{2,1}$ are unit space like vectors. We call the two vectors v_1 and v_2 :

- ultraparallel iff $\mathbf{P}(v_1) \cap \mathbf{P}(v_2)$ is a space like line or equivalently $\mathbb{B}(v_1 \boxtimes v_2, v_1 \boxtimes v_2) > 0$.
- asymptotic iff $\mathbf{P}(v_1) \cap \mathbf{P}(v_2)$ is a null line or equivalently $\mathbb{B}(v_1 \boxtimes v_2, v_1 \boxtimes v_2) = 0$.
- crossing iff $\mathbf{P}(v_1) \cap \mathbf{P}(v_2)$ is a time like line or equivalently $\mathbb{B}(v_1 \boxtimes v_2, v_1 \boxtimes v_2) < 0$.

We describe the intersection of two stems separately for the above mentioned three different cases.

Lemma 5.4.1. *Let v_1 and v_2 be ultraparallel unit space like vectors. The stems $\mathscr{S}(v_1, p_1)$ and $\mathscr{S}(v_2, p_2)$ are disjoint if and only if*

$$|\mathbb{B}(p_2 - p_1, v_1 \boxtimes v_2)| > |\mathbb{B}(p_2 - p_1, v_1)| + |\mathbb{B}(p_2 - p_1, v_2)|.$$

Otherwise $\mathscr{S}(v_1, p_1) \cap \mathscr{S}(v_2, p_2)$ is a line segment whose end points lie on two distinct lines contained in $\partial\mathscr{S}(v_1, p_1) \cup \partial\mathscr{S}(v_2, p_2)$.

Lemma 5.4.2. *Let v_1 and v_2 be asymptotic unit space like vectors such that $x^-(v_1) = x^+(v_2)$. The stems $\mathscr{S}(v_1, p_1)$ and $\mathscr{S}(v_2, p_2)$ are disjoint if and only if*

$$\mathbb{B}(p_2 - p_1, x^-(v_2) \boxtimes x^+(v_1)) \text{ has a different sign from } \\ \mathbb{B}(p_2 - p_1, v_1) \text{ and } \mathbb{B}(p_2 - p_1, v_2).$$

Otherwise $\mathscr{S}(v_1, p_1) \cap \mathscr{S}(v_2, p_2)$ is

- *a point if and only if $\mathbb{B}(p_2 - p_1, x^-(v_2) \boxtimes x^+(v_1)) = 0$ and $\mathbb{B}(p_2 - p_1, v_1)$ and $\mathbb{B}(p_2 - p_1, v_2)$ have the same signs.*
- *a line segment if and only if $\mathbb{B}(p_2 - p_1, x^-(v_2) \boxtimes x^+(v_1))$ and $\mathbb{B}(p_2 - p_1, v_1)$ and $\mathbb{B}(p_2 - p_1, v_2)$ have the same signs.*
- *a ray if and only if either $\mathbb{B}(p_2 - p_1, v_i) = 0$ for only one i or $\mathbb{B}(p_2 - p_1, v_1)$ and $\mathbb{B}(p_2 - p_1, v_2)$ have opposite signs.*
- *a line if and only if $\mathbb{B}(p_2 - p_1, v_i) = 0$ for both i.*

Lemma 5.4.3. *Let v_1 and v_2 be crossing unit space like vectors. The intersection of the stems $\mathscr{S}(v_1, p_1)$ and $\mathscr{S}(v_2, p_2)$ lies on a line and is either a line, the union of two disjoint rays, or two rays and a line segment, where all end points lie on the set $\partial\mathscr{S}(v_1, p_1) \cup \partial\mathscr{S}(v_2, p_2)$.*

5.5 Intersection of Crooked Planes

In this section we describe the intersection of two crooked planes. While considering the intersection of two crooked planes, some terminology regarding the orientation of the vectors defining the crooked planes is useful. Say that space like vectors $v_1, v_2, ...v_n$ are consistently oriented if

$$\mathbb{B}(v_i, v_j) < 0 \text{ and } \mathbb{B}(v_i, x^\pm(v_j)) \leqslant 0$$

for $i \neq j$.
Geometrically, consistent orientation means the following:
A unit space like vector v defines a half plane $\mathscr{H}(v)$ in the hyperbolic plane $\mathbb{H}^2$ as follows:

$$\mathscr{H}(v) := \{u \in \mathbf{U}^+ : \mathbb{B}(u,v) > 0\}.$$

Suppose $v_1, v_2 \in \mathbb{R}^{2,1}$ are ultraparallel (respectively asymptotic) unit space like vectors. Then the half planes $\mathscr{H}(v_1), \mathscr{H}(v_2)$ are bounded by ultraparallel (respectively asymptotic) geodesics in $\mathbb{H}^2$. A pair $v_1, v_2 \in \mathbb{R}^{2,1}$ of ultraparallel (respectively asymptotic) unit space like vectors are consistently oriented if and only if the half planes $\mathscr{H}(v_1), \mathscr{H}(v_2)$ are disjoint. In that case $\mathscr{H}(v_1) \cap \mathscr{H}(v_2)$ is a strip with two ideal boundary components and boundary components $\partial\mathscr{H}(v_1), \partial\mathscr{H}(v_2)$.

Lemma 5.5.1. *If $v_1, v_2 \in \mathbb{R}^{2,1}$ are two consistently oriented unit space like vectors, then*

- $x^+(v_1) \boxtimes x^+(v_2)$ *is a positive scalar multiple of* $(v_1 \boxtimes v_2) - v_1 + v_2$.
- $x^+(v_1) \boxtimes x^-(v_2)$ *is a positive scalar multiple of* $(v_1 \boxtimes v_2) + v_1 + v_2$.
- $x^-(v_1) \boxtimes x^+(v_2)$ *is a positive scalar multiple of* $(v_1 \boxtimes v_2) - v_1 - v_2$.
- $x^-(v_1) \boxtimes x^-(v_2)$ *is a positive scalar multiple of* $(v_1 \boxtimes v_2) + v_1 - v_2$.

Proof. We know that $\mathbf{G}_0$ acts transitively on the set of Lorentz unit vectors so without loss of generality we can take

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } x^\pm(v_1) = 1/\sqrt{2} \begin{bmatrix} 0 \\ \mp 1 \\ 1 \end{bmatrix}.$$

Applying a hyperbolic element with v_1 as a fixed eigenvector we get,

$$v_1 = \begin{bmatrix} -a \\ 0 \\ c \end{bmatrix}$$

where $|c| = \sqrt{a^2-1}$. Furthermore,

$$x^\pm(v_2) = 1/\sqrt{2} \begin{bmatrix} -c/a \\ \mp 1/a \\ 1 \end{bmatrix}.$$

Since v_1, v_2 are consistently oriented, $a, c \geqslant 0$. Thus $a > 1$ and $c = \sqrt{a^2-1}$. And for $j, k \in \{-,+\}$ we have

$$v_1 \boxtimes v_2 - kv_1 + jv_2 = \begin{bmatrix} -k - ja \\ -c \\ jc \end{bmatrix}$$

$$x^j(v_1) \boxtimes x^k(v_2) = \begin{bmatrix} -j - k/a \\ -c/a \\ jc/a \end{bmatrix}$$

$x^j(v_1) \boxtimes x^k(v_2)$ is a multiple of $v_1 \boxtimes v_2 - kv_1 + jv_2$, since $\mathbb{B}(x^j(v_1), v_1 \boxtimes v_2 \pm v_1 + jv_2) = 0$, $\mathbb{B}(x^k(v_2), v_1 \boxtimes v_2 - kv_1 \pm v_2) = 0$.
Furthermore, $x^j(v_1) \boxtimes x^k(v_2)$ is a positive multiple of $v_1 \boxtimes v_2 - kv_1 + jv_2$ since $\mathbb{B}(x^j(v_1) \boxtimes x^k(v_2), v_1 \boxtimes v_2 - kv_1 + jv_2) = (a \pm 1)^2/a > 0$. Hence we have our desired result.

□

Theorem 5.5.2. *Let $v_1, v_2 \in \mathbb{R}^{2,1}$ be two consistently oriented ultraparallel unit space like vectors and $p_1, p_2 \in \mathbb{R}^{2,1}$ be two points. The positively oriented crooked planes $\mathscr{C}(v_1, p_1)$ and $\mathscr{C}(v_2, p_2)$ are disjoint if and only if*

$$|\mathbb{B}(p_2 - p_1, v_1 \boxtimes v_2)| > |\mathbb{B}(p_2 - p_1, v_1)| + |\mathbb{B}(p_2 - p_1, v_2)|.$$

Otherwise the intersection is either a single point or a polygon.

Proof. The result follows from the lemma 5.4.1.

□

Theorem 5.5.3. *Let $v_1, v_2 \in \mathbb{R}^{2,1}$ be two consistently oriented asymptotic unit space like vectors such that $x^-(v_1) = x^+(v_2)$ and $p_1, p_2 \in \mathbb{R}^{2,1}$ be two points. The positively oriented crooked planes $\mathscr{C}(v_1, p_1)$ and $\mathscr{C}(v_2, p_2)$ are disjoint if and only if*

- $\mathbb{B}(p_2 - p_1, v_1) < 0$,
- $\mathbb{B}(p_2 - p_1, v_2) < 0$,
- $\mathbb{B}(p_2 - p_1, x^-(v_2) \boxtimes x^+(v_1)) > 0$.

Proof. The result follows from applying the lemma 5.4.2.

□

Theorem 5.5.4. *Let v_1 and v_2 be two crossing unit space like vectors. The intersection of crooked planes of indeterminate orientation with spines parallel to v_1 and v_2 is always nonempty.*

Proof. In this case, since the stems intersect, the orientations are irrelevant and the proof follows easily from the lemma 5.4.3.

□

Now we consider the intersection of two crooked planes of different orientations.

Theorem 5.5.5. *Let v_1 and v_2 be two unit space like vectors which are not parallel. The intersection of the positively oriented crooked plane $\mathscr{C}(v_1, p_1)$ and the negatively oriented crooked plane $\mathscr{K}(v_2, p_2)$ is always nonempty.*

Proof. The result follows easily from the lemmas in the previous sections.

□

Chapter 6

The Extended Margulis Invariant and Its Applications

In this chapter we take an alternate approach to attack the original problem. We describe an extension of the Margulis invariant to a continuous function on higher dimensions given by Labourie and following the recent works of Margulis, Goldman and labourie, use the extended Margulis invariant to give an equivalent criterion for properness.

6.1 Flat Bundles associated to Affine Deformations

In this section we make the necessary ground work to extend the Margulis invariant. Let us consider the Lie group $\mathbf{G}_0$, the vector space $\mathbb{V} = \mathbb{V}_r$ and a linear representation, $\mathbb{L} : \mathbf{G_0} \longrightarrow \mathbf{GL}(\mathbb{V})$. Let $\mathbf{G}$ be the corresponding semi-direct product $\mathbb{V} \rtimes \mathbf{G}_0$. Multiplication in $\mathbf{G}$ is defined by

$$(v_1, g_1) \centerdot (v_2, g_2) := (v_1 + g_1 v_2, g_1 g_2).$$

The projection

$$\begin{gathered} \Pi : \mathbf{G} \longrightarrow \mathbf{G}_0 \text{ given by} \\ (v, g) \longmapsto g \end{gathered}$$

defines a trivial bundle with fiber $\mathbb{V}$ over $\mathbf{G}_0$. We note that when $r = 1$, that is, $\mathbb{V} = \mathbb{R}^{2,1}$, $\mathbf{G}$ is the tangent bundle of $\mathbf{G}_0$ with its natural Lie group structure.
Since the fiber of Π equals the vector space $\mathbb{V}$, the fibration Π can be given the structure of a (trivial) affine bundle over $\mathbf{G}_0$. Futhermore, this structure is $\mathbf{G}$-invariant. Denote the total space of this $\mathbf{G}$-homogeneous affine bundle over $\mathbf{G}_0$ by $\tilde{\mathcal{E}}$.
Note that we can also consider Π as a (trivial) vector bundle. This structure is then $\mathbf{G}_0$-invariant. Via $\mathbb{L}$, this $\mathbf{G}_0$-homogeneous vector bundle becomes a $\mathbf{G}$-homogeneous vector bundle $\tilde{\mathcal{V}}$ over $\mathbf{G}_0$. This vector bundle underlies $\tilde{\mathcal{E}}$. The $\mathbf{G}$-homogeneous affine bundle $\tilde{\mathcal{E}}$ and the $\mathbf{G}$-homogeneous vector bundle $\tilde{\mathcal{V}}$ admit flat connections. We denote both of these connections by $\tilde{\nabla}$.
We have seen in chapter two that $\mathbb{L}$ preserves a bilinear form $\mathbb{B}$ on $\mathbb{V}$. The $\mathbf{G}_0$ invariant bilinear form $\mathbb{B} : \mathbb{V} \times \mathbb{V} \longrightarrow \mathbb{R}$ defines a bilinear pairing

$$\mathbb{B} : \tilde{\mathcal{V}} \times \tilde{\mathcal{V}} \longrightarrow \mathbb{R}$$

of vector bundles. We note that, if $\mathbb{L}$ preserves a bilinear form $\mathbb{B}$ on $\mathbb{V}$, the bilinear pairing $\mathbb{B}$ on $\tilde{\mathcal{V}}$ is parallel with respect to $\tilde{\nabla}$.
Now let

$$a(t) := \begin{bmatrix} 1 & 0 & 0 \\ 0 & cosh(t) & sinh(t) \\ 0 & sinh(t) & cosh(t) \end{bmatrix}$$

for $t \in \mathbb{R}$. We notice that for $p \in \mathbb{H}^2$, $a(t)p$ describes the geodesic through p and tangent to $\dot{p}$.
Right multiplication by $a(-t)$ on $\mathbf{G}_0$ identifies with the geodesic flow $\tilde{\varphi}_t$ on $U\mathbb{H}^2$ where $U\mathbb{H}^2$ denotes the unit tangent bundle on $\mathbb{H}^2$. We denote the vector field corresponding to the geodesic flow by $\tilde{\varphi}$. Similarly, right multiplication by $a(-t)$ on $\mathbf{G}$ identifies with the geodesic flow $\tilde{\Phi}_t$ on $\tilde{\mathcal{E}}$. We note that this flow covers the flow φ_t on $\mathbf{G}_0$ defined by right multiplication by $a(-t)$ on $\mathbf{G}_0$. Also the vector field $\tilde{\Phi}$ on $\tilde{\mathcal{E}}$ generating $\tilde{\Phi}_t$ covers the vector field $\tilde{\varphi}$ generating $\tilde{\varphi}_t$.
We note that the flow $\tilde{\Phi}_t$ commutes with the action of $\mathbf{G}$. Thus $\tilde{\Phi}_t$ is a flow on the flat $\mathbf{G}$-homogeneous affine bundle $\tilde{\mathcal{E}}$ covering φ_t. We also note that identifying $\tilde{\mathcal{V}}$ as the vector bundle underlying $\tilde{\mathcal{E}}$, the $\mathbb{R}$-action is just the linearization $D\tilde{\Phi}_t$ of the action $\tilde{\Phi}_t$.
The $\mathbf{G}$-action and the flow $D\tilde{\Phi}_t$ on $\tilde{\mathcal{V}}$ preserve a section $\tilde{\nu}$ of the bundle $\tilde{\mathcal{V}}$. This section is called the neutral section. We note that although $\tilde{\nu}$ is not parallel in every direction, it is parallel along the flow $\tilde{\varphi}_t$.
Let $\Gamma \subset \mathbf{G}$ be an affine deformation of a discrete subgroup $\Gamma_0 \subset \mathbf{G}_0$. We denote the quotient manifold $\mathbb{H}^2/\Gamma_0$ by Σ. Since Γ is a discrete subgroup of $\mathbf{G}$, the quotient

$$\mathcal{E} := \tilde{\mathcal{E}}/\Gamma$$

is an affine bundle over $U\Sigma = U\mathbb{H}^2/\Gamma_0$ and inherits a flat connection ∇ from the flat connection $\tilde{\nabla}$ on $\tilde{\mathcal{E}}$. Furthermore, the flow $\tilde{\Phi}_t$ on $\tilde{\mathcal{E}}$ descends to a flow Φ_t on $\mathcal{E}$ which is the horizontal lift of the flow φ_t on $U\Sigma$. The vector bundle $\mathcal{V}$ underlying $\mathcal{E}$ is the quotient

$$\mathcal{V} := \tilde{\mathcal{V}}/\Gamma = \tilde{\mathcal{V}}/\Gamma_0$$

and inherits a flat linear connection ∇ from the flat linear connection $\tilde{\nabla}$ on $\tilde{\mathcal{V}}$. The flow $D\tilde{\Phi}_t$ on $\tilde{\mathcal{V}}$ covering $\tilde{\varphi}_t$ and the neutral section $\tilde{\nu}$ both descend to a flow $D\Phi_t$ and a section ν respectively. We denote $U_{rec}\Sigma \subset U\Sigma$ to be the union of all recurrent orbits of φ.
Let us define a geodesic current as a Borel probability measure on the unit tangent bundle $U\Sigma$ of Σ invariant under the geodesic flow φ_t. We denote the set of all geodesic currents on Σ by $\mathcal{C}(\Sigma)$ and the subset of $\mathcal{C}(\Sigma)$ consisting of measures supported on periodic orbits by $\mathcal{C}_{per}(\Sigma)$. We note that $\mathcal{C}(\Sigma)$ has the structure of a topological space with the weak $\star$-topology and has a notion of convexity.

6.2 Labourie's diffusion of the Margulis Invariant

We recall that the Margulis invariant $\alpha = \alpha_u$ is an $\mathbb{R}$ valued class function on Γ_0 whose value on $\gamma \in \Gamma_0$ equals

$$\mathbb{B}(\rho_u(\gamma)O - O, x^0(\gamma))$$

where O is the origin and $x^0(\gamma) \in \mathbb{V}$ is the neutral vector of γ. Now the origin O will be replaced by a section s of $\mathcal{E}$, the neutral vector will be replaced by the neutral section ν of $\mathcal{V}$, and the linear action of Γ_0 on $\mathbb{V}$ will be replaced by the geodesic flow on $U\Sigma$.
Let s be a C^∞ section of $\mathcal{E}$. Its covariant derivative with respect to φ is a smooth section $\nabla_\varphi(s)$. And let ν be the null section. We define a function,

$$F_{u,s} : U\Sigma \longrightarrow \mathbb{R} \text{ where}$$
$$F_{u,s} := \mathbb{B}(\nabla_\varphi(s), \nu).$$

Let $S(\mathcal{E})$ denote the space of continuous sections s of $\mathcal{E}$ over $U_{rec}\Sigma$ which are differentiable along φ and the covariant derivative $\nabla_\varphi(s)$ is continuous. If $s \in S(\mathcal{E})$, then $F_{u,s}$ is continuous.

Now let us define,

$$\Psi_{[u],s}(\mu) := \int_{U\Sigma} F_{u,s} d\mu.$$

We note that $\Psi_{[u],s}(\mu)$ is independent of the section s which was used to define it. So we get a well defined function,

$$\Psi_{[u]} : \mathcal{C}(\Sigma) \longrightarrow \mathbb{R} \text{ given by}$$
$$\Psi_{[u]}(\mu) = \Psi_{[u],s}(\mu)$$

where s is a section.
We also note that $\Psi_{[u]}$ is continuous in the weak $\star$-topology on $\mathcal{C}(\Sigma)$. The following theorem establishes the link between this invariant and the Margulis invariant. It shows that $\Psi_{[u]}$ is indeed an extension of the Margulis invariant.

Theorem 6.2.1. *Let $\gamma \in \Gamma_0$ be hyperbolic and let $\mu_\gamma \in \mathcal{C}_{per}(\Sigma)$ be the corresponding geodesic current. Then*

$$\alpha(\gamma) = \ell(\gamma) \int_{U\Sigma} F_{u,s} d\mu$$

where $\ell(\gamma)$ is the length of the closed geodesic corresponding to the element $\gamma \in \Gamma_0$.

6.3 An equivalent condition for Properness

In this section we give an equivalent condition for properness using the extended Margulis invariant and we conclude this section by giving a conceptual proof of the opposite sign lemma.

Theorem 6.3.1. *Let Γ_u denote an affine deformation of Γ_0. Then Γ_u acts properly if and only if $\Psi_{[u]}(\mu) \neq 0$ for all $\mu \in \mathcal{C}(\Sigma)$.*

Now using this theorem we give a proof of the Opposite sign lemma.

Corollary 6.3.2 (Opposite Sign Lemma)**.** *Let $\gamma_1, \gamma_2 \in \Gamma_0$ be such that $\alpha(\gamma_1)$ and $\alpha(\gamma_2)$ has opposite sign. Then Γ does not act properly.*

Proof. As $\alpha(\gamma_1)$ and $\alpha(\gamma_2)$ has opposite sign, we can without loss of generality assume that $\alpha(\gamma_1) < 0 < \alpha(\gamma_2)$. Using theorem 6.2.1 we get

$$\Psi_{[u]}(\mu_{\gamma_1}) < 0 < \Psi_{[u]}(\mu_{\gamma_2}).$$

Now convexity of $\mathcal{C}(\Sigma)$ implies that there exists a continuous path $\mu_t \in \mathcal{C}(\Sigma)$ with $t \in [1,2]$, for which $\mu_1 = \mu_{\gamma_1}$ and $\mu_2 = \mu_{\gamma_2}$. We know that the function

$$\Psi_{[u]} : \mathcal{C}(\Sigma) \longrightarrow \mathbb{R}$$

is continuous. Therefore the intermediate value theorem implies that $\Psi_{[u]}(\mu_t) = 0$ for some $t \in (1,2)$. Now theorem 6.3.1 implies that Γ does not act properly.

$\square$

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