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The Hodge-Tate decomposition theorem for Abelian Varieties over p-adic fields

following J.M. Fontaine

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Introduction

0.1. For a smooth complex projective variety X or, more generally, a compact Kähler manifold X, a fundamental result is the so-called "Hodge decomposition" of its singular cohomology with complex coefficients. More precisely, we have a decomposition of the cohomology groups

$$\mathrm{H}^n(X,\mathbb{C}) = \bigoplus_{p+q=n} \mathrm{H}^p(X,\Omega^q_X) \tag{1}$$

where Ω^q_X is the sheaf of holomorphic q-differential forms on X. This decomposition behaves well with respect to the action of the Galois group of $\mathbb{C}$ over $\mathbb{R}$: if we denote by σ the complex conjugation, i.e. the unique non trivial element of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$, then σ acts on $\mathrm{H}^n(X,\mathbb{C})$ and transforms a holomorphic q-form in an anti-holomorphic q-form, inducing a map on the cohomology groups that satisfies $\overline{\mathrm{H}^p(X,\Omega^q_X)} = \mathrm{H}^q(X,\Omega^p_X)$.

If X is an abelian variety over $\mathbb{C}$, the Hodge decomposition (1) reduces to give the following canonical isomorphism

$$\mathrm{H}^1(X,\mathbb{C}) = \mathrm{H}^0(X,\mathcal{O}_X) \oplus \mathrm{H}^0(X,\Omega^1_X), \tag{2}$$

since the cup-product pairings identify $\mathrm{H}^r(X,\mathbb{C})$ with the r-th exterior power of $\mathrm{H}^1(X,\mathbb{C})$ and (see [**Ser59**, VII, Th. 10])

$$\mathrm{H}^q(X,\Omega^p_X) = \bigwedge^q \mathrm{H}^1(X,\mathcal{O}_X) \otimes \bigwedge^p \mathrm{H}^0(X,\Omega^1_X).$$

0.2. In the late sixties, Tate asked if a similar result could hold for the p-adic étale cohomology of a proper and smooth variety over a complete discrete valuation field K of characteristic 0 and perfect residue field of characteristic $p > 0$. In [**Tat67**], he established a "Hodge-like" decomposition for an abelian variety with good reduction over K, after extending the scalars to the p-adic completion of an algebraic closure of K.

More precisely, let $\mathcal{O}_K$ be the valuation ring of K, $S = \mathrm{Spec}(\mathcal{O}_K)$, η the generic point of S and $\overline{\eta}$ the geometric point corresponding to an algebraic closure $\overline{K}$ of K. Let $\mathcal{O}_{\mathbf{C}}$ be the p-adic completion of $\mathcal{O}_{\overline{K}}$, $\mathbf{C}$ its fraction field. Let $G_K = \mathrm{Gal}(\overline{K}/K)$ be the absolute Galois group of K. For every $r \in \mathbb{N}$, we denote by $\mathbf{C}(r)$ the Galois module $\mathbf{C}$ twisted by the action of the r-power of the p-adic cyclotomic character χ_p and by $\mathbf{C}(-r)$ its dual. Let X be an abelian variety over η with good reduction. Tate proved the existence of a canonical G_K-equivariant isomorphism

$$\mathbf{C} \otimes_{\mathbb{Q}_p} \mathrm{H}^1_{\text{ét}}(X_{\overline{\eta}},\mathbb{Q}_p) \xrightarrow{\sim} \mathrm{H}^0(X,\Omega^1_{X/\eta}) \otimes_K \mathbf{C}(-1) \oplus \mathrm{H}^1(X,\mathcal{O}_X) \otimes_K \mathbf{C}, \tag{3}$$

now called the Hodge-Tate decomposition.

We know that there is a canonical isomorphism

$$\mathrm{H}^1_{\text{ét}}(X_{\overline{\eta}},\mathbb{Z}_p) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{T}_p(X_{\overline{\eta}}),\mathbb{Z}_p),$$

where $\mathrm{T}_p(X_{\overline{\eta}})$ is the p-adic Tate module of the abelian variety $X_{\overline{\eta}}$. In this case, (3) is equivalent to the existence of canonical isomorphisms

$$\begin{array}{l} \mathrm{H}^1(X, \mathcal{O}_X) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}_p[G_K]}(\mathrm{T}_p(X_{\overline{\eta}}), \mathbf{C}) \\ \mathrm{H}^0(X, \Omega^1_{X/\eta}) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}_p[G_K]}(\mathrm{T}_p(X_{\overline{\eta}}), \mathbf{C}(1)). \end{array} \tag{4}$$

The theorem was proved more generally in [**Tat67**] for p-divisible groups. Using the semi-stable reduction theorem, Raynaud proved in [**SGA 7**] (Exposé 9 Th. 3.6 and Prop. 5.6) the conjecture for all abelian varieties over K, while the proof for the most general statement was established in 1988 by Faltings in [**Fal88**].

In this *mémoire* we present a different proof, due to Fontaine [**Fon82**], of the theorem of Tate and Raynaud as a consequence of a sophisticated, although relatively elementary, analysis of the module of Kähler differentials $\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}$. The main advantage of this argument is that it avoids completely the notion of p-divisible group as well as the notion of Néron model and it does not involve the semi-stable reduction theorem.

We give an overview of the content of the different chapters.

0.3. Let K be a complete discrete valuation field of characteristic 0, with perfect residue field of characteristic $p > 0$. In the first chapter, following [**Fon04**], we present some classical results of Tate and Sen: they rely on a fine analysis of the ramification in the cyclotomic $\mathbb{Z}_p$-extension of K, i.e. the unique $\mathbb{Z}_p$-extension K_∞ of K contained in the field generated over K by all the p^n-th roots of 1.

Let $\mathfrak{m}_{K_\infty}$ be the maximal ideal of $\mathcal{O}_{K_\infty}$, $H_K = \mathrm{Gal}(\overline{K}/K_\infty)$, Γ_K the quotient G_K/H_K. Let L be the fraction field of the p-adic completion of $\mathcal{O}_{K_\infty}$. The crucial point is the fundamental theorem of Tate 1.2.6, that states that for every finite extension M of K_∞, we have $\mathrm{Tr}_{M/K_\infty}(\mathcal{O}_M) \supseteq \mathfrak{m}_{K_\infty}$. Using this result, we will show that $L^{\Gamma_K} = \mathbf{C}^{G_K} = K$ and that we have an isomorphism, for every $h \in \mathbb{N}$

$$\mathrm{H}^1_{\mathrm{cont}}(\Gamma_K, \mathrm{GL}_h(K_\infty)) \cong \mathrm{H}^1_{\mathrm{cont}}(G_K, \mathrm{GL}_h(\mathbf{C})).$$

Furthermore, we prove that $\mathrm{H}^0_{\mathrm{cont}}(G_K, \mathbf{C}(1)) = \mathrm{H}^1_{\mathrm{cont}}(G_K, \mathbf{C}(1)) = 0$.

In the next section we study the category of $\mathbf{C}$-representations of G_K, that is the category of finite dimensional $\mathbf{C}$-vector spaces equipped with a continuous and semi-linear action of G_K. They form an abelian category, that we denote by $\mathbf{Rep}_{\mathbf{C}}(G_K)$. In a similar way we define the notion of L-representation and K_∞-representation of Γ_K. According to Sen, we have canonical $\otimes$-equivalences of categories of representations

$$\mathbf{Rep}_{\mathbf{C}}(G_K) \xrightarrow{\sim} \mathbf{Rep}_L(\Gamma_K) \to \mathbf{Rep}_{K_\infty}(\Gamma_K),$$

that can be described as follows.

By a first theorem of Sen, every $\mathbf{C}$-representation of G_K, the $\mathbf{C}$-linear morphism $\mathbf{C} \otimes_L W^{H_K} \to W$ is an isomorphism. Hence the functor $W \mapsto W^{H_K}$ is a $\otimes$-equivalence between $\mathbf{Rep}_{\mathbf{C}}(G_K)$ and $\mathbf{Rep}_L(\Gamma_K)$.

Let $X \in \mathbf{Rep}_L(G_K/H_K)$ and let X_f be the K_∞-vector space obtained by taking the union of all finite dimensional K-subspaces of X that are stable by G_K. A second theorem of Sen proves that the functor $X \mapsto X_f$ defines a $\otimes$-equivalence between $\mathbf{Rep}_{K_\infty}(G_K/H_K)$ and $\mathbf{Rep}_L(G_K/H_K)$, quasi-inverse of the functor $Y \mapsto Y \otimes_{K_\infty} L$.

Let $Y \in \mathbf{Rep}_{K_\infty}(\Gamma_K)$. We will prove that there exists a unique endomorphism s of the K_∞-vector space Y such that, for every $y \in Y$, there exists an open subgroup Γ_y of Γ_K such that

$$\gamma(y) = \exp(\log \chi_p(\gamma) s)(y)$$

for every $\gamma \in \Gamma_y$. The endomowphism s is now called the Sen endomorphism of Y. We will see that s provides enough information to classify the representations up to isomorphisms. We conclude the chapter by giving the abstract definition of Hodge-Tate representations.

0.4. In the second chapter we give the proof of Fontaine of the Theorem of Tate and Raynaud. Let K be as in 0.3. Let $\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}$ be the module of $\mathcal{O}_K$-differentials of $\mathcal{O}_{\overline{K}}$. The first part of the chapter is dedicated to the study of this Galois module: we will construct a surjective, G_K-equivariant and $\mathcal{O}_{\overline{K}}$-linear morphism

$$\xi \colon \overline{K} \otimes \mathrm{T}_p(\mathbb{G}_m) \to \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}$$

where $\mathrm{T}_p(\mathbb{G}_m)$ denotes the p-adic Tate module of the multiplicative group over $\overline{K}$. The kernel of ξ is given by $\mathfrak{a} \otimes \mathrm{T}_p(\mathbb{G}_m)$, where

$$\mathfrak{a} = \left\{ a \in \overline{K} \mid v(a) \geq -v(\mathcal{D}) - \frac{1}{q-1} \right\}$$

and $\mathcal{D}$ is the absolute different of K. By passing to the limit, we will get a G_K-isomorphism

$$\mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Q}_p, \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}) \xrightarrow{\sim} \mathbf{C}(1). \tag{5}$$

This will be obtained as a particular case of more general results on Lubin-Tate formal groups, that hold also when K is a complete discrete valuation field of characteristic $p > 0$ and perfect residue field.

Let X be an abelian variety over η. In section 2.4, we will use the results presented so far to give Fontaine's proof of the decomposition (4). The idea goes as follows: the theorem can be reduced to showing the existence of a K-linear injective morphism

$$\mathrm{H}^0(X, \Omega^1_{X/\eta}) \to \mathrm{Hom}_{\mathbb{Z}_p[G_K]}(\mathrm{T}_p(X), \mathbf{C}(1)). \tag{6}$$

The first step is to consider a proper model $\mathfrak{X}/S$ of finite type for the abelian variety X/η. The group scheme structure on X induces a group structure on the set $\mathfrak{X}(\mathcal{O}_{\overline{K}})$, identified with $X(\overline{K})$, and the translation action of $X(\overline{K})$ induces a morphism

$$\widehat{\varrho} = \widehat{\varrho}_{X,\mathfrak{X},r} \colon p^r \mathrm{H}^0(\mathfrak{X}, \Omega^1_{\mathfrak{X}/S}) \to \mathrm{Hom}_{\mathbb{Z}[G_K]}(X(\overline{K}), \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K})$$

for a suitable non negative integer r. More precisely, given $\omega \in p^r \mathrm{H}^0(\mathfrak{X}, \Omega^1_{\mathfrak{X}/S})$, we set $\widehat{\varrho}(\omega)$ to be the $\mathbb{Z}[G_K]$-linear morphism

$$\widehat{\varrho}(\omega) \colon u \mapsto u^*(\omega).$$

Let $V_p(X) = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[p^{-1}], X(\overline{K}))$. By composing with

$$\mathrm{Hom}_{\mathbb{Z}[G_K]}(X(\overline{K}), \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}) \to \mathrm{Hom}_{\mathbb{Z}[G_K]}(V_p(X), \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Q}_p, \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}))$$

and extending the scalars to K, we get a K-linear map that eventually restricts to

$$\varrho = \varrho_{X,\mathfrak{X},r} \colon \mathrm{H}^0(X, \Omega^1_{X/\eta}) \to \mathrm{Hom}_{\mathbb{Z}_p[G_K]}(\mathrm{T}_p(X), \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Q}_p, \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K})).$$

This is the required injective morphism (6), if we take into account the isomorphism (5). It does not depend on the choice of r and of $\mathfrak{X}$.

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Io sono qui
sono venuto a suonare
sono venuto a danzare
e di nascosto ad amare...
(P. Conte)

CHAPTER 1

C-representations: the theory of Tate and Sen

1.1. Review of group cohomology

1.1.1. Let G be a topological group. Let M be a topological G-module, i.e. a topological abelian group endowed with a liner and continuous action of G. Let $\mathcal{C}^n_{\mathrm{cont}}(G,M)$ be the group of continuous n-cochains of G with values in M. Let

$$d_n\colon \mathcal{C}^n_{\mathrm{cont}}(G,M) \to \mathcal{C}^{n+1}_{\mathrm{cont}}(G,M)$$

be the boundary map

$$d_n f(g_1,\dots,g_{n+1}) = g_1 f(g_2,\dots,g_{n+1}) + \sum_{j=1}^{n} (-1)^j f(g_1,\dots,g_j g_{j+1},\dots,g_{n+1}) + (-1)^{n+1} f(g_1,\dots g_n).$$

The sequence $\mathcal{C}^*_{\mathrm{cont}}(G,M)$ is a cochain complex. We denote by $\mathrm{H}^n_{\mathrm{cont}}(G,M)$ the n-th cohomology group of this complex: it is called the n-th continuous cohomology group of G with coefficients in M.

1.1.2. Given a short exact sequence of topological G-modules

$$0 \to M \to M' \to M'' \to 0$$

we have a six-terms-long exact sequence

$$0 \to M^G \to M'^G \to M''^G \to \mathrm{H}^1_{\mathrm{cont}}(G,M) \to \mathrm{H}^1_{\mathrm{cont}}(G,M') \to \mathrm{H}^1_{\mathrm{cont}}(G,M'').$$

1.1.3. We can still define the groups H^0 and H^1 even when we drop the abelian hypothesis on M, as in [**Ser62**], Appendix to chap. VII. Let M be a topological group, written multiplicatively, endowed with a continuous action of G. $\mathrm{H}^0_{\mathrm{cont}}(G,M)$ is defined as the group M^G of elements of M fixed by G. We denote by $Z^1_{\mathrm{cont}}(G,M)$ the subset of the set of continuous functions of G into M such that

$$f(g_1g_2) = f(g_1)g_1(f(g_2))$$

for $g_1,g_2 \in G$: we call $f \in Z^1_{\mathrm{cont}}(G,M)$ a continuous cocycle. We say that two cocycles f and f' are cohomologous and write $f \sim f'$ if there exists $a \in M$ such that

$$f'(g) = a^{-1} f(g) g(a)$$

for every $g \in G$. This defines an equivalence relation on the set of cocycles. The quotient set has a structure of pointed set: it contains a distinguished element equal to the class of the unit cocycle $f(g) = 1$ for every $g \in G$. We denote its class by 1. We denote $Z^1_{\mathrm{cont}}(G,M)/\sim$ by $\mathrm{H}^1_{\mathrm{cont}}(G,M)$ and we call it the cohomology set of G with values in M. This definition coincides (if we retain just the structure of pointed sets) with the usual one in the abelian case.

1.1.4. Let G be a topological group and let H be a closed normal subgroup of G. Any topological G-module M (abelian or not) can be regarded as H-module, as well as M^H can be regarded as G/H-module. Then we can naturally define the restriction map

$$\mathrm{res}\colon \mathrm{H}^1_{\mathrm{cont}}(G, M) \to \mathrm{H}^1_{\mathrm{cont}}(H, M)$$

and the inflation map

$$\mathrm{Inf}\colon \mathrm{H}^1_{\mathrm{cont}}(G/H, M^H) \to \mathrm{H}^1_{\mathrm{cont}}(G, M).$$

One has the following inflation-restriction exact sequence of pointed sets (resp. of abelian groups if M is abelian):

(1.1.4.1) $$1 \to \mathrm{H}^1_{\mathrm{cont}}(G/H, M^H) \xrightarrow{Inf} \mathrm{H}^1_{\mathrm{cont}}(G, M) \xrightarrow{res} \mathrm{H}^1_{\mathrm{cont}}(H, M).$$

There is a direct proof, valid for the abelian as well as for the non abelian case, in [**Ser62**], chap. VII, §6.

1.2. Statement of the theorems of Tate and Sen

1.2.1. Let K be a complete discrete valuation field of characteristic 0, with perfect residue field of characteristic $p > 0$. We fix an algebraic closure $\overline{K}$ of K and we denote by G_K the Galois group of $\overline{K}$ over K. We denote by $\mathcal{O}_K$ the ring of integers of K and by $\mathcal{O}_{\overline{K}}$ the ring of integers of $\overline{K}$. Let $\mathcal{O}_{\mathbf{C}}$ be the p-adic completion of $\mathcal{O}_{\overline{K}}$, $\mathbf{C}$ its field of fractions. We denote by v_p the valuation of $\mathbf{C}$ extending the valuation of $\overline{K}$ normalized by $v_p(p) = 1$, and by $|.|$ the p-adic absolute value.

For any subfield M of $\mathbf{C}$, we denote by $\mathcal{O}_M$ its valuation ring and by $\mathfrak{m}_M$ the maximal ideal of $\mathcal{O}_M$. If M is a finite extension of K we denote by v_M the unique valuation of $\mathbf{C}$ normalized by $v_M(M^\times) = \mathbb{Z}$ and by $e_M = v_M(p)$ the absolute ramification index of M.

1.2.2. Let χ_p be the cyclotomic character of K, i.e. the continuous homomorphism

$$\chi_p\colon G_K \to \mathbb{Z}_p^\times$$

that gives the action of G_K on the group of units of order a power of p. Let log be the p-adic logarithm, $\log\colon \mathbb{Z}_p^\times \to \mathbb{Z}_p$. We denote by H_K its kernel and by Γ_K the quotient G_K/H_K. Notice that $\Gamma_K \cong \mathbb{Z}_p$ as abelian groups.

Let K_∞ be the cyclotomic $\mathbb{Z}_p$-extension of K: it is the unique $\mathbb{Z}_p$ extension of K contained in the subfield of $\overline{K}$ generated by the roots of unity of order a power of p. By construction we have that $H_K = \mathrm{Gal}(\overline{K}/K_\infty)$ and $\Gamma_K = \mathrm{Gal}(K_\infty/K)$. Let L be the closure of K_∞ in $\mathbf{C}$.

The goal of the first part of this chapter is to present the proof of the following theorems (originally due to Tate and Sen).

1.2.3. Theorem. *We have:*

i) $\mathrm{H}^0_{\mathrm{cont}}(H_K, \mathbf{C}) = \mathbf{C}^{H_K} = L$*;*
ii) *For every* $n \geq 1$, $\mathrm{H}^1_{\mathrm{cont}}(H_K, \mathrm{GL}_h(\mathbf{C})) = 1$.

As a corollary, we have $\mathbf{C}^{G_K} = L^{\Gamma_K}$ and $\mathrm{H}^1_{\mathrm{cont}}(G_K, \mathrm{GL}_h(\mathbf{C})) = \mathrm{H}^1_{\mathrm{cont}}(\Gamma_K, \mathrm{GL}_h(L))$. Indeed, $\mathbf{C}^{G_K} = (\mathbf{C}^{H_K})^{\Gamma_K} = L^{\Gamma_K}$ while the second statement follows from the inflation-restriction exact sequence (1.1.4.1).

1.2.4. Theorem. *We have:*

i) $\mathrm{H}^0_{\mathrm{cont}}(G_K, \mathbf{C}) = \mathrm{H}^0_{\mathrm{cont}}(\Gamma_K, L) = K$;
ii) *For every $h \geq 1$, the map*

$$\mathrm{H}^1_{\mathrm{cont}}(\Gamma_K, \mathrm{GL}_h(K_\infty)) \to \mathrm{H}^1_{\mathrm{cont}}(\Gamma_K, \mathrm{GL}_h(L)) = \mathrm{H}^1_{\mathrm{cont}}(G_K, \mathrm{GL}_h(\mathbf{C}))$$

induced by $\mathrm{GL}_h(K_\infty) \subset \mathrm{GL}_h(\mathbf{C})$ is bijective.

1.2.5. The proof of theorems 1.2.3 and 1.2.4 relies on the following important result of Tate, whose proof is a consequence of a detailed analysis of a ramified $\mathbb{Z}_p$-extension of K (not necessarily the cyclotomic $\mathbb{Z}_p$-extension of 1.2.2).

1.2.6. Theorem (Tate, cf. [**Fon04**], Théorème 1.8). *We keep the notations of 1.2.1. Let K_∞ be a ramified $\mathbb{Z}_p$ extension of K and let M be a finite extension of K_∞. Let $\mathrm{Tr}_{M/K_\infty} : M \to K_\infty$ be the trace map. Then $\mathrm{Tr}_{M/K_\infty}(\mathcal{O}_M) \supseteq \mathfrak{m}_{K_\infty}$.*

1.3. The proof of Tate's Theorem 1.2.6

1.3.1. Let K be as in 1.2.1. Let E be a finite extension of K, J the Galois group $\mathrm{Gal}(E/K)$, $\alpha \in \mathcal{O}_E$ such that $\mathcal{O}_E = \mathcal{O}_K[\alpha]$ [**Ser62**, chap. III, Prop. 12]. We denote by J_i the i-th higher ramification group of K of J [**Ser62**, chap. IV, §1]. We have

$$J_i = \{g \in J \mid i_J(g) \geq i+1\}$$

where $i_J(g) = v_E((g-1)\alpha)$ for every $g \in J$. We call the integers i such that $J_i \neq J_{i+1}$ the ramification numbers of the extension E/K.

1.3.2. Proposition. *Let E be a cyclic ramified extension of K of degree p. Let i be the unique ramification number of the extension E/K. Then we have $i \leq \frac{e_E}{p-1}$ and, for every $x \in E$, $v_E(\mathrm{Tr}_{E/K}(x)) \geq v_E(x) + (p-1)i$.*

Proof. Let τ be a generator of $J = \mathrm{Gal}(E/K)$. We have, for every $x \in E$, $v_E((\tau-1)x) \geq v_E(x) + i$, and the equality holds if an only if $v_E(x)$ is prime to p. Let $P(T) \in \mathbb{Z}[T]$ be a polynomial such that

$$pP(T) = \sum_{j=0}^{p-1} T^j - (T-1)^{p-1}. \tag{1.3.2.1}$$

Hence, for every $x \in E$, we have

$$\mathrm{Tr}_{E/K}(x) = (\tau-1)^{p-1}(x) + pP(\tau)(x) \tag{1.3.2.2}$$

and

$$v_E(pP(\tau)(x)) = e_E + v_E(x), \tag{1.3.2.3}$$

since

$$pP(\tau)(x) = px + \sum_{j=1}^{p-1}(1+\tau+\ldots+\tau^{j-1})(1-\tau)(x) - (1-\tau)^{p-1}(x).$$

Suppose that p divides i and let $\pi \in E$ such that $v_E(\pi) = 1$. We have $v_E((\tau-1)^{p-1}(\pi)) = (p-1)i+1$ and $v_E(pP(\tau)(\pi)) = e_E+1$ (by (1.3.2.3)), that are both prime to p (as e_E is divisible by p). On the other hand, $v_E(\mathrm{Tr}_{E/K}(\pi)) = pv_K(\mathrm{Tr}_{E/K}(\pi))$ is divisible by p. Therefore we have the equality $e_E + 1 = (p-1)i+1$ (using (1.3.2.2)).

Suppose that p does not divide i and let $y \in E$ such that $v_E(y) = i$. We have

$$v_E((\tau - 1)^{p-1}(y)) = (p-1)i + i = pi,$$

while $v_E(pP(\tau)(y)) = e_E + i$ is prime to p. As we have again that $v_E(\mathrm{Tr}_{E/K}(y))$ is divisible by p, we must have $pi < e_E + i$.

By (1.3.2.2) we have, in both cases,

$$v_E(\mathrm{Tr}_{E/K}(x)) \geq v_E(x) + \min\{(p-1)i, e_E\} \geq v_E(x) + (p-1)i$$

for every $x \in E$. □

1.3.3. Lemma. *Let m, n be integers verifying $n \geq m - 1 \geq 0$. Let $i_0, i_1, \ldots, i_{m-1}$ be integers verifying $i_r \equiv i_{r-1} \mod p^r$ for $1 \leq r \leq m-1$. Then the integers $j + i_{v_p(j)}$ for $j \in \mathbb{Z}$ verifying $0 < j < p^n$ and $v_p(j) < m$ are all distinct mod p^n.*

Proof. Suppose, by contradiction, that there exist $j, j' \in \mathbb{Z}$ as above and verifying $j' + i_{v_p(j'} = j + i_{v_p(j)} + p^n a$. Let $s = v_p(j)$, $s' = v_p(j')$. We can suppose $s < s'$, so that $0 \leq s \leq m-2$. But then $v_p(j' - j) = s$, while

$$v_p((i_s - i_{s'}) + p^n a) \geq \min\{s+1, n\} = s+1,$$

which is a contradiction, as $j' - j = (i_s - i_{s'}) + p^n a$. □

1.3.4. Proposition. *Let n be an integer ≥ 1, E a cyclic totally ramified extension of K of degree p^n. Let γ be a generator of the Galois group* $\mathrm{Gal}(E/K)$. *Then*

i) *The extension E/K has exactly n distinct ramification numbers*

$$0 < i_0 < i_1 < \ldots < i_{n-1}.$$

ii) *For $1 \leq r \leq n-1$, we have $i_r \equiv r_{r-1} \mod p^r$.*

iii) *For every $y \in E^\times$ there exists $\lambda \in K$ such that*

$$v_p(y - \lambda) \geq v_p((\gamma - 1)y) - \frac{1}{p-1} \tag{1.3.4.1}$$

Proof. Let K' (resp. E') be the unique extension of degree p (resp. of degree p^{n-1}) of K contained in E. We argue by induction on n. The ramification numbers of E/K' are $i_1, i_2, \ldots, i_{n-1}$, since the lower numbering is compatible with the passage to subgroups. Using [**Ser62**, chap. IV, Prop. 3], we get that for $n \geq 2$, the ramification numbers of E'/K are $i_0, i_1, \ldots i_{n-2}$, and i) follows.

Let π be a uniformizer of E, so that $v_E(\pi) = 1$. Let $J = \mathrm{Gal}(E/K)$. For every $r \in \mathbb{N}$ verifying $1 \leq r < p^n$, we have $i_J(\gamma^r) = i_{v_p(r)}$ and $v_E(\gamma^r - 1)(\pi) = i_{v_p(r)} + 1$. For every $s \in \mathbb{Z}$ verifying $v_p(s) < n$, there exists $\pi_s \in E$ such that $v_E(\pi_s) = s$ and $v_E((\gamma - 1)(\pi_s)) = s + i_{v_p(s)}$. Indeed, set $\pi_0 = 1$ and define, for every $1 \leq r < p^n$, $\pi_r = \pi\gamma(\pi)\ldots\gamma^{r-1}(\pi)$. Then $v_E(\pi_r) = r$ and $(\gamma-1)(\pi_r) = \pi\gamma(\pi)\ldots\gamma^{r-1}(\pi)(\gamma^r(\pi)-\pi)/\pi$, so that $v_E((\gamma-1)(\pi_r)) = r + i_{v_p(r)}$. For $s \geq p^n$, let r be the remainder of the division of s by p^n. Then there exists $\lambda_s \in K$ such that $v_E = s - r$, and we can take $\pi_s = \lambda_s \pi_r$. By substituting K with K', we see that for every $s \in \mathbb{Z}$ verifying $v_p(s) < n-1$, there exists $z_s \in E$ such that $v_E(z_s) = s$ and $v_E((\gamma^p - 1)(z_s)) = s + i_{v_p(s)+1}$.

We show ii) by induction on n. For $n = 1$ there is nothing to prove, so we can assume $n \geq 2$. The induction hypothesis applied to the extension E'/K shows that

$$i_r \equiv i_{r-1} \mod p^r \quad \text{for} \quad 1 \leq r \leq n-2. \tag{1.3.4.2}$$

On the other hand, the induction hypothesis applied to E/K' shows that $i_{n-1} \equiv i_{n-2} \mod p^{n-2}$. Let $s = i_{n-2} - i_{n-1}$. To conclude we need to show that $v_p(s) \neq n-2$.

We argue by contradiction. Let z_s be as above, so that $v_E((\gamma^p - 1)(z_s)) = s + i_{n-1} = i_{n-2}$. Let $x_s = (1 + \gamma + \gamma^2 + \ldots + \gamma^{p-1})(z_s)$. By (1.3.2.1) and (1.3.2.3) we have $v_E(x_s) > s$ and $v_E((\gamma-1)(x_s)) = v_E((\gamma^p - 1)(z_s)) = i_{n-2}$. Since the extension E/K is totally ramified of degree p^n, $\{\pi_r\}_{1 \leq r < p^n}$ is a basis of E over K. Write $x_s = \sum_{r=0}^{p^n-1} b_r \pi_r$ for $b_r \in K$. Hence

$$v_E(x_s) = \min_{0 \leq r < p^n} \{p^n v_K(b_r) + r\}$$

so that $p^n v_K(b_r) + r > s$ for every r. As $(\gamma - 1)(x_s) = \sum_{r=1}^{p^n} b_r(\gamma - 1)(\pi_r)$, if $v_p(r) = n-1$ we have $v_E(b_r(\gamma-1)(\pi_r)) > s + i_{n-1} = i_{n-2}$. By 1.3.3 (for $m = n-1$) we have

$$i_{n-2} = v_E((\gamma - 1)(x_s)) = \min_{0 \leq r < p^n;\, v_p(r) < i_{n-1}} \{p^n v_K(b_r) + r + i_{v_p(r)}\}.$$

Therefore there exists r such that $i_{n-2} \equiv r + i_{v_p(r)} \mod p^r$, which is impossible as

$$v_p(i_{n-2} - i_{v_p(r)}) \geq v_p(r) + 1$$

by (1.3.4.2).

We finally prove iii). For $y \in E^\times$ we have $y = \sum_{r=0}^{p^n-1} b_r \pi_r$, $b_r \in K$ and we can take $\lambda = b_0$. Indeed, there exists a unique r_0, $0 < r_0 < p^n$, such that $v_E(y - \lambda) = v_E(b_{r_0} \pi_{r_0})$. By 1.3.3 (for $m = n$) we have

$$v_E((\gamma - 1)(y)) = \min_{0 < r < p^n} \{v_E(b_r \pi_r + i_{v_p(r)} \leq v_E(y - \lambda) + i_{n-1}$$

so $v_E(y - \lambda) \geq v_E((\gamma - 1)(y)) - i_{n-1}$. Hence

$$v_p(y - \lambda) \geq v_p((\gamma - 1)(y)) - \frac{i_{n-1}}{e_E} \geq v_p((\gamma - 1)(y)) - \frac{1}{p-1}$$

by 1.3.2 applied to the extension E/E'. □

1.3.5. Proposition. *Let n be an integer ≥ 1, E a cyclic totally ramified extension of K of degree p^n. Then for every $x \in E$ we have*

$$v_p(\mathrm{Tr}_{E/K}(x)) \geq v_p(x) + \frac{n(p-1)}{pe_K}$$

Proof. Let $i_0 < i_1 \ldots < i_{n-1}$ be the ramification numbers of the extension E/K. From 1.3.2 we deduce that

$$v_p(\mathrm{Tr}_{E/K}(x)) \geq v_p(x) + \frac{(p-1)}{e_K}\Big(\frac{i_0}{p} + \frac{i_1}{p^2} + \ldots + \frac{i_{n-1}}{p^n}\Big).$$

and the result follows, since by 1.3.4 ii) we have $i_r \geq p^r$ for every r. □

1.3.6. Let K_∞ be a ramified $\mathbb{Z}_p$ extension of K. For every $r \in \mathbb{N}$, we denote by K_r the unique extension of degree p^r of K contained in K_∞. If $\Gamma_K = \mathrm{Gal}(K_\infty/K)$, we denote by Γ_r the Galois group $\mathrm{Gal}(K_\infty/K_r)$. We fix a topological generator γ_0 of Γ_K and we let $\gamma_r = \gamma_0^{p^r}$ be a topological generator of Γ_r.

By 1.3.4, there exists a unique non negative integer $r_0 \geq 0$ and a strictly increasing sequence of positive integers

$$i_0 < i_1 < \ldots < i_{r-1} < i_r < \ldots$$

such that K_{r_0} is the maximal unramified extension of K contained in K_∞ and that, for every $r > r_0$, the ramification numbers of the extension K_r/K_{r_0} are precisely $i_0, i_1, \dots, i_{r-r_0-1}$. Moreover, we have $i_r \equiv i_{r-1} \mod p^r$. The sequence $(i_r)_{r\in\mathbb{N}}$ is called the sequence of ramification numbers of the extension K_∞/K.

1.3.7. Let F be a finite Galois extension of K such that $F \cap K_\infty = K$. For every $r \geq 0$, let $F_r = K_r F$ and $F_\infty = K_\infty F$. Let $J = \mathrm{Gal}(F_\infty/K_\infty)$, $J_r = \mathrm{Gal}(F_r/K_r)$. Let ϖ_r be the canonical isomorphism $J \xrightarrow{\varpi_r} J_r$. For $\tau \in J$, we set $i_r(\tau) = i_{J_r}(\varpi_r(\tau))$.

1.3.8. PROPOSITION. *Under the assumptions of 1.3.7, for every $\tau \in J$ the sequence $\{i_r(\tau)\}$ is stationary.*

PROOF. Up to replacing K with K_m for a sufficiently large m, we can suppose that the extension F_∞/F is totally ramified. Let (j_r) be the sequence of ramification numbers of this extension. Using [**Ser62**, chap. IV, Prop. 3] we have

$$i_r(\tau) = \begin{cases} i_{r+1}(\tau) & \text{if } i_{r+1}(\tau) \leq j_r \\ \frac{1}{p}(i_{r+1}(\tau) + (p-1)j_r) & \text{if } i_{r+1}(\tau) > j_r \end{cases}$$

or, equivalently,

$$i_{r+1}(\tau) = \begin{cases} i_r(\tau) & \text{if } i_r(\tau) \leq j_r \\ p i_r(\tau) - (p-1)j_r & \text{if } i_r(\tau) > j_r. \end{cases} \tag{1.3.8.1}$$

Therefore we have to show that there exists r such that $i_r(\tau) \leq j_r$. Otherwise we would have $i_r(\tau) > j_r$, so that $i_{r+1}(\tau) = p i_r(\tau) - (p-1)j_r$ by (1.3.8.1). Hence, by induction,

$$i_r(\tau) = p^r i_0(\tau) - (p-1)(j_{r-1} + p j_{r-2} + \dots p^{r-1} j_0)$$

so that

$$j_0 + \frac{j_1 - j_0}{p} + \frac{j_2 - j_1}{p^2} + \dots + \frac{j_r - j_{r-1}}{p^r} < i_0(\tau).$$

The right-hand term is independent from r, but the left-hand term is $\geq r+1$, since it is the sum of $r+1$ integers ≥ 1 by 1.3.4, which is a contradiction. ☐

1.3.9. Let E be a finite extension of K. Let r be the unique integer such that $\mathfrak{m}_K^r = \mathcal{D}_{E/K} \cap \mathcal{O}_K$. We have $\mathfrak{m}_K^r \mathcal{D}_{E/K}^{-1} \subset \mathcal{O}_E$. Let $\{a_1, \dots, a_d\}$ be a basis of $\mathcal{O}_E$ over $\mathcal{O}_K$, $\{a_1^*, \dots, a_d^*\}$ the dual basis with respect to the trace form $\mathrm{Tr}\colon E \times E \to K$, b a generator of $\mathfrak{m}_K^r$. Then $(ba_i)a_i^* \in \mathcal{O}_E$ and $\mathrm{Tr}_{E/K}((ba_i)a_i^*) = b$ for every $1 \leq i \leq d$. As $\mathrm{Tr}_{E/K}(\mathcal{O}_E)$ is an ideal of $\mathcal{O}_K$, we deduce that

$$\mathfrak{m}_K^r \subset \mathrm{Tr}_{E/K}(\mathcal{O}_E). \tag{1.3.9.1}$$

PROOF OF 1.2.6. Up to replacing M with a finite extension, we can suppose that M is a Galois extension of K_∞. Up to replacing K with a finite extension contained in K_∞, we can suppose that $M = K_\infty F$, for a finite Galois extension F of K such that $K_\infty \cap F = K$. Using the notations of 1.3.8, we have by [**Ser62**, chap. IV, Prop. 4]

$$v_{F_r}(\mathcal{D}_{F_r/K_r}) = \sum_{\tau \in J,\, \tau \neq 1} i_r(\tau)$$

for every $r \in \mathbb{N}$. By 1.3.8 there exist an integer r_0 and a constant $c \geq 0$ such that $v_{F_r}(\mathcal{D}_{F_r/K_r}) = c$ for $r \geq r_0$.

Let e be the ramification number of F_r/K_r for every $r \geq r_0$. Let $n \in \mathbb{N}$ be the smallest integer such that $en \geq c$. We have

$$\mathfrak{m}_{K_r}^n \subset \mathrm{Tr}_{F_r/K_r}(\mathcal{O}_{F_r}) \subset \mathrm{Tr}_{M/K_\infty}(\mathcal{O}_M).$$

The first inclusion follows from (1.3.9.1). For the second inclusion, notice that $M = FK_\infty$ and that $J = \mathrm{Gal}\,(M/K_\infty)$ is isomorphic to $J_r = \mathrm{Gal}(F_r/K_r)$. Hence, for $x \in \mathcal{O}_F \subset \mathcal{O}_M$, we have $\mathrm{Tr}_{F_r/K_r}(x) = \sum_{g\in J_r} g(x) = \sum_{\tau\in J} \tau(x) = \mathrm{Tr}_{M/K_\infty}(x)$ using the isomorphism ϖ_r.

Since $v_p(\mathfrak{m}_{K_r}^n) = n/e_{K_r}$ goes to 0 as r goes to ∞, we have that $\cup_{r\geq r_0} = \mathfrak{m}_{K_\infty}$ and we conclude that $\mathrm{Tr}_{M/K_\infty}(\mathcal{O}_M) \supseteq \mathfrak{m}_{K_\infty}$. □

1.4. The cohomology of $\mathrm{Gal}(\overline{K}/K_\infty)$: the proof of Theorem 1.2.3

1.4.1. We keep the notations of 1.2.1—1.2.2. Let M be a finite Galois extension of K_∞ and let $J = \mathrm{Gal}(M/K_\infty)$ be the Galois group of M over K_∞.

1.4.2. Lemma. *Let c be a real number > 1. For every $\lambda \in M$ there exists $a \in K_\infty$ such that*

$$|\lambda - a| < c \sup_{g\in J} |(g-1)\lambda|$$

Proof. By 1.2.6 the elements in $\mathrm{Tr}_{M/K_\infty}(\mathcal{O}_M)$ have arbitrary small valuation. Therefore, we can find $y \in \mathcal{O}_M$ such that $x = \mathrm{Tr}_{M/K_\infty}(y)$ satisfies $|x| > \frac{1}{c}$. Let $\mu = \frac{\lambda y}{x}$ and let $a = \mathrm{Tr}_{M/K_\infty}(\mu)$. We have:

$$a = \frac{\mathrm{Tr}(y\lambda)}{x} = \frac{1}{x}\sum_{g\in J} g(y)g(\lambda) = \lambda + \frac{1}{x}\sum_{g\in J} g(y)(g-1)\lambda$$

and

$$|\lambda - a| \leq \sup_{g\in J} |\frac{1}{x} g(y)[(g-1)\lambda]| < c \sup_{g\in J} |(g-1)\lambda|$$

as $|g(y)| \leq 1$, being $y \in \mathcal{O}_M$. □

Proof of part i) of 1.2.3. Let $\lambda \in \mathrm{H}^0_{\mathrm{cont}}(H_K, \mathbf{C}) = C^{H_K}$ and write λ as limit of a sequence $\{\lambda_n\}_{n\in\mathbb{N}} \subset \overline{K}$ such that $|\lambda - \lambda_n| < p^{-n}$. As λ is fixed by H_K we have, for every $h \in H_K$,

$$|(h-1)\lambda_n| = |h(\lambda - \lambda_n) + (\lambda - \lambda_n)| \leq |h(\lambda - \lambda_n)| = |(\lambda - \lambda_n)| < p^{-n}. \tag{1.4.2.1}$$

For every $n \in \mathbb{N}$, let M_n be a finite Galois extension of K_∞ containing λ_n. Let $J_n = \mathrm{Gal}(M_n/K_\infty)$. By (1.4.2.1) we have$|(g-1)\lambda_n| < p^{-n}$ for every $g \in J_n$ (as $J_n \leq H_K$). By 1.4.2 with $c = p$, we have that there exists $a_n \in K_\infty$ such that $|\lambda_n - a_n| < p^{1-n}$. Hence the sequences $\{\lambda_n\}_{n\in\mathbb{N}}$ and $\{a_n\}_{n\in\mathbb{N}}$ have the same limit λ. Hence $\lambda \in L$. □

1.4.3. Let $\mathrm{M}_h(\mathcal{O}_\mathbf{C})$ be the ring of $h \times h$ square matrices with coefficients in $\mathcal{O}_\mathbf{C}$. We equip $\mathrm{M}_h(\mathcal{O}_\mathbf{C})$ with the p-adic topology. Let $|.|$ be the p-adic absolute value on $\mathrm{M}_h(\mathcal{O}_\mathbf{C})$: we have $|A| \leq p^{-r}$ if and only if $A \in p^r\mathrm{M}_h(\mathcal{O}_\mathbf{C})$.

1.4.4. Lemma. *Let H be an open subgroup of H_K and let m be an integer ≥ 2. Let $f_m \in Z^1_{\mathrm{cont}}(H, \mathrm{GL}_h(\mathbf{C}))$ be a continuous cocycle verifying $|f_m(s) - 1| \leq p^{-m}$ for every $s \in H$. Then there exists $b_m \in \mathrm{GL}_h(\mathbf{C})$ with $|b_m - 1| \leq p^{1-m}$ such that the continuous cocycle f_{m+1} defined by*

$$f_{m+1}(s) = (b_m)^{-1} f_m(s) s(b_m)$$

satisfies $|f_{m+1}(s) - 1| \leq p^{-m-1}$ for every $s \in H$.

PROOF. We can reduce to the case $H = H_K$. Indeed, if $K'_\infty = \overline{K}^H$, we can find a finite Galois extension K' of K such that K'_∞ is a ramified $\mathbb{Z}_p$ extension of K'.

Being f_m continuous, we can find an open normal subgroup N of H_K such that $|f_m(g)-1| \leq p^{-m-2}$ for $g \in N$ (it's enough to take for N the pre-image of the open ball of radius p^{-m-2} and center 1). Let $J = H_K/N$ and let $M = \overline{K}^N$ be the corresponding finite Galois extension of K_∞. By Theorem 1.2.6, there exists $y \in \mathcal{O}_M$ such that $\sum_{\tau\in J}\tau(y) = p$. If T is a system of representatives for J in H_K, we let

$$b_m = \frac{1}{p}\sum_{g\in T} f_m(g)g(y).$$

As $f_m(g) \in 1 + p^m\mathrm{M}_h(\mathcal{O}_\mathbf{C})$, we can write $f_m(g) = 1 + p^m a_m(g)$ for $a_m(g) \in \mathrm{M}_h(\mathcal{O}_\mathbf{C})$, so that

$$b_m = \frac{1}{p}\sum_{g\in T}(1+p^m a_m(g))g(y) = \frac{1}{p}\sum_{g\in T} g(y) + p^{m-1}\sum_{g\in T} a_m(g)g(y).$$

Hence $b_m \in 1 + p^{m-1}\mathrm{M}_h(\mathcal{O}_\mathbf{C})$. In particular, $b_m \in \mathrm{GL}_h(\mathcal{O}_\mathbf{C})$. For every $s \in H_K$ we have

$$s(b_m) = \frac{1}{p}\sum_{g\in T} s(f_m(g))((sg)(y)) = \frac{1}{pf_m(s)}\sum_{g\in T} f_m(sg)((sg)(y)). \tag{1.4.4.1}$$

By the cocycle condition we also have $f_m(sg) \equiv f_m(g) \pmod{p^{m+2}}$ when $s \in N$ and $g \in H_K$, and (1.4.4.1) implies

$$s(b_m) \equiv f_m(s)^{-1}b_m \pmod{p^{m+1}}$$

i.e. $f_{m+1} = (b_m)^{-1}f_m(s)s(b_m) \equiv 1 \pmod{p^{m+1}}$. □

PROOF OF PART ii) of 1.2.3. Let $f \in Z^1_{\mathrm{cont}}(H_K, \mathrm{GL}_h(\mathbf{C}))$. Being f continuous, we can find an open normal subgroup N of H_K such that $|f(s) - 1| \leq p^{-2}$ for every $s \in N$ (notice that if f is a cocycle, then $f(1) = 1$, so that the inverse image of an open ball centred in 1 is not empty). Let f_2 be the restriction of f to N. By 1.4.4 we can find a sequence $\{f_m\}_{m\geq 2}$ of continuous cocycles verifying $|f_m(s) - 1| \leq p^{-m}$ for every $s \in N$ and a sequence $\{b_m\}_{m\geq 2} \subseteq \mathrm{GL}_h(\mathbf{C})$ verifying $|b_m - 1| \leq p^{1-m}$ such that

$$f_{m+1}(s) = b_m^{-1}f_m(s)s(b_m)$$

for every $s \in N$. Let $\{\beta_m = \prod_{k=2}^m b_k\}$ the sequence of products. Then, for every $s \in N$,

$$f_{m+1}(s) = \beta_m^{-1}f(s)s(\beta_m).$$

Let b be the limit of the sequence $\{\beta_m\}_{m\geq 2}$; since $\lim_{m\to\infty} f_m = 1$, b is an element of $\mathrm{GL}_h(\mathbf{C})$ satisfying $1 = b^{-1}f(s)s(b)$ for every $s \in N$. In other words, the restriction of f to N is cohomologous to the trivial cocycle. The inflation-restriction exact sequence (1.1.4.1) implies that f is in the image of $\mathrm{H}^1_{\mathrm{cont}}(H_K/N, (\mathrm{GL}_h(\mathbf{C}))^N)$. But H_K/N is the galois group J of the finite Galois extension $\mathbf{C}^N/\mathbf{C}^{H_K}$ and

$$\mathrm{H}^1_{\mathrm{cont}}(J, (\mathrm{GL}_h(\mathbf{C}))^N) = \mathrm{H}^1(J, \mathrm{GL}_h(\mathbf{C}^N))$$

which is trivial by Hilbert's Theorem 90 [**Ser62**, chap. X, Prop. 3]. □

1.5. The cohomology of $\mathrm{Gal}(K_\infty/K)$: the proof of Theorem 1.2.4

1.5.1. Throughout this section, we denote by K_∞ a ramified $\mathbb{Z}_p$ extension of K. We keep the notations of 1.3.6. We say that the $\mathbb{Z}_p$ extension K_∞/K is regular if it is totally ramified and if the sequence $(i_r)_{r\in\mathbb{N}}$ of ramification numbers verifies

$$i_r - i_{r-1} = p^r e_K \quad \text{for every } r \geq 1.$$

We say that the extension K_∞/K is potentially regular if there exists $r_0 \geq 0$ such that K_∞/K_{r_0} is regular. In this case, for every $r \geq r_0$, K_∞/K_r is regular.

1.5.2. Lemma ([**Fon04**, Prop. 1.11]). *The cyclotomic $\mathbb{Z}_p$ extension of K considered in 1.2.2 is potentially regular.*

1.5.3. Lemma ([**Fon04**, Prop. 1.12]). *Let F be a finite extension of K. Then a $\mathbb{Z}_p$ extension K_∞/K is potentially regular if and only if FK_∞/F is potentially regular.*

1.5.4. For every $r \in \mathbb{N}$, let $\mathrm{Tr}_{K_r/K} \colon K_r \to K$ be the trace map. For $x \in K_\infty$, let $r \in \mathbb{N}$ such that $x \in K_r$; let

$$t_K(x) = \frac{1}{p^r}\,\mathrm{Tr}_{K_r/K}(x).$$

The map $t_K \colon K_\infty \to K$ does not depend on the choice of r: it's a projector from the K-vector space K_∞ to its subspace K. Indeed, let $x \in K_r \subseteq K_{r'}$. We have

$$\frac{1}{p^{r'}}\,\mathrm{Tr}_{K_{r'}/K}(x) = \frac{1}{p^r}\Big(\frac{1}{p^{r'-r}}\,\mathrm{Tr}_{K_r/K}(\mathrm{Tr}_{K_{r'}/K_r}(x))\Big) = \frac{1}{p^r}\Big(\frac{1}{p^{r'-r}}\sum_i \mathrm{Tr}_{K_r/K}(\bar{\gamma}^i(x))\Big)$$

where $\bar{\gamma}$ is a generator of $\mathrm{Gal}(K_{r'}/K_r) \leq \mathrm{Gal}(K_{r'}/K)$, so that $\mathrm{Tr}_{K_r/K}(\bar{\gamma}^i(x)) = \mathrm{Tr}_{K_r/K}(x)$, repeated exactly $p^{r'-r}$ times.

1.5.5. Proposition ([**Fon04**, Prop. 1.13]). *Suppose that K_∞/K is regular. Then there exists $c \in \mathbb{R}_{>0}$ such that for every $x \in K_\infty$ we have*

$$|t_K(x) - x| \leq c.|(\gamma_0 - 1)x|.$$

1.5.6. Proposition. *Let K_∞/K be a potentially regular $\mathbb{Z}_p$ extension. Then the map $t_K \colon K_\infty \to K$ is continuous. If $\widehat{t}_K \colon L \to K$ denotes the extension of t_K by continuity and L_0 denotes the kernel of $\widehat{t}_K$, we have a decomposition $L = K \oplus L_0$. The operator $\gamma_0 - 1$ is bijective on L_0, with a continuous inverse.*

Proof. Let r_0 be an integer such that the extension K_∞/K_{r_0} is regular. We have

$$t_K = p^{-r_0}\,\mathrm{Tr}_{K_{r_0}/K} \circ t_{K_{r_0}}$$

by transitivity of the norm maps: $p^{-r_0}\,\mathrm{Tr}_{K_{r_0}/K}$ is clearly continuous (being K_{r_0}/K finite) and $t_{K_{r_0}}$ is continuous by 1.5.5.

For the second assertion, suppose firstly that K_∞/K is regular. If $x \in K$, then $\widehat{t}_K(x) = x$, so that $\widehat{t}_K^2 = \widehat{t}_K$ and we can write L as sum $K \oplus L_0$. For every $x \in L$ we clearly have $(\gamma_0 - 1)(x) \in L_0$ and, in particular, $(\gamma_0 - 1)(L_0) \subset L_0$. Let $K_{\infty,0} = K_\infty \cap L_0$ and let, for every $r \in \mathbb{N}$, $K_{r,0} = K_r \cap L_0$: with this notation $K_{\infty,0}$ is the union of $K_{r,0}, r \in \mathbb{N}$ $(K_{r,0} \subset K_{r+1,0} \subset \ldots)$ and L_0 is the closure of $K_{\infty,0}$ in L. As the operator $\gamma_0 - 1$ is injective (hence bijective) on every finite-dimensional K-vector space $K_{r,0}$, it is also bijective on their union $K_{\infty,0}$. Let ϱ be its inverse. For every $y \in K_{\infty,0}$, as $t_K(\varrho(y)) = \widehat{t}_K(\varrho(y)) = 0$, we have by 1.5.5

$$|\varrho(y)| \leq c.|y|$$

and ϱ is continuous. We can extend it to a continuous map, denoted again by ϱ, from L_0 to itself, which is a continuous inverse of $\gamma_0 - 1$.

For the general case, let r_0 be an integer such that the extension K_∞/K_{r_0} is regular and let $\widehat{t}_{K_{r_0}}$ be the continuous extension of $t_{K_{r_0}}$ to L. Let L_{r_0} be its kernel, $\varrho_{r_0} \colon L_{r_0} \to L_{r_0}$ the inverse of the restriction of $\gamma_{r_0} - 1$. We have

$$L = K \oplus L_0 = K_{r_0} \oplus L_{r_0}$$

and, since $L_{r_0} \subset L_0$, we can write

$$L_0 = L_0 \cap K_{r_0} \oplus L_{r_0}.$$

The map $\gamma_0 - 1$ is injective on L_0, as $L_0 \cap K = 0$. Since K_{r_0} is a finite-dimensional K-vector space, $L_0 \cap K_{r_0}$ is of finite dimension over K, so that $\gamma_0 - 1$ is bijective with continuous inverse on it. As

$$\gamma_{r_0} - 1 = \gamma_0^{p_0^r} - 1 = (\gamma_0 - 1)A(\gamma_0)$$

for $A \in \mathbb{Z}[\gamma_0]$, we see that $\gamma_0 - 1$ is bijective on L_{r_0}, with continuous inverse $A(\gamma_0)\varrho_{r_0}$. □

1.5.7. Proposition. *Suppose that K_∞/K is regular. Let λ be a principal unit of $\mathcal{O}_K$ (i.e. $|\lambda - 1| < 1$) but not a root of unity, then $\gamma_0 - \lambda$ is bijective with continuous inverse on L.*

Proof. Since $\gamma_0 - \lambda$ is obviously bijective on K if $\lambda \neq 1$, we can use the decomposition $L = K \oplus L_0$ and prove the statement for L_0. Let ϱ be the inverse of $\gamma_0 - 1$. We have:

$$\varrho \circ (\gamma_0 - \lambda) = \varrho \circ ((\gamma_0 - 1) - \lambda + 1) = 1 - (\lambda - 1)\varrho. \tag{1.5.7.1}$$

Let c be the constant in 1.5.5. If $|\lambda - 1|c < 1$, we have $|(\lambda - 1)\varrho(y)| < |y|$ for all $y \in L_0$ (see the proof of 1.5.6), and consequently $1 - (\lambda - 1)\varrho$ is an automorphism of L_0, with inverse given by the (convergent) geometric series

$$\sum_{r \geq 0} [(\lambda - 1)\varrho]^r.$$

Hence, by (1.5.7.1), $\gamma_0 - \lambda$ has a continuous inverse on L_0. If $|\lambda - 1|c \geq 1$, we replace γ_0 by $\gamma_r = \gamma_0^{p^r}$ and λ by λ^{p^r}, where r is large so large that $|\lambda^{p^r} - 1|c < 1$ (notice that such r exists, since $\lambda = 1 + x$, where $v(x) \geq 1$). We then replace K by K_r, so that $\gamma_r - \lambda^{p^r}$ has a continuous inverse on L_0. Hence the map

$$(\gamma_0 - \lambda)^{p^r} - \gamma_r - \lambda^{p^r}$$

has a continuous inverse, so the same is true for $(\gamma_0 - \lambda)^{p^r}$ and hence for $(\gamma_0 - \lambda)$ too. □

1.5.8. Remark. Using exactly the same argument as in the proof of 1.5.6, we can prove 1.5.7 assuming only that K_∞/K is potentially regular.

1.5.9. From now on, we suppose that the $\mathbb{Z}_p$ extension K_∞/K is potentially regular. We denote by L the closure of K_∞ in $\mathbf{C}$, $\Gamma_K = \mathrm{Gal}(K_\infty/K)$, $H_K = \mathrm{Gal}(\overline{K}/K_\infty)$. We will prove Theorem 1.2.4 as a particular case of the same statement for any potentially regular $\mathbb{Z}_p$-extension.

PROOF OF PART i) of 1.2.4. It's an immediate consequence of 1.5.6. Indeed we have

$$\mathrm{H}^0_{\mathrm{cont}}(\Gamma_K, L) = L^{\Gamma_K} = \{x \in L \mid (\gamma_0 - 1)x = 0\} = \mathrm{Ker}(\gamma_0 - 1),$$

but $L = K \oplus L_0$ and $\gamma_0 - 1$ is bijective on L_0, so that $\mathrm{Ker}(\gamma_0 - 1) = K$. □

1.5.10. THEOREM ([**Sen80**, Prop. 3]). *Let V be a finite dimensional K-vector space, $V \subset L$. If V is stable by γ_0, then $V \subset K_\infty$.*

PROOF. Let $u \in \mathrm{End}_K(V)$ be the restriction of γ_0 to V and let $f_u(T)$ be its characteristic polynomial: we can reduce to the case $f_u(T)$ has all its roots in K. Indeed, let K' be the extension of K obtained by adding the roots of $f_u(T)$ in $\overline{K}$. Let $K'_\infty = K'K_\infty$. Then the extension K'_∞/K' is potentially regular (see Remark 1.5.3) and we can substitute K by K', V by $K' \otimes_K V$ and so on. Moreover, we can suppose that u has only one eigenvalue, say a, by taking the decomposition of V as direct sum of its generalized eigenspaces.

Let v be a non zero eigenvector of u. We have $\gamma_0(v) = av$, so that $\gamma_0^{p^r}(v) = a^{p^r}v$. We have that $|(\gamma_0 - 1)x| \le |x|$, being the action of Γ_K on L is continuous, so that a must be a principal unit (i.e. congruent to 1 mod p). By 1.5.7 a must be a root of unity (cfr [**Tat67**], Prop. 7). Up to replacing K by a finite extension contained in K_∞, we can suppose that $a = 1$. Up to replacing V by $V + K$ (if V does not contain K), we may assume that $V = K \oplus V'$, with $V' \subset L_0 = \mathrm{Ker}\,\widehat{t}_K$. But then $\gamma_0 - 1$ is bijective on L_0, so that $V' = 0$ and $V = K \subset K_\infty$. □

PROOF OF PART ii) of 1.2.4. Let ι be the map

$$\iota\colon \mathrm{H}^1_{\mathrm{cont}}(\Gamma_K, \mathrm{GL}_h(K_\infty)) \to \mathrm{H}^1_{\mathrm{cont}}(\Gamma_K, \mathrm{GL}_h(L))$$

We first prove that ι is injective: let $f, f' \in Z^1_{\mathrm{cont}}(\Gamma_K, \mathrm{GL}_h(K_\infty))$ be two continuous cocycles that become cohomologous in $\mathrm{GL}_h(L)$. Then there exists $b \in \mathrm{GL}_h(L)$ such that

$$f'(\gamma_0) = b^{-1} f(\gamma_0)\gamma_0(b) \tag{1.5.10.1}$$

and it's enough to show that $b \in \mathrm{GL}_h(K_\infty)$. We can rewrite (1.5.10.1) as

$$\gamma_0(b) = f(\gamma_0)^{-1} b f'(\gamma_0). \tag{1.5.10.2}$$

Let K' be the extension of K generated by the coefficients of $f(\gamma_0)$ and $f'(\gamma_0)$: it is a finite extension of K contained in K_∞. Let V be the K'-vector space generated by the coefficients of b: it's a finite dimensional K-vector space, contained in L, and (1.5.10.2) shows that it is stable by γ_0. Being V closed in L, we can apply Theorem 1.5.10 to get $V \subset K_\infty$, so that $b \in \mathrm{GL}_h(K_\infty)$.

To prove the surjectivity we need an auxiliary technical result:

1.5.11. LEMMA. *For every matrix $A \in \mathrm{M}_h(L)$, let $v(A)$ be the minimum of the p-adic valuations of its coefficients. Let r be an integer such that the extension K_∞/K_r is regular and let m be an integer ≥ 5. Let $A_m \in \mathrm{GL}_h(L)$, $X_m \in \mathrm{GL}_h(K_r)$ be matrices verifying*

$$v(A_m - 1) \ge \frac{3p}{p-1}, \quad v(A_m - X_m) \ge \frac{mp}{p-1}.$$

Then there exist $B_m \in \mathrm{GL}_h(L)$ *verifying* $v(B_m - 1) \geq \frac{(m-2)p}{p-1}$ *and* $X_m \in \mathrm{GL}_h(K_r)$ *such that the matrix*

$$A_{m+1} = B_m^{-1} A_m \gamma_r(B_m)$$

verifies $v(A_{m+1} - 1) \geq \frac{3p}{p-1}$ *and* $v(A_{m+1} - X_{m+1}) \geq \frac{p(m+1)}{p-1}$.

The proof of the lemma is a direct computation similar to 1.4.4, using 1.5.6, and we omit it. See [**Fon04**, Lemme 1.17].

We can now prove that ι *is surjective*: let $f \in Z^1_{\mathrm{cont}}(\Gamma_K, \mathrm{GL}_h(L))$. Being f continuous, there exists an integer r — that we can choose big enough so that the extension K_∞/K_r is regular — such that $v(f(\gamma_r) - 1) \geq \frac{5p}{p-1}$. Let $a_5 = f(\gamma_r)$ and let $x_5 = 1$. Using the previous lemma, we can produce three sequences of matrices: $\{a_m\}_{m\geq 5}$ and $\{b_m\}_{m\geq 5}$ in $\mathrm{GL}_h(L)$ and $\{x_m\}_{m\geq 5}$ in $\mathrm{GL}_h(K_r)$ such that, for every $m \geq 5$:

$$\begin{aligned} v(a_m - 1) &\geq \frac{3p}{p-1}; \\ v(a_m - x_m) &\geq \frac{mp}{p-1}; \\ v(b_m - 1) &\geq \frac{(m-2)p}{p-1}; \\ a_{m+1} &= b_m^{-1} a_m \gamma_r(b_m). \end{aligned}$$

The sequence $\{\beta_m = \prod_{k=5}^m b_k\}_{m\geq 5}$ converges to a matrix $b \in \mathrm{GL}_h(L)$ and the sequences $\{a_m\}_{m\geq 5}$ and $\{x_m\}_{m\geq 5}$ both converge to the same limit $x \in \mathrm{GL}_h(K_r)$ and we have

$$x = b^{-1} f(\gamma_r) \gamma_r(b).$$

Let f' be the continuous cocycle, cohomologous to f, defined by $f'(\gamma) = b^{-1} f(\gamma)\gamma(b)$ for every $\gamma \in \Gamma_K$: by construction we have $f'(\gamma_r) = x \in \mathrm{GL}_h(K_r)$. For every $\gamma \in \Gamma_K$, $\gamma_r\gamma = \gamma\gamma_r$, so that

$$f'(\gamma)\gamma(f'(\gamma_r)) = f'(\gamma_r)\gamma_r(f'(\gamma))$$

or, equivalently

$$\gamma_r(f'(\gamma)) = f'(\gamma_r)^{-1} f'(\gamma)\gamma(f'(\gamma_r)) = x^{-1} f'(\gamma)\gamma(x).$$

Hence, the K_r subspace V of L generated by the coefficients of $f'(\gamma)$ is stable by γ_r. Since V is finite dimensional over K we can use again Theorem 1.5.10 to deduce $f'(\gamma) \in \mathrm{GL}_h(K_\infty)$ for every $\gamma \in \Gamma_K$, i.e. f is cohomologous to a cocycle with values in $\mathrm{GL}_h(K_\infty)$ and it is therefore in the image of ι. □

1.5.12. Corollary. *We have* $\mathrm{H}^0_{\mathrm{cont}}(\Gamma_K, L_0) = \mathrm{H}^1_{\mathrm{cont}}(\Gamma_K, L_0) = 0$.

Proof. Indeed, $L_0^{\Gamma_K} = 0$ as we have seen in the proof of part i) of 1.2.4. Let $f \in Z^1_{\mathrm{cont}}(\Gamma_K, L_0)$ be a cocycle. Being f continuous, it is determined by $f(\gamma_0)$ and under this identification the group $B^1_{\mathrm{cont}}(\Gamma_K, L_0)$ of continuous coboundaries is a subgroup of the image of $\gamma_0 - 1$. Hence $\mathrm{H}^1_{\mathrm{cont}}(\Gamma_K, L_0) \subset \mathrm{Coker}(\gamma_0 - 1) = 0$ by 1.5.6. □

1.5.13. Let K_∞/K be the cyclotomic $\mathbb{Z}_p$ extension of K. Let χ be a continuous character of Γ_K into the group of units of $\mathcal{O}_K$. We can define the space L with a twisted action of Γ_K

$$^{s}x = \chi(s)(sx)$$

for all $s \in \Gamma_K$ and all $x \in L$. Following Tate [**Tat67**], we denote this space by $L(\chi)$. Let $\lambda = \chi(\gamma_0)$ and suppose that λ satisfies the assumptions of Prop. 1.5.7: this is the case, for example, when $\chi(L)$ is infinite.

1.5.14. PROPOSITION. *We have* $\mathrm{H}^0(\Gamma_K, L(\chi)) = \mathrm{H}^1_{\mathrm{cont}}(\Gamma_K, L(\chi)) = 0$

PROOF. Indeed, $\mathrm{H}^0(\Gamma_K, L(\chi)) \subset \mathrm{Ker}(\gamma_0 - \lambda) = 0$ by 1.5.7. We can identify $\mathrm{H}^1_{\mathrm{cont}}(\Gamma_K, L(\chi))$ with a subgroup of $\mathrm{Coker}(\gamma_0 - \lambda)$, which is trivial, again by 1.5.7. □

1.5.15. Let χ_p be the cyclotomic character $\chi_p \colon G_K \to \mathbb{Z}_p^\times$ and consider the field $\mathbf{C}$ with the action of G_K twisted by χ_p. We have

$$\mathrm{H}^0_{\mathrm{cont}}(G_K, \mathbf{C}(\chi_p)) = \mathrm{H}^1_{\mathrm{cont}}(G_K, \mathbf{C}(\chi_p)) = 0.$$

Indeed, we have $\mathbf{C}(\chi_p)^{G_K} = (\mathbf{C}(\chi_p)^{H_K})^{\Gamma_K} = L(\chi_p)^{\Gamma_K} = 0$ by Prop 1.5.14, as the kernel of χ_p is contained in H_K by definition. The statement for H^1 follows from 1.5.14 together with the inflation-restriction exact sequence (1.1.4.1).

1.6. Galois Representations

1.6.1. Let G be a topological group and let F be a field endowed with a linear topology and a continuous action of G, compatible with the field structure. A finite-dimensional F-vector space V endowed with a semi-linear action of G is called an F-representation of G. We form a category, denoted $\mathbf{Rep}_F(G)$, with morphisms given by the G-equivariant maps.

We call unit representation the field F with the given action of G. If $V \in \mathbf{Rep}_F(G)$ we call the dual representation of V the F-vector space V^* (dual of V) with the action $g((\varphi)(v)) = g(\varphi(g^{-1}(v)))$ for every $g \in G$, $v \in V$, $\varphi \in V^*$. Finally, given $V_1, V_2 \in \mathbf{Rep}_F(G)$, we can form the tensor product representation $V_1 \otimes V_2$ where the action of G is given by $g(v_1 \otimes v_2) = g(v_1) \otimes g(v_2)$ for every $g \in G, v_i \in V_i$ $(i = 1, 2)$. If $E = F^G$ is the subfield of F fixed by G, the category $\mathbf{Rep}_F(G)$ is a Tannakian category over E.

1.6.2. PROPOSITION. *For every* $V \in \mathbf{Rep}_F(G)$*, the* F*-linear morphism*

$$\varrho_F(V) \colon F \otimes_E V^G \to V$$

induced by the inclusion $V^G \subset V$ *is injective.*

PROOF. By contradiction, let m be the smallest positive integer such that there exist $v_1, v_2, \ldots, v_m \in V^G$ linearly independent over E but not over F. By the minimality of m, there exist $a_1 = 1, \ldots, a_m \in F^\times$ such that $\sum_{i=1}^m a_i v_i = 0$. For every $g \in G$ we have

$$0 = g\Big(\sum_{i=1}^m a_i v_i\Big) = v_1 + \sum_{i=2}^m g(a_i) v_i$$

so that $\sum_{i=2}^m (g(a_i) - a_i) v_i = 0$. Hence, again by the minimality of m, $g(a_i) - a_i = 0$ for every $i = 2, \ldots m$, i.e. $a_i \in E$, that contradicts the independence of the v_i's over E. □

1.6.3. Remark. We can prove in a similar way the following strengthened version of 1.6.2. Let B be an integral E-algebra endowed with a linear topology and a continuous action of G, compatible with the ring structure. Suppose that $B^G = \mathrm{Frac}(B)^G = E$. Then for every $V \in \mathbf{Rep}_F(G)$, the F-linear morphism

$$\varrho_{B,F}(V) \colon B \otimes_E (B \otimes_F V)^G \to B \otimes_F V$$

is injective.

1.6.4. We say that $V \in \mathbf{Rep}_F(G)$ is trivial if $V \cong F^n$ for some $n \in \mathbb{N}$ (isomorphism as F-representations of G). By 1.6.2, we see that V is trivial if and only if the map $\varrho_F(V)$ is bijective or, equivalently, if and only if we have the equality $\dim_E(V^G) = \dim_F V$.

1.6.5. We keep the notations of 1.2.1—1.2.2: K_∞ is the cyclotomic $\mathbb{Z}_p$ extension of K contained in $\overline{K}$, $L = \mathrm{Frac}(\widehat{\mathcal{O}}_{K_\infty})$, the completion taken with respect to the p-adic topology. For every $r \in \mathbb{N}$ we denote by K_r the unique extension of degree p^r over K contained in K_∞. We have $H_K = \mathrm{Gal}(\overline{K}/K_\infty)$ and $\Gamma = \Gamma_K$ is the Galois group $\mathrm{Gal}(K_\infty/K)$. If γ_0 is a topological generator of Γ, $\gamma_r = \gamma_0^{p^r}$ is a topological generator of $\Gamma_r = \mathrm{Gal}(K_\infty/K_r)$.

We naturally have two $\otimes$-functors

$$\begin{aligned} \mathbf{Rep}_{K_\infty}(\Gamma) &\to \mathbf{Rep}_L(\Gamma) \\ V &\mapsto L \otimes_{K_\infty} V \end{aligned} \tag{1.6.5.1}$$

and

$$\begin{aligned} \mathbf{Rep}_L(\Gamma) &\to \mathbf{Rep}_{\mathbf{C}}(G_K) \\ W &\mapsto \mathbf{C} \otimes_L W. \end{aligned} \tag{1.6.5.2}$$

The object of the theory of Sen is to construct two functors in the opposite direction defining $\otimes$-equivalences of categories

$$\mathbf{Rep}_{K_\infty}(\Gamma) \xrightarrow{\sim} \mathbf{Rep}_L(\Gamma) \xrightarrow{\sim} \mathbf{Rep}_{\mathbf{C}}(G_K).$$

1.6.6. Theorem ([**Sen80**, Th. 2]). *Every $\mathbf{C}$-representation of H_K is trivial*

Proof. By 1.6.2 we have to show that, for every $W \in \mathbf{Rep}_{\mathbf{C}}(H_K)$, the map $\varrho_{\mathbf{C}}(W)$ is bijective. Let $\{w_1, \ldots, w_h\}$ be a $\mathbf{C}$-basis of W. We can define a continuous cocycle $f \colon H_K \to \mathrm{GL}_h(\mathbf{C})$ by the assignment $g \mapsto M_g$, where M_g is the matrix representing the action of g on W in the basis $\{w_1, \ldots, w_h\}$, so that the i-th column is given by the coefficients of $g(w_i)$. Let b be the matrix of base-change for another basis of W: the corresponding cocycle is given by the formula $f'(g) = bf(g)b^{-1}$, so that f and f' are cohomologous and the map does not depend on the choice of $\{w_1, \ldots, w_h\}$. By 1.2.3 (ii), $\mathrm{H}^1_{\mathrm{cont}}(H_K, \mathrm{GL}_h(\mathbf{C}))$ is trivial, so that we can choose a basis formed by elements $\{w_i\}_{i=1}^h$ fixed by H_K. Hence, given $w = \sum_{i=1}^h b_i w_i \in W$, we have $w \in W^{H_K}$ if and only $b_i \in \mathbf{C}^{H_K} = L$ (by 1.2.3 (i)). Therefore W^{H_K} is the L-vector space of basis $\{w_i\}_{i=1}^h$ and the statement follows. □

1.6.7. Corollary. *The functor $W \mapsto W^{H_K}$ defines a $\otimes$-equivalence between the category $\mathbf{Rep}_{\mathbf{C}}(G_K)$ and the category $\mathbf{Rep}_L(\Gamma)$, quasi-inverse of the functor* (1.6.5.2).

PROOF. By 1.6.6, the functor $W \mapsto W^{H_K}$ defines a $\otimes$-equivalence between $\mathbf{Rep}_{\mathbf{C}}(H_K)$ and the category of finite-dimensional L-vector spaces, where a quasi inverse given by

$$X \mapsto \mathbf{C} \otimes_L X.$$

If $W \in \mathbf{Rep}_{\mathbf{C}}(G_K)$, W^{H_K} is naturally an L-representation of $\Gamma = G_K/H_K$ and $\mathbf{C} \otimes_L W^{H_K}$ is isomorphic to W as (trivial) representation of H_K, but also as representation of G_K. If $Y \in \mathbf{Rep}_L(\Gamma)$, $(\mathbf{C} \otimes_L Y)^{H_K} \cong C^{H_K} \otimes_L Y = Y$, by definition of the action of G_K on a tensor product. □

1.6.8. Let $V \in \mathbf{Rep}_{K_\infty}(\Gamma)$ and let $\{v_1, \dots, v_h\}$ be a K_∞-basis of V as vectors space. Let M_0 be the matrix representing the action of γ_0 on V in the basis $\{v_i\}$. Let K_r be the field generated over K by the coefficients of M_0: the integer r is called the degree of the basis $\{v_1, \dots, v_h\}$. Since K_r is complete and the action of Γ over V is continuous, the K_r-vector space generated by $\{v_1, \dots, v_h\}$ and contained in V is stable for Γ.

1.6.9. THEOREM ([**Sen80**, Th. 3]). *Let $X \in \mathbf{Rep}_L(\Gamma)$. Let X_f be the union of the sub-K-vector spaces of finite dimension of X that are stable by Γ. The L-linear map*

$$L \otimes_{K_\infty} X_f \to X$$

induced by the inclusion $X_f \subset X$ is bijective.

PROOF. As in the proof of 1.6.6, we fix a basis $\{x_1, \dots, x_h\}$ of X over L and we consider the continuous cocycle $f : \Gamma \to \mathrm{GL}_h(L)$ that maps $\gamma \in \Gamma$ to M_γ, where M_γ represents the action of γ on X in the basis $\{x_1, \dots, x_h\}$: f does not depend on the choice of the basis. By 1.2.4 (ii), the map

$$\mathrm{H}^1_{\mathrm{cont}}(\Gamma, \mathrm{GL}_h(K_\infty)) \to \mathrm{H}^1_{\mathrm{cont}}(\Gamma, \mathrm{GL}_h(L))$$

is surjective, so we can suppose that f takes value in $\mathrm{GL}_h(K_\infty)$. In other words, we can choose the x_i's such that the sub-K_∞-vector space Y of X is stable for Γ; in particular $Y \in \mathbf{Rep}_{K_\infty}(\Gamma)$. Since the L-linear map $L \otimes_{K_\infty} Y \to X$ induced by the inclusion $Y \subset X$ is clearly bijective, to complete the proof of the theorem it is enough to show that $Y = X_f$.

First of all, we have $Y \subset X_f$. Indeed, let r be the degree of the basis $\{x_1, \dots, x_h\}$. For every $s \geq r$, the K_s-vector space generated by the x_i's is of finite dimension over K, stable by Γ and Y is clearly equal to the union of those space.

Let $x \in X_f$, $x = \sum_{i=1}^h c_i x_i$ with $c_i \in L$. For every $\gamma \in \Gamma$, $\gamma(x) = \sum_{i=1}^h c_i(\gamma) x_i$, for suitable coefficients $c_i(\gamma) \in L$. Let V be the K_r-subspace of L generated by $c_i(\gamma)$ for $i = 1, \dots, h$ and $\gamma \in \Gamma$ is of finite dimension over K. Write $(a_{i,j}(\gamma))_{1 \leq i,j \leq h}$ for the matrix M_γ. Then $(a_{i,j}(\gamma))_{1 \leq i,j \leq h} \in \mathrm{GL}_h(K_r)$ and

$$\gamma(x) = \sum_{i=1}^h (c_i(\gamma)) x_i = \sum_{i=1}^h \gamma(c_i) a_{i,j}(\gamma) x_i$$

so that V is also the K_r vector space generated by $\gamma(c_i)$ for $i = 1, \dots, h$ and $\gamma \in \Gamma$. It is therefore stable by Γ and, being finite-dimensional, it is contained in K_∞ by 1.5.10. Hence $c_i \in K_\infty$ and $x \in Y$. □

1.6.10. COROLLARY. *The functor $X \mapsto X_f$ defines a $\otimes$-equivalence between the category $\mathbf{Rep}_L(\Gamma)$ and the category $\mathbf{Rep}_{K_\infty}(\Gamma)$, quasi-inverse of the functor* (1.6.5.1).

PROOF. It follows directly from 1.6.9 that the functor defined by the composition $X \mapsto X_f \mapsto X_f \otimes_{K_\infty} L$ is naturally isomorphic to the identity functor. On the other hand, $(L \otimes_{K_\infty} V)_f$ is isomorphic to V for every $V \in \mathbf{Rep}_{K_\infty}(\Gamma)$ by construction, since given a K_∞-basis $\{v_1, \dots, v_h\}$ of V, $\{1 \otimes v_i\}_{i=1}^h$ is an L-basis of $L \otimes_{K_\infty} V$ such that the K_∞ sub-vector space that they generate is stable by Γ. □

1.7. The study of $\mathbf{Rep}_{K_\infty}(\Gamma)$

1.7.1. THEOREM ([**Sen80**, Th. 4]). *Let $Y \in \mathbf{Rep}_{K_\infty}(\Gamma)$. There exists a unique K_∞-linear endomorphism s of Y such that, for every $y \in Y$, there is an open subgroup Γ_y of Γ satisfying*

$$\gamma(y) = \exp\left(\log \chi_p(\gamma).s\right)(y)$$

for every $\gamma \in \Gamma_y$. Moreover, the characteristic polynomial of s has coefficients in K.

PROOF. Let $\{y_1, \dots, y_h\}$ be a K_∞ basis of Y. We first prove the uniqueness of s. Let s, s' be two endomorphisms of Y having the required properties. Then there exists an open subgroup Γ_r of Γ such that for every $\gamma \in \Gamma_r$

$$\gamma(y_i) = \exp\left(\log \chi_p(\gamma).s\right)(y_i) = \exp\left(\log \chi_p(\gamma).s'\right)(y_i)$$

for $i = 1, \dots, h$. Hence $\exp\left(\log \chi_p(\gamma).s\right) = \exp\left(\log \chi_p(\gamma).s'\right)$ for every $\gamma \in \Gamma_r$ and $s = s'$.

Let r_0 be the degree of the basis $\{y_1, \dots, y_h\}$, Y' the K_{r_0}-sub-vector space of Y generated by the y_i's and stable by Γ: Γ_{r_0} acts linearly on Y' (since Γ_{r_0} fixes K_{r_0}) and the action on the y_i's is given by a continuous homomorphism $\Gamma_{r_0} \to \mathrm{GL}_h(K_{r_0})$, $\gamma \mapsto M_\gamma$. For γ sufficiently close to 1 (but different from 1), M_γ is close to I_h in $\mathrm{GL}_h(K_{r_0})$ and the series $\log(M_\gamma)$ converges to an endomorphism $\log(\gamma) \in \mathrm{End}_{K_{r_0}}(Y')$. The endomorphism $s_0 = \frac{\log \gamma}{\log(\chi_p(\gamma))}$ does not depend on the choice of γ. Indeed, let γ_0 be a topological generator of Γ. Let $\gamma = \gamma_0^t$ and let $\gamma' = \gamma_0^{t'}$ be another element in Γ such that $\log \gamma'$ is defined. Then

$$\log(\gamma') = \log(\gamma_0^{\log \chi_p(\gamma')}) = \log \chi_p(\gamma') \log \gamma_0$$

so that the quotient $\frac{\log(\gamma')}{\log \chi_p(\gamma')} = \frac{\log(\gamma)}{\log \chi_p(\gamma)}$ is independent from γ.

Let s be the unique K_∞ endomorphism of Y that restricts to s_0 on Y'.

1.7.2. LEMMA. *There exists $r \geq r_0$ such that the endomorphism $\exp(\log \chi_p(\gamma).s)$ of Y is well defined for every $\gamma \in \Gamma_r$.*

We postpone the proof of the lemma. Writing out the definition of s, for every $y \in Y'$, we have

$$\gamma(y) = \exp\left(\log \chi_p(\gamma).s\right)(y).$$

For a general $y = \sum_{i=1}^h c_i y_i$ with $c_i \in K_\infty$, the formula is satisfied if $\gamma \in \Gamma_y = \Gamma' \cap \Gamma_r$, where Γ' is an open subgroup of Γ which fixes all the c_i's. This proves the existence part of the theorem.

Let M be the matrix of s in the basis $\{y_1, \dots, y_h\}$. For every $\gamma \in \Gamma_r$ we have

$$(\gamma(y_1), \dots, \gamma(y_h)) = \exp(\log \chi_p(\gamma) M)(y_1, \dots, y_h).$$

As $\gamma\gamma_0 = \gamma_0\gamma$ we have, for every $\gamma \in \Gamma_r$

$$(\gamma_0(\gamma(y_1)), \dots, \gamma_0(\gamma(y_h))) = \exp(\log \chi_p(\gamma)\gamma_0(M))(y_1, \dots, y_h)$$

so that M and $\gamma_0(M)$ are similar, that implies that the characteristic polynomial of s is fixed by γ_0, i.e. it's coefficients are in K. □

Proof of Lemma 1.7.2. It's enough to show that there exists an open subgroup Γ_r of Γ such that the series

$$\exp(\log \chi_p(\gamma).s) = \sum_{n\geq 0} \frac{(\log(\chi_p(\gamma)))^n}{n!} s^n$$

converges in the ring $\mathrm{End}_{K_\infty}(Y)$ for every $\gamma \in \Gamma_r$.

Let $\{y_1, \dots, y_h\}$ be a K_∞-basis of Y. For every $b \in \mathbb{Q}$, let Y_b be the $\mathcal{O}_{K_\infty}$-sub module of Y defined by

$$Y_b = \Big\{ \sum_{i=1}^h c_i y_i \in Y \mid v_p(c_i) \geq b \Big\}.$$

Let $a \in \mathbb{Q}$ be such that $s(Y_0) \subseteq Y_a$. Recall that (see [**NS99**, chap. II, Prop. 5.5])

$$\log \mathbb{Z}_p^\times = \begin{cases} p\mathbb{Z}_p & \text{if } p \neq 2 \\ p^2\mathbb{Z}_p & \text{if } p = 2. \end{cases}$$

Let r_K be the unique integer such that $\log \chi_p(\Gamma_K) = p^{r_K}\mathbb{Z}_p$: we have $r_K \geq 1$ if $p \neq 2$ (resp. $r_K \geq 2$ if $p = 2$) and the equality holds if and only if K is absolutely unramified, i.e. $v_K(p) = e_K = 1$ (see [**NS99**, chap. II, Prop. 5.4-5.5]). Let r be the smallest non-negative integer such that $r + r_k + a > \frac{1}{p-1}$. Then for every $n \in \mathbb{N}$ and $\gamma \in \Gamma_r$ we have

$$\frac{(\log(\chi_p(\gamma)))^n}{n!} s^n(Y_0) \subset Y_{n(r+r_K-\frac{1}{p-1}+a)}$$

as $v_p(n!) = \frac{1}{p-1}\sum_{i=1}^r a_i(p^i - 1)$ if $n = \sum_{i=0}^r a_i p^i$, $0 \leq a_i < p$ is the p-adic expansion of n ([**NS99**, chap. II, Lemma 5.6]). Therefore the series $\exp(\log \chi_p(\gamma).s)$ converges. □

1.7.3. Let E be any field. We denote by $\mathcal{S}_E$ the category whose objects are couples (Y, s), where Y is a finite-dimensional E-vector space and $s \in \mathrm{End}_E(Y)$, and morphisms $f\colon (Y_1, s_1) \to (Y_2, s_2)$ are E-linear maps from Y_1 to Y_2 such that $s_2 \circ f = f \circ s_1$.

We set the unit object to be $(E, 0)$ and we define the tensor product $(Y_1, s_1) \otimes (Y_2, s_2)$ by $(Y_1 \otimes_E Y_2, s_1 \otimes id_{Y_2} + id_{y_1} \otimes s_2)$. The dual of (Y, s) is $(Y^*, -s^t)$ where Y^* is the dual vector space of Y and s^t is the transpose homomorphism of s. With these definitions $\mathcal{S}_E$ has a structure of Tannakian category over E.

1.7.4. Let E be a field containing K_∞. Let $Y \in \mathbf{Rep}_{K_\infty}(\Gamma)$. Let $Y_E = E \otimes_{K_\infty} Y$ and let s_E be the E-endomorphism of Y_E deduced by scalar extension from the endomorphism s of 1.7.1. We have therefore defined a $\otimes$-functor

$$Y \mapsto (Y_E, s_E)$$

from $\mathbf{Rep}_{K_\infty}(\Gamma)$ to $\mathcal{S}_E$.

1.7.5. Theorem. *In the notations of 1.7.4, let* $Y_1, Y_2 \in \mathrm{End}_{K_\infty}(\Gamma)$. *The canonical* E*-linear map*

$$E \otimes_K \mathrm{Hom}_{\mathbf{Rep}_{K_\infty}(\Gamma)}(Y_1, Y_2) \to \mathrm{Hom}_{\mathcal{S}_E}((Y_{1,E}, s_{1,E}), (Y_{2,E}, s_{2,E}))$$

is an isomorphism.

PROOF. We can reduce to the case $Y_1 = K_\infty$. Indeed we have the following canonical isomorphisms:

$$\mathrm{Hom}_{\mathbf{Rep}_{K_\infty}(\Gamma)}(Y_1, Y_2) = \mathrm{Hom}_{\mathbf{Rep}_{K_\infty}(\Gamma)}(K_\infty, Y_1^* \otimes Y_2)$$
$$\mathrm{Hom}_{\mathcal{S}_E}(Y_{1,E}, Y_{2,E}) = \mathrm{Hom}_{\mathcal{S}_E}(E, Y_{1,E}^* \otimes Y_{2,E}).$$

We put $Y = Y_2$. For every $\xi \in \mathrm{Hom}_{\mathbf{Rep}_{K_\infty}(\Gamma)}(K_\infty, Y)$, the map $\xi \mapsto \xi(1)$ allow us to identify the K-vector space $\mathrm{Hom}_{\mathbf{Rep}_{K_\infty}(\Gamma)}(K_\infty, Y)$ with $\mathrm{H}^0_{\mathrm{cont}}(\Gamma, Y) = Y^\Gamma$. Moreover, we can identify $\mathrm{Hom}_{\mathcal{S}_E}(E, Y_E)$ with $\mathrm{Ker}\, s_E$. Indeed, if $\varphi\colon (E, 0) \to (Y_E, s_E)$ is a $\mathcal{S}_E$-morphism, then $s_E \circ \varphi = \varphi \circ 0 = 0$, so that $\varphi(1) \in \mathrm{Ker}\, s_E$. We are therefore reduced to prove that the canonical map

$$\varrho\colon E \otimes_K Y^\Gamma \to \mathrm{Ker}\, s_E$$

is bijective. By definition of s_E, we see that it is enough to prove the statement for $E = K_\infty$. Up to replacing Y by $\mathrm{Ker}\, s$ we can assume $s = 0$, $\mathrm{Ker}\, s = Y$. By 1.6.2 ϱ is injective. We fix a K_∞-basis $\{y_1, \ldots, y_h\}$ of Y. Let r_0 be its degree. Being $s = 0$, by 1.7.1, there exists r —that we may assume $r \geq r_0$— such that $\gamma(y_i) = y_i$ for $i = 1, \ldots, h$ and $\gamma \in \Gamma_r$. Let Y_r be the K_r-sub-vector space of Y generated by $y_1, \ldots, y_h$: by construction, Y_r is stable by Γ, that acts on it by means of the finite quotient $\mathrm{Gal}(K_r/K)$. As in the proof of 1.6.6, we can define a 1-cocycle $f\colon \mathrm{Gal}(K_r/K) \to \mathrm{GL}_h(K_r)$ describing the action of $\mathrm{Gal}(K_r/K)$ on Y_r with respect to $\{y_1, \ldots, y_h\}$. By Hilbert's Theorem 90 [**Ser62**, chap. X, Prop. 3], we have

$$\mathrm{H}^1(\mathrm{Gal}(K_r/K), \mathrm{GL}_h(K_r)) = 1.$$

Hence we can assume that $\mathrm{Gal}(K_r/K)$ acts trivially on $y_1, \ldots, y_h$, so that Γ fixes a basis of Y and the map ϱ is therefore surjective. □

1.7.6. LEMMA. *Let E be a field and let Z_1, Z_2 be finite-dimensional E-vector spaces. Let E_0 be an infinite subfield of E, L a sub-E_0-vector space of the E-vector space $\mathcal{L}_E(Z_1, Z_2)$ of E-linear applications from Z_1 to Z_2. The E-vector space $L_E = E \otimes L$ contains an isomorphism if and only if L already contains one.*

PROOF. Let $f \in L_E$ be an isomorphism, $f\colon Z_1 \to Z_2$. Let $\{f_1, \ldots, f_n\}$ be an E-basis of L_E formed by elements of L. Let h be the dimension $\dim_E Z_1 = \dim_E Z_2$ and fix an E-basis of Z_1 and an E-basis of Z_2. For $j = 1, \ldots, n$, let $A_j \in \mathrm{M}_h(E)$ be the matrix of f_j with respect to those basis. Let $P(X_1, \ldots, X_n)$ be the polynomial

$$P(X_1, \ldots, X_n) = \det(X_1 A_1 + X_2 + \ldots + X_n A_n) \in E[X_1, \ldots, X_n].$$

If $f = \sum_{i=1}^n \lambda_i f_i$, $\lambda_i \in E$, we have $P(\lambda_1, \ldots, \lambda_n) \neq 0$, so that P is not identically zero. Being E_0 an infinite field, there exist $\mu_1, \ldots, \mu_n \in E_0$ such that $P(\mu_1, \ldots, \mu_n) \neq 0$ and the element $\sum_{i=1}^n \mu_i f_i$ is an element of L, isomorphism of Z_1 over Z_2. □

1.7.7. COROLLARY. *Two K_∞-representations of Γ, Y_1 and Y_2, are isomorphic in $\mathbf{Rep}_{K_\infty}(\Gamma)$ if and only if $(Y_{1,E}, s_{1,E})$ and $(Y_{2,E}, s_{2,E})$ are isomorphic in $\mathcal{S}_E$.*

1.7.8. Let $W \in \mathbf{Rep}_{\mathbf{C}}(G_K)$. Then we dispose of the L-representation of Γ W^{H_K} and of the K_∞-representation of Γ $(W^{K_K})_f$. We denote by $\Delta_{\mathrm{Sen}}(W)$ the object of $\mathcal{S}_{K_\infty}$ formed by the K_∞-vector space underlying $(W^{K_K})_f$ and by the endomorphism $s_{W,f}$ defined in 1.7.1.

Δ_{Sen} defines a faithful $\otimes$-functor

$$\Delta_{\mathrm{Sen}}\colon \mathbf{Rep}_{\mathbf{C}}(G_K) \to \mathcal{S}_{K_\infty}.$$

By 1.7.7 we see that the knowledge of $\Delta_{\mathrm{Sen}}(W)$ determines — up to isomorphisms — W as $\mathbf{C}$-representation of G_K,

1.8. Classification of C-representations

1.8.1. We keep the notations of 1.2.1—1.2.2. Let W be a $\mathbf{C}$-representation of G_K. In the notations of 1.7.8, we call Sen weights of W the eigenvalues of the endomorphism $s_{W,f}$ in $\overline{K}$. By 1.7.1, the characteristic polynomial of $s_{W,f}$ has coefficients in K. Hence the set of Sen weights of W is stable by G_K.

Let X be a subset of $\overline{K}$ which is stable by G_K. We say that a $\mathbf{C}$-representation W of G_K is of type S_X if its Sen weights are in X. We say that W is of type S_X^m if it is of type S_X and if $s_{W,f}$ is semi-simple.

1.8.2. We denote by $\mathcal{C}(\overline{K})$ the set of the orbists of $\overline{K}$ for the action of G_K. For every indecomposable object W in $\mathbf{Rep_C}(G_K)$, there exists a unique $A \in \mathcal{C}(\overline{K})$ such that W is of type S_A.

Let W be an indecomposable object of type S_A. We can write its Sen endomorphism $s_{W,f}$ as $s_0.s_u = s_u.s_0$, with s_0 semi-simple and s_u unipotent. Let V be the K_∞-vector space underlying $(W^{H_K})_f$, $\overline{V}$ the vector space $V \otimes_{K_\infty} \overline{K}$. We denote again by s_0 the endomorphism of $\overline{V}$ deduced by scalar extension. Then we have:

i) a decomposition of $\overline{V}$ as a direct sum of the eigenspaces of s_0;
ii) a nilpotent endomorphism $\log s_u$ of V.

1.8.3. The $\mathbf{C}$-representations of G_K of type $S_{\{0\}}$ correspond to representations of the additive group $\mathbb{G}_a$. Indeed, to give an action of the additive group $\mathbb{G}_a$ over a K_∞-vector space V comes down to give a nilpotent endomorphism ν of V (so that $\lambda \in K_\infty = \mathbb{G}_a(K_\infty)$ acts over V via $\exp(\lambda\nu)$). Let $K_\infty[\log t]$ be the algebra of polynomials in the variable $\log t$ and coefficients in K_∞. For every $d \geq 1$, we denote by $\mathbb{Z}_p(0;d)$ the sub $\mathbb{Z}_p$-module of $K_\infty[\log t]$ formed by the polynomials in $\log t$ of degree $< d$ with coefficients in $\mathbb{Z}_p$. Hence we see that, up to isomorphisms, there exists a unique indecomposable $\mathbf{C}$-representation of G_K of type $S_{\{0\}}$ of dimension d over $\mathbf{C}$, namely

$$C^K(0;d) = \mathbf{C} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(0;d)$$

where the nilpotent endomorphism ν is $-\frac{\partial}{\partial \log t}$.

Notice that $C^K(0;d)$ is not simple, as $C^K(0;d) \supset C^K(0;d-1) \supset \ldots \supset C^K(0;1)$.

1.8.4. Let W be a simple object of $\mathbf{Rep_C}(G_K)$ and let A be the unique conjugacy class of $\overline{K}$ such that W is of type S_A. Then, for every $d \geq 1$, we can define the indecomposable object of type S_A

$$W \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(0;d).$$

On the other hand, we see that a $\mathbf{C}$-representation W' of G_K is indecomposable of type S_A if and only if there exists $d \in \mathbb{N}^*$ (necessarily unique) such that $W' \cong W \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(0;d)$. Then W' is simple if and only if $d = 1$.

1.8.5. We fix a topological generator γ_0 of Γ. For every $r \in \mathbb{N}$, let $\mathfrak{a}_r$ be the $\mathcal{O}_{\overline{K}}$-sub-module of $\overline{K}$

$$\mathfrak{a}_r = \{\alpha \in \overline{K} \mid v_p(\alpha) > -r - r_K + \frac{1}{p-1}\}$$

where r_K is the integer defined in the proof of 1.7.2. Let $A \in \mathcal{C}(\overline{K})$ and set $P_A(X) = \prod_{\alpha \in A}(X - \alpha) \in K[X]$ be the minimal polynomial of any $\alpha \in A$ over K. Let $K_A \subset \overline{K}$ be the field $K[X]/(P_A(X))$ and denote by β the image of X in K_A. Let d_A be the number of elements in A. Let r_A be the smallest integer r such that an element $\alpha \in A$ belongs to $\mathfrak{a}_r$. By construction, it is the smallest $r \in \mathbb{N}$ such that

$$v_p(\beta \log \chi_p(\gamma)) = v_p(\beta) + v_p(\log(\chi_p(\gamma)) > \frac{1}{p-1}$$

for every $\gamma \in \Gamma_r$. We can therefore define a continuous homomorphism $\varrho_A \colon \Gamma_{r_A} \to K_A^\times$ by

$$\varrho_A(\gamma) = \exp(\beta \log \chi_p(\gamma)).$$

We denote by $M[A]$ the field K_A endowed with the linear and continuous action of Γ_{r_A} given by ϱ_A.

Let $N[A] = K_A[\Gamma] \otimes_{K_A[\Gamma_{r_A}]} M[A]$ be the induced K_A-linear representation of Γ. It is a K_A-vector space of dimension p^{r_A}, since $\{\gamma_0^i \otimes 1\}_{0 \leq i < p^{r_A}}$ is a basis of $N[A]$ over K_A. We denote by $N_\infty[A] = K_\infty \otimes_K N[A]$ the K_∞-representation of Γ deduced by $N[A]$ by scalar extension. We choose a simple sub-object of $N_\infty[A]$ in $\mathbf{Rep}_{K_\infty}(\Gamma)$ and we denote it by $K_\infty[A]$. We set $\mathbf{C}[A]$ to be the $\mathbf{C}$-representation of G_K corresponding to $K_\infty[A]$, i.e.

$$\mathbf{C}[A] = \mathbf{C} \otimes_{K_\infty} K_\infty[A].$$

1.8.6. Theorem. *In the notations 1.8.5, let W be a $\mathbf{C}$-representation of G_K.*

i) *W is simple if and only if there exists $A \in \mathcal{C}(\overline{K})$ such that $W \cong \mathbf{C}[A]$; then W is of type S_A^m and has dimension $d_A p^{s_A}$ over $\mathbf{C}$, where s_A is an integer $0 \leq s_A \leq r_A$ verifying* $\dim_{K_\infty}(K_\infty[A]) = \dim_{\mathbf{C}}(\mathbf{C}[A])$.

ii) *W is indecomposable if and only if there exists $A \in \mathcal{C}(\overline{K})$ such that $W \cong \mathbf{C}[A; d] = \mathbf{C}[A] \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(0; d)$; then W is of type S_A and has dimension $d.d_A p^{s_A}$ over $\mathbf{C}$.*

iii) *There exist natural integers $(h_{A,d}(W))_{A \in \mathcal{C}(\overline{K}), d \in \mathbb{N}^*}$, almost all zero, uniquely determined, such that*

$$W \cong \bigoplus_{A \in \mathcal{C}(\overline{K}), d \in \mathbb{N}^*} \mathbf{C}[A; d]^{h_{A,d}(W)}$$

1.8.7. Lemma ([**Fon04**, Prop. 2.12]). *Let F be a field, E a subfield of F, $\overline{E}$ a separable closure of E, $G_E = \mathrm{Gal}(\overline{E}/E)$, $\eta \colon G_E \to \mathbb{Q}/\mathbb{Z}$ a continuous homomorphism and $b \in F$. Let $E' = \overline{E}^{\operatorname{Ker} \eta}$, N the degree of the cyclic extension E'/E, σ the generator of* $\mathrm{Gal}(E'/E)$ *such that $\eta(\sigma) \equiv 1/N \mod \mathbb{Z}$. Let $\Lambda_{E,F}(\eta, b)$ be the associative and unitary $E' \otimes_E F$-algebra generated by an element c satisfying*

$$c^N = 1 \otimes b; \tag{1.8.7.1}$$

$$c(u \otimes x) = (\sigma(u) \otimes x)c \quad \text{if } u \in E' \text{ and } x \in F.$$

Then the algebra $\Lambda_{E,F}(\eta, b)$ is a central simple algebra. The center of $\Lambda_{E,F}(\eta, b)$ is F and of dimension N^2 over its center. $\Lambda_{E,F}(\eta, b)$ isomorphic to an algebra of square matrices with coefficients in a skew field $D_{E,F}(\eta, b)$.

Theorem 1.8.6 is then a consequence of the previous discussion and of the following

1.8.8. PROPOSITION. *In the notations of 1.8.5, let* $\eta\colon G_K \to \mathbb{Q}/\mathbb{Z}$ *be the unique character of* G_K *that factors through* Γ *and that maps* γ_0 *to* $\frac{1}{p^{r_A}}$. *Let* $b = \varrho_A(\gamma_0^{p^{r_A}})$. *The* K_A*-algebra*

$$E_A = \mathrm{End}_{\mathbf{Rep}_{K_\infty}(\Gamma)}(N_\infty[A])$$

is identified with $\Lambda_{K,K_A}(\eta, b)$. *The skew field* $D_A = D_{K,K_A}(\eta, b)$ *has rank* p^{2s_A}, *where* s_A *is an integer verifying* $0 \le s_A \le r_A$. *We have*

$$\dim_{K_\infty}(K_\infty[A]) = \dim_{\mathbf{C}} \mathbf{C}[A] = d_A p^{s_A}.$$

Moreover, $\mathbf{C}[A]$ *is a simple object of* $\mathbf{Rep}_{\mathbf{C}}(G_K)$ *of type* S_A *and*

$$\mathrm{End}_{\mathbf{Rep}_{K_\infty}(\Gamma)}(K_\infty[A]) = \mathrm{End}_{\mathbf{Rep}_{\mathbf{C}}(G_K)}(\mathbf{C}[A]) = D_A.$$

PROOF. Let $M_\infty[A] = K_\infty \otimes_K M[A]$. For every $s \in \mathbb{N}$ we set $M_s[A] = K_s \otimes_K M[A]$ and $N_s[A] = K_s \otimes_K N[A]$. We have then the following inclusions:

$$\begin{array}{ccccc} M[A] & \subset & M_s[A] & \subset & M_\infty[A] \\ \cap & & \cap & & \cap \\ N[A] & \subset & N_s[A] & \subset & N_\infty[A]. \end{array}$$

To simplify the notation, we set $r = r_A$. For every $s \ge r$ we have the topological generator $\gamma_s = \gamma_0^{p^s}$ of $\Gamma_s \subset \Gamma_r$. By construction, γ_s acts on $M[A]$ by multiplication with the element $b^{p^{s-r}} = \exp(\beta \log \chi_p(\gamma_0^{p^s}))$.

Let $f \in \mathrm{End}_{\mathbf{Rep}_K(\Gamma_s)}(M[A])$. Then for every $\gamma \in \Gamma_s$, we have $f(\gamma(x)) = \gamma f(x)$ if and only if $f(\gamma_0^{p^s}(x)) = \gamma_0^{p^s} f(x)$, i.e. f satisfies $f(b^{p^{s-r}}x) = b^{p^{s-r}} f(x)$ for every $x \in K[A]$. But we have $K_A = K(b^{p^{s-r}})$, since

$$\beta = \frac{\log b^{p^{s-r}}}{\log(\chi_p(\gamma_0^{p^s}))}$$

and $\log(\chi_p(\gamma_0^{p^s})) \in K^\times$. Hence, the natural injection $K_A \to \mathrm{End}_{\mathbf{Rep}_K(\Gamma_s)}(M[A])$ is an isomorphism.

Let $\{e_1, \ldots, e_d\}$ be a basis of K_A over K, seen as ring of endomorphisms $\mathrm{End}_{\mathbf{Rep}_K(\Gamma_s)}(M[A])$. Let $f \in \mathrm{End}_{\mathbf{Rep}_{K_\infty}(\Gamma_r)}(M_\infty[A])$. Then there exists $s \ge r$ such that $f(M[A]) \subset M_s[A]$. Since f is K_∞-linear, we also have $f(M_s[A]) \subset M_s[A]$, so that the restriction f_s of f to $M_s[A]$ is an element of $\mathrm{End}_{\mathbf{Rep}_{K_s}(\Gamma_r)}(M_s[A])$. Since Γ_s acts trivially on K_s, we have

$$\mathrm{End}_{\mathbf{Rep}_{K_s}(\Gamma_s)}(M_s[A]) = K_s \otimes_K \mathrm{End}_{\mathbf{Rep}_K(\Gamma_s)}(M[A]) = K_s \otimes_K K_A.$$

We can therefore find $\lambda_1, \ldots, \lambda_d \in K_s$ such that f_s, as element of $\mathrm{End}_{\mathbf{Rep}_{K_s}(\Gamma_s)}(M_s[A])$, can be written as $f_s = \sum_{i=1}^d \lambda_i \otimes e_i$. Adding the further condition that f_s commutes with the action of γ_r, we have $\gamma_r(\lambda_i) = \lambda_i$ for every $i = 1, \ldots, d$, i.e. $\lambda_i \in K_r$, so that

$$\mathrm{End}_{\mathbf{Rep}_{K_\infty}(\Gamma_r)}(M_\infty[A]) = K_r \otimes_K K_A. \tag{1.8.8.1}$$

By construction, every element of $N_\infty[A]$ can be written in a unique way as $x = \sum_{i=0}^{p^r-1} \gamma_0^i(x_i)$, with $x_i \in M_\infty[A]$. Let $f \in E_A$. Then $f(x) = \sum_{i=0}^{p^r-1} \gamma_0^i(\varphi(x))$ where φ is the restriction of f to $M_\infty[A]$. Therefore the application

$$E_A \to \mathrm{Hom}_{\mathbf{Rep}_{K_\infty}(\Gamma_r)}(M_\infty[A], N_\infty[A]), \quad f \mapsto \varphi$$

is bijective. Let c be the unique element of E_A defined by $c(x) = \gamma_0(x)$ for every $x \in M[A]$ (so that $c(\lambda x) = \lambda c(x) = \lambda\gamma_0(x)$ for every $\lambda \in K_\infty$, $x \in M[A]$). Then, using (1.8.8.1), every element of E_A can be written in a unique way as $\sum_{i=0}^{p^r-1} c^i f_i$, for $f_i \in K_r \otimes_K K_A$. We see therefore that E_A is an algebra over $K_r \otimes_K K_A$ generated by an element c satisfying the conditions (1.8.7.1) of 1.8.7. As $c^{p^r} = b$, we have that the dimension of E_A over its center K_A is p^{2r}. The skew field D_A has rank p^{2s_A} over K_A for a suitable $0 \leq s_A \leq r$ and for any simple sub-object $K_\infty[A]$ we have therefore

$$\mathrm{End}_{\mathbf{Rep}_{K_\infty}(\Gamma)}(K_\infty[A]) = D_A$$

and $\dim_{K_\infty}(K_\infty[A]) = d_A p^{s_A}$. The statement for $\mathbf{C}[A]$ is clear. □

1.9. Hodge-Tate representations

1.9.1. We keep the notations of 1.8.5. Let $W \in \mathbf{Rep}_{\mathbf{C}}(G_K)$. We say that W is deployed (fr. *déployée*) over K if the Sen weights of W are in K. Let $\mathfrak{a}_0^K = \mathfrak{a}_0 \cap K$ be the fractional ideal of $\mathcal{O}_K$ formed by the elements of p-adic valuation $> -r_K + \frac{1}{p-1}$. Every simple $\mathbf{C}$-representation of G_K of type $S_{\mathfrak{a}_0^K}$ has dimension 1 over $\mathbf{C}$ and the ring of its endomorphisms is reduced to K (see 1.8.8).

Among the representations of type $S_{\mathfrak{a}_0^K}$ we have the representations of type $S_{\mathbb{Z}}^m$. These latter are called $\mathbf{C}$-representation of type Hodge-Tate (or simply $\mathbf{C}$-representation Hodge-Tate). Thus $W \in \mathbf{Rep}_{\mathbf{C}}(G_K)$ is Hodge Tate if it is semi-simple and its Sen weights are in $\mathbb{Z}$.

Let V be a p-adic representation of G_K, i.e. $V \in \mathbf{Rep}_{\mathbb{Q}_p}(G_K)$. By base-change we get the corresponding $\mathbf{C}$-representation, namely

$$\mathbf{C} \otimes_{\mathbb{Q}_p} V \in \mathbf{Rep}_{\mathbf{C}}(G_K).$$

We say that V is Hodge-Tate if $\mathbf{C} \otimes_{\mathbb{Q}_p} V$ is Hodge-Tate.

1.9.2. We fix a generator t of the Tate module $\mathbb{Z}_p(1) = \mathrm{T}_p(\mathbb{G}_m)(\overline{K})$. For every $i \in \mathbb{N}$ we denote by $\mathbb{Z}_p(i)$ the i-th power $\mathbb{Z}_p(1)^{\otimes^i}$ and by $\mathbb{Z}_p(-i)$ its $\mathbb{Z}_p$-dual. For every $\mathbb{Z}_p$-module M, we denote by $M(i)$ the i-th Tate twist of M, i.e. $M(i) = M \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(i)$. For $x \in M$ and $u \in \mathbb{Z}_p(i)$, we write xu for $x \otimes u \in M(i)$. The map $x \mapsto xt^i$ is a $\mathbb{Z}_p$-linear bijection between M and $M(i)$, depending on the choice of t.

The group G_K acts over $Z_p(i)$ for every $i \in \mathbb{Z}$: we have $g.u = \chi_p^i(g)u$ for every $g \in G_K$ and $u \in \mathbb{Z}_p(i)$. Similarly, if M is a topological $\mathbb{Z}_p$-module endowed with a linear and continuous action of G_K, we have an induced linear and continuous action on $M(i)$. Namely, we have

$$g(xt^i) = \chi_p^i(g)g(x)t^i \quad \text{for every } g \in G_K, x \in M.$$

We can therefore identify $\mathbf{C}(i) = \mathbf{C} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(i)$ with $\mathbf{C}[\{i\}]$ defined in 1.8.5 for every $i \in \mathbb{Z}$. Indeed, for $A = \{i\}$ we have that Γ acts on $K = K_A$ via ϱ_A, that turns out to be χ_p^i. This identification is not canonical, depending on the choice of a generator t of $\mathbb{Z}_p(1)$, but is G_K-equivariant. Similarly, for every $d \in \mathbb{N}$, $\mathbf{C}[\{i\}; d]$ is isomorphic to $\mathbf{C}(i) \otimes_{\mathbb{Z}_p} Z_p(0; d)$.

Hence, by Fontaine's classification theorem 1.8.6, for any Hodge-Tate object W in $\mathbf{Rep}_{\mathbf{C}}(G_K)$ there exist non negative integers $h_q(W)$, almost always zero and uniquely determined by W, such that

$$W \cong \sum_{q \in \mathbb{Z}} \mathbf{C}(q)^{h_q(W)}.$$

The integer $h_q(W)$ is called the multiplicity of q as a Hodge-Tate weight of W.

1.9.3. Let $B_{\mathrm{HT}} = \mathbf{C}[t^{(1)}, 1/t^{(1)}]$ be the polynomial algebra in the variable $t^{(1)}$. Let t be a generator of $\mathbb{Z}_p(1)$. Then $t = (\varepsilon_n)_{n\in\mathbb{N}}$ where ε_n is a primitive p^n-th root of 1 in $\overline{K}$ and $\varepsilon_{n+1}^p = \varepsilon_n$. For $p \neq 2$, we denote by π_t the unique uniformizer of $\mathbb{Q}_p(\varepsilon_1)$ such that

$$(\pi_t)^{p-1} + p = 0, \quad v_p(\varepsilon_1 - 1 - \pi_t) \geq \frac{2}{p-1}.$$

If $p = 2$ we set $\pi_t = \varepsilon_2 - 1$. Then the map

$$\mathbb{Z}_p(1) = \mathbb{Z}_p t \to B_{\mathrm{HT}}, \quad \lambda t \mapsto \lambda \pi_t t^{(1)}$$

is injective and commutes with the action of G_K. We can identify B_{HT} with $\mathbf{C}[t, t^{-1}] = \bigoplus_{i\in\mathbb{Z}} \mathbf{C}(i)$. By 1.2.4, we have $B_{\mathrm{HT}}^{G_K} = \mathrm{Frac}(B_{\mathrm{HT}})^{G_K} = K$.

1.9.4. For every $\mathbf{C}$-representation W of G_K, we set $D_{\mathrm{HT}}(W) = (B_{\mathrm{HT}} \otimes_{\mathbf{C}} W)^{G_K}$. By 1.6.2—1.6.3, the canonical map

$$\varrho\colon B_{\mathrm{HT}} \otimes_K D_{\mathrm{HT}}(W) \to B_{\mathrm{HT}} \otimes_{\mathbf{C}} W \tag{1.9.4.1}$$

is injective and $\dim_{\mathbf{C}}(D_{\mathrm{HT}}(W)) \leq \dim_{\mathbf{C}}(W)$. Therefore, the representation W is Hodge-Tate if and only if $\dim_{\mathbf{C}}(D_{\mathrm{HT}}(W)) = \dim_{\mathbf{C}}(W)$, that is if and only if (1.9.4.1) is an isomorphism.

1.9.5. Let $V \in \mathbf{Rep}_{Q_p}(G_K)$. Then the dimension inequality in 1.9.4 can be stated as

$$\sum_{i\in\mathbb{Z}} \dim_K(\mathbf{C}(i) \otimes_{\mathbb{Q}_p} V)^{G_K} \leq \dim_{\mathbf{C}}(\mathbf{C} \otimes_{\mathbb{Q}_p} V) = \dim_{\mathbb{Q}_p} V. \tag{1.9.5.1}$$

V is Hodge-Tate if and only if the equality holds in (1.9.5.1).

CHAPTER 2

The Hodge-Tate decomposition Theorem for Abelian Varieties

2.1. Lubin-Tate formal groups and differential modules

2.1.1. Let K be a complete discrete valuation field with perfect residue field k of characteristic $p > 0$, $\mathcal{O}_K$ the ring of integers of K. We fix a separable closure $\overline{K}$ of K and we denote by G_K the absolute Galois group of $\overline{K}$ over K. Let $\mathcal{O}_{\mathbf{C}}$ be the p-adic completion of $\mathcal{O}_{\overline{K}}$ and let $\mathbf{C}$ be its field of fractions.

Let E and K_0 be discrete valuation fields and let $E \to K_0 \to K$ an injective homomorphism such that E has finite residue field k_E, a uniformizer of E is a uniformizer of K_0, K is a finite, separable and totally ramified extension of K_0. Namely,

i) If K has characteristic 0, we take for E any finite extension of $\mathbb{Q}_p$ contained in K. If π is a uniformizer of E, then K_0 is the subfield of K obtained by adjoining π to the fraction field of the ring of Witt vectors $W(k)$.
ii) If K has characteristic p, we have $E = k_E((T)) \subseteq k((T)) = K_0 = K$.

We fix a uniformizer π of E. We denote by v the valuation of $\mathbf{C}$, extending the valuation of $\overline{K}$, normalized by $v(\pi) = 1$. Given any subfield L of $\mathbf{C}$, we denote by $\mathcal{O}_L = \{x \in L \mid v(x) \geq 0\}$ its valuation ring, by $U_L = \{x \in L \mid v(x) = 0\}$ the group of units of $\mathcal{O}_L$ and by $\mathfrak{m}_L = \{x \in L \mid v(x) > 0\}$ the maximal ideal. If I is a sub-$\mathcal{O}_L$-module of L which is free of rank 1, we denote by $v(I)$ the valuation of a generator of I.

2.1.2. Let $\Gamma \in \mathcal{O}_K[[X, Y]]$ be a formal power series in the variables X and Y and coefficients in $\mathcal{O}_K$. We say that Γ is a one-parameter commutative formal group law over $\mathcal{O}_K$ if the following identities are satisfied:

(1) $\Gamma(X, \Gamma(Y, Z)) = \Gamma(\Gamma(X, Y), Z)$ [associativity];
(2) $\Gamma(X, 0) = X$, $\Gamma(Y, 0) = Y$;
(3) $\Gamma(X, Y) = \Gamma(Y, X)$ [commutativity];

It follows immediately that there exist a unique $G(X) \in \mathcal{O}_K[[X]]$ such that $\Gamma(X, G(X)) = 0$ and that $\Gamma(X, Y) = X + Y \mod (X, Y)^2$. If Γ and Γ' are one-parameter commutative formal group laws over $\mathcal{O}_K$, a morphism from Γ to Γ' is a power series f in one variable over $\mathcal{O}_K$ with no constant term such that $f(\Gamma(X, Y)) = \Gamma'(f(X), f(Y))$.

2.1.3. Let Γ be a one-parameter commutative formal group law over $\mathcal{O}_K$ and let $x, y \in \mathfrak{m}_K$. Then the series $\Gamma(x, y)$ converges and its sum belongs to $\mathfrak{m}_K$. Under this composition law, $\mathfrak{m}_K$ is a group which we denote $\Gamma(\mathfrak{m}_K)$. We put

$$\Gamma(\mathfrak{m}_{\overline{K}}) = \varinjlim_{\substack{\overline{K} \supset L \\ L/K \text{ finite}}} \Gamma(\mathfrak{m}_L)$$

If we equip $\mathcal{O}_K[[T]]$ with the T-adic topology and we consider $\mathcal{O}_K$ with the π-adic topology, we have a canonical isomorphism

$$\mathfrak{m}_K \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{cont},\mathcal{O}_K}(\mathcal{O}_K[[T]], \mathcal{O}_K), \quad x \mapsto \varphi_x(T \mapsto x),$$

the identification being compatible with the group structure induced by Γ. By passage to the inductive limit from the finite case we get

$$\Gamma(\mathfrak{m}_{\overline{K}}) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{cont},\mathcal{O}_K}(\mathcal{O}_K[[T]], \mathcal{O}_{\overline{K}}). \tag{2.1.3.1}$$

2.1.4. We equip $\mathcal{O}_K[[T]]$ with the T-adic topology. Let $\widehat{\Omega}^1_{\mathcal{O}_K[[T]]/\mathcal{O}_K}$ be the module of continuous $\mathcal{O}_K$-differentials of $\mathcal{O}_K[[T]]$: it is a free $\mathcal{O}_K[[T]]$-module of basis $\mathrm{d}T$. Let Γ be a one-parameter commutative formal group law over $\mathcal{O}_K$. An invariant differential with respect to the formal group law Γ is a differential form

$$\omega = \alpha(T)\mathrm{d}T \in \widehat{\Omega}^1_{\mathcal{O}_K[[T]]/\mathcal{O}_K}$$

satisfying

$$\alpha(\Gamma(X,Y))\mathrm{d}\Gamma(X,Y) = \alpha(X)\mathrm{d}X + \alpha(Y)\mathrm{d}Y \tag{2.1.4.1}$$

or, equivalently,

$$\alpha(\Gamma(X,Y))\Gamma_X(X,Y) = \alpha(X) \tag{2.1.4.2}$$

where $\Gamma_X(X,Y)$ is the partial derivative of Γ with respect to the first variable. We denote by ω_Γ the sub module of $\widehat{\Omega}^1_{\mathcal{O}_K[[T]]/\mathcal{O}_K}$ of the invariant differentials. We say that $\alpha(X)\mathrm{d}T \in \omega_\Gamma$ is normalized if $\alpha(0) = 1$.

2.1.5. PROPOSITION. *We keep the assumptions of 2.1.4. There exists a unique normalized invariant differential with respect to the formal group law Γ, given by the formula*

$$\omega = \frac{\mathrm{d}T}{F_X(0,T)}.$$

ω_Γ is a free $\mathcal{O}_K$-module of rank 1, generated by ω.

PROOF. Suppose $\alpha(T)\mathrm{d}T$ is an invariant differential on Γ. Putting $X = 0$ in (2.1.4.2) gives

$$\alpha(Y)\Gamma_X(0,Y) = \alpha(0)$$

as $\Gamma(0,Y) = Y$. Since $\Gamma_X(0,T) \equiv 1 \mod (T)$, we see that $\Gamma_X(0,T)^{-1} \in \mathcal{O}_K[[T]]$. Hence $\alpha(T)$ is determined by $\alpha(0)$ and every invariant differential is of the form $a\omega$ with $a \in \mathcal{O}_K$ and

$$\omega = \Gamma_X(0,T)^{-1}\mathrm{d}T.$$

Since ω is normalized, it only remains to show that it is invariant. To prove this, we differentiate the relation

$$\Gamma(X, \Gamma(Y,Z)) = \Gamma(\Gamma(X,Y),Z)$$

with respect to X to obtain

$$\Gamma_X(X,\Gamma(Y,Z)) = \Gamma_X(\Gamma(X,Y),Z)\Gamma_X(X,Y).$$

Putting $X = 0$ gives the desired result. □

2.1.6. Let q be the cardinality of the residue field k_E and let $\mathcal{F}_\pi$ be the set of formal power series $f \in \mathcal{O}_E[[T]]$ such that $f(T) \equiv \pi T \mod (T^2)$ and $f(T) \equiv T^q \mod (\pi)$.

2.1.7. THEOREM ([**LT65**, Th. 1 and 2]). (i) *For each $f \in \mathcal{F}_\pi$ there exists a unique $F_f(X,Y) \in \mathcal{O}_E[[X,Y]]$ such that*

$$F_f(X,Y) \equiv X + Y \mod (X,Y)^2,$$
$$f(F_f(X,Y)) = F_f(f(X), f(Y)).$$

The series F_f defines a one-parameter commutative formal group law over $\mathcal{O}_E$.

(ii) *For each $a \in \mathcal{O}_E$ and $f, g \in \mathcal{F}_\pi$ there exists a unique $[a]_{f,g}(T) \in \mathcal{O}_E[[T]]$ such that*

$$[a]_{f,g}(T) \equiv aT \mod (X,Y)^2 \quad \text{and} \quad f([a]_{f,g}(T)) = [a]_{f,g}(g(T)).$$

The series $[a]_{f,g}$ is a formal homomorphism from F_g to F_f.

(ii) *The map $a \mapsto [a]_f = [a]_{f,f}$ defines an isomorphism from $\mathcal{O}_E$ to $\mathrm{End}_{\mathcal{O}_E}(F_f)$, inverse of the morphism $\sum_{i \geq 1} c_i X^i \mapsto c_1$. Under this isomorphism,*

$$[\pi]_f(T) = f(T).$$

The F_f's for $f \in \mathcal{F}_\pi$ are canonically isomorphic by means of the isomorphisms $[1]_{f,g}$. We call any one-parameter commutative formal group law over $\mathcal{O}_E$ of the form F_f, for $f \in \mathcal{F}_\pi$, a Lubin-Tate formal group over $\mathcal{O}_E$.

2.1.8. Let $f \in \mathcal{F}_\pi$ and let $\Gamma = F_f$ be the corresponding Lubin-Tate formal group. By 2.1.7, $\Gamma(\mathfrak{m}_{\overline{K}})$ is canonically equipped with an $\mathcal{O}_E$-module structure. For $a \in \mathcal{O}_E$, $x \in \mathfrak{m}_{\overline{K}}$ we write $a.x = [a]_f(x)$. For every $n \geq 0$, let

$$\Gamma_{\pi^n}(\mathfrak{m}_{\overline{K}}) = \{x \in \Gamma(\mathfrak{m}_{\overline{K}}) \mid \pi^n.x = 0\}$$

be the set of π^n-torsion points of $\Gamma(\mathfrak{m}_{\overline{K}})$. It is naturally an $\mathcal{O}_E/\pi^n\mathcal{O}_E$-module. Moreover, the maps $\Gamma_{\pi^{n+1}}(\mathfrak{m}_{\overline{K}}) \to \Gamma_{\pi^n}(\mathfrak{m}_{\overline{K}})$ given by $x \mapsto \pi.x$ are $\mathcal{O}_E$-linear and $\Gamma_{\pi^0}(\mathcal{O}_{\overline{K}}) = 0$. We call the projective limit

$$\mathrm{T}_\pi(\Gamma) = \varprojlim \Gamma_{\pi^n}(\mathfrak{m}_{\overline{K}})$$

the Tate module of Γ.

2.1.9. PROPOSITION. *Under the assumptions of 2.1.8, $\mathrm{T}_\pi(\Gamma)$ is a free $\mathcal{O}_E$-module of rank* 1.

PROOF. According to 2.1.7, we may choose $f(X) = \pi X + X^q$. Firstly, we prove that $\Gamma(\mathfrak{m}_{\overline{K}})$ is π-divisible. With this choice of f, the map

$$\Gamma(\mathfrak{m}_{\overline{K}}) \xrightarrow{\pi} \Gamma(\mathfrak{m}_{\overline{K}})$$

is given by $x \mapsto \pi x + x^q$. For every $\alpha \in \mathfrak{m}_{\overline{K}}$, the polynomial $f(X) - \alpha$ is separable and so solvable in $\overline{K}$. All its solutions belong clearly to $\mathfrak{m}_{\overline{K}}$. To prove that $\mathrm{T}_\pi(\Gamma)$ is a free $\mathcal{O}_E$-module of rank 1, it's enough to show that, for every $n \geq 1$, $\Gamma_{\pi^n}(\mathfrak{m}_{\overline{K}})$ is isomorphic to $\mathcal{O}_E/(\pi^n)$ as $\mathcal{O}_E$-module. We proceed by induction on n. For $n = 1$, $\Gamma_\pi(\mathfrak{m}_{\overline{K}})$ is the set of solutions of the equation $f(X) = 0$: it has therefore q elements and it is isomorphic to $\mathcal{O}_E/(\pi)$. Consider the sequence

$$0 \to \Gamma_\pi(\mathfrak{m}_{\overline{K}}) \to \Gamma_{\pi^n}(\mathfrak{m}_{\overline{K}}) \xrightarrow{\pi} \Gamma_{\pi^{n-1}}(\mathfrak{m}_{\overline{K}}) \to 0. \tag{2.1.9.1}$$

Since $\Gamma(\mathfrak{m}_{\overline{K}})$ is π-divisible, (2.1.9.1) is exact. By induction hypothesis, $\Gamma_{\pi^{n-1}}(\mathfrak{m}_{\overline{K}}) \cong \mathcal{O}_E/(\pi^{n-1})$, and the sequence (2.1.9.1) cannot split, since $\Gamma_{\pi^n}(\mathfrak{m}_{\overline{K}})$ contains an element of order exactly π^n: it is enough to divide a generator of $\Gamma_{\pi^{n-1}}(\mathfrak{m}_{\overline{K}})$ by π. $\square$

2.1.10. Let Γ be a Lubin-Tate formal group law over $\mathcal{O}_E$. Let $u \in \Gamma(\mathfrak{m}_{\overline{K}})$: according to 2.1.3.1, u corresponds to $\varphi_u \in \mathrm{Hom}_{\mathrm{cont},\mathcal{O}_E}(\mathcal{O}_E[[T]], \mathcal{O}_{\overline{K}})$. Let

$$\omega = \alpha(T)\mathrm{d}T \in \widehat{\Omega}^1_{\mathcal{O}_E[[T]]/\mathcal{O}_E}$$

be a continuous differential form. We denote by $u^*(\omega)$ the pull-back $\varphi_u(\alpha(T))\mathrm{d}\varphi_u(T)$: it is a well defined element in $\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}$. Indeed, by construction, the $\mathcal{O}_E$-linear and continuous morphism φ_u factors through a finite extension L/K

$$\varphi_u \colon \mathcal{O}_E[[T]] \to \mathcal{O}_L$$

where $u = \varphi_u(T) \in \mathfrak{m}_L \subset \mathfrak{m}_{\overline{K}}$. Since L is complete, $\varphi_u(\alpha(T)) = \alpha(u)$ converges in $\mathcal{O}_L$ and we can consider $\alpha(u)\mathrm{d}u$ as an element in $\Omega^1_{\mathcal{O}_L/\mathcal{O}_K}$. We denote by $u^*(\omega)$ its image by the canonical map

$$\Omega^1_{\mathcal{O}_L/\mathcal{O}_K} \to \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}.$$

Restricting to the sub-module of invariant differentials, we have a map:

$$\langle\ ,\ \rangle \colon \Gamma(\mathfrak{m}_{\overline{K}}) \times \omega_\Gamma \to \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K} \quad (u,\omega) \mapsto \langle u,\omega\rangle = u^*(\omega).$$

2.1.11. Proposition. *The pairing $\langle\ ,\rangle$ is $\mathcal{O}_E$-bilinear and it is compatible with the action of G_K, i.e. for any $g \in G_K$, $u \in \Gamma(\mathfrak{m}_{\overline{K}})$, $\omega \in \omega_\Gamma$ we have $\langle g(u),\omega\rangle = g(\langle u,\omega\rangle)$.*

Proof. Indeed, for $u, u' \in \Gamma(\mathfrak{m}_{\overline{K}})$ and $\omega \in \omega_\Gamma$, $\langle u+u',\omega\rangle = \langle u,\omega\rangle + \langle u',\omega\rangle$ by (2.1.4.1). The fact that $\langle au,\omega\rangle = a\langle u,\omega\rangle$ for any $a \in \mathcal{O}_E$, $\omega \in \omega_\Gamma$, $u \in \Gamma(\mathfrak{m}_{\overline{K}})$ follows from the identification of $\mathcal{O}_E$ with $\mathrm{End}_{\mathcal{O}_E}(\Gamma)$ in 2.1.7. The linearity in the second variable and the compatibility with the action of G_K are clear. $\square$

2.1.12. Let Γ be a Lubin-Tate formal group over $\mathcal{O}_E$. Let G_K act trivially on ω_Γ and consider the $\overline{K}$-vector space

$$\overline{K} \otimes_{\mathcal{O}_E} \mathrm{T}_\pi(\Gamma) \otimes_{\mathcal{O}_E} \omega_\Gamma.$$

By 2.1.9 and 2.1.5, it is a $\overline{K}$-vector space of dimension 1, endowed with a semilinear continuous action of G_K.

Let $\alpha \in \overline{K} \otimes_{\mathcal{O}_E} \mathrm{T}_\pi(\Gamma) \otimes_{\mathcal{O}_E} \omega_\Gamma$. Then α can be written (in a non-unique way) as

$$\alpha = \frac{a}{\pi^r} \otimes u \otimes \omega$$

with $u = (u_n)_{n\in\mathbb{N}} \in \mathrm{T}_\pi(\Gamma)$, $a \in \mathcal{O}_{\overline{K}}$, $r \in \mathbb{N}$ and $\omega \in \omega_\Gamma$. It follows immediately from 2.1.11 and from the definition of $\mathrm{T}_\pi(\Gamma)$ that the element $au_r^*(\omega)$ depends only on α, so that the map

(2.1.12.1)
$$\xi_{K,\Gamma} = \xi \colon \overline{K} \otimes_{\mathcal{O}_E} \mathrm{T}_\pi(\Gamma) \otimes_{\mathcal{O}_E} \omega_\Gamma \to \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}$$
$$\alpha = \frac{a}{\pi^r} \otimes u \otimes \omega \mapsto au_r^*(\omega)$$

is well defined, $\mathcal{O}_{\overline{K}}$-linear and compatible with the action of G_K.

Let $\mathcal{D}_{K/K_0}$ be the different of the extension K/K_0 and let $\mathfrak{a}_{K,\Gamma}$ be the $\mathcal{O}_{\overline{K}}$-module

$$\mathfrak{a}_{K,\Gamma} = \Big\{a \in \overline{K} \mid v(a) \geq -v(\mathcal{D}_{K/K_0}) - \frac{1}{q-1}\Big\}.$$

2.1.13. THEOREM ([**Fon82**, Thm. 1]). *Under the assumptions of 2.1.12, the map ξ is surjective and*

$$\mathrm{Ker}(\xi) = \mathfrak{a}_{K,\Gamma} \otimes_{\mathcal{O}_E} T_\pi(\Gamma) \otimes_{\mathcal{O}_E} \omega_\Gamma.$$

2.2. The proof of Theorem 2.1.13

2.2.1. Let K be as in 2.1. For any field extension L/K, we denote by $\mathcal{D}_{L/K}$ the different of L/K and by $\mathrm{d}_{L/K}\colon \mathcal{O}_L \to \Omega^1_{\mathcal{O}_L/\mathcal{O}_K}$ the universal derivation.

2.2.2. LEMMA. *Let $K \subseteq M \subseteq L$ be a tower of finite and separable field extensions, u the canonical map $\Omega^1_{\mathcal{O}_L/\mathcal{O}_K} \xrightarrow{u} \Omega^1_{\mathcal{O}_L/\mathcal{O}_M}$. Then, for any $\omega \in \Omega^1_{\mathcal{O}_L/\mathcal{O}_K}$, we have:*

$$v(\mathrm{Ann}(u(\omega))) = \max\{0, v(\mathrm{Ann}(\omega)) - v(\mathcal{D}_{M/K})\}.$$

PROOF. Let b be a generator of $\mathcal{O}_L$ as an $\mathcal{O}_K$-algebra and let $\omega = a\mathrm{d}_{L/K}b \in \Omega^1_{\mathcal{O}_L/\mathcal{O}_K}$ be a non-zero differential form. Since $\Omega^1_{\mathcal{O}_L/\mathcal{O}_K}$ is generated by $\mathrm{d}_{L/K}b$ and is killed by $\mathcal{D}_{L/K}$, we have $v(\mathrm{Ann}(\omega)) = v(\mathcal{D}_{L/K}) - v(a)$. By definition $u(\omega) = a\mathrm{d}_{L/M}b$, hence

$$v(\mathrm{Ann}(u(\omega))) = \max\{0, v(\mathcal{D}_{L/M}) - v(a)\}.$$

By [**Ser62**, chap. III, Prop. 8], we have

$$v(\mathcal{D}_{L/M}) = v(\mathcal{D}_{L/K}) - v(\mathcal{D}_{M/K})$$

and we can conclude. □

2.2.3. LEMMA. *Let $K \subseteq M \subseteq L$ be a tower of finite and separable field extensions. Let $\iota\colon \Omega^1_{\mathcal{O}_M/\mathcal{O}_K} \to \Omega^1_{\mathcal{O}_L/\mathcal{O}_K}$ be the map induced by the inclusion $\mathcal{O}_M \subset \mathcal{O}_L$. Then, for every $\omega \in \Omega^1_{\mathcal{O}_M/\mathcal{O}_K}$, we have*

$$\mathrm{Ann}_{\mathcal{O}_L}(\iota(\omega)) = \mathcal{O}_L\,\mathrm{Ann}_{\mathcal{O}_M}(\omega).$$

PROOF. It is enough to consider the case where L/M is unramified or totally ramified. If $\mathcal{O}_L/\mathcal{O}_M$ is étale, then $\Omega^1_{\mathcal{O}_M/\mathcal{O}_K} \otimes_{\mathcal{O}_M} \mathcal{O}_L \cong \Omega^1_{\mathcal{O}_L/\mathcal{O}_K}$ by [**EGA IV**, 0.20.5.8] and the statement is clear.

Suppose now that L/M is totally ramified. Let b' be a uniformizer for L: it is a root of an Eistenstein polynomial $P(X) = \sum_{i=0}^n a_i X^i \in \mathcal{O}_M[X]$, with $-a_0 = b$ a uniformizer for M. Let $\omega = a\mathrm{d}_{M/K}b \in \Omega^1_{\mathcal{O}_M/\mathcal{O}_K}$ be a non-zero differential and let $\mathfrak{a}$ be its annihilator. Let $\iota(\omega) = a\mathrm{d}_{L/K}b \in \Omega^1_{\mathcal{O}_L/\mathcal{O}_K}$ be the image of ω. As $b = \sum_{i=1}^n a_i (b')^i$, we have

$$\mathrm{d}_{L/K}b = (a_1 + 2a_2 b' + \ldots + n(b')^{n-1})\mathrm{d}_{L/K}b' = P'(b')\mathrm{d}_{L/K}b',$$

so that $\iota(\omega) = aP'(b')\mathrm{d}_{L/K}b'$. Hence $c \in \mathrm{Ann}_{\mathcal{O}_L}(\iota(\omega))$ if and only if

$$v(caP'(b')) \geq v(\mathcal{D}_{L/K}). \tag{2.2.3.1}$$

Since $\mathcal{D}_{L/M} = (P'(b'))$ by [**Ser62**, chap. III, Cor. 2 to Prop. 11] and since $\mathcal{D}_{L/K} = \mathcal{D}_{L/M}\mathcal{D}_{M/K}$ by [**Ser62**, chap. III, Prop. 8], (2.2.3.1) is equivalent to $v(c) \geq v(\mathcal{D}_{M/K}) - v(a) = v(\mathfrak{a})$, i.e.

$$\mathrm{Ann}_{\mathcal{O}_L}(\iota(\omega)) = \mathcal{O}_L\mathfrak{a}.$$

□

2.2.4. The modules $\Omega^1_{\mathcal{O}_L/\mathcal{O}_K}$ for $K \subseteq L$ varying in the set of finite and separable extensions of K contained in $\overline{K}$ form an inductive system and we have

$$\varinjlim \Omega^1_{\mathcal{O}_L/\mathcal{O}_K} = \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K},$$

that makes clear the fact that $\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}$ is a torsion $\mathcal{O}_{\overline{K}}$-module. By 2.2.3, the canonical map $\Omega^1_{\mathcal{O}_L/\mathcal{O}_K} \to \varinjlim \Omega^1_{\mathcal{O}_L/\mathcal{O}_K}$ is injective.

Let $\omega \in \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}$, L a finite and separable extension of K such that $\omega \in \Omega^1_{\mathcal{O}_L/\mathcal{O}_K} \subset \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}$, $\mathfrak{a}$ the annihilator $\mathrm{Ann}_{\mathcal{O}_L}(\omega) \subset \mathcal{O}_L$. Then the annihilator $\mathrm{Ann}(\omega)$ of ω in $\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}$ is simply given by $\mathcal{O}_{\overline{K}}\mathfrak{a}$: in particular $\mathrm{Ann}(\omega)$ is a principal ideal of $\mathcal{O}_{\overline{K}}$ and its valuation is the valuation of $\mathfrak{a}$.

2.2.5. Lemma. *Let $\omega, \omega' \in \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}$. Then we have $\mathrm{Ann}(\omega) \subseteq \mathrm{Ann}(\omega')$ if and only if there exists $c \in \mathcal{O}_{\overline{K}}$ such that $\omega' = c\omega$.*

Proof. It is clear that $\omega' = c\omega$ for some $c \in \mathcal{O}_{\overline{K}}$ implies the inclusion between the annihilators.

Assume $\mathrm{Ann}(\omega) \subseteq \mathrm{Ann}(\omega')$. The case $\omega' = 0$ is trivial, so we can assume ω' and ω both non-zero: indeed $\omega' \neq 0$ implies $\mathrm{Ann}(\omega')$ — and a fortiori $\mathrm{Ann}(\omega)$ — different from $\mathcal{O}_{\overline{K}}$, so that also ω is non-zero. Let L be a finite and separable extension such that $\omega, \omega' \in \Omega^1_{\mathcal{O}_L/\mathcal{O}_K}$. If b is a uniformizer of L, we can write $\omega = a\mathrm{d}b$ and $\omega' = a'\mathrm{d}b$, with $a, a' \in \mathcal{O}_L$.

As ω' and ω are both non-zero, we have $v(a) < v(\mathcal{D}_{L/K})$ and $v(a') < v(\mathcal{D}_{L/K})$, while $v(\mathrm{Ann}(\omega)) = v(\mathcal{D}_{L/K}) - v(a)$ and $v(\mathrm{Ann}(\omega')) = v(\mathcal{D}_{L/K}) - v(a')$. The assumption $\mathrm{Ann}(\omega) \subseteq \mathrm{Ann}(\omega')$ implies

$$v(\mathcal{D}_{L/K}) - v(a) \geq v(\mathcal{D}_{L/K}) - v(a') \quad \text{hence} \quad v(a') \geq v(a)$$

so that $a' \in a\mathcal{O}_L$, i.e. there exists a $c \in \mathcal{O}_{\overline{K}}$ such that $\omega' = c\omega$. □

2.2.6. We consider again the notations of 2.1.12: Γ is a Lubin-Tate formal group over $\mathcal{O}_E$ and $\mathrm{T}_\pi(\Gamma)$ is its Tate module. We fix a generator $(\pi_r)_{r\in\mathbb{N}}$ of $\mathrm{T}_\pi(\Gamma)$ over $\mathcal{O}_E$: for every $r \geq 1$, π_r is a generator of the rank one $\mathcal{O}_E/\pi^r$-module $\Gamma_{\pi^r}(\mathfrak{m}_{\overline{K}})$.

Let E_r be the field $E[\pi_r]$. From [**LT65**, Theorem 2] and [**CF67**, VI, §3], we know that the field extensions $E_r = E[\Gamma_{\pi^r}(\mathfrak{m}_{\overline{K}})]$ of E depend only on the uniformizer π of E and are totally ramified, finite, abelian Galois extensions of E. Moreover, π_r is a uniformizer of E_r.

2.2.7. Proposition. *For every $r \geq 1$ we have $v(\mathcal{D}_{E_r/E}) = r - \frac{1}{q-1}$.*

Proof. By [**CF67**, p. 152], we have:

i) the Galois group $\mathrm{Gal}(E_r/E)$ is canonically isomorphic to the quotient

$$U_E/U_E^{(r)} = U_E/(1 + \pi^r\mathcal{O}_E);$$

ii) $e_{E_r/E} = [E_r : E] = q^{r-1}(q-1)$;

Under the isomorphism $U_E/U_E^{(r)} \xrightarrow{\sim} G = \mathrm{Gal}(E_r/E)$, the subgroup $U_E^{(i)}/U_E^{(r)}$ maps onto the ramification group G_{q^i-1}. Hence, from the filtration

$$U_E/U_E^{(r)} \supset U_E^{(1)}/U_E^{(r)} \supset \cdots U_E^{(r)}/U_E^{(r)} = 1,$$

we get that a complete set of ramification groups for the extension E_r/E is given by

$$\begin{aligned} G &= G_0; \\ G_1 = \ldots = G_{q-2} &= G_{q-1}; \\ G_q = \ldots &= G_{q^2-1}; \\ &\ldots \\ 1 &= G_{q^r-1}. \end{aligned}$$

The corresponding upper numbering is $G^i = G_{q^i-1}$ and

$$[G^0 : G^1] = q-1 \quad [G^i : G^{i+1}] = q.$$

By [**Ser62**, chap. IV, Prop. 4], we have

$$v_{E_r}(\mathcal{D}_{E_r/E}) = \sum_{s \neq 1} i_G(s)$$

where $i_G(s) = v_{E_r}(s(\pi_r) - \pi_r)$ for $s \in G$. Moreover:

$$v_{E_r}(\mathcal{D}_{E_r/E}) = \sum_{i=0}^{r-1} \sum_{s \in G^i \setminus G^{i+1}} i_G(s)$$

and the function $i_G(s)$ is constant for $s \in G^i \setminus G^{i+1}$ and equal to q^i for every i. For $i \geq 1$ we have that $\#G^i = q^{r-i}$ and that $\#G^i \setminus G^{i+1} = (q-1)q^{r-i-1}$, where $\#S$ denotes the cardinality of the (finite) set S. Hence:

$$\sum_{i=0}^{r-1} \sum_{s \in G^i \setminus G^{i+1}} i_G(s) = (q-2)q^{r-1} + \sum_{i=1}^{r-1} q^i (q-1) q^{r-i-1} = q^{r-1}(r(q-1)-1).$$

As $v(\mathcal{D}_{E_r/E}) = \frac{1}{e_{E_r/E}} v_{E_r}(\mathcal{D}_{E_r/E})$, we deduce that

$$v(\mathcal{D}_{E_r/E}) = \frac{1}{q^{r-1}(q-1)} q^{r-1}(r(q-1)-1) = r - \frac{1}{q-1}.$$

□

2.2.8. Corollary. *Let ω_0 be a generator of the module of invariant differentials ω_Γ. Then for any non-negative integer r we have:*

(2.2.8.1) $$v(\mathrm{Ann}(\pi_r^*(\omega_0))) = \max\left\{0, r - \frac{1}{q-1} - v(\mathcal{D}_{K/K_0})\right\}$$

Proof. The statement is evident for $r = 0$ (since $u_0 = 0$), so we can assume $r \geq 1$. By passing to the limit in 2.2.2, we have

$$v(\mathrm{Ann}(\nu(\omega))) = \max\{0, v(\mathrm{Ann}(\omega)) - v(\mathcal{D}_{K/K_0})\}$$

where ν is the canonical map $\nu \colon \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_{K_0}} \to \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}$. We can therefore assume that $K = K_0$.

Let P_r be the minimal polynomial of π_r over E: it is an Eisenstein polynomial. Since the uniformizer π of E is a uniformizer of K, then $K_r = K[\pi_r] = K \otimes_E E_r$ is a field extension of K, totally ramified, with π_r as uniformizer.

Since $\mathcal{O}_{K_r} = \mathcal{O}_K[\pi_r]$, $\mathrm{d}\pi_r$ generates $\Omega^1_{\mathcal{O}_{K_r}/\mathcal{O}_K}$ and we have:

$$v(\mathrm{Ann}(\mathrm{d}\pi_r)) = v(P_r'(\pi_r)) = v(\mathcal{D}_{E_r/E}).$$

By 2.1.5, we know that ω_0 is of the form $\alpha(T)\mathrm{d}T$ with $\alpha(T)$ invertible in $\mathcal{O}_E[[T]]$. Hence, for every $r \geq 1$,

$$\pi_r^*(\omega_0) = \alpha(\pi_r)\mathrm{d}\pi_r$$

with $\alpha(\pi_r)$ unit. Therefore $v(\mathrm{Ann}(\pi_r^*(\omega_0))) = v(\mathcal{D}_{E_r/E})$ and the statement follows from 2.2.7. □

Proof of Theorem 2.1.13. We first prove the surjectivity of the map ξ. Let ω_0 be a generator of ω_Γ and let $u = (\pi_n)_{n\in\mathbb{N}}$ be a generator of $\mathrm{T}_\pi(\Gamma)$. Let $\omega \in \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}$ and let r be an integer such that

$$v(\mathrm{Ann}(\omega)) \leq r - \frac{1}{q-1} - v(\mathcal{D}_{K/K_0}) \leq v(\mathrm{Ann}(\pi_r^*(\omega_0)))$$

by 2.2.8. Hence $\mathrm{Ann}(\omega) \supseteq \mathrm{Ann}(\pi_r^*(\omega_0))$, so that there exists $c \in \mathcal{O}_{\overline{K}}$ such that $\omega = c.\pi_r^*(\omega_0)$ (by 2.2.5) and

$$\omega = \xi\Big(\frac{c}{\pi^r} \otimes u \otimes \omega_0\Big),$$

proving the surjectivity of ξ.

We now determine the kernel: any element $\alpha \in \overline{K} \otimes \mathrm{T}_\pi(\Gamma) \otimes \omega_\Gamma$ can be written in a unique way as $a \otimes u \otimes \omega_0$, with $a \in \overline{K}$. Let $r \in \mathbb{N}$ such that $r \geq \frac{1}{q-1} + v(\mathcal{D}_{K/K_0})$ and such that $\pi^r a \in \mathcal{O}_{\overline{K}}$. The element α is in $\mathrm{Ker}(\xi)$ if and only if $v(\mathrm{Ann}(\xi(\alpha))) \leq 0$ (the annihilator taken in $\overline{K}$). Hence

$$v(\pi^r a) \geq r - \frac{1}{q-1} - v(\mathcal{D}_{K/K_0}),$$

so that $\alpha \in \mathrm{Ker}\,\xi$ if and only if $\alpha \in \mathfrak{a} \otimes \mathrm{T}_\pi(\Gamma) \otimes \omega_\Gamma$. □

2.3. Consequences and corollaries

2.3.1. We keep the assumptions of 2.1.12. Let $\mathrm{T}_\pi(\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_E})$ be the π-Tate module of the $\mathcal{O}_E$-module $\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}$, i.e.

$$\mathrm{T}_\pi(\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_E}) = \mathrm{Hom}_{\mathcal{O}_E}(E/\mathcal{O}_E, \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K})$$

and let $V_\pi(\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K})$ be the E-vector space

$$V_\pi(\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}) = \mathrm{Hom}_{\mathcal{O}_E}(E, \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}).$$

2.3.2. Corollary. *Let $\widehat{\mathfrak{a}}$ be the π-adic completion of $\mathfrak{a}$. We have the following canonical isomorphisms of $\mathcal{O}_{\overline{K}}$-modules (resp. $\mathcal{O}_{\mathbf{C}}$-modules, $\mathbf{C}$-vector spaces)*

$$\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K} \cong (\overline{K}/\mathfrak{a}) \otimes_{\mathcal{O}_E} \mathrm{T}_\pi(\Gamma) \otimes_{\mathcal{O}_E} \omega_\Gamma, \tag{2.3.2.1}$$

$$\mathrm{T}_\pi(\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_E}) \cong \widehat{\mathfrak{a}} \otimes_{\mathcal{O}_E} \mathrm{T}_\pi(\Gamma) \otimes_{\mathcal{O}_E} \omega_\Gamma, \tag{2.3.2.2}$$

$$V_\pi(\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}) \cong \mathbf{C} \otimes_{\mathcal{O}_E} \mathrm{T}_\pi(\Gamma) \otimes_{\mathcal{O}_E} \omega_\Gamma \tag{2.3.2.3}$$

that commute with the action of G_K.

Proof. Isomorphism (2.3.2.1) simply follows from 2.1.13. As $E/\mathcal{O}_E = \varinjlim(\frac{1}{\pi^n}\mathcal{O}_E)/\mathcal{O}_E$ we have:

$$\mathrm{T}_\pi(\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_E}) = \varprojlim \mathrm{Hom}_{\mathcal{O}_E}\Big(\frac{1}{\pi^n}\mathcal{O}_E/\mathcal{O}_E, \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}\Big).$$

Moreover

$$\mathrm{Hom}_{\mathcal{O}_E}\Big(\frac{1}{\pi^n}\mathcal{O}_E/\mathcal{O}_E, \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}\Big) \cong \Big(\frac{1}{\pi^n}\mathfrak{a}/\mathfrak{a}\Big) \otimes_{\mathcal{O}_E} \mathrm{T}_\pi(\Gamma) \otimes_{\mathcal{O}_E} \omega_\Gamma$$

using (2.3.2.1) together with the fact that $\mathrm{T}_\pi(\Gamma)$ and ω_Γ are free rank one $\mathcal{O}_E$-modules (hence torsion-free) and that the morphisms are $\mathcal{O}_E$-linear. Therefore

$$\mathrm{T}_\pi(\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_E}) = \varprojlim \Big(\frac{1}{\pi^n}\mathfrak{a}/\mathfrak{a}\Big) \otimes_{\mathcal{O}_E} \mathrm{T}_\pi(\Gamma) \otimes_{\mathcal{O}_E} \omega_\Gamma = \widehat{\mathfrak{a}} \otimes_{\mathcal{O}_E} \mathrm{T}_\pi(\Gamma) \otimes_{\mathcal{O}_E} \omega_\Gamma.$$

Finally, for (2.3.2.3) we write $E = \varinjlim \frac{1}{\pi^n}\mathcal{O}_E$. As above we have:

$$V_\pi(\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_E}) = \mathrm{Hom}_{\mathcal{O}_E}(E, \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}) = \varprojlim \mathrm{Hom}_{\mathcal{O}_E}\Big(\frac{1}{\pi^n}\mathcal{O}_E, \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}\Big).$$

To get the isomorphism with $\mathbf{C} \otimes_{\mathcal{O}_E} \mathrm{T}_\pi(\Gamma) \otimes_{\mathcal{O}_E} \omega_\Gamma$, we use again (2.3.2.1). The morphisms ξ of Theorem 2.1.13 is compatible with the action of G_K, so isomorphisms (2.3.2.1), (2.3.2.2) and (2.3.2.3) commute clearly with the action of G_K. □

2.3.3. Assume that K is of characteristic 0, that $E = \mathbb{Q}_p$ and $\pi = p$, so that $q = p$ and $K_0 = \mathrm{Frac}(W(k))$. For this special case (see [**LT65**, §1, p. 380]), the Lubin-Tate formal group Γ over $\mathbb{Q}_p$ is the formal multiplicative group $\widehat{\mathbb{G}}_m$, i.e. the completion along the unit section of the multiplicative group $\mathbb{G}_m$ over $\mathbb{Z}_p$. For $f(T) = (1+T)^p - 1 \in \mathbb{Z}_p[[T]]$, the group law $\Gamma = \Gamma_f(X, Y)$ is the power series $X + Y + XY$. By 2.1.5, we have a canonical generator of ω_Γ, namely the unique normalized invariant differential form $\omega_0 = \frac{\mathrm{d}T}{1+T}$.

We can identify the Tate module $\mathrm{T}_p(\Gamma)$ with the points in $\overline{K}$ of the Tate module of the multiplicative group $\mathbb{G}_m$. More precisely we have, for any $n \in \mathbb{N}$,

$$1 \to \mu_{p^n}(\overline{K}) \to \overline{K}^* \xrightarrow{\cdot p^n} \overline{K}^* \to 1$$

and $\mathrm{T}_p(\mathbb{G}_m)$ is the projective limit $\varprojlim \mu_{p^n}(\overline{K})$, where the transition maps are given by raising to the p-th power. As the map

$$a \mapsto 1 + a \colon \mathfrak{m}_{\overline{K}} \to 1 + \mathfrak{m}_{\overline{K}} \qquad (2.3.3.1)$$

is an isomorphism between the group $\Gamma(\mathfrak{m}_{\overline{K}})$ and $U^{(1)}_{\overline{K}}$ (with standard multiplication), the points of p^n-torsion with respect to the formal group law correspond to the point of p^n-torsion with respect to the standard multiplication in $\overline{K}$. Therefore

$$\mathrm{T}_p(\Gamma) = \mathrm{T}_p(\mathbb{G}_m) = \varprojlim \mu_{p^n}(\overline{K})$$

is the free $\mathbb{Z}_p$-module of rank 1 formed by the sequences $(\varepsilon_n)_{n\in\mathbb{N}}$ of elements of $\mathcal{O}_{\overline{K}}$ such that $\varepsilon_0 = 1$ and $\varepsilon_{n+1}^p = \varepsilon_n$.

Notice that, by definition, the character $\chi\colon G_K \to \mathrm{Aut}_{\mathbb{Z}_p}(\mathrm{T}_p(\Gamma)) \cong \mathbb{Z}_p^\times$ giving the action of G_K on the Tate module of Γ is nothing else but the cyclotomic character χ_p, giving the action of G_K on the group of units of order (a power of) p.

2.3.4. For any $\mathbb{Z}_p$-module M endowed with a linear action of G_K and any $i \in \mathbb{Z}$, we write $M(i)$ for the tensor product

$$M \otimes_{\mathbb{Z}_p} \mathrm{T}_p(\mathbb{G}_m)^{\otimes^i}$$

with the convention $\mathrm{T}_p(\mathbb{G}_m)^{\otimes^0} = \mathbb{Z}_p$ and, for $i > 0$, $\mathrm{T}_p(\mathbb{G}_m)^{\otimes^{-i}}$ is the dual of $\mathrm{T}_p(\mathbb{G}_m)^{\otimes^i}$.

In this setting, we can reformulate Theorem 2.1.13 in the following way:

2.3.5. THEOREM. *The map* $\xi\colon \overline{K}(1) \to \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}$ *defined by*

$$p^{-r}a \otimes (\varepsilon_n)_{n\in\mathbb{N}} \mapsto a.\frac{\mathrm{d}\varepsilon_r}{\varepsilon_r}$$

for $a \in \mathcal{O}_{\overline{K}}$, $r \in \mathbb{N}$ *is surjective with kernel* $\mathfrak{a}(1)$ *and induces canonical isomorphisms:*

(2.3.5.1) $$\Omega^1_{\mathcal{O}_{\overline{K}}/K} \cong (\overline{K}/\mathfrak{a})(1),$$

(2.3.5.2) $$\mathrm{T}_p(\Omega^1_{\mathcal{O}_{\overline{K}}/K}) = \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Q}_p/\mathbb{Z}_p, \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}) \cong \widehat{\mathfrak{a}}(1),$$

(2.3.5.3) $$V_p(\Omega^1_{\mathcal{O}_{\overline{K}}/K}) = \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Q}_p, \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}) \cong \mathbf{C}(1).$$

2.4. Applications to Abelian Varieties

2.4.1. Let K be a complete discrete valuation field of characteristic 0 with perfect residue field k of characteristic $p > 0$, $\mathcal{O}_K$ the valuation ring of K, $S = \mathrm{Spec}(\mathcal{O}_K)$. We note by η the generic point of S and by $\overline{\eta}$ a geometric point corresponding to an algebraic closure $\overline{K}$ of K. We denote by G_K the absolute Galois group of $\overline{K}$ over K. $f\colon X \to \mathrm{Spec}(K)$ be a morphism of schemes. We call proper $\mathcal{O}_K$-model of X any scheme $\mathfrak{X}$ proper over S such that $\mathfrak{X}_\eta = X$.

2.4.2. PROPOSITION ([**EGA IV**, 2.8.5]). *Let* $f\colon X \to S$ *be a morphism of schemes and let* $X_\eta = f^{-1}(\eta)$ *be the generic fibre of* X. *Let* $\iota\colon X_\eta \to X$ *be the canonical morphism. Let* Z *be a closed subscheme of* X_η. *Then there exists a unique closed subscheme* $\mathfrak{Z}$ *of* X, *flat over* S *and such that* $\iota^{-1}(\mathfrak{Z}) = Z$.

The scheme $\mathfrak{Z}$ is the schematic closure of Z by the composite morphism $Z \to X_\eta \xrightarrow{\iota} X$, where the first arrow is the canonical injection; its underlying space is the closure in X of Z.

2.4.3. From now on, let X be an abelian variety over K and let $\varphi\colon X \to \mathbb{P}^n_K$ be a closed immersion. Let $\iota\colon \mathbb{P}^n_K \to \mathbb{P}^n_{\mathcal{O}_K}$ be the canonical morphism. By 2.4.2, there exists a unique scheme $\mathfrak{X}$, flat and proper over S, such that $i^{-1}(\mathfrak{X}) = X$.

2.4.4. Let $u\colon \mathrm{Spec}(\mathcal{O}_{\overline{K}}) \to \mathfrak{X}$ and let $\omega \in \mathrm{H}^0(\mathfrak{X}, \Omega^1_{\mathfrak{X}/\mathcal{O}_{\overline{K}}})$. We denote by $u^*(\omega) \in \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}$ the image of $u^*\omega$ by the canonical $\mathcal{O}_K$-linear map

$$u^*\Omega^1_{\mathfrak{X}/\mathcal{O}_K} \xrightarrow{v} \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}.$$

In this way we obtain a pairing:

(2.4.4.1) $$\mathrm{H}^0(\mathfrak{X}, \Omega^1_{\mathfrak{X}/\mathcal{O}_K}) \times \mathfrak{X}(\mathcal{O}_{\overline{K}}) \to \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}$$

by

$$(\omega, u) \mapsto \langle \omega, u\rangle = u^*(\omega).$$

The map (2.4.4.1) is clearly $\mathcal{O}_K$-linear in the first variable and it is compatible with the action of G_K. More precisely, for any $g \in G_K$, $\omega \in \mathrm{H}^0(\mathfrak{X}, \Omega^1_{\mathfrak{X}/\mathcal{O}_K})$, $u \in \mathfrak{X}(\mathcal{O}_{\overline{K}})$ we have

$$\langle \omega, g.u\rangle = g(\langle \omega, u\rangle) = g(u^*(\omega)).$$

2.4.5. By construction, we have the fibre product diagram:

$$\begin{array}{ccc} X & \xrightarrow{q_1} & \mathfrak{X} \\ {\scriptstyle q_2}\downarrow & & \downarrow \\ \eta & \longrightarrow & S \end{array}$$

that allow us to identify $\mathrm{H}^0(X, \Omega^1_{X/K})$ with $K \otimes_{\mathcal{O}_K} \mathrm{H}^0(\mathfrak{X}, \Omega^1_{\mathfrak{X}/\mathcal{O}_K})$. Indeed, let $(U_i)_{i \in I}$ be an affine open covering of $\mathfrak{X}$ and consider the canonical exact sequence:

$$0 \to \mathrm{H}^0(\mathfrak{X}, \Omega^1_{\mathfrak{X}/\mathcal{O}_K}) \to \prod_i \mathrm{H}^0(U_i, \Omega^1_{\mathfrak{X}/\mathcal{O}_K}) \to \prod_{i,j} \mathrm{H}^0(U_{i,j}, \Omega^1_{\mathfrak{X}/\mathcal{O}_K}) \tag{2.4.5.1}$$

where $U_{ij} = U_i \cap U_j$. Since K is flat over $\mathcal{O}_K$, the latter induces an exact sequence

$$0 \to \mathrm{H}^0(\mathfrak{X}, \Omega^1_{\mathfrak{X}/\mathcal{O}_K}) \otimes_{\mathcal{O}_K} K \to \prod_i \mathrm{H}^0(U_i, \Omega^1_{\mathfrak{X}/\mathcal{O}_K}) \otimes_{\mathcal{O}_K} K \to \prod_{i,j} \mathrm{H}^0(U_{i,j}, \Omega^1_{\mathfrak{X}/\mathcal{O}_K}) \otimes_{\mathcal{O}_K} K \tag{2.4.5.2}$$

On the other hand, $(U_i \cap X = U_i \otimes_{\mathcal{O}_K} K)_{i \in I}$ is an affine open covering of X and we have, for every $i \in I$,

$$\mathrm{H}^0(U_i, \Omega^1_{\mathfrak{X}/\mathcal{O}_K}) \otimes_{\mathcal{O}_K} K = \mathrm{H}^0(U_i \otimes_{\mathcal{O}_K} K, \Omega^1_{X/K}).$$

Hence (2.4.5.2) implies that

$$\mathrm{H}^0(X, \Omega^1_{X/K}) = K \otimes_{\mathcal{O}_K} \mathrm{H}^0(\mathfrak{X}, \Omega^1_{\mathfrak{X}/\mathcal{O}_K}).$$

2.4.6. By the Valuative Criterion of Properness [**EGA II**, 7.3.8] we have a canonical identification of $X(\overline{K})$ with $\mathfrak{X}(\mathcal{O}_{\overline{K}})$: in this way $\mathfrak{X}(\mathcal{O}_{\overline{K}})$ inherits a structure of abelian group, even though $\mathfrak{X}$ is not a group scheme over S.

2.4.7. PROPOSITION. *Under the assumptions of 2.4.4, there exists a non negative integer r_0 such that for every $\omega \in p^{r_0}\mathrm{H}^0(\mathfrak{X}, \Omega^1_{\mathfrak{X}/\mathcal{O}_K})$ and every $u_1, u_2 \in \mathfrak{X}(\mathcal{O}_{\overline{K}}) = X(\overline{K})$ we have:*

$$\langle \omega, u_1 + u_2 \rangle = \langle \omega, u_1 \rangle + \langle \omega, u_2 \rangle$$

PROOF. Let $\mathfrak{Y}$ be an $\mathcal{O}_K$-model of $X \times X$ over K such that the canonical projections $p_1, p_2 \colon X \times_\eta X \rightrightarrows X$ and the group multiplication $m \colon X \times_\eta X \to X$ extend to maps from $\mathfrak{Y}$ to $\mathfrak{X}$. We can construct $\mathfrak{Y}$ as follows: if $\psi \colon X \times_\eta X \to \mathbb{P}^m_K$ is a projective embedding of the product $X \times_\eta X$, we can consider the composite map

$$X \times_\eta X \xrightarrow{id \times m} X \times_\eta X \times_\eta X \to \mathfrak{X} \times_S \mathfrak{X} \times_S \mathfrak{X}.$$

Let $\mathfrak{Y}$ be schematic closure of the composite morphism, so that we have the diagram

$$\begin{array}{ccccccc} X \times_\eta X & \longrightarrow & X \times_\eta X \times_\eta X & \longrightarrow & \mathbb{P}^m_K & \longrightarrow & \eta \\ {\scriptstyle\sigma}\downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathfrak{Y} & \longrightarrow & \mathfrak{X} \times_S \mathfrak{X} \times_S \mathfrak{X} & \longrightarrow & \mathbb{P}^m_{\mathcal{O}_K} & \longrightarrow & S \end{array} \tag{2.4.7.1}$$

We get the required extensions

$$p_{1,\mathfrak{X}}, p_{2,\mathfrak{X}}, m_{\mathfrak{X}} \colon \mathfrak{Y} \to \mathfrak{X}$$

by mean of the other projections.

We know ([**BLR90**, §4.2, Prop.1]) that the everywhere regular differential forms on X are precisely the invariant forms, so that for any $\omega \in \mathrm{H}^0(X, \Omega^1_{X/K})$ we have:

$$m^*\omega - p_1^*\omega - p_2^*\omega = 0$$

in $\mathrm{H}^0(X, \Omega^1_{X\times X/K})$. Let $\omega \in \mathrm{H}^0(\mathfrak{X}, \Omega^1_{\mathfrak{X}/\mathcal{O}_K})$ and consider the form $\omega' \in \mathrm{H}^0(\mathfrak{Y}, \Omega^1_{\mathfrak{Y}/\mathcal{O}_K})$ defined by

$$\omega' = m_{\mathfrak{X}}^*\omega - p_{1,\mathfrak{X}}^*\omega - p_{2,\mathfrak{X}}^*\omega.$$

The natural map

$$\mathrm{H}^0(\mathfrak{Y}, \Omega^1_{\mathfrak{Y}/\mathcal{O}_K}) \to \mathrm{H}^0(X\times_K X, \Omega^1_{X\times X/K}) = K \times_{\mathcal{O}_K} \mathrm{H}^0(\mathfrak{Y}, \Omega^1_{\mathfrak{Y}/\mathcal{O}_K}) \tag{2.4.7.2}$$

corresponds to taking the pull-back of a differential form on $\mathfrak{Y}$ via the map σ of (2.4.7.1). Let q_1 be the canonical map $X \to \mathfrak{X}$. Then, by definition, $m_{\mathfrak{X}} \circ \sigma = q_1 \circ m$. Similarly,

$$\begin{aligned} p_{1,\mathfrak{X}} \circ \sigma &= q_1 \circ p_1 \\ p_{2,\mathfrak{X}} \circ \sigma &= q_1 \circ p_2, \end{aligned}$$

so that

$$1 \otimes \omega' = \sigma^*\omega' = m^*(q_1^*\omega) - p_1^*(q_1^*\omega) - p_2^*(q_1^*\omega) = 0.$$

The kernel of (2.4.7.2) is the torsion submodule of the $\mathcal{O}_K$-module $\mathrm{H}^0(\mathfrak{Y}, \Omega^1_{\mathfrak{Y}/\mathcal{O}_K})$. Since $\mathfrak{Y} \to S$ is proper and the sheaf of differentials $\Omega^1_{\mathfrak{Y}/\mathcal{O}_K}$ is coherent, $\mathrm{H}^0(\mathfrak{Y}, \Omega^1_{\mathfrak{Y}/\mathcal{O}_K})$ is of finite type. Therefore there exists an integer $r_0 \geq 0$ such that

$$p^{r_0}[\mathrm{H}^0(\mathfrak{Y}, \Omega^1_{\mathfrak{Y}/\mathcal{O}_K})_{\mathrm{Tors}}] = 0.$$

The restriction

$$p^{r_0}\mathrm{H}^0(\mathfrak{X}, \Omega^1_{\mathfrak{X}/\mathcal{O}_K}) \to p^{r_0}\mathrm{H}^0(\mathfrak{Y}, \Omega^1_{\mathfrak{Y}/\mathcal{O}_K}), \quad \omega \mapsto \omega' = m_{\mathfrak{X}}^*\omega - p_{1,\mathfrak{X}}^*\omega - p_{2,\mathfrak{X}}^*\omega \tag{2.4.7.3}$$

vanishes.

Let $u_1, u_2 \in \mathfrak{X}(\mathcal{O}_{\overline{K}})$ and denote by $u_{1,X}$ and $u_{2,X}$ the corresponding $\overline{K}$-points of X. Let v_X

$$v_X\colon \mathrm{Spec}(\overline{K}) \xrightarrow{\Delta} \mathrm{Spec}(\overline{K}) \times \mathrm{Spec}(\overline{K}) \xrightarrow{u_{1,X}\times u_{2,X}} X \times_K X$$

and let $v \in \mathfrak{Y}(\mathcal{O}_{\overline{K}})$ be the corresponding point of $\mathfrak{Y}$. We have:

$$\begin{aligned} &u_1 = p_{1,\mathfrak{X}} \circ v; \quad u_2 = p_{2,\mathfrak{X}} \circ v; \\ &u_{1,X} = p_1 \circ v_X; \quad u_{2,X} = p_2 \circ v_X; \\ &u_1 + u_2 = m_{\mathfrak{X}} \circ v. \end{aligned}$$

By (2.4.7.3), we get for any $\omega \in p^{r_0}\mathrm{H}^0(\mathfrak{X}, \Omega^1_{\mathfrak{X}/\mathcal{O}_K})$,

$$(u_1+u_2)^*\omega = v^*(m_{\mathfrak{X}}^*\omega) = v^*(p_{1,\mathfrak{X}}^*\omega + p_{2,\mathfrak{X}}^*\omega) = u_1^*\omega + u_2^*\omega.$$

□

2.4.8. Let $r \geq r_0$ be a non negative integer such that $p^r \mathrm{H}^0(\mathfrak{X}, \Omega^1_{\mathfrak{X}/\mathcal{O}_K})$ is torsion free or, so that the restriction of the canonical map

$$\mathrm{H}^0(\mathfrak{X}, \Omega^1_{\mathfrak{X}/\mathcal{O}_K}) \to K \otimes_{\mathcal{O}_K} \mathrm{H}^0(\mathfrak{X}, \Omega^1_{\mathfrak{X}/\mathcal{O}_K}) = \mathrm{H}^0(X, \Omega^1_{X/K})$$

to $p^r \mathrm{H}^0(\mathfrak{X}, \Omega^1_{\mathfrak{X}/\mathcal{O}_K})$ is injective. We can restrict the map (2.4.4.1) to

$$p^r \mathrm{H}^0(\mathfrak{X}, \Omega^1_{\mathfrak{X}/\mathcal{O}_K}) \times X(\overline{K}) \to \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}.$$

By 2.4.7, this pairing is $\mathbb{Z}[G_K]$-linear in the second variable. The associated homomorphism

$$p^r \mathrm{H}^0(\mathfrak{X}, \Omega^1_{\mathfrak{X}/\mathcal{O}_K}) \to \mathrm{Hom}_{\mathbb{Z}[G_K]}(X(\overline{K}), \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}) \tag{2.4.8.1}$$

is $\mathcal{O}_K$-linear.

2.4.9. Let

$$\mathrm{T}_p(X) = \mathrm{T}_p(X_{\overline{\eta}}) = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}_p/\mathbb{Z}_p, X(\overline{K}))$$

be the p-adic Tate module of X. Let $V_p(X) = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[p^{-1}], X(\overline{K}))$. We have a natural inclusion of $\mathrm{T}_p(X)$ in $V_p(X)$: given any $\alpha = (a_n)_{n\in\mathbb{N}} \in \mathrm{T}_p(X)$ we can define a map $\varphi_\alpha \colon \mathbb{Z}[p^{-1}] \to X(\overline{K})$ by the assignment $p^{-n} \mapsto a_n$ for $n \geq 0$. Let $V_p(\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K})$ be $\mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Q}_p, \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K})$ as in 2.3. We have the isomorphism

$$\mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Q}_p, \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[p^{-1}], \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}). \tag{2.4.9.1}$$

We can compose the $\mathcal{O}_K$-homomorphism (2.4.8.1) with the map:

$$\mathrm{Hom}_{\mathbb{Z}[G_K]}(X(\overline{K}), \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}) \xrightarrow{\psi} \mathrm{Hom}_{\mathbb{Z}[G_K]}(V_p(X), V_p(\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K})) \tag{2.4.9.2}$$

to get

$$p^r \mathrm{H}^0(\mathfrak{X}, \Omega^1_{\mathfrak{X}/\mathcal{O}_K}) \to \mathrm{Hom}_{\mathbb{Z}[G_K]}(V_p(X), V_p(\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}))$$

and then, by extending the scalars to K:

$$\widehat{\varrho} = \widehat{\varrho}^0_{X,\mathfrak{X},r} \colon \mathrm{H}^0(X, \Omega^1_{X/K}) = K \otimes_{\mathcal{O}_K} p^r \mathrm{H}^0(\mathfrak{X}, \Omega^1_{\mathfrak{X}/\mathcal{O}_K}) \to \mathrm{Hom}_{\mathbb{Z}[G_K]}(V_p(X), V_p(\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K})).$$

2.4.10. Remark. The map ψ in (2.4.9.2) is injective, as $X(\overline{K})$ is a p-divisible group (in the classical sense).

2.4.11. For any $\omega \in \mathrm{H}^0(X, \Omega^1_{X/K})$ we can take the restriction of the morphism of $\mathbb{Z}[G_K]$-modules

$$\widehat{\varrho}(\omega) \colon V_p(X) \to V_p(\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K})$$

to $\mathrm{T}_p(X) \subset V_p(X) \to V_p(\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K})$. By continuity, $\widehat{\varrho}(\omega)|_{\mathrm{T}_p(X)}$ is $\mathbb{Z}_p$ linear and, in the end, we get a K-linear map:

$$\varrho^0_X = \varrho^0_{X,\mathfrak{X},r} \colon \mathrm{H}^0(X, \Omega^1_{X/K}) \to \mathrm{Hom}_{\mathbb{Z}_p[G_K]}(\mathrm{T}_p(X), V_p(\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K})) = \mathrm{Hom}_{\mathbb{Z}_p[G_K]}(\mathrm{T}_p(X), \mathbf{C}(1))$$

since $V_p(\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K})$ is $\mathbb{Z}_p[G_K]$-isomorphic to $\mathbf{C}(1)$ by Theorem 2.3.5.

2.4.12. Proposition. *The restriction map*

$$\mathrm{Hom}_{\mathbb{Z}[G_K]}(V_p(X), \mathbf{C}(1)) \to \mathrm{Hom}_{\mathbb{Z}[G_K]}(\mathrm{T}_p(X), \mathbf{C}(1))$$

induced by the inclusion $\mathrm{T}_p(X) \subset V_p(X)$ *is injective*

PROOF. Let $X[p^\infty]$ be the subgroup of p-primary torsion of $X(\overline{K})$. The quotient $D_p(X) = X(\overline{K})/X[p^\infty]$ is a uniquely p-divisible abelian group and we have a canonical isomorphism between $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[p^{-1}], D_p(X))$ and $D_p(X)$ given by

$$\varphi \mapsto \varphi(1), \quad x \in D_p(X) \mapsto (\varphi_x \colon 1 \mapsto x).$$

Therefore, the exact sequence

$$0 \to X[p^\infty] \to X(\overline{K}) \to D_p(X) \to 0$$

leads to the exact sequence

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[p^{-1}], X[p^\infty]) \to V_p(X) \to D_p(X) \to 0. \tag{2.4.12.1}$$

Moreover, we have a canonical isomorphism:

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[p^{-1}], X[p^\infty]) \cong \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathrm{T}_p(X). \tag{2.4.12.2}$$

Indeed, given any $\mathbb{Z}$-linear map $\varphi \colon \mathbb{Z}[p^{-1}] \to X[p^\infty]$, let $x_0 \in X[p^r]$ be $\varphi(1)$. Then for any $n \in \mathbb{N}$, $x_n = \varphi(1/p^n) \in X[p^{r+n}]$, with $px_n = x_{n-1}$, defining in this way the element $p^{-r} \otimes (p^r x_n)_{n\in\mathbb{N}} \in \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathrm{T}_p(X)$: it is easy to check that the map is an isomorphism.

By applying $\mathrm{Hom}_{\mathbb{Z}[G_K]}(-, \mathbf{C}(1))$ to (2.4.12.1) we get

$$0 \to \mathrm{Hom}_{\mathbb{Z}[G_K]}(D_p(X), \mathbf{C}(1)) \to \mathrm{Hom}_{\mathbb{Z}[G_K]}(V_p(X), \mathbf{C}(1)) \xrightarrow{\alpha} \mathrm{Hom}_{\mathbb{Z}[G_K]}(\mathrm{T}_p(X), \mathbf{C}(1)) \tag{2.4.12.3}$$

as

$$\mathrm{Hom}_{\mathbb{Z}[G_K]}(\mathrm{T}_p(X), \mathbf{C}(1)) = (\mathrm{Hom}_{\mathbb{Z}}(\mathrm{T}_p(X), \mathbf{C}(1)))^{G_K} = \mathrm{Hom}_{\mathbb{Z}[G_K]}(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathrm{T}_p(X), \mathbf{C}(1)),$$

so that $\mathrm{Hom}_{\mathbb{Z}[G_K]}(D_p(X), \mathbf{C}(1))$ is identified with the kernel of α.

Since

$$X(\overline{K}) = \bigcup_{\substack{\overline{K} \supset L \supseteq K \\ L \text{ finite, Galois}}} X(L) = \bigcup_{\substack{H \trianglelefteq G_K \\ H \text{ open}}} X(\overline{K})^H,$$

also $D_p(X) = \bigcup (D_p(X))^H$ for H varying in the set of open normal subgroups of G_K. Given $f \in \mathrm{Hom}_{\mathbb{Z}[G_K]}(D_p(X), \mathbf{C}(1))$ we have

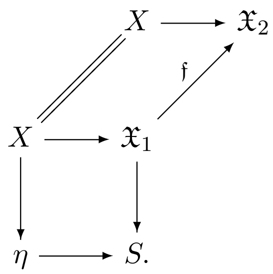
$$f((D_p(X))^H) \subseteq (\mathbf{C}(1))^H = 0$$

by Tate's Theorem (cfr. 1.5.15), for any open normal subgroup H of G_K. Hence $f(D_p(X)) = \bigcup f((D_p(X))^H) = 0$. □

2.4.13. PROPOSITION. *The maps ϱ_X^0 and $\widehat{\varrho}$ do not depend on the choice of r and on the choice of the $\mathcal{O}_K$-model $\mathfrak{X}$.*

PROOF. The K-linearity gives immediately the independence from r. It is clearly enough to check the independence of the map $\widehat{\varrho}$ from the choice of $\mathfrak{X}$. Let $\mathfrak{X}_1$ and $\mathfrak{X}_2$ be two proper

$\mathcal{O}_K$-model of X and suppose that the identity map id_X extends to a morphism $\mathfrak{f}\colon \mathfrak{X}_1 \to \mathfrak{X}_2$:



In this situation we say that $\mathfrak{X}_1$ dominates $\mathfrak{X}_2$. The commutativity of the above diagram implies that

$$\begin{array}{ccccc} \widehat{\varrho}^0_{X,\mathfrak{X}_2}\colon \mathrm{H}^0(X,\Omega^1_{X/K}) & \xrightarrow{\sim} & K\otimes_{\mathcal{O}_K} p^r\mathrm{H}^0(\mathfrak{X}_2,\Omega^1_{\mathfrak{X}_2/\mathcal{O}_K}) & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}[G_K]}(V_p(X),V_p(\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K})) \\ \| & & \Big\downarrow{\scriptstyle \mathfrak{f}^*} & & \| \\ \widehat{\varrho}^0_{X,\mathfrak{X}_1}\colon \mathrm{H}^0(X,\Omega^1_{X/K}) & \xrightarrow{\sim} & K\otimes_{\mathcal{O}_K} p^r\mathrm{H}^0(\mathfrak{X}_1,\Omega^1_{\mathfrak{X}_1/\mathcal{O}_K}) & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}[G_K]}(V_p(X),V_p(\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K})) \end{array}$$

also commutes, proving that $\widehat{\varrho}^0_{X,\mathfrak{X}_2} = \widehat{\varrho}^0_{X,\mathfrak{X}_1}$. In the general case, if $\mathfrak{X}_1$ and $\mathfrak{X}_2$ are two models of X, we can construct a third $\mathcal{O}_K$-model of X, say $\mathfrak{X}_3$, forcing the existence of maps $\mathfrak{X}_3 \xrightarrow{f_{3,1}} \mathfrak{X}_1$ and $\mathfrak{X}_3 \xrightarrow{f_{3,2}} \mathfrak{X}_2$ extending the identity id_X. Indeed, let $\varphi\colon X \to \mathbb{P}^n_K$ be a projective embedding of X. Arguing as in (2.4.7.1), we can consider the composite map

$$X \xrightarrow{\Delta} \times_K X \times_K X \to \mathfrak{X}_1 \times_{\mathcal{O}_K} \mathfrak{X}_2.$$

and we let $\mathfrak{X}_3$ be the schematic closure of the composite morphism. □

2.4.14. Theorem. *Let X be an abelian variety over K. Then*

$$\varrho^0_X\colon \mathrm{H}^0(X,\Omega^1_{X/K}) \to \mathrm{Hom}_{\mathbb{Z}_p[G]}(T_p(X),\mathbf{C}(1))$$

defined in 2.4.11 is an injective K-linear map, functorial in X.

2.4.15. The same argument used in the proof of 2.4.13 allow us to prove that the map ϱ^0_X just defined is actually functorial in X: given any homomorphism of abelian varieties $f\colon X \to Z$, it's enough to choose two $\mathcal{O}_K$-models for X and Z respectively, say $\mathfrak{X}$ and $\mathfrak{Z}$, such that f extends to a morphism $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{Z}$.

2.4.16. The map ϱ^0_X is K linear by construction and functorial by 2.4.15. Since the restriction map $\mathrm{Hom}_{\mathbb{Z}[G_K]}(V_p(X),\mathbf{C}(1)) \to \mathrm{Hom}_{\mathbb{Z}[G_K]}(\mathrm{T}_p(X),\mathbf{C}(1))$ is injective by 2.4.12, it's enough to prove that $\widehat{\varrho}$ defined in (2.4.9) is injective. On the other hand, $\widehat{\varrho}$ is the scalar extension to K of the composition between the map (2.4.8.1) and the injective map ψ of (2.4.9.2). Hence, we are reduced to prove the following

2.4.17. Proposition. *The map*

$$p^r\mathrm{H}^0(\mathfrak{X},\Omega^1_{\mathfrak{X}/\mathcal{O}_K}) \to \mathrm{Hom}_{\mathbb{Z}[G_K]}(X(\overline{K}),\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K})$$

defined in (2.4.8.1) *is injective.*

We dedicate section 2.5 to the proof of this result.

2.5. The Proof of Proposition 2.4.17

The independence from the choice of the $\mathcal{O}_K$-model $\mathfrak{X}$ given by 2.4.13, allow us to use the following desingularization lemma:

2.5.1. Lemma. *Let X be a projective variety over K, of dimension d. Let $u \in X(K)$ be a regular point of X. Then there exists a proper $\mathcal{O}_K$-model $\mathfrak{X}$ of X such that if $\overline{u}$ denotes the closed point in the closure of u in $\mathfrak{X}$, the $\mathfrak{m}_{\overline{u}}$-adic completion of $\mathcal{O}_{\mathfrak{X},\overline{u}}$ is isomorphic to the ring of formal powers series in d variables over $\mathcal{O}_K$.*

Proof. Let φ be a closed immersion $\varphi\colon X \to \mathbb{P}^n_K$, so that:

$$X = \mathrm{Proj}(K[X_0, \dots, X_n]/I)$$

for a homogeneous ideal I of $K[X_0, \dots, X_n]$. We choose homogeneous coordinates $(X_0; \dots; X_n)$ of $\mathbb{P}^n_K$ so that u is the point $(1 : 0 \dots : 0)$: being u a regular point of X, the Jacobian criterion implies — up to a variable reordering — that we can find homogeneous polynomials $F_1, \dots, F_{n-d}$ in I, locally defining X, such that the $(n-d) \times (n-d)$ minor

$$\left(\frac{\partial F_i}{\partial X_{d+j}}(u)\right)_{1 \le i,j \le n-d} \tag{2.5.1.1}$$

of the Jacobian matrix at u is invertible. By a linear change of variables we can further assume that such minor is the identity matrix I_{n-d}.

Let J be the homogeneous ideal of $K[X_0, \dots, X_n]$ generated by

$$\begin{array}{ll} X_i & \text{for } 1 \le i \le d \\ X_i X_j & \text{for } i, j \ge 1. \end{array} \tag{2.5.1.2}$$

If $r_i = \deg F_i$, $1 \le i \le n-d$, we have

$$F_i \equiv X_0^{r_i - 1} X_{d+i} \pmod{J}, \qquad \text{for } 1 \le i \le n-d. \tag{2.5.1.3}$$

by (2.5.1.1) and (2.5.1.2). Let π be a uniformizer of $\mathcal{O}_K$. We choose non negative integers s_i such that

$$\pi^{s_i} F_i \in \mathcal{O}_K[X_0, \dots, X_n], \qquad \text{for } 1 \le i \le n-d.$$

Let $s \in \mathbb{N}$ such that $s \ge s_i$ for every $1 \le i \le n-d$ and we set:

$$\begin{array}{ll} X_0 = X_0', & \\ X_i = \pi^{2s} X_i' & \text{for } 1 \le i \le d, \\ X_i = \pi^s X_i' & \text{for } d+1 \le i \le n. \end{array} \tag{2.5.1.4}$$

With this choice, a straightforward computation shows that we can find $(n-d)$ homogeneous polynomials G_i in the variables X_i' such that:

$$\begin{array}{l} F_i = \pi^s G_i \\ G_i \equiv (X_0')^{r_i - 1} X_{d+i}' \pmod{\pi \mathcal{O}_K[X_0', \dots X_n']} \end{array} \qquad \text{for } 1 \le i \le n-d. \tag{2.5.1.5}$$

We adopt the linear change of coordinates (2.5.1.4) in $\mathbb{P}^n_K$ and consider the open immersion

$$\mathbb{P}^n_K = \mathrm{Proj}(K[X_0', \dots X_n']) \to \mathrm{Proj}(\mathcal{O}_K[X_0', \dots X_n']) = \mathbb{P}^n_{\mathcal{O}_K} \tag{2.5.1.6}$$

Let $\mathfrak{X}$ be the schematic closure of $X \to \mathbb{P}^n_{\mathcal{O}_K}$ via (2.5.1.6). Let $\overline{u}$ be the closed point of the closure of u in $\mathfrak{X}$. We place ourselves in the principal affine open neighbourhood of $\overline{u}$ (resp. u) $D_+(X_0') = \mathbb{P}^n_{\mathcal{O}_K} \setminus V_+(X_0')$ (resp. $D_+(X_0') \cap \mathbb{P}^n_K$), so to have affine coordinates $x_i = X_i'/X_0'$.

Let $\mathfrak{m}_u \subset \mathcal{O}_{X,u}$ be the maximal ideal of the local ring of X at u. The ring $\mathcal{O}_{X,u}$ is regular and local of dimension d. By construction, the K-vector space $\mathfrak{m}_u/\mathfrak{m}_u^2$ is generated by $x_1, \ldots, x_d$.

Let $\mathfrak{m}_{\overline{u}} \subset \mathcal{O}_{\mathfrak{X},u}$ be the maximal ideal of the local ring of $\mathfrak{X}$ at $\overline{u}$. Let $I_{0,\mathcal{O}_K}$ be the ideal of $\mathcal{O}_K[x_1, \ldots, x_n]$ defining $\mathfrak{X}$ in $D_+(X_0') = \mathbb{A}^n_{\mathcal{O}_K}$. It is generated locally at $\overline{u}$ by the de-homogenized polynomials ${X_0'}^{-r_i} G(X_i')$, written in the variables x_i. Then $\mathfrak{m}_{\overline{u}}$ is generated by π together with the images of $x_1, \ldots, x_n$ modulo $I_{0,\mathcal{O}_K}$. The local ring $\mathcal{O}_{\mathfrak{X},\overline{\mathrm{u}}}$ is a regular local ring of dimension $d+1$. Indeed, $\mathcal{O}_{\mathfrak{X},\overline{\mathrm{u}}}$ has dimension at least $d+1$, since when we invert π we obtain a ring of dimension d. The equality in the dimension and the regularity follow from the fact that $\mathfrak{m}_{\overline{u}}/\mathfrak{m}_{\overline{u}}^2$ is generated by $\pi, x_1, \ldots, x_d$ by (2.5.1.5).

We have

$$\widehat{\mathcal{O}}_{\mathfrak{X},\overline{u}} \cong \mathcal{O}_K[[x_1, \ldots, x_d]]$$

Indeed, any element of $\widehat{\mathcal{O}}_{\mathfrak{X},\overline{u}}$ can be expanded as a power series in the x_i with coefficients in $\mathcal{O}_K$, so we have a surjective map

$$\mathcal{O}_K[[x_1, \ldots, x_d]] \to \widehat{\mathcal{O}}_{\mathfrak{X},\overline{u}}$$

and we conclude by [**EGA IV**, 0.17.3.5 (ii)], being $\mathcal{O}_K[[x_1, \ldots, x_d]]$ a regular local ring of dimension $d+1 = \dim \mathcal{O}_{\mathfrak{X},\overline{\mathrm{u}}} = \dim \widehat{\mathcal{O}}_{\mathfrak{X},\overline{\mathrm{u}}}$. □

2.5.2. Let $e \in X(K)$ be the unit section of X and let $\mathfrak{X}$ be the proper $\mathcal{O}_K$-model of X provided by Lemma 2.5.1, so that

$$\widehat{\mathcal{O}}_{\mathfrak{X},\overline{e}} = \mathcal{O}_K[[T_1, \ldots, T_g]]$$

where $g = \dim X$ and $\overline{e}$ is the closed point of the closure of e in $\mathfrak{X}$. Let $\widehat{\Omega}^1_{\widehat{\mathcal{O}}_{\mathfrak{X},\overline{e}}/\mathcal{O}_K}$ be the module of continuous $\mathcal{O}_K$-differentials of $\widehat{\mathcal{O}}_{\mathfrak{X},\overline{e}}$, i.e. the separated completion of the $\widehat{\mathcal{O}}_{\mathfrak{X},\overline{e}}$-module of $\mathcal{O}_K$-differentials $\Omega^1_{\widehat{\mathcal{O}}_{\mathfrak{X},\overline{e}}/\mathcal{O}_K}$ (see [**EGA IV**, 0.20.7.14.2]). By [**EGA IV**, 0.20.4.5], we have the canonical isomorphism

$$\widehat{\Omega}^1_{\widehat{\mathcal{O}}_{\mathfrak{X},\overline{e}}/\mathcal{O}_K} = \varprojlim \Omega^1_{\mathcal{O}_{\mathfrak{X},\overline{e}}/\mathcal{O}_K}/\mathfrak{m}_{\overline{e}}^n \Omega^1_{\mathcal{O}_{\mathfrak{X},\overline{e}}/\mathcal{O}_K}.$$

If we take the composition with the (injective) canonical map

$$\Omega^1_{\mathcal{O}_{\mathfrak{X},\overline{e}}/\mathcal{O}_K} \to \widehat{\Omega}^1_{\widehat{\mathcal{O}}_{\mathfrak{X},\overline{e}}/\mathcal{O}_K} \tag{2.5.2.1}$$

we have an injective $\mathcal{O}_K$-linear morphism

$$p^r \mathrm{H}^0(\mathfrak{X}, \Omega^1_{\mathfrak{X}/\mathcal{O}_K}) \to \widehat{\Omega}^1_{\widehat{\mathcal{O}}_{\mathfrak{X},\overline{e}}/\mathcal{O}_K}. \tag{2.5.2.2}$$

Indeed, a global section $\omega \in p^r \mathrm{H}^0(\mathfrak{X}, \Omega^1_{\mathfrak{X}/\mathcal{O}_K}) \subset \mathrm{H}^0(X, \Omega^1_{X/K})$ is mapped to 0 in the stalk $\Omega^1_{\mathcal{O}_{\mathfrak{X},\overline{e}}/\mathcal{O}_K}$ if and only if it is mapped to 0 in $\Omega^1_{\mathcal{O}_{X,e}/K}$, that implies $\omega = 0$, since the everywhere defined 1-form over an abelian variety are determined by the value in e.

2.5.3. We equip $\widehat{\mathcal{O}}_{\mathfrak{X},\overline{e}}$ with the $\mathfrak{m} = (T_1, \ldots, T_g)$-adic topology and $\mathcal{O}_{\overline{K}}$ with the p-adic topology. To give a continuous $\mathcal{O}_K$-linear map $f \colon \widehat{\mathcal{O}}_{\mathfrak{X},\overline{e}} \to \mathcal{O}_{\overline{K}}$ amounts to give g elements $x_{f,1}, \ldots, x_{f,g}$ in the maximal ideal $\mathfrak{m}_{\overline{K}}$ of $\mathcal{O}_{\overline{K}}$. Therefore we have a canonical map

$$\widehat{\Omega}^1_{\widehat{\mathcal{O}}_{\mathfrak{X},\overline{e}}/\mathcal{O}_K} \to \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_{\mathrm{cont},\mathcal{O}_K}(\widehat{\mathcal{O}}_{\mathfrak{X},\overline{e}}, \mathcal{O}_{\overline{K}}), \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}) \tag{2.5.3.1}$$

given by

$$\omega = \sum_{i=1}^{d} \alpha_i(T_1, \ldots, T_g)\mathrm{d}T_i \in \widehat{\Omega}^1_{\widehat{\mathcal{O}}_{\mathfrak{X},\bar{e}}/\mathcal{O}_K} \mapsto (f \mapsto \sum_{i=1}^{d} \alpha_i(x_{f,1}, \ldots, x_{f,g})\mathrm{d}x_{f,i})$$

as $\alpha_i(x_{f,1}, \ldots, x_{f,g})$ converges in $\mathcal{O}_{\overline{K}}$ for every i and f.

Let ϑ be the composition of (2.5.2.2) with (2.5.3.1):

$$\vartheta\colon p^r\mathrm{H}^0(\mathfrak{X}, \Omega^1_{\mathfrak{X}/\mathcal{O}_K}) \to \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_{\mathrm{cont},\mathcal{O}_K}(\widehat{\mathcal{O}}_{\mathfrak{X},\bar{e}}, \mathcal{O}_{\overline{K}}), \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K})$$

Using the natural inclusion

$$\mathrm{Hom}_{\mathrm{cont},\mathcal{O}_K}(\widehat{\mathcal{O}}_{\mathfrak{X},\bar{e}}, \mathcal{O}_{\overline{K}}) \subset \mathfrak{X}(\mathcal{O}_{\overline{K}}) = X(\overline{K})$$

we see that for every $\omega \in p^r\mathrm{H}^0(\mathfrak{X}, \Omega^1_{\mathfrak{X}/\mathcal{O}_K})$, $\vartheta(\omega)$ corresponds to the restriction to the subset $\mathrm{Hom}_{\mathrm{cont},\mathcal{O}_K}(\widehat{\mathcal{O}}_{\mathfrak{X},\bar{e}}, \mathcal{O}_{\overline{K}})$ of $\langle \omega, - \rangle \in \mathrm{Hom}_{\mathbb{Z}[G_K]}(X(\overline{K}), \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K})$, image of ω through (2.4.8.1). To complete the proof of 2.4.17 is therefore enough to establish the following

2.5.4. Lemma. *The canonical map*

$$\widehat{\Omega}^1_{\widehat{\mathcal{O}}_{\mathfrak{X},\bar{e}}/\mathcal{O}_K} \to \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_{\mathrm{cont},\mathcal{O}_K}(\widehat{\mathcal{O}}_{\mathfrak{X},\bar{e}}, \mathcal{O}_{\overline{K}}), \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K})$$

is injective.

2.5.4 can be restated in the following purely algebraic form:

2.5.5. Lemma. *Let $\omega = \sum_{i=1}^{d} \alpha_i(T_1, \ldots, T_g)\mathrm{d}T_i$ be a formal power series in d variables with coefficients in $\mathcal{O}_K$. be a non-zero continuous differential form. Then there exist $x_1, \ldots, x_g \in \mathfrak{m}_{\overline{K}}$ such that*

$$\sum_{i=1}^{d} \alpha_i(x_1, \ldots, x_g)\mathrm{d}x_i$$

is a non-zero element of $\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}$.

Proof. We first verify the statement for $g = 1$. Let $\omega = \alpha(T)\mathrm{d}T = \sum_{i\geq 0} a_i T^i \mathrm{d}T$ with $a_i \in \mathcal{O}_K$. Let v be the valuation of $\overline{K}$ normalized by $v(K^\times) = \mathbb{Z}$ and let

$$s = \inf_{i\in\mathbb{N}} v(a_i) \in \mathbb{N}.$$

As $s \in \mathbb{N}$, there exists a smallest non negative integer i_0 satisfying $v(a_{i_0}) = s$. Then, for any $x \in \mathfrak{m}_{\overline{K}}$ such that $v(x) < \frac{1}{i_0}$ we have:

$$v(\alpha(x)) = s + i_0 v(x) < s + 1.$$

It's enough to choose x to be a uniformizer for a finite (ramified) extension L of K, contained in $\overline{K}$ such that $v(\mathcal{D}_{L/K}) \geq s+1$. Then by 2.2.3 the annihilator of $\mathrm{d}x$ in $\mathcal{O}_{\overline{K}}$ is $\mathcal{O}_{\overline{K}}\mathcal{D}_{L/K}$, so that $\alpha(x)\mathrm{d}x$ is not zero as element of $\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}$. □

The general case is a consequence of the following statement:

2.5.6. LEMMA. *Let $\alpha_1, \dots, \alpha_g$ be g formal power series in g variables, $\alpha_i \in \mathcal{O}_K[[T_1, \dots, T_g]]$ and suppose that at least one of them is non zero. Then there exist g formal power series $\varphi_1, \dots, \varphi_g$ in one variable T over $\mathcal{O}_K$ with no constant terms such that*

$$\sum_{i=1}^{g} \alpha_i(\varphi_1, \dots, \varphi_g)\varphi_i'$$

is a non zero element of $\mathcal{O}_K[[T]]$, where φ_i' denotes the formal derivative of φ_i with respect to the variable T.

PROOF. We look for the φ_i's of the form $\varphi_i = a_iT + b_iT^2$ with $a_i, b_i \in \mathcal{O}_K$. Let $\lambda = \sum_{i=1}^{g} \alpha_i(\varphi_1, \dots, \varphi_g)\varphi_i'$; we have

$$\lambda = \sum_{i=1}^{g} \alpha_i(a_1T + b_1T^2, \dots, a_gT + b_gT^2)(a_i + 2b_iT).$$

Write α_i in the form $\alpha_i = \sum_{m \geq 0} \alpha_{i,m}$ with $\alpha_{i,m}$ homogeneous of degree m in the variables $T_1, \dots, T_g$. If r is the smallest integer such that there exists j with $\alpha_{j,r} \neq 0$, we have the following expansion for λ:

$$\lambda = \Big(\sum_{i=1}^{g} a_i\alpha_{i,r}(a_1, \dots, a_g)\Big)T^r + \Big(\sum_{i=1}^{g} a_i\alpha_{i,r+1}(a_1, \dots, a_g) + \\ + \sum_{j=1}^{g} 2b_j\alpha_{j,r}(a_1, \dots, a_g) + \sum_{i,j} a_ib_j\frac{\partial\alpha_{i,r}}{\partial T_j}(a_1, \dots, a_g)\Big)T^{r+1} + \dots$$

We now have three possibilities:

i) If $F = \sum_{i=1}^{g} T_i\alpha_{i,r}(T_1, \dots, T_g) \neq 0$, being $\mathcal{O}_K$ infinite, we can find $a_1, \dots, a_g$ in $\mathcal{O}_K$ such that $F(a_1, \dots, a_g) \neq 0$. For this choice of the a_i's, $\lambda \neq 0$ for any choice of the b_j's.

ii) If $F = 0$ we look at the next term in the expansion of λ: if

$$G = \sum_{i=1}^{g} T_i\alpha_{i,r+1}(T_1, \dots, T_g) \neq 0,$$

we can use again the fact that $\mathcal{O}_K$ is infinite to find a_i's such that $G(a_1, \dots, a_g) \neq 0$. If we set $b_j = 0$ for every j we see that $\lambda \neq 0$.

iii) If $F = G = 0$, we have, by taking the derivative of F with respect to T_j:

$$\alpha_{j,r}(T_1, \dots, T_g) + \sum_{i=1}^{g} T_i\frac{\partial\alpha_{i,r}}{\partial T_j}(T_1, \dots, T_g) = 0 \qquad (2.5.6.1)$$

for every $1 \leq j \leq g$. Moreover

$$\lambda = \Big(\sum_{j=1}^{g} b_j\Big(2\alpha_{j,r}(a_1, \dots, a_g) + \sum_{i=1}^{g} a_i\frac{\partial\alpha_{i,r}}{\partial T_j}(a_1, \dots, a_g)\Big)\Big)T^{r+1} + \dots$$

so that if we substitute (2.5.6.1), we get

$$\lambda = \Big(\sum_{j=1}^{g} b_j\alpha_{j,r}(a_1, \dots, a_d)\Big)T^{r+1} + \dots$$

It is enough to choose a j such that $\alpha_{j,r} \neq 0$ to find a_i's in $\mathcal{O}_K$ such that $\alpha_{j,r}(a_1, \dots, a_g) \neq 0$. If we set $b_j = 1$ and $b_i = 0$ for $i \neq j$ we see that $\lambda \neq 0$.

□

2.6. Connections with Tate's conjecture

2.6.1. Let K be as in 2.4.1, X an abelian variety over K, $\mathrm{T}_p(X) = \mathrm{T}_p(X_{\overline{\eta}})$ the p-adic Tate module of X.

2.6.2. THEOREM (Tate-Raynaud). *Under the assumptions 2.6.1, there exist canonical, bijective, K-linear homomorphisms*

$$\varrho^1_X : \mathrm{H}^1(X, \mathcal{O}_X) \to \mathrm{Hom}_{\mathbb{Z}_p[G_K]}(\mathrm{T}_p(X), \mathbf{C}),$$
$$\varrho^0_X : \mathrm{H}^0(X, \Omega^1_{X/K}) \to \mathrm{Hom}_{\mathbb{Z}_p[G_K]}(\mathrm{T}_p(X), \mathbf{C}(1))$$

where ϱ^0_X is the homomorphism defined in 2.4.11.

PROOF. Let g be the dimension of X. By 2.4.14 we have:

$$d = \dim_K(\mathrm{Hom}_{\mathbb{Z}_p[G_K]}(T_p(X), \mathbf{C}(1))) \geq \dim_K \mathrm{H}^0(X, \Omega^1_{X/K}) = g. \tag{2.6.2.1}$$

Equality holds in (2.6.2.1) if and only if ϱ^0_X is an isomorphism. Let $\hat{X}$ be the dual abelian variety of X. If we interchange the roles of X and $\hat{X}$, we get from the injection

$$\varrho^0_{\hat{X}} : \mathrm{H}^0(\hat{X}, \Omega^1_{\hat{X}/K}) \to \mathrm{Hom}_{\mathbb{Z}_p[G_K]}(\mathrm{T}_p(\hat{X}), \mathbf{C}(1))$$

the inequality

$$d' = \dim_K(\mathrm{Hom}_{\mathbb{Z}_p[G_K]}(\mathrm{T}_p(\hat{X}), \mathbf{C}(1))) \geq g.$$

The Weil pairing

$$\mathrm{T}_p(X) \times \mathrm{T}_p(\hat{X}) \to \mathbb{Z}_p(1)$$

is a perfect $\mathbb{Z}_p$-linear pairing, compatible with the action of G_K (see [**Mum70**, p. 186]). It induces a canonical isomorphism

$$\mathrm{T}_p(X) \cong \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{T}_p(\hat{X}), \mathbb{Z}_p(1)). \tag{2.6.2.2}$$

Let $W = \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{T}_p(X), \mathbf{C}(1))$ and $\hat{W} = \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{T}_p(\hat{X}), \mathbf{C}(1))$. By (2.6.2.2) we have $W \cong \mathrm{T}_p(\hat{X}) \otimes_{\mathbb{Z}_p} \mathbf{C}$ and $\hat{W} \cong \mathrm{T}_p(X) \otimes_{\mathbb{Z}_p} \mathbf{C}$, so that there is a canonical non-degenerate G_K-pairing

$$W \times \hat{W} \to \mathbf{C}(1). \tag{2.6.2.3}$$

By (1.5.15), we have $\mathrm{H}^0_{\mathrm{cont}}(G_K, \mathbf{C}(1)) = \mathrm{H}^1_{\mathrm{cont}}(G_K, \mathbf{C}(1)) = 0$. By 1.6.2, $\hat{W}^{G_K} \otimes_K \mathbf{C}$ and $W^{G_K} \otimes_K \mathbf{C}$ are $\mathbf{C}$-subspaces of $\hat{W}$ and W. Since they are paired into $\mathbf{C}(1)^{G_K}$, they are orthogonal with respect to the pairing (2.6.2.3). Their dimensions are d' and d respectively, and by (1.9.5.1) we have $d + d' \leq 2g = \dim_{\mathbf{C}}(T_p(X) \otimes_{\mathbb{Z}_p} \mathbf{C})$, as required.

In order to get the morphism ϱ^1_X we use again duality for abelian varieties. First of all, recall that there is a canonical isomorphism between the tangent space at 0 to the dual abelian variety $\hat{X}$ and $\mathrm{H}^1(X, \mathcal{O}_X)$ ([**Mum70**], Corollary 3, p. 130). Hence

$$\mathrm{H}^1(X, \mathcal{O}_X) = \mathrm{Hom}_K(\mathrm{H}^0(\hat{X}, \Omega^1_{\hat{X}/K}), K).$$

The spaces $\mathrm{H}^0(X, \Omega^1_{X/K})$ and $\mathrm{H}^0(\hat{X}, \Omega^1_{\hat{X}/K})$ are mapped injectively onto subspaces of W and $\hat{W}$ which are orthogonal with respect to the pairing to $\mathbf{C}(1)$. Hence we have

$$\mathrm{Hom}_{\mathbf{C}}(\mathrm{H}^0(\hat{X}, \Omega^1_{\hat{X}/K}), \mathbf{C}(1)) = W^{G_K},$$

so that $\mathrm{H}^1(X, \mathcal{O}_X) = W^{G_K} \otimes_K \mathbf{C}(-1)$. But then

$$W^{G_K} \otimes_K \mathbf{C}(-1) = \mathrm{Hom}_{\mathbb{Z}_p[G_K]}(\mathrm{T}_p(X), \mathbf{C}(1)) \otimes_K \mathbf{C}(-1) \cong \mathrm{Hom}_{\mathbb{Z}_p[G_K]}(\mathrm{T}_p(X), \mathbf{C})$$

providing the required isomorphism

$$\varrho^1_X \colon \mathrm{H}^1(X, \mathcal{O}_X) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}_p[G_K]}(\mathrm{T}_p(X), \mathbf{C}).$$

$\square$

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