Research project GSAQF

Geometric properties of group schemes in the theory of quadratic forms

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Research project short description:

A classical result of Gauss says in how many ways a positive integer n can be written as a sum $x^2+y^2+z^2$ with x, y and z integers with $\gcd(x,y,z)=1$. In other words, let $r(n)$ be the number of elements in $\mathbf{Z}^3$ that are at distance $\sqrt{n}$ from the origin. Then one has:

$$r(n) = 0 \qquad \text{if } n \equiv 0,\ 4 \text{ or } 7 \bmod 8$$

$$r(n) = 48\#\mathrm{Pic}(O_{-n})/\#(O_{-n}^{\times}) \qquad \text{if } n \equiv 3 \bmod 8$$

$$r(n) = 24\#\mathrm{Pic}(O_{-4n})/\#(O_{-4n}^{\times}) \qquad \text{if } n \equiv 1 \text{ or } 2 \bmod 4,$$

where O_d denotes the quadratic order of discriminant d, $\#\mathrm{Pic}(O_d)$ the number of elements of its Picard group, and $\#(O_d^{\times})$ the number of elements of its multiplicative group.

It turns out that, given existence of a solution, the number of solutions can be very nicely explained from the action of the group scheme $\mathrm{SO}_{3,\mathbf{Z}}$ on the closed subscheme of $\mathbf{A}^3$ given by the equation $x^2+y^2+z^2=n$, in terms of the first Zariski cohomology group of the stabiliser of one solution.

The project consists of investigating more systematically the principle of using symetries in the form of actions of group schemes over $\mathbf{Z}$ and their geometric properties in the theory of integral quadratic forms. In particular, how does bad reduction of these group schemes relate to properties of integral quadratic forms?

References:

Bas Edixhoven. *Gauss's theorem on sums of* 3 *squares, via group schemes.* Lecture in Bordeaux, 2011/11/15.
http://www.math.leidenuniv.nl/~edix/talks/2011_11_15_gauss_3_squares.pdf