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The Gauss-Manin connection and regular singular points.

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1. Abstract

In this paper we provide an outline of the *algebraic* proof of the classical theorem that a certain local system has regular singular points everywhere. The proof is due to Katz [K]. In particular, a theory of nilpotency is developed and then coupled with a theorem of Turrittin to deduce the statement on regular singularities. Furthermore, work of Manin is subsequently used to show that in fact the local monodromy admits rational exponents. The outline of this paper is as follows. After some preliminaries on smooth morphisms and Kahler differentials we introduce connections. In Section 3 the Gauss-Manin connection is defined and is calculated for the case of a family of affine curves. Section 4 provides the basic theory of nilpotent connnections. Sections 5,6, and 7 treat regular singular points and Turrittin's theorem and in Section 8 we briefly present the relevant notions of monodromy. Section 9 brings the theory of nilpotent connections and Turrittin's theorem together to give the desired theorem on regular singular points. The only novel aspect of this paper is the explicit calculation in the case of curves and possibly, the exposition.

The only prerequisite for reading this paper is a first course in algebraic geometry. This paper closely follows [K] and we have included various examples in hopes of illuminating concepts. I thank my family for their constant support and my masters advisor J.P.P. dos Santos for his help and encouragement.

To Jin, for everything

2. Preliminaries

2.1. Smooth morphisms. We refer the reader to [AK, Ch. VII, §1] as the main reference for this subsection.

DEFINITION 2.1. A morphism of schemes $f : X \to Y$ is said to be *smooth at* $x \in X$ if X and Y are locally noetherian and there exists a neighborhood U of x such that there is a commutative diagram

$$\begin{tikzcd} U \arrow[r, "g"] \arrow[dr, "f"'] & \mathbb{A}^n_Y \arrow[d, "q"] \\ & Y \end{tikzcd}$$

where g is étale and q is the projection $q : \mathbb{A}^n_Y = \mathbb{A}^n_{\mathbb{Z}} \times Y \to Y$. Moreover, f is said to be *smooth* if it is smooth at every point $x \in X$.

Since étale morphisms are flat we have that the relative dimension of X over Y is *constant*, i.e. that $\dim_x X_{f(x)}$ is a constant function of $x \in X$. As étale morphisms are also stable under base change, then assuming $Y = Spec\, k(y)$ and using that $\dim(\mathbb{A}^n_{k(y)}) = n$ we have the

LEMMA 2.2. If $f : X \to Y$ is smooth, then the relative dimension of X over Y is equal to n (the n appearing in the defintion).

2.2. Good properties of smooth morphisms. Here are two of the main reasons for why we restrict ourselves to smooth schemes. Suppose that we have smooth morphisms $X \xrightarrow{\pi} S \xrightarrow{f} T$.

(i). The sheaf of relative differentials $\Omega^1_{X/T}$ is then locally free of finite rank (composition of smooth is smooth)

(ii). If f is *smooth* then in fact the sequence

$$0 \to f^*\Omega^1_{S/T} \to \Omega^1_{X/T} \to \Omega^1_{X/S} \to 0 \tag{1}$$

is locally split and short exact.

Let us recall why the canonical differential map $d : \Omega^r_{X/T} \to \Omega^{r+1}_{X/T}$ makes $\Omega^\bullet_{X/T}$ into a complex. It suffices to verify this claim locally, so working over affine open subsets we may suppose that $\omega \in \Omega^r_{X/T}$ is given by $\omega = f d\alpha_1 \wedge \cdots \wedge d\alpha_r$ where f and the a_i are sections $\Omega^r_{X/T}$ of such an open set. Then

$$d(\omega) = df \wedge d\alpha_1 \wedge \cdots \wedge d\alpha_r.$$

Applying d again we have

$$d(df) = d(1 \cdot df \wedge d\alpha_1 \wedge \cdots \wedge d\alpha_r) = 0$$

since $d1 = 0$.

Item (ii) gives a particulary simple way of calculating graded objects associated to the complex $\Omega^\bullet_{X/T}$. Define a filtration on $\Omega^\bullet_{X/T}$:

$$F^i(\Omega^\bullet) = image(\pi^*\Omega^i_{S/T} \otimes_{O_X} \Omega^{\bullet - i}_{X/T} \to \Omega^\bullet_{X/T}).$$

The graded objects are

$$gr^i = F^i/F^{i+1}$$

and the short exact sequence (1) yields that

$$gr^i = \pi^*\Omega^i_{S/T} \otimes_{O_X} \Omega^{\bullet-i}_{X/S}.$$

This result will be particularly useful in calculating the first term of the spectral sequence associated to the above filtration.

2.3. de Rham Cohomology. In the above subsection we showed that $\Omega^\bullet_{X/T}$ forms a complex. This fact allows us to define cohomology on it, which is called the *de Rham* cohomology and denoted by $H^\bullet_{dR}(X/T)$. We will write $H^\bullet_{dR}$ for short when there is no confusion.

EXAMPLE 2.3. Here we compute $H^\bullet_{dR}$ for an elliptic curve defined by $y^2 = x^3 + ax + b$ over a field k of characteristic zero in the affine case. Write $P(x) = x^3 + ax + b$,

$$R = k[x, y]/(y^2 - P(x)),$$

and define

$$X = Spec\ R.$$

Being in the affine case, we identify $\Omega^1_{X/k}$ with $\Omega^1_{R/k}$. Clearly, $H^0_{dR} = \langle 1\rangle$ and $H^i_{dR} = 0$ for $i > 1$ since X is a curve ($\dim X = 1$). It remains to calculate H^1_{dR}.

On differentiating P we obtain the relation

$$2ydy = (3x^2 + a)dx$$

in $\Omega^1_{R/k}$. Since P and P' are relatively prime, there are polynomials $A, B \in k[x]$ satisfying $AP + BP' = 1$. Set

$$\omega := Aydx + 2Bdy$$

and observe that

$$\begin{aligned} y\omega &= Ay^2dx + 2yBdy \\ &= A(x^3 + ax + b)dx + (3x^2 + a)dxB \\ &= APdx + BP'dx \\ &= dx. \end{aligned}$$

Therefore, $dx = y\omega$ and $dy = \frac{P'}{2}\omega$ and hence any element $\gamma \in \Omega^1_{R/k}$ can be written in the form $\gamma = (C + Dy)\omega$, for some $C, D \in k[x]$. In order to calculate the cohomology, we need to study when $(C + Dy)\omega$ is an exact form. If this is the case, then for some $E, F \in k[x]$, with E corresponding to D and F corresponding to C, we calculate

$$\begin{aligned}(C+Dy)\omega &= d(E+Fy)\\ &= E'dx + Fdy + yF'dx\\ &= E'y\omega + \frac{P'}{2}F\omega + F'y^2\omega\\ &= (E'y + PF' + \frac{P'}{2}F)\omega.\end{aligned}$$

Clearly, the term $Dy\omega$ is exact since it corresponds to the term $E'y\omega$; necessarily, the class of ω is an element of H^1_{dR}. Moreover, $C\omega$ corresponds to the term $(PF' + \frac{P'}{2}F)\omega$. Suppose that F has leading term αx^r, then the leading term of $\frac{1}{2}P'F + F'P$ is equal to

$$\begin{aligned}\left(\frac{1}{2}3x^2\right)(\alpha x^r) + (r\alpha x^{r-1})(x^3) &= \frac{3}{2}\alpha x^{r+2} + r\alpha x^{r+2}\\ &= \left(\frac{3}{2}+r\right)\alpha x^{r+2}.\end{aligned}$$

Therefore, F and $\frac{1}{2}P'F + F'P$ together generate polynomials with leading terms x^r and x^{r+2}; the missing term x^{r+1} is accounted for by multiplying x to the elements of H^1_{dR}; necessarily, the class of $x\omega$ is an element of H^1_{dR}. Therefore,

$$H^1_{dR} = \langle \omega, x\omega \rangle.$$

2.4. Connections. Let $f : S \to T$ be a smooth T-scheme, $\mathcal{E}$ a quasi-coherent sheaf of $\mathcal{O}_S$-modules.

Definition 2.4. A *connection* on $\mathcal{E}$ is a homomorphism

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_{S/T}$$

of abelian sheaves satisfying the "Leibniz rule "

$$\nabla(ge) = g\nabla(e) + dg \otimes e$$

where g and e are sections of $\mathcal{O}_S$ and $\mathcal{E}$, respectively, over an open subset of S and $d : \mathcal{O}_S \to \Omega^1_{S/T}$ canonical exterior differentiation.

Example 2.5. Consider $f : S = \mathbb{A}^1_k \to T = Spec\ k$, then $\Omega^1_{S/T} \cong \{\frac{d}{dt}\} \cdot k[t]$. Suppose $\mathcal{E}$ is a free $k[t]$-module of rank 1, so $\mathcal{E} \cong k[t] \cdot e$. A connection $\nabla : \mathcal{E} \to \Omega^1_{S/T} \otimes e$ is determined by declaring $\nabla(e) = \omega \otimes e$, for some $\omega \in \Omega^1_{S/T}$.

We will want connections to act not only on quasi-coherent sheaves $\mathcal{E}$ of $\mathcal{O}_S$-modules, but also on the sheaves $\Omega^i_{S/T} \otimes_{\mathcal{O}_S} \mathcal{E}$. To this end, a T-connection is extended to a homomorphism of abelian sheaves

$$\nabla_i : \Omega^i_{S/T} \otimes_{\mathcal{O}_S} \mathcal{E} \to \Omega^{i+1}_{S/T} \otimes_{\mathcal{O}_S} \mathcal{E}$$

satisfying

$$\nabla_i(\omega \otimes e) = d\omega \otimes e + (-1)^i \omega \wedge \nabla(e)$$

where ω and e are sections of $\Omega^i_{S/T}$ and $\mathcal{E}$, respectively, over an open subset of S, and where $\omega \wedge \nabla(e)$ denotes the image of $\omega \otimes \nabla(e)$ under the canonical map

$$\Omega^i_{S/T} \otimes_{\mathcal{O}_S} \left(\Omega^1_{S/T} \otimes_{\mathcal{O}_S} \mathcal{E}\right) \to \Omega^{i+1}_{S/T} \otimes_{\mathcal{O}_S} \mathcal{E}$$

defined by $\omega \otimes \tau \otimes e \mapsto (\omega \wedge \tau) \otimes e$.

2.5. Connections and derivations. Here we establish a correspondence between connections and derivations, which we will implicitly use in subsequent sections.

Suppose that S/T is smooth. Let $Der(S/T)$ denote the sheaf of germs of T-derivations of $\mathcal{O}_S$ into itself. Let $End_T(\mathcal{E})$ denote the sheaf of $f^{-1}(\mathcal{O}_T)$-linear endomorphisms of $\mathcal{E}$.

Let $MC(S/T)$ denote the category whose objects are pairs $(\mathcal{E}, \nabla)$ and whose morphisms between objects $(\mathcal{E}, \nabla)$ and $(\mathcal{F}, \nabla')$ are $\mathcal{O}_S$-linear mappings

$$\phi : \mathcal{E} \to \mathcal{F}$$

with the property that

$$\phi(\nabla(D)(e)) = \nabla'(D)(\phi(e))$$

where D and e are sections of $Der(S/T)$ and $\mathcal{E}$, respectively, defined over an open subset of S.

First we show how to associate a derivation to a connection. Given a T-connection ∇ on $\mathcal{E}$, this connection gives rise to an $\mathcal{O}_S$-linear mapping:

$$\nabla : Der(S/T) \to End_T(\mathcal{E})$$

defined by

$$D \mapsto \nabla(D)$$

where $\nabla(D)$ is defined as the composite mapping:

$$\mathcal{E} \xrightarrow{\nabla} \Omega^1_{S/T} \otimes_{\mathcal{O}_S} \mathcal{E} \xrightarrow{D \otimes 1} \mathcal{O}_S \otimes_{\mathcal{O}_S} \mathcal{E} \cong \mathcal{E}.$$

Recalling the Leibniz rule for a connection, we obtain:

$$\nabla(D)(fe) = D(f)e + f\nabla(D)(e) \tag{2}$$

where D, f, and e are sections of $Der(S/T), \mathcal{O}_S$, and $\mathcal{E}$, respectively, over an open subset of S.

Conversely, given an $\mathcal{O}_S$-linear mapping $\tilde{\nabla} : Der(S/T) \to End_T(\mathcal{E})$ which satisfies the Leibniz rule (3), then because S/T is smooth, $\tilde{\nabla}$ arises from a unique T-conneciton ∇.

2.6. Integrable Connections.

Definition 2.6. The *curvature*, $K = K(\mathcal{E}, \nabla)$, of the T-connection ∇ is the $\mathcal{O}_S$-linear map

$$K = \nabla_1 \circ \nabla : \mathcal{E} \to \Omega^2_{S/T} \otimes_{\mathcal{O}_S} \mathcal{E}.$$

In fact,

$$(\nabla_{i+1} \circ \nabla_i)(\omega \otimes e) = \omega \wedge K(e). \tag{3}$$

Definition 2.7. The T-connection ∇ is called *integrable* if $K = 0$.

Therefore, if ∇ is integrable then we define the following *complex*, the de Rham complex of $(\mathcal{E}, \nabla)$:

$$0 \to \mathcal{E} \xrightarrow{\nabla} \Omega^1_{S/T} \otimes_{\mathcal{O}_S} \mathcal{E} \xrightarrow{\nabla_1} \Omega^2_{S/T} \otimes_{\mathcal{O}_S} \mathcal{E} \xrightarrow{\nabla_2} \cdots$$

which we denote by $\Omega^\bullet_{S/T} \otimes_{\mathcal{O}(S)} \mathcal{E}$.

Define $MIC(S/T)$ to be the full subcategory of $MC(S/T)$ consisting of sheaves of quasi-coherent $\mathcal{O}_S$-modules with *integrable* connections.

3. The Gauss-Manin connection

3.1. The definition via spectral sequences. Suppose that $\pi : X \to S$ and $f : S \to T$ are smooth morphisms of schemes. Filter $\Omega^\bullet_{X/S}$ by locally free subsheaves

$$F^i(\Omega^\bullet_{X/S}) := \textit{image of } (\pi^*(\Omega^i_{X/S}) \otimes_{\mathcal{O}_X} \Omega^{\bullet - i}_{X/S} \to \Omega^\bullet_{X/S})$$

so that the filtration is of the form

$$\Omega^\bullet_{X/S} = F^0(\Omega^\bullet_{X/S}) \supset F^1(\Omega^\bullet_{X/S}) \supset \cdots .$$

There is then a spectral sequence associated to the finitely filtered object $\Omega^\bullet_{X/S}$ and we denote this spectral sequence by $(E_r^{p,q}, d_r^{p,q})$; it abutes to $\mathbb{R}^q \pi_*(\Omega^\bullet_{X/S})$.

Definition 3.1. For a smooth morphism $\pi : X \to S$ define the *de Rham cohomology sheaves* as the hyperderived functors of π_* of the relative differentials, that is

$$H^q_{dR}(X/S) := \mathbb{R}^q \pi_*(\Omega^\bullet_{X/S}).$$

Given an object $(\mathcal{E}, \nabla)$ of $MIC(X/T)$, consider the natural forgetful functor

$$MIC(X/T) \to MIC(X/S)$$

sending $(\mathcal{E}, \nabla) \mapsto (\mathcal{E}, \nabla | Der(X/S))$. Denote the associated de Rham complex of $(\mathcal{E}, \nabla | Der(X/S))$ by $\Omega^\bullet_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E}$. The associated filtration on $\Omega^\bullet_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E}$ is given by

$$F^i(\Omega^\bullet_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E}) = F^i(\Omega^\bullet_{X/S}) \otimes_{\mathcal{O}_X} \mathcal{E}$$

and the associated graded objects are given by

$$gr^i(\Omega^\bullet_{X/S}\otimes_{\mathcal{O}_X}\mathcal{E})=\pi^*(\Omega^i_{S/T})\otimes_{\mathcal{O}_X}(\Omega^{\bullet-i}_{X/S}\otimes_{\mathcal{O}_X}\mathcal{E}).$$

Similarly, we get a spectral sequence associated to the filtered object $\Omega^\bullet_{X/S}\otimes_{\mathcal{O}_X}\mathcal{E}$. From [K, (3.2.5)] we have that the terms $E_1^{p,q}$ of this spectral sequence are given by

$$E_1^{p,q}=\Omega^p_{S/T}\otimes_{\mathcal{O}_S}H^q_{dR}(X/S,(\mathcal{E},\nabla)).$$

Therefore, the de Rham complex of $H^q_{dR}(X/S,(\mathcal{E},\nabla))$ is the complex $(E_1^{\bullet,q},d_1^{\bullet,q})$, i.e. the q-th row of E_1 terms.

Definition 3.2. The *Gauss-Manin connection* on the relative de Rham cohomology sheaf $H^q_{dR}(X/S,(\mathcal{E},\nabla))$ is the differential map $d_1^{0,q}$ of the associated spectral sequence.

Remark. Taking $\mathcal{E}=\mathcal{O}_X$ and ∇ equal to the canonical differential $d:\mathcal{O}_X\to\Omega^1_{X/S}$, that is

$$(\mathcal{E},\nabla)=(\mathcal{O}_X,d)$$

then

$$H^q_{dR}(X/S,(\mathcal{E},\nabla))=H^q_{dR}(X/S)$$

which shows that the Gauss-Manin connection defined above (i.e. as in [K]) *does* generalize the case taken in [KO68].

3.2. The definition via Čech cohomology. We can explicate the construction of the Gauss-Manin connection via Čech cohomology. Since defining a connection is a local question, assume that S is affine. Then, since X/S is assumed smooth, X admits a finite covering $\{U_\alpha\}$ such that each U_α is étale over $\mathbb{A}^n_S$. On U_α, the sheaf $\Omega^1_{X/S}$ is a free $\mathcal{O}_X$-module with basis $\{dx_1^\alpha,\dots,dx_n^\alpha\}$.

Let $(\mathcal{E},\nabla)$ be a given object of $MIC(X/T)$. The S-modules

$$\mathbb{R}^i\pi_*(\Omega^\bullet_{X/S}\otimes_{\mathcal{O}_X}\mathcal{E})$$

may be calculated as the total cohomology of the bicomplex of $\mathcal{O}_S$-modules

$$C^{p,q}:=C^p(\{U_\alpha\},\Omega^q_{X/S}\otimes_{\mathcal{O}_X}\mathcal{E})$$

of alternating Čech cochains on the nerve of $\{U_\alpha\}$. That is,

$$\mathbb{R}^i\pi_*(\Omega^\bullet_{X/S}\otimes_{\mathcal{O}_X}\mathcal{E})=H^i(Tot(C^{p,q}(\mathcal{E})).$$

We now calculate a T-connection on the total complex $Tot(C^{p,q}(\mathcal{E}))$. The Gauss-Manin connection on the sheaves $H^q_{dR}(X/S,(\mathcal{E},\nabla))$ is then defined on passage to cohomology.

To begin, let D by any T-derivation of the coordinate ring of S. For a fixed α, let $D_\alpha\in Der_T(U_\alpha,U_\alpha)$ be the unique *extension* of D satisfying

$$D_\alpha(dx_i^\alpha)=0,\ (1\le i\le n),$$

i.e. the uniqe extension of D which "kills" the basis elements $dx_1^\alpha, \dots, dx_n^\alpha$. Further, D_α induces a T-linear endomorphism of sheaves

$$D_\alpha : \Omega^q_{U_\alpha/S} \to \Omega^q_{U_\alpha/S}$$

defined by setting

$$D_\alpha(h dx_1^\alpha \wedge \cdots \wedge dx_n^\alpha) = D_\alpha(h) dx_1^\alpha \wedge \cdots \wedge dx_n^\alpha$$

where h is a section of $\mathcal{O}_X$ over an open subset of U_α. In addition, $D_\alpha \in Der_T(U_\alpha, U_\alpha)$ also induces a T-linear endomorphism of sheaves

$$D_\alpha : \Omega^q_{U_\alpha/S} \otimes_{\mathcal{O}_{U_\alpha}} \mathcal{E} \to \Omega^q_{U_\alpha/S} \otimes_{\mathcal{O}_{U_\alpha}} \mathcal{E}$$

defined by

$$D_\alpha(\omega \otimes e) = D_\alpha(\omega) \otimes e + \omega \otimes \nabla(D_\alpha)(e)$$

where ω and e are sections of $\Omega^q_{X/S}$ and $\mathcal{E}$, respectively, defined over an open subset of U_α.

Choose a total ordering on the indexing set of the collection $\{U_\alpha\}$ and deonote this totally ordered set by $(I, <)$. We employ the notation

$$U_{i_0, \dots, i_k} := U_{i_0} \cap \cdots \cap U_{i_k}$$

for elements $U_{i_0}, \dots, U_{i_k} \in \{U_\alpha\}$. Define a T-linear endomorphism $\tilde{D}$ on the $\mathcal{O}_S$-module $C^{p,q} = C^p(\{U_\alpha\}, \Omega^q_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E})$ by setting

$$\tilde{D} | \Gamma \left(U_{\alpha_0, \dots \alpha_p}, \Omega^q_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E} \right) = D_{\alpha_0}$$

whenever $\alpha_0 < \cdots < \alpha_p$ for $\alpha_i \in I, 1 \leq i \leq p$.

For each pair of indices $\alpha, \beta \in I$, define an $\mathcal{O}_X$-linear mapping of sheaves

$$\lambda(D)_{\alpha,\beta} : \Omega^q_{X/S} | U_{\alpha,\beta} \to \Omega^{q-1}_{X/S} | U_{\alpha,\beta}$$

by

$$\lambda(D)_{\alpha,\beta}(h dx_1 \wedge \cdots \wedge dx_q) = h \sum_{i=1}^{q} (-1)^i (D_\alpha - D_\beta)(x_i) dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_q$$

where $h, x_1, \dots, x_q$ are sections of $\mathcal{O}_X$ over an open subset of $U_{\alpha,\beta}$. Moreover, put $\lambda(D)_{\alpha,\beta} = 0$ on $\mathcal{O}_X$. Similarly, define an $\mathcal{O}_X$-linear map

$$\lambda(D)_{\alpha,\beta} : \left(\Omega^q_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E} \right) | U_{\alpha,\beta} \to \left(\Omega^{q-1}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E} \right) | U_{\alpha,\beta}$$

by

$$\lambda(D)_{\alpha,\beta}(\omega \otimes e) = \lambda(D)_{\alpha,\beta}(\omega) \otimes e.$$

Finally, define an $\mathcal{O}_X$-linear endomorphism $\lambda(D)$ of bidegree $(+1, -1)$ on the $\mathcal{O}_S$-module $C^{\bullet,\bullet}(\mathcal{E})$:

$$\lambda(D) : C^p(\{U_\alpha\}, \Omega^q_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E}) \to C^{p+1}(\{U_\alpha\}, \Omega^{q-1}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E})$$

by

$$(\lambda(D)(\sigma))_{\alpha_0, \dots, \alpha_{p+1}} = (-1)^q \lambda(D)_{\alpha_0, \alpha_1} (\sigma_{\alpha_1, \dots, \alpha_{p+1}})$$

whenever $\alpha_0 < \cdots < \alpha_{p+1}$ and where σ is the alternating p-cochain whose value on $U_{\alpha_0,\ldots,\alpha_p}$ is equal to $\sigma_{\alpha_0,\ldots,\alpha_p}$.

The Gauss-Manin connection on the bicomplex $C^{\bullet,\bullet}(\mathcal{E})$ is given by

$$\Psi : D \in Der_T(\mathcal{O}_S, \mathcal{O}_S) \mapsto \Psi(D) = \tilde{D} + \lambda(D).$$

In the case that X is étale over $\mathbb{A}^n_S$, with $\Omega^1_{X/S}$ free with basis $\{dx_1, \ldots, dx_n\}$, then

$$H^q_{dR}(X/S, (\mathcal{E}, \nabla)) = H^q\left(\Gamma(X, \Omega^\bullet_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E})\right).$$

Then for $D \in Der_T(\mathcal{O}_S, \mathcal{O}_S)$, the action of $\Psi(D)$ on $H^q_{dR}(X/S, (\mathcal{E}, \nabla))$ is the action induced from the T-endomorphism $\tilde{D}$ of $\Gamma(X, \Omega^\bullet_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E})$:

$$\tilde{D}(\omega \otimes e) = D_0(\omega) \otimes e + \omega \otimes \nabla(D_0)(e)$$

where $D_0 \in Der_T(\mathcal{O}_X, \mathcal{O}_X)$ is the unique extension of D which kills the basis elements $dx_1, \ldots, dx_n$, and where ω and e are sections of $\Omega^\bullet_{X/S}$ and $\mathcal{E}$, respectively, defined over an open subset of X (cf. [K, 3.5.3]).

3.3. The case of curves. Parts of the following setup were inspired from [I]. Let k be any field and let S/k be a smooth *curve* and $f : X \to S$ a smooth morphism. It suffices to define the Gauss-Manin connection over an affine open subset of S, so further suppose that S is affine. By smoothness, we have an exact sequence

$$0 \to gr^1 \to F^0/F^2 \to gr^0 \to 0$$

where the filtration $F^\bullet$ is defined as above and the grading is $gr^i := F^i/F^{i+1}$. Since S/k is a curve, we have $F^2 = 0$ hence the filtration is equivalent to the exact sequence

$$0 \to f^*\Omega^1_{X/S} \otimes_{\mathcal{O}_X} \Omega^{\bullet-1}_{X/S} \to \Omega^\bullet_{X/k} \to \Omega^\bullet_{X/S} \to 0. \tag{4}$$

This exact sequence of complexes gives rise to a long exact sequence in hypercohomology by hyperderiving with the left exact morphism f_*. Write $\mathbb{H}^i(U, \Omega^\bullet_{X/S}|U) := H^i_{dR}(X/S) = (\mathbb{R}^i f_* \Omega^\bullet_{X/S})(U)$.

Definition 3.3. The Gauss-Manin connection, ∇_i, is equal to the boundary map in the long exact sequence (4):

$$\nabla_i : H^i_{dR} = \mathbb{H}^i(\Omega^\bullet_{X/S}) \to \mathbb{H}^{i+1}\left(f^{-1}\Omega^\bullet_{S/k} \otimes_{f^{-1}(\mathcal{O}_S)} \Omega^{\bullet-1}_{X/S}\right).$$

Notice that since $f^{-1}(\Omega^1_{S/k})$ is locally free and the differential of the above complex is $f^{-1}(\mathcal{O}_S)$-linear, then in fact

$$\mathbb{H}^{i+1}\left(f^{-1}\Omega^\bullet_{S/k} \otimes_{f^{-1}(\mathcal{O}_S)} \Omega^{\bullet-1}_{X/S}\right) = \Omega^\bullet_{S/k} \otimes_{\mathcal{O}_S} \mathbb{H}^{i+1}(\Omega^{\bullet-1}_{X/S}) = \Omega^\bullet_{S/k} \otimes_{\mathcal{O}_S} H^i_{dR}(X/S).$$

Therefore, the Gauss-Manin connection is the connecting homomorphism:

$$\nabla_i : H^i_{dR}(X/S) \to \Omega^1_{S/k} \otimes_{\mathcal{O}_S} H^i_{dR}(X/S).$$

We now calculate the Čech cohomology in this case. Let $\{U_i\}_{i \in I}$ be a cover of X by open affines with the indexing set I well-ordered.

Let us make the total complexes $Tot(C^{p,q})$ more explicit. Put

$$C^{p,q} = \prod_{i_0<\cdots<i_q} \Gamma(U_{i_0,\ldots,i_q}, \Omega^p_{X/S}).$$

On the bicomplex $C^{p,q}$ we have horizontal differentials

$$d : C^{p,q} \to C^{p+1,q}$$

given by the differential of $(\Omega^\bullet_{X/S}, d)$ and vertical differentials

$$\delta : C^{p,q} \to C^{p,q+1}$$

obtained from the classical Čech construction. Recall that

$$H^i_{dR}(X/S) = H^i(Tot(C^{\bullet,\bullet})).$$

In this subsection, we are primarily interested in the first two terms $H^0_{dR}(X/S)$ and $H^1_{dR}(X/S)$.

Since $C^{0,0} = \prod_{i\in I} \mathcal{O}_X(U_i)$, then $s \in C^{0,0}$ if and only if s is a global section of $\mathcal{O}_X$. Therefore $ker(\delta) = \mathcal{O}_X(X)$. As

$$H^0_{dR}(X/S) = ker(d+\delta) = ker(d) \cap \ker(\delta),$$

then restricting $d : \mathcal{O}_X(X) \to \Omega^1_{X/S}(X)$ to $ker(\delta)$ we find that

$$H^0_{dR}(X/S) = ker(d : \mathcal{O}_X(X) \to \Omega^1_{X/S}(X)).$$

Similar arguments show that

$$H^1_{dR}(X/S) = Z^1/B^1$$

where Z^1 is the set of 1-hypercocycles and B^1 is the set of 1-hypercoboundaries. Explicitly,

$$Z^1 = \{(\omega_i, f_{ij}) \in \left(\prod_{i\in I} \Omega^1_{X/S}(U_i)\right) \times \left(\prod_{i<j} \mathcal{O}_X(U_{i,j})\right) :$$

$$d\omega_i = 0, \omega_i|U_{i,j} - \omega_j|U_{i,j} = df_{ij},\ f_{ij}|U_{i,j,k} - f_{ik}|U_{i,j,k} + f_{jk}|U_{i,j,k} = 0\}$$

and

$$B^1 = \{(dx_i, x_i|U_{i,j} - x_j|U_{i,j}) \in \left(\prod_{i\in I} \Omega^1_{X/S}(U_i)\right) \times \left(\prod_{i<j} \mathcal{O}_X(U_{i,j})\right) : x_i \in \mathcal{O}_X(U_i)\}.$$

We can now compute the Gauss-Manin connection

$$\nabla_1 : H^1_{dR}(X/S) \to \Omega^1_{S/k} \otimes_{\mathcal{O}_S} H^1_{dR}(X/S).$$

Let $S = Spec\, R$ and $U_i = Spec\, B_i$. Localizing if necessary, assume that $\Omega^1_{S/k} = R \cdot dt$ for some $t \in R$.

Let $[\omega] \in H^1_{dR}(X/S)$. In order to calculate the image $\nabla_1[\omega]$ consider the following diagram, where T^i denotes total cohomology and K^i denotes the cokernel by the corresponding differential:

$$\begin{tikzcd}
 & & Z^1(\Omega^\bullet_{X/S}) \arrow[d] \arrow[dddll] \\
 & T^1(\Omega^\bullet_{X/k}) \arrow[r] \arrow[d, "D"] & T^1(\Omega^\bullet_{X/S}) \\
T^2(dt \otimes \Omega^{\bullet -1}_{X/S}) \arrow[r] \arrow[d] & T^2(\Omega^\bullet_{X/k}) & \\
K^2(dt \otimes \Omega^{\bullet -1}_{X/S}) & &
\end{tikzcd}$$

Following the above diagram, let $[\omega]$ be represented by

$$((\omega_i)_{i\in I}, (f_{ij})_{i<j}) \in T^1(\Omega^\bullet_{X/S})$$

for $\omega_i \in \Omega^1_{B_i/R}$ and $f_{ij} \in B_{ij}$. Now lift the ω_i to elements $\overline{\omega_i}$ so that we get an element $((\overline{\omega_i})_{i\in I}, (f_{ij})_{i<j}) \in T^1(\Omega^\bullet_{X/k})$. Next, apply D and then write

$$D((\overline{\omega_i})_{i\in I}, (f_{ij})_{i<j}) = dt \otimes ((\tau_i)_{i\in I}, (g_{ij})_{i<j}).$$

Therefore, we express $\nabla_1([\omega])$ as

$$\nabla_1([\omega]) = dt \otimes ((\tau_i)_{i\in I}, (g_{ij})_{i<j}) \in \Omega^1_{S/k} \otimes_{\mathcal{O}_S} H^1_{dR}(X/S).$$

Example 3.4. Consider $k = \mathbb{C}$, $A = \mathbb{C}[t]$, and $B = \mathbb{C}[x, y, t]/(F(x, y, t))$ such that we have a sequence of smooth morphsisms

$$X = Spec\ B \to S = Spec\ A \to Spec\ k.$$

We may regard X/S as a family of (nonsingular) curves parametrized by the indeterminent t, e.g. for $\alpha \in \mathbb{C}$ the fibre X_α corresponds to the curve given by the locus $F(x, y, t = \alpha) = 0$.

First, we compute the relative differentials,

$$\Omega^1_{X/k} = \frac{\mathcal{O}_X\{dx, dy, dt\}}{\left(\frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial t}dt\right)},\ \ \Omega^1_{X/S} = \frac{\mathcal{O}_X\{dx, dy\}}{\left(\frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy\right)},\ \ \Omega^1_{S/k} = \mathcal{O}_S\{dt\}.$$

We will use Čech cohomology to compute de Rham cohomology. In particular, since each of our schemes is affine, we take the entire space as the affine cover in each case, hence $C^{\bullet,q} = 0$ whenever $q \geq 1$ since there is only one open set in each case. (More concisely, Čech cohomology is concentrated in degree zero.) Moreover, since we are in the affine setting throughout, we will identify sheaves and modules of differentials, e.g. $\mathcal{O}_X$ "=" A. We have:

$$Z^1_{(X/S)} = \{\omega \in \Omega^1_{B/A} : d\omega = 0\},\ B^1_{X/S} = \{df_i : f_i \in A\}$$

and

$$T^i(\Omega^\bullet_{X/S}) = \Omega^i_{X/S},\ T^i(\Omega^\bullet_{X/k}) = \Omega^i_{X/k}.$$

The total complexes, T^i, are equal to the i-th differential since $C^{\bullet,q} = 0, \forall q \geq 1$ and $T^i = C^{i,0} = \Omega^i$ in each case.

So now let $[\omega] \in H^1_{dR}(X/S) = Z^1_{X/S}/B^1_{X/S}$ and represent it by $\omega \in Z^1_{(X/S)}$. In fact, we can represent $[\omega]$ by $\omega = f_1 dx + f_2 dy$, for some $f_1, f_2 \in \mathbb{C}[x,y]$. Now lift this element to $T^1(\Omega^\bullet_{X/k}) = \Omega^1_{X/k}$ and express this lift as $\bar{\omega} = \bar{f}_1 dx + \bar{f}_2 dy$, for some $\bar{f}_1, \bar{f}_2 \in B$.

Finally, the map D in our example is just the canonical differential

$$d : \Omega^1_{X/k} \to \Omega^2_{X/k}.$$

(NB: $\Omega^2_{X/k}$ is not necessarily zero since X is a surface!)

In particular,

$$d(\bar{\omega}) = \frac{\partial \bar{f}_1}{\partial y} dy \wedge dx + \frac{\partial \bar{f}_1}{\partial t} dt \wedge dx + \frac{\partial \bar{f}_2}{\partial x} dx \wedge dy + \frac{\partial \bar{f}_2}{\partial t} dt \wedge dy.$$

Since $d(\omega) = 0$ (it is closed by assumption), then in fact

$$\left(\frac{\partial \bar{f}_2}{\partial x} - \frac{\partial \bar{f}_1}{\partial y} \right) dx \wedge dy = 0,$$

hence,

$$d(\bar{\omega}) = dt \wedge \left(\frac{\partial \bar{f}_1}{\partial t} dx + \frac{\partial \bar{f}_2}{\partial t} dy \right).$$

Therefore, the Gauss-Manin connection in this case is given by

$$\nabla_1([\omega]) = dt \otimes \left(\frac{\partial \bar{f}_1}{\partial t} dx + \frac{\partial \bar{f}_2}{\partial t} dy \right).$$

Remark. Observe that calculating the kernel of ∇_1 comes down to finding a solution to the separable differential equation

$$\frac{\partial \bar{f}_1}{\partial t} dx + \frac{\partial \bar{f}_2}{\partial t} dy = 0.$$

4. Global Nilpotence

In this section we begin with a definition which is local-to-global in flavor.

Definition 4.1. Let R be an integral domain which is finitely generated *as a ring* over $\mathbb{Z}$, and whose field of fractions has characteristic zero (for example the ring of integers in a number field, or even $\mathbb{Z}[X_1, \ldots, X_n]$; the latter is not module finite). Let $T = Spec\ R$. Then T is said to be a *global affine variety.*

Let $f : S \to T$ be a smooth morphism. Let p be a prime number which is *not* invertible in S. There are only finitely many prime numbers which are invertible in S, hence the existence of infinitely-many non-invertible primes in S.

EXAMPLE 4.2. The most trivial example where all primes numbers are not invertible in S is given by taking $S = T = \mathit{Spec}\ \mathbb{Z}$. On the other hand, consider the morphism $f : \mathit{Spec}\ \mathbb{Q} \to \mathit{Spec}\ \mathbb{Z}$ induced by the natural inclusion $\mathbb{Z} \to \mathbb{Q}$. Every nonzero prime in $\mathbb{Z}$ is invertible in $\mathbb{Q}$. But since $\mathbb{Q}$ is not finitely generated as a $\mathbb{Z}$-module, then the morphism f is not locally of finite type and hence *not* smooth.

The reason we take a non-invertible prime p is so that we can reduce (the equations defining) T modulo p. Write

$$T \otimes \mathbb{F}_p := \mathit{Spec}\ (R/pR) = \mathit{Spec}\ (R \otimes_{\mathbb{Z}} \mathbb{F}_p)$$

and likewise for $S \otimes \mathbb{F}_p$. For otherwise, if p were invertible in S, then $S \otimes \mathbb{F}_p$ corresponds to a tensor product of a p-divisible module with a p-torsion module which implies that $S \otimes \mathbb{F}_p = 0$.

EXAMPLE 4.3. Consider a smooth family of plane curves

$$S = \mathit{Spec}\ \mathbb{Z}[x,t]/(x^2 + px + t) \to \mathit{Spec}\ T = \mathit{Spec}\ \mathbb{Z}[t],$$

then $S \otimes \mathbb{F}_p = \mathit{Spec}\ \mathbb{F}_p[t^{1/2}]$ and $T \otimes \mathbb{F}_p = \mathit{Spec}\ \mathbb{F}_p[t]$.

Focusing our attention now in positive characteristic, let T be a scheme of characterstic $p > 0$ and $S \to T$ smooth. Consider an object $(\mathcal{E}, \nabla)$ of $MIC(S/T)$.

DEFINITION 4.4. The *p-curvature* of the connection ∇ is the mapping

$$\psi : \mathit{Der}(S/T) \to \mathit{End}_T(\mathcal{E})$$

given by

$$\psi(D) = (\nabla(D))^p - \nabla(D^p)$$

where D is a section of $\mathit{Der}(S/T)$ over an open subset of S.

It is straightforward to check that ψ is actually S-linear. We now come to the definition of nilpotence in positive characterstic:

DEFINITION 4.5. The pair $(\mathcal{E}, \nabla)$ is *nilpotent* if there is an integer $n \geq 0$ such that for any n-tuple of sections $D_1, \ldots, D_n$ of $\mathit{Der}(S/T)$ over an open subset of S, we have that

$$\psi(D_1)\psi(D_2)\cdots\psi(D_n) = 0.$$

In this case, $(\mathcal{E}, \nabla)$ is said to be *nilpotent of exponenent* $\leq n$.

Denote by $\mathit{Nilp}(S/T)$ the full subcategory of $MIC(S/T)$ of nilpotent objects and $\mathit{Nilp}^n(S/T)$ those with exponenent $\leq n$.

Going back to characteristic zero, suppose that $(\mathcal{M}, \nabla)$ is an object of $MIC(S/T)$ where T is a global affine variety, S/T is smooth, and $\mathcal{M}$ is locally free of finite rank on S. Let p be a prime number which is not invertible in S. Consider the following diagram where each square commutes:

$$\begin{array}{ccc} S \otimes \mathbb{F}_p & \longrightarrow & S \\ \downarrow & & \downarrow \\ T \otimes \mathbb{F}_p & \longrightarrow & T \\ \downarrow & & \downarrow \\ Spec\, \mathbb{F}_p & \longrightarrow & Spec\, \mathbb{Z} \end{array}$$

Consider the inverse image (given by the pull-back) of $(\mathcal{M}, \nabla)$ in $MIC(S \otimes \mathbb{F}_p / T \otimes \mathbb{F}_p)$; we denote it by $(\mathcal{M} \otimes \mathbb{F}_p, \nabla \otimes \mathbb{F}_p)$. Let us be more precise about what it means to take an inverse image.

In general, suppose that $f : S \to T$ and $f' : S' \to T'$ are smooth morphisms. Suppose that the diagram

$$\begin{array}{ccc} S' & \xrightarrow{g} & S \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle f} \\ T' & \xrightarrow{h} & T \end{array}$$

is commutative. Let $(\mathcal{E}, \nabla)$ be an object of $MC(S/T)$. The mapping

$$\nabla : \mathcal{E} \to \Omega^1_{S/T} \otimes_{\mathcal{O}_S} \mathcal{E}$$

gives a mapping

$$g^*(\mathcal{E}) \to g^*(\Omega^1_{S/T}) \otimes_{\mathcal{O}_{S'}} g^*(\mathcal{E}). \tag{5}$$

The canonical mapping

$$g^*(\Omega^1_{S/T}) \to \Omega^1_{S'/T'}$$

gives a map

$$g^*\Omega^1_{S/T} \otimes_{\mathcal{O}_{S'}} g^*(\mathcal{E}). \tag{6}$$

The composition of (5) and (6) is a mapping, denoted by $g^*(\nabla)$:

$$g^*(\nabla) : g^*(\mathcal{E}) \to \Omega^1_{S'/T'} \otimes_{\mathcal{O}_{S'}} g^*(\mathcal{E})$$

which is a T'-connection on $g^*(\mathcal{E})$.

Definition 4.6. The *inverse image* of $(\mathcal{E}, \nabla)$ is $(g^*(\mathcal{E}), g^*(\nabla))$.

Definition 4.7. The object $(\mathcal{M}, \nabla)$ is *globally nilpotent on* S/T if, for each prime number p not invertible in S, we have that

$$(\mathcal{M} \otimes \mathbb{F}_p, \nabla \otimes \mathbb{F}_p) \in Nilp(S \otimes \mathbb{F}_p / T \otimes \mathbb{F}_p).$$

We need one more definition in order to completely state the main theorem of this section.

DEFINITION 4.8. The *Hodge cohomology* of X/S with $\pi : X \to S$ a proper and smooth morphism, $H^n_{Hodge}(X/S)$, is defined as

$$H^n_{Hodge}(X/S) = \prod_{p+q=n} R^q\pi_*(\Omega^p_{X/S}).$$

We now have all of the ingredients to state the main theorem of this section.

THEOREM A. Let T be a global affine variety, $f : S \to T$ a smooth morphsim, with S connected, and $\pi : X \to S$ proper and smooth. Suppose that each of the coherent sheaves on S : $H^{p,q}_{Hodge}(X/S)(p, q \geq 0), H^n_{dR}(X/S)(n \geq 0)$ is locally free on S. For each integer $i \geq 0$, let $h(i)$ be number of intgers a such tht $H^{a,i-a}_{Hodge}(X/S)$ is nonzero. Then, for each integer $i \geq 0$, the locally free sheaf $H^i_{dR}(X/S)$, with the Gauss-Manin connection, is globally nilpotent of exponent $h(i)$ on S/T.

Let us give a brief sketch as to how this theorem is proved in [K].

For the characteristic p-part of the problem (cf. [K, §7]), suppose that S is of characteristic $p > 0$ and that $\pi : X \to S$ is smooth. One considers the commutative diagram

$$\begin{tikzcd} X^{(p)} \arrow[r, "W"] \arrow[d, "\pi^{(p)}"'] & X \arrow[d, "\pi"] \\ S \arrow[r, "F_{abs}"] & S \end{tikzcd}$$

where F_{abs} is the absolute Frobenius (p-th power mapping on $\mathcal{O}_S$), $X^{(p)}$ is the fibre product of $\pi : X \to S$ and $F_{abs} : S \to S$, and W is the canonical projection. Further, let $F : X \to X^{(p)}$ be the relative Frobenius. The spectral sequences for the de Rham cohomology of X/S are

$$E_2^{p,q} = R^p\pi^{(p)}_*(\mathcal{H}^q(F_*(\Omega^\bullet_{X/S}))) \Rightarrow \mathbb{R}^{p+q}\pi_*(\Omega^\bullet_{X/S}) = H^{p+q}_{dR}(X/S)$$

where $\mathcal{H}^q$ denotes the sheaf of cohomology.

These E_2 terms are simplified by a theorem of Deligne using a Cartier operation ([K, 7.2.0]) from which it is deduced that

$$\mathcal{H}^i(F_*(\Omega^\bullet_{X/S})) \cong \Omega^i_{X^{(p)}/S}.$$

From this result, one obtains

$$E_2^{p,q} = R^p\pi^{(p)}_*\left(\Omega^q_{X^{(p)}/S}\right) = F^*_{abs}\left(R^p\pi_*(\Omega^q_{X/S})\right). \tag{7}$$

From the isomorphisms in (7) one deduces that the E_2 terms are in fact *quasi-coherent* sheaves on $S^{(p)}$ and hence, by a theorem accredited to Cartier (cf. [K] 5.1) one obtains the Gauss-Manin connection on the E_2 terms, which in fact, has p-curvature zero on the terms $E_r^{p,q}$ for $r \geq 2$. Consequently, the sheaves $E_r^{p,q}$, for $r \geq 2$, are *nilpotent of exponent 1*. Using the additional fact that nilpotence is stable under higher direct images (cf. [K, 5.10]), one further deduces that if $h(i)$ is the number of integers k such that $E_2^{k,i-k}$ is nonzero, then for any smooth morphism $f : S \to T$ we have

$$H^i_{dR}(X/S) \in \; Nilp^{\,h(i)}(S/T).$$

Results on base-changing are then provided in [K, §8] which then give Theorem A.

5. Classical theory of regular singular points.

Let k be a field of characteristic 0 and K a function field in one variable over k. Let W be a finite dimensional vector space over K and ∇ a k-connection on W. That is, ∇ is an additive mapping

$$\nabla : W \to \Omega^1_{K/k} \otimes_K W$$

satisfying

$$\nabla(fw) = f\nabla(w) + df \otimes w$$

for $f \in K, w \in W$. It will be useful to regard ∇ as a K-linear mapping

$$\nabla : Der(K/k) \to End_k(W)$$

satisfying

$$(\nabla(D))(fw) = f(\nabla(D))(w) + D(f)w$$

for $D \in Der(K/k), f \in K$, and $w \in W$.

As $K/k(t)$ is a finite algebraic extension for some transcendental element t, then using that $dt \wedge dt = 0$ we see that $\Omega^2_{K/k} = 0$ which implies that any k-connection on W is necessarily integrable.

We have the category $MC(K/k)$ with objects (W, ∇) and arrows given by K-linear mappings $\phi : W \to W'$ satisfying

$$\phi(\nabla(D)(w)) = \nabla'(D')(\phi(w)).$$

This category is equal to $MIC(K/k)$ by the above remark that $\Omega^2_{K/k} = 0$.

Definition 5.1. For $\mathfrak{p}$ a place of K we say that (W, ∇) has a *regular singular point at* $\mathfrak{p}$ if there is a K-basis $\mathbf{e} = (e_1, \ldots, e_n)$ of W such that

$$\nabla\left(h\frac{d}{dh}\right)\mathbf{e} = B\mathbf{e},\ \exists B \in M_n(\mathcal{O}_{\mathfrak{p}})$$

where h is a uniformizing parameter of $\mathfrak{p}$.

6. Turrittin's theorem

Here we present one part of Turrittin's theorem and an application.

Turrittin's theorem tells us that a connection has a regular singular point if and only if it is nilpotent as a matrix over a certain residue field. This statement appears in precise form as Thereom B.

THEOREM B. (Turrittin) Let (W, ∇) be an object of $MC(K/k), \mathfrak{p}$ a place of K/k, h a uniformizing parameter of $\mathfrak{p}$, and $n = dim_K(W)$. We have:

(W, ∇) does *not* have a regular singular point at $\mathfrak{p}$ *if and only if* for every integer a with $n!|a$, there exists a base $\mathbf{f}$ of $W \otimes_K K(h^{1/a})$ in terms of which the connection is expressed as (setting $t = h^{1/a}$)

$$\nabla \left(t \frac{d}{dt} \right) \mathbf{f} = B\mathbf{f}$$

where $B = t^{-v} B_{-v}$ for some integer $v \geq 1$ and $B_{-v} \in M_n \left(\mathcal{O}_{\mathfrak{p}^{1/a}} \right)$ has *non-nilpotent* image in $M_n(k(\mathfrak{p}))$.

In order to prove the converse of this statement, "Jurkat's estimate" is used in [K, §11] and we omit this auxillary method and so content ourselves with proving sufficiency only.

Application: from section 13 of [K]:

THEOREM 6.1. Let T be a global affine variety, $f : S \to T$ a smooth morhpish of relative dimension one, whose generic fibre is geometrically connected. Let $(\mathcal{M}, \nabla)$ be an object of $MIC(S/T)$, with $\mathcal{M}$ locally free of finite rank on S. Let k denote the function field of T, K the function field of S.

Suppose that $(\mathcal{M}, \nabla)$ is globally nilpotent of exponent v on S/T. Then the inverse image of $(\mathcal{M}, \nabla)$ in $MC(K/k)$ has a regular singular point at **every** place $\mathfrak{p}$ of K/k.

PROOF. First, assume that S is a principal open subset of $\mathbb{A}^1_T$. Since $T = Spec\ R[t]$, then a principal affine open is given by $D(g) = Spec\ R[t] - V(g)$ where $g(t) \in R[t]$. As $\mathcal{O}_T(D(g)) = R[t]_g = R[t] \left[\frac{1}{g(t)} \right]$, it follows that we can take $S = Spec\ R[t] \left[\frac{1}{g(t)} \right]$. By way of contradiction, assume that the place defined by $t = 0$ of K is a not a regular singular point. Further assume that $g(t) = t^j h(t)$ for some $j \geq 1$, otherwise there is no singularity at $t = 0$. Further, by localizing at $h(0)$ if necessary, we may assume $h(0)$ to be a unit in R.

By assumption, (M, ∇) is globally nilpotent, where M is a free module of finite rank, say n, over $R[t] \left[\frac{1}{g(t)} \right]$. Make the base change

$$R[t] \left[\frac{1}{g(t)} \right] \to R[z] \left[\frac{1}{g(z)} \right]$$

defined by $z = t^{1/n!}$.

By Turrittin's theorem there exists a basis $\mathbf{m}$ of M over an open subset of S, which in fact we can and do take to be all of S, in terms of which the connection is expressed as

$$\nabla\left(z\frac{d}{dz}\right)\mathbf{m} = B\mathbf{m}$$

where $B = t^{-v}B'$ with $v \geq 1$ for some $B' \in M_n(\mathcal{O}_{\mathfrak{p}^{1/n!}}) = M_n(R[z][\frac{1}{g(z)}])$ (since $\mathfrak{p} = (0)$) and such that B' has non-nilpotent image modulo $\mathfrak{p}$, i.e. in $R[z][\frac{1}{g(z)}]$ (recall that R is an is an integral domain).

In fact, we can write

$$\nabla\left(z\frac{d}{dz}\right)\mathbf{m} = z^{-\mu}(A + zB)\mathbf{m}$$

where $A \in M_n(R)$ is non-nilpotent and $B \in M_n(R[z]\left[\frac{1}{g(z)}\right]$ (recall that $h(0)$ is invertible in R). Applying the operator $\nabla(z\frac{d}{dz})$ repeatedly we find that

$$\nabla\left(z\frac{d}{dz}\right)^j\mathbf{m} = z^{-\mu j}(A^j + zB^j)\mathbf{m}$$

so setting $B^j = B_j$ we have

$$\nabla\left(z\frac{d}{dz}\right)^j\mathbf{m} = z^{-\mu j}(A^j + zB_j)\mathbf{m}$$

with $B_j \in M_n\left(R[z]\left[\frac{1}{g(z)}\right]\right)$. We now apply mod p arguments to show that in fact A must be nilpotent by the global nilpotence hypothesis.

For a fixed prime number p, we have that in $Der(\mathbb{F}_p[z]/\mathbb{F}_p)$ the following relation holds: $(z\frac{d}{dz})^p = z\frac{d}{dz}$, being in characteristic p. Therefore, the hypothesis of global nilpotence is that there exists an integer $\alpha(p)$ such that the following inclusion holds:

$$\left(\left(\nabla\left(z\frac{d}{dz}\right)\right)^p - \nabla\left(z\frac{d}{dz}\right)\right)^{\alpha(p)} M \subset pM$$

equivalently,

$$\left(z^{-\mu p}(A^p + zB_p) - z^{-\mu}(A + zB)\right)^{\alpha(p)} \in pM_n\left(R[z]\left[\frac{1}{g(z)}\right]\right).$$

Expanding the above expression, the term with the greatest negative power in z, "the most polar term", is $A^{p\alpha(p)}$ and since $A \in M_n(R)$, it follows that the nilpotent condition is then that $A^{p\alpha(p)} \in pM_n(R)$.

Since p was an arbitrary prime number, then $A^{p\alpha(p)} \in pM_n(R)$ holds for *every* p. Now look at the characteristic polynomial of A, $\det(XI_n - A)$. This polynomial's reduction at each maximal ideal of

$T = Spec(R)$, i.e., its value at *every* closed point of T, is equal to X^n, hence

$$\det(XI_n - A) = X^n$$

which implies that some power of A is zero, hence A is nilpotent, a contradiction to the assumption that the place defined by $t = 0$ was not a regular singular point.

□

7. Proof of Turritin's theorem

LEMMA 7.1. Suppose

$$0 \to (V, \nabla') \to (W, \nabla) \to (U, \nabla'') \to 0$$

is an exact sequence in $MC(K/k)$. Then (W, ∇) has a regular singular point at $\mathfrak{p}$ if and only if both (V, ∇') and (U, ∇'') have a regular singular point at $\mathfrak{p}$.

PROOF. We first prove sufficiency. Suppose $(\mathbf{e}, \mathbf{f}) = (e_1, \ldots, e_{n_1}, f_1, \ldots f_{n_2})$ is a K-base of W; as $W/V \cong U$ we can assume $\mathbf{e}$ to be a basis of V and $\mathbf{f}$ to project to a base of U. There are matrices $A \in M_{n_1}(\mathcal{O}_\mathfrak{p}), C \in M_{n_2}(\mathcal{O}_\mathfrak{p})$ such that the connection on W is expressed as

$$\nabla\left(h\frac{d}{dh}\right) = \begin{pmatrix} \mathbf{e} \\ \mathbf{f} \end{pmatrix} = \begin{pmatrix} A & O \\ B & C \end{pmatrix}\begin{pmatrix} \mathbf{e} \\ \mathbf{f} \end{pmatrix}$$

so it remains to show that we can take B as "holomorphic at $\mathfrak{p}$", i.e. that $B \in M_n(\mathcal{O}_\mathfrak{p})$. To this end, observe that the operation which scales f to $h^v f$ changes the matrix of $\nabla(h\frac{d}{dh})$ to

$$\begin{pmatrix} I_{n_1} & O \\ O & h^v I_{n_2} \end{pmatrix}\begin{pmatrix} A & O \\ B & C \end{pmatrix}\begin{pmatrix} I_{n_1} & O \\ O & h^{-v} I_{n_2} \end{pmatrix} = \begin{pmatrix} A & O \\ h^v B & C \end{pmatrix}.$$

For $B = (b_{i,j})$ $(1 \leq i, j, \leq n_1)$ take $v = max_{1 \leq i,j \leq n_1}|ord_\mathfrak{p}(b_{ij})|$; this change of basis then makes $h^v B$ holomorphic at $\mathfrak{p}$.

For the converse, first note that any $\mathcal{O}_\mathfrak{p}$-lattice $W_\mathfrak{p}$ is a finitely generated module over the PID $\mathcal{O}_\mathfrak{p}$. So using the corresponding structure theorem in this case, it follwows that for any non-zero $\mathcal{O}_\mathfrak{p}$ submodule W'_p of $W_\mathfrak{p}$ there exists a a basis $(e_1, \ldots, e_n)$ of $W_\mathfrak{p}$ and $a_i \in \mathcal{O}_\mathfrak{p}$ such that $(a_1 e_1, \ldots, a_q e_q)$ is a basis of $W'_\mathfrak{p}$ with $a_i | a_{i+1}$ $(1 \leq i < q \leq n)$. As a consequence, if $W_\mathfrak{p}$ "works", then $V \cap W_\mathfrak{p}$ is an $\mathcal{O}_\mathfrak{p}$-lattice in V which "works". Likewise, $U \cap$ (image of $W_\mathfrak{p}$ in U) "works". □

We introduce the following terminology which will provide useful reductions in the proof Turrittin's theorem.

DEFINITION 7.2. We say that (W, ∇) is *cyclic* if there is a vector $w \in W$ such that for some $D \in Der(K/k)$ the set of vectors $w, \nabla(D)(w), (\nabla(D))^2 w, \ldots$ span W over K.

Remark. The K-span of $w, \nabla(D)(w), (\nabla(D))^2 w, \ldots$ is in fact independent of the choice of D, hence this span is a $Der(K/k)$-stable subspace of W.

Using the above Lemma and the fact than any finite dimensional vector space is a quotient of a direct sum of finitely many of its cyclic subspaces, we have the

COROLLARY 7.3. Let (W, ∇) be an object of $MC(K/k)$. Then (W, ∇) has a regular singular point at $\mathfrak{p}$ if and only if every cyclic subobject of (W, ∇) has a regular singular point at $\mathfrak{p}$.

LEMMA 7.4. Suppose that (W, ∇) is *cyclic* and does *not* have a regular singular point at $\mathfrak{p}$.

Then for every integer a with $n!|a$, the inverse image of (W, ∇) in $MC(K(h^{1/a})/k)$ admits a base $\mathbf{f}$ such that for $t = h^{1/a}$ the connection is expressed as

$$\nabla\left(t\frac{d}{dt}\right)\mathbf{f} = B\mathbf{f}$$

where for some integer $v \geq 1$ we have

$$B = t^{-v}B_{-v},\ B_{-v} \in M_n(\mathcal{O}_{\mathfrak{p}^{1/a}}),$$

and B_{-v} has non-nilpontent image modulo $\mathfrak{p}^{1/a}$.

PROOF. The cyclic hypothesis means that we have a base $\mathbf{e} = \{w, \nabla\left(h\frac{d}{dh}\right)(w), \ldots, \nabla\left(h\frac{d}{dh}\right)^{n-1}(w)\}$ for W. Using the fact that $\nabla\left(h\frac{d}{dh}\right)^n$ is a linear combination of the $\nabla\left(h\frac{d}{dh}\right)^j$ $(0 \leq j \leq n-1)$, say

$$\nabla\left(h\frac{d}{dh}\right)^n(w) + a_{n-1}\nabla\left(h\frac{d}{dh}\right)^{n-1}(w) + \cdots + a_0 \cdot w = 0$$

for some $a_i \in K$ $(0 \leq i \leq n-1)$, and that the endomorphism $\nabla\left(h\frac{d}{dh}\right)$ acts as multiplication on W, it follows that the connection is expressed as

$$\nabla\left(h\frac{d}{dh}\right)\mathbf{e} = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -a_0 & -a_1 & -a_2 & \ldots & -a_{n-1} \end{pmatrix}\mathbf{e} =: C\mathbf{e}.$$

Since it is assumed that (W, ∇) does *not* have a regular singular point at $\mathfrak{p}$, then for at least one of the a_i we have $ord_{\mathfrak{p}}(a_i) < 0$.

To go into $MC(K(h^{1/a}/k))$ make the change of base $K \to K(t)$ with $t^a = h$; this "substitution" changes the above connection by a factor of a, so the new connection is given by

$$\frac{1}{a}\nabla\left(t\frac{d}{dt}\right)\mathbf{e} = C\mathbf{e}.$$

Moreover, since $t^a = h$, $f_i \in K$, and $n!|a$, then the *integers* $ord_{\mathfrak{p}^{1/a}}(f_i)$ are divisible by $n!$. Consider then the strictly positive integer

$$v = \max_{0 \leq j \leq n-1} (-ord_{\mathfrak{p}^{1/a}}(f_j)/(n-j)).$$

Our candidate basis $\mathbf{f}$ for $W \otimes_K K(t)$ is given by scaling the basise $\mathbf{e}$ of W by successive powers of t^v:

$$\mathbf{f} = \begin{pmatrix} 1 & & & \\ & t^v & & \\ & & \ddots & \\ & & & t^{(n-1)v} \end{pmatrix} \mathbf{e} =: A\mathbf{e}$$

Using the Leibniz rule along with the equalities $\nabla\left(t\frac{d}{dt}\right)\mathbf{e} = aC\mathbf{e}$ and $\mathbf{e} = A^{-1}\mathbf{f}$ we calculate:

$$\begin{aligned} \nabla\left(t\frac{d}{dt}\right)\mathbf{f} &= \nabla\left(t\frac{d}{dt}\right)(A\mathbf{e}) \\ &= A \cdot \nabla\left(t\frac{d}{dt}\right)(\mathbf{e}) + \left(\left(t\frac{d}{dt}\right)(A)\right) \cdot \mathbf{e} \\ &= \left(A(aC)A^{-1} + \left(t\frac{d}{dt}(A)\right)A^{-1}\right)\mathbf{f} \\ &=: B\mathbf{f}. \end{aligned}$$

Thus,

$$\frac{1}{a}\nabla\left(t\frac{d}{dt}\right)\mathbf{f} = B\mathbf{f}$$

where

$$B = \begin{pmatrix} 0 & t^{-v} & 0 & \dots & 0 \\ 0 & 0 & t^{-v} & \dots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & t^{-v} \\ -a_0 t^{(n-1)v} & -a_1 t^{(n-2)v} & -a_2 t^{(n-3)v} & \dots & -a_{n-1} \end{pmatrix} + \frac{1}{a}\begin{pmatrix} 0 & & & \\ & v & & \\ & & \ddots & \\ & & & (n-1)v \end{pmatrix}$$

and it is an easy task to check that B has the required properties. □

We now present the proof of Turrittin's theorem as stated above:

Proof. The proof is by induction on the the dimension n of W over K. If $n = 1$, we appeal to Corollary 7.3 above. Now if (W, ∇) does not have any proper non-zero objects, it is necessarily cyclic. So suppose that (V, ∇') is a nonzero proper cyclic subobject. We then obtain a short exact sequence

$$0 \to (V, \nabla') \to (W, \nabla) \to (U, \nabla'') \to 0$$

where both $n_1 = dim_K(V)$ and $n_2 = dim_K(U)$ are strictly less than n. Applying the induction hypothesis to V and U, there exists a basis of $W \otimes_K K(h^{1/a})$ of the form $(\mathbf{e}, \mathbf{f})$ where $\mathbf{e}$ is a basis of $V \otimes_K K(h^{1/a})$ and where $\mathbf{f}$ projects to a basis of $U \otimes_K K(h^{1/a})$ in terms of which the connection ∇ is expressed as, by setting $t = h^{1/a}$,

$$\nabla\left(t\frac{d}{dt}\right)\begin{pmatrix}\mathbf{e}\\ \mathbf{f}\end{pmatrix} = \begin{pmatrix}A & O\\ B & C\end{pmatrix}\begin{pmatrix}\mathbf{e}\\ \mathbf{f}\end{pmatrix}$$

where $A = t^{-v}A_{-v}$ for some $v \geq 0$ and $A_{-v} \in M_n(\mathcal{O}_{\mathfrak{p}^{1/a}})$ with the corresponding statement for $C = t^{-\tau}C_{-\tau}, \tau \geq 0$.

By Lemma 7.1, (V, ∇') or (U, ∇'') does not have a reuglar singular point at $\mathfrak{p}$. Without loss of generality, assume that (V, ∇'') does not have a regular singularity at $\mathfrak{p}$, so that $v > 0$ *and* A_{-v} has *non-nilpotent* image in $M_{n_1}(k(\mathfrak{p}))$.

Applying the same change of basis trick to the above matrix, we conclude that $t^N B$ is holomorphic at $\mathfrak{p}^{1/a}$, for some integer $N > 0$. Furthermore, this connection has a pole of order $\sup(v, \tau)$ at $\mathfrak{p}^{1/a}$.

Hence,

$$t^{\sup(v,\tau)}\begin{pmatrix}A & O\\ t^N B & C\end{pmatrix}$$

is holomorphic at $\mathfrak{p}^{1/a}$ and has non-nilpotent image in $M_n(k(\mathfrak{p}))$. $\square$

8. Local monodromy and exponents

From [K, §12] we have the

THEOREM 8.1. Let K/k be a function field in one variable, with k of characteristic zero. Let $\mathfrak{p}$ be a place of K/k which is rational, i.e. with residue field $k(\mathfrak{p}) = k$. Suppose that (W, ∇) is an object of $MC(K/k)$ which has a regular singular point at $\mathfrak{p}$. In terms of a uniformizing parameter t at $\mathfrak{p}$ and a basis $\mathbf{e}$ of an $\mathcal{O}_\mathfrak{p}$-lattice $W_\mathfrak{p}$ of W which is stable under $\nabla\left(t\frac{d}{dt}\right)$, the connection is expressed as

$$\nabla\left(t\frac{d}{dt}\right)\mathbf{e} = B\mathbf{e}$$

where $B \in M_n(\mathcal{O}_\mathfrak{p})$. Suppose that the matrix $B(\mathfrak{p}) \in M_n(k)$ has all of its eigenvalues in k. Then

(i). The set of images of the eigenvalues of $B(\mathfrak{p})$ in the additive group $k^+/\mathbb{Z}$, called the *exponents of* (W, ∇) at $\mathfrak{p}$ is independent of the choice of the $\nabla\left(t\frac{d}{dt}\right)$-stable $\mathcal{O}_\mathfrak{p}$-lattice $W_\mathfrak{p}$ in W.

(ii). For a fixed set theoretic section $\phi : k^+/\mathbb{Z} \to k^+$ of the canonical projection $k^+ \to k^+/\mathbb{Z}$, there exists a *unique* $\mathcal{O}_\mathfrak{p}$-lattice $W'_\mathfrak{p}$, stable under $\nabla\left(t\frac{d}{dt}\right)$, in which the connection is expressed in terms of a base $\mathbf{e}'$ as

$$\nabla\left(t\frac{d}{dt}\right)\mathbf{e}' = C\mathbf{e}'$$

for some $C \in M_n(\mathcal{O}_\mathfrak{p})$ such that $C(\mathfrak{p}) \in M_n(k)$ has all of its eigenvalues *fixed* by the composition $k^+ \overset{proj}{\to} k^+/\mathbb{Z} \overset{\phi}{\to} k^+$. In particular, non-equal eigenvalues of $C(\mathfrak{p})$ do *not* differ by integers.

Since the eigenvalues of $C(\mathfrak{p})$ are contained in k, we have a Jordan decomposition

$$C(\mathfrak{p}) = D + N \tag{8}$$

where $[D, N] = 0$, D is a semi-simple matrix, and N is nilpotent.

DEFINITION 8.2. Let K/k be a function field in one variable, with k of characteristic zero. Let $\mathfrak{p}$ be a place of K/k which is rational, (W, ∇) an object of $MC(K/k)$ which has a regular singular point at $\mathfrak{p}$. The local monodromy is *quasi-unipotent* if the exponents at $\mathfrak{p}$ (defined above in (i)) are rational numbers. If the local monodromy at $\mathfrak{p}$ is quasi-unipotent, then we say that its *exponent of nilpotence is* $\leq v$ if, in the notation of (8) we have $N^v = 0$.

9. Local Monodromy theorem

Let $S/\mathbb{C}$ be a smooth connected curve, and let $\pi : X \to S$ be a proper and smooth morphism. The following objects exist:

(i). A subring R of $\mathbb{C}$ which is finitely generated over $\mathbb{Z}$.
(ii). A smooth connected curve S/R which gives $S/\mathbb{C}$ via the base change $R \to \mathbb{C}$.
(iii). A proper and smooth morphism $\tilde{\pi} : \tilde{X} \to \tilde{S}$ which gives $\pi : X \to S$ via the base change $S \to \tilde{S}$.

Combining theorems A (pg. 14) and B (pg. 15) along with results based on the previous section, we have the following theorem (cf. [K, 14.1])

THEOREM 9.1. Local Monodromy Theorem.
Let $S/\mathbb{C}$ be a smooth connected curve with function field $K/\mathbb{C}$, $\pi : X \to S$ a proper and smooth morphism, and X_K/K the generic fibre of π.

For each integer $i \geq 0$, let $h(i)$ be the number of pairs (p, q) of integers satisfying $p+q = i$ such that $h^{p,q}(X_K/K) := \dim_K H^q(X_K, \Omega^p_{X_K/K}) = rank_{\mathcal{O}_S} R^q\pi_*(\Omega^p_{X_K/K})$ is non-zero. Then the inverse image of $H^i_{dR}(X/S)$, with the Gauss-Manin connection, in $MC(K/\mathbb{C})$ has regular singular points at every place of $K/\mathbb{C}$ and quasi-unipotent local monodromy, whose exponent of nilpotence is $\leq h(i)$.

10. References

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