Università degli Studi di Padova
Dipartimento di Matematica Pure ed Applicata
Facoltà di Scienze Matematiche, Fisiche e Naturali
Laurea Specialistica in Matematica
Relatore: Professore Luca Barbieri-Viale

Homotopy theory of schemes and A^1-fundamental groups

Utsav Choudhury

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Contents

Chapter 1

Introduction

The aim of this thesis is to study homotopy theory of schemes and to give a good basis to study the A^1-fundamental groups. We will define a good homotopy theory on the category Sm/S (smooth S-schemes), where S is noetherian.

1.1 Nisnevich topology and stable homotopy category

To define this theory we need a good Grothendieck topology on Sm/S. Zariski topology turns out to be too coarse. For example, $X \to Y$, S-closed immersion, is not locally isomorphic to a closed immersion of the form $(A^n \times \{0\}) \bigcap U \to A^{n+m} \bigcap U$, where U is an open subscheme of A^{n+m}. This kind of property exists if we take U to be étale over A^{n+m}. We also want the cohomological dimension to be equal to the Krull dimension, so we can not use étale topology. For example, the étale cohomological dimension of the small étale site over field is not zero in general. So we use Nisnevich topology. In Chapter 2.3 we define Nisnevich topology and show that the site $(Sm/S)_{Nis}$ has enough points. We will give a charecterisation of Nisnevich sheaves on Sm/S using elementary distinguish square. We also show that the Nisnevich cohomological dimension is equal to the Krull dimension.

We will follow the usual way of defining closed model category structure to obtain suitable homotopy theories. For that in Chapter 2.1 we breifly recall the definition of closed model categories and some homotopy theory of closed model categories. Then we briefly recall the properties of simplicial sets.

For a closed model category structure we need to know the notion of weak equivalence, which gives us the idea of all the morphism we invert in our homotopy category. First on the category of simplicial sheaves over $(Sm/S)_{Nis}$, we will define a morphism to be a weak equivalence if it induces weak equivalence for every point. This will give us the the unstable homotopy category of S in 2.4.

In Chapter 2.5, we will invert the morphism $A^1_S \to S$, i.e, $A^1_S \to S$ will become a weak equivalence. Following the construction of Morel and Voevodsky in [MV], we will define the A^1 homotopy category. Also we will show that there exists an A^1-localisation functor L_{A^1} which we will use in later sections.

1.2 Properties of A^1-fundamental groups

Let k be a field (mostly infinite), $\Delta^{op}Shv(Sm_k)$ will denote the category of simplicial sheaves over the site $(Sm_k)_{Nis}$. By $\tilde{Sm_k}$, we will denote the category where the objects are smooth separated finite type k-schemes and the morphisms are smooth morphisms between smooth schemes. Note that a sheaf of sets over $(Sm_k)_{Nis}$ is always a sheaf of sets over $(\tilde{Sm_k})_{Nis}$.

Given a simplcial sheaf of sets on Sm_k, $\mathbb{B}$, we will denote by $\pi_0^{A^1}(\mathbb{B})$ the associated sheaf in the Nisnevich topology to the presheaf $U \mapsto Hom_{H(k)}(U, \mathbb{B})$, where $H(k)$ denotes the A^1-homotopy category of smooth k-schemes ([MV]). Moreover if $\mathbb{B}$ is pointed, given an integer $n \geq 1$, we denote by $\pi_n^{A^1}(\mathbb{B})$ the associated sheaf of groups in the Nisnevich topology to the presheaf of groups $U \mapsto Hom_{H_\bullet(k)}(\sum_s^n(U_+), \mathbb{B})$, where $H_\bullet(k)$ denotes the pointed A^1-homotopy category over $(Sm_k)_{Nis}$ and $\sum_s$ the simplicial suspension (for pointed homotopy category see [MV, section 3], and for simplicial suspensions see [MV, page 83]).

Let A^1 be the affine line over k.

Definition 1.2.1.

1. A sheaf of sets over Sm_k is said to be A^1 invariant if for any $X \in Sm_k$, the map $S(X) \to S(A^1 \times X)$ induced by the projection $A^1 \times X \to X$ is a bijection.
2. A sheaf of groups G on Sm_k is said to be strongly A^1 invariant if for any $X \in Sm_k$, the map $H^i_{Nis}(X; G) \to H^i_{Nis}(X \times A^1; G)$ induced by the projection is a bijection for $i \in \{0, 1\}$.
3. A sheaf M of abelian groups on Sm_k is said to be strictly A^1 invariant if for any $X \in Sm_k$, the map $H^i_{Nis}(X; M) \to H^i_{Nis}(X \times A^1; M)$ induced by the projection is a bijection for any $i \in \mathbb{N}$.

The definition is influenced from the fact that if A is an abelian group (or a group). Consider the constant sheaf A_X over topological spaces, for a connected topological space X, $A(X) = A$. It has the property that for all X, we have isomorphisms on cohomology groups $H^i(X, A) = H^i(X \times [0, 1], A)$ for all $i \in \mathbb{N}$ if A is abelian. If A is non-abelian then $H^i(X, A) = H^i(X \times [0, 1], A)$ for $i \in \{0, 1\}$.

On the other hand if $O = C^0(-; \mathbb{R})$, that is the sheaf of $\mathbb{R}$-valued continuous function on a topological space X. The cohomology groups $H^0(X, O)$ and $H^0(X \times [0, 1], O)$ are not same in general.

In both cases, we have a topological group. In the first case, A_X is discrete, in the other, O is not.

Our main objective in Chapters 3, 4, 5 is to prove that $\pi_1^{A^1}(\mathbb{B})$ is strongly A^1-invariant. and for $n \geq 2$ the sheaf $\pi_n^{A^1}(\mathbb{B})$ is strongly A^1-invariant. So the homotopy sheaves will be A^1-discrete. It is not known whether the sheaf of sets $\pi_0^{A^1}(\mathbb{B})$ is A^1-invariant or not.

It can be shown that for $n \geq 2$ the sheaf $\pi_n^{A^1}(\mathbb{B})$ is strictly A^1-invariant.

It is belived that A^1 fundamental group sheaf will play a fundamental role in the understanding of A^1-connected projective smooth varieties as the usual fundamental group plays a fundamental role in the classification of compact connected differentiable manifolds.

In Chapter 6, we will define A^1-coverings and give the relation of it to the A^1-fundamental sheaves of groups. We will give a sketch of the following theorem ([MO1, page 119, theorem 4.8]

Theorem 1.2.2. *Any pointed A^1-connected space $\mathfrak{X}$ admits a universal pointed A^1-covering $\tilde{\mathfrak{X}} \to \mathfrak{X}$ in the category of pointed covering of $\mathfrak{X}$. It is (up to unique isomorphism) the unique pointed A^1-covering whose source is A^1-simply connected. It is a $\pi_1^{A^1}(\mathfrak{X})$-torsor over $\mathfrak{X}$ and the cannonical morphism $\pi_1^{A^1}(\mathfrak{X}) \to Aut_{\mathfrak{X}}(\tilde{\mathfrak{X}})$ is an isomorphism.*

1.3 Conclusion

The work can be roughly divided into three parts. One is developing the theory, so that we have all the notions properly understood, which is done in Chapter 2. The second part is to understand some of the algebraic geometry tools used to understand the A^1-homotopy groups. This is done in Chapters 3, 4, 5. The beautiful arguement of F.Morel to prove that A^1-homotopy groups are strongly A^1-invariant is completely given. The third and the last part is to understand the tricks coming from the Homotopy theory. It is done in the last chapter. Given these three different ways of understanding, it will give us enough tools to understand the more recent works in this area.

This work can help to understand different parts related to the A^1-homotopy theory. Firstly understanding the A^1 homotopy category will give us the option to study the results of [MO1]. It can be shown that ([MO1, page 104, theorem 3.38])

Theorem 1.3.1. *Let $\mathfrak{X}$ be a pointed simplicial sheaf and $n \geq 0$ an integer. If $\mathfrak{X}$ is simplicially n-connected then it is A^1-n-connected, i.e $\pi_1^{A^1}(\mathfrak{X})$ is trivial for $i \leq n$.*

Also to find analogous tools from algebraic topology in A^1-algebraic topology can be done. We can study Hopf maps ([MO1]), obstruction theory ([MOREL]) also A^1-homotopy classification of vector bundles over smooth affine schemes.

There is a notion of A^1-homology theory $H_n^{A^1}$, described in [MO1, section 3]. One can show that $H_n^{A^1}(\mathfrak{X})$ of a simplcial set vanishes for $n < 0$ and are strictly A^1 invariant sheaves for $n \geq 0$([MO1, page 102, corollary 3.31]. This is true for simplcial sheaves on Sm/k, where k is field, and not true in general for arbitrary base . It is shown in [JY] that over a base of dimension ≥ 2, this is not true. For dimension one base, it is still an open problem. Also we can expect the follwoing conjecture to be true :

Conjecture 1. Let X be a smooth quasi projective variety of dimension d. Then $H_n^{A^1}(X) = 0$ for $n > 2d$ and if X is affine then $H_n^{A^1}(X) = 0$ for $n > d$.

Understanding and calculating A^1-fundamental group sheaf for smooth projective algebraic groups can be inresting. Calculation of higher A^1-homotopy groups are not yet done for non trivial cases and it turns out to be very difficult without using Milnor or Bloch-Kato conjecture.

The other aspect of this study can be understanding Milnor-Witt K theory and unramified Milnor-Witt K theory. The results of Chapters 3, 4, and 5 will be useful, specially the tricks of unramified sheaf of sets. This is described in of [MO1, section 2.2].

We can also describe analogue of Brower degree in our setting of Homotopy theory using the following result from [MO1]

Theorem 1.3.2. *For $n \geq 2$, the canonical morphism $[A^{n+1}\setminus\{0\}, A^{n+1}\setminus\{0\}]_{H_\bullet(k)} \to K_0^{MW}(k) = GW(k)$ is an isomorphism.*

Thus in A^1-homotopy theory we have all the relevant tools compared to the algebraic topology: degree, homology, fundamental group, cobordism groups, classification of vector bundles etc. This tools are used to construct surgery theory in algebraic topology, so it will be natural to ask for surgery theory in A^1-homotopy setting. But till now there is no obvious analogues for surgery. Since the A^1 fundamental group of a pointed projective smooth scheme is almost never trivial we can not have h-cobordism theorem. A major step will be to find the analogue of the "s-cobordism" theorem, the generalization of the h-cobordism theorem in the presence of A^1-fundamental group.

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Chapter 2

Basic Notions

In this chapter our main aim is to understand the A^1-homotopy category. First we will discuss the results from closed model category and simplicial sets which we want to extend for unstable homotopy category, then we will construct the A^1-homotopy category. The main references being [GJ] for the simplcial sets and closed model ategory part, for the Nisnevich topology part [JR] and for the homotopy theory part [MV].

2.1 Closed Model Category

Let C be a category having three classes of morphisms Fib, $Cofib$, W which are called fibration, cofibration and weak equivalence respectively.

Definition 2.1.1. A trivial fibration is a map which is a fibration and weak equivalence both. A trivial cofibration is a map which is cofibration and a weak equivalence both.

Definition 2.1.2. A morphism $f : X \to Y$ is a retract of a morphism $g : X' \to Y'$ if there exist a commutative diagram of the following form.

$$\begin{array}{ccccc} X & \xrightarrow{p} & X' & \xrightarrow{q} & X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{s} & Y' & \xrightarrow{t} & Y \end{array}$$

such that $q \circ p = id$ and $t \circ s = id$.

So in the case of the homotopy category of topological spaces we have retraction in the usual sense and in this case the map g is inclusion and f is the retraction of g .

Definition 2.1.3. Suppose we have a commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle p} \\ B & \longrightarrow & Y \end{array}$$

A lifting is a morphism $\varphi : B \to X$ such that the two triangles commute.

1. Let $i : A \to B$ a morphism and $\mathfrak{C}$ a class of morphism in C. We say that i has the left lifting proerty with respect to $\mathfrak{C}$ if for all commutative diagram of the above form, such that the left vertical morphism is i and the right vertical moprhism is in $\mathfrak{C}$, there exists a lifting.

2. Let $p : X \to Y$ a morphism and $\mathfrak{C}$ a class of morphism in C. We say that p has the right lifting property by $\mathfrak{C}$ if for all commutative diagram of the above form such that the right vertical map is p and left vertical map is in $\mathfrak{C}$, there exists a lifting.

Definition 2.1.4. A category C, with three classes of morphisms Fib, $Cofib$ and W is called a closed model category if it satisfies the following axioms :

1. C has all limits and colimits.

2. If f and g are two composable morphisms and two of f, g or $g \circ f$ are weak equivalences, then so is the third.

3. If the morpshism f is a retract of g and g is a weak equivalence, cofibration or fibration then so is f.

4. Any fibration has the right lifting property with respesct to trivial cofibrations and any cofibration has left lifting property with respect to trivial fibration.

5. Any morphism f can be functorially factorised as a composition $p \circ i$ where p is a fibration and i is a trivial cofibration. It can also be factoried as $q \circ j$ where q is a trivial fibration and j a cofibration.

In particular for a closed model category C we have ϕ and $\bullet$ as the initial and the final object respectively.

Definition 2.1.5. Let X be an object of C. X is called cofibrant if the morphism $\phi \to X$ is a cofibration and X is called fibrant of the morphism $X \to \bullet$ is a fibration.

Definition 2.1.6. Let D be a category and W a class of morphism in D. Suppose there exists a category $D[W^{-1}]$ and a functor $Q : D \to D[W^{-1}]$, such that for all $w \in W$, $Q(w)$ is an isomorphism and for all category Γ with a functor $R : D \to \Gamma$ such that for all $w \in W$, $R(w)$ is an isomorphism, then there exists a unique functor $\Phi : D[W^{-1}] \to \Gamma$ with $\Phi \circ Q = R$. Then the category $D[W^{-1}]$ is called the strict localised category of D by W.

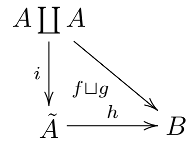
Definition 2.1.7.

1. Let A be an object of C. A cylinder object for A is given by a commutative triangle of the follwoing form

$$\begin{array}{ccc} A \coprod A & & \\ {\scriptstyle i=i_0 \sqcup i_1}\downarrow & \searrow^{\bigtriangledown} & \\ \tilde{A} & \xrightarrow{\sigma} & A \end{array}$$

where i is cofibration, σ is a weak equivalence and $\bigtriangledown$ is the codiagonal.

2. Let f and g be two morphism $A \to B$ in C. A left homotopy of f and g relative to a cylinder object $(\tilde{A}, i, \sigma)$ of A is a morphism $h : \tilde{A} \to B$ such that the following diagram commutes



3. Let f and g two morphisms $A \to B$ in C. We say that f and g are left homotopic if there exists a cylinder object $(\tilde{A}, i, \sigma)$ and a left homotopy of f and g relative to $(\tilde{A}, i, \sigma)$.

It can be shown by [GJ, lemma 1.5, page 69] that for A and B objects of C, the left homotopy relation on $Hom_C(A, B)$ is an equivalence relation if A is cofibrant. We can dually define path objects, right homotopy with respect to a path object and the right homotopy relation.

Lemma 2.1.8. *Let A be cofibrant and B be fibrant. If f and g are two morphisms $A \to B$ in C. Then the following are equivalent :*

1. *f and g are left homotopic;*

2. *f and g are right homotopic*

We denote by $\pi(A, B)$, the quotient of $Hom_C(A, B)$ by the homotopy relation when A is cofibrant and B is fibrant.

Proof. [GJ, corollary 1.9, page 72]. □

So we can define an unique category πC_{cf}, whose objects are cofibrant and fibrant objects of C and the set of morphisms between X and Y in πC_{cf} is $\pi(X, Y)$. Compositions are induced by the composition of the morphisms in C.

Lemma 2.1.9. 1. *The strict localised category of C by W (weak equivalence) exists. It is denoted by $Ho(C)$ and the functor is $\gamma : C \to Ho(C)$.*

2. *If X is cofibrant and Y is fibrant, then the functor γ induces a bijection*

$\pi(X, Y) = Hom_{Ho(C)}(\gamma X, \gamma Y)$.

Proof. [GJ, theorem 1.11, page 75]. □

2.1.1 Derived Functors

Definition 2.1.10. Let C be a category, W be a class of morphisms. Let $F : C \to A$ be a functor to any category A. Let $C[W^{-1}]$ exists and $\gamma : C \to C[W^{-1}]$ be the functor of localisation. A total right derived functor of F is a functor $RF : C[W^{-1}] \to A$ with a natural transformation $\varepsilon : F \rightsquigarrow RF \circ \gamma$, which satisfies the following universal property : For all functor $G : C[W^{-1}] \to A$, with a natural transformation $\varepsilon_G : F \rightsquigarrow G \circ \gamma$, there exists a unique natural transformation $\theta : RF \rightsquigarrow G$ such that $\varepsilon_G = (\theta \star \gamma) \circ \varepsilon$.

Let $(C, Fib, Cofib, W)$ be a closed model category, A be any other category and $F : C \to A$ be a functor. Let F transforms weak equivalences of fibrant objects into isomorphisms (or trivial fibrations of fibrant objects into isomorphisms). Then we have the following proposition

Proposition 2.1.11. *F has a total right derived functor $RF : Ho(C) \to A$. Moreover if X is a fibrant object in C, then the morphism $F(X) \to RF(X)$, induced by ε is an isomorphism.*

.

Proof. [JR, page 28, Prop 2.43].

□

We can dually define total left derived functor LF of a functor F.

Definition 2.1.12. Let C and D be two closed model categories, $F : C \to D$ and $G : D \to C$ be adjoint functors. (G, F) is called adjunction of Quillen if G preserves cofibration and F preserves fibrations. (G, F) is called equivalence of Quillen if moreover for all cofibrant object X in D and all fibrant object Y in C, if $f : X \to FY$ and $g : GX \to Y$ are the morphisms corresponding to the adjunction (G, F), then f is a weak equivalence in D if and only if g is a weak equivalence in C.

(G, F) is an adjunction of Quillen, then G preserves weak equivalences of cofibrant objects and F preserves weak equivalences of fibrant objects.

$LG : Ho(D) \to Ho(C)$ denotes the total left derived functor of the functor $\gamma \circ G : D \to Ho(C)$ and $RF : Ho(C) \to Ho(D)$ denotes the total right derived functor of $\gamma \circ F : C \to Ho(D)$.

Theorem 2.1.13. *If (G, F) is adjunction of Quillen, then the functors (LG, RF) are adjoint. Moreover, LG and RF gives equivalence of categories if and only if (G, F) is an equivalence of Quillen.*

Proof. [JR, page 29, thm 2.47]. □

2.2 Simplicial Sets

The category of simplicial sets are the basic building blocks of homotopy theory. This will be the first example of closed model category (on simplicial sets) and we will use this category to define the homotopy theory of schemes.

Definition 2.2.1. Let $\triangle$ denote the category whose objects are denoted by $[n]$ for every non-negative integer n and morphisms $[n] \to [m]$ is non-decreasing functions $\{0, ..., n\} \to \{0, ..., m\}$.

Definition 2.2.2. For C any category, $\triangle^{op}C$ is the category of covariant functors $F : \triangle^{op} \to C$. For category of sets we will use $\triangle^{op}Sets$, call it the category of simplicial sets.

For every positive integer n and $0 \leq i \leq n$ we have the unique injective non decreasing morphism $d^i : [n-1] \to [n]$ whose image does not contain i. For every non negative integer n and $0 \leq i \leq n$, we have the unique surjective non decreasing morphism $s^i : [n+1] \to [n]$ which collapses i and $i+1$ to i. The d^i's are called coface and s^i's are called codegeneracies. In case of simplicial set $F : \triangle^{op} \to Sets$, $d_i := F(d^i)$ and $s_i := F(s^i)$, are called face and degeneracy morphisms respectively.

For all non-negative integer n, let $\mid \triangle^n \mid := \left\{(x_0, x_1, ..., x_n) \in R^{n+1} \mid \sum_{i=0}^{n} x_i = 1\right\}$. So we have cosimplicial objects in Top, where Top is the category of topological spaces, defined by $\mid \triangle^\bullet \mid : \triangle \to Top$, which sends $[n]$ to $\mid \triangle^n \mid$.

Definition 2.2.3. Let X be a topological space. SX is the simplicial set which associates to every integer n, the set $Hom_{Top}(\mid \triangle^n \mid, X)$. We have canonical face and degeneracy maps coming from the coface and degeneray map of the cosimplcial object $\mid \triangle^\bullet \mid$. SX is called the singular simplicial set of X.

Proposition 2.2.4. *The functor $S : Top \to \triangle^{op}Sets$ admits a right adjoint $\mid - \mid$, called the topological realisation.*

Proof. [GJ, prop 2.2, page 7].

□

Definition 2.2.5. For every non-negative integer n, $\triangle^n$ is defined to be the simplcial set , such that $(\triangle^n)_m := Hom_\triangle([n],[m])$, the faces and degeneracies are those coming from the coface and codegeneracies of $\triangle$.

Definition 2.2.6. $\partial\triangle^n$ is the subsimplicial set of $\triangle^n$, generated by the non degenerated simplices of dimension $n-1$ of $\triangle^n$. For all positive integers n and $0 \leq k \leq n$, $\bigwedge_k^n$ (the k-th horn) is the subsimplicial set of $\triangle^n$ generated by the $n-1$-th simplices of $\triangle^n$ os the form $d_i(id_{[n]})$ for $i \neq k$.

Definition 2.2.7. Let $f : X \to Y \in Top$. Then f is called weak equivalence of topological spaces if $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ is bijective and for all $x \in X$ and $n \geq 1$, $\pi_n(f) : \pi_n(X,x) \to \pi_n(Y, f(x))$ is isomorphism.

Definition 2.2.8. In the category of $\triangle^{op}Sets$, cofibrations are defined to be the monomorphisms, weak equivalences is a morphism f such that $\mid f \mid$ is a weak equivalence of topological spaces, fibration are the morphism which has unique right lifting property with respect to all the inclusions $\bigwedge_k^n \to \triangle^n$, for n positive integers, and $0 \leq k \leq n$.

Theorem 2.2.9. *The category $\triangle^{op}Sets$ with the fibration, cofibration and weak equivalences defined in the previous definition, is a closed model category. Moreover, a morphism is trivial fibration if and only if it has right lifting property with respect to all inclusions $\partial\triangle^n \to \triangle^n$ for all n.*

Proof. [GJ, 11.2,11.3, page 61-62].

□

Let $X, Y \in \triangle^{op}Sets$, $hom(X,Y)$ is the simplicial set that associates to each $[n]$, $Hom_{\triangle^{op}Sets}(X \times \triangle^n, Y)$.

Lemma 2.2.10.

1. *For $X, Y, Z \in \triangle^{op}Sets$, we have a canonical isomorphism $hom(X, hom(Y,Z)) \cong hom(X \times Y, Z)$.*
2. *For $X, Y \in \triangle^{op}Sets$, we have a canonical bijection of sets $hom(X,Y)_0 = Hom_{\triangle^{op}Sets}(X,Y)$.*
3. *If $i : A \to B$ is a cofibration and $p : X \to Y$ a fibration in $\triangle^{op}Sets$, then the following morphism $hom(B,X) \to hom(A,X) \times_{hom(A,Y)} hom(B,Y)$ is a fibration and it is a trivial fibration if one of i or p is a weak equivalence.*

Proof. [GJ, proposition 5.1, 5.2, page 21, 22].

□

H^{top} will denote the homotopy category of $\triangle^{op}Sets$ with the given closed model category structure in this section.

2.2.1 Dold-Kan correspondance

Let $D^{\leq 0}(Ab)$ be the full subcatgeory of the derived category $D(Ab)$ formed by the complexes concentrated in degrees ≤ 0. Let $comp_+(Ab)$ is the category of complexes of abelian groups (differential of degree -1 or homological complexes) concentrated in degrees ≥ 0.

Definition 2.2.11. Let $A \in \triangle^{op}Ab$, the Moore complex associated to A is the object in $comp_+(Ab)$ whose degree n abelian group is A_n, and for all $n \geq 1$, $\partial = \sum_{i=0}^{n}(-1)^i d_i$. The normalized complex $NA \in comp_+(Ab)$, associated to A is the subcomplex of the Moore complex of A, such that $NA_n = \bigcap_{i=0}^{n-1} kerd_i \subset A_n$ and the differentials of Moore complex generates the differential of the normalized complex. If X is a simplicial set, $C_*(X) \in comp_+(Ab)$ is the Moore complex of the simplicial abelian group ZX, where $Z : \triangle^{op}Sets \to \triangle^{op}Ab$ is the right adjoint to the forgetful functor $\triangle^{op}Ab \to \triangle^{op}Sets$.

Theorem 2.2.12. *1. The functor $N : \triangle^{op}Ab \to comp_+(Ab)$ is an equivalence of categories.*

2. The inclusion $NA \to A$ in $comp_+(Ab)$ is a homotopy equivalence for all $A \in comp_+(Ab)$.

Proof. [GJ, corollary 2.3, page 149]. □

Lemma 2.2.13. *1. The functors $X \to C_*(X)$ transfers weak equivalences to quasi isomorphisms.*

2. All simplicial groups are simplicially fibrant.

3. For all simplicial abelian groups A and all $n \geq 1$, we have a canonical isomorphism $\pi_n(A, 0) = H_n(NA)$.

Proof. [GJ, lemma 3.4, page 12, corollary 2.7 page 153]. □

2.3 Nisnevich topology

2.3.1 Charecterisation of Nisnevich Sheaves

Let S be a noetherian scheme of finite dimension. Sch/S (resp Sm/S) the category of schemes (resp. smooth schemes) of finite type over S. Let $O_{X,x}$ (resp $O^h_{X,x}$) be the local ring (resp. the henselisation of the local ring) of X at x, where $x \in X$.

Proposition 2.3.1. *Let X be a scheme of finite type over S and $\mathfrak{U} = \{U_i \to X\}$ a finite family of étale morphisms in Sch/S. Then the following are equivalent*

1. *For any point x of X there is an i and a point u of U_i over x such that the corresponding morphism of residue fields is an isomorphism which maps to x with same residue field.*
2. *For any point x of X, the morphism of S schemes $\sqcup_i(U_i \times_X SpecO^h_{X,x}) \to SpecO^h_{X,x}$ admits a section.*

Proof. [MV, page 95, proposition 1.1] □

Remark 1. The collection of families of *étale* morphisms $\{U_i \to X\}$ in Sm/S satisfying the equivalent conditions of the proposition forms a pretopology on the category Sm/S. The corresponding topology is called the Nisnevich topology on Sm/S.The corresponding site will be denoted by $(Sm/S)_{Nis}$ and called the grand Nisnevich site of S . Similarly suppose X_{Nis} denotes the category of seprated, finite type, *étale* X schemes then the collection of families of *étale* morphisms $\{U_i \to Y\}$ in X_{Nis} satisfying the equivalent conditions of the proposition forms a pretopology on the category X_{Nis}. The corresponding topology is called the Nisnevich topology on X_{Nis}.The corresponding site will be denoted by X_{Nis} and called the small Nisnevich site of X

Definition 2.3.2. Let X be a scheme , the family of $\{U_i \to X\}$ satisfying the properties of the proposition 2.3.1 of this section is called covering of X and is denoted by $Cov_{Nis}(X)$.

Example 1. When char $k \neq 2$, the two morphisms $j : U_0 = A^1 \setminus \{a\} \hookrightarrow A^1$ and $z \mapsto z^2 : U_1 = A^1 \setminus \{0\} \to A^1$ forms a Nisnevich covering of A^1 if and only if $a \in (k^*)^2$. They form an étale covering of A^1 for any nonzero $a \in k$.

Lemma 2.3.3. *Let $\{U_i \to X\}$ is a Nisnevich covering then there is a nonempty open $V \subset X$ and an index i such that $U_i \times_X V \to V$ has a section.*

Proof. Let X be reduced. Then we can take any generic point $x \in X$, such that by hypothesis $\exists u \in U_i$ for some$i$ over x, such that $\kappa(x) \cong \kappa(u)$. So $U_i \to X$ induces a birational morphism between a closed subscheme of U_i and X. Hence we have a nonempty open subset $V \subset X$ and an index i such that $U_i \times_X V \to V$ has a section. Since $\{U_i \to X\}$ is an étale cover and by [MILNE, page 30, theorem 3.23] , there is an equivalence of categories of $X_{ét}$ and $(X_{red})_{ét}$, we have the result. □

Theorem 2.3.4. *For noetherian scheme X , all family in $Cov_{Nis}(X)$ admits a finite subfamily in $Cov_{Nis}(X)$.*

Proof.

Definition 2.3.5. A splitting sequence of length $n \geq 0$ for a morphism of S-schemes $p : U \to X$ is a decreasing sequence of closed subschemes of X ($\phi = Z_{n+1} \subset Z_n \subset ... \subset Z_0 = X$) such that for all $0 \leq i \leq n$, the morphism of S-schems $U \times_X (Z_i - Z_{i+1}) \to Z_i - Z_{i+1}$ induced by p after base change admits a S-section.

Lemma 2.3.6. *Let X noetherian scheme and $\mathfrak{U} = \{f_i : U_i \to X\}_{i \in I} \in Cov_{Nis}(X)$. Let $W = \coprod_{i \in I} U_i$, and p the canonical morphism $W \to X$ induced by the morphisms f_i. Then p has a split sequence.*

Proof. Let $Z_0 = X$. Now suppose we have constructed Z_i such that $Z_i \neq \phi$ then we have the étale morphism $W \times_X Z_i \to Z_i$, hence there exists and open dence set of U_{i+1} of Z_i such that the morphism $W \times_X U_{i+1} \to U_{i+1}$ splits . Take $Z_{i+1} = (Z_i - U_{i+1})_{red}$. Since X is noetherian any decreasing sequence of closed sets stabilizes after finite steps hence we get a split sequence for p. □

Let $\phi = Z_{n+1} \subset Z_n \subset ... \subset Z_0 = X$ be the splitting sequence for the morphism $p : W \to X$. We have the section s_i of the morphism $U \times_X (Z_i - Z_{i+1}) \to Z_i - Z_{i+1}$ for all $0 \leq i \leq n$, moreover $Z_i - Z_{i+1}$ is noetherian and so the image of all these $Z_i - Z_{i+1}$ is noetherian , hence it is inside an open set of W of the form $\coprod_{i \in J} U_i$, where J is finite, and hence the subfamily $(U_j \to^{f_j} X)_{j \in J} \in Cov_{Nis}(X)$. □

Definition 2.3.7. A presheaf of sets on a site T is a contravariant functor $T \to$ Sets. The category of sets can be replaced by any category desired.

Definition 2.3.8. Suppose $F : T \to$ Sets is a presheaf of sets on a site T. If for every object U, every covering $\mathcal{U} = \{U_i \to U\}$ two sections $s, t \in F(U)$ agree on every restriction $s|_{U_i} = t|_{U_i}$ if and only if $s = t$ in $F(U)$ then the presheaf is called separated.

Definition 2.3.9. A sheaf is a presheaf on a site T that satisfies the following condition: Given an object U, a covering $\mathcal{U} = \{p_i : U_i \to U\}$ of U and a set of elements $s_i \in U_i$ such that for each i, j we have $p_i^* s_i = p_j^* s_j \in F(U_i \times_U U_j)$ there is a unique element $s \in U$ such that $s|_{U_i} = s_i$ for each i.

Definition 2.3.10. Let P be a presheaf. For each U define two elements of $P(U)$ to be equivalent if there is a covering $\{U_i \xrightarrow{p_i} U\}$ such that $p_i^*(a) = p_i^*(b)$ for each i.

For each U, Let $P'(U) := (P(U)/ \backsim)$ where $\backsim$ is the equivalence relation defined in the previous definition.

Lemma 2.3.11. *P' is a well-defined presheaf.*

Proof. If $V \xrightarrow{f} U$ is an arrow in the site, a, b two different representatives from an equivalence class of $P'(U)$ and $\{U_i \xrightarrow{p_i} U\}$ is a covering on which a and b agree. The set $\{U_i \times_U V \xrightarrow{\pi} V\}$ is a covering of V and by commutivity of

$$\begin{array}{ccc} U_i \times_U V & \longrightarrow & V \\ \downarrow & & \downarrow \\ U_i & \longrightarrow & U \end{array}$$

$f^*(a)$ and $f^*(b)$ in $P(V)$ agree on each restriction to $P(U_i \times_U V)$. Hence, the map $f' : P'(U) \to P'(V)$ induced by f^* is well-defined. □

Definition 2.3.12. Given a presheaf P, for each U in the site and each covering of $\mathcal{U}$ denote by $H^0(\mathcal{U}, P)$ the equalizer of the maps

$$\Pi P(U_i) \rightrightarrows \Pi P(U_i \times_U U_j)$$

which are induced by projections.

Definition 2.3.13. For each refinement $\mathcal{U}'$ of $\mathcal{U}$ there is a well-defined map $H^0(\mathcal{U}, P) \to H^0(\mathcal{U}', P)$. For U an object in the site and P a presheaf, set

$$aP(U) = \varinjlim_{\mathcal{U}} H^0(\mathcal{U}, P')$$

Lemma 2.3.14. *Let functor $a : PSh(T) \to Sh(T)$ associating a sheaf to a presheaf is left adjoint to the inclusion $Sh(T) \to PSh(T)$. That is, for a presheaf F and a sheaf G there is a natural bijection $Hom_{Sh(T)}(aF, G) \to Hom_{PSh(T)}(F, G)$. In particular the inductive limit esists in $Sh(T)$ and the functor a commutes with it. Projective limits exist in the category $Sh(T)$ and the the inclusion functor i commutes with it. The functor a commutes with every finite projective limit.*

Proof. [SGA4, 2 , 3.4] .

□

Definition 2.3.15. T be a small site. A simplicial presheaf is a contravariant functor from $T \to \triangle^{op}Sets$. Equivalently a simplicial presheaf can be defined as the simplicial object in the category of presheaves on T .

Definition 2.3.16. A simplicial sheaf over a site T is a simplicial object in the category of sheaves over T .

S be a noetherian scheme .

Definition 2.3.17. An elementary distinguished square in $(Sm/S)_{Nis}$ is a cartesian square of the form

$$\begin{array}{ccc} U \times_X V & \longrightarrow & V \\ \downarrow & & \downarrow p \\ U & \xrightarrow{j} & X \end{array}$$

such that the p is *étale* and j is an open immersion and $p^{-1}(X - U) \to (X - U)$ is an isomorphism where $X - U$ and $p^{-1}(X - U)$ is considered with reduced structures.

Example 2. When $chark \neq 2$, the two morphisms $j : U_0 = A^1 \setminus \{a\} \hookrightarrow A^1$ and $z \mapsto z^2 : U_1 = A^1 \setminus \{a', 0\} \to A^1$ forms an elementary distinguished square of A^1, where a' is one of the roots of the equation $x^2 - a$, if and only if $a \in (k^*)^2$. More generally if $\dim X \leq 1$, any Nisnevich covering of X admits a refinement $\{U, V\}$ such that

$$\begin{array}{ccc} U \times_X V & \longrightarrow & V \\ \downarrow & & \downarrow p \\ U & \xrightarrow{j} & X \end{array}$$

is an elementary distinguished square. Let $i : E \to F$ separable extension, and v is a discrete valuation of F which restricts to a discrete valuation w on E with ramification index 1,and moreover if the induced map $\bar{i} : \kappa(w) \to \kappa(v)$ is an isomorphism, then

$$\begin{array}{ccc} SpecE & \longrightarrow & SpecO_w \\ \downarrow & & \downarrow p \\ SpecF & \xrightarrow{j} & SpecO_v \end{array}$$

is an elementary distinguished square.

The following theorem describes the Nisnevich sheaves with respect to elementary distinguished square.

Theorem 2.3.18. *A presheaf of sets F on Sm/S is a sheaf in the Nisnevich topology if and only if for any elementary distinguished square as in Definition 2.3.17 ,the square of sets*

$$\begin{array}{ccc} F(X) & \longrightarrow & F(U) \\ \downarrow & & \downarrow \\ F(V) & \longrightarrow & F(U \times_X V) \end{array}$$

is cartesian

Proof. For the if case, to prove that F is a sheaf of sets in the Nisnevich topology fix a Nisnevich covering $\{U_i \to X\}$ for any $X \in Sm/S$. An open scheme $V \subset X$ is said to be good with respect to the given covering if $F(V) \to \prod F(U_i \times_X V) \rightrightarrows \prod F(U_i \times_X U_j \times_X V)$ is an equilizer diagram. We have to show that X is itself good. Suppose $V \subset X$ is the maximal open subset which is good. Now if V is not equal to X then take the closed scheme $Z = X \setminus V$, by lemma 6.3.1 there exists an index i and a nonempty open set $W \subset Z$ such that $U_i \times_Z W \to W$ splits. Let $X' \subset X$ be the compliment of the closed set $Z \backslash W$. Then V and $U'_i = U_i \times_X X'$ forms an elementary distinguished square over X' . Pulling back along each $U'_j = U_j \times_X X'$ gives elementary distinguished squares. So we have cartesian squares of the following forms

$$\begin{array}{ccc} F(X') & \longrightarrow & F(U'_i) \\ \downarrow & & \downarrow \\ F(V) & \longrightarrow & F(U'_i \times_X V) \end{array}$$

$$\begin{array}{ccc} F(U'_j) & \longrightarrow & F(U'_i \times_X U'_j) \\ \downarrow & & \downarrow \\ F(U'_j \times_X V) & \longrightarrow & F(U'_i \times_X U'_j \times_X V) \end{array}$$

Now we know that V is good. To show X' is good too, we observe the follwoing diagram

$$\begin{array}{ccccc} F(X') & \longrightarrow & \prod F(U'_j) & \rightrightarrows & \prod F(U'_i \times_X U'_j) \\ \downarrow & & \downarrow & & \downarrow \\ F(V) & \longrightarrow & \prod F(U'_j \times_X V) & \rightrightarrows & \prod F(U'_i \times_X U'_j \times_X V) \end{array}$$

Suppose $\{b_j\} \in \prod F(U'_j)$ such that $b_j \mid F(U'_k \times_X U'_j) = b_k \mid F(U'_k \times_X U'_j)$ and $\{b_j\}$ gets maped to $\{b'_j\} \in \prod F(U'_j \times_X V)$, then $b'_j \mid F(U'_k \times_X U'_j \times_X V) = b'_k \mid F(U'_k \times_X U'_j \times_X V)$. So there exists $a \in F(V)$ which maps to $\{b'_j\}$. But by the first cartesian diagram we get a $b \in F(X')$ mappaing to a and whose restriction to $F(U'_i)$ is b_i. By the secodn cartesian diagram we get that b restricted to each U'_j is b_j . Hence X' is also good.

For the only if case let F be Nisnevich sheaf on Sm/S and we have the following elementary distinguished square

$$\begin{array}{ccc} B & \longrightarrow & Y \\ \downarrow & & \downarrow p \\ A & \xrightarrow{j} & X \end{array}$$

Where j is an open immersion p is étale and $p^{-1}(X - A) \to (X - A)$ is an isomorphism, $B = Y \times_X A$. We need to prove that $F(X) = F(Y) \times_{F(B)} F(A)$. Since F is a Nisnevich sheaf, by the separated property we have $F(X) \to F(Y) \times_{F(B)} F(A)$ injective. We also have the following equalizer diagram

$F(V) \to F(A) \times F(Y) \rightrightarrows F(A) \times F(B) \times F(Y \times_X Y)$. Since $(\triangle(Y), B \times_A B)$ is a Nisnevich covering of $Y \times_X Y$, we have $F(Y \times_X Y) \to F(Y) \times F(B \times_A B)$ is injective. Now, if $(y, a) \in F(Y) \times_{F(B)} F(A)$, then the two restriction maps from $F(Y) \times F(A)$ to $F(B)$ maps (y, a) to the same element in $F(B)$, also the two maps sends (y, a) to the same element in $F(A)$. It is enough to show that the two maps from $F(Y) \to F(Y \times_X Y)$ maps (y, a) to the same element, or more precisely the two maps $F(Y) \to F(B \times_A B)$ maps (y, a) to the same element, but then these two maps factors through $F(B)$, so by hypothesis our claim is proved.

□

Theorem 2.3.19. *Let X be any smooth scheme over S , then the presheaf represented by $Y \mapsto Hom_{Sm/S}(Y, X)$, for $Y \in Sm/S$, is a sheaf. ((Sm/S) endowed with Nisnevich topology).*

So the category Sm/S(resp. X_{Nis}) is a full subcategory of $Sh(Sm/S)$(resp $Sh(X_{Nis})$.

Lemma 2.3.20. *For any elementary distinguished as in the definition 2.3.17 , the cannonical morphism of Nisnevich sheaves $V/(U \times_X V) \to X/U$ is an isomorphism .*

Proof. By Yonedas lemma and previous two theorems we get that the diagram of representable sheaves coming from any elementary distinguished square is a cocartesian square in the category $Sh(Sm/S)_{Nis}$. Again by Yonedas lemma for the covering coming from elementary distinguished square $U \bigsqcup V \to X$ gives an epimorphism of sheaves. Also the maps of sheaves $U \to X$ for any open immersion gives a monomorphism of sheaves. So we have the isomorphism of sheaves. □

2.3.2 Functoriality of the small Nisnevich site

Let X and Y be Noetherian schemes and $f : X \to Y$ be a morphism of schemes. There exists a functor $- \times_Y X : Y_{Nis} \to X_{Nis}$. We can define a direct image functor $f_\sharp : PSh(X_{Nis}) \to PSh(Y_{Nis})$ by the formula $f_\sharp F(Z) = F(Z \times_Y X)$ for all $F \in PSh(X_{Nis})$ and $Z \in Y_{Nis}$. For all $F \in Sh(X_{Nis})$, we have $f_\sharp F \in Sh(Y_{Nis})$. So $f_\sharp$ induces a functor $f_* : Sh(X_{Nis}) \to Sh(Y_{Nis})$. The functors $f_\sharp$ and f_* admits left adjoints $f^\sharp : PSh(Y_{Nis}) \to PSh(X_{Nis})$ and $f^* : Sh(Y_{Nis}) \to Sh(X_{Nis})$ respectively and we have $a \circ f^\sharp \cong f^* \circ a$. Observe that f^* commutes with all finite projective limit by [SGA4] IV 4.9.2, since Y_{Nis} has all finite projective limits and $- \times_Y X$ is a continuous functor commuting with all finite projective limit. So we have the following proposition.

Proposition 2.3.21. *The functor $- \times_Y X : Y_{Nis} \to X_{Nis}$ induces a morphism of sites $X_{Nis} \to Y_{Nis}$. Moreover the couple (f^*, f_*) is a morphism of topos $Sh(X_{Nis}) \to Sh(Y_{Nis})$*

Proof. [JR] page 9, prop 1.17. □

Suppose $f : X \to Y$ is étale , by composing with f we get a functor $X_{Nis} \to Y_{Nis}$ which gives a functor $f_?^\sharp : PSh(Y_{Nis}) \to PSh(X_{Nis})$.

Let $F \in PSh(Y_{Nis})$ and $G \in PSh(X_{Nis})$, Let $\alpha : Hom(F, f_\sharp G) \to Hom(f_?^\sharp F, G)$ be the morphism which sends $\psi : F \to f_\sharp G$, to $\alpha(\psi) : f_?^\sharp F \to G$, such that for all $W \in X_{Nis}$, $\alpha(\psi)_W$ is the compostion $\psi_W : f_?^\sharp F(W) = F(W) \to f_\sharp G(W) = G(W \times_Y X)$ and the structtural morphism $G(W \times_Y X) \to G(W)$. By Yonedas lemma $Hom(f^\sharp F, G) = Hom(F, f_\sharp G) \to Hom(f_?^\sharp F, G)$ gives a morphism $f_?^\sharp \to f^\sharp$. It can be shown that it is an isomorphism and we have the following lemma.

Lemma 2.3.22. *There exists a canonical morphism of functors $f_?^\sharp \to f^\sharp$ which is an isomorphism. Moreover if $F \in Sh(Y_{Nis})$, then $f^\sharp F \in Sh(X_{Nis})$ and the restriction of $f^\sharp$ to $Sh(Y_{Nis})$ induces the inverse image functor $f^* : Sh(Y_{Nis}) \to Sh(X_{Nis})$.*

Proof. [JR] page 10, lemma 1.18. □

Now let $x \in X$ and $j_x : SpecO_{X,x} \to X$ be the canonical morphism of schemes. The open subschemes of X containing x is ordered (by the reverse of inclusion). So we have a projective system $(X_\lambda)_{\lambda \in L}$, where X_λ's are open subschemes of X containing x. So we can identify $SpecO_{X,x}$ with $\lim_{\longleftarrow \lambda \in L} X_\lambda$. Let $f_\mu : Y_\mu \to Z_\mu$ be morphisms of X_μ schemes of finite type and $Y_\lambda := Y_\mu \times_{X_\mu} X_\lambda$ if $\lambda \geq \mu$, $Y_x := Y_\mu \times_{X_\mu} SpecO_{X,x}$, Then $Y_x \to Z_x$ si one of isomorphism, monomorphism, immersion, étale, finite, separated, affine, proper, surjective, qasiprojective iff for all λ sufficiently big $Y_\lambda \to Z_\lambda$ satisfies the property.

Let $Z \in (SpecO_{X,x})_{Nis}$, we can choose $\lambda \in L$, $Z_\lambda \in (X_\lambda)_{Nis}$, and an isomorphism $Z \cong Z_\lambda \times_{X_\lambda} SpecO_{X,x}$.

For all $F \in PSh(X_{Nis})$ and $Z \in (Spec(O_{X,x}))_{Nis}$ let $j_{x,?}^\sharp F(Z) := \varinjlim_{\mu \geq \lambda} F(Z_\lambda \times_{X_\lambda} X_\mu$. This gives a morphism $j_{x,?} : PSh(X_{Nis}) \to PSh((SpecO_{X,x})_{Nis})$.

Lemma 2.3.23. *There exists a canonical isomorphism of the functor $j_{x,?} \to j_x^\sharp$. Moreover if $F \in Sh(Y_{Nis})$, then $j_x^\sharp F \in Sh(X_{Nis})$ and the restriction of $j_x^\sharp$ to $Sh(Y_{Nis})$ induces the inverse image functor $j_x^* : Sh(Y_{Nis}) \to Sh(X_{Nis})$.*

Proof. [JR], page 10, lemma 1.19 □

2.3.3 Points of Nisnevich Sites

Definition 2.3.24. A fiber functor (or points) on the site $(Sm/S)_{Nis}$ (or X_{Nis}) is a functor $\Phi : Sh((Sm/S)_{Nis}) \to Sets$ (or $Sh(X_{Nis}) \to Sets$) which commutes with all inductive limits and all finite projective limits.

Let k be a field. $F \in PSh((Speck)_{Nis})$. Then it can be shown that $F \in Sh((Speck)_{Nis})$ if and only if $F(\varnothing)$ is singleton and $F(X \coprod Y) \to F(X) \times F(Y)$ is bijective for all $X, Y \in (Speck)_{Nis}$. This property will give us that the functor $\Gamma : Sh((Speck)_{Nis}) \to Sets$, that associates to each $F \in Sh((Speck)_{Nis})$, the set $F(Speck)$, is a fiber functor.

Let X be noetherian scheme. Let y be k point of X, where k is a field. We can define a functor $-_y : Sh(X_{Nis}) \to Sets$, that associates to each $F \in Sh(X_{Nis})$, the set $\Gamma(y^* F)$, where $\star(y^*, y_*)$ is the morphism of topos $Sh((Speck)_{Nis}) \to Sh(X_{Nis})$. It can be shown easily that for any k point y of X, $-_y$ is a fiber functor over X_{Nis}.

Let $Nbd_{x,X}^{Nis}$ be the category whose objects are (V, v), where $V \in X_{Nis}$, and $v \in V$, over $x \in X$, such that the induced morphism $\kappa(x) \to \kappa(v)$ is an isomorphism. Morphisms in $Nbd_{x,X}^{Nis}$ between $(V, v) \to (V', v')$ are morphisms $V \to V'$ in X_{Nis} which sends v to v' over x. We have the follwoing result :

Proposition 2.3.25. *Let $x \in X$, and the canonical morphism $x : \kappa(x) \to X$. Then the functor $-_x : Sh(X_{Nis}) \to Sets$ can be identified with the functor $F \mapsto \varinjlim_{(V,v)\in Nbd_{x,X}^{Nis}} F(V)$.*

Proof. [JR], page 12, Prop 1.26. □

Definition 2.3.26. A family $(\Phi_i)_{i\in I}$ of fiber functors on a site T is called conservative if for all morphism f in $Sh(T)$, f is an isomorphism if and only if for $\forall i \in I$, $\Phi_i(f)$ is bijective. We say that T has enough points if T has a conservative family of fiber functors.

It is easy to verify that if $(\Phi_i)_{i\in I}$ is a conservative family of fiber functors on a site T and $f : F \to G \in Sh(T)$ is a morphism, then f is monomorphism (resp. epimorphism) if and only if $\forall i \in I$, Φ_i is injective (resp. surjective). By using the previous description of the fiber functor $-_y$, where y is a k point of X, inducing finite separable extension of fields, we have the following:

Theorem 2.3.27. *Let X be a noetherian scheme. The family of fiber functors $-_y$, where y is a k point of X inducing finite seprable field extension, is a conservative family of fiber functors on the site X_{Nis}. Hence X_{Nis} has enough points.*

Proof. [JR], page 13, Thm 1.30. □

Now let S be a noetherian schemeand $X \in (Sm/S)_{Nis}$. We have a canonical functor $\pi_X : X_{Nis} \to (Sm/S)_{Nis}$. So we can define a functor $(\pi_X)_\sharp : PSh((Sm/S)_{Nis}) \to PSh(X_{Nis})$, which associates to every $F \in PSh((Sm/S)_{Nis})$, a presheaf $(\pi_X)_\sharp(F)$ defined by , $(\pi_X)_\sharp(F)(Z) = F(\pi_X(Z))$, $\forall Z \in X_{Nis}$. By [SGA4] IV 4.9.2 and III 2.1, we have that $(\pi_X)_\sharp$ induces a functor $(\pi_X)_* : Sh((Sm/S)_{Nis}) \to Sh(X_{Nis})$ and $(\pi_X)_*$ has a left adjoint $(\pi_X)^*$ such that $((\pi_X)^*, (\pi_X)_*)$ defines a morphism of topos $Sh((Sm/S)_{Nis}) \to Sh(X_{Nis})$. Moreover by [SGA4] III 2.3.3 $(\pi_X)_*$ commutes with all inductive limits. Now, let $X \in (Sm/S)_{Nis}$, and y is a k-point, where k is a field. We can define a fiber functor using our previous descriptions $-_y : Sh((Sm/S)_{Nis}) \to Sets$, which associates to every $F \in Sh((Sm/S)_{Nis})$, the set $\Gamma(y^*(\pi_X)_*F)$. The family $-_x$ of functors, $\forall x \in X$ and $\forall X \in (Sm/S)_{Nis}$gives a conservative family of fiber functors on the site $(Sm/S)_{Nis}$. So $(Sm/S)_{Nis}$ has enough points.

2.3.4 Cohomology for Nisnevich Sheaves

Definition 2.3.28. Let $S = (C, T)$ be a site. Let $X \in C$. For all integer $n \geq 0$ we define $H^n_S(X; -)$ the n-th right derived functor of the functor $F \to F(X)$ where F is a sheaf of abelian groups over S.

Lemma 2.3.29. *Let S is a Noetherian scheme and $X \in Sm/S$. For all sheaf of abelian groups F over Sm/S_{Nis}, there exist a canonical isomorphism $H^n_{X_{Nis}}(X; (\pi_X)_*F) \cong H^n_{Sm/S_{Nis}}(X; F)$ for all integer $n \geq 0$.*

Proof. Since $(\pi_X)_*$ is an exact functor and it has a left adjoint $(\pi_X)^*$, hence $(\pi_X)_*$ sends injective sheaves to injective sheaves, so we have the isomorphism since $(\pi_X)_*$ is left exact.

□

Lemma 2.3.30. *Let F is a sheaf of abelian groups over X_{Nis}. Suppose for all open imersions $V \to U$ in X_{Nis}, the map $F(U) \to F(V)$ is surjective. Then , for all $n \geq 0$ and for all $Y \in X_{Nis}$ we have $H^n_{X_{Nis}}(Y; F) = 0$.*

Proof. F is flasque on X. To show that $H^1_{X_{Nis}}(Y; F) = 0$, it is enough to show that if $0 \to F \to G \to H \to 0$ be an exact sequence of sheaves on X_{Nis} of abelian groups, then for all $Y \in X_{Nis}$ we have $G(Y) \to H(Y)$ is surjective. We can assume that $Y = X$. Let $h \in H(X)$. There exist a maximal open set $U \subset X$ and $s \in G(U)$ such that image of $s = h \mid_U$ (By noetherian property). Suppose $U \neq X$. Let x be a generic point of $X - U$ and let $F = (X - U)_{red}$. Now by the exactness condition of the Nisnevich sheaves there exists $V \in X_{Nis}$ and $t \in G(V)$ such that t gets mapped to $h \mid_V$ and $V \times_X \kappa(x) \to \kappa(x)$ is an isomorphism. Since $F \times_{SpecO_{x,X}} SpecO_{X,x}$ is closed subscheme of $SpecO_{x,X}$ and it is reduced with the generic point x, we have $F \times_X SpecO_{X,x} \cong \kappa(x)$, so we have an isomorphism $V \times_X F \times_X SpecO_{X,x} \cong F \times_X SpecO_{X,x}$. So there exists an open set W such that $x \in W$ and we have an isomorphism $V \times_X F \times_X W \to F \times_X W$. Let $U' = U \bigcup W$. Then we have $(U, V \times_X U')$ an elementary Nisnevich covering of U'. And since $F(U') \to F(U)$ is surjective, we can get $s' \in G(U')$ such that s' gets mapped to $h \mid_{U'}$. So we get a contradiction. Now it can be shown that all injective sheaves are flasque and if $0 \to F \to G \to H \to 0$ exact, moreover F and G are flasque then H is flasque. By the previous arguement we have $H^1_{X_{Nis}}(Y; F) = 0$ for F flasque, hence by induction on $n > 0$ we get $H^n_{X_{Nis}}(Y; F) = 0$.

□

Proposition 2.3.31. *Let F be a sheaf of abelian groups over X_{Nis} and $(U \to X; V \to X)$ be an an elementary Nisnevich covering of X. Then there exist a long exact sequence of the following form $H^{n-1}_{Nis}(U \times_X V; F) \to H^n_{Nis}(X; F) \to H^n_{Nis}(U; F) \oplus H^n_{Nis}(V; F) \to H^n_{Nis}(U \times_X V; F) \to H^{n+1}_{Nis}(X; F)$.*

Proof. Let $\mathbb{Z}X$ be the sheaf of free abelian groups generated by the representable sheaf X. So $H^n_{Nis}(X; F) \cong Ext^n(\mathbb{Z}X, F)$.And we also have the folwoing cocartesian diagram coming from the elementary Nisnevich square property of Nisnevich sheaves.

$$\begin{array}{ccc} \mathbb{Z}(U \times_X V) & \longrightarrow & \mathbb{Z}(U) \\ \downarrow & & \downarrow \\ \mathbb{Z}(V) & \longrightarrow & \mathbb{Z}(X) \end{array}$$

So we have the proof.

□

Theorem 2.3.32. *Let X be a Noetherian scheme of finite Krull dimension. Then for all sheaves of abelian groups F over X_{Nis} and for all integer $n > dimX$, we have $H^n_{Nis}(X; F) = 0$.*

Proof. Proof by induction : Let the theorem be true for all Noetherian scheme of dimension $< n$. Let $\pi : X_{Nis} \to X_{Zar}$ be the canonical morphism and $R^q\pi_*$ be the q-th derived functor of the functor π_*. So by Leray spectral sequence we have $E_2^{pq} = H^p_{Zar}(X; R^q\pi_*F) \Longrightarrow H^{p+q}_{Nis}(X; F)$. Now if X is noetherian scheme of finite Krull dimension, n be any non negative integer and F is a sheaf of abelian groups on X_{Zar}. If for all $x \in X$, $dimO_{x,X} < dimX - n$, we have $F_x = 0$, then for all $q > n$ we have $H^q_{Zar}(X; F) = 0$. So for our case we have to show that $E_2^{pq} = 0$ for $p + q > n$, which is by the previous statement same as showing $(R^q\pi_*F)_x$ is zero for all $x \in X$ such that $dimO_{x,X} < q$. But then by 2.3.30 $(R^q\pi_*F)_x \cong H^q_{Nis}(SpecO_{x,X}, j^*F)$, where $j : Spec\,O_{x,X} \to X$ is the canonical morphism. So now let X be local noetherian scheme of Krull dimension n and closed point x. The sheaf associated to the presheaf over X_{Nis} which associates to every $U \in X_{Nis}$ the sheaf of abelian groups $H^{n+1}(U; F\mid_U)$ is zero sheaf. So for all element $a \in H^{n+1}_{Nis}(X; F)$, $\exists V \in X_{Nis}$ such that $V \times_X \kappa(x) \cong \kappa(x)$ and $a \mid_{H^{n+1}_{Nis}(V;F)}$ is zero. Let $U = X - x$. Using proposition 2.3.31 for the elementary Nisnevich covering (U, V) of X gives $H^{n+1}_{Nis}(X; F) \to H^{n+1}_{Nis}(X; F)$ is injective (since U and $U \times_X V$ has dimension $< n$ and by the induction hypothesis). So $a = 0$, Hence $H^{n+1}_{Nis}(X; F) = 0$. □

2.4 Homotopy Category of a site with interval

2.4.1 Simplicial structure on $\triangle^{op}Sh(Sm/S)_{Nis}$

[JR] and [MV]. Let $S = (C, T)$ be a site with enough points.

Definition 2.4.1. Let $\mathfrak{X} \in \triangle^{op}PSh(S)$ and $n \geq 1$. The n-th homotopy presheaf of sets $\prod_n \mathfrak{X}$ is defined to be the presheaf that associates for every $U \in C$,

$(\prod_n \mathfrak{X})(U) := \{(y, u), u \in X(U)_0, y \in \pi_n(\mathfrak{X}(U), u)\}$, with obvious restrictions as morphisms. There exists an obvious morphism of presheaves of sets $\prod_n \mathfrak{X} \to \mathfrak{X}_0$. The 0-th homotopy presheaf of sets of $\mathfrak{X}$ (denoted by $\prod_0 \mathfrak{X}$) is defined to be the presheaf which associates for every $U \in C$, $(\prod_0 \mathfrak{X})(U) = \pi_0(\mathfrak{X}(U))$. For $\mathfrak{X} \in \triangle^{op}Sh(S)$ and $n \geq 0$, we define $\prod^T_n \mathfrak{X} = a_T(\prod_n \mathfrak{X})$. We have a morphism(functorial) of sheaves $\prod^T_n \mathfrak{X} \to \mathfrak{X}_0$ for $n \geq 1$. For all $U \in C$ and $n \geq 1$, $\prod^T_n \mathfrak{X}(U)$ is a group over $\mathfrak{X}(U)_0$. In particular , over all section $u \in \mathfrak{X}(U)_0$, we have a group denoted by $\pi_{n,U}(\mathfrak{X}, u)$.

Definition 2.4.2. Let $f : \mathfrak{X} \to \mathfrak{Y}$ morphism in $\triangle^{op}Sh(S)$, then f is a weak equivalence if,

1. The morphism $\prod^T_0 f : \prod^T_0 \mathfrak{X} \to \prod^T_0 \mathfrak{Y}$ is isomorphism;
2. For all $n \geq 1$, the following commutative diagram is cartesian ;

$$\begin{array}{ccc} \prod^T_n \mathfrak{X} & \xrightarrow{\prod^T_n f} & \prod^T_n \mathfrak{Y} \\ \downarrow & & \downarrow \\ \mathfrak{X}_0 & \xrightarrow{f_0} & \mathfrak{Y}_0 \end{array}$$

If $\Phi : Sh(S) \to Sets$ is a fiber functor on S. For all $\mathfrak{X} \in \triangle^{op}Sh(S)$, there exist a bijection $\Phi(\prod^T_0 \mathfrak{X}) \to \pi_0(\Phi(\mathfrak{X}))$. For all $n \geq 1$, the map $\Phi(\prod^T_n \mathfrak{X}) \to \Phi(\mathfrak{X}_0)$ can be identified with the map $\coprod_{f_\Phi \in \Phi(\mathfrak{X}_0)} \pi_n(\Phi(\mathfrak{X}, f_\Phi) \to \Phi(\mathfrak{X}_0)$. Using this description we have the following result:

Proposition 2.4.3. *Let $(\Phi_i)_{i \in I}$ a conservative family of fiber functors on the site S, and $f : \mathfrak{X} \to \mathfrak{Y}$ a morphism in $\triangle^{op}Sh(S)$. Then the following conditions are equivalent :*

1. *f is a simplicial weak equivalence.*

2. *For all $i \in I$, $\Phi_i(f) : \Phi_i(\mathfrak{X}) \to \Phi_i(\mathfrak{Y})$ is a weak equivalence of simplicial sets.*

Now if we have $(f^*, f_*) : Sh(S) \to Sh(S')$, morphism of topos, S and S' has enough points, then for a fiber functor Φ of S, $\Phi \circ f^*$ is a fiber functor of S'. So if $g : \mathfrak{X} \to \mathfrak{Y}$ a morphism in $Sh(S')$, such that g is an simplicial weak equivalence, then $f^*(g)$ is a simplicial weak equivalence too.

Definition 2.4.4. Cof_s is the class of monomorphism in $\triangle^{op}Sh(S)$, W_s is the class of simplicial weak equivalences and Fib_s is the class of morphism having right lifting property with repect to $Cof_s \bigcap W_s$. The classes Cof_s and Fib_s are called simplicial cofibration and simplicial fibration respectively.

Theorem 2.4.5. *$(\triangle^{op}Sh(S), Cof_s, Fib_s, W_s)$ is a closed model category.*

Proof. [JR] page 36, theorem 3.7.

□

Remark 2. Since fibration in $\triangle^{op}Sh(S)$ induces fibration in each fiber and the closed model category structure on $\triangle^{op}Sets$ is proper, we have that the previous model category structure on $\triangle^{op}Sh(S)$ is proper.

If $(f^*, f_*) : Sh(S) \to Sh(S')$ is a morphism of topos, then since f^* is left exact, f^* preserves monomorphisms and also f^* preserves weak equivalences. Hence (f^*, f_*) is an adjunction of Quillen. Moreover we have a couple of adjoint functors $(f^*, Rf_*) : Ho_s(S) \to Ho_s(S')$.

2.4.2 Adjunction

Let $S = (C, T)$ is a site with sufficient points. Let $\mathsf{H}om(-,-)$ is the internal hom on $\triangle^{op}Sh(S)$ for the monoidal structure $- \times -$, that is the right adjoint bifunctor of $- \times -$. We will denote $hom(-,-)$ as the global section of $\mathsf{H}om(-,-)$.

Lemma 2.4.6. *Let $\mathfrak{A} \to^i \mathfrak{B}$ a cofibration and $\mathfrak{X} \to^p \mathfrak{Y}$ a fibration in $\triangle^{op}Sh(S)$;*

1. *The morphism $\mathsf{H}om(\mathfrak{B}, \mathfrak{X}) \to \mathsf{H}om(\mathfrak{A}, \mathfrak{X}) \times_{\mathsf{H}om(\mathfrak{A},\mathfrak{Y})} \mathsf{H}om(\mathfrak{B}, \mathfrak{Y})$ is a fibration which is trivial if i or p is a simplicial weak equivalence.*

2. *The morphism $hom(\mathfrak{B}, \mathfrak{X}) \to hom(\mathfrak{A}, \mathfrak{X}) \times_{hom(\mathfrak{A},\mathfrak{Y})} hom(\mathfrak{B}, \mathfrak{Y})$ is a fibration which is trivial if i or p is simplicial weak equivalence.*

Proof.

1. If $i : \mathfrak{A} \to \mathfrak{B}$ and $j : \mathfrak{X} \to \mathfrak{Y}$ are two cofibration, then the morphism $\mathfrak{A} \times \mathfrak{Y} \bigsqcup_{\mathfrak{A} \times \mathfrak{X}} \mathfrak{B} \times \mathfrak{X} \to \mathfrak{B} \times \mathfrak{Y}$ is a cofibration which is trivial if i or j is a weak equivalence. Now by the definition of $\mathsf{H}om(-,-)$ and the right lifting property of trivial fibration and fibration we have the result.

2. Let, for any $U \in C$, $\Gamma : \triangle^{op}Sh(S) \to \triangle^{op}Sets$ be the global section functor. We have left adjoint $\triangle^{op}Sets \to \triangle^{op}Sh(S)$, called the constant sheaf functor. This constant sheaf functor transforms cofibration (resp. trivial cofibration) in $\triangle^{op}Sets$ to cofibration (resp. trivial cofibration) in $\triangle^{op}Sh(S)$. The global section functor sends fibration (resp. trivial

fibration) in $\triangle^{op}Sh(S)$ to fibration (resp. trivial fibration) in $\triangle^{op}Sets$. Hence by 1 we have the result.

□

Lemma 2.4.7. *Let $\mathfrak{X} \to^f \mathfrak{Y}$ a morphism in $\triangle^{op}Sh(S)$ with $\mathfrak{X}$ and $\mathfrak{Y}$ simplicially fibrant. The following conditions are equivalent.*

1. *f is a homotopy equivalence.*
2. *f is a simplicial weak equivalence.*
3. *For $U \in C$, the morphism $\mathfrak{X}(U) \to \mathfrak{Y}(U)$ is a weak equivalence of simplicial sets.*

Proof. 1 implies 3 is evident from the previous result and the fact that $\mathfrak{X} \times \triangle^1$ is the cylinder object in $\triangle^{op}Sh(S)$ for $\mathfrak{X}$. 1 and 2 are equivalent by the hypothesis of closed model category. And from the definition 2.4.1 and 2.4.2 3 implies 2.

□

2.4.3 Homotopical classification of G-torsors

Let $S = (C, T)$ be a site and G be a sheaf of simplicial groups on S. A right (resp. left) action of G on a simplicial sheaf $\mathfrak{X}$ is a morphism $a : \mathfrak{X} \times G \to \mathfrak{X}$ (resp $a : G \times \mathfrak{X} \to \mathfrak{X}$) such that the diagram for associativity commutes and the action of identity of G on $\mathfrak{X}$ fixes $\mathfrak{X}$.

Definition 2.4.8. A (left) action is called free if the morphism $G \times \mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ of the form $(g, x) \mapsto (a(g, x), x)$ is a monomorphism.

Definition 2.4.9. For any right action of G on $\mathfrak{X}$ the quotient $\mathfrak{X}/G$ is defined as the coequilizer of the morphism pr_1 and a from $\mathfrak{X} \times G \to X$.

Definition 2.4.10. A principal G-bundle (a G-torsor) over $\mathfrak{X}$ is a morphism $\mathfrak{Y} \to \mathfrak{X}$ together with a free right action of G on $\mathfrak{Y}$ over $\mathfrak{X}$ such that the canonical morphism $\mathfrak{Y}/G \to \mathfrak{X}$ is an isomorphism.

We will denote the set of isomorphism classes of principal G-bundles over $\mathfrak{X}$ by $P(\mathfrak{X}, G)$. This set is pointed by the trivial G-bundle $G \times \mathfrak{X} \to \mathfrak{X}$. If $\mathfrak{X}' \to \mathfrak{X}$ is a morphism of simplicial sheaves and $\mathfrak{Y} \to \mathfrak{X}$ is a principal G-bundle over $\mathfrak{X}$ then $\mathfrak{Y} \times_{\mathfrak{X}} \mathfrak{X}'$ has canonical structure of a principal G-bundle over $\mathfrak{X}'$. This can be used to give a contravariant functor from $\triangle^{op}Sh(S)$ to the category of pointed sets by mapping $\mathfrak{X}$ to $P(\mathfrak{X}, G)$.

Definition 2.4.11. Let $\mathfrak{X}$ be sheaf on S. $E(\mathfrak{X})$ is the simplicial sheaf of sets with n-th term $\mathfrak{X}^{n+1}$ (product of $\mathfrak{X}$ $n+1$ times) and with faces (resp. degeneracies) induced by the partial projections (resp. diagonals).

If G is a sheaf of simplcial groups on S, then $E(G)$ becomes a simplicial sheaf of groups such that $E(G)_0 = G$ and it has right and left action of G. The morphism $E(G) \to B(G)$, which sends $(g_0, g_1, ..., g_n)$ to $(g_0 g_1^{-1}, g_1 g_2^{-1}, ..., g_{n-1} g_n^{-1}, g_n)$ induces an isomorphism $E(G)/G \cong B(G)$.

Definition 2.4.12. Let G be a simplicial sheaf of groups. The diagonal of the bisimplicial group $(n, m) \mapsto E(G_n)_m$ defines a sheaf of simplicial groups, which is denoted by $E(G)$.

Again we have a morphism from $E(G) \to B(G)$ such that $E(G)/G \cong B(G)$. This G torsor $E(G) \to B(G)$ is called the universal G-torsor over $B(G)$.

Lemma 2.4.13. *Let G be a simplicial sheaf of groups on S, and let $\mathfrak{E}$ a G-torsor over a simplicial sheaf $\mathfrak{X}$. Then there is a trivial local fibration $\mathfrak{Y} \to \mathfrak{X}$ and a morphism $\mathfrak{Y} \to B(G)$ such that the pullback of $\mathfrak{E}$ to $\mathfrak{Y}$ is isomorphic to the pullback of $E(G)$ to $\mathfrak{Y}$.*

Proof. [MV] , page 128, lemma 1.12. □

A morphism of simplcial sheaves $f : \mathfrak{X} \to \mathfrak{Y}$ is called a local fibration (resp. trivial local fibration) if for any fiber functor Φ of S the corresponding morphisms of simplcial sets $\Phi(\mathfrak{X}) \to \Phi(\mathfrak{Y})$ is Kan fibration (resp. a Kan fibration and a weak equivalence).

The lemma above is the first step of classifying G-torsor over any simplcial set $\mathfrak{X}$ using the universal G-torsor $E(G) \to B(G)$ as we can locally classify G-torsor as a pullback of $E(G) \to B(G)$ by previous lemma.

Lemma 2.4.14. *Suppose G has simplicial diension zero and $f : \mathfrak{X} \to \mathfrak{Y}$ is a trivial local fibration. Then the corresponding map $P(\mathfrak{Y}, G) \to P(\mathfrak{X}, G)$ is a bijection.*

Proof. [MV], page 129, lemma 1.13. □

Lemma 2.4.15. *If G has simplicial dimension zero, then for any simplical sheaf $\mathfrak{X}$, the map $P(\mathfrak{X}, G) \to P(\mathfrak{X} \times \triangle^1, G)$ is a bijection. In particular, the functor $P(-, G)$ is homotopy invariant.*

Proof. [MV], page 129, lemma 1.14. □

Now let $\mathfrak{E} \in P(\mathfrak{X}, G)$, where G has simplcial dimensio zero, by 2.4.13 there exist a trivial local fibration $p : \mathfrak{Y} \to \mathfrak{X}$ such that there exists a map $f : \mathfrak{Y} \to BG$. But in the simplcial homotopy category $H_s(S)$, p is invertible so we get a map in $H_s(S)$ given by $f \circ p^{-1} : \mathfrak{X} \to BG$. So we get a natural transformation (by 2.4.14) from $P(\mathfrak{X}, G) \to Hom_{H_s(S)}(\mathfrak{X}, BG)$.

Proposition 2.4.16. *For any G of simplicial dimension zero the natural map $P(\mathfrak{X}, G) \to Hom_{H_s(S)}(\mathfrak{X}, BG)$ is a bijection. Suppose $BG \to \mathbb{B}G$ is a trivial cofibration such that $\mathbb{B}G$ is fibrant. Then there exists a principal G-bundle $\mathbb{E}G \to \mathbb{B}G$ such that for any $\mathfrak{X}$ the map $Hom(\mathfrak{X}, \mathbb{B}G) \to P(\mathfrak{X}, G)$ given by $f \mapsto f^*(\mathbb{E}G \to \mathbb{B}G)$ defines a bijection $Hom_{H_s(S)}(\mathfrak{X}, BG) \cong P(\mathfrak{X}, G)$*

Proof. [MV], page 130, proposition 1.15. □

2.5 The A^1-homotopy category of schemes over a base

Let S be Noetherien scheme. We denote by $H_s((Sm/S)_{Nis})$, the homotopy category associated to the simplicial closed model category structure on $\triangle^{op}Sh((Sm/S)_{Nis})$. We denote the final object of the category Sm/S by $\bullet$ and $\iota : \bullet \to A^1$ is the zero section.

Definition 2.5.1. Let $\mathfrak{X}$ be an object of $\triangle^{op}Sh((Sm/S)_{Nis})$. $\mathfrak{X}$ is called A^1-local if for all $\mathfrak{Y} \in Sh((Sm/S)_{Nis})$, the map $Hom_{H_s((Sm/S)_{Nis})}(\mathfrak{Y} \times A^1, \mathfrak{X}) \to Hom_{H_s((Sm/S)_{Nis})}(\mathfrak{Y}, \mathfrak{X})$ induced by ι is bijective.

Definition 2.5.2. Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a morphism in $\triangle^{op}Sh((Sm/S)_{Nis})$, f is a A^1-weak equivalence if for all A^1-local object $\mathfrak{Z}$ of $H_s((Sm/S)_{Nis})$, the map $Hom_{H_s(Sm/S)_{Nis})}(\mathfrak{Y}, \mathfrak{Z}) \to Hom_{H_s((Sm/S)_{Nis})}(\mathfrak{X}, \mathfrak{Z})$ is bijective.

It is clear that any simplicial weak equivalence is an A^1 weak equivalence. Also for any $\mathfrak{X} \in \triangle^{op}Sh((Sm/S)_{Nis})$, the morphism $\mathfrak{X} \times A^1 \to \mathfrak{X}$ is an A^1-weak equivalence.

Definition 2.5.3. We denote the class of A^1-weak equivalences in $\triangle^{op}Sh(Sm/S_{Nis})$ by W_{A^1}, the class of all monomorphisms in $\triangle^{op}Sh(Sm/S_{Nis})$ by Cof_{A^1} and class of morphisms in $\triangle^{op}Sh(Sm/S_{Nis})$ having the right lifting property by $Cof_{A^1} \bigcap W_{A^1}$ by Fib_{A^1}.

Lemma 2.5.4. *Let $\mathfrak{X} \in \triangle^{op}Sh((Sm/S)_{Nis})$. If $\mathfrak{X}$ is simplicially fibrant, then the following are equivalent*

1. *$\mathfrak{X}$ is A^1-local;*
2. *The simplicial fibration $\mathsf{H}om(A^1, \mathfrak{X}) \to \mathfrak{X}$ induced by ι is a simplicial weak equivalence;*
3. *For all $U \in Sm/S$, the fibration $\mathfrak{X}(A^1_U) \to \mathfrak{X}(U)$ is an weak equivalence.*

Proof.

1. By lemma 2.4.6 $\mathsf{H}om(A^1, \mathfrak{X})$ is simplicially fibrant. By lemma 2.4.7 2 and 3 are equivalent.
2. Suppose 2, then $\mathsf{H}om(A^1, \mathfrak{X}) \to \mathsf{H}om(\bullet, \mathfrak{X})$ is simplicial trivial fibration. By 2.4.6, for all simplicially cofibrant object $\mathfrak{Y} \in \triangle^{op}Sh((Sm/S)_{Nis})$, the morphism of simplicial sets $hom(\mathfrak{Y}, \mathsf{H}om(A^1, \mathfrak{X})) \to hom(\mathfrak{Y}, \mathsf{H}om(\bullet, \mathfrak{X}))$ is a trivial fibration, which implies $hom(\mathfrak{Y} \times A^1, \mathfrak{X}) \to hom(\mathfrak{Y}, \mathfrak{X})$ is a trivial fibration. But then by taking π_0 and 2.1.9 $Hom_{H_s((Sm/S)_{Nis})}(\mathfrak{Y} \times A^1, \mathfrak{X}) \to Hom_{H_s((Sm/S)_{Nis})}(\mathfrak{Y}, \mathfrak{X})$ is bijective, so $\mathfrak{X}$ is A^1-local.
3. Suppose 1, to show 3 by Yoneda's lemma it is enough to show that for any $U \in Sm/S$, the morphism of simplicially fibrant sets $hom(U \times A^1, \mathfrak{X}) \to hom(U, \mathfrak{X})$ induced by ι is an weak equivalence in $\triangle^{op}Sets$. By 2.1.9 it is enough to show that for any $K \in \triangle^{op}Sets$ the morphism $hom(K, hom(\mathfrak{Y} \times A^1, \mathfrak{X})) \to hom(K, hom(\mathfrak{Y}, \mathfrak{X})$, induces bijection on π_0. $hom(K, hom(\mathfrak{Y} \times A^1, \mathfrak{X})) \to hom(K, hom(\mathfrak{Y}$ can be identified with the morphism $hom(K \times \mathfrak{Y} \times A^1, \mathfrak{X}) \to hom(K \times \mathfrak{Y}, \mathfrak{X})$, by the hyothesis and 2.1.9 this morphism gives bijection on π_0.

□

Lemma 2.5.5. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ a morphism in $\triangle^{op}Sh(Sm/S_{Nis})$. The following conditions are equivalent*

1. *f is an A^1-weak equivalence;*
2. *For all $W \in Sh(Sm/S_{Nis})$ simplicially fibrant and A^1-local, the morphism of simplicial sheaves $Hom(\mathfrak{Y}, W) \to Hom(\mathfrak{X}, W)$ is a simplicial weak equivalence;*
3. *For all $W \in Sh(Sm/S_{Nis})$ simplicially fibrant and A^1-local, the morphism of simplicial sets $hom(\mathfrak{Y}, W) \to hom(\mathfrak{X}, W)$ ia a weak equivalence.*

Proof. By 2.4.7 2 is equivalent to say that for all $U \in Sm/S$ and simplicially fibrant A^1-local $W \in \triangle^{op}Sh(Sm/S_{Nis})$, the morphism of simplicial set $\mathsf{H}om(\mathfrak{Y}, W)(U) \to \mathsf{H}om(\mathfrak{X}, W)(U)$ is a weak equivalence. By adjunction , it is equivalent to say that $hom(\mathfrak{Y}, \mathsf{H}om(U, W)) \to hom(\mathfrak{X}, \mathsf{H}om(U, W))$ is a weak equivalence.Moreover for W, simplicially fibrant and A^1-local and if $W' \in \triangle^{op}Sh(Sm/S_{Nis})$, then $\mathsf{H}om(W', W)$ is simplicially fibrant and A^1-local. So showing 2 is equivalent to show for all $W \in \triangle^{op}Sh(Sm/S_{Nis})$, simplicially fibrant and A^1-local, the morphism $hom(\mathfrak{Y}, W) \to hom(\mathfrak{X}, W)$ is a weak equivalence. So 2 and 3 are equivalent. By the same way as in the proof of last lemma, we get 3 is equivalent to show that for all $W \in \triangle^{op}Sh(Sm/S_{Nis})$ simplicially fibrant and A^1-local, the morphism $hom(\mathfrak{Y}, W) \to hom(\mathfrak{X}, W)$ induces a bijection on π_0. So by 2.1.9 we get 3 and 1 are equivalent.

□

Lemma 2.5.6. *1. The product of two A^1-weak equivalence is an A^1-weak equivalence;*

2. Direct sum of A^1-weak equivalence is a A^1-weak equivalence.

3. If $i : A \to B$ and $j : C \to D$ be two cofibration in $\triangle^{op}Sh(Sm/S_{Nis})$, then the morphism $A \times D \coprod_{A\times C} B \times C \to B \times D$ is a cofibration which is an A^1-weak equivalence if i or j is a A^1-weak equivalence.

Proof. 1. By 2.5.5, if f is an A^1 weak equivalence then for any $\mathfrak{X} \in \triangle^{op}Sh(Sm/S_{Nis})$, $f \times id_{\mathfrak{X}}$ is an A^1-weak equivalnce. Since composition of A^1 weak equivalences is an A^1-weak equivalence, we have the result.

2. Since product of simplicial weak equivalences of simplicially fibrant objects is weak equivalence, by 2.5.5 we have the result.

3. By 1 and [JR] , page 43 lemma 3.28.

□

Definition 2.5.7. We denote the full subcategory of A^1-local objects of $H_s((Sm/S)_{Nis})$ by $H_{s,A^1-loc}((Sm/S)_{Nis})$.

Theorem 2.5.8. *The inclusion functor $H_{s,A^1-loc}((Sm/S)_{Nis}) \to H_s((Sm/S)_{Nis})$ has a left adjoint L_{A^1}.*

Proof. By [JR], page 36, lemma 3.9, there exists a set $B$of monomorphisms in $\triangle^{op}Sh(Sm/S_{Nis})$ such that the simplicial fibrations are exactly the morphisms having the right lifting property with respect to B. Let B' be the set of morphisms of the form $U \times \triangle^n \coprod_{U\times\partial\triangle^n} A^1_U \times \partial\triangle^n \to A^1_U \times \triangle^n$, induced by $\iota : \bullet \to A^1$, for $n \geq 0$ and $U \in Sm/S$. By lemma 2.5.5, $\mathfrak{X} \in \triangle^{op}Sh(Sm/S_{Nis})$ is A^1-local and simplicially fibrant if and only if $\mathfrak{X} \to \bullet$ has the right lifting property with respect to $B \bigcup B'$. By 2.5.6 $B \bigcup B' \subset Cof_{A^1} \bigcap W_{A^1}$.

By [JR], page 25, thm 2.28, there exists a functor $\phi : \triangle^{op}Sh(Sm/S_{Nis}) \to \triangle^{op}Sh(Sm/S_{Nis})$, with a natural transformation $Id \to \phi$, such that for $\mathfrak{X} \in \triangle^{op}Sh(Sm/S_{Nis})$, the morphism $\mathfrak{X} \to \phi(\mathfrak{X})$ is a transfinite composition of direct images of the direct sums of $B \bigcup B'$ and such that $\phi(\mathfrak{X}) \to \bullet$ has right lifting property with respect to the morphisms of $B \bigcup B'$. So for all $\mathfrak{X} \in \triangle^{op}Sh(Sm/S_{Nis})$, $\phi(\mathfrak{X})$ is simplicially fibrant and A^1-local.

By 2.5.6 and 2.5.5, direct image of direct sum of elements of $B \bigcup B'$ are inside $Cof_{A^1} \bigcap W_{A^1}$. So by ([GJ], page 44, lemma 3.32) transfinite composition of trivial A^1-cofibration is a trivial A^1-cofibration. So $\mathfrak{X} \to \phi(\mathfrak{X})$ is a trivial cofibration and an A^1-weak equivalence. Moreover ϕ transforms A^1-weak equivalence to A^1-weak equivalence. Since the image of ϕ is A^1-local objects, so ϕ transforms A^1-weak equivalences to simplicial weak equivalence (An A^1-weak equivalence between A^1-local objects is a simplicial weak equivalence). Since simplicial weak equivalences are A^1-weak equivalence, ϕ induces a functor $H_s((Sm/S)_{Nis} \to H_{s,A^1-loc}((Sm/S)_{Nis})$, denoted by L_{A^1}.

We have the inclusion functor $i : H_{s,A^1-loc}((Sm/S)_{Nis}) \to H_s((Sm/S)_{Nis})$. Using this functor we can define a morphism of functors $Id_{H_s((Sm/S)_{Nis})} \to i \circ L_{A^1}$. So we have a canonical morphism for $\mathfrak{X} \in H_s((Sm/S)_{Nis})$ and $\mathfrak{Y} \in H_{s,A^1-loc}((Sm/S)_{Nis})$: $Hom_{H_{s,A^1-loc}((Sm/S)_{Nis})}(L_{A^1}\mathfrak{X}, \mathfrak{Y}) \to Hom_{H_s((Sm/S)_{Nis})}(\mathfrak{X}, i\mathfrak{Y})$. This morphism is bijective since $\mathfrak{X} \to \phi(\mathfrak{X})$ is an A^1-weak equivalence for all $\mathfrak{X} \in \triangle^{op}Sh(Sm/S_{Nis})$. So L_{A^1} is adjoint to the functor $i : H_{s,A^1-loc}((Sm/S)_{Nis}) \to H_s((Sm/S)_{Nis})$. □

Theorem 2.5.9. *Let S be a Noetherian scheme. The category $\triangle^{op}Sh(Sm/S_{Nis})$ with morphisms $(Cof_{A^1}, Fib_{A^1}, W_{A^1})$ forms a closed model category. Moreover this closed model category structure is proper. The homotopy category of this closed model category is denoted by $H(S)$ and it is called the homotopy category of S.*

Proof. [JR], page 47, theorem 3.40. □

2.5.1 Pointed category and model category structure

A pointed category is a category is a category which has same initial and final object . Given a category C which has final object $\bullet$, we denote $C_\bullet$ the comma category $(\bullet \downarrow C)$. We have a basepoint forgetful functor $U : C_\bullet \to C$. If C has finite direct sums then this functor has a left adjoint $-_+ : C \to C_\bullet$ which associates $X \bigsqcup \bullet$ to every $X \in C$.

Definition 2.5.10. Let C be closed model category. A morphism in $C_\bullet$ is a weak equivalence (resp. cofibration, resp. a fibration) if and only if $U(f)$ is a weak equivalence (resp. a cofibration , resp. a fibration) in C.

Proposition 2.5.11. *If C is a closed model category, then $C_\bullet$ is also a closed model category. Moreover, if C is proper then $C_\bullet$ is proper too. Moreover*

Proof. [JR], page 52, proposition 3.56. □

We can apply the proposition to the closed model categories $\triangle^{op}Sets$, $Sh(Sm/S_{Nis})$ and $\triangle^{op}Sh(Sm/S_{Nis})$ (for both simplicial and A^1-local structure). We denote $H_\bullet^{top}$ as the homotopy category of pointed simplicial sets , $Ho_{s,\bullet}(Sm/S_{Nis})$ as the homotopy category of pointed simplicial sets and $H_\bullet(S)$ as the pointed homopoty category of S-schemes.

Definition 2.5.12. Let C be a closed model category (left proper). If $A \to^i X$ is a cofibration in C, we define a cofibrant object X/A of $C_\bullet$ by the following cocartesian diagram

$$\begin{array}{ccc} A & \xrightarrow{i} & X \\ \downarrow & & \downarrow \\ \bullet & \longrightarrow & X/A \end{array}$$

Let Cbe a closed model category. For all objects X and Y of $C_\bullet$, let $X \vee Y$ denote the direct sum in $C_\bullet$. We have morphism $X \vee Y \to X \times Y$, so we have a cocartesian diagram

$$\begin{array}{ccc} X \vee Y & \longrightarrow & X \times Y \\ \downarrow & & \downarrow \\ \bullet & \longrightarrow & X \wedge Y \end{array}$$

Lemma 2.5.13. *If, in C, for all object X, the functor $- \times X$ commutes with all finite inductive limits , then the bifunctor $- \wedge -$ defines a symmetric monoidal category structure on $C_\bullet$, and the neutral object is $S_s^0 := \bullet_+$.*

Proof. [JR] , page 53, lemma 3.58.

□

Definition 2.5.14. For all $n \in \mathbb{N}$, $S_s^n = (S_s^1)^n$, where $S_s^1 = \triangle^1/\partial\triangle^1 \in \triangle^{op}Sets_\bullet$.

Lemma 2.5.15. *For all $U \in Sm/S$, the functor $\triangle^{op}Sh(Sm/S_{Nis})_\bullet \to \triangle^{op}Sets_\bullet$ which associates $F(U)$ to F has a right adjoint $K \mapsto K \wedge U_+$.*

Proof. [JR], page 53, lemma 3.62.

□

Chapter 3

Unramified sheaves and strongly A^1-invariance of unramified sheaves

3.1 Main Idea

Let F_k be the category of finite type separable field extension of k. By a discrete valuation v on $F \in F_k$, we mean a discrete valuation coming from a codimension 1 point in a smooth model for F, $O_v \subset F$ will be its discrete valuation ring, $m_v \subset O_v$ its maximal ideal and $\kappa(v)$ its residue field.

We will denote by $\tilde{Sm}_k$ the category of smooth k-schemes where the morphisms are only smooth k morphisms. If X is a smooth finite type seprated k-scheme we denote X^n by the set of all codimension n points of X . In this chapter we will define unramified presheaf (resp. sheaf of sets) 3.2.1.

Then it will turn out that giving an unramified sheaf of sets on Sm_k is same as giving for all $X \in Sm_k$ irreducible with function field F and for any codimension one point $x \in X$, two sets and an inclusion $S(O_{x,X}) \subset S(F)$ (which is related to the third condition of unramified sheaves, also a specialisation map $s_v : S((O_{x,X}) \to S(\kappa(v))$ (related to closed immersion of codimension one). This data will satisfy some axioms (see 3.2.8 and 3.2.6), the axioms $A1$ is related to the Nisnevich square property, $A2$ is related to the second condition of unramified sheaves. $A4$ is to inductively factorise any codimension $d > 0$ morphism such that the composition of the structural sheaf map is independent of of the factorisation and $A3$ captures the functorial propeperty of sheaf of sets for closed immersion .

Atlast we will show the following lemma, which will be used in later sections to prove $\pi_1^{A^1}$ is strongly A^1-invariant.

Result 1. 1. Let S be an unramified sheaf of sets on $\tilde{Sm}_k$. Then S is A^1-invariant if and only if it satisfies the following : For any k -smooth local ring B (or localisation of a smooth scheme at codimension 1 point) of *dimension* ≤ 1 the canonical map $S(B) \to S(A^1_B)$ is bijective .

2. Let S be an unramified sheaf of sets on Sm_k . Then S is A^1-invariant if and only if it satisfies the following : For any $F \in F_k$ the canonical map $S(F) \to S(A^1_F)$ is bijective.

3.2 Unramified sheaves and A^1-invariance

Definition 3.2.1. An unramified presheaf of sets S on Sm_k is a presheaf of sets such that the following holds

1. for any $X \in Sm_k$ with irreducible components $\bar{\alpha}$, $\alpha \in X^0$ the map $S(X) \to \Pi_{\alpha \in X^0} S(\bar{\alpha})$ is a bijection. Where $\bar{\alpha}$ is the closure of the point α inside X.

2. for any $X \in Sm_k$ and for any open subshceme U of X the map $S(X) \to S(U)$ is injective if U is dense in X.

3. for any $X \in Sm_k$ irreducible with function filed F, The injective map $S(X) \to \cap_{x \in X^1} S(O_{X,x})$ is a bijection. The intersection is computed inside S(F).

Lemma 3.2.2. *An unramified presheaf S (on Sm_k) is automatically a sheaf of sets in Zariski topology.*

Proof. Let $\mathcal{U} = \{U_i \to U\}$ be a covering of an irreducible scheme $X \in Sm_k$(by axiom 1 it is enough to show for irreducible schemes). By 3 $S(X) \to \cap_{x \in X^1} S(O_{X,x})$ is a bijection and we have to show that $\cap_{x \in X^1} S(O_{X,x}) \to \Pi_i(\cap_{x \in U_i^1} S(O_{U_i,x})) \rightrightarrows \Pi_{ij}(\cap_{x \in U_{ij}^1} S(O_{U_{ij},x}))$ is exact(where $U_{ij} = U_i \times_X U_j$). The presheaf is separated comes from axiom 2. Note that $x \in X^1$ iff such that $x \in U_i^1$ for some U_i iff $x \in U_{ij}^1$ for some j. And moreover $O_{X,x} = O_{U_i,x} = O_{U_ij,x}$. Now the exactness follows easily. □

Lemma 3.2.3. *Suppose S is an unramified presheaf, condition 3 holds for X (localisation of a smooth k-scheme) with function field F .*

Proof. For any irreducible k-smooth scheme X with generic point η(corresponding to the function field F) , if s is an element of $S(F)$, U be the maximal open subset of X to which s extends, the closed set $Z = X - U$ is purely of codimension 1, which means that if x is the generic point of a irreducible component of Z, then x is a codimension one point in X. Indeed if codimension of the generic point x of some irreducible component of Z is greater than 1, then there exist a codimension 1 point $y \in U$ and $x \in X$ such that $x \in \bar{y}$. There exist an open subscheme U' containing both y and x and such that $U \cap U'$ contains all the codimension one point(by noetherian property of Z and from the fact that $\bar{x}$ is an irreducible component) of U' hence s can be extended to $U \cup U'$ using (3) which gives a contradiction as U is maximal. Now let X be a smooth scheme with generic point η with field of function F , x a point in X , s an element of $S(F)$, we assume that for any codimension one point y of Spec $O_{X,x}$ (i.e. a codimension one point y of X such that x belongs to the closure of y) s extends to an element in $S(O_{X,y})$, we want to prove that s extends to an element of $S(O_{X,x})$. Let U be the maximal open subset of X to which s extends. Let $Z = X - U$. We have shown that the generic points of the irreducible components of Z are of codimension one in X. If z is such a generic point of some irreducible component of Z of codimension 1 in X, then z does not belong to $SpecO_{X,x}$ by hypothesis(the maximality condition of U), so the closure of z does not contain x. Z is the union of the closure of its maximal points, so x does not belong to Z , which means that s extends to an open neighbourhood of x and thus is an element of $S(O_{X,x})$.

□

Lemma 3.2.4. *Let S be the sheaf of sets in Zariski topology on Sm_k satisfying properties 1 and 2 of the previous definition then it is unramified iff , for any $X \in Sm_k$ and any open subscheme U of X the restriction map $S(X) \to S(U)$ is bijective if $X - U$ is everywhere of codimension > 1 in X .*

Proof. If S is unramified then take any open subscheme U of a irreducible scheme X such that $X - U$ is everywhere of codimension > 1 , this implies that all the codimension one points of X are inside U and by the property three of unramified sheaves it follows that $S(X) \to S(U)$ is bijection .

For the converse for any irreducible scheme X , let $s \in \cap_{x \in X^1} S(O_{X,x})$. There exists a maximal open set $U \subset X$ such that s is induced by some element in $S(U)$. Hence by the property of this open set we have $\forall x \in X^1$, $x \in U$. So the closed set $X - U$ is of *codimension* ≥ 2 . So by the bijection between $S(X) \to S(U)$ we get $s \in S(X)$. □

Lemma 3.2.5. *Any strictly A^1-invariant sheaf of abelian groups M on Sm_k is unramified.*

Proof. [MO2] page 67 lemma 6.4.11 we have M is pure (see [MO2] page 66 definition 6.4.9). So for $X \in Sm_k$ we have isomorphism $H^n_{Zar}(X; M) = H^n_{Nis}(X; M)$ and if U is a dense open set of X (so $codim(X - U) \geq 1$) ,$M(X) = H^0_{Zar}(X; M) = H^0_{Nis}(X; M) \to H^0_{Nis}(U; M) = H^0_{Zar}(U; M) = M(U)$ is injective.Which proves the property 2 of unramified sheaves .

Suppose M satisfies property 1 , suppose U be any open subscheme of $X \in Sm_k$ such that $codim(X - U) \geq 2$, we have $M(X) = H^0_{Zar}(X; M) = H^0_{Nis}(X; M) \to H^0_{Nis}(U; M) = H^0_{Zar}(U; M) = M(U)$ is bijective , so by previous lemma M will satisfy property 3 of unramified sheaves.

To show the property 1 it is enough to show that $X \in Sm_k$ the irreducible components of X are same as the connected components of X. (irreducible components are connected and for smooth scheme X the local rings are regular,hence integral, so if X is connected then it is irreducible).

□

Base Change Let $K \in F_k$ and $\pi : Spec(K) \to Spec(k)$ be the structural morphism , this gives a morphism between the sites $\pi : Sm_K \to Sm_k$, let S be a sheaf of sets on Sm_k then we can pullback S to the sheaf $S |_K := \pi^* S$ on Sm_K . For finite type separable extension F of K we can show that $S(F) = \pi^* S(F)$. $\pi^* S$ is a sheaf in Nisnevich topology so it is a sheaf in Zariski topology and it satisfies property one and two of uramified sheaves , and it satisfies the condition of the lemma 3.2.4 so it satisfies 3 too.

Definition 3.2.6. An unramified $\tilde{F_k}$ set consists of

(D1) A functor $S : \mathrm{F}_k \longrightarrow Set$;

(D2) For any $F \in \mathrm{F}_k$ and any discrete valuation v on F , a subset $S(O_v) \subset S(F)$;

The previous data are moreover supposed to satisfy the following axioms

(A1) If $i : E \subset F$ is a separable extension in F_k, and v is a discrete valuation on F which restricts to a discrete valuation w on E with ramification index 1 and $\kappa(v)$ is separable over $\kappa(w)$, then $S(i)$ maps $S(O_w)$ into $S(O_v)$ and moreover if the induced extension $\tilde{i} : \kappa(w) \to \kappa(v)$ is an isomorphism , then the following square of sets is cartesian :

$$\begin{array}{ccc} S(O_w) & \longrightarrow & S(O_v) \\ \downarrow & & \downarrow \\ S(E) & \longrightarrow & S(F) \end{array}$$

(A2) Let $X \in Sm_k$ be irreducible with function field F. If $x \in S(F)$, then x lies in all but a finite number of $S(O_x)$'s , where x runs over the set X^1 of points of codimension one of X.

Theorem 3.2.7. *The category of unramified sheaves on $\tilde{Sm}/k$ is equivalent to the category of unramified $\tilde{\mathrm{F}}_k$-sets .*

Proof. There exists a functor from the category of unramified sheaf of sets on $\tilde{Sm}_k$ to the category of unramified $\tilde{F}_k$ sets. Indeed given an unramified sheaf of sets S on $\tilde{Sm}_k$ we can take a smooth model for any $F \in \tilde{F}_k$ and then evaluate S at F . For any discrete valuation v on F , there exists $X \in Sm_k$ irreducible with function field F and the discrete valuation comes from a codimension 1 point of X . Now using property 2 of unramified set we get $S(O_v) \subset S(F)$. If $E \subset F$,where $E, F \in F_k$ and moreover F is finite type separable extension over E , there exists $X, Y \in Sm_k$ irreducible with function field E and F respectively and $f : Y \to X$ smooth which maps the generic point to the generic point . So we get a map from $S(f) : S(X) \to S(Y)$ which induces the map $S(f) : S(E) \to S(F)$.

Axiom (A1) can be checked using smooth models over k for $Spec(F)$ and $Spec(Ov)$. We can have smooth models X and Y for $Spec(F)$ and $Spec(E)$ respectively , such that there exists a smooth morphism $f : X \to Y$ which maps the generic point to the generic point and the codimension one point corresponding to v to the codimension one point corresponding to w . So we have the following diagram

$$\begin{array}{ccc} S(U) & \longrightarrow & S(f^{-1}(U)) \\ \downarrow & & \downarrow \\ S(E) & \longrightarrow & S(F) \end{array}$$

Where U is any open subscheme of X containing w. After taking the colimit of this diagram we get that $S(i)$ maps $S(O_w)$ to $S(O_v)$. Now the following diagram is an elementary distinguished square over $Spec(O_w)$.

$$\begin{array}{ccc} Spec(F) & \longrightarrow & Spec(O_v) \\ \downarrow & & \downarrow \\ Spec(E) & \longrightarrow & Spec(O_w) \end{array}$$

which is the colimit of the diagram

$$\begin{array}{ccc} U \times_X V & \longrightarrow & V \\ \downarrow & & \downarrow p \\ U & \xrightarrow{j} & X \end{array}$$

where V is smooth model for $Spec(F)$ and X is smooth model for $Spec(E)$. Now using the Theorem 2.3.18 we get $A1$.

For axiom (A2) , firstly for any irreducible scheme X, by the noetherian property the compliment of any open subscheme U of X contains only finitely many $x \in X^1$. By definition any $f \in S(F)$ comes from an element $f \in S(U)$ where $U \in Sm_k$ and it is an open subscheme of X. So any $f \in S(F)$ lies in all the $S(O_x)$ for $x \in X^1$ and $x \in U$, but there are finitely many $x \in X^1, x \notin U$.

Now let us define a functor form the category of unramified $\tilde{F_k}$ sets to the category of unramified sheaves on $\tilde{Sm_k}$. First given an unramified $\tilde{F_k}$ set S, and $X \in Sm_k$ irreducible with function field F, define $S(X) \subset S(F)$ as $\cap_{x \in X^1} S(O_{X,x}) \subset S(F)$. It can be extended to any smooth scheme such that the first property of unramified sheaves is satisfied . Now given a smooth morphism $f : Y \to X$ we have to define a map $S(f) : S(X) \to S(Y)$. We can assume (by the first property of unramified sheaves and the fact that image of a irreducible set is irreducible) that X and Y are irreducible with function field E and F respectively. Moreover we can assume that f is dominant (since image of f is open) . If $x \in X^1$ then $f^{-1}(x)$ has finitely many irreducible components and the generic points of those irreducible components are codimension 1 points in Y . Now using A1, field inclusion $E \subset F$ gives the desired map . The fact that this gives a sheaf of sets in Nisnevich topology comes from axiom (A1) and 2.3.18 of Nisnevich sheaves . It is unramified by construction and A2 . And it is the inverse to the restriction functor . □

So from now on if S is an unramified sheaf of sets over $\tilde{F_k}$ we will denote the associated sheaf of sets over $\tilde{Sm_k}$ by S also.

Definition 3.2.8. An unramified F_k-set S is an unramified $\tilde{\mathrm{F}}_k$ set together with the following additional data :

(D3) For any $F \in \mathrm{F}_k$ and any discrete valuation v on F such that the residue field $\kappa(v)$ is separable over k , a map $s_v : S(O_v) \to S(\kappa(v))$, called the specialization map associated to v. And this data satisfies the additional conditions

(A3) (a) If $i : E \subset F$ is an extension in F_k, and v is a discrete valuation on F which restricts to a discrete valuation w on E with ramification index 1, then $S(i)$ maps $S(O_w)$ to $S(O_v)$ and if the two residue fields are separable over k the following diagram is commutative

:

$$\begin{array}{ccc} S(O_w) & \longrightarrow & S(O_v) \\ \downarrow & & \downarrow \\ S(\kappa(w)) & \longrightarrow & S(\kappa(v)) \end{array}$$

(b) If $i : E \subset F$ is an extension in F_k, and v is a discrete valuation on F which restricts to 0 on E then the map $S(i) : S(E) \to S(F)$ has its image contained in $S(O_v)$.

(c) If moreover $\kappa(v)$ is separable over k, then if we let $j : E \subset \kappa$ then the composition $S(E) \to S(O_v) \overset{s(v)}{\to} S(\kappa(v))$ is equal to $S(j)$.

(A4) (a) For any $X \in Sm_k$, and any point $z \in X^2$ of codimension 2, and for any point $y_0 \in X^1$ such that $z \in \bar{y_0}$ and such that $\bar{y_0} \in Sm_k$, the map $s_{y_0} : S(O_{y_0}) \to S(\kappa(y_0))$ maps $\cap_{y \in (X_z)^1} S(O_y)$ into $S(O_{\bar{y_0},z}) \subset S(\kappa(y_0))$. Where X_z is X localised at z.

(b) If $\kappa(z)$ is seprable over k then the composition

$\cap_{y \in X^1} S(O_y) \to S(O_{\bar{y_0},z}) \to S(\kappa(z))$

does not depend on the choice of the point y_0 .

Theorem 3.2.9. *The category of unramified sheaf of sets on Sm_k is equivalent to the category of unramified F_k sets .*

Proof. Let us define a functor from unramified sheaf of sets on Sm_k to the category of unramified F_k sets . By the previous theorem we have an unramified $\tilde{F_k}$ set constructed from an unramified sheaf of sets S on Sm_k . If v is a discrete valuation on $F \in F_k$ with residue filed $\kappa(v)$ separable over k, then by choosing smooth model for the closed immersion $Spec(\kappa(v)) \to Spec(O_v)$ will give the specialisation map s_v. We have a smooth X and x a codimension one point with $O_v = O_{X,x}$, $k(v)$ the residue field . Let Z be the closure of x in X . We may assume that Z is smooth (in (D3), we assume that $k(v)$ is separable, so there is a dense open subset of Z that is smooth over k . Since O_v is the inductive limit of the ring of functions of U where U vary in the ordered set of (affine) neighbourhoods of x in X . For any such U, there is a map of smooth schemes $Z \cap U \to U$, and so, the original data gives a map $S(U) \to S(Z \cap U)$. Now, we can take the inductive limit of these maps where U goes through the neighbourhoods of x in X . By construction, we can map $S(O_v) \to S(k(v))$. [$k(v)$ is the function field of Z]. To show that it satisfies A3 (a) We can assume there exist X and Y smooth irreducible schemes with function field F and E respectively, which has codimension one points v and w respectively, and a smooth map $f : X \to Y$ such that f maps the generic point to the generic point and v to w, moreover take the closed subscheme generated by v and w. So the composite map from $\bar{v} \to X \to Y$ factors through $\bar{w}$. Now the commutativity of the squae in A3 (a) is same as the commutativity of the following square

$$\begin{array}{ccc} \bar{v} & \longrightarrow & X \\ \downarrow & & \downarrow \\ \bar{w} & \longrightarrow & Y \end{array}$$

To show A3 (b) and (c), actually we can assume that f maps v to the generic point of Y.

Now rest is to show A4 (a). First of all notice that for X as in A4 we have a commutative diagram

$$\begin{array}{ccc} \bar{y_0} \cap U & \longrightarrow & U \\ \downarrow & & \downarrow \\ \bar{y_0} & \longrightarrow & X \end{array}$$

This gives the following commutative diagram

$$\begin{array}{ccc} S(X) & \longrightarrow & S(O_{y_0}) \\ \downarrow & & \downarrow \\ S(\bar{y_0}) & \longrightarrow & S(\kappa(y_0)) \end{array}$$

We can thus replace X by any open subscheme U containig z, and $\bar{y_0}$ by $U \cap \bar{y_0}$. We get

$$\begin{array}{ccc} S(O_z) & \longrightarrow & S(O_{y_0}) \\ \downarrow & & \downarrow \\ S(O_{\bar{y_0},z}) & \longrightarrow & S(\kappa(y_0)) \end{array}$$

which proves A4 a. Again notice that we have the following commutative diagram

$$\begin{array}{ccccc} \bar{z} \cap U & \longrightarrow & \bar{y_0} \cap U & \longrightarrow & U \\ \downarrow & & \downarrow & & \downarrow \\ \bar{z} & \longrightarrow & \bar{y_0} & \longrightarrow & X \end{array}$$

and from this we get a commutaive diagram

$$\begin{array}{ccccc} S(X) & \longrightarrow & S(\bar{y_0}) & \longrightarrow & S(\bar{z}) \\ \downarrow & & \downarrow & & \downarrow \\ S(U) & \longrightarrow & S(\bar{y_0} \cap U) & \longrightarrow & S(\kappa(z)) \end{array}$$

since every open set containing z contains all the codimension one point y such that $z \in \bar{y}$ and for any such y and U open such that $z \in U$ we have $S(U) \to S(\bar{y_0} \cap U)$ is the composition of the maps $S(U) \to S(\bar{y} \cap U) \to S(\bar{y_0} \cap U)$. So now the colimit over the open sets V (replacing X by V in the previous diagram) containing z gives $A4(b)$.

Now to finish the proof of our theorem it is sufficient to prove the following.

Lemma 3.2.10. *Given an unramified F_k set S , there is a unique way to extend the unramified sheaf of sets $S : (\tilde{Sm_k})^{op} \to Set$ to sheaf $S : (Sm_k)^{op} \to Set$, such that for any discrete valuation v on $F \in F_k$ with separable residue field , the map $S(O_v) \to S(\kappa(v))$ induced by the sheaf structure map is the specialization map $s_v : S(O_v) \to S(\kappa(v))$. This sheaf is automatically unramified .*

Proof. Let $i : Y \to X$ be a closed immersion of codimension one in Sm_k. To define a map $S(i) : S(X) \to S(Y)$ we can assume that X and Y is irreducible. Indeed , if $Y = \amalg_\alpha Y_\alpha$ be the irreducible decomposition of Y then by property 1 of unramified sheaves $S(Y) = \Pi_\alpha S(Y_\alpha)$. Hence the map $s(i)$ should be product of $s(i_\alpha)$, so Y can be chosen as irreducible, similarly using the fact that image of irreducible set is irreducible we can assume that X is irreducible too.

We can show that there exists a (unique) map $s(i) : S(X) \to S(Y)$ which makes the following diagram commutative

$$\begin{array}{ccc} S(X) & \xrightarrow[s(i)]{} & S(Y) \\ \downarrow & & \downarrow \\ S(O_{X,y}) & \xrightarrow[s_y]{} & S(\kappa(y)) \end{array}$$

where y is the generic point of Y . If such map exists then by the commutativity of the previous diagram and the property 3 of unramified sheaves , s_y will map $S(X)$ inside $S(O_{Y,z})$ for all $z \in Y^1$. So to get the above map it is sufficient to prove that for any $z \in Y^1$, the image of $S(X)$ through s_y is contained in $S(O_{Y,z})$. Observe that z has codimension 2 in X . Hence by axiom (A4 a) s_y maps $\cap_{x \in X^1} S(O_{X,x}) \subset \cap_{y \in (X_z)^1} S(O_y)$ into $S(O_{Y,z})$.

Lemma 3.2.11. *Let $i : Z \to X$ be a closed immersion in Sm_k of codimension $d > 0$. Assume that there exists a factorisation*

$$Z \overset{j_1}{\to} Y_1 \overset{j_2}{\to} Y_2 \to .... \overset{j_d}{\to} Y_d = X$$

of i into a composition of codimension 1 closed immersions, with Y_i closed subschemes of X each of which is smooth over k. Then the composition

$$S(X) \overset{s(j_d)}{\to} ... \to S(Y_2) \overset{s(j_2)}{\to} S(Y_1) \overset{s(j_1)}{\to} S(Z)$$

does not depend on the choice of the above factorisation of i. We denote this composition by $S(i)$.

Proof. Proof is by induction (induction on d) . For $d = 1$ there is nothing to prove . Assume $d \geqslant 2$. By the arguement of previous lemma we can reduce it to the case where Z is irreducible with generic point z . We have to show that the composition does not depend on the flag

$$Z \overset{j_1}{\to} Y_1 \overset{j_2}{\to} Y_2 \to .... \overset{j_d}{\to} Y_d = X$$

. First of all we can assume X is irreducible and using the commutativity of the next diagram we can substitute X by any open subset Ω containing z.

$$\begin{array}{ccccccc} S(X) & \longrightarrow & S(Y_1) & \longrightarrow & \cdots & \longrightarrow & S(Z) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ S(\Omega) & \longrightarrow & S(Y_1 \cap \Omega) & \longrightarrow & \cdots & \longrightarrow & S(Z \cap \Omega) \end{array} \tag{3.1}$$

That is to show that the composition is irrespective of the flag

$$Z \cap \Omega \overset{j_1}{\to} Y_1 \cap \omega \overset{j_2}{\to} Y_2 \cap \Omega \to .... \overset{j_d}{\to} Y_d = X \cap \Omega$$

. Now when $d = 2$ This foillows from axiom A4. As $\kappa(z)$ is separable and over k, so the composition $\cap_{y \in X^1} S(O_y) \to S(O_{\bar{y_0},z}) \to S(\kappa(z))$ doesn't depend on the choice of y_0.

Now for the general case $O_{X,z}$ is regular local ring of dimension d. So by Nakayamas lemma there exists an open neighborhood Ω of z in X and a sequence of elements $(x_1, x_2, ..., x_d) \in O(\Omega)$

which generates the maximal ideal m_v of $A := O_{X,z}$ and such that the flag $Spec(A/(x_1, ..., x_d)) \to Spec(A/(x_2, ..., x_d) \to ......Spec(A/(x_d)) \to Spec(A)$ is the induced flag :

$$Z \cap \Omega \xrightarrow{j_1} Y_1 \cap \omega \xrightarrow{j_2} Y_2 \cap \Omega \to .... \xrightarrow{j_d} Y_d = X \cap \Omega$$

Thus we have to show that given $z \in X^d$, with separable residue field, in a smooth k-scheme X, and with $A = O_{X,z}$, and given a sequence $(x_1, ..., x_d)$ whose associated flag of closed subschemes of $Spec(A)$ consists of smooth k-schemes, the composition $S(SpecA) \to S(Spec(A/(x_d))) \to ....S(Spec(A/(x_2, ..., x_d)) \to S(\kappa)$ doesn't depend on the choice of $(x_1, x_2, ..., x_d)$.

As $\kappa(v)$ is separable over k , the condition of smoothness on the members of the associated flag to the sequence $(x_1, ..., x_d)$ is equivalent to the fact that the family $(x_1, ..., x_d)$ forms a basis of the $\kappa(v)$ vector space $m_v/(m_v)^2$. Now if, $M \in GL_d(A)$, then the sequence $M.x_i$ satisfies this condition. If we permute x_i and x_{i+1} then from the case of $d = 2$ the composition $S(A) \to S(\kappa(v))$ remains same after the permutation. So it shows that any permutation of $(x_1, ..., x_d)$ keeps the composition invariant.

If $(\tilde{x_1}, ..., \tilde{x_d})$ is another sequence in A satisfying the same assumption. Then $\tilde{x_i}$ can be written as linear combination in x_j. We get a matrix $M \in M_d(A)$ with $\tilde{x_i} = Mx_j$. This matrix reduces in $M_d(\kappa)$ to an invertible matrix , thus M is itself invertible. If we multiply an elemnt x_i in a sequence $(x_1, ..., x_d)$ by a unit of A then it won't change the flag so the composition $S(A) \to S(\kappa(v))$ remains unchanged . So we can assume $det(M) = 1$. For a local ring A the group $SL_d(A)$ is the group $E_d(A)$ of elementary matrices in A ([KNUS] chapter VI ,corollary 1.5.3) . So M can be written as a product of elementary matrices in $M_d(A)$.

The composite map doesn't depend on the permutation of $(x_1, ..., x_d)$, So we have to show that given a seqeunce $(x_1, ..., x_d)$ and $a \in A$, the regular sequence $(x_1 + ax_2, ..., x_d)$ induces the same compostion $S(A) \to S(\kappa(v))$ as $(x_1, ..., x_d)$. But this induces same flag . □

Let $i : Z \to X$ be a closed immersion in Sm_k . So X can be covered by U_i's such that $Z \cap U_i \to U_i$ admits a factorization as in the previous lemma. Thus for each such U_i we get a canonical map $s_{U_i} : S(U_i) \to S(Z \cap U_i)$. But then we can apply the lemma to the intersection $U_i \cap U_j$, with U_j another open set for which the factorization exists, so s_{U_i} are compatible, hence defines a canonnical map $s(i) : S(X) \to S(Z)$.

If $f : Y \to X$ be any morphism between smooth k-schemes. Then f is the composition $Y \hookrightarrow Y \times_k X \to X$ of the closed immersion graph of f, i.e $\Gamma_f : Y \hookrightarrow Y \times_k X$ and the smooth projection $p_X : Y \times_k X \to X$. So we can now define $s(f) :=$

$$S(X) \xrightarrow{s(p_X)} S(Y \times_k X) \xrightarrow{\Gamma_f} S(Y)$$

.

If we have a smooth morphism $\pi : X' \to X$ and closed immersion $i : Z \to X$ in Sm_k, Let $p_{X'} : Z \times_X X' \to X'$ and $p_Z : Z \times_X X' \to Z$. Then the following diagram is commutative

$$\begin{array}{ccc} S(X) & \xrightarrow{s(\pi)} & S(X') \\ \downarrow{\scriptstyle s(i)} & & \downarrow{\scriptstyle s(p_{X'})} \\ S(Z) & \xrightarrow{s(p_Z)} & S(Z \times_X X') \end{array}$$

(We can reduce to the case using the proof of previous lemma that the closed immersion is of codimension 1 and both X and Z are irreducible . But then the commutativity of the diagram follows from A3 (a)).

Let $Z \to Y \to X$ be two composable morphism we get the following commutative diagram

$$\begin{array}{ccccc} Z & \longrightarrow & Z \times_k Y & \longrightarrow & Z \times_k Y \times_k X \\ \downarrow & & \downarrow & & \downarrow \\ Z & \longrightarrow & Y & \longrightarrow & Y \times_k X \\ \downarrow & & \downarrow & & \downarrow \\ Z & \longrightarrow & Y & \longrightarrow & X \end{array} \tag{3.2}$$

Then applying S and s gives a commutative diagram . Restriction of this presheaf to $\tilde{Sm}_k$ gives an unramified sheaf hence it is itself unramified sheaf on Sm_k . □

□

Corollary 3.2.12. *Let S and G be sheaves of sets on Sm_k , with S unramified and G satisfying conditions 1 and 2 of unramified presheaves . Then to give a morphism of sheaves $\phi : G \to S$ is equivalent to give a natural transformation $\varphi : G\mid_{F_k} \to S\mid_{F_k}$ such that :*

1. *for any discrete valuation v on $F \in F_k$, the image of $G(O_v) \subset G(F)$ through φ is contained inside $S(O_v) \subset S(F)$.*
2. *If moreover the residue field of v is separable over k , then the induced square commutes :*

$$\begin{array}{ccc} G(O_v) & \xrightarrow[s_v]{} & G(\kappa(v)) \\ \varphi\downarrow & & \varphi\downarrow \\ S(O_v) & \longrightarrow & S(\kappa(v)) \end{array}$$

Proof. If such a ϕ exist then we can easily construct φ and show that it satisfies 1 and 2 (by elemntary properties of sheaves) .

Suppose φ exist and satisfies 1 and 2 then for any $X \in Sm_k$ irreducible with function field F we can define a morphism $G(X) \to S(X)$ using property 1 , And if $Z \to X$ is a codimension one closed immersion then porperty 2 implies that the following square is commutaive

$$\begin{array}{ccc} G(X) & \longrightarrow & G(Z) \\ \downarrow & & \downarrow \\ S(X) & \longrightarrow & S(Z) \end{array}$$

Now following the proof of the previous two lemmas there exists a morphism of sheaves since S is unramified .

□

Lemma 3.2.13. *1. Let S be an unramified sheaf of sets on $\tilde{Sm}_k$. Then S is A^1-invariant if and only if it satisfies the following : For any k -smooth local ring B (or localisation of a smooth scheme at codimension 1 point) of dimension ≤ 1 the canonical map $S(B) \to S(A^1_B)$ is bijective .*

2. *Let S be an unramified sheaf of sets on Sm_k . Then S is A^1-invariant if and only if it satisfies the following : For any $F \in F_k$ the canonical map $S(F) \to S(A^1_F)$ is bijective .*

Proof. 1. If S is A^1-invariant then for any smooth k-scheme X , we have bijection $S(X) \to S(A^1_X)$ hence the claim follows . To show the opposite let $X \in Sm_k$ irreducible with function field F . We have the following commutative square :

$$\begin{array}{ccc} S(X) & \longrightarrow & S(A^1_X) \\ \downarrow & & \downarrow \\ S(F) & \longrightarrow & S(F(T)) \end{array}$$

Each map is injective in this square . Now $S(A^1_X) \to S(F(T))$ factors as $S(A^1_X) \to S(A^1_F) \to S(F(T))$. By assumption $S(F) = S(A^1_F)$, so $S(A^1_X) \subset S(F)$. It is sufficient to show that for any $x \in X^1$ there is an inclusion $S(A^1_X) \subset S(O_{X,x}) \subset S(F)$. But $S(A^1_X) \subset S(A^1_{O_{X,x}})$ and by our assumption $S(O_{X,x}) = S(A^1_{O_{X,x}})$. So S is A^1-invariant .

2. Again by the property of A^1-invariant sheaves it is clear that $S(F) \to S(A^1_F)$ (since for any irredicible scheme X with function field F we have $S(X) \to S(A^1_X)$ is bijective) is bijective. To show the converse, let $X \in Sm_k$ irreducible with function field F . We have the following commutative square

$$\begin{array}{ccc} S(A^1_X) & \longrightarrow & S(A^1_F) \\ \downarrow & & \downarrow \\ S(X) & \longrightarrow & S(F) \end{array}$$

Now the upper horizontal, right vertical and the lower horizontal maps are injective hence the left vertical map is also injective and since X is a retract of A^1_X we have the left vertical map is surjective . Which gives the proof .

□

Chapter 4

Unramified Sheaves of groups and strong A^1-invariance

4.1 Main idea

Let G be an unramified sheaf of groups on Sm_k (or $\tilde{Sm_k}$). For any discrete valuation v on $F \in F_k$ let $H^1_v(O_v; G) := G(F)/G(O_v)$. For y a point of codimension 1 in $X \in Sm_k$, we set $H^1_y(X;G) = H^1_y(O_{X,y};G)$. By the axiom $A2$ of unramified sets on F_k if X is irreducible with function field F the induced action of $G(F)$ on $\Pi_{y\in X^1} H^1_y(X;G)$ preserves the weak product $\Pi'_{y\in X^1} H^1_y(X;G) \subset \Pi_{y\in X^1} H^1_y(X;G)$ and by definition the isotropy subgroup of the action of $G(F)$ on the basepoint of $\Pi_{y\in X^1} H^1_y(X;G)$ is exactly $G(X) = \bigcap_{y\in X^1} G(O_{X,y})$. This shows that the following sequence

$$1 \to G(X) \to G(F) \implies \Pi'_{y\in X^1} H^1_y(X;G)$$

is exact (in the sense of definition 4.2.3). The main idea in this section is to understand this exact sequence when G is A^1-invariant. We will give necessary and sufficient conditions for a sheaf of A^1-invariant groups G to be strongly A^1-invariant.

4.2 The conditions of strongly A^1-invariance

Definition 4.2.1. For any point z of codimension 2 in smooth k-scheme X, we denote $H^2_z(X;G)$ the orbit set of $\Pi'_{y\in X^1_z} H^1_y(X;G)$ under the left action of $G(F)$, where $F \in F_k$ denotes the field of functions of X_z (where X_z is the localisation of X at the point z).

If X is irreducible smooth k-scheme with function field F we have a $G(F)$-equivariant map $\Pi'_{y\in X^1} H^1_y(X;G) \to \Pi'_{y\in X^1_z} H^1_y(X;G) \to H^2_z(X;G)$. So we can define a $G(F)$-equivariant map

$$\Pi'_{y\in X^1} H^1_y(X;G) \to \Pi'_{z\in X^2} H^2_z(X;G) \tag{4.1}$$

But it is not clear that the image is inside $\Pi'_{z\in X^2} H^2_z(X;G)$. So we will use another axiom depending on G and an integer d

(A2') For any smooth k-scheme , irreducible of dimension d , the image of the boundary map 4.1 is contained in the weak product $\Pi'_{z\in X^2} H^2_z(X;G)$.

If we assume that G satisfies $(A2')$, for $X \in Sm_k$ irreducible with function field F , we get a complex $C^*(X;G)$ of groups , action and pointed sets :

$$1 \to G(X) \to G(F) \implies \Pi'_{y\in X^1} H^1_y(X;G) \to \Pi_{z\in X^2} H^2_z(X;G)$$

If $X \in Sm_k$, define :
$G^0(X) := \Pi'_{x\in X^0} G(\kappa(x))$;
$G^1(X) := \Pi'_{y\in X^1} H^1_y(X;G)$;
$G^2(X) := \Pi'_{z\in X^2} H^2_z(X;G)$.

Lemma 4.2.2. *The correspondence $X \mapsto G^i(X)$, $i \le 2$ can be extended to presheaf of sets on $\tilde{Sm}_k$. Moreover G^0 is an unramified sheaf in Nisnevich topology.*

Proof. If $f : X \to Y$ is a smooth dominant morphism between irreducible smooth k-schemes . By the same reasoning as in theorem 3.2.7 we can show that it is sufficient to define the map $G^i(f) : G^i(Y) \to G^i(X)$. If $x \in Y^d$, then the generic points of $f^{-1}(x)$ are in X^d . This gives the desired maps which makes $X \mapsto G^i(X)$ presheaves (using the A1 axiom of unramified $\tilde{F}_k$ sets).

G^0 is a Nisnevich sheaf follows from elementary distinguish squares. And it is unramified from the definition of G^0.

□

Definition 4.2.3. Let $1 \to H \subset G \Rightarrow E \to F$ be a sequence with G a group acting on the set E which is pointed as a set , with $H \in G$ a subgroup and $E \to F$ a G-equivariant map of sets , with F endowed with the trivial action . This sequence is exact if the isotropy subgroup of the base point of E is H and if the kernel of the pointed map $E \to F$ is equal o the orbit under G of the base point of E . This sequence is called exact in the strong sense if moreover the map $E \to F$ induces an injection into F of the left quotient set $G \setminus E \subset F$.

Let $C^*(X;G)$ is the complex

$$1 \to G(X) \to G^0(X) \implies G^1(X) \to G^2(X)$$

If X is localisation of a smooth scheme at point of *codimension* ≤ 2 , then X has atmost one codimension 2 point z, now $G^2(X) = H^2_z(X;G)$, so $G^1(X) \to G^2(X)$ is surjective, the exactness at $G^0(X)$, $G^1(X)$ and $G(X)$ follows directly from our previous discussions. So $C^*(X;G)$ is exact in the strong sense.

Let $Z^1(-;G) \subset G^1$ be the sheaf theoretic orbit of the base point under the action of G^0 in the Zariski topology on $\tilde{Sm}_k$. So we will have an exact sequence of sheaves on $\tilde{Sm}_k$ in the Zariski topology

$1 \to G \subset G^0 \Rightarrow Z^1(-;G) \to *$ (the exact sequenceof sheaves comes from the local exactness).

G^0 is flasque, so $H^1_{Zar}(X;G^0)$ is trivial. So for any $X \in Sm_k$ we have an exact sequence of groups and pointed sets in the strong sense

$1 \to G(X) \subset G^0(X) \Rightarrow Z^1(X;G) \to H^1_{Zar}(X;G) \to *$.

Remark 3. If X is a localisation of smooth k-scheme at a point of *codimension* ≤ 1, Then $Z^1(X;G) = G^1(X)$ and hence $H^1_{Zar}(X;G) = G^0(X) \setminus G^1(X)$.We can generalise the result for X smooth of dimension 1 or any open subset of *dimension* ≤ 1 of the localisation of a smooth scheme X (since $G^1(X)$ is a sheaf in Zariski topology and for localisaion at the *codimension* ≤ 1 point $Z^1(X;G) = G^1(X)$). So if X is a smooth local k-scheme of diemnsion 2 , and $V \subset X$ is the compliment of the closed point , a k-scheme of dimension 1 . We have $G^i(X) = G^i(V)$ for $i \leq 1$ and $H^2_z(X;G) = H^1_{Zar}(V;G)$.

Let $K^1(X;G) \subset \Pi'_{y \in X^1} H^1_y(X;G)$ be the kernel of the boundary map $\Pi'_{y \in X^1} H^1_y(X;G) \to \Pi_{z \in X^2} H^2_z(X;G)$ for any $X \in Sm_k$. Being the kernel of Zariski sheaves the sheaf $X \mapsto K^1(X;G)$ is a sheaf in the Zariski topology on $\tilde{Sm_k}$. If X is a localisation of a smooth scheme then $Z^1(X;G) \to K^1(X;G)$ is injective hence on $\tilde{Sm_k}$ we have an injective morphism $Z^1(-;G) \to K^1(-;G)$. Since for any X local of *dimension* ≤ 2 the complex $C^*(X;G)$ is exact hence for X smooth of *dimension* ≤ 2 we have bijection between $Z^1(X;G) \to K^1(X;G)$, since we have bijection at all the localisations.

Remark 4. It follows from the previous descriptions that for any smooth(or localisation of a smooth scheme) scheme of *dimension* ≤ 2 the H^1 of the complex $C^*(X;G)$ is $H^1_{Zar}(X;G)$.

We will add two more axioms on G

(A5) (a) For any separable field extension $E \subset F$ in F_k , any discrete valuation v on F which restricts to a discrete valuation w on E wih ramification index 1 , and such that the induced extension $\bar{i} : \kappa(w) \to \kappa(v)$ is an isomorphism , the commutative square of groups

$$\begin{array}{ccc} G(O_w) & \longrightarrow & G(O_v) \\ \downarrow & & \downarrow \\ G(E) & \longrightarrow & G(F) \end{array}$$

induces a bijection between $H^1_v(O_v;G) \to H^1_w(O_w;G)$.

(b) For any *étale* morphism $X' \to X$ between smooth local k-schemes of dimension 2 , with closed points z' and z respectively , such that the induced morphism $\kappa(z) \to \kappa(z')$ is an isomorphism , the pointed map
$H^2_z(X;G) \to H^2_{z'}(X';G)$ has trivial kernel .

Lemma 4.2.4. *Let G be an unramified sheaf of groups which satisfies (A2)'. Then the following are equivalent :*

1. *The Zariski sheaf $X \mapsto K^1(X;G)$ is a sheaf in Nisnevich topology on $\tilde{Sm_k}$;*

2. *For any smooth k-scheme X of dimension ≤ 2 the comparison map $H^1_{Zar}(X;G) \to H^1_{Nis}(X;G)$ is a bijection;*

3. *G satisfies axiom A5 .*

Proof. 1 $\Rightarrow$ 2 As $Z^1(X;G) \to K^1(X;G)$ is a bijection for $X \in Sm_k$ and $dim(X) \leq 2$, we have $X \mapsto Z^1(X;G)$ is a sheaf in the nisnevich topology on smooth k-schemes of $dimension \leq 2$. So the exact sequence in the Zariski topology

$1 \to G \subset G^0 \rightrightarrows Z^1(-;G) \to *$

is then a exact sequence in the Nisnevich topology on the site smooth schemes over k of $dimension \leq 2$. Now $H^1_{Nis}(X;G^0)$ is trivial (A.0.8). Hence the comparison map $H^1_{Zar}(X;G) \to H^1_{Nis}(X;G)$ is a bijection (by remark 4).

2 $\Rightarrow$ 3 A5 a

By 2. of the lemma and by Appnedix A implies that this H^1 set can also be defined using Nisnevich-torsors. So we get the canonical isomorphism $H^1_v((O_v)_{Nis}, G) \to H^1_w((O_w)_{Nis}; G)$.

A5 b Let $V = X - z$ and $V' = X' - z'$. The follwoing square is distinguished

$$\begin{array}{ccc} V' & \longrightarrow & X' \\ \downarrow & & \downarrow \\ V & \longrightarrow & X \end{array}$$

We have shown that $H^2_z(X;G) = H^1_{Zar}(V;G)$, so the kernel of the map $H^2_z(X;G) \to H^2_{z'}(X';G)$ is the set of G-torsors over V (which are indifferent as $H^1_{Zar}(X;G) \cong H^1_{Nis}(X;G)$) which becomes trivial over V' , but such a torsor can be extended to the trvial torsor of X' , so this torsor can be extended from V to X (by gluing the torsor over X and torsor over V). Since X is local , $H^1_{Nis}(X;G) = H^1_{Zar}(X;G)$ is zero, hence the extension to X is trivial.

3 $\Rightarrow$ 1 A 5 a implies that $X \mapsto G^1(X)$ is a sheaf in Nisnevich topology (using elementary dintinguished Nisnevich squares) , and A5B implies that G^2 is very closed to being a separated presheaf in Nisnevich topology (a section that vanishes locally vanishes). Since $K^1(-;G)$ is the kernel of the map $G^1 \to G^2$ is then a sheaf in Nisnevich topology.

□

Lemma 4.2.5. *Assume G satisfies A5. Fix an integer $d \geq 0$.The following conditions are equivalent :*

1. *For any smooth k-scheme X of dimension $\leq d$ the map $Z^1(X;G) \to K^1(X;G)$ is bijective;*
2. *For any local smooth k-scheme of dimension $\leq d$ the map $Z^1(X;G) \to K^1(X;G)$ is bijective;*
3. *For any local smooth k-scheme U of dimension $\leq d$ with function field F, the complex $C^*(X;G)$*

$$1 \to G(U) \to G^0(U) \implies G^1(U) \to G^2(U)$$

is exact.

Moreover if G satisfies any one of the above we have $H^1_{Zar}(X;G) \to H^i_{Nis}(X;G)$ is a bijection for any smooth k-scheme of dimension d.

Proof. $1 \iff 2$ is clear as both are Zariski sheaves so the map is bijective iff sections over localisation at any point is bijective . $2 \Rightarrow 3$ comes from the fact that the kernel of the map $G^1(U) \to G^2(U)$ is equal to $Z^1(U;G)$. $3 \Rightarrow 1$ First of all by the exactness of the sequence we have for any X (localisation of a smooth k-scheme at a point of *codimension* $\leq d$) the kernel $K^1(X;G) = G^0(X) \setminus G^1(X) = Z^1(X;G)$, So the injective map between Zariski sheaves $Z^1 \to K^1$ over the site of smooth k scheme of *dimension* $\leq d$ in Zariski topology induces isomorphism at every stalk hence it is an isomorphism.

We have an exact sequence $1 \to G \subset G^0 \Rightarrow K^1(-;G) \to *$ of Nisnevich sheaves , and moreover $H^1_{Nis}(X;G^0)$ is trivial , so we get an exact sequence

$1 \to G(X) \subset G^0(X) \Rightarrow K^1(X;G) \to H^1_{Nis}(X;G) \to *$. Hence $H^1_{Nis}(X;G) = G^0 \setminus K^1(X;G)$, and we have $K^1(X;G) = Z^1(X;G)$. So we have the following

$G^0(X) \setminus Z^1(X;G) = H^1_{Zar}(X;G) \to H^i_{Nis}(X;G) = G^0 \setminus K^1(X;G)$ is a bijection. □

Let us give another axiom related to A^1-invariance property :

(A6) For any localisation U of a smooth k-scheme at some point u of *codimension* ≤ 1, the complex

$$1 \to G(A^1_U) \to G^0(A^1_U) \implies G^1(A^1_U) \to G^2(A^1_U)$$

is exact. Moreover the morphism $G(U) \to G(A^1_U)$ is an isomorphism .

Theorem 4.2.6. *Assume k is infinite . Let G be an unramified sheaf of groups on Sm_k that satisfies the axioms (A2 ') , (A5) , (A6) . Then it is strongly A^1-invariant . Moreover , for any smooth k-scheme the comparison map $H^1_{Zar}(X;G) \to H^1_{Nis}(X;G)$ is a bijection .*

Proof. The proof will follow from next three lemmas.

Lemma 4.2.7. *Assume G is A^1-invariant . Fix an integer $d \geq 0$. The following conditions are equivalent :*

1. *For any smooth k-scheme X of dimension $\leq d$ the map $G^0(X) \setminus Z^1(X;G) = H^1_{Zar}(X;G) \to H^1_{Zar}(A^1_X;G) = G^0(A^1_X) \setminus Z^1(A^1_X;G)$ is bijective .*
2. *For any local smooth k-scheme U of dimension $\leq d$*

   $G^0(A^1_U) \setminus Z^1(A^1_U;G) = *$

Proof. To show $1 \Rightarrow 2$, we notice that U is a local k-scheme, so $H^1_{Zar}(X;G) = G^0(U) \setminus Z^1(U;G) = *$.

To show $2 \Rightarrow 1$,

If $U = \{U_i\}$ is an open covering of X, then $H^1(\{U_i\};G) \to H^1(\{A^1 \times U_i\};G)$ is a bijection, because for any open V of X (for instance, $V = U_i$, $V = U_i \cap U_j$) we have $G(V) = G(A^1 \times V)$

so the description of these two sets with cocycles are the same. ($H^1(\{U_i\};G)$ is the subset of $H^1(X;G)$ made of isomorphism classes of G-torsors that are trivial over each U_i. For any G-torsor, there exists such a covering (U_i).) We have to show that for any G-torsor T over $A^1 \times X$, there exists $\{U_i\}$ an open covering of X such that T is trivial over $A^1 \times U_i$. But (2) shows that this is true locally, so there exists such a covering. □

Lemma 4.2.8. *Let G be an unramified sheaf of groups satisfying (A2 '), (A5), (A6).*

1. *Let v be a discrete valuation on $F \in F_k$. Let $v[T]$ denote the discrete valuation in $F(T)$ corresponding to the kernel of $O_v[T] \to \kappa(v)[T]$. Then the map*
   $H^1_v(O_v;G) \to H^1_{v[T]}(A^1_{O_v};G)$
   is injective and its image is exactly the kernel of
   $H^1_{v[T]}(A^1_{O_v};G) \to \Pi'_{z\in(A^1_{\kappa(v)})^1} H^2_z(A^1_{O_v};G)$.
2. *For any k-smooth local scheme U of dimension 2 , with closed point u , the kernel of the map*
   $H^2_u(U;G) \to H^2_{u[T]}(U;G)$
   is trivial .

Proof. [MO1] lemma 1.30 page 33. □

For any integer $d \geq 0$ and for G as in theorem 4.2.6 we introduce two new properties

(H1)(d) For any local smooth scheme of *dimension* $\leq d$ the complex

$$1 \to G(U) \to G^0(U) \implies G^1(U) \to G^2(U)$$

is exact.

(H2)(d) For any localisation U of a smooth k-scheme at some point u of *codimension* $\leq d$, the complex

$$1 \to G(A^1_U) \to G^0(A^1_U) \implies G^1(A^1_U) \to G^2(A^1_U)$$

is exact.

H1(d) is proved for $d \leq 2$, and by A6, H2 1 holds.

Lemma 4.2.9. 1. *$(H1)(d) \Rightarrow (H2)(d)$.*

2. *If k is infinite : $(H2)(d) \Rightarrow (H1)(d+1)$.*

Proof. To prove 1 let us assume that U be an irreducible smooth k-scheme with function field F.

$$
\begin{array}{ccccccc}
G(U) & \longrightarrow & G(F) & \longrightarrow & \Pi'_{y\in U^1} H^1_y(U;G) & \longrightarrow & \Pi'_{z\in U^2} H^2_z(U;G) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
G(A^1_U) & \longrightarrow & G(F(T)) & \longrightarrow & \Pi'_{y\in (A^1_U)^1} H^1_y(A^1_U;G) & \longrightarrow & \Pi'_{z\in (A^1_U)^2} H^2_z(A^1_U;G) \\
\downarrow & & \downarrow & & \downarrow & & \\
G(F) & \longrightarrow & G(F(T)) & \longrightarrow & \Pi'_{y\in (A^1_F)^1} H^1_y(A^1_F;G) & &
\end{array}
\tag{4.2}
$$

By the axiom A6 the bottom row is exact. G is A^1-invariant by axiom A6 and lemma 3.2.13, so $G(U) \to G(A^1_U)$ is a bijection. If U is local of *dimension* $\leq d$, then by (H1)(d) the topmost horizontal row is exact. The points y of codimension 1 in A^1_U are of two types, either the image of y in U is a generic point of U or it is a codimension 1 point of U. Set of those points of codimension 1 in A^1_U such that the image is the generic point is in bijection with $(A^1_F)^1$ and the set of points whose image is codimension 1 point in U is in bijection with U^1 given by the map $y \in U^1 \mapsto y[T] := A^1_{\bar{y}} \subset A^1_U$(that is the generic point of $A^1_{\bar{y}}$). So for y of the first type the set $H^1_y(U;G) = H^1_y(A^1_F;G)$ and the set $\Pi'_{y\in(A^1_U)^1} H^1_y(A^1_U;G)$ is the product of $\Pi'_{y\in(A^1_F)^1} H^1_y(A^1_F;G)$ with $\Pi'_{y\in(U)^1} H^1_{y[T]}(A^1_U;G)$.

To prove (H2)(d) we have to prove exactness of the middle row, or to show that the action of $G(F(T))$ on $K^1(A^1_U;G)$ is transitive.

Let $a \in K^1(A^1_U;G)$. The bottom horizontal row is exact, so $\Pi'_{y\in(A^1_F)^1} H^1_y(A^1_F;G)$ one orbit under the action of $G(F(T))$, now the map $\Pi'_{y\in(A^1_U)^1} H^1_y(A^1_U;G) \to \Pi'_{y\in(A^1_F)^1} H^1_y(A^1_F;G)$ is $G(F(T))$ invariant and the kernel is $\Pi'_{y\in(U)^1} H^1_{y[T]}(A^1_U;G)$, hence there exists a $g \in G(F(T))$ such that $g.a \in \Pi'_{y\in(U)^1} H^1_{y[T]}(A^1_U;G)$.

So $g.a$ is inside $K^1(A^1_U;G) \cap \Pi'_{y\in(U)^1} H^1_{y[T]}(A^1_U;G) \subset \Pi'_{y\in(A^1_U)^1} H^1_y(A^1_U;G)$. By the first part of the previous lemma $\Pi'_{y\in U^1} H^1_y(U;G) \to \Pi'_{y\in(U)^1} H^1_{y[T]}(A^1_U;G)$ is injective and the image is exactly the kernel of the composition of the boundary map $\Pi'_{y\in(U)^1} H^1_{y[T]}(A^1_U;G) \to \Pi_{z\in(A^1_U)^2} H^2_z(A^1_U;G)$ and the projection $\Pi_{z\in(A^1_U)^2} H^2_z(A^1_U;G) \to \Pi_{y\in U^1, z\in(A^1_{\bar{y}})^1} H^2_z(A^1_U;G)$. This shows that $K^1(A^1_U;G) \cap \Pi'_{y\in U^1} H^1_{y[T]}(A^1_U;G)$ is contained in $\Pi'_{y\in U^1} H^1_y(U;G)$. The right vertical map in 4.2, $\Pi'_{z\in U^2} H^2_z(U;G) \to \Pi'_{z\in(A^1_U)^2} H^2_z(A^1_U;G)$ is induced by the correspondance $z \in U^2 \mapsto A^1_{\bar{z}}$. By 2 of the previous lemma this map has trivial kernel. So by the commutativity of the top rightmost square of 4.2 we have $K^1(A^1_U;G) \cap \Pi'_{y\in U^1} H^1_{y[T]}(A^1_U;G)$ is contained in $K^1(U;G)$. So $g.a$ lies in $K^1(U;G)$, but as the topmost row is exact by (H1)(d) we have an $h \in G(F)$ such that $hg.a = *$. Hence the action is transitive.

To prove 2 assume (H2)(d) holds. Let X be an irreducible smooth k-scheme of *dimension* $\leq d+1$ with function field F, let $u \in X$ be a point of codimension $d+1$, let U be the localised scheme at u , F its function field.

Let $a \in K^1(U;G) \in \Pi'_{y\in U^1} H^1_y(U;G)$. We have to show that there exists $g \in G(F)$ such that $a = g.*$. Let $y_i \in U$ the points of codimension one in U such that a has trivial component in $H^1_{y_i}(U;G)$. Now by definition $H^1_y(U;G) = H^1(X;G)$ for $y \in U^1$. Let $a_X \in \Pi'_{y\in X^1} H^1_y(X;G)$ be the unique element with same support y_i's and the same component as a. Though a_X may not be in $K^1(X;G)$, but by axiom $(A2)'$, its boundary is trivial except on finitely many z_j of codimension 2 point of X. Now this points are not in U^2 either. So we can remove the closure of this z_j's to get an open subscheme Ω' in X which contains u and the y_i's and the corresponding element induced by a denoted by $a_{\Omega'} \in \Pi'_{y \in \Omega'^1} H^1_y(X; G)$ is in $K^1(\Omega', G)$.

Since k is inifinite , by Gabber's geometric presentation lemma (see 5.2.8) there exists an open subscheme Ω of Ω', containing u and the y_i's and an *étale* morphism $\Omega \to A^1_V$, With V smooth of dimension d, such that if $Y \subset \Omega$ denotes the reduced closed subscheme whose generic point are the y_i, the composition $Y \to \Omega \to A^1_V$ is still a closed immersion and the composition $Y \to \Omega \to A^1_V \to V$ is a finite morphism.

As U is the localization of Ω at u, the *étale* morphism $U \to A^1_V$ induces a morphism of complexes of the form

$$\begin{array}{ccccc} G(E(T)) & \longrightarrow & \Pi'_{y \in (A^1_V)^1} H^1_y(A^1_V; G) & \longrightarrow & \Pi'_{z \in (A^1_V)^2} H^2_z(A^1_V; G) \\ \downarrow & & \downarrow & & \downarrow \\ G(F) & \longrightarrow & \Pi'_{y \in U^1} H^1_y(U; G) & \longrightarrow & \Pi'_{z \in U^2} H^2_z(U; G) \end{array} \tag{4.3}$$

Here E is the function field of V. Let y'_i are the images of y_i in A^1_V , these are points of codimension 1 and since $Y \to A^1_V$ is a closed immersion they have same residue field. By the axiom (A5)(a), for each i the map $H^1_{y'_i}(A^1_V; G) \to H^1_{y_i}(U; G)$ is a bijection, so there exists an element $a' \in \Pi'_{y \in (A^1_V)^1} H^1_y(A^1_V; G)$ whose image is a. If $z \in (A^1_V)^2$ is not in image of Y then by definition of the boundary map and a' the boundary of a' has trivial component in $H^2_z(A^1_V; G)$. If $z \in (A^1_V)^2$ is in the image of Y in A^1_V, there is an unique point z' of codimension 2 in Ω, lying in Y mapping to z. It has the same residue field as z. So z' gives a codimension 2 point in U. By the commutativity of the rightmost square and (A5)(b) a' has a trivial component in $H^2_z(A^1_V; G)$. Which proves that the boundary of a' is trivial.

As (H2)(d) is satisfied we have $a' = h.*$ for some $h \in G(E(T))$. If g is the image of h in $G(F)$ we have $a = g.*$.

□

Now lets prove theorem 4.2.6 . As k is infinite , using the lemma 4.2.9 and by induction arguement ((H1)(1), (H1)(2) and (H2)(1) is true) it is clear that G satisfies (H1)(d) and (H2)(d) for all d. By (H1)(d) G satisfies the property three of the lemma 4.2.5 hence the map $H^1_{Zar}(X; G) \to H^1_{Nis}(X; G)$ is a bijection for all smooth scheme X of dimension d for all d. So we have to show that condition 1 of lemma 4.2.7 is satisfied. For any smooth scheme X we have $K^1(X; G) = Z^1(X; G)$. Also as G satisfies A6, G is A^1-invariant and it satisfies H2(d), so it satisfies the condition 2 of lemma 4.2.7. Hence the condition 1 is satisfied.

□

Chapter 5

Strongly A^1 invariance of the sheaves $\pi_n^{A^1}, n \geq 1$

5.1 Main Idea

Using the properties of last two chapters we will show that

Result 2. For any pointed space $\mathbb{B}$, its A^1 fundamental sheaf of groups $\pi_1^{A^1}$ is strongly A^1 invariant .

To show that $\pi_1^{A^1}(\mathbb{B})$ satisfies the properties of strongly A^1-invariantness, described in last two chapters, we will study the local properties of $\pi_1^{A^1}(\mathbb{B})$. The only non trivial theorem used here without proof is the following :

Lemma 5.1.1 (Gabbers presentation lemma)**.** *Let X be a smooth , affine , irreducible variety of dimension d over an infinite field k , let $t_1, ..., t_r \in X$ be a finite set of points and Z a closed subvariety of codimension > 0 . Then there exists a map $\varphi = (\psi, \upsilon) : X \to A^{d-1} \times A^1$, an open set $V \subset A^{d-1}$, and an open set $U \subset \psi^{-1}(V)$ containing $t_1, ..., t_r$ such that*

1. *$Z \cap U = Z \cap \psi^{-1}(V)$.*
2. *$\psi \mid_Z$ is finite .*
3. *$\varphi \mid_U$ is étale and defines a closed immersion $Z \cap U \to A^1_V$.*
4. *$\varphi(t_i) \notin \varphi(Z)$ if $t_i \notin Z$.*
5. *$\varphi^{-1}(\varphi(Z \cap U)) \cap U = Z \cap U$.*

5.2 The proof of srongly A^1-invariance of $\pi_1^{A^1}$

Definition 5.2.1. A B.G class of objects in Sm/k is a class A of objects in Sm/k such that :

1. For any object $X \in A$ and any open immersion $U \to X$ we have $U \in A$.

2. Any smooth k-scheme X has a nisnevich covering which consists of objects in A.

In this section by a B.G class we will mean all the objects of Sm_k.

Definition 5.2.2. A simplicial presheaf S on $(Sm/k)_{Nis}$ is said to have the B.G property with respect to A (B.G class of objects) if for any elementary distinguish square (2.3.17) such that X and V belong to A the square of simplicial sets

$$\begin{array}{ccc} S(X) & \longrightarrow & S(V) \\ \downarrow & & \downarrow \\ S(U) & \longrightarrow & S(U \times_X V) \end{array}$$

is homotopy cartesian .

Theorem 5.2.3. *For any pointed space $\mathbb{B}$, its A^1 fundamental sheaf of groups $\pi_1^{A^1}$ is strongly A^1 invariant .*

Proof. We will directly show that $\pi_1^{A^1}$ is unramified and satisfies the axioms (A2'),(A5) and (A6) of section 2. □

Theorem 5.2.4. *Let $\mathbb{B}$ be a pointed simplicial presheaf of sets on Sm_k which satisfies the B.G properties in the nisnevich topology and the A^1 invariance property (see [MOREL] definition A.1.5) . Then the associated sheaf of groups to the presheaf $U \mapsto \pi_1(\mathbb{B}(U))$ is strongly A^1 invariant .*

Lemma 5.2.5. *The last two theorems are equivalent .*

Proof. (using results from [MOREL] appendix A.1 and [MV]) Suppose theorem 5.2.4 is true. The simplicial fibrant resolution $L_{A^1}(\mathbb{B})$ is A^1 local , so it satisfies the propersties of theorrem 5.2.4.

Now the sheaf associated to the presheaf $U \mapsto Hom_{H_\bullet(k)}(\sum(U_+), \mathbb{B})$ is same as the sheaf associated to the presheaf $U \mapsto Hom_{H_s(k)}(\sum(U_+), L_{A^1}(\mathbb{B}))$. But we have equality of sheaves between the sheaf associated to the presheaf $U \mapsto \pi_1(L_{A^1}(\mathbb{B})$ and the sheaf associated to the presheaf $U \mapsto Hom_{H_s(k)}(\sum(U_+), L_{A^1}(\mathbb{B}))$. Hence theorem 5.2.3 is proved .

Now suppose theorem 5.2.3 is true, then for any presheaf $\mathbb{B}$ satisfying properties of theorem 5.2.4, take the sheafification $a(\mathbb{B})$, take the fibrant resolution $L_{A^1}a(\mathbb{B})$, because of [MV], Prop 1.16, page 100, this is simplicially equivalent to $a(\mathbb{B})$ which is simplicially equivalent to $\mathbb{B}$. So using the previous description of different homotopy sheaves of simplicial sets we can prove theorem 5.2.4.

□

Corollary 5.2.6. *For any pointed space $\mathbb{B}$, and any integer $n \geqslant 1$, the A^1- homotopy sheaf of groups $\pi_n^{A^1}(\mathbb{B})$ is strongly A^1-invariant.*

Proof. Take $L_{A^1}(\mathbb{B})$ and take the $(n-1)$th iterated simplicial sheaf $\Omega_s^{n-1}(L_{A^1}(\mathbb{B}))$ (see [MV]). It can be shown that this satisfies the conditions of theorem 5.2.4. Hence applying theorem 5.2.4 we get that the sheaf associated to presheaf $U \mapsto \pi_1(\Omega_s^{n-1}(L_{A^1}(\mathbb{B})))(U))$ is strongly A^1 invariant, but the previous sheaf is isomorphic to the sheaf associated to the presheaf $U \mapsto \pi_n(L_{A^1}\mathbb{B}(U))$. So $\pi_n^{A^1}(\mathbb{B})$ is strongly A^1 invariant.

□

Lemma 5.2.7. *Given a pointed A^1-local space $\mathbb{B}$, the connected component of the base point $\mathbb{B}^0$ is also A^1-local and the morphism*

$\pi_1^{A^1}(\mathbb{B}^0) \to \pi_1^{A^1}(\mathbb{B})$ *is an isomorphism .*

Proof. $L_{A^1}\mathbb{B}^0$ is zero connected if $\forall X \in Sm_k$ and $\forall x \in X$ the simplicial set $L_{A^1}\mathbb{B}^0(O^h_{X,x})$ is connected, as $L_{A^1}\mathbb{B}^0$ is simplicially equivalent to $\mathbb{B}^0$, this shows that $L_{A^1}\mathbb{B}^0$ is 0-connected. We have the following commutative square

$$\begin{array}{ccc} \mathbb{B}^0 & \longrightarrow & \mathbb{B} \\ \downarrow & & \wr\downarrow \\ L_{A^1}\mathbb{B}^0 & \longrightarrow & L_{A^1}\mathbb{B} \end{array}$$

and $L_{A^1}\mathbb{B} = \mathbb{B}$. By assumption $\mathbb{B}^0$ is the connected component of the base point. So the induced map $L_{A^1}\mathbb{B}^0 \to \mathbb{B}$ induces a map $L_{A^1}\mathbb{B}^0 \to \mathbb{B}^0$ giving an inverse to $\mathbb{B}^0 \to L_{A^1}\mathbb{B}^0$. So $\mathbb{B}^0$ is a retract in $H(k)$ of the A^1-local space $L_{A^1}\mathbb{B}^0$, so it is A^1 local .

□

So we can assume that $\mathbb{B}$ is A^1-local and 0-connected . For an open immersion $U \subset X$ and any $n \geqslant 0$ we set

$\Pi_n(X,U) := [S^n \bigwedge (X/U), \mathbb{B}]_{H_*(k)} = \pi_n(\mathbb{B}(X/U))$,

where S^n denotes the simplicial n-sphere . We may extend this to an open immersion $U \to X$ between essentially smooth k-schemes by passing to the colimit.

Lemma 5.2.8 (Gabbers presentation lemma)**.** *Let X be a smooth , affine , irreducible variety of dimension d over an infinite field k , let $t_1, ..., t_r \in X$ be a finite set of points and Z a closed subvariety of codimension > 0 . Then there exists a map $\varphi = (\psi, \upsilon) : X \to A^{d-1} \times A^1$, an open set $V \subset A^{d-1}$, and an open set $U \subset \psi^{-1}(V)$ containing $t_1, ..., t_r$ such that*

1. *$Z \cap U = Z \cap \psi^{-1}(V)$.*
2. *$\psi\mid_Z$ is finite .*
3. *$\varphi\mid_U$ is étale and defines a closed immersion $Z \cap U \to A^1_V$.*
4. *$\varphi(t_i) \notin \varphi(Z)$ if $t_i \notin Z$.*
5. *$\varphi^{-1}(\varphi(Z \cap U)) \cap U = Z \cap U$.*

Lemma 5.2.9. *Assume k is infinite . Let X be a smooth k-scheme , $S \subset X$ be a finite set of points and $Z \subset X$ be a closed subscheme of codimension > 0. Then there exists an open subscheme $\Omega \subset X$ containing S and a closed subscheme $\tilde{Z} \subset \Omega$, of codimension $d-1$, containing $Z_\Omega := Z \cap \Omega$ such that the map of the pointed sheaves $\Omega/(\Omega - \tilde{Z}) \to \Omega/(\Omega - Z_\Omega)$ is the trivial map in $H_\bullet(k)$.*

Proof. By Gabber's geometric presentation lemma there exists an open neighborhood Ω of S , a morphism $\varphi : \Omega \to A^1_V$ with V some open subset in some affine space over k and the morphism is *étale* by 3 of the previous lemma . By 3 again we have $Z_\Omega := Z \cap \Omega \to A^1_V$ closed immersion . By 5 we have $\varphi^{-1}(\varphi(Z_\Omega)) \cap \Omega = Z_\Omega$ and by 2 we have $Z_\Omega \to V$ is a finite morphism . Let F denotes the image of Z_Ω in V . Let $\tilde{Z} := \varphi^{-1}(A^1_F)$. Now $dim(F) = dim(Z)$, thus $codimension(\tilde{Z}) = d-1$.

Since the following square is an elementary distinguish square in Nisnevich topology

$$\begin{array}{ccc} (\Omega - Z_\Omega) & \longrightarrow & \Omega \\ \downarrow & & \downarrow \\ (A^1_V - Z_\Omega) & \longrightarrow & A^1_V \end{array}$$

We have , the isomorphism of sheaves

$\Omega/(\Omega - Z_\Omega) \cong A^1_V/(A^1_V - Z_\Omega)$.

The commutativity of the following square

$$\begin{array}{ccc} \Omega/(\Omega - \tilde{Z}) & \longrightarrow & \Omega/(\Omega - Z_\Omega) \\ \downarrow & & \sim\downarrow \\ A^1_V/(A^1_V - A^1_F) & \longrightarrow & A^1_V/(A^1_V - Z_\Omega) \end{array}$$

shows that it is sufficient to show that the map of pointed sheaves

$A^1_V/(A^1_V - A^1_F) \to A^1_V/(A^1_V - Z_\Omega)$ is the trivial map in $H_\bullet(k)$. Now $Z_\Omega \to F$ is finite , hence the composition $Z_\Omega \to A^1_F \subset P^1_F$ is still a closed immersion , so it has empty intersection with the section at infinity $s_\infty : V \to P^1_V$. Since the following square ,

$$\begin{array}{ccc} (A^1_V - Z_\Omega) & \longrightarrow & A^1_V \\ \downarrow & & \downarrow \\ (P^1_V - Z_\Omega) & \longrightarrow & P^1_V \end{array}$$

is an elementary disnguish square we have $A^1_V/(A^1_V - Z_\Omega) \cong P^1_V/(P^1_V - Z_\Omega)$. So it is sufficient to prove that

$A^1_V/(A^1_V - A^1_F) \cong P^1_V/(P^1_V - Z_\Omega)$ is the trivial map in $H_\bullet(k)$. Now the morphism $s_0 : V/(V - F) \to A^1_V/(A^1_V - A^1_F)$ induced by the zero section is an A^1-weak equivalence.The composition $s_0 : V/(V - F) \to A^1_V/(A^1_V - A^1_F) \to P^1_V/(P^1_V - Z_\Omega)$ is A^1 homotopic to the section at infinity $s_\infty : V/(V - F) \to P^1_V/(P^1_V - Z_\Omega)$. But the image of s_∞ is disjoint from Z_Ω , so $s_\infty : V/(V - F) \to P^1_V/(P^1_V - Z_\Omega)$ is equal to the point. □

Corollary 5.2.10. *Assume k is inifinite . Let X be a smooth (or localisation of a smooth k-scheme) k-scheme, $S \in X$ be a finite set of points and $Z \subset X$ be a closed subscheme of codimension $d > 0$. Then there exists an open subscheme $\Omega \subset X$ containing S and a closed subscheme $\tilde{Z} \subset \Omega$, of codimension $d - 1$, containing $Z_\Omega := Z \cap \Omega$ and such that for any $n \in \mathbb{N}$ the map $\Pi_n(\Omega, \Omega - Z_\Omega) \to \Pi_n(\Omega, \Omega - \tilde{Z})$ is the trivial map . In particular , observe that if Z has codimension 1 and X is irreducible , then $\tilde{Z}$ must be Ω . Thus for any $n \in \mathbb{N}$ the map*

$\Pi_n(\Omega, \Omega - Z_\Omega) \to \Pi_n(\Omega)$

is the trivial map .

Proof. The map between $\Pi_n(\Omega, \Omega - Z_\Omega) \to \Pi_n(\Omega)$ is obatained from the map between $\Omega/(\Omega - \tilde{Z}) \to \Omega/(\Omega - Z_\Omega)$. By the previous theorem $\Omega/(\Omega - \tilde{Z}) \to \Omega/(\Omega - Z_\Omega)$ is the trivial map in $H_\bullet(k)$

. Hence $\Pi_n(\Omega, \Omega - Z_\Omega) \to \Pi_n(\Omega)$ is the trivial map . If X is irreducible and Z has codimension 1 then codimension $\tilde{Z}$ is 0 , and since U is irreducible so it implies $U = \tilde{Z}$. □

Now let X be localisation of a smooth scheme . For any flag of open subschemes $V \subset U \subset X$ of X and from the cartesian daigram

$$\begin{array}{ccc} \mathbb{B}(U/V) & \longrightarrow & \mathbb{B}(X/V) \\ \downarrow & & \downarrow \\ pt & \longrightarrow & \mathbb{B}(X/U) \end{array}$$

We get the long exact sequence of homotopy groups

$$\cdots \to \Pi_1(X,U) \to \Pi_1(X,V) \to \Pi_1(U,V) \to \Pi_0(X,U) \to \Pi_0(X,V) \to \Pi_0(U,V) \tag{5.1}$$

where the exactness at $\Pi_0(X,V)$ is the exactness in the sense of pointed sets , and at $\Pi_0(X,U)$ there is an action of the group $\Pi_1(X,U)$ on the set $\Pi_0(X,U)$, so the exactness there is usual exactness of group action . The exactness elsewhere is exactness for group diagrams .

Let X be the localisation of a smooth k-scheme at a point x, and x be its closed point . For any flag $\mathbb{F} : Z^2 \subset Z^1 \subset X$ of closed reduced subschemes , with Z^i of codimension at least i . Let $U_i = X - Z^i$, so we get a corresponding flag of open subschemes $U_1 \subset U_2 \subset X$. The set $\mathbb{F}$ of flags is ordered by increasing inclusion of closed subschemes. So if $U = U_1$ and $V = \emptyset$ we get an exact sequence :

$$\cdots \to \Pi_1(X,U_1) \to \Pi_1(X) \to \Pi_1(U_1) \to \Pi_0(X,U_1) \to \Pi_0(X) \to \Pi_0(U_1) \tag{5.2}$$

Let S is the set containing only the closed point x and, using the previous lemma, we see that $\Omega = X$ (since X local and Ω contains the closed point), and we get that the maps for any n $\Pi_n(X,U_1) \to \Pi_n(X)$ are trivial .

So we get a short exact sequence

$$1 \to \Pi_1(X) \to \Pi_1(U_1) \to \Pi_0(X,U_1) \to * \tag{5.3}$$

and a map of pointed sets $\Pi_0(X) \to \Pi_0(U_1)$ which has trivial kernel . Taking the right filtering colimit on flags we get a short exact sequence

$$1 \to \Pi_1(X) \to \Pi_1(F) \to colim(\Pi_0(X,U_1)) \to * \tag{5.4}$$

and a pointed map with trivial kernel $\Pi_0(X) \to \Pi_0(F)$, where F is the field of functions of X . Now $\mathbb{B}$ is 0-conected so we have $\Pi_0(F) = *$ and hence $\Pi_0(X) = *$.

Consider for any flag of open subschemes $U_1 \subset U_2 \subset X$

$$\cdots \to \Pi_0(X,U_2) \to \Pi_0(X,U_1) \to \Pi_0(U_2,U_1) \tag{5.5}$$

Again we can apply the previous corollary to X, with S as the set with the closed point and to the closed subset $Z^2 \subset X$, again $\Omega = X$ and there exists $\tilde{Z} \subset X$ of codimension 1 , containig Z^2 such that $\Pi_0(X,U_2) \to \Pi_0(X, X - \tilde{Z})$ is the trivial map . Define a new flag $\bar{\mathbb{F}} : \bar{Z}^2 \subset \bar{Z}^1 \subset X$ by setting $\bar{Z}^2 = Z^2$ and $\bar{Z}^1 = Z^1 \bigcup \tilde{Z}$, then the map $colim\Pi_0(X, U_2) \to colim\Pi_0(X, U_1)$ is trivial. So using the exact sequence 5.5 we find that the map

$$colim\Pi_0(X, U_1) \to colim\Pi_0(U_2, U_1) \tag{5.6}$$

has trivial kernel. Now using the exact sequence related to the flags of opensets of the form $\emptyset \subset U_1 \subset U_2$ we can get a action of $\Pi_1(F)$ on $colim\Pi_0(U_2, U_1)$ such that the map 5.6 is $\Pi_1(F)$ equivariant . But by equation 5.4 $colim\Pi_0(X, U_1)$ is one orbit under $\Pi_1(F)$ hence the map $colim\Pi_0(X, U_1) \to colim\Pi_0(U_2, U_1)$ is injective.

So we have shown that if k is an infinite field and X is localisation of a smooth k-scheme with function field F . The natural sequence

$$1 \to \Pi_1(X) \to \Pi_1(F) \Rightarrow colim\Pi_0(U_2, U_1) \tag{5.7}$$

is exact, where the double arrow means group action .

If X is localization at a point x of codimension 1. There exists only one non empty closed subset of positive codimension, which is the closed point x. Hence the set $colim\Pi_0(U_2, U_1)$ reduces to the $\Pi_1(F)$-set $\Pi_0(X, X - x)$. Using the exact sequence 5.3 we find that the action of $\Pi_1(F)$ on $\Pi_0(X, X - x)$ is transitive and $\Pi_0(X, X - x) = \Pi_1(F)/\Pi_1(X)$. Let us denote $\Pi_0(X, X - x)$ by $H^1_y(X; \Pi_1)$.

Let $X' \to X$ be *étale* morphism between smooth local k-schemes . This will induce morphism of corresponding associated exact sequences

$$\begin{array}{ccccccc} 1 & \longrightarrow & \Pi_1(X) & \longrightarrow & \Pi_1(F) \Rightarrow & & colim\Pi_0(U_2, U_1) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \Pi_1(X') & \longrightarrow & \Pi_1(F') \Rightarrow & & colim\Pi_0(U'_2, U'_1) \end{array} \tag{5.8}$$

Now if we allow X' to be the localisation of X at codimension one points then we get a $\Pi_1(F)$ invariant map

$$colim\Pi_0(U_2, U_1) \to \Pi_{y \in X^1} H^1_y(X; \Pi_1) \tag{5.9}$$

Lemma 5.2.11. *The above map is injective and its image is the weak product , giving a bijection :* $colim\Pi_0(U_2, U_1) \to \Pi'_{y \in X^1} H^1_y(\Pi_1)$

Proof. Let $x \in colim\Pi_0(U_2, U_1)$. Hence x is represented by say $(x', \mathbb{F})$ where $\mathbb{F}$ is a flag but now for the flag $\mathbb{F}$ we have $U_1 \subset U_2 \subset X$ and if U_1 contains a point of codimnsion one y then the inverse image of U_1 is the whole X' (where X' is the localisation of X at y) , Hence $(x', \mathbb{F})$ gets map to $\Pi_0(X', X') = *$ whenever $y \in U_1$, but by the noetherian property there are only finitely many codimensional one point outside U_1 hence the image of the map is inside $\Pi'_{y \in X^1} H^1_y(\Pi_1)$. ∎

Corollary 5.2.12. *Let k is an inifinte field*

1. *Let X be smooth local k-scheme with function field F . Then the natural sequence :*

   $1 \to \Pi_1(X) \to \Pi_1(F) \Rightarrow \Pi'_{y \in X^1} H^1_y(X; \Pi_1)$ *is exact .*

2. *The Zariski sheaf associated with* $X \mapsto \Pi_1(X)$ *is a sheaf in the nisnevcih topology and coincides with* $\pi_1^{A^1}(\mathbb{B})$ *, which is thus unramified .*

Proof. 1. this follows directly from the previous lemma and the the exact sequence 5.7 .

2. Let G be the sheaf $(\Pi_1)_{Zar}$. If X is local then by the property of Zariski sheaves $G(X) = \Pi_1(X)$. By 1 , take the sheaf associated to the presheaf $X \mapsto \Pi_{y \in X^0} G(\kappa(y))$ and the sheaf $X \mapsto \Pi'_{y \in X^1} H^1_y(X; \Pi_1)$. After localisation by part 1 we get the exact sequence $1 \to \Pi_1(X) \to \Pi_1(F) \Rightarrow \Pi'_{y \in X^1} H^1_y(X; \Pi_1)$, so the exact sequence of sheaves , global sections are left exact, hence for any k-smooth X irreducible with function field F we get an exact sequence :
$$1 \to G(X) \to G(F) \Rightarrow \Pi'_{y \in X^1} H^1_y(X; G). \tag{5.10}$$
If X is local of dimension 1 with closed point y, by exact sequence 5.3 we get that $H^1_y(X; G) = H^1_y(X; \Pi_1) = H^1_{Nis}(X, X - y; \pi_1(\mathbb{B}))$.

If $V \to X$ is an *étale* morphism between local k-schemes(localisation of smooth k-schemes) of dimension 1, with closed point y' and y respectively , and with the same residue fields $\kappa(y) = \kappa(y')$, the map
$$H^1_{Nis}(X, X - y; \pi_1(\mathbb{B})) \to H^1_{Nis}(V, V - y'; \pi_1(\mathbb{B})) \tag{5.11}$$
is bijective .

Using elementary distinguish square and the last isomorphism it can be shown that $X \mapsto \Pi'_{y \in X^1} H^1_y(X; \Pi_1)$ is a sheaf in the Nisnevich topology on $\tilde{Sm}_k$.

Using the exact sequence 5.10 we see that $X \to G(X)$ is a sheaf in nisnevich topology since it is the kernel of the morphism between two Nisnevich sheaves . But now the map $\Pi_1 \to G$ induces isomorphism between $a(\Pi_1)_{Zar} \to G$, so it induces isomorphism between $a(\Pi_1)_{Nis} \to G_{Nis}$ but G is already a Nisnevich sheaf and $a(\Pi_1)_{Nis} = \pi_1^{A^1}(\mathbb{B})$. Hence the morphism $\pi_1^{A^1}(\mathbb{B}) \to G$ is an isomorphism .

To show that G is unramified , we can show that G defines an unramified F_k set . D1 can be constructed as in the proof of theorem 3.2.7 section 1. It satisfies D2 follows from exact sequence 5.7 . Moreover it satisfies A1 follows from 5.11 . To show A2 consider the exact sequence 5.10 , If $y \in \cap_{x \in X^1} G(O_{X,x})$ then y fixes the point of $\Pi'_{y \in X^1} H^1_y(X; \Pi_1)$, hence y is in $G(X)$. So like the proof of axiom A2 in theorem 3.2.7 we can prove that G satisfies the axiom A2. So G is unramified.

□

Theorem 5.2.13. *G is strongly A^1 invariant .*

Proof. To prove the theorem we will directly show that G satisfies (A2)' , (A5) and (A6) . Axiom (A5)(a) follows directly from the bijection 5.11 and the description of $H^1_v(X; G)$ for X local of dimension 1 with closed point v. Now we have the following exact sequence

$1 \to G(X) \to G(F) \Rightarrow \Pi'_{y \in X^1} H^1_y(X; G)$ for X dimension 1 smooth k schemes, so this gives an exact sequence of sheaves of *dimension* ≤ 1 in Nisnevich and Zariski topology. By 5.11

$X \mapsto \Pi'_{y\in X^1} H^1_y(X;G)$ is flasque in Nisnevich topology. So we get for any smooth k-scheme V of *dimension* ≤ 1 a bijection

$H^1_{Zar}(V;G) = H^1_{Nis}(V;G) = G(F) \setminus \Pi'_{y\in X^1} H^1_y(X;G)$.

For X a smooth local k-scheme of dimension 2 with closed point z and $V = X - z$, We get $H^1_{Nis}(V;G) = H^2_z(X;G)$. Using the same method as in lemma 4.2.4 section 3 we get Axiom (A5) (b)

To prove (A2') observe that

$colim \Pi_0(U_2, U_1) \cong \Pi'_{y\in X^1} H^1_y(X;G)$

for any smooth k-scheme X .

Let $z \in X^2$ and X_z be the localisation of X at z , moreover let $V_z = X_z - z$. We have shown earlier that $H^1_{Zar}(V;G) = H^1_{Nis}(V;G) = H^2_z(X;G)$. And by part 2 of the next lemma we have $H^1_{Nis}(V;G) = \Pi_0(V_z) = [(V_z)_+, \mathbb{B}]_{H_\bullet(k)}$, since $\mathbb{B}$ is A^1 local and fibrant and V_z is a smooth scheme of dimension 1.

For a fixed flag $\mathbb{F}$ in X , by definition , the composition $\Pi_0(U_2, U_1) \to \Pi(U_2) \to H^2_z(X;G)$ is trivial if $z \in U_2$. So given an element of $\Pi'_{y\in X^1} H^1_y(X;G)$ which comes from $\Pi_0(U_2, U_1)$, its boundary to $H^z_2(X;G)$ at points of codimension 2 are trivial except for those z not in U_2 , but there are only finitely many such z's . This proves $(A2')$.

To show A6 we first observe that using the first part of the next lemma for any field $F \in F_k$, the map $[\sum((A^1_F)_+), \mathbb{B}]_{H_\bullet(k)} \to [\sum((A^1_F)_+), B(G)]_{H_\bullet(k)} = G(A^1_F)$ is onto. As $\mathbb{B}$ is A^1 local, $[\sum((A^1_F)_+), \mathbb{B}]_{H_\bullet(k)} = [\sum((F)_+), \mathbb{B}]_{H_\bullet(k)} = G(F)$ and this show that the map $G(F) \to G(A^1_F)$ is onto, it is injective, so it is an isomorphism. So G is A^1 invariant.

Now using the second part of the next lemma for any k-scheme X which comes from the localisation of a smooth scheme at a point of *codimension* ≤ 1, the map $[((A^1_X)_+), \mathbb{B}]_{H_\bullet(k)} \to [((A^1_X)_+), B(G)]_{H_\bullet(k)} = H^1_{Nis}(A^1_X;G)$ is onto. As $\mathbb{B}$ is 0-connected and A^1-local, this shows that $H^1_{Nis}(A^1_X;G) = *$. Now G satisfies A5 hence $H^1_{Nis}(A^1_X;G) = * = H^1_{Zar}(A^1_X;G)$. So the complex $C^*(X;G)$ (described in chapter 3) is exact which proves A6.

□

Lemma 5.2.14. *1. For any smooth k-scheme X of dimension ≤ 1 , the map $[\sum((X)_+), \mathbb{B}]_{H_\bullet(k)} \to [\sum((X)_+), B(G)]_{H_\bullet(k)} = G(X)$*

2. For any smooth k-scheme X of dimension ≤ 2 the map

$[(X)_+), \mathbb{B}]_{H_{s,\bullet(k)}} \to [(X)_+), B(G)]_{H_{s,\bullet(k)}} = H^1_{Nis}(X;G)$

is onto and it is surjective if $dim(X) \leq 1$.

Where $G = \pi_1(\mathbb{B})$.

Proof. Appendix B [MOREL]

□

Chapter 6

A^1-coverings, $\pi_1^{A^1}(P^n)$ and $\pi_1^{A^1}(SL_n)$

In this chapter we will first define simplicial covering and A^1 covering for sheaves of simplicial sets on Sm/k (6.1.1). Then we will show that like normal algebraic topology there exists universal simplicial and A^1 covering by 6.1.5 and 6.1.8, we will also include a weak version of Van Kampen's theorem in A^1 setting (6.1.10). Our main aim is to compute some A^1 fundamental groups, and for that we show the following two theorems:

Result 3. For $n \geq 2$ the canonical $\mathbb{G}_m$-torsor $\mathbb{A}^{n+1} \setminus \{0\}) \to \mathbb{P}^n$ is the universal A^1-covering of $\mathbb{P}^n$. This defines a canonical isomorhism $\pi_1^{A^1}(\mathbb{P}^n) \cong \mathbb{G}_m$.

Where G_m is the simplcial sheaf concentrated at degree 0 represented by $A^1 \setminus \{0\}$. And we also have

Result 4. We have $\pi_1^{A^1}(SL_2) \cong \underline{K}_2^{MW}$.

Using the last result and lemma 6.2.4 and lemma 6.2.5 we can describe the A^1-fundamental groups of SL_n, for all $n \geq 2$.

Also we will show that:

Result 5. The fundamental group sheaf $\pi_1^{A^1}(P^1)$ is not abelian.

For the computation part we use $\underline{K}_2^{MW}$ and $\underline{K}_2^{M}$, which are Milnor Witt K theory sheaves of weight 2 and Milnor K-theory sheaves of weight 2 respectively. For details see [MO1, section 2].

6.1 A^1-coverings, universal A^1-coverings and $\pi_1^{A^1}$

Definition 6.1.1. A simplicial covering (resp. an A^1-covering) $\mathfrak{Y} \to \mathfrak{X}$ is a morphism of spaces which has the unique right lifting property with respect to simplicially (resp. A^1) trivial cofibrations.

Lemma 6.1.2. *A morphism $\mathfrak{Y} \to \mathfrak{X}$ is a simplicial (resp A^1) covering if and only if it has the unique right lifting property with respect to any simplicial(resp. A^1) weak equivalences.*

Proof. First of all it is enough to show that a simplicial covering has unique right lifting property with respect to any simplicial weak equivalences. Let $\mathfrak{Y} \to \mathfrak{X}$ be the map and let $\mathfrak{A} \to \mathfrak{B}$ is a simplicial weak equivalence , also let us assume that we have the following commutative diagram

$$\begin{array}{ccc} \mathfrak{A} & \longrightarrow & \mathfrak{Y} \\ \downarrow & & \downarrow \\ \mathfrak{B} & \longrightarrow & \mathfrak{X} \end{array}$$

Now using the properties of closed model category we can write the map $\mathfrak{A} \to \mathfrak{B}$ as composition of $\mathfrak{A} \to \mathfrak{C}$ a trivial cofibration and $\mathfrak{A} \to \mathfrak{B}$ a trivial fibration. Since $\mathfrak{Y} \to \mathfrak{X}$ is a simplicial covering so it has the unique right lifting property with respect to trivila cofibartion, so we can assume that $\mathfrak{A} \to \mathfrak{B}$ is a trivial fibration. Uniqueness of the lifting map comes from the fact that trivial fibrations are epimorphisms of spaces (?). Next we have that $\mathfrak{A}$ and $\mathfrak{B}$ are cofibrant(by definition of the model category structure i both cases). Again using the property of trivial fibration (right lifting property with respect to cofibration) we can get a map $i : \mathfrak{B} \to \mathfrak{A}$ which is a section of $\mathfrak{A} \to \mathfrak{B}$ and i is a trivial cofibration. We have the following diagram

$$\begin{array}{ccc} \mathfrak{B} & \xrightarrow{f \circ i} & \mathfrak{Y} \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle g} \\ \mathfrak{A} & \xrightarrow{g \circ f \circ i \circ \pi} & \mathfrak{X} \end{array}$$

Where $\pi : \mathfrak{A} \to \mathfrak{B}$. As i is a trivial cofibration we get $f \circ i \circ \pi = f$ hence fro the diagram

$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow{f} & \mathfrak{Y} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle g} \\ \mathfrak{B} & \longrightarrow & \mathfrak{X} \end{array}$$

has a solution which is $f \circ i : \mathfrak{B} \to \mathfrak{Y}$. □

Remark 5. A morphism $Y \to X$ in Sm_k, with X irreducible, is a simplicial covering if and only if Y is a disjoint union of copies of X maping identically to X.

6.1.1 The simplicial theory

Lemma 6.1.3. *If $\mathfrak{Y} \to \mathfrak{X}$ is a simplicial covering then for each $x \in X \in Sm_k$ the morphism of simplicial sets $\mathfrak{Y}_x \to \mathfrak{X}_x$ is a covering of simplicial sets.*

Proof. Let $\wedge^{n,i} \subset \triangle^n$ be the union of all faces of $\triangle^n$ except the i-th face for $i \in \{0, .., n\}$. By the theory of simplicial sets we know that $\wedge^{n,i} \subset \triangle^n$ is an weak equivalence and moreover we have to show that $\mathfrak{Y}_x \to \mathfrak{X}_x$ has right lifting property with respect to $\wedge^{n,i} \subset \triangle^n$. Let $x \in X$ and U be a Nisnevich neighborhood of x then since $\mathfrak{Y} \to \mathfrak{X}$ is a simplicial covering we have the desired right lifting property of $\mathfrak{Y} \to \mathfrak{X}$ with respect to $\wedge^{n,i} \times U \subset \triangle^n \times U$. Hence the map $\mathfrak{Y}_x \to \mathfrak{X}_x$ has the right lifting property with respect to $\wedge^{n,i} \subset \triangle^n$.

□

Definition 6.1.4. Let $\mathfrak{X}$ be a simplicial sheaf over $(Sm/k)_{Nis}$. A simplicial sheaf $\tilde{\mathfrak{X}} \to \mathfrak{X}$ over $(Sm/k)_{Nis}$ is called a simplicial universal covering of $\mathfrak{X}$ if $\tilde{\mathfrak{X}} \to \mathfrak{X}$ is a simplicial covering and $\tilde{\mathfrak{X}}$ simplicially 1-connected, moreover if $\mathfrak{X}' \to \mathfrak{X}$ is a covering such that $\mathfrak{X}'$ is 1-connected then $\mathfrak{X}'$ is isomorphic (canonically) to $\tilde{\mathfrak{X}}$. Also it is the universal object (initail) in the category of pointed simplcial covering of $\mathfrak{X}$.

Proposition 6.1.5. *Let $\mathfrak{X}$ be a simplicial sheaf over $(Sm/k)_{Nis}$ such that $\mathfrak{X}$ is connected. there exists an universal covering of $\mathfrak{X}$.*

Proof. Sketch of the proof: Main point of the proof is to use 6. By Postnikov tower, for any pointed simplicially connected space $\mathfrak{X}$, there exist a canonical morphism on $H_{s,\bullet}(k)$ of the form $\mathfrak{X} \to BG$, where G is the group sheaf $\pi_1(\mathfrak{X})$, so by **??** there is a canonical isomorphism class $\tilde{\mathfrak{X}} \to \mathfrak{X}$ of G-torsors. By we can point $\tilde{\mathfrak{X}}$ by lifting the base point of $\mathfrak{X}$. $\tilde{\mathfrak{X}}$ is simplicially 1-connected, which can be shown calculating the stalks and using the classical theory of silmplicial sets. We need to show that this $\tilde{\mathfrak{X}} \to \mathfrak{X}$ gives the universal simplicial covering. Let $\mathfrak{X}' \to \mathfrak{X}$ is any other simplicially 1-connected covering. We need to show that this is canonically isomorphic to the G-torsor $\tilde{\mathfrak{X}} \to \mathfrak{X}$. First of all by the properties of 2.4.3 we can show that $\tilde{\mathfrak{X}} \to \mathfrak{X}$ is a simplcial covering (infact any G-torsor is a simplcial covering). Now observe that $\mathfrak{X}' \to \mathfrak{X} \to BG \to \mathbb{B}G$ is homotopically trivial, since $\mathfrak{X}'$ is 1-connected. We have a canonical covering $\mathbb{E}G \to \mathbb{B}G$, which is a simplicial fibration , so $\mathbb{E}G$ is simplicially fibrat. So there is a lifting $\mathfrak{X}' \to \mathbb{E}G$, and the following sqaure commutes

$$\begin{array}{ccc} \mathfrak{X}' & \longrightarrow & \mathbb{E}G \\ \downarrow & & \downarrow \\ \mathfrak{X} & \longrightarrow & \mathbb{B}G \end{array}$$

By the theory of simplcial sets (covering of simplicial set) and using the previous lemma , this square is cartesian on each stalk, hence cartesian. So $\mathfrak{X}' \to \mathfrak{X}$ is as covering isomorphic to the covering $\tilde{\mathfrak{X}} \to \mathfrak{X}$ and also as a pointed covering. Given any pointed simplcial covering $\mathfrak{X}' \to \mathfrak{X}$, we have the connected component of the base point $\mathfrak{X}^{0'}$ of $\mathfrak{X}'$. $\mathfrak{X}^{0'} \to \mathfrak{X}$ is still a pointed simplicial covering. Now we construct universal covering of $\mathfrak{X}^{0'}$, by the above porcedure which is also universal covering of $\mathfrak{X}$. So there exists an unique isomorphism form the pointed universal covering of $\mathfrak{X}$ to the pointed universal covering of $\mathfrak{X}^{0'}$. So the composition $\tilde{\mathfrak{X}} \to \mathfrak{X}'$ is the unique morphism of pointed covering spaces. So we have $\tilde{\mathfrak{X}} \to \mathfrak{X}$ as the universal object in the category of pointed simplicial covering of $\mathfrak{X}$. □

6.1.2 The A^1- theory

A simplical trivial cofibration is a A^1-trivial cofibration by definition. Hence a A^1-covering is also a simplicial covering.

Lemma 6.1.6. 1. *A G-torsor $\mathfrak{Y} \to \mathfrak{X}$ with G a strongly A^1-invariant sheaf of groups is an A^1-covering.*

2. *Any G-torsor $\mathfrak{Y} \to \mathfrak{X}$ in the étale topology, with G a finite étale k-group of order prime to the characteristic, is an A^1-covering.*

Proof. 1. The set $P(\mathfrak{X}, G)$ of isomorphism classes of G-torsors over a space $\mathfrak{X}$ is in one to one correspondence with $[\mathfrak{X}, BG]_{H_s(k)}$. Now G is a strongly A^1-invariant sheaf of groups implies BG is A^1-local (?). So we have $P(\mathfrak{X}, G) = [\mathfrak{X}, BG]_{H_s(k)} = [\mathfrak{X}, BG]_{H(k)}$. Now we have the following commutative diagram

$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow{f} & \mathfrak{Y} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle g} \\ \mathfrak{B} & \longrightarrow & \mathfrak{X} \end{array}$$

where $\mathfrak{A} \to \mathfrak{B}$ is a A^1-trivial cofibration. So we have a bijection $[\mathfrak{B}, G]_{H(k)} \to [\mathfrak{A}, G]_{H(k)}$. Commutativity of the above square shows that if we pullback the G-torsor $\mathfrak{Y} \to \mathfrak{X}$ over $\mathfrak{B}$ and then restrict it to $\mathfrak{A}$, it becomes a trivial G-torsor over $\mathfrak{A}$(?) , but since $[\mathfrak{B}, BG]_{H(k)} \to [\mathfrak{A}, BG]_{H(k)}$ is a bijection the pullback to $\mathfrak{B}$ itself is the trivial G-torsor $\mathfrak{B} \times G$, so we have a section $s : \mathfrak{B} \to \mathfrak{B} \times G \to \mathfrak{Y}$ of the map $\mathfrak{Y} \to \mathfrak{X}$. The composition $s \circ \pi : \mathfrak{A} \to \mathfrak{Y}$ might not be equal to the morphism $f : \mathfrak{A} \to \mathfrak{Y}$. But by properties of G-torsors there exists $g : \mathfrak{A} \to G$ such that $s \circ \pi = g.f$(?). Now G is A^1 invariant so the map $Hom(\mathfrak{B}, G) \to Hom(\mathfrak{A}, G)$ is a bijection. Let $\tilde{g} : \mathfrak{B} \to G$ be the extension of g. Now we have $\tilde{g}^{-1}.s : \mathfrak{B} \to \mathfrak{Y}$ whose composition with π gives the map f. Now suppose there exists another map $s' : \mathfrak{B} \to \mathfrak{Y}$ such that $s' \circ \pi = f$ then there exists $\tilde{g}' : \mathfrak{B} \to G$ such that $\tilde{g}^{-1}.s = \tilde{g}'.s'$ but then we have $s' \circ \pi = \tilde{g}'.s' \circ \pi = g'.f = f$ which implies $g' = id \in Hom(\mathfrak{A}, G)$, but $Hom(\mathfrak{B}, G) \to Hom(\mathfrak{A}, G)$ is an isomorphism, hence $\tilde{g}' = id \in Hom(\mathfrak{B}, G)$. So the uniqueness follows.

2. The space $B_{et}(G) = R\pi_*(BG)$ is A^1-local where $\pi : (Sm_k)_{et} \to (Sm_k)_{Nis}$ is the canonical morphism of sites.But then $[\mathfrak{X}, B_{et}(G)]_{H(k)} = [\mathfrak{X}, B_{et}(G)]_{H_s(k)} = Hom_{H_s(Sm_k)_{et}}(\pi^*(\mathfrak{X}, BG) = H^1_{et}(\mathfrak{X}; G)$. Moreover if $\mathfrak{A} \to \mathfrak{B}$ is a A^1-trivial cofibration then $H^1_{et}(\mathfrak{B}; G) \to H^1_{et}(\mathfrak{A}; G)$ is a bijection. Again since G is A^1 invariant $Hom(\mathfrak{B}, G) \to Hom(\mathfrak{A}, G)$ is an isomorphism. Hence the same reasoning as in part 1 will give the proof.

□

Remark 6. In particular any G_m torsor is an A^1 covering . Let $A = k[x_0, ..., x_n]$. So $P^n = Proj(A)$. We have $P^n = \bigcup_{i=0}^n D_+(x_i)$ and $D_+(x_i) \cong Spec((A_{x_i})_0)$. This gives a map $(A_{x_i})_0 \to A_{x_i}$, which gives map $A^{n+1} \setminus \{0\} = \bigcup_{i=0}^n Spec(A_{x_i}) \to D_+(x_i) = P^n$. This map gives a G_m torsor $A^{n+1} \setminus \{0\} \to P^n$. In particular for any smooth projective variety of non zero dimension has nontrivial A^1 covering by pulling back the G_m torsor $A^{n+1} \setminus \{0\} \to P^n$ over the variety.

Lemma 6.1.7. *1. Any pull back of an A^1-covering is an A^1-covering.*

2. The composition of two A^1-coverings is an A^1-covering.

3. Any A^1-covering is an A^1-fibration.

4. A morphism $\mathfrak{Y}_1 \to \mathfrak{Y}_2$ of A^1-coverings $\mathfrak{Y}_i \to \mathfrak{X}$ which is an A^1-weak equivalence is an isomorphism.

Proof. 1. Suppose $\mathfrak{Y} \to \mathfrak{X}$ is an A^1-covering and $\mathfrak{Z} \to \mathfrak{X}$ is any morphism suppose there exists a commutative diagram

$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow{f} & \mathfrak{Y} \times_{\mathfrak{X}} \mathfrak{Z} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle g} \\ \mathfrak{B} & \longrightarrow & \mathfrak{Z} \end{array}$$

such that the map $\mathfrak{A} \to \mathfrak{B}$ is an A^1-weak equivalence then we have the following commutative diagram

$$\begin{array}{ccccc} \mathfrak{A} & \xrightarrow{f} & \mathfrak{Y} \times_{\mathfrak{X}} \mathfrak{Z} & \longrightarrow & \mathfrak{Y} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle g} & & \downarrow \\ \mathfrak{B} & \longrightarrow & \mathfrak{Z} & \longrightarrow & \mathfrak{X} \end{array}$$

Now $\mathfrak{Y} \to \mathfrak{X}$ is an A^1-covering implieas there exists a unique map $\mathfrak{B} \to \mathfrak{Y}$ comuuting the above diagram , but then using the universal property of pullbacks we get a unique map from $\mathfrak{B} \to \mathfrak{Y} \times_{\mathfrak{X}} \mathfrak{Z}$.

2. Let $f : \mathfrak{Y} \to \mathfrak{X}$ and $g : \mathfrak{Y}' \to \mathfrak{Y}$ be two A^1-covering. And let we have the following diagram

$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow{p} & \mathfrak{Y}' \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle g \circ f} \\ \mathfrak{B} & \xrightarrow{q} & \mathfrak{X} \end{array}$$

Where $\mathfrak{A} \to \mathfrak{B}$ is an A^1-weak equivalence. Then we have the following commutative diagram

$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow{p \circ g} & \mathfrak{Y} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle f} \\ \mathfrak{B} & \xrightarrow{q} & \mathfrak{X} \end{array}$$

Which by definition gives a unique morphism $q' : \mathfrak{B} \to \mathfrak{Y}$ such that the two triangles commutes. Now using this q' we have the following diagram

$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow{p} & \mathfrak{Y}' \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle g} \\ \mathfrak{B} & \xrightarrow{q'} & \mathfrak{Y} \end{array}$$

Which again by definition gives a unique morphism $q'' : \mathfrak{B} \to \mathfrak{Y}'$.

3. Follows from the definition of A^1-fibration.

4. $\mathfrak{Y}_1 \to \mathfrak{Y}_2$ We have the following diagram

$$\begin{array}{ccc} \mathfrak{Y}_1 & \xrightarrow{id} & \mathfrak{Y}_1 \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle p_1} \\ \mathfrak{Y}_2 & \xrightarrow{p_2} & \mathfrak{X} \end{array}$$

So we have a unique map $g : \mathfrak{Y}_2 \to \mathfrak{Y}_1$, which is also an A^1-weak equivalence by the properties of model category, and $g \circ h = id$. Using this g we have the follwoing diagram

$$\begin{array}{ccc} \mathfrak{Y}_2 & \xrightarrow{id} & \mathfrak{Y}_2 \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle p_2} \\ \mathfrak{Y}_1 & \xrightarrow{p_1} & \mathfrak{X} \end{array}$$

So by definition we have a unique map $g' : \mathfrak{Y}_1 \to \mathfrak{Y}_2$, such that $g' \circ g = id$. So $g' \circ g \circ h = g'$, which implies $g' = h$.

□

Theorem 6.1.8. *Any pointed A^1-connected space $\mathfrak{X}$ admits a universal pointed A^1-covering $\tilde{\mathfrak{X}} \to \mathfrak{X}$ in the category of pointed covering of $\mathfrak{X}$. It is (up to unique isomorphism) the unique pointed A^1-covering whose source is A^1-simply connected. It is a $\pi_1^{A^1}(\mathfrak{X})$-torsor over $\mathfrak{X}$.*

Proof. Sketch of the proof (for details see [MO1] page 119, Theorem 4.8.: Let $\mathfrak{X}$ be a pointed A^1-connected space. Let $\mathfrak{X} \to L_{A^1}(\mathfrak{X}$ be its A^1-localisation. Let $\tilde{\mathfrak{X}}_{A^1}$ be the simplicial universal covering of $L_{A^1}(\mathfrak{X})$ constructed in simplicial case. It is a $\pi_{A^1}(\mathfrak{X})$ torosor from the way we constructed it. By lemma 6.1.6 $\tilde{\mathfrak{X}}_{A^1} \to L_{A^1}(\mathfrak{X})$ is an A^1 covering, since $\pi_{A^1}(\mathfrak{X})$ is strongly A^1 invariant. Let $\tilde{\mathfrak{X}} \to \mathfrak{X}$ be its pullback to $\mathfrak{X}$. This is a pointed $\pi_{A^1}(\mathfrak{X})$-torsor and also a pointed A^1-covering by previous lemma. Our aim is to prove that $\tilde{\mathfrak{X}} \to \mathfrak{X}$ is the pointed A^1-covering. Then it can be shown that to prove the universal property , it is enough to show that $\tilde{\mathfrak{X}} \to \mathfrak{X}$ is the universal object in the category of pointed A^1 connected A^1 coverings of $\mathfrak{X}$. Let $\mathfrak{Y} \to \mathfrak{X}$ be pointed A^1-covering of pointed A^1-connected simplcial sheaves of sets. An let $\mathfrak{X} \to L_{A^1}(\mathfrak{X})$ be the A^1-localisation (hence it is A^1-weak equivalence). By lemma 6.1.9 we have a cartesian square of pointed simplicial sheaves

$$\begin{array}{ccc} \mathfrak{Y} & \longrightarrow & \mathfrak{Y}' \\ \downarrow & & \downarrow \\ \mathfrak{X} & \longrightarrow & L_{A^1}(\mathfrak{X}) \end{array}$$

with $\mathfrak{Y}' \to L_{A^1}(\mathfrak{X})$ a pointed A^1-covering. By simplicial theory we have a unique map $\tilde{\mathfrak{X}_{A^1}} \to L_{A^1}(\mathfrak{X})$. By pulling back over $\mathfrak{X}$ gives a commutative diagram of A^1-coverings:

$$\begin{array}{ccc} \tilde{\mathfrak{X}} & \longrightarrow & \mathfrak{Y} \\ \downarrow & & \downarrow \\ \mathfrak{X} & \xrightarrow{id} & \mathfrak{X} \end{array}$$

We have to show there exists a unique map $\tilde{\mathfrak{X}} \to \mathfrak{Y}$, making the above diagram commutative. Suppose there exist two such maps f_i for $i \in \{0, 1\}$ then using the lemma 6.1.9 we have the follwoing diagram

$$\begin{array}{ccc} \tilde{\mathfrak{X}} & \xrightarrow{\tilde{f}_i} & \tilde{\mathfrak{X}}_i \\ {\scriptstyle f_i}\downarrow & & \downarrow \\ \mathfrak{Y} & \xrightarrow{id} & \mathfrak{Y}' \end{array}$$

Since $\tilde{\mathfrak{X}}_i$ is $1 - A^1$ connected and $\mathfrak{Y}'$ is A^1-local, $\tilde{\mathfrak{X}}_i \to \mathfrak{Y}'$ is the simplcial universal pointed covering of $\mathfrak{Y}'$ (and also of $L_{A^1}(\mathfrak{X})$). So we have a unique isomorphism $\phi : \tilde{\mathfrak{X}}_0 \to \tilde{\mathfrak{X}}_1$ of pointed simplcial covering. To show that $f_0 = f_1$, then becomes equivalent to $\phi \circ \tilde{f}_0 = \tilde{f}_1$. But then we have a morphism $\psi : \tilde{X} \to \pi_1^{A^1}(\mathfrak{X})$, such that $\tilde{f}_1 = \psi.(\phi \circ \tilde{f}_0)$, where . means the action induced by ψ. Since $\tilde{X}$ is A^1-connected, so ψ factors as $\tilde{X} \to * \to \pi_1^{A^1}(\mathfrak{X})$. But all the morphisms are pointed, so $* \to \pi_1^{A^1}(\mathfrak{X})$ is the neutral element hence $\phi \circ \tilde{f}_0 = \tilde{f}_1$.

Now if $\mathfrak{Y} \to \mathfrak{X}$ is a pointed A^1-covering with $\mathfrak{Y}$ $1 - A^1$ connected, then the morphism $\tilde{\mathfrak{X}} \to \mathfrak{Y}$ is an A^1-weak equivalence (using the arguement above), and thus by lemma 6.1.7 it is an isomorphism. □

Lemma 6.1.9. *Let $\mathfrak{Y} \to \mathfrak{X}$ be a pointed A^1-covering between pointed A^1-connected spaces. Then for any A^1-weak equivalence $\mathfrak{X} \to \mathfrak{X}'$, there exists a cartesian square of spaces*

$$\begin{array}{ccc} \mathfrak{Y} & \longrightarrow & \mathfrak{Y}' \\ \downarrow & & \downarrow \\ \mathfrak{X} & \longrightarrow & \mathfrak{X}' \end{array}$$

in which the right vertical morphism is an A^1-covering (and thus the top horizontal morphism is an A^1-weak equivalence).

Proof. [MO1], page 121, lemma 4.9. □

By Postnikov tower we have the follwoing bijection for any pointed connected simplcial set $\mathfrak{X}$: $[\mathfrak{X}, BG]_{H_{s,\bullet}(k)} \cong Hom_{Gr}(\pi_1(\mathfrak{X}, G)$. Where Gr is the vategory of sheaves of groups. If G is a strongly A^1-invariant sheaf, we get in the same way $[\mathfrak{X}, BG]_{H_{\bullet}(k)} \cong Hom_{Gr_{A^1}}(\pi_1^{A^1}(\mathfrak{X}, G)$, where Gr_{A^1} is the category of strongly A^1-invariant sheaves of groups. It can be shown that the inclusion $Gr_{A^1} \subset Gr$ admits a left adjoint $G \mapsto G_{A^1}$, with $G_{A^1} := \pi_1^{A^1}(BG) = \pi_1(L_{A^1}(BG))$. So, Gr_{A^1} has all colimits, so we have a sum too, denoted by $*^{A^1}$ and defined by , $*^{A^1} G_i$ for a family of strongly A^1-invariant sheaves of groups, $*_i^{A^1} G_i := (*_i G_i)_{A^1}$, where $*$ is the usual sum in the category Gr.

Theorem 6.1.10. *Let X be an A^1-connected smooth scheme. Let $\{U_i\}_{i \in I}$ be an one covering of X by A^1-connected one subscheme which contains the base point. Assume further that each intersection $U_i \bigcap U_j$ is still A^1-connected. Then for any strongly A^1-invariant sheaf of groups G, the follwoing diagram $*_{i,j}^{A^1} \pi_1^{A^1}(U_i \bigcap U_j) \rightrightarrows *_i^{A^1}(U_i) \to \pi_1^{A^1}(X) \to *$ is right exact in the category Gr_{A^1}.*

Proof. [MO1], page 123, theorem 4.12. □

Theorem 6.1.11. *For $n \geq 2$ the canonical $\mathbb{G}_m$-torsor $\mathbb{A}^{n+1} \setminus \{0\}) \to \mathbb{P}^n$ is the universal A^1-covering of $\mathbb{P}^n$. This defines a canonical isomorhism $\pi_1^{A^1}(\mathbb{P}^n) \cong \mathbb{G}_m$.*

Proof. by remark 6 we know that $\mathbb{A}^{n+1} \setminus \{0\}) \to \mathbb{P}^n$ gives a G_m torsor, so it is an A^1 covering. As $\mathbb{A}^{n+1} \setminus \{0\})$ is simplcially 1-connected, we know by [MO1], theorem 3.38, page 104, that $\mathbb{A}^{n+1} \setminus \{0\})$ is 1-A^1-connected. Hence we have our result.

□

6.2 The fundamental group sheaf $\pi_1^{A^1}(SL_n)$

In this section we will compute $\pi_1^{A^1}(SL_n)$

Theorem 6.2.1. *We have $\pi_1^{A^1}(SL_2) \cong \underline{K}_2^{MW}$.*

Proof. Sketch of the proof : For $n = 1$, $A^2 \setminus \{0\}$ is not A^1-connected, since by ([MO1, page 104, Thm 3.40], $\pi_1^{A^1}(A^2 \setminus \{0\}) \cong \underline{K}_2^{MW}$. Now we have map $SL_2 \to A^2 \setminus \{0\}$, given by $SL_2(F) \to A^2 \setminus \{0\}(F)$, for any field F over k, the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ mapping to (a, c). But then we have a map $A^2 \setminus \{0\} \times A^1 \to SL_2$, defined by matrix $(a, c), t$ maps to matrix $\begin{pmatrix} a & t \\ c & (1+ct)/a \end{pmatrix}$ if a is nonzero, else $((0, c), t)$ mapping to the matrix $\begin{pmatrix} 0 & (-1)/c \\ c & t \end{pmatrix}$. The composition $A^2 \setminus \{0\} \times A^1 \to SL_2 \to A^2 \setminus \{0\}$, being the canonical projection $A^2 \setminus \{0\} \times A^1 \to A^2 \setminus \{0\}$, it is A^1 weak equivalence. Moreover the map $A^2 \setminus \{0\} \times A^1 \to SL_2$ is a simplicial equivalence, hence A^1 weak equivalence. So the morphism $SL_2 \to A^2 \setminus \{0\}$ is an A^1 weak equivalence. Hence $\pi_1^{A^1}(SL_2) \cong \pi_1^{A^1}(A^2 \setminus \{0\}) \cong \underline{K}_2^{MW}$. □

Lemma 6.2.2. *Let G be a group space which is A^1-connected. Then there exists a unique group structure on the pointed space $\tilde{G}$ for which the A^1-covering $\tilde{G} \to G$ is an (epi)-morphism of group spaces. The kernel is central and canonically isomorphic to $\pi_1^{A^1}(G)$.*

Theorem 6.2.3. *The universal A^1-covering of SL_2 admits a group structure and we get a central extension of sheaves of groups $0 \to \underline{K}_2^{MW} \to \tilde{SL}_2 \to SL_2 \to 1$.*

Theorem 6.2.4. *The canonical isomorphism $\pi^{A^1}(SL_2) \cong \underline{K}_2^{MW}$ induces through the inclusions $SL_2 \to SL_n$, $n \geq 3$, an isomorphism $\underline{K}_2^M = \underline{K}_2^{MW}/\eta \cong \pi_1^{A^1}(SL_n) = \pi_1^{A^1}(SL_\infty)$.*

[MO1], page 127, theorem 4.20.

Proof. □

Lemma 6.2.5.
1. *For $n \geq 3$, the inclusion $SL_n \subset Sl_{n+1}$ induces an isomorphism $\pi_1^{A^1}(SL_n) \cong \pi_1^{A^1}(SL_{n+1})$.*
2. *The inclusion $SL_2 \subset SL_3$ induces an epimorphism $\pi_1^{A^1}(SL_2) \to \pi_1^{A^1}(SL_3)$.*

Proof. Sketch of the proof: Let $SL'_n \subset SL_{n+1}$ the subgroup formed by the matrices of the form :

$$\begin{pmatrix} 1 & 0 & \dots & 0 \\ * & & & \\ \vdots & & M & \\ * & & & \end{pmatrix}$$

with $M \in SL_n$. The inclusion $SL_n \subset SL'_n$ gives us SL'_n as the semidirect product of SL_n and A^n, hence as simplcial set SL'_n is the product $A^n \times SL_n$. The group SL'_n is the isotropy subgroup of $(1, 0, ..., 0)$ under the right action of SL_{n+1} on $A^{n+1} \setminus \{0\}$. So we have an SL'_n Zariski torsor $SL_{n+1} \to A^{n+1} \setminus \{0\}$. Where the map $SL_{n+1} \to A^{n+1} \setminus \{0\}$, assigns to each matrix its first row. This is an simplcial fibration sequence and by [MO1, page 111, theorem 3.53] this is also an A^1-fibration sequence. From this fibration sequence we get long exact sequence of A^1-homotopy groups. Now using the facts that $A^{n+1} \setminus \{0\}$ is $n-1$-A^1-connected and $SL_n \subset SL'_n$ is an A^1-weak equivalence we get our result.

□

6.3 Computation of $\pi_1^{A^1}(P^1)$

Let $G_m := A^1 \setminus \{0\}$. Then by [MV, page 112, corollary 2.8, page 111, lemma 2.5], we have a canonical isomorphism $P^1 \cong \sum_s(G_m)$ in $H_\bullet(k)$, which is induced by the covering of P^1 by its two standard A^1's intersecting to G_m. So we have $\pi_1^{A^1}(P^1) \cong \pi_1^{A^1}(\sum_s(G_m))$. We will denote the category of sheaves of pointed sets on Sm_k by $Shv_\bullet$. The canonical map from $S \to \pi_1(\sum_s(S))$ composed with the map $\pi_1(\sum_s(S)) \to \pi_1^{A^1}(\sum_s(S))$, for $S \in Shv_\bullet$ will be denoted by θ_S.

Lemma 6.3.1. *The morphism θ_S induces for any strongly A^1-invariant sheaf of groups G a bijection $Hom_{Gr}(\pi_1^{A^1}(\sum_s(S)), G) \cong Hom_{Shv_\bullet}(S, G)$.*

Proof. [MO1, page 129, lemma 4.23]. □

From now on we will denote $\pi_1^{A^1}(\sum_s((G_m)^n))$ by $F_{A^1}(n)$. Our aim is to calculate $F_{A^1}(1)$ and show that it is not abelian.

Remark 7.
1. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be two pointed simplicial set. We denote $\mathfrak{X} * \mathfrak{Y}$ (the reduced join of $\mathfrak{X}$ and $\mathfrak{Y}$) by the quotient of $\triangle^1 \times \mathfrak{X} \times \mathfrak{Y}$ by the relations $(0, x, y) = (0, x, y')$, $(1, x, y) = (1, x', y)$ and $(t, x_0, y_0) = (t, x_0, y_0)$ where x_0 (resp y_0) is the base point of $\mathfrak{X}$ (resp. $\mathfrak{Y}$).

2. We can cover $A^2 \setminus \{0\}$ by $G_m \times A^1$ and $A^1 \times G_m$, such that the intersection is $G_m \times G_m$, which gives that $A^2 \setminus \{0\}$ is A^1-equivalent to $G_m * G_m$.

3. The join $C(\mathfrak{X}) := \mathfrak{X} * (\bullet)$ is called the cone of $\mathfrak{X}$. It is the smash product of $\triangle^1 \wedge \mathfrak{X}$, where $\triangle^1$ is pointed by 1. Let $\mathfrak{X} \subset C(\mathfrak{X})$ be the canonical inclusion. The quotient is $(\sum_s(\mathfrak{X}))$. We dfine $C'(\mathfrak{X})$ as the smash product $\triangle^1 \wedge \mathfrak{X}$, where $\triangle^1$ is pointed by 0. The join $\mathfrak{X} * \mathfrak{Y}$ contains the wedge $C(\mathfrak{X}) \vee C'(\mathfrak{X})$ and the quotient $(\mathfrak{X} * \mathfrak{Y})/(\mathfrak{X} \vee \mathfrak{Y})$ is $(\sum_s(\mathfrak{X}) \times \mathfrak{Y})$. Also the quotient $(\mathfrak{X} * \mathfrak{Y})/(C(\mathfrak{X}) \vee C'(\mathfrak{X}))$ is $(\sum_s(\mathfrak{X}) \vee \mathfrak{Y})$. So the morphism of pointed spaces $\mathfrak{X} * \mathfrak{Y} \to \sum_s(\mathfrak{X}) \vee \mathfrak{Y})$ is a simplcial weak equivalence. So using the map $\sum_s(\mathfrak{X}) \times \mathfrak{Y}) \to \sum_s(\mathfrak{X}) \vee \mathfrak{Y})$ and the map $\mathfrak{X} * \mathfrak{Y} \to \sum_s(\mathfrak{X}) \times \mathfrak{Y})$, we get a map in $H_{s,\bullet}(k) : \omega_{\mathfrak{X},\mathfrak{Y}} : \sum_s(\mathfrak{X}) \times \mathfrak{Y}) \to \sum_s(\mathfrak{X}) \times \mathfrak{Y})$.

Lemma 6.3.2. *The morphism $\omega_{\mathfrak{X},\mathfrak{Y}}$ is equal to the morphism $(p_1)^{-1}.Id_{\sum_s(\mathfrak{X}) \times \mathfrak{Y})}.(p_2)^{-1}$ in $H_{s,\bullet}(k)$, where p_1 is the map $\sum_s(\mathfrak{X}) \times \mathfrak{Y}) \to \sum_s(\mathfrak{X}) \to \sum_s(\mathfrak{X}) \times \mathfrak{Y})$ induced by the first projection, and p_2 is induced by the second projection.*

Proof. [MO1, page 130, lemma 4.25]. □

Remark 8.
1. Let G be a sheaf of groups . There is a map of pointed simplicial sheaf of groups $(\mu_G)' : G \times G \to G$, given by $(g, h) \mapsto g^{-1}.h$. This morphism induces a morphism $\triangle^1 \times G \times G \to \triangle^1 \times G$, which gives a morphism $\eta_G : G * G \to \sum_s(G)$. This map is called the geometric Hopf map of G. Now in $H_{s,\bullet}(k)$ the map $G * G \to \sum_s(G \wedge G)$ is invertible, so we get a map $(\eta_G)' : \sum_s(G \wedge G) \sum_s(G)$.

2. By 7 the map $A^2 \setminus \{0\} \to P^1$ is same as the map $(\eta_{G_m})' : \sum_s(G_m \wedge G_m) \sum_s(G_m)$ in $H_\bullet(k)$. Now G acts diagonally on $G * G$ and $\eta_G : G * G \to \sum_s(G)$ is a G-torsor. By [MO1, page 111, theorem 3.53] the simplcial fibration $G * G \to \sum_s(G) \to BG$ is also an A^1-fibration sequence if $\pi_0^{A^1}(G)$ is a strongly A^1-invariant sheaf.

So by the previous theorem we have the follwoing corollary:

Corollary 6.3.3. *For any sheaf of groups G, the composition $\sum_s(G\times G) \to \sum_s(G\wedge G) \to \sum_s(G)$ where the last map is $(\eta_G)'$, is equal in $Hom_{H_{s,\bullet}(k)}(\sum_s(G\times G), \sum_s(G))$ to $(\sum_s(\chi_1))^{-1}.\sum_s(\mu').(\sum_s(pr_2))^{-1}$, where χ_1 is the composition $pr_1 : G \times G \to G$; $(g \mapsto g^{-1}) : G \to G$ and pr_2 is just the second projection from $G \times G \to G$.*

Remark 9. Now for our case $G = G_m$. We have an A^1-fibration sequence $G_m * G_m \to \sum_s(G_m) \to BG_m$, which is equivalent to $A^2 \setminus \{0\} \to P^1 \to P^\infty$ in $H_\bullet(k)$. Now the simplcial sets P^1 and P^∞ are A^1-connected, so from the long exact sequence of homotopy sheaves we get a short exact sequence: $1 \to \underline{K}_2^{MW} \to F_{A^1}(1) \to G_m \to 1$. We have $\theta_{G_m} : G_m \to F_{A^1}(1)$ the section coming from 6.3.1. As the sheaf of pointed sets $F_{A^1}(1)$ is the product $\underline{K}_2^{MW} \times G_m$, by the section θ_{G_m}.

The following result entirely describes the structure of $F_{A^1}(1)$

Theorem 6.3.4. *1. The morphism of sheaves of sets $G_m \times G_m \to \underline{K}_2^{MW}$ induced by the morphism $(U, V) \mapsto (\theta(U^{-1}))^{-1}\theta(U^{-1}V)(\theta(V))^{-1}$ is equal to the symbol $(U, V) \mapsto [U][V]$.*

2. The follwoing short exact sequence
$1 \to \underline{K}_2^{MW} \to F_{A^1}(1) \to G_m \to 1$ is a central extension.

Proof. [MO1, page 132, theorem 4.29]. □

Theorem 6.3.5. *The sheaf of groups $F_{A^1}(1)$ is not abelian.*

Proof. By [MO1, page 132, theorem 4.29], we get $\theta(U)\theta(V) =< -1 > [U][V]\theta[UV]$. Which implies $\theta(U)\theta(V)\theta(U)^{-1} = h([U][V])\theta(V)$. Let for any field k, the field $F = k(U, V)$ be the filed of rational functions in U and V over k. The composition of residue morphism ∂_U and ∂_V commutes with multiplication by h. The image of the symbol $[U][V]$ is one, and $h \in K_0^{MW}(k)$ is nonzero hence we have $h([U][V]) \in \bar{K}_2^{MW}(F)$ nonzero. □

Appendix A

G-torsors and non-abelian cohomology

Definition A.0.6. Let C be a Grothendieck site, G is a sheaf of groups on C. A G-torsor is defined to be a sheaf of set S with a free G-action $G \times S \to S$ such that $S/G = *$ in the sheaf category. The requirement $G \times S \to S$ is free means that the isotropy subgroups of G for the action are trivial in all sections.

Let $X \in Sm_k$ and X_{Nis}(or X_{Zar}) denotes the small site whose elements are the *étale* separated finite type X schemes (covering of X in Zariski topology) and morphisms are morphism between schemes (for $X \in \tilde{Sm}_k$ we can also consider the morphism to be smooth).

Let G' is a sheaf of groups on $(Sm_k)_{Nis}$, $X \in Sm_k$, $G' \mid X_{Nis}$(or $G' \mid X_{Zar}$) is a sheaf of groups on X_{Nis}(or X_{Zar}).

Definition A.0.7. The first cohomology $H^1_{Nis}(X; G)$(or $H^1_{Zar}(X; G)$) is equal to the isomorphism classes of G-torsors on X_{Nis}(or X_{Zar}).

Lemma A.0.8. *If G is a Nisnevich sheaf of groups such that for any open immersion $U \to V$, the map $G(V) \to G(U)$ is surjective, then $H^1_{Nis}(X; G) = 0$.*

Proof. The main step is step 1 , after that the proof follows from induction arguement.

step 1 if

$$\begin{array}{ccc} p^{-1}(U) & \longrightarrow & V \\ \downarrow & & \downarrow p \\ U & \xrightarrow{j} & X \end{array}$$

is a Nisnevich distinguished square, and T a G-torsor on X that is trivial on U and on V, then T is trivial. Let s in $T(V)$ and t in $T(U)$. There exists g in $G(p^{-1}(U))$ such that $g.(t \mid p^{-1}(U)) = s \mid p^{-1}(U)$. As $G(V) \to G(p^{-1}(U))$ is onto, there exists g' in $G(V)$ inducing g. Define $t' = g'.t$ in $T(V)$. The restriction of t' and s on $p^{-1}(U)$ are the same, they glue to give an element in $T(X)$, so the torsor T is trivial.

step 2 Then starting from a G-torsor T on a Noetherian scheme X, it is enough to show that the restriction of T to the local schemes of X, then for any $x \in X$ there will exist an open neighbourhood U_x of x such that the restriction of T to U_x is trivial. Then we can choose a finite set of these U_x that cover X and by induction using step 1 above we can show that T is trivial over any finite union of these U_x.

step 3 Let X local of dimension d with closed point x. Then by induction on the dimension, for smooth (or local or open subschemes of a local scheme) of *dimension* $< d$ the restriction of T on those schemes is trivial. This implies that for any $U \to X - x$ *étale*, the inverse image of T over U is trivial. By hypothesis T is locally trivial for the Nisnevich topology, so there exists $p : V \to X$ *étale* with a rational point x' in V over x such that T restricted to V is trivial. We can choose V so that $p^{-1}(x) = x'$. Then, we have a Nisnevich distinguished square:

$$\begin{array}{ccc} (V - x') & \longrightarrow & V \\ \downarrow & & \downarrow{\scriptstyle p} \\ (X - x) & \xrightarrow{j} & X \end{array}$$

The torsor T is trivial on V and on $X - x$, so step 1 implies that T is trivial over X.

□

Definition A.0.9. If G is a sheaf of groups on X_{Zar} (or X_{Nis}) and Z a closed subset of X, then $H^1_Z(X_{Zar}; G)$(or $H^1_Z(X_{Nis}; G)$) [first cohomology group with supports in Z] can be defined as the set of isomorphism class of G-torsors equipped with a trivialisation on $X - Z$.

Remark 10. If G is a Nisnevich sheaf such that for any scheme X, $H^1_{Zar}(X; G) = H^1_{Nis}(X; G)$, then for any X, Z, we also have $H^1_Z(X_{Zar}; G) = H^1_Z(X_{Nis}; G)$.

Remark 11. Now, if we have a Nisnevich distinguished square, then the canonical map $H^1_{X-U}(X_{Nis}, G) \to H^1_{V-p^{-1}(U)}(V_{Nis}; G)$ is bijective.

Remark 12. Now, in the situation considered in chapter 3 lemma 44, we have $H^1_v(O_v; G) = G(F)/G(O_v)$. This set is isomorphic to the set $H^1_v((O_v)_{Zar}; G)$(using torsors). Consider G torsors on *Spec*(O_v) equipped with a trivialisation on the generic point *SpecF*. For any s in $G(F)$, consider the trivial torsor T on O_v equipped with the trivialisation on *SpecF* given by the element s of $T(F)$. This defines a map $G(F) \to H^1_v((O_v)_{Zar}; G)$ which induces a bijection $G(F)/G(O_v) \to H^1_v((O_v)_{Zar}; G)$.

Appendix B

B.G properties and A^1 local properties

Definition B.0.10. A simplicial sheaf $\mathbb{B}$ on Sm_k is called A^1-local if for any simplicial sheaf $\mathbb{Y}$ the map $Hom_{H_s}(Sm_k)(\mathbb{Y}, \mathbb{B}) \to Hom_{H_s}(Sm_k)(\mathbb{Y} \times A^1, \mathbb{B})$, induced by the projection $\mathbb{Y} \times A^1 \to \mathbb{Y}$ is a bijection. (A^1 is the simplicial sheaf represented by the affine line).

Definition B.0.11. Let $\mathbb{B}$ be a presheaf of simplicial sets on Sm_k, we say that it satisfies the A^1-B.G property in the Nisnevich topology if it satisfies the B.G property(5.2.2) described in chapter 4 (where the B.G class of objects are all the objects of Sm_k) and if moreover, for any $X \in Sm_k$ the map $\mathbb{B}(X) \to \mathbb{B}(A^1 \times X)$ induced by the projection $A^1 \times X \to X$ is a weak equivalence.

Lemma B.0.12. *Let $\mathbb{B}$ be a simplicial presheaf which satisfies the A^1-B.G property in the Nisnevich topology and let $a_{Nis}(\mathbb{B})$ be its sheafification in the Nisnevich topology. Then $a_{Nis}(\mathbb{B})$ is A^1-local.*

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