Algant Master Thesis

Étale Cohomology over $Spec(k)$

Candidate:
Sergio Zarabara

Advisor:
Dr. Robin S. de Jong

Coadvisor:
Prof. Marco Garuti

Università degli Studi di Padova

Universiteit Leiden

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Ai miei genitori
per avermi educato alla libertà

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The lyf so short, the craft so longe to lerne.
Th' assay so hard, so sharp the conquerynge,
The dredful joye, alwey that slit so yerne;
Al this mene I be love.

— Chaucer G.
Parlement of Foules

Introduction

The aim of this thesis is to give a proof of a fundamental result which relates the classical Galois Cohomology theory with the "broader" Étale Cohomology theory. More specifically we will prove that the category of continuous modules for the action of the absolute Galois group G_k of some field k is equivalent to that of the abelian sheaves on the étale site $Spec(k)_{ét}$. It will follow that the Galois cohomology groups can be equivalently computed as étale (sheaf) cohomology groups on the spectrum of k. In order to properly define the étale cohomology groups we will have to develop a rather abstract machinery. In chapter I we will introduce the fundamental notion of *étale morphism of schemes* which somehow mimics the notion of local diffeomorphism of differentiable manifolds. If we were to express this concept by means of a formula we would write

$$\textit{étaleness} = \textit{unramifiedness} \cap \textit{flatness}.$$

We will first give the "classical" definition via *local morphisms of local rings*. At a later time the *functor of points* will enter the scene enabling us to rephrase étaleness for scheme morphisms in a wholly categorical fashion. Chapter II will deal with the delicate matter of widening our standard conception of *topology on a space* in favour of the much more abstract notion of *Grothendieck topology on a site*. This last tool will overshadow the role of points (of primary importance in Top) emphasizing on the other hand the role of the objects of the chosen category which will act like open subsets in a topological space. We will define *sheaves on a site* and we will develop the particularly suitable notion of *canonical site* (on a category). Armed with these tools we will prove a couple of equivalences that involve the category of sets with a left action of a group G and the category $Sh(\mathcal{T}_G)$ of sheaves of sets on the canonical site on the former category itself. This result will easily extend to the case of G-modules and abelian sheaves. We will be especially interested in the case of continuous G-modules and abelian sheaves on the *continuous canonical G-site.* In chapter III we will consider the *étale site* $S_{ét}$ of a scheme S, whose underlying category $\acute{E}t/S$ has objects the étale schemes over S and we will introduce the key notion of this thesis, namely the *étale cohomology of a scheme with values in a sheaf on the site $S_{ét}$*. Next we will consider the case when $S = Spec(k)$ with k a field. We will prove the essential result which gives us an isomorphism of sites between $Spec(k)_{\acute{e}t}$ and the continuous canonical site $\mathcal{T}_{G_k}^{\mathscr{C}}$, where G_k denotes the absolute Galois group of k. We will end this chapter giving a proof of the main theorem of this thesis which relates the category of abelian sheaves on the site $Spec(k)_{\acute{e}t}$ and the category of continuous G_k-modules. From this we will be able to recover "classical" Galois Cohomology theory, which originates as way of measuring the non-exactness of the fixed-point functor with source the category of continuous G_k-modules, in terms of the étale cohomology of the one-point scheme $Spec(k)$ with values in an abelian sheaf over the étale site associated to this scheme. In chapter IV we will present the very basics of Galois Cohomology in its classical fashion, mainly following the homonymous cornerstone of the subject edited by Serre. Lastly in Chapter V we will propose a nice application of the main theorem; namely we will recover *Hilbert Theorem 90* (one of the fundamental results of classical Galois Cohomology) in an étale way.

Notation

Everytime we write "ring" we mean "commutative ring with 1" and we denote with just $CRing$ the category of commutative unitary rings. Everytime we write $X \in \mathcal{C}$ (where $\mathcal{C}$ here denotes a category) we mean $X \in \mathrm{Ob}(\mathcal{C})$. When we have an arrow $f \in \mathrm{Mor}(\mathcal{C})$ we simply write $f \in \mathcal{C}$. We denote with $\mathrm{Iso}(\mathcal{C}) \subseteq \mathrm{Mor}(\mathcal{C})$ the set of isomorphisms in the category $\mathcal{C}$. In the sequel we will also use the terms "map" or "arrow" to refer to the notion of morphism in a certain category $\mathcal{C}$. Everytime we say "F is a functor" we mean "F is a covariant functor". Given two functors $F, G : \mathcal{C} \to \mathcal{D}$, we write $F \simeq G$ to denote an isomorphism of functors between the two. Also we denote with $\mathrm{Nat}(F, G)$ the collection of all natural transformations (or "morphisms of functors") between F and G. Given two categories $\mathcal{C}$ and $\mathcal{D}$, we denote with $\mathrm{Func}(\mathcal{C}, \mathcal{D})$ the collection of all functors from $\mathcal{C}$ to $\mathcal{D}$. For any $X \in \mathcal{C}$, we set $h^X := \mathrm{Hom}_{\mathcal{C}}(X, -)$ and $h_X := \mathrm{Hom}_{\mathcal{C}}(-, X)$. We set $\mathrm{Maps}(A, B) := \mathrm{Hom}_{Set}(A, B)$, for any $A, B \in Set$. Whenever we write "neigh" we mean "neighbourhood". Fld denotes the category of fields and field extensions. Given R a ring, $J(R)$ denotes its Jacobson radical (ideal). The symbol $\approx$ is used to denote homeomorphisms (i.e. the isos in Top). Given $X \in Top$ we denote with $PSh(X)$ and $Sh(X)$ resp. the category of presheaves and of sheaves of sets on X. More generally, given $X \in Top$ and $\mathcal{C}$ a category, we denote with $\mathcal{C}(X)$ and with $P\mathcal{C}(X)$ the category of $\mathcal{C}$-valued presheaves and sheaves on X. When we deal with a site $T = T_{\mathcal{C}}$, we use the same kind of notation just with T in place of X. We write *G-Set* to denote the category of left G-sets, whereas with *Set-G* we denote the right G-sets. We use the terms *mono*, *epi* and *iso* respectively for monomorphism, epimorphism and isomorphism. Given $G \in Grp$ and $A \in G$-Mod we denote with A^G the set of G-invariant elements of A. Given $X \in Top$ we denote with Op_X the category whose objects are the open subsets of X and whose arrows defined setting $\mathrm{Hom}_{Op_X}(V, U) := \emptyset$ if $V \nsubseteq U$ and $\mathrm{Hom}_{Op_X}(V, U) := \{pt\}$ otherwise.

Chapter 1

Algebra

"Non devo ascoltarla
o non terminerò la [mia] rivoluzione.".

— Lenin V.I.U.

1.1 Finiteness conditions

Definition 1.1. cp. [Bourb, ch. I, §2, no.8, p. 20] Let A be a ring. Let E be a left A-module (the definition can be given analogously for right modules). We define a **presentation** for E to be an exact sequence of left A-modules

$$L_1 \to L_0 \to E \to 0 \tag{1.1}$$

where L_1 and L_0 are free A-modules.

Proposition 1.2. *Every (left) A-module E admits a presentation.*

Proof. Every A-module E can be written as a quotient of a free A-module L_0, hence we have a morphism of A-modules $\varphi_0 : L_0 \twoheadrightarrow E$; similarly we get a morphism $\psi : L_1 \twoheadrightarrow ker(\varphi_0)$, with L_1 free A-module; composing ψ with the inclusion $ker(\varphi_0) \hookrightarrow L_0$, we get the second desired A-module morphism $\varphi_1 : L_1 \to L_0$. $\square$

Definition 1.3. Given a (left) A-module E and $\mathscr{P} = (L_1, L_0)$ a presentation of E, we say that $\mathscr{P}$ is **finite**, if both L_1 and L_0 have finite bases as free A-modules. We call E a **finitely presented** A-module, if E admits a finite presentation $\mathscr{P}$.

Here below we give some more *concrete* definitions for finiteness conditions on ring morphisms:

Definition 1.4. Let $f : A \to B$ be a morphism of rings. We say that:

1. f is **finite**, if B is finitely generated as an A-module;
2. f is of **finite type**, if B is finitely generated as an A-algebra;
3. f is of **finite presentation**, if B is finitely generated and finitely related as an A-algebra.

Remark 1.5. Rephrasing the above definitions:

1. $B \cong_{\text{A-Mod}} A^{\oplus n}/I \qquad (I \subseteq_{submod} A^{\oplus n},\ n \in \mathbb{N});$
2. $B \cong_{\text{A-Alg}} A[x_1, \dots, x_n]/I \qquad (I \subseteq_{ideal} A[x_1, \dots, x_n],\ n \in \mathbb{N});$
3. $B \cong_{\text{A-Alg}} A[x_1, \dots, x_n]/(f_1, \dots, f_m) \qquad (f_i \in A[x_1, \dots, x_n],\ m, n \in \mathbb{N}).$

A choice of a surjection $A^{\oplus n} \twoheadrightarrow B$ or $A[x_1, \dots, x_n] \twoheadrightarrow B$ is called a **presentation** of B.

Let's now recall some basic notions about *schemes*. Everytime we are given an arrow in Sch, we are indeed dealing with a pair $f = (f, f^\sharp) : (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$, where

- $(f : X \to S) \in Top$
- $(f^\# : \mathcal{O}_S \to f_*\mathcal{O}_X) \in CRing(S)$

such that the induced ring morphisms on the stalks $\mathcal{O}_{S,f(x)} \xrightarrow{f_x^\#} \mathcal{O}_{X,x}$ are local, (which means $f_x^\#(\mathfrak{m}_{f(x)}) \subseteq \mathfrak{m}_x$). Given $(X, \mathcal{O}_X) \in Sch$ and $(S, \mathcal{O}_S) \in AffSch$ we have the following contravariant adjunction on the right:

$$\mathrm{Hom}_{CRing}(A, \mathcal{O}_X(X)) \cong \mathrm{Hom}_{Sch}(X, Spec(A))$$

where $(S, \mathcal{O}_S) \cong (Spec(A), \mathcal{O}_{Spec(A)})$, in $LRSp$. On the level of affine schemes we have the above adjunction takes the form

$$\mathrm{Hom}_{CRing}(A, B) \cong \mathrm{Hom}_{LRSp}(Spec(B), Spec(A))$$

where $X \cong Spec(B)$ in $LRSp$. This can be alternatively stated saying that the functor $Spec : CRing^{op} \to AffSch$ is fully faithful; specifically we have the following categorical duality:

$$\begin{array}{rcl} CRing^{op} & \xrightarrow{\sim} & AffSch \\ A & \longmapsto & Spec(A) \\ \mathcal{O}_X(X) & \longleftarrow & X \end{array}$$

recalling that for $S = Spec(A) \in AffSch$, the ring $\mathcal{O}_X(X)$ is canonically identified with A. Moreover given an arrow $f : X \to S$ in Sch and open affines $U = Spec(B) \subseteq X$ and $V = Spec(A) \subseteq S$ such that $f(U) \subseteq V$, (i.e. $U \subseteq f^{-1}(V)$), we get a ring morphism

$$\mathcal{O}_S(V) \to (f_*\mathcal{O}_X)(V) = \mathcal{O}_X(f^{-1}(V)) \to \mathcal{O}_X(U)$$

that makes $\mathcal{O}_X(U)$ into an $\mathcal{O}_S(V)$-algebra. Now, since $\mathcal{O}_V = \mathcal{O}_S|_V$ and $\mathcal{O}_S|_V(V) = \mathcal{O}_S(V)$, we have that $\mathcal{O}_S(V) = \mathcal{O}_V(V) = A$ and analogously $\mathcal{O}_X(U) = \mathcal{O}_U(U) = B$. So the ring morphism we are dealing with is nothing but the arrow $A \to B$, (hence $B \in A$-Alg).

After these considerations let's define some notions about scheme morphisms.

Definition 1.6. Let $f : X \to S$ be a scheme morphism and let $x \in X$. We say that f is of **finite type at** x, if there exist an affine open neighbourhood $U = Spec(B) \subseteq X$ of x and an affine open $V = Spec(A) \subseteq S$ with $f(U) \subseteq V$, such that the induced ring morphism $\mathcal{O}_S(V) = A \to B = \mathcal{O}_X(U)$ is of finite type. We say that f is **locally of finite type**, if it is of finite type at x, for every $x \in X$.

Definition 1.7. Let $f : X \to S$ be a scheme morphism and let $x \in X$. We say that f is of **finite presentation at** x, if there exist an affine open neighbourhood $U = Spec(B) \subseteq X$ of x and an affine open $V = Spec(A) \subseteq S$ with $f(U) \subseteq V$, such that the induced ring morphism $\mathcal{O}_S(V) = A \to B = \mathcal{O}_X(U)$ is of finite presentation. We say that f is **locally of finite presentation**, if it is of finite presentation at x, for every $x \in X$.

Remark 1.8. If a scheme morphism is locally of finite presentation then is clearly also locally of finite type.

1.2 Étaleness

The first aim of this section is to introduce the key notion of *étale morphism of schemes*. In order to do this we will introduce the concepts of *flatness* and of *unramifiedness* in Sch.

Definition 1.9. Let $f : A \to B$ be a morphism of rings. We say that f is **flat**, if the functor $- \otimes_A B$ is exact. In this case we can also say that B is a **flat** A-algebra.

Definition 1.10. Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ and let $s = f(x)$. We say that f is **flat at** x, if the natural morphism $f_x^\# : \mathcal{O}_{S,s} \to \mathcal{O}_{X,x}$ is flat, (i.e. $- \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{X,x}$ is exact). We say that f is **flat**, if f is flat at x, for every $x \in X$.

Definition 1.11. Let $f : A \to B$ be a morphism of rings. Let $\mathfrak{p} \in Spec(A)$ and $\mathfrak{q} \in Spec(B)$ such that $f^{-1}(\mathfrak{q}) \supseteq \mathfrak{p}$ (i.e. $\mathfrak{q}$ *lies over* $\mathfrak{p}$). We say that f is **unramified at** $\mathfrak{q}$, if

- f is of finite type
- $\mathfrak{p}B_{\mathfrak{q}} = \mathfrak{q}B_{\mathfrak{q}}$
- $\kappa(\mathfrak{p}) \hookrightarrow \kappa(\mathfrak{q})$ is finite and separable.

Remark 1.12. Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ and let $s = f(x)$. The induced ring morphism on stalks $f_x^{\#} : \mathcal{O}_{S,s} \to \mathcal{O}_{X,x}$ canonically induces the commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{S,s} & \xrightarrow{\quad f_x^{\#} \quad} & \mathcal{O}_{X,x} \\ \downarrow & & \downarrow \\ \mathcal{O}_{S,s}/\mathfrak{m}_s & \hookrightarrow & \mathcal{O}_{X,x}/\mathfrak{m}_s\mathcal{O}_{X,x} \end{array}$$

where $\mathfrak{m}_s\mathcal{O}_{X,x}$ denotes the pushforward of the ideal $\mathfrak{m}_s$ to the ring $\mathcal{O}_{X,x}$ and the bottom arrow is nothing but the projection of $f_x^{\#}$, namely $[t]_{\mathfrak{m}_s} \mapsto [f_x^{\#}(t)]_{\mathfrak{m}_s\mathcal{O}_{X,x}}$. (It is injective as $\mathcal{O}_{S,s}/\mathfrak{m}_s$ is a field.).

Definition 1.13. Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ and let $s = f(x)$. We say that f is **unramified at** x, if

- f is of finite type at x
- $\mathfrak{m}_s\mathcal{O}_{X,x} = \mathfrak{m}_x$
- $\kappa(s) \hookrightarrow \kappa(x)$ is finite and separable.

We say that f is **unramified**, if f is unramified at x, for every $x \in X$.

Remark 1.14. The 2^{nd} and the 3^{rd} condition in the previous definition can be merged together requiring that $\kappa(s) \hookrightarrow \mathcal{O}_{X,x}/\mathfrak{m}_s\mathcal{O}_{X,x}$ is finite and separable in *Fld*.

We will now give a *concrete* definition of étale scheme morphism:

Definition 1.15. Let $f : X \to S$ be a morphism of schemes and let $x \in X$. We say that f is **étale at** x, if f is finitely presented, flat and unramified at x. We say that f is **étale**, if f is étale at x, for every $x \in X$.

1.3 Functor of points

We will now reformulate (almost) all the notions that we introduced in the two previous sections in terms of the *language of functors of points.*

Definition 1.16. Let's consider the (covariant) functor

$$h : Sch \longrightarrow Func(Sch^{op}, Set)$$
$$X \longmapsto \mathrm{Hom}_{Sch}(-, X)$$

defined on the arrows in obviuos manner. For any given $X \in Sch$ the contravariant functor

$$h_X : Sch^{op} \longrightarrow Set$$
$$Y \longmapsto \mathrm{Hom}_{Sch}(Y, X)$$

is called the **functor of points** of the scheme X. An arrow $(Y \to X) \in Sch$ is said to be a **Y-valued point** of X. The set $h_X(Y)$ is called the *set of Y-valued points of X*. When $Y = Spec(A)$, with $A \in CRing$, the above arrow is said to be an **A-valued point** of X and $h_X(Y)$ is called the *set of A-valued points of X*.

Let's recall here a powerful categorical tool: the *Yoneda's Lemma.*

Lemma 1.17. [*Yoneda*] *Let $\mathcal{C}$ be a locally small category (i.e. the hom-sets are not proper classes). Then we have that:*

1. *for any $F \in Func(\mathcal{C}, Set)$ and any $X \in \mathcal{C}$, there is a isomorphism (in Set)*

$$Nat(h_X, F) \longrightarrow F(X)$$
$$\eta \longmapsto \eta_X(id_X)$$

   natural in both F and X. In other words there is a 1-1 correspondence between the natural transformations $h_X \Rightarrow F$ and the elements of $F(X)$;

2. *given $X, Y \in \mathcal{C}$, we have that*

$$h_X \simeq h_Y \quad \Rightarrow \quad X \cong Y.$$

Proof. Part (1): see for example [MacL]. Part (2): follows immediately from part (1) applied to the case $F = h^Y$. □

Remark 1.18. By Yoneda's Lemma (part 2), given $X, X' \in Sch$ we have that $h_X \simeq h_{X'}$ implies $X \cong_{Sch} X'$. Hence the functor of points h_X actually determines the scheme X.

Definition 1.19. Given $F \in Func(Sch^{op}, Set)$ we say that F is **co-representable**, if there exists $X \in Sch$ such that $F \simeq h_X$. Again by Yoneda's Lemma (part 2), if such X does exist then it is unique (up to iso).

Totally analogue definitions apply to the case when we consider schemes in the slice category Sch/S, where S denotes some scheme.

Notation 1.20. Given $S \in Sch$ and $X \in Sch/S$ we donote with $X_S := \mathrm{Hom}_S(-, X)$ the **slice functor of points** of X. (Here we set $\mathrm{Hom}_S(-, X) := \mathrm{Hom}_{Sch/S}(-, X)$.) When $Y = Spec(A)$, with $A \in CRing$, we set $X_S(A) := X_S(Y)$.

Definition 1.21. Let $(A \xrightarrow{f} B) \in Ring$. We say that f is **formally unramified**, if for every commutative Ring-diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & \ulcorner & \downarrow \\ R & \twoheadrightarrow & R/I \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & \overset{\varphi}{\swarrow} & \downarrow \\ R & \twoheadrightarrow & R/I \end{array}$$

there exist *at most one* arrow φ that makes the two triangles commute. Here I is a square-zero ideal of R. Analogously we say that f is **formally étale**, if there exist a *unique* such φ.

Remark 1.22. Observe that asking $(A \to B) \in Ring$ to be formally unramified, resp. formally étale, is equivalent to ask the natural set-map

$$\mathrm{Hom}_A(B, R) \longrightarrow \mathrm{Hom}_A(B, R/I)$$

to be injective, resp. bijective, (where $\mathrm{Hom}_A(-,-) : (A\text{-Alg})^2 \to Set$).

The following is an equivalent definition of unramifiedness for ring morphisms.

Definition 1.23. Let $(A \xrightarrow{f} B) \in Ring$. We say that f is **unramified**, if

- f is of finite type;
- f is formally unramified.

Definition 1.24. Let $(A \xrightarrow{f} B) \in Ring$. We say that f is **étale**, if

- f is of finite presentation;
- f is formally étale.

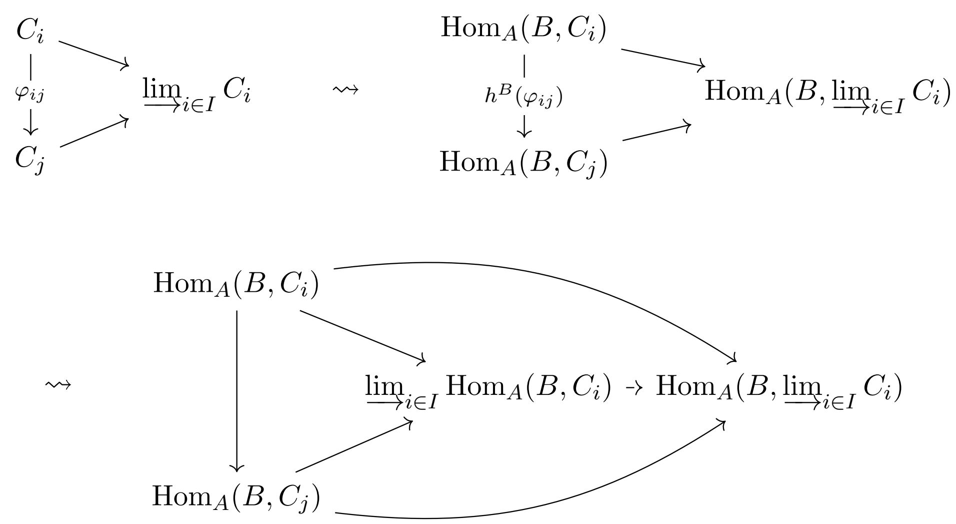
Here we will give an alternative definition of finitely presented ring map.

Definition 1.25. [Vez] Let $(A \xrightarrow{f} B) \in \mathit{Ring}$. We say that f is **finitely presented**, if the functor $\mathrm{Hom}_A(B, -) : A\text{-}\mathit{Alg} \to \mathit{Set}$ commutes with filtered colimits, i.e. if for every filtered diagram $C : I \to A\text{-}\mathit{Alg}$ (where I is a filtrant category) the natural set-map

$$\varinjlim_{i\in I} \mathrm{Hom}_A(B, C_i) \longrightarrow \mathrm{Hom}_A(B, \varinjlim_{i\in I} C_i)$$

is a bijection.

Remark 1.26. The above canonical map is "schematically" constructed as follows:



where the φ_{ij}'s denote the transition maps of the filtered diagram.

Remark 1.27. This last definition is equivalent to the one given in Def 1.4. For a proof (of this equivalence) see [EGAIV,3, Cor 8.14.2.2, p. 53].

Let's talk about *schemes* now:

Definition 1.28. Let $(X \xrightarrow{f} S) \in \mathit{Sch}$. We say that f is **formally unramified**, resp. **formally étale**, if for every $Y \in \mathit{AffSch}/S$ and for every closed immersion $Y_0 \hookrightarrow Y$ defined by a square-zero ideal we have that the natural set-map

$$\mathrm{Hom}_S(Y, X) \longrightarrow \mathrm{Hom}_S(Y_0, X)$$

is injective, resp. bijective. Equivalently we can ask the map

$$\mathrm{Hom}_S(\mathit{Spec}(C), X) \longrightarrow \mathrm{Hom}_S(\mathit{Spec}(C/I), X)$$

to be injective, resp. bijective, for every $C \in CRing/S$ and every square-zero ideal $I \subseteq C$.

Remark 1.29. Observe that in the case of an affine scheme morphism $Spec(B) \to Spec(A)$ the above is nothing but the "Spec"-ed version of Def. 1.21:

$$
\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & \overset{\varphi}{\swarrow} & \downarrow \\ R & \twoheadrightarrow & R/I \end{array}
\quad \rightsquigarrow \quad
\begin{array}{ccc} Spec(A) & \longleftarrow & Spec(B) \\ \uparrow & \overset{Spec(\varphi)}{\nearrow} & \uparrow \\ Spec(R) & \hookleftarrow & Spec(R/I). \end{array}
$$

Definition 1.30. Let $S \in Sch$. We define a **ring over** S to be a scheme morphism $Spec(C) \to S$, (where $C \in CRing$).

Notation 1.31. Given $S \in Sch$ we denote with $CRing/S$ the category of rings over S and with $AffSch/S$ the category of affine schemes over S.

Definition 1.32. [Vez] Let $(X \xrightarrow{f} S) \in Sch$. We say that f is **locally finitely presented**, if the functor $h_X : AffSch/S \to Set$ commutes with filtered colimits, i.e. if for every filtered diagram $C : I \to CRing/S$ the natural set-map

$$\varinjlim_{i \in I} \mathrm{Hom}_S(Spec(C_i), X) \longrightarrow \mathrm{Hom}_S(Spec(\varinjlim_{i \in I} C_i), X)$$

is a bijection.

Remark 1.33. This last definition is equivalent to the one given in Def 1.7. For a proof (of this equivalence) see [EGAIV,3, Prop 8.14.2, p. 52].

Remark 1.34. In the case of an affine morphism $Spec(B) \xrightarrow{\varphi} Spec(A)$, asking φ to be (locally) finitely presented in the sense of the above definition equals asking the corresponding map of rings $A \to B$ being finitely presented in the sense of Def 1.25.

Remark 1.35. No such a functorial characterization is known for a scheme morphism to be locally of finite type.

Definition 1.36. Let $(X \xrightarrow{f} S) \in Sch$. We say that f is **étale**, if

- f is formally étale;
- f is locally finitely presented.

Remark 1.37. This new definition is equivalent to the one given in Def 1.15. For a proof (of this equivalence) see [S.P., Tag 02HM].

So we characterized the notion of *étaleness* of a scheme morphism $X \to S$ purely in terms of its slice functor of points X_S. The two conditions it has to satisfy can be (just briefly) resumed as

- $X_S(C) = X_S(C/I)$ $\qquad (I^2 = 0)$;
- $\varinjlim_{i \in I} X_S(C_i) = X_S(\varinjlim_{i \in I} C_i)$.

1.4 Sorites

Definition 1.38. Let $f : X \to S$ be a morphism of schemes and let $s \in S$. We define the **fibre of** f **at the point** $s \in S$ to be the fiber product $X_s := X \times_S Spec(\kappa(s))$. We refer to it also calling it the *s-fibre of f*.

Next we characterize the notion of *unramifiedness* for a scheme map by several different formulations, some of which we already encountered. Before doing this we introduce the notion of *separable* k-algebra and we give a useful lemma.

Definition 1.39. Let $k \in Fld$ and $A \in k\text{-}Alg$. Let's denote with $\overline{k}$ an algebraic closure of k and set $A_{\overline{k}} := A \otimes_k \overline{k}$. We say that A is **separable**, if $J(A_{\overline{k}}) = 0$.

Lemma 1.40. *Let $k \in Fld$ and $A \in k\text{-}Alg$. Assume A is finite over k. The following assertions are equivalent:*

1. *A is separable;*
2. *$A \cong \prod_{i=1,\dots,n} k_i$, where the k_i's are (finite and) separable field extension of k and $n \in \mathbb{N}$;*
3. *$A_{\overline{k}} \cong \prod_{finite} \overline{k}$.*

Proof. See [Mil1, Prop. 3.1, p.20]. □

Proposition 1.41. *Let $f : X \to S$ be a morphism of schemes. The following assertions are equivalent:*

1. *f is unramified;*
2. *for every $s \in S$, the s-fibre decomposes as $X_s = \coprod_{i \in I} Spec(k_i)$, where $k_i/\kappa(s)$ is a finite and separable field extension, for every $i \in I$;*
3. *for every $s \in S$, the fiber X_s admits an open covering by spectra of finite and separable $\kappa(s)$-algebras;*
4. *for every $s \in S$, the fiber morphism $X_s \to Spec(\kappa(s))$ is unramified;*
5. *for every morphism $Spec(k) \to S$, with k separably closed, the morphism $X \times_S Spec(k) \to Spec(k)$ is unramified, (i.e. all the **geometric fibers of** f are unramified);*
6. *the diagonal morphism $\Delta_{X/S} : X \to X \times_S X$ is an open immersion;*
7. *f is formally unramified and locally of finite type.*

Proof. See for instance [Mil1, Prop. 3.2, p.21]. □

Here below we state some properties of the *étale morphisms*.

Corollary 1.42. *Let $k \in Fld$ and $X \in Sch/k$. Then X/k is étale if and only if we can write X as a disjoint union of spectra of finite and separable field extensions of k.*

Proof. Follows immediately from Prop. 1.41 and from the fact that $- \otimes_k \mathcal{O}_{X,x}$ is exact for every $x \in X$. □

Corollary 1.43. *The category of étale schemes over $k \in Fld$ admits arbitrary coproducts.*

Proposition 1.44. *Let $S \in Sch$.*

1. *Open immersions of schemes are étale;*
2. *given $f : X \to Y$ and $g : Y \to Z$ in Sch we have that*
   - *f and g étale $\quad\Rightarrow\quad$ $g \circ f$ étale;*
   - *$g \circ f$ and g étale $\quad\Rightarrow\quad$ f étale;*
3. *given $f : X \to X'$ and $g : Y \to Y'$ in Sch/S we have that*

$$f \text{ and } g \text{ étale} \quad \Rightarrow \quad f \times_S g : X \times_S Y \to X' \times_S Y' \quad \text{étale.}$$

Proof. See [EGAIV,4, Prop. 17.3.3, 17.3.4, pp. 61-62]. □

Chapter 2

Topology

Cos'è l'amore?
Il bisogno di uscire da sé stessi.

— Baudelaire C.
Il mio cuore messo a nudo

2.1 Sites & Sheaves

Definition 2.1. We define a **site** to be a pair $T_{\mathcal{C}} = (\mathcal{C}, Cov(\mathcal{C})$, where $\mathcal{C}$ is a category and $Cov(\mathcal{C})$ is a collection of families $\{U_i \to U\}_{i \in I}$ of morphisms in $\mathcal{C}$ satisfying the four following axioms:

1. (*existence of fiber products*) given $\{U_i \to U\}_{i \in I} \in Cov(\mathcal{C})$ and $(V \to U) \in \mathcal{C}$, the fiber product $U_i \times_U V$ exists, for every $i \in I$;

2. (*stability under base change*) given $\{U_i \to U\}_{i \in I} \in Cov(\mathcal{C})$ and $(V \to U) \in \mathcal{C}$, $\{U_i \times_U V \to V\}_{i \in I}$ lies in $Cov(\mathcal{C})$;

3. (*transitivity*) given $\{U_i \to U\}_{i \in I} \in Cov(\mathcal{C})$ and, for every $i \in I$, an element $\{V_{ij} \to U_i\}_{j \in J(i)} \in Cov(\mathcal{C})$, the composite element $\{V_{ij} \to U\}_{i \in I, j \in J(i)}$ lies in $Cov(\mathcal{C})$;

4. (*iso-cover*) given $(U' \to U) \in \mathrm{Iso}(\mathcal{C})$, we have that $\{U' \to U\}$ lies in $Cov(\mathcal{C})$.

The elements of $Cov(\mathcal{C})$ are called **coverings** of elements of $\mathcal{C}$ and the collection $Cov(\mathcal{C})$ itself is said to be a **Grothendieck topology** on the category $\mathcal{C}$. The four axioms above will be called the *covering-axioms*.

Example 2.2. Let $X \in Top$. Let also $\mathcal{C} := Op_X$ be the category whose objects are the open subsets in X and whose arrows are the usual inclusion maps between opens. Then we have that $T_X := T_{\mathcal{C}}$ is a site, where $Cov(T_X)$ consists of the usual coverings made of open subsets in Top.

Definition 2.3. Given two sites $T = T_{\mathcal{C}}$ and $T' = T_{\mathcal{C}'}$, we define a **morphism of sites** $f : T \to T'$ to be a functor $f : \mathcal{C} \to \mathcal{C}'$ satisfying the two following properties w.r.t. the coverings:

- (*cover-compatibility*) given $\{U_i \to U\}_{i\in I} \in Cov(\mathcal{C})$, the element $\{f(U_i) \to f(U)\}_{i\in I}$ lies in $Cov(\mathcal{C}')$;
- (*fiber product compatibility*) given $\{U_i \to U\}_{i\in I} \in Cov(\mathcal{C})$ and $(V \to U) \in \mathcal{C}$, we have a $\mathcal{C}'$- isomorphism

$$f(U_i \times_U V) \xrightarrow{\sim} f(U_i) \times_{f(U)} f(V)$$

for every $i \in I$.

Here we denoted with f both the morphism of sites and the functor between the underlying categories, for convenience.

Definition 2.4. Let $T = T_{\mathcal{C}}$ and $T' = T_{\mathcal{C}'}$ be two sites and $f : T \to T'$ be a morphism of sites. We say that f is an **isomorphism of sites** if:

- the underlying functor $f : \mathcal{C} \xrightarrow{\sim} \mathcal{C}'$ is an equivalence of categories;
- $\{W_j \to W\}_{j\in J} \in Cov(\mathcal{C}') \Rightarrow \{g(W_j) \to g(W)\}_{j\in J} \in Cov(\mathcal{C})$.

Here g denotes a quasi-inverse functor to f. Restricting our attention to the collections of coverings this is also called an **equivalence of Grothendieck topologies**.

Example 2.5. Let $(X \xrightarrow{f} Y) \in \mathrm{Mor}(Top)$. We have that f induces a morphism of sites $f^{-1} : T_Y \to T_X$, which acts on the objects as

$$Op_Y \ni \ U \mapsto f^{-1}(U) \ \in Op_X.$$

Definition 2.6. Let $T = T_{\mathcal{C}}$ be a site and let $\mathcal{D}$ be a category with arbitrary products, (*Set* or *Ab* for instance). We define a **presheaf on** T **values in** $\mathcal{D}$ to be a functor $\mathscr{F} : \mathcal{C}^{op} \to \mathcal{D}$.

Definition 2.7. Let $T = T_{\mathcal{C}}$ be a site and let $\mathscr{F}, \mathscr{G} : \mathcal{C}^{op} \to \mathcal{D}$ be two presheaves (on T with values in $\mathcal{D}$). We define a **morphism of presheaves** to be a morphism of functors $\eta : \mathscr{F} \to \mathscr{G}$.

Notation 2.8. We denote with $PSh(T)$ and with $P\mathcal{D}(T)$ the categories whose objects are the preshaves on the site T respectively with values in *Set* and in $\mathcal{D}$ and whose arrows are the morphisms between them.

Before starting to define the notion of *sheaf on a site*, we will make clear the construction of the arrows which will be involved. Given a site $T = T_{\mathcal{C}}$, a presheaf $\mathscr{F} \in P\mathcal{D}(T)$ and an element $\{U_i \to U\}_{i\in I} \in Cov(\mathcal{C})$, the set of pullback maps $\{\mathscr{F}(U) \to \mathscr{F}(U_i)\}_{i\in I}$ naturally induces a map $\varphi : \mathscr{F}(U) \to \prod_{i\in I} \mathscr{F}(U_i)$ (unique up to iso) that makes the below diagram commute

$$\begin{array}{ccc} & & \prod_{i\in I}\mathscr{F}(U_i) \\ & \overset{\varphi}{\nearrow} & \downarrow{\scriptstyle \pi_i} \\ \mathscr{F}(U) & \xrightarrow{p_i} & \mathscr{F}(U_i) \end{array}$$

for every $i \in I$. Moreover we define two symmetric maps (1) and (2) by requiring the diagrams

$$\begin{array}{ccccc} \prod_{i\in I}\mathscr{F}(U_i) & \xrightarrow{(1)} & \prod_{(i,j)\in I^2}\mathscr{F}(U_i \times_U U_j) & \xleftarrow[(2)]{} & \prod_{i\in I}\mathscr{F}(U_i) \\ \downarrow{\scriptstyle \pi_i} & & \downarrow{\scriptstyle \pi_{i,j}} & & \downarrow{\scriptstyle \pi_j} \\ \mathscr{F}(U_i) & \xrightarrow{\mathscr{F}p_i} & \mathscr{F}(U_i \times_U U_j) & \xleftarrow[\mathscr{F}p_j]{} & \mathscr{F}(U_i) \end{array}$$

to commute for every $i, j \in I$, where p_i and p_j come from the pullback diagram

$$\begin{array}{ccc} U_i \times_U U_j & \xrightarrow{p_j} & U_j \\ \downarrow{\scriptstyle p_i} & \ulcorner & \downarrow{\scriptstyle \varphi_j} \\ U_i & \xrightarrow{\varphi_i} & U \end{array}$$

.

Definition 2.9. Let $T = T_{\mathcal{C}}$ be a site and let $\mathscr{F} \in P\mathcal{D}(T)$. We say that $\mathscr{F}$ is a **sheaf**, if the arrow φ in the first diagram is the equalizer of arrows (1) and (2) above constructed, i.e. if $\varphi = Eq((1), (2))$. This condition can be restated saying that for every element $\{U_i \to U\}_{i\in I} \in Cov(\mathcal{C})$ we require the diagram

$$\mathscr{F}(U) \dashrightarrow^{\varphi} \prod_i \mathscr{F}(U_i) \overset{(1)}{\underset{(2)}{\rightrightarrows}} \prod_{(i,j)} \mathscr{F}(U_i \times_U U_j)$$

to be exact, (with i varying in I and (i, j) in I^2).

Definition 2.10. Let $T = T_{\mathcal{C}}$ be a site and let $\mathscr{F}, \mathscr{G} : \mathcal{C}^{op} \to \mathcal{D}$ be two sheaves on T with values in $\mathcal{D}$. We define a **morphism of sheaves** to be a morphism of functors $\eta : \mathscr{F} \to \mathscr{G}$ (i.e. η is simply a morphism of presheaves).

Notation 2.11. We will denote with $Sh(T)$ and with $\mathcal{D}(T)$ the categories whose objects are the shaves on the site T respectively with values in Set and in $\mathcal{D}$ and whose arrows are the morphisms between them. A sheaf in $Ab(T)$ will be called an *abelian sheaf*.

Example 2.12. Let $T = T_{\mathcal{C}}$ be a site. For any $Z \in \mathcal{C}$, we have that $\mathrm{Hom}(-, Z) : \mathcal{C}^{op} \to Set$ is a covariant functor and so $\mathrm{Hom}(-, Z) \in PSh(T)$.

Definition 2.13. Let $T = T_{\mathcal{C}}$ be a site and let $Z \in \mathcal{C}$. Given a presheaf $\mathscr{F} \in PSh(T)$, we say that $\mathscr{F}$ is **(co-)representable**, if we have that $\mathscr{F} \simeq \mathrm{Hom}_{\mathcal{C}}(-, Z)$ for some $Z \in \mathcal{C}$.

We will now introduce the key notion of *canonical site* on a category $\mathcal{C}$. In the sequel we will assume $\mathcal{C}$ to be a *category with fiber products*. Recall that a given arrow $(U \to V) \in \mathcal{C}$ is said to be an *epimorphism in* $\mathcal{C}$, if the canonical map $\mathrm{Hom}(V, Z) \to \mathrm{Hom}(U, Z)$ is injective for every $Z \in \mathcal{C}$. Let's denote with $\mathrm{Epi}(\mathcal{C}) \subseteq \mathrm{Mor}(\mathcal{C})$ the collection of all epis in $\mathcal{C}$.

Definition 2.14. Let $\mathcal{C}$ be a category and let $(U \xrightarrow{\varphi} V) \in \mathrm{Epi}(\mathcal{C})$. We say that

- φ is **effective**, if the natural diagram

$$\mathrm{Hom}_{\mathcal{C}}(V, Z) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(U, Z) \rightrightarrows \mathrm{Hom}_{\mathcal{C}}(U \times_V U, Z)$$

  (arising from the pullback diagram) is exact;

- φ is **universally effective**, if it is effective and for every $(V' \to V) \in \mathcal{C}$ the morphism $U \times_V V' \to V'$ is an effective epi. (This condition equals requiring effectiveness to be stable under base change.).

Let's now extend these notions of *effectiveness* to families of morphisms, in order to address *coverings*. Recall that a collection $\{U_i \to V\}_{i \in I}$ of morphisms in $\mathcal{C}$ is defined to be a *family of epimorphism in* $\mathcal{C}$, if the canonical map

$$\mathrm{Hom}_{\mathcal{C}}(V, Z) \longrightarrow \prod_i \mathrm{Hom}_{\mathcal{C}}(U_i, Z)$$

is injective for every $Z \in \mathcal{C}$.

Definition 2.15. Let $\mathcal{U} := \{U_i \to V\}_{i \in I}$ be a family of epis in $\mathcal{C}$. (Not necessarily $\mathcal{U}$ satisfies the covering-axioms.) We say that

- $\mathcal{U}$ is **effective**, if the diagram

$$\mathrm{Hom}_{\mathcal{C}}(V, Z) \longrightarrow \prod_i \mathrm{Hom}_{\mathcal{C}}(U_i, Z) \rightrightarrows \prod_{(i,j)} \mathrm{Hom}_{\mathcal{C}}(U_i \times_V U_j, Z)$$

  is exact for every $Z \in \mathcal{C}$;

- $\mathcal{U}$ is **universally effective**, if it is effective and for every $(V' \to V) \in \mathcal{C}$ the collection $\{U_i \times_V V' \to V'\}_{i \in I}$ is an effective family of epis. (This condition equals requiring effectiveness of the collection to be stable under base change.).

Definition 2.16. We define the **canonical site** $\mathcal{T}_{\mathcal{C}}$ by endowing the category $\mathcal{C}$ with the Grothendieck topology $Cov(\mathcal{C})$ consisting of all the elements $\{U_i \to U\}_{i \in I} \in \mathrm{Mor}(\mathcal{C})$ which are universally effective families epis. The pair $\mathcal{T}_{\mathcal{C}} = (\mathcal{C}, Cov(\mathcal{C})$ so defined is indeed a site: the first and the third axioms are trivially verified; for what concerns the second axiom is sufficient to extend the result in [SGA3, exp. IV, Prop. 1.8, p. 180] to universally effective families of epis.

Remark 2.17. Observe that:

- by construction of the canonical site, the (co-)representable presheaf of sets
$$U \mapsto \mathrm{Hom}_{\mathcal{C}}(U, Z)$$
lies in $Sh(\mathcal{T}_{\mathcal{C}})$, for every $Z \in \mathcal{C}$. Whenever the functor $\mathrm{Hom}_{\mathcal{C}}(-, Z)$ is a sheaf we denote it with $\mathscr{H}om_{\mathcal{C}}(-, Z)$.
- $\mathcal{T}_{\mathcal{C}}$ is the finest site on $\mathcal{C}$ such that $\mathrm{Hom}_{\mathcal{C}}(-, Z)$ is a sheaf for every $Z \in \mathcal{C}$. (In other words $\mathcal{T}_{\mathcal{C}}$ is the terminal object in the *category of sites on* $\mathcal{C}$. In fact given a site $T = T_{\mathcal{C}}$ such that $\{\mathrm{Hom}_{\mathcal{C}}(-, Z)\}_{Z \in \mathcal{C}} \subseteq Sh(T)$, we have that $\mathcal{U} \in Cov(T)$ implies that $\mathcal{U}$ is a universally effective family of epis and so the identity functor $id_{\mathcal{C}}$ induces a morphism of sites $T \to \mathcal{T}_{\mathcal{C}}$.). In this case we say that the site T is **subcanonical**.

2.2 *G-Set*

Notation 2.18. Given $G \in Grp$ we define the **canonical G-site** $\mathcal{T}_G$ to be the canonical site on the category *G-Set*. (Morphisms in the underlying category are the G-equivariant maps.).

Remark 2.19. The following characterization holds: given an element $\mathcal{U} = \{\varphi_i : U_i \to U\}_{i \in I} \in Cov(G\text{-}Set)$, we have that $\mathcal{U}$ is a family of universally effective epis if and only if we can write $U = \bigcup_{i \in I} \varphi_i(U_i)$. A family $\mathcal{U}$ of maps in *G-Set* satisfying the last condition is said to be **jointly surjective**.

Remark 2.20. a morphism $(G \xrightarrow{f} H) \in Grp$ naturally induces a morphism H-Set $\to$ G-Set which in turn induces a morphism of sites $\mathcal{T}_H \to \mathcal{T}_G$. Given $Z \in H$-Set we can make G act on Z by setting $(g, z) \mapsto f(g).z$ for every $z \in Z$.

We know that by construction of the canonical site $\mathcal{T}_G$ the presheaf $\mathrm{Hom}_{G\text{-}Set}(-, Z)$ is a sheaf on $\mathcal{T}_G$ for every $Z \in G\text{-}Set$. The next result will establish that all the sheaves of sets on $\mathcal{T}_G$ arise in this way.

Notation 2.21. For shortness we set $\mathrm{Hom}_G(-, Z) := \mathrm{Hom}_{G\text{-}Set}(-, Z)$ and $\mathscr{H}om_G(-, Z) := \mathscr{H}om_{G\text{-}Set}(-, Z)$.

Remark 2.22. Given $G \in Grp$ we can define a G-structure on $\mathscr{F}(G)$, by setting $g.s := \mathscr{F}(\cdot g)(s)$, for every $g \in G$ and every $s \in \mathscr{F}(G)$. Here $\cdot g$ denotes the map "right multiplication by g". By functoriality of $\mathscr{F}$, this is indeed an action of G on $\mathscr{F}(G)$.

Theorem 2.23. *Let $G \in Grp$. The functors*

$$\begin{aligned} G\text{-}Set &\longrightarrow Sh(\mathcal{T}_G) \\ Z &\overset{\phi}{\longmapsto} \mathscr{H}om_G(-, Z) \\ \mathscr{F}(G) &\overset{\psi}{\longleftarrow\!\shortmid} \mathscr{F} \end{aligned}$$

yield an equivalence of categories.

Proof. We already observed that the presheaf $\mathrm{Hom}_G(-, Z)$ lies in $Sh(\mathcal{T}_G)$, so ϕ is well-defined. By Rmk 2.22 ψ is well-defined too.
Step 1: $\psi \circ \phi \simeq id_{G\text{-}Set}$. This means that for every $Z \in G\text{-}Set$ we want an isomorphism $Z \cong \mathrm{Hom}_G(G, Z)$, functorial in Z. **First way:** by hands. Let's define a map τ by setting:

$$\begin{aligned} Z &\xrightarrow{\tau} \mathrm{Hom}_G(G, Z) & \qquad f_z : G &\longrightarrow Z \\ z &\longmapsto f_z & g &\longmapsto g.z \end{aligned}$$

for every $z \in Z$ and every $g \in G$. We have that $f_z \in \mathrm{Hom}_G(G, Z)$, for every $z \in Z$. In fact, given $g, h \in G$, we have that $h.f_z(g) = h.(g.z) = (hg).z = f_z(hg)$. ($f_z$ is obviously well-defined as Z is a G-set.). Hence τ is well-defined. Define now a map

$$\begin{aligned} G \times \mathrm{Hom}_G(G, Z) &\xrightarrow{\phi} \mathrm{Hom}_G(G, Z) \\ (g, f) &\longmapsto g_* f \qquad : G \longrightarrow Z \\ & \qquad\qquad\qquad h \longmapsto f(hg). \end{aligned}$$

We have that:

- ϕ is well-defined: given $h, t \in G$, it holds $t.((g_* f)(h)) = t.(f(hg)) = f(t(hg)) = f((th)g) = (g_* f)(th)$;
- ϕ is a (left) group action: given $g, h, t \in G$, it holds $(t_*(g_* f))(h) = (g_* f)(ht) = f((ht)g) = f(h(tg)) = ((tg)_* f)(h)$ (and trivially also $1_* f = f$).

Hence indeed $\mathrm{Hom}_G(G, Z) \in G\text{-}Set$. (The key point here is that G lies both in G-*Set* and in *Set*-G.). Let's now prove that τ is a G-isomorphism:

- injectivity: given $z, z' \in Z$, with $g.z = g.z'$, we have that acting by g^{-1} we get $z = z'$;
- surjectivity: given $f \in \mathrm{Hom}_G(G, Z)$, we have that $f(g) = f(g \cdot 1) = g.f(1)$, for any $g \in G$ (hence f is totally determined by its value at 1). Therefore $f = f_{f(1)}$;
- G-compatibility: given $z \in Z$ and $h, g \in G$, we have that $(h_* f_z)(g) = f_z(gh) = (gh).z = g.h.z = f_{h.z}(g)$, hence $h_*\tau(z) = \tau(h.z)$.

So we showed that τ is a bijective G-morphism and by properties of group actions this implies τ^{-1} is a G-morphism, hence $\tau \in \mathrm{Iso}_G(Z, \mathrm{Hom}_G(G, Z))$.
Second way: categorically. Referring to the notation used in the statement of Yoneda's Lemma, we take $\mathcal{C} := \mathcal{G}$, where here we see the group G as a category $\mathcal{G}$ "in itself" with $\mathrm{Ob}(\mathcal{G}) := \{pt\}$ (one-object category) and $\mathrm{Mor}(\mathcal{G}) := G$, (here every arrow is an iso). As there is an isomorphism G-Set $\cong \mathrm{Func}(\mathcal{G}, Set)$,we can interpret the G-set Z as the image of a certain $F \in \mathrm{Func}(\mathcal{G}, Set)$, namely $F(pt) = Z$. Let $X := pt$. By Yoneda's Lemma we get a bijection $\mathrm{Nat}(h^{pt}, F) \cong F(pt) = Z$. Moreover we can see the group G as the image of the functor h^{pt}, namely $h^{pt}(pt) = \mathrm{Hom}_{\mathcal{G}}(pt, pt) = G$, so by the above isomorphism of categories we have that $\mathrm{Hom}_G(G, Z) \cong \mathrm{Nat}(h^{pt}, F)$, hence our iso. By Yoneda's Lemma this iso is natural in F, hence in Z.
Step 2: $\phi \circ \psi \simeq id_{Sh(\mathcal{T}_G, Set)}$. So what we want is a natural isomorphism $\theta = \{\theta_{\mathscr{F}} : \mathscr{F} \xrightarrow{\sim} \phi(\psi(\mathscr{F}))\}_{\mathscr{F} \in Sh(\mathcal{T}_G, Set)}$. This means that given any sheaf $\mathscr{F} \in Sh(\mathcal{T}_G)$, we have to show that for every $U \in G$-Set there is an iso $\mathscr{F}(U) \cong (\phi(\psi(\mathscr{F})))(U) = \mathrm{Hom}_G(U, \mathscr{F}(G))$.

So let's assume that $\mathscr{F} \in Sh(\mathcal{T}_G)$. We want to show that $\mathscr{F}(U) \cong \mathrm{Hom}_G(U, \mathscr{F}(G))$. Consider the collection of G-maps $\mathcal{U} = \{G \overset{\varphi_u}{\to} U\}_{u \in U}$ defined as $\varphi_u : g \mapsto g.u$. We have that $\mathcal{U} \in Cov(\mathcal{T}_G)$. In fact by Rmk 2.19, it suffices to prove that $U = \bigcup_{u \in U} \varphi_u(G)$. Now for every $u \in U$, we have that $\varphi_u(G) = \mathcal{O}_u$ (i.e. the orbit of u under G) and so, as $\{\mathcal{O}_u\}_{u \in U}$ is a partition of U, we are done. For every $u, v \in U$ we have the two commutative diagrams below

$$\begin{array}{ccc} G \times_U G & \xrightarrow{p_u} & G \\ \downarrow{\scriptstyle p_v} & & \downarrow{\scriptstyle \varphi_u} \\ G & \xrightarrow{\varphi_v} & U \end{array} \qquad \begin{array}{ccc} \mathscr{F}(G \times_U G) & \xleftarrow[\mathscr{F}p_u]{} & \mathscr{F}(G) \\ {\scriptstyle \mathscr{F}p_v}\uparrow & & \uparrow{\scriptstyle \mathscr{F}\varphi_u} \\ \mathscr{F}(G) & \xleftarrow[\mathscr{F}\varphi_v]{} & \mathscr{F}(U). \end{array}$$

(where the first one is cartesian). As $\mathscr{F}$ is a sheaf on $\mathcal{T}_G$ we have the exact Set-diagram

$$\mathscr{F}(U) \overset{\varphi}{\hookrightarrow} \prod_u \mathscr{F}(G) \underset{(2)}{\overset{(1)}{\rightrightarrows}} \prod_{(u,v)} \mathscr{F}(G \times_U G)$$

We will now prove that the condition $\varphi = \mathrm{Eq}((1),(2))$ implies that our iso holds. Let's first observe a few facts:

- for the fiber product in G-set there is the description

$$G \times_U G = G \times_{U,u,v} G = \{(g,h) \in G^2 \mid g.u = h.v\};$$

- as we are working with sheaves of sets we can write $\prod_{u\in U} \mathscr{F}(G) = \mathrm{Hom}_{Set}(U, \mathscr{F}(G))$;
- as we are working with sheaves of sets the isomorphism we are looking for is just a bijection;
- 1^{st} case: $\mathscr{O}_u \cap \mathscr{O}_v = \emptyset$. In this case $G \times_{U,u,v} G = \emptyset$.
- 2^{nd} case: $\mathscr{O}_u = \mathscr{O}_v$. In this case there exist $t \in G$ such that $v = t.u$. So a pair $(g,h) \in G^2$ satisfies $g.u = h.v = h.t.u$ iff $g^{-1}ht \in St_G(u)$ iff $g^{-1}h \in St_G(u)t^{-1}$.

Now considering the diagrams

$$\begin{array}{ccccc}
\prod_{u\in U} \mathscr{F}(G) & \xrightarrow{(1)} & \prod_{(u,v)\in U^2} \mathscr{F}(G \times_{U,u,v} G) & \xleftarrow{(2)} & \prod_{u\in U} \mathscr{F}(G) \\
\downarrow{\scriptstyle \pi_u} & \ulcorner & \downarrow{\scriptstyle \pi_{u,v}} & \ulcorner & \downarrow{\scriptstyle \pi_v} \\
\mathscr{F}(G) & \xrightarrow{\mathscr{F}p_u} & \mathscr{F}(G \times_{U,u,v} G) & \xleftarrow{\mathscr{F}p_v} & \mathscr{F}(G)
\end{array}$$

we have that the "equalizer condition" means that the image of the injection φ, namely $\mathrm{Im}(\varphi) = \{\varphi(a) = (a_u)_{u\in U} \mid a \in \mathscr{F}(U)\}$ coincides with the subset of $\prod_{u\in U} \mathscr{F}(G)$

$$\begin{aligned}
&\{(a_u)_{u\in U} \in \prod_{u\in U} \mathscr{F}(G) \mid (1)((a_u)_u) = (2)((a_u)_u)\} \\
=&\{(a_u)_{u\in U} \in \prod_{u\in U} \mathscr{F}(G) \mid \forall u,v \in U,\ \pi_{u,v}((1)((a_u)_u)) = \pi_{u,v}((2)((a_u)_u))\} \\
=&\{(a_u)_{u\in U} \in \prod_{u\in U} \mathscr{F}(G) \mid \forall u,v \in U,\ (\mathscr{F}p_u)(a_u) = (\mathscr{F}p_v)(a_v)\}
\end{aligned}$$

which by the last above observation is the subset of $\prod_{u\in U} \mathscr{F}(G)$ with elements $(a_u)_{u\in U}$ such that for every $u, v \in U$ with $\mathscr{O}_u = \mathscr{O}_v$ it holds $(\mathscr{F}p_u)(a_u) = (\mathscr{F}p_v)(a_v)$. Hence this subset can be reparametrized as the one with elements $(a_u)_{u\in U}$ such that for every $u \in U$ and every $g \in G$ it holds $(\mathscr{F}p_u)(a_u) = (\mathscr{F}p_v)(a_{g.u})$. On the other hand we have that

$$\mathrm{Hom}_G(U, \mathscr{F}(G)) = \{f : U \to \mathscr{F}(G) \mid f \text{ is } G\text{-equivariant}\}$$

coincides with

$$\begin{aligned}&\{(a_u)_{u\in U} \in \prod_{u\in U} \mathscr{F}(G) \mid \forall u \in U, \forall g \in G,\, g.a_u = a_{g.u}\,\}\\ =&\{(a_u)_{u\in U} \in \prod_{u\in U} \mathscr{F}(G) \mid \forall u \in U, \forall g \in G,\, \mathscr{F}(\cdot g)(a_u) = a_{g.u}\}.\end{aligned}$$

Now noticing that the set-map $\mathscr{F}(\cdot g)$ lands in $\mathscr{F}(G)$, if we reinterpret the maps $\mathscr{F}p_u, \mathscr{F}p_v : \mathscr{F}(G) \to \mathscr{F}(G \times_{U,u,v} G)$ as maps $\mathscr{F}(G) \to \mathscr{F}(G)$, namely $\mathscr{F}(\cdot g)$ on the left and "the identity" on the right, we see that the two above sets do coincide. This way we get the wanted natural set-iso $\mathscr{F}(U) \cong \mathrm{Hom}_G(U, \mathscr{F}(G))$ which is functorial in both $U \in G\text{-}Set$ and in $\mathscr{F} \in Sh(\mathcal{T}_G)$, whence the thesis. □

Here below we obtain as a "corollary" an analogous equivalence which relates G-modules and sheaves of abelian groups on the G-site, given by the "same" functors:

Theorem 2.24. *Let $G \in Grp$. We have the following equivalence (of categories)*

$$G\text{-}Mod \xrightarrow{\sim} Ab(\mathcal{T}_G).$$

2.3 $\mathscr{C}G$-*Set*

Let's now consider the case when G is a *profinite group* and the action is *continuous*. Here below we give two equivalent ways to define a profinite group.

Definition 2.25. Let $G \in Grp$ be a topological group. We say that G is **profinite** if

1. G has the algebraic structure $\varprojlim_{i\in I} G_i$ of an inverse limit (of an inverse system) of finite groups and it is endowed with the topology induced by the product topology on the product of the discrete finite groups involved;

2. G it is totally disconnected, compact and T_2.

Remark 2.26. In a profinite group the set $\mathcal{H} := \{H\}_{H \trianglelefteq_{open} G}$ is a fundamental system of neighbourhoods of 1_G (i.e. for any neighbourhood $U \subseteq G$ of 1_G in G we can find a finite number of elements $H_1, \ldots, H_n \in \mathcal{H}$ such that $1_G \in \bigcap_{i=1,\ldots,n} H_i \subseteq U$). We can canonically identify G with $\varprojlim_{H \trianglelefteq_{open} G} G/H$.

Let's now move to continuous actions.

Definition 2.27. Let $G \in Grp$ be a topological group and let $U \in Top$. We define a **continuous action** of G on U to be a continuous map $G \times U \to U$ satisfying the axioms of a group action.

In our case we endow the set U with the discrete topology and we call it a **continuous** G-set. G for us will be a profinite group.

Remark 2.28. The continuity of the action of G on U can be characterized, for example, in the two following ways:

- $\{St_G(u) \,|\, u \in U\} \subseteq Op_G$
- $U = \bigcup_{H \in \mathcal{H}} U^H$.

Notation 2.29. Let's denote with $\mathscr{C}G$-Set the category with Ob($\mathscr{C}G$-Set) := {continuous G-sets} and Mor($\mathscr{C}G$-Set) := Mor(G-Set). We define the **canonical continuous** G**-site** $\mathcal{T}_G^{\mathscr{C}}$ to be the canonical site on the category $\mathscr{C}G$-Set. Also in this case we set $\mathscr{H}om_{\mathscr{C}G}(-,U) := \mathscr{H}om_{\mathscr{C}G\text{-}Set}(-,U)$ for every $U \in \mathscr{C}G\text{-}Set$.

Remark 2.30. The result of Rmk 2.19 holds also for continuous G-sets.

We have that also the sheaves of sets on the continuous G-site arise all as representable (pre)sheaves.

Theorem 2.31. *Let $G \in Grp$ and assume G is profinite. The functors*

$$\mathscr{C}G\text{-}Set \longrightarrow Sh(\mathcal{T}_G^{\mathscr{C}})$$

$$U \xmapsto{\ \phi\ } \mathscr{H}om_{\mathscr{C}G}(-,U)$$

$$\varinjlim_{H \trianglelefteq_{open} G} \mathscr{F}(G/H) \xleftarrow{\ \psi\ } \mathscr{F}.$$

yield a categorical equivalence.

Proof. Given $H \trianglelefteq_{open} G$, we have that $G \times G/H \to G/H$, $(g, tH) \mapsto gtH$ makes G/H into a continuous G-set. We make G act on $\mathscr{F}(G/H)$ by setting $g.s := \mathscr{F}(\cdot gH)(s)$ for every $g \in G$ and every $s \in \mathscr{F}(G/H)$ where $\cdot gH : G/H \to G/H$ is the map $tH \to tHgH$. The set $\{H\}_{H \trianglelefteq_{open} G}$ is a directed system, setting $K \succeq H$ iff $K \subseteq H$. The canonical G-morphisms $G/K \twoheadrightarrow G/H$ induce G-morphisms $\mathscr{F}(G/H) \to \mathscr{F}(G/K)$, which will be the transition maps in the direct system considered. The maps $G \times \mathscr{F}(G/H) \to \mathscr{F}(G/H)$ (that define the action of G on $\mathscr{F}(G/H)$) give rise to the canonical map

$$\varinjlim_H (G \times \mathscr{F}(G/H)) \to \varinjlim_H \mathscr{F}(G/H).$$

Now since $(G\times -, Hom_{Set}(G, -))$ is an adjoint pair we have that the functor $G \times -$ is right exact, hence it preserves colimits. So we have a map

$$G \times \varinjlim_H \mathscr{F}(G/H) \to \varinjlim_H \mathscr{F}(G/H)$$

which endows our direct limit with a continuous G-structure. (Notice that here the direct limit is in *Set*, hence is nothing but $\bigcup_{H \trianglelefteq_{open} G} \mathscr{F}(G/H)$.). Hence ψ is well-defined.
Step 1: $\psi \circ \phi \simeq id_{\mathscr{C}G\text{-}Set}$. Let $U \in \mathscr{C}G$-*Set*. What we is want a G-iso

$$\varinjlim_H \mathrm{Hom}_{\mathscr{C}G}(G/H, U) \cong U.$$

First let's observe that there is a (set-)isomorphism $\mathrm{Hom}_{\mathscr{C}G}(G/H, U) \cong U^H$ given by setting $f \mapsto f(1_{G/H}) = f(1_G \cdot H)$. From this we deduce that

$$\varinjlim_H \mathrm{Hom}_{\mathscr{C}G}(G/H, U) \cong \varinjlim_H U^H = \bigcup_H U^H = U$$

Step 2: $\phi \circ \psi \simeq id_{Sh(\mathcal{T}_G^{\mathscr{C}})}$. Let $\mathscr{F} \in Sh(\mathcal{T}_G^{\mathscr{C}})$. What we want is an iso of sheaves

$$\mathscr{F} \simeq \mathrm{Hom}_{\mathscr{C}G}(-, \varinjlim_H \mathscr{F}(G/H)).$$

Let $U \in \mathscr{C}G$-*Set*. Then we can write $U = \bigcup_H U^H$. Let's write U as $\varinjlim_H U^H$ taking the colimit over the directed set of the open normal subgroups of G with the order relation above defined and let's call $\mathcal{H}$ this system. We have that $\{U^H \hookrightarrow U\} \in Cov(\mathscr{C}G\text{-}Set)$ and so we have that the sheaf diagram

$$\mathscr{F}(U) \overset{\varphi}{\hookrightarrow} \prod_{H \in \mathcal{H}} \mathscr{F}(U^H) \underset{(2)}{\overset{(1)}{\rightrightarrows}} \prod_{(H,K) \in \mathcal{H}^2} \mathscr{F}(U^H \times_U U^K)$$

is exact. Observe that

- $U^H \times_U U^K = U^H \cap U^K$, for every $H, K \in \mathcal{H}$
- $K \succeq H$ iff $K \subseteq H$ iff $U^H \supseteq U^K \rightsquigarrow \mathscr{F}(U^H) \to \mathscr{F}(U^K)$

so $\{\mathscr{F}(U^H)\}_{H \in \mathcal{H}}$ is an inverse system in *Set* with transition maps $\tau_{K,H} = \mathscr{F}(\rho_{K,H})$ where $\rho_{K,H} : U^K \hookrightarrow U^H$ for any $K \succeq H$. We claim that exactness of the above sheaf diagram translates into the isomorphism

$$\mathscr{F}(U) \cong \varprojlim_H \mathscr{F}(U^H).$$

On one hand we have that

$$\begin{aligned}
\mathrm{Eq}((1),(2)) &= \{(a_H)_{H \in \mathcal{H}} \in \prod_{H \in \mathcal{H}} \mathscr{F}(U^H) \mid (1)((a_H)_H) = (2)((a_H)_H)\} \\
&= \{(a_H)_H \mid \forall H, K \in \mathcal{H},\ \pi_{H,K}((1)((a_H)_H)) = \pi_{H,K}((2)((a_H)_H))\} \\
&= \{(a_H)_H \mid \forall H, K \in \mathcal{H},\ (\mathscr{F} i_H)(a_H) = (\mathscr{F} i_K)(a_K)\}
\end{aligned}$$

where the diagrams involved are

$$\begin{array}{ccc} U^H \cap U^K & \xhookrightarrow{i_K} & U^K \\ \Big\downarrow{\scriptstyle i_H} & & \Big\downarrow{\scriptstyle \varphi_K} \\ U^H & \xhookrightarrow{\varphi_H} & U \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} \mathscr{F}(U^H \cap U^K) & \xleftarrow[\mathscr{F} i_K]{} & \mathscr{F}(U^K) \\ {\scriptstyle \mathscr{F} i_H}\Big\uparrow & & {\scriptstyle \mathscr{F}\varphi_K}\Big\uparrow \\ \mathscr{F}(U^H) & \xleftarrow[\mathscr{F}\varphi_H]{} & \mathscr{F}(U). \end{array}$$

and

$$\begin{array}{ccccc} \prod_{H\in\mathcal{H}} \mathscr{F}(U^H) & \xrightarrow{(1)} & \prod_{(H,K)\in\mathcal{H}^2} \mathscr{F}(U^H \cap U^K) & \xleftarrow[(2)]{} & \prod_{H\in\mathcal{H}} \mathscr{F}(U^H) \\ \Big\downarrow{\scriptstyle \pi_H} & \ulcorner & \Big\downarrow{\scriptstyle \pi_{H,K}} & \ulcorner & \Big\downarrow{\scriptstyle \pi_K} \\ \mathscr{F}(U^H) & \xrightarrow{\mathscr{F} i_H} & \mathscr{F}(U^H \cap U^K) & \xleftarrow[\mathscr{F} i_K]{} & \mathscr{F}(U^K). \end{array}$$

Now assuming for example $H, K \in \mathcal{H}$ such that $K \succeq H$ we have that from $i_H = \rho_{K,H} \circ i_K$ we get $\mathscr{F} i_H = \mathscr{F}\rho_{K,H} \circ i_K = \mathscr{F} i_K \circ \mathscr{F}\rho_{K,H}$ and so the condition $(\mathscr{F} i_H)(a_H) = (\mathscr{F} i_K)(a_K)$ can be rewritten as $(\mathscr{F} i_K)((\mathscr{F}\rho_{K,H})(a_H)) = (\mathscr{F} i_K)(a_K)$. On the other hand we have that

$$\begin{aligned} \varprojlim_H \mathscr{F}(U^H) &= \{(a_H)_H \in \prod_{H\in\mathcal{H}} \mathscr{F}(U^H) \mid \forall K \succeq H,\, \tau_{K,H}(a_H) = a_K\} \\ &= \{(a_H)_H \mid \forall K \succeq H,\, (\mathscr{F}\rho_{K,H})(a_H) = a_K\}. \end{aligned}$$

Then by the same kind of "trick" used in the proof of Thm 2.23 we get that $\mathrm{Eq}((1),(2)) = \varprojlim_H \mathscr{F}(U^H)$ and so we are done.

Next consider the collection of maps $\{G/H \xrightarrow{\varphi_u} U^H\}_{u\in U^H}$. We have that this lies indeed in $Cov(\mathscr{C}G\text{-}Set)$. First notice that given $u \in U^H$, $gH \in G/H$ and $h \in H$ we have that $h.g.u = g.u$ iff $g^{-1}hg.u = h^g.u = u$ which is true as $H \trianglelefteq G$. So the φ_u's are well-defined and they are clearly G-homomorphisms. Moreover given an orbit-decomposition of U, namely $U = \bigcup_{u\in U} \mathcal{O}_u$, we have that

$$U^H = (\textstyle\bigcup_{u\in U} \mathcal{O}_u) \cap U^H = \bigcup_{u\in U} (\mathcal{O}_u \cap U^H) = \bigcup_{u\in U^H} \mathcal{O}_u$$

where last equality holds since $G \curvearrowright U^H$. Hence we are done, observing that for every $u \in U^H$ we have $Im(\varphi_u) = \{g.u \mid gH \in G/H\} = \mathcal{O}_u$. Consider now the exact sheaf Set-diagram

$$\mathscr{F}(U^H) \xhookrightarrow{\varphi} \prod_u F(G/H) \underset{(2)}{\overset{(1)}{\rightrightarrows}} \prod_{(u,v)} \mathscr{F}(G/H \times_U G/H)$$

where u varies in U^H and (u, v) in $(U^H)^2$. By the same kind of reasoning explained in the proof of Thm 2.23 we get an set-iso

$$\mathscr{F}(U^H) \cong \mathrm{Hom}_{\mathscr{C}G}(U^H, \mathscr{F}(G/H)) \cong \mathrm{Hom}_{\mathscr{C}G/H}(U^H, \mathscr{F}(G/H))$$

where the last iso follows from the fact that the G-structures on U^H and $\mathscr{F}(G/H)$ coincide with their G/H-structures in this case. Resuming what we found till now we have that for every continuous G-set U it holds $\mathscr{F}(U) \cong \varprojlim_H \mathscr{F}(U^H)$ and for every $H \in \mathcal{H}$ we have that

$$\mathscr{F}(U^H) \cong \mathrm{Hom}_{\mathscr{C}G/H}(U^H, \mathscr{F}(G/H)).$$

What we want is to prove that

$$\mathscr{F}(U) \cong \mathrm{Hom}_{\mathscr{C}G}(U, \varinjlim_H \mathscr{F}(G/H)).$$

In order to achieve this we need one more intermediate "techinical" result, namely the set-iso

$$\mathrm{Hom}_{\mathscr{C}G/H}(U^H, \mathscr{F}(G/H)) \cong \mathrm{Hom}_{\mathscr{C}G}(U^H, \varinjlim_H \mathscr{F}(G/H)).$$

To this aim let's consider the one-map G-covering given by the projection $G/K \xrightarrow{\pi_H} G/H$ defined for $K \succeq H$, i.e. $K \subseteq H$. From the associated exact Set-diagram

$$\mathscr{F}(G/H) \hookrightarrow \mathscr{F}(G/K) \rightrightarrows \mathscr{F}(G/K)$$

we deduce, once more by the same "trick" used in the proof of Thm 2.23, that

$$\begin{aligned}\mathscr{F}(G/H) \cong& \mathrm{Hom}_{\mathscr{C}G}((G/K)/(H/K), \mathscr{F}(G/K))\\ \cong& \mathrm{Hom}_{\mathscr{C}G}(G/H, \mathscr{F}(G/K))\\ \cong& \mathrm{Hom}_{\mathscr{C}G/K}(G/H, \mathscr{F}(G/K))\end{aligned}$$

where the last iso follows again from the fact that in this case the G-structures on G/H and $\mathscr{F}(G/H)$ coincide with their G/K-structures. Now by the same kind of iso that we used in Step 1 we have that

$$\begin{aligned}\mathrm{Hom}_{\mathscr{C}G/K}((G/H), \mathscr{F}(G/K)) \cong& \mathrm{Hom}_{\mathscr{C}G/K}((G/K)/(H/K), \mathscr{F}(G/K))\\ \cong& \mathscr{F}(G/K)^{H/K}\end{aligned}$$

and so $\mathscr{F}(G/H) \cong \mathscr{F}(G/K)^{H/K}$. From this we deduce that the natural map $\mathscr{F}(G/H) \to \varinjlim_{H'} \mathscr{F}(G/H'))$ induces the iso

$$\mathscr{F}(G/H) \cong (\varinjlim_{H'} \mathscr{F}(G/H'))^H.$$

To get our result we must now observe that there is an adjunction between the functors

$$\mathscr{C}G\text{-}Set \underset{(-)^H}{\overset{G(-)}{\rightleftarrows}} \mathscr{C}G/H\text{-}Set$$

where ${}_G(-)$ is the "pullback functor" built along the projection $G \twoheadrightarrow G/H$; it is defined by setting $U \mapsto {}_GU$ which denotes the set U endowed with its natural G-structure. In the opposite direction acts the functor $(-)^H$ which is just the "fixed-points functor". (See for instance [GrMay, Lemma 1.1, p.4].). This implies that for every $Z \in \mathscr{C}G$-*Set* and every $U \in \mathscr{C}G/H$-*Set* we have a set-iso

$$\mathrm{Hom}_{\mathscr{C}G}(U, Z) \cong \mathrm{Hom}_{\mathscr{C}G/H}(U, Z^H).$$

Applying it to the $\mathscr{C}G$-set $Z := \varinjlim_H \mathscr{F}(G/H))$ we get

$$\begin{aligned}\mathrm{Hom}_{\mathscr{C}G/H}(U^H, \mathscr{F}(G/H)) \cong & \mathrm{Hom}_{\mathscr{C}G/H}(U^H, (\varinjlim_{H'} \mathscr{F}(G/H')))^H) \\ \cong & \mathrm{Hom}_{\mathscr{C}G}(U^H, \varinjlim_{H'} \mathscr{F}(G/H')).\end{aligned}$$

Finally we can write natural isos

$$\begin{aligned}\mathscr{F}(U) &\cong \varprojlim_H \mathscr{F}(U^H) \\ &\cong \varprojlim_H \mathrm{Hom}_{\mathscr{C}G/H}(U^H, \mathscr{F}(G/H)) \\ &\cong \varprojlim_H \mathrm{Hom}_{\mathscr{C}G}(U^H, \varinjlim_{H'} \mathscr{F}(G/H')) \\ &\cong \mathrm{Hom}_{\mathscr{C}G}(\varinjlim_H U^H, \varinjlim_{H'} \mathscr{F}(G/H')) \\ &\cong \mathrm{Hom}_{\mathscr{C}G}(U, \varinjlim_{H'} \mathscr{F}(G/H')).\end{aligned}$$

which are functorial both in $U \in \mathscr{C}G$-*Set* and in $\mathscr{F} \in Sh(\mathcal{T}_G^{\mathscr{C}})$. Hence we won. □

Also in this case we have an analogous equivalence which relates continuous G-modules with sheaves of abelian groups on the continuous G-site, given by the "same" functors (of the previous theorem):

Corollary 2.32. *Let $G \in Grp$. We have the following equivalence (of categories)*

$$\mathscr{C}G\text{-}Mod \xrightarrow{\sim} Ab(\mathcal{T}_G^{\mathscr{C}}).$$

Chapter 3

Étale cohomology

"[...] in a world that he did not create, but he will go through it as if it was his own making [...]."

— Rose A.
(*live at The Ritz New York - 1988*)

3.1 $H^q_{ét}(X, \mathscr{F})$

In this section we will deal with the *étale topology on a scheme*.

Notation 3.1. By implication 1) in Prop. 1.44 we can consider the category $\acute{E}t/S$ of the étale schemes over $S \in Sch$. Morphisms in this category can be seen as commuting triangles

$$\begin{array}{ccc} X & \longrightarrow & Y \\ & {\scriptstyle ét}\searrow \quad \swarrow{\scriptstyle ét} & \\ & S & \end{array}.$$

(By implication 2) in Prop. 1.44, it follows that the horizontal arrow is étale too.)

Remark 3.2. By the last implication in Prop. 1.44 applied to the case $X' = Y' = S$, we have that $\acute{E}t/S$ has (finite) fiber products (and $id_S : S \to S$ is its terminal object).

Definition 3.3. A family of morphisms $\{\varphi_i : X_i \to X\}_{i \in I}$ in Sch is said to be **jointly surjective**, if $X = \bigcup_{i \in I} \varphi_i(X_i)$.

Remark 3.4. Given $S \in Sch$, the collection of all the onto families of étale S-morphisms satisfies the covering-axioms.

Definition 3.5. Let $S \in Sch$. We define the **étale site** to be the pair $S_{ét} := (Ét/S, Cov(Ét/S))$, where $Cov(Ét/S)$ is defined to be the collection of all the onto families of morphisms in $Ét/S$. (By the above Rmk, $S_{ét}$ is indeed a site.).

Remark 3.6. The category $Sh(S_{ét})$ is called the **étale topos** of S.

Proposition 3.7. *Given $T = T_{\mathcal{C}}$ a site:*

- *$PAb(T)$ and $Ab(T)$ are abelian categories;*
- *$PAb(T)$ and $Ab(T)$ have enough injectives;*
- *$\mathscr{F} \to \mathscr{G} \to \mathscr{H}$ is exact in $PAb(T)$ iff $\mathscr{F}(U) \to \mathscr{G}U) \to \mathscr{H}(U)$ is exact in Ab, $\forall U \in \mathcal{C}$.*

Proof. See for instance [Tam, Prop. 2.1.1, p. 31] □

Remark 3.8. Given $U \in \mathcal{C}$, the evaluation functor $\Gamma_U : PAb(T) \to Ab$, sending $\mathscr{F} \mapsto \mathscr{F}(U)$ is exact, by the second point of Prop. 3.7. Consider the composition of the two functors

$$Ab(T) \xrightarrow{i} PAb(T) \xrightarrow{\Gamma_U} Ab.$$

Since Γ_U is exact and the inclusion functor i is left exact (for a proof of these two facts see for instance [Tam, Thm. 3.2.1, p. 50]), then the composite of the two, denoted again with Γ_U, is left exact.

Definition 3.9. Let $U \in \mathcal{C}$ and let $\mathscr{F} \in Ab(T)$. By Prop. 3.7 and Rmk 3.8, we can right-derive the functor $\Gamma_U : Ab(T) \to Ab$. So for every $q \geq 0$ we define the **q-th cohomology group of** U **with values in** F to be

$$H^q(U, \mathscr{F}) := (R^q\Gamma_U)(\mathscr{F}) = H^q(\Gamma_U(\mathcal{I}^\bullet))$$

where $0 \to \mathscr{F} \to \mathcal{I}^\bullet$ denotes an injective resolution of $\mathscr{F}$ in $Ab(T)$.

Remark 3.10. The case when $T = S_{ét}$, $U = X \in Ét/S$ and $\mathscr{F} \in Ab(S_{ét})$ gives us the **q-th étale cohomology group** $H^q_{ét}(X, \mathscr{F})$ **of** X **with values in** $\mathscr{F}$.

3.2 The case $S = Spec(k)$

We will now consider the case when $k \in Fld$ and $S = Spec(k)$. Let k_s be a separable closure of k and set $G_k = \mathcal{G}al(k_s/k)$ (which is a profinite group).

Notation 3.11. For brevity we set $Ét/k := Ét/Spec(k)$ and $Sch/k := Sch/Spec(k)$.

Definition 3.12. Let $S \in Sch$. We define a **geometric point** of S to be a morphism $Spec(\Omega) \to S$, where $\Omega \in Fld$ is separably closed. Alternatively it can be defined as a point $s \in S$ together with an injection $k(s) \hookrightarrow \Omega$, where $\Omega \in Fld$ is separably closed.

Notation 3.13. Let $X \in Sch/k$. We denote with $X(k_s) := X_{Spec(k)}(k_s)$ the set of all k_s**-valued points** of X, i.e. the set of all the morphisms $(Spec(k_s) \to X) \in Sch/k$. This is the set of all geometric points of X.

Notation 3.14. Given an k-automorphism $\varphi \in \mathcal{G}al(k_s/k)$, we will denote with $\overline{\varphi} = Spec(\varphi) : Spec(k_s) \to Spec(k_s)$ the associated morphism of schemes.

We define an action $G_k \curvearrowright X(k_s)$ by setting $(\overline{\varphi}, f) \mapsto f \circ \overline{\varphi}$, for every $\varphi \in G_k$ and every $f \in X(k_s)$. On the level of the topological spaces the elements of G_k just send a point to itself; the core of the action is on the level of the structure sheaves.

Remark 3.15. Let's now take an open subgroup $H \leq G_k$. It holds then the identification $X(k_s)^H = X(k_s^H)$. Let's consider the affine case $X = Spec(A)$, where $A \in k$-Alg. We have that

$$\begin{aligned} X(k_s)^H &= \{f : Spec(k_s) \to Spec(A) \,|\, f \circ \overline{\varphi} = f, \forall \varphi \in H\} \\ &= \{f^\# : A \to k_s \,|\, \varphi \circ f^\# = f^\#, \forall \varphi \in H\} \\ &= \{f^\# : A \to k_s \,|\, Im(f^\#) \subseteq k_s^H\} \\ &= \{f : Spec(k_s^H) \to Spec(A)\} \\ &= X(k_s^H), \end{aligned}$$

where $f^\#$ denotes the sheaf morphism $\mathcal{O}_{Spec(A)} \to f_* \mathcal{O}_{Spec(k_s)}$ evaluated on the whole space $X = Spec(A)$ (i.e. we are using the duality $CRing^{op} \simeq AffSch$). The morphisms of schemes considered in the above equalities are all over $Spec(k)$ and the morphisms of rings are all over k.

Remark 3.16. Let $H \leq_{open} G_k$. Since G_k is profinite this means H is a closed subgroup of G_k with $[G_k : H] < \infty$. By Galois Theory this implies $[k_s^H : k] < \infty$. Moreover for every such H we have that $X(k_s^H) \subseteq X(k_s)$ by pre-composing arrows with the canonical morphism $Spec(k_s) \to Spec(k_s^H)$. Actually it holds the decomposition

$$X(k_s) = \bigcup_{H \subseteq_{open} G_k} X(k_s^H).$$

Indeed consider a morphism $f : Spec(k_s) \to X\ (\to Spec(k))$. This f locates a point $x \in X$ and we get a tower of fields $k \to \kappa(x) \to k_s$. By Galois Theory there exists a closed subgroup $H \subseteq G_k$ such that $\kappa(x) = k_s^H$. If we now assume that X is locally of finite type we have that the extension

$\kappa(x)/k$ is finite, since it is algebraic (see for example [G.-W., Prop. 3.33, p. 79]). This means that H is closed and of finite index in G_k, hence open. Let now $U = Spec(A) \subseteq X$ an open affine containing x. Denoting with $\mathfrak{p}$ the ideal in A corresponding to x we have canonical composition of ring morphisms $A \to A_{\mathfrak{p}} \to A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} = \kappa(x)$ from which we get a scheme morphism $Spec(\kappa(x)) = Spec(k_s^H) \to X$ that composed with the canonical arrow $Spec(k_s) \to Spec(k_s^H)$ gives us f. Therefore $f \in X(k_s^H) = X(k_s)^H$, for some $H \subseteq_{open} G_k$. Hence $G_k \curvearrowright X(k_s)$ continuously, since the directed system of normal open subgroups of a profinite group is cofinal to the one of its open subgroups.

We will now state and prove a theorem which will turn out to be the second main ingredient in the proof of the main theorem of this thesis.

Theorem 3.17. *Let $k \in Fld$. The functors*

$$\acute{E}t/k \longrightarrow \mathscr{C}G_k\text{-}Set$$

$$X/k \xmapsto{\ \phi\ } X(k_s)$$

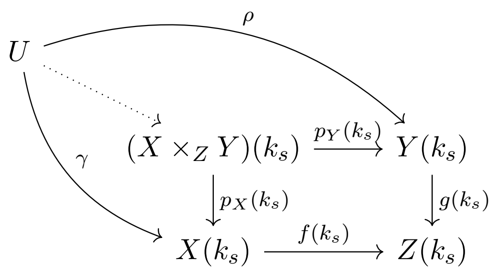
$$\coprod_{i\in I} Spec(k_s^{H_{u_i}}) \xleftarrow{\ \psi\ } U$$

yield an equivalence of categories. Here $|I|$ is the cardinality of the collection of all the G_k-orbits in U, u_i is a chosen point in the i-th orbit and H_{u_i} denotes the stabilizer $St_{G_k}(u_i)$. This equivalence induces an isomorphism

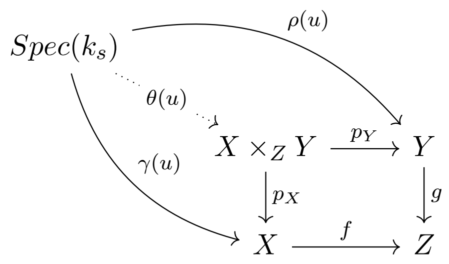
$$Spec(k)_{\acute{e}t} \xrightarrow{\ \sim\ } \mathcal{T}^{\mathscr{C}}_{G_k}$$

of sites.

Proof. **First step**: we have to check that ψ is well-defined (on the objects). So let $U \in \mathscr{C}G_k\text{-}Set$ and let's write $U = \coprod_{i\in I} \mathcal{O}_{u_i}$ as the coproduct of its G_k-orbits. Now as G_k is profinite we have that $H_{u_i} = St_{G_k}(u_i) \leq_{open} G_k$ and so it is closed and finite. This implies that $k_s^{H_{u_i}}/k$ is a finite (and separable) field extension, hence the morphism $\varphi_i : Spec(k) \to Spec(k_s^{H_{u_i}})$ is unramified. As it is also clearly flat, we have that φ_i is étale. Since by Cor. 1.43 the category $\acute{E}t/k$ has arbitrary coproducts we have that $\psi(U) = \coprod_{i\in I} \varphi_i$ is étale over k. Let's show moreover that the map ψ does not depend on the choice of the points in the orbits. Choose an index $i \in I$ and let u and v be two different points in the G_k-orbit $\mathcal{O}$. Then there exist $g \in G_k$ such that $g.u = v$. From this it follows that $H_v = St_{G_k}(v) = St_{G_k}(u)^{g^{-1}} = H_u^{g^{-1}}$. Via some elementary computations we can also see that $k_s^{H_u^{g^{-1}}} = g(k_s^{H_u})$. So in practice we found that a choice of a different point in $\mathcal{O}$ equals substituting the field extension $k_s^{H_u}$ with one of its conjugates (under G_k). Clearly all remains the same passing to spectra and so we are done. **Second step**: we prove that the above functors are mutually quasi-inverse. *First direction*: $\psi \circ \phi \simeq id_{\acute{E}t/k}$. Let X be an étale scheme over k. By Prop. 1.41 we have that X can be written as $\bigsqcup_{i \in I} Spec(k_i)$, where k_i/k is a finite and separable field extension of k, for every $i \in I$. Let's fix a separable closure k_s of k so that we can see all in towers, namely $k \subseteq k_i \subseteq k_s$. Now $\phi(X) = X(k_s) = (\bigsqcup_{i \in I} Spec(k_i))(k_s) = \bigsqcup_{i \in I} Spec(k_i)(k_s)$. The essential fact to observe here is that "surprisingly" the set of indices I parametrizes also the orbits of the G_k-set $\phi(X)$. In fact for every $i \in I$ we have that $Spec(k_i)(k_s) \cong \mathrm{Hom}_k(k_i, k_s)$ and G_k acts transitively on this set on the right permuting the k-immersions of k_i in k_s. So we have that $\phi(X)$ is sent by ψ to the étale k-scheme $\bigsqcup_{i \in I} Spec(k_s^{H_{f_i}})$, where f_i is a chosen point in $Spec(k_i)(k_s)$ for every $i \in I$. Now $H_{f_i} = St_{G_k}(f_i)$ is nothing but the stabilizer of k_i for the action of G_k on k_s, i.e. $H_{f_i} = \mathcal{G}al(k_s/k_i)$. So $k_s^{H_{f_i}} = k_s^{\mathcal{G}al(k_s/k_i)} = k_i$ and we are done. *Second direction*: $\phi \circ \psi \simeq id_{\mathscr{C}G_k\text{-}Set}$. Let U be a continuous G_k-set. First we apply ψ obtaining the étale k-scheme $\bigsqcup_{i \in I} Spec(k_s^{H_{u_i}})$. Then we apply ϕ passing to its k_s-valued points. Since ϕ clearly commutes with arbitrary coproducts we can restrict our attention to the G_k-set $U_i := Spec(k_s^{H_{u_i}})(k_s)$. Now for every choice of an index $i \in I$ we have that $U_i \cong \mathrm{Hom}_k(k_s^{H_{u_i}}, k_s)$ and this last set is G_k-isomorphic to the G_k-set $[G_k : H_{u_i}]$ of the cosets of H_{u_i} in G_k. Finally $[G_k : H_{u_i}]$ is G_k-isomorphic to the orbit $\mathcal{O}_{u_i}$ of u_i under G_k. So for every i we have that $U_i \cong \mathcal{O}_{u_i}$ and taking the disjoint union we recover U. **Third step**: the equivalence ϕ induces a morphism of sites. Given $(X \xrightarrow{f} Z), (Y \xrightarrow{g} Z) \in \acute{E}t/k$, we have that $(X \times_Z Y)(k_s) \cong X(k_s) \times_{Z(k_s)} Y(k_s)$. Consider the $(\mathscr{C}G_k\text{-}Set)$ - diagram



where the two squares commute. We want to prove that there exist a "dotted map" such that "triangles" commute. Let $u \in U$. Then we have that $g(k_s)(\rho(u)) = f(k_s)(\gamma(u))$ i.e. $g \circ \rho(u) = f \circ \gamma(u)$ i.e. in the $\acute{E}t/k$ - diagram



the outer square commutes and so there exists a unique mediating arrow $\theta(u)$ that makes the two "triangles" commute. Now if we define the dotted arrow $\theta : U \to (X \times_Z Y)(k_s)$ in the first diagram by setting $u \mapsto \theta(u)$ we are done since we have that

$$\begin{cases} \gamma(u) = p_X \circ \theta(u) \\ \rho(u) = p_Y \circ \theta(u) \end{cases} \rightsquigarrow \begin{cases} \gamma(u) = p_X(k_s)(\theta(u)) \\ \rho(u) = p_Y(k_s)(\theta(u)) \end{cases} \rightsquigarrow \begin{cases} \gamma = p_X \circ \theta \\ \rho = p_Y \circ \theta \end{cases}$$

where in the first two brackets we mean "for every $u \in U$". So the object $(X \times_Z Y)(k_s)$ satisfies the universal property of the fiber product, whence our iso. From the fact that the functor ϕ preserves fiber products it follows immediately that the 2^{nd} property of morphisms of sites is verified. Notice now that given an arrow $(\varphi : Y \to X) \in \acute{E}t/k$ its image via our functor ϕ is nothing but the post-composition $\varphi \circ -$. So now given $\{X_i \xrightarrow{\varphi_i} X\}_{i\in I} \in Cov(\acute{E}t/k)$, i.e. $X = \bigcup_{i\in I} \varphi_i(X_i)$, we have that

$$X(k_s) = (\textstyle\bigcup_{i\in I} \varphi_i(X_i))(k_s) = \bigcup_{i\in I}(\varphi_i(X_i))(k_s) = \bigcup_{i\in I} \varphi_i(k_s)(X_i(k_s))$$

where the second equality follows from the equality

$$(\varphi_i(X_i))(k_s) = \varphi_i(k)(X_i(k_s))$$

which holds for every $i \in I$. This is just because the data of a morphism $Spec(k) \to \varphi_i(X_i)$ is equivalent to the data of a morphism $Spec(k) \to X_i$ post-composed with the i-th arrow of our covering. So ϕ maps coverings to coverings. **Fourth step**: this is indeed an site isomorphism. Let $\{f_i : U_i \to U\}_{i\in I} \in Cov(\mathscr{C}G_k\text{-}Set)$. By essential surjectivity of ϕ, we have that $U_i = X_i(k_s)$ and $U = X(k_s)$, for some X and X_i's in $\acute{E}t/k$. By fully faithfulness of ϕ we have moreover that for every $i \in I$ there exist (a unique) $\varphi_i \in \mathrm{Hom}_{\acute{E}t/k}(X_i, X)$ such that $f_i = \varphi_i(k_s)$. So our covering is nothing but the covering $\{\varphi_i(k_s) : X_i(k_s) \to X(k_s)\}_{i\in I}$. The covering condition has then the form $X(k_s) = \bigcup_{i\in I} \varphi_i(k)(X_i(k_s))$. By the same kind of reasoning shown above we have that

$$X(k_s) = \textstyle\bigcup_{i\in I} \varphi_i(k_s)(X_i(k_s)) = \bigcup_{i\in I}(\varphi_i(X_i))(k_s) = (\bigcup_{i\in I} \varphi_i(X_i))(k_s).$$

From this we can conclude that $X = \bigcup_{i\in I} \varphi_i(X_i)$. In fact take any étale k-map $f : Y \to X$ such that $f(k_s) : Y(k_s) \to X(k_s)$ is onto. Well now if we take a point $x \in X$ we have that the map $k \to \kappa(x)$ is finite and separable and so the composition $Spec(k_s) \to Spec(\kappa(x)) \to X$ gives us a geometric point x' of X associated to x. Since $f(k_s)$ is onto there is a geometric point $y' : Spec(k_s) \to Y$ lying over x'. The corresponding point $y \in Y$ lies over x. $\square$

Remark 3.18. Before stating the previous theorem giving explicitly the two functors in both directions we were following another path. The idea was to first just "abstractly" prove the existence of a left adjoint to ϕ and then show that indeed the associated adjoint morphisms (below defined) are indeed isomorphisms. While pursuing the second task we recognized that it would have been necessary to anyway give an "explicit way back" from continuous G_k-sets to étale k-schemes. So we abandoned the first approach and removed it from the body of the proof as otherwise (possibly) redundant. We insert it anyway here below, since it seems a nice criterion that can be used in order to prove "existence of (left) adjoints". First we will explain how the "method" works, then we will apply to our case above. **Method**: given a functor $F : \mathcal{C} \to \mathcal{D}$, a *left adjoint* to F is a functor $G : \mathcal{D} \to \mathcal{C}$ such that for every $C \in \mathcal{C}$ and $D \in \mathcal{D}$ we have an isomorphism

$$\mathrm{Hom}_{\mathcal{D}}(D, F(C)) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(G(D), C)$$

functorial in C and in D. (The functor G is determined up to a unique iso.) The adjointness condition induces a pair of canonical natural transformations

$$id_{\mathcal{D}} \longrightarrow F \circ G$$

$$G \circ F \longrightarrow id_{\mathcal{C}}$$

called *adjoint morphisms*. In order to prove the existence of a left adjoint G to F it suffices to show that the functor $\mathcal{C} \to Set$, defined by $C \mapsto \mathrm{Hom}_{\mathcal{D}}(D, F(C))$ is co-representable, i.e. there exist an iso

$$\mathrm{Hom}_{\mathcal{D}}(D, F(C)) \cong \mathrm{Hom}_{\mathcal{C}}(G(D), C)$$

functorial in C. In fact if we assume this last condition to hold, we have that given $D \in \mathcal{D}$ there is a canonical arrow $\phi_D : D \to F(G(D))$. Moreover given an arrow $(f : D \to E) \in \mathcal{D}$ we get an arrow $\phi_E \circ f : D \to F(G(E))$, which gives us a last arrow $G(f) : G(D) \to G(E)$, by the above iso. So finally we got a functor $G : \mathcal{D} \to \mathcal{C}$ which, by construction, is such that the iso $\mathrm{Hom}_{\mathcal{D}}(D, F(C)) \cong \mathrm{Hom}_{\mathcal{C}}(G(D), C)$ holds and is functorial also in D. Hence $G \dashv F$. **Case of Thm 3.17**: the functor ϕ admits a left adjoint ψ. Thanks to Rmk 3.18 in order to do this it suffices to show that the functor $\acute{E}t/k \to Set$ defined by

$$X/k \longmapsto \mathrm{Hom}_{G_k}(U, X(k_s))$$

is co-representable for every $U \in \mathscr{C}G_k\text{-}Set$. Now let's observe a few facts:

1. every $U \in \mathscr{C}G_k\text{-}Set$ can be decomposed as the disjoint union of its G_k-orbits, namely $U = \bigsqcup_{u \in U} \mathcal{O}_u$;

2. every orbit $\mathcal{O}_u$ is isomorphic, as continuous G_k-set, to some quotient space G_k/H, for some $H \leq_{open} G_k$;

3. the category $\acute{E}t/k$ admits arbitrary coproducts.

We can then restrict to show the co-representability of the functor $X \mapsto \mathrm{Hom}_{G_k}(G_k/H, X(k_s))$. In fact, once proven this, i.e. once we found a certain scheme $\psi(G_k/H) \in \acute{E}t/k$ such that

$$\mathrm{Hom}_{G_k}(G_k/H, X(k_s)) \cong \mathrm{Hom}_{\acute{E}t/k}(\psi(G_k/H), X)$$

we have that given any continuous G_k-set U it holds

$$\begin{aligned}\mathrm{Hom}_{G_k}(U, X(k_s)) &\cong \mathrm{Hom}_{G_k}(\bigsqcup_{u \in U} \mathcal{O}_u, X(k_s)) \\ &\cong \mathrm{Hom}_{G_k}(\bigsqcup_{u \in U} G_k/H, X(k_s)) \\ &\cong \prod_{u \in U} \mathrm{Hom}_{G_k}(G_k/H, X(k_s)) \\ &\cong \prod_{u \in U} \mathrm{Hom}_{\acute{E}t/k}(\psi(G_k/H), X) \\ &\cong \mathrm{Hom}_{\acute{E}t/k}(\bigsqcup_{u \in U} \psi(G_k/H), X),\end{aligned}$$

hence, by point (3) above, we would be done. So now consider the field k_s^H, for some $H \leq_{open} G_k$. We have that $Spec(k_s^H)/k \in \acute{E}t/k$. In fact $Spec(k_s^H) \to Spec(k)$ being étale equals $k \hookrightarrow k_s^H$ being so, i.e. being flat and unramified. But being H open in G_k, the extension k_s^H/k is finite and obviously separable, hence unramified. Moreover the tensor functor $k_s^H \otimes_k -$ is exact (barely because $k \in Fld$). Now by Rmk 3.15 we have that

$$\mathrm{Hom}_{G_k}(G_k/H, X(k_s)) \cong X(k_s)^H \cong X(k_s^H) \cong \mathrm{Hom}_{\acute{E}t/k}(Spec(k_s^H), X)$$

and these isos are functorial in X. So our functor is co-represented.

Remark 3.19. Observe that the above result gives us a site isomorphism $Spec(k)_{\acute{e}t} \cong \mathcal{T}_{\acute{E}t/k}$. This kind of result does not hold when considering the category $\acute{E}t/S$, where $S \in Sch$ is an arbitrary scheme.

Now we have finally all the tools to state and prove the main theorem of this thesis. Before doing this we observe an important fact that already sheds a light on the interrelation between Étale and Galois Cohomology.

Remark 3.20. Let G be a profinite group. By Cor 2.32 we know that there is an categorical equivalence $\mathscr{C}G\text{-}Mod \simeq Ab(\mathcal{T}_G^{\mathscr{C}})$ given by the mutually quasi-inverse functors $A \mapsto \mathscr{H}om_{\mathscr{C}G}(-, A)$ and $\mathscr{F} \mapsto \varinjlim_{H \trianglelefteq_{open} G} \mathscr{F}(G/H)$. Let's denote with P the set with one element, endowed with its unique (continuous) left G-structure. Observe that given $A \in \mathscr{C}G\text{-}Mod$ we have that

$$\Gamma_P(\mathscr{H}om_{\mathscr{C}G}(-, A)) = \mathrm{Hom}_{\mathscr{C}G}(P, A) \cong A^G$$

so the functors $\Gamma_P : Ab(\mathcal{T}_G^{\mathscr{C}}) \to Ab$ and $-^G : \mathscr{C}G\text{-}Mod \to Ab$ get identified. So, since the q-th étale cohomology group and the q-th Galois cohomology group are respectively obtained right-deriving the first and the second functor above, we have that

$$H^q(P, \mathscr{H}om_{\mathscr{C}G}(-, A)) \cong H^q(G, A).$$

Theorem 3.21. *Let $k \in Fld$. Denote with Σ the set of all field inter-extensions $k \subseteq l \subseteq k_s$ such that $[l : k] < \infty$. The functor*

$$\begin{array}{ccc} Ab(Spec(k)_{\acute{e}t}) & \xrightarrow{\quad\phi\quad} & \mathscr{C}G_k\text{-}Mod \\ \mathscr{F} & \longmapsto & \varinjlim_{l \in \Sigma} \mathscr{F}(Spec(l)) \end{array}$$

yields an equivalence of categories. For every $\mathscr{F} \in Ab(Spec(k)_{\acute{e}t})$ and every $q \geq 0$ this induces a (∂-functorial) isomorphism

$$H^q_{\acute{e}t}(Spec(k), \mathscr{F}) \cong H^q(G_k, \varinjlim_{l \in \Sigma} \mathscr{F}(Spec(l)))$$

in Ab, (where the l.h.s. denotes the étale cohomology and the r.h.s. denotes the Galois cohomology).

Proof. **Equivalence**. Let's first observe the following facts:

- in the above statement the notation $Spec(l)$ stands for $Spec(l)/k$;
- the set Σ is a directed system in Fld;
- $\varinjlim_{l \in \Sigma} \mathscr{F}(Spec(l)) \in Ab$ and there is a natural continuous action of G_k on it (by the same reasoning used in the proof of last theorem of chapter II).

So ϕ is well-defined (on the objects). The equivalence follows immediately from Cor 2.32 and Thm 3.17. Indeed by the first we have the categorical equivalence $\mathscr{C}G_k\text{-}Mod \simeq Ab(\mathcal{T}_{G_k}^{\mathscr{C}})$ and by the second we have the site iso $\mathcal{T}_{G_k}^{\mathscr{C}} \simeq Spec(k)_{\acute{e}t}$. So we conclude that $Ab(Spec(k)_{\acute{e}t}) \simeq \mathscr{C}G_k\text{-}Mod$.
Cohomology. Let's first observe that the above direct limit can be equivalently taken over the directed subsystem $\Sigma' := \{l \in \Sigma \mid l/k \text{ normal}\} \subseteq \Sigma$.

By Cor 2.32 we have that every sheaf $\mathscr{F}$ on $\mathcal{T}_{G_k}^{\mathscr{C}}$ corresponds to a unique (continuous) G_k-module A (up to iso), via the mutually quasi-inverse functors $A \mapsto Hom_{\mathscr{C}G_k}(-, A)$ and $\mathscr{F} \mapsto \varinjlim_{H \trianglelefteq_{open} G_k} \mathscr{F}(G_k/H)$. Let then $\mathscr{F}$ be an abelian sheaf on $Spec(k)_{ét}$. By Rmk 3.20 applied to the case when $P := Spec(k)$ we have that

$$\begin{aligned} H^q_{ét}(Spec(k), \mathscr{F}) &\cong H^q(P, \mathscr{F}') \\ &\cong H^q(P, \mathrm{Hom}_{\mathscr{C}G_k}(-, A)) \\ &\cong H^q(G_k, A) \\ &\cong H^q(G_k, \varinjlim_{H \trianglelefteq_{open} G_k} \mathscr{F}'(G_k/H) \\ &\cong^{(1)} H^q(G_k, \varinjlim_{l \in \Sigma'} \mathscr{F}'(Spec(l)(k_s))) \\ &\cong^{(2)} H^q(G_k, \varinjlim_{l \in \Sigma'} \mathscr{F}(Spec(l))) \end{aligned}$$

where

- $\mathscr{F}'$ denotes the sheaf that corresponds (up to iso) to the sheaf $\mathscr{F}$ via the site iso $Ab(Spec(k)_{ét}) \simeq Ab(\mathcal{T}_{G_k}^{\mathscr{C}})$;
- iso (1) follows from the G_k-isos

$$\begin{aligned} G_k/H &\cong \mathrm{Hom}_k(k_s^H, k_s) \\ &\cong \mathrm{Hom}_{Ét/k}(Spec(k_s), Spec(k_s^H)) \\ &= Spec(k_s^H)(k_s); \end{aligned}$$

- iso (2) follows from the categorical equivalence given in Thm 3.17.

□

This theorem provides us an alternative way to compute the classical group cohomology of continuous G_k-modules and highlights the fact that Galois Cohomology theory can be somehow "embedded" in the étale one as the Étale Cohomology over $Spec(k)$.

Chapter 4

Classical theory

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In this chapter we will give a summary of the main definitions, results and some applications of the classical Galois Cohomology theory. We will mainly follow [Ser], [T.] and [C.-F.]. Let's start with some notions about *group cohomology*.

4.1 Group Cohomology

Notation 4.1. Let G be profinite. We will denote with $\mathscr{C}G$-Mod the category of the discrete abelian groups on which G acts continuously. This category is abelian (in particular we can take kers and cokers) and is full inside G-Mod. As in the case of continuous G-sets we have the two equivalent conditions:

- $\{St_G(a) \mid a \in A\} \subseteq Op_G$
- $A = \bigcup_{H \trianglelefteq_{\text{open}} G} A^H$.

Any $A \in \mathscr{C}G$-Mod will be called a *discrete G-module.*

Remark 4.2. "A is a G-module", in this context, means that there is a map $G \times A \to A$ such that it is continuous, it is linear in the second argument, 1_G acts trivially and $g \circ h$ acts as "g acts after h".

Now let's pass to define cohomology of our profinite group G with values in $\mathscr{C}G$-Mod. There are essentially two ways to define it. **First way**: denote by $C^n(G, A)$ the "set of all continuous maps $G^n \to A$ ", (as A is discrete we are just dealing with the locally constant maps here). We define the coboundary map

$$d^n : C^n(G, A) \to C^{n+1}(G, A)$$

setting

$$\begin{aligned}(df)(g_1, \ldots, g_{n+1}) &:= g_1 f(g_2, \ldots, g_{n+1}) \\ &+ \sum_{i=1}^{n} (-1)^i f(g_1, \ldots, g_i g_{i+1}, \ldots, g_{n+1}) \\ &+ (-1)^{n+1} f(g_1, \ldots, g_n)\end{aligned}$$

for every $f \in C^n(G, A)$. We obtain then a complex $(C^\bullet(G, A), d^\bullet)$. The cohomology of this complex, defined as $H^q(G, A) := ker(d^q)/Im(d^{q-1})$ is what we will call "the q-th cohomology group of G with values in A". (When $|G| < \infty$, we recover the usual cohomology of finite groups.) The following result points out the way profinite cohomology can be seen in terms of the finite one:

Proposition 4.3. *Let $\{G_i\}_{i \in I}$ be an inverse system of profinite groups and let $\{A_i\}_{i \in I}$ be a direct system of abelian groups, where $A_i \in \mathscr{C}G$-Mod for every $i \in I$. Assume that the morphisms in these two systems are compatible in the sense that the diagram*

$$\begin{array}{ccc} G_i \times A_i & \longrightarrow & A_i \\ \downarrow & & \downarrow \\ G_j \times A_j & \longrightarrow & A_j \end{array}$$

commutes. Then we have that

$$H^q(\varprojlim_i G_i, \varinjlim_i A_i) = \varinjlim_i H^q(G_i, A_i),$$

for each $q \geq 0$.

Proof. See for example [Ser, Prop. 8, p. 11]. □

Corollary 4.4. *Given $A \in \mathscr{C}G$-Mod, with G profinite, it holds*

$$H^q(\varprojlim_H G/H, \varinjlim_H A^H) = \varinjlim_H H^q(G/H, A^H)$$

with H varying among the open normal subgroups of G.

Second way: we can define the q-th cohomology group of G with values in A, by setting

$$H^q(G, A) := \mathrm{Ext}^q_{\mathbb{Z}[G]}(\mathbb{Z}, A) = R^q(\mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, -))(A).$$

Now since we can naturally make $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, A)$ into a (continuous) G-module and it holds $\mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, A) = (\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, A))^G = A^G$ we have that we are indeed taking the right derived functors of the functor $-^G$. This construction makes sense since $\mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, -)$ is left exact and the category $\mathscr{C}G$-Mod is equivalent to the module category $\mathscr{C}\mathbb{Z}[G]$-Mod and so it has enough injectives.

Remark 4.5. The "low degree" cohomology groups can be interpreteted in some nice ways:

- $H^0(G, A) = A^G$ ($\rightsquigarrow$ it corresponds to "taking G-invariants in A");
- $H^1(G, A) =$ "classes of continuous crossed-homomorphisms $G \to A$";
- $H^2(G, A) =$ "classes of continuous cocycles (or *factor systems*) $G^2 \to A$".

If $G \curvearrowright A$ trivially, we have that $H^1(G, A) = \mathrm{Hom}_{Grp}(G, A)$.

Let's now introduce two very relevant maps relating cohomology groups. Consider a morphisms $(G' \xrightarrow{f} G) \in Grp$ and let $A \in \mathscr{C}G$-Mod and $A' \in \mathscr{C}G'$-Mod. Let $A \xrightarrow{h} A'$ be a map such that $h(f(g).a) = g.h(a)$, $\forall a \in A$ and $\forall g \in G'$, which means some sort of commutativity of the below diagram

$$\begin{array}{ccc} G \times A & \longrightarrow & A \\ \uparrow \quad \downarrow & & \downarrow \\ G' \times A' & \longrightarrow & A'. \end{array}$$

Via h, we can consider A as a G'-module. Passing to cohomology, this compatible pair (f, h) defines a morphism of groups that we will denote with the same notation:

$$(f, h) : H^q(G, A) \to H^q(G', A').$$

The map here is so defined: given an element $[\sigma] \in H^q(G, A)$ represented by the cocycle $\sigma : G^q \to A$ we define $(f, h)([\sigma])$ to be the element of $H^q(G', A')$ represented by the cocycle $\sigma' : (G')^q \to A$, defined setting

$$(g'_1, \ldots, g'_q) \mapsto h(\sigma(f(g'_1), \ldots, f(g'_q))).$$

If we apply the above definition to the case when $G' = H \hookrightarrow G$ is a closed subgroup of G and $A = A'$ we get the well-known *restriction map*:

$$\mathrm{Res} : H^q(G, A) \to H^q(H, A),$$

for every $q \geq 0$. When we have an open subgroup $H \hookrightarrow G$ we can define the *corestriction map*

$$\mathrm{Cor} : H^q(H, A) \to H^q(G, A),$$

for every $q \geq 0$, (i.e. the "dual" map to Res). This second map is defined by a limit process starting from finite groups.

Proposition 4.6. *We have that Cor $\circ$ Res $= n$, where $n = [G : H]$.*

Proof. See for instance [R.-Z., Thm 6.7.3, p. 226]. □

4.2 Galois Cohomology

Galois Cohomology theory is nothing but a specialization of the "profinite group cohomology theory", above introduced, given by regarding the involved profinite groups as automorphism groups of Galois extensions of fields. Namely the usual approach consists in:

1. fixing some base field k;
2. taking some Galois extension l/k;
3. associating some discrete (abelian) group $M(l)$ to the group $\mathcal{G}al(l/k)$.

This "construction" leads us to consider the (well-defined) cohomology groups $H^q(\mathcal{G}al(l/k), M(l))$, for any $q \geq 0$, (only $q \in \{0, 1\}$ in the non-abelian case). However there is a more efficient way to introduce the Galois cohomology groups. Fix an object $k \in Fld$ and denote with Sep_k the category of separable field extensions of the base field k. Let's consider a functor $M : Sep_k \to Grp$. Assume M satisfies the following four conditions:

- l/k Galois extension $\rightsquigarrow \mathcal{G}al(l/k) \curvearrowright M(l)$;
- $M(l) = \varinjlim_{k_i} M(k_i)$, where k_i varies among the inter-extensions $k \subseteq k_i \subseteq l$ that are finite over k;
- $l \hookrightarrow l' \Rightarrow M(l) \hookrightarrow M(l')$;
- l'/l Galois extension $\Rightarrow M(l) = H^0(\mathcal{G}al(l'/l), M(l'))$.

Remark 4.7. It's enough to assume the first condition in order to have an action $\mathcal{G}al(l'/l) \curvearrowright M(l')$, anytime we meet a Galois extension l'/l. In fact if $\mathcal{G}al(l'/k) \curvearrowright M(l')$ then we can just consider this action restricted to the subgroup $\mathcal{G}al(l'/l) \leq \mathcal{G}al(l'/k)$.

Notation 4.8. The following notation is commonly used:

- $H^q(l/k, M) := H^q(\mathcal{G}al(l/k), M(l))$;
- $H^q(k, M) := H^q(k_s/k, M)$.

Example 4.9. The preminent example we will consider in terms of the étale cohomology will be the one associated to the extension k_s/k, (here k_s denotes a separable closure of k).

Example 4.10. The functor M can be defined enlarging the source category to Fld_k (i.e. we can drop out the assumptions of algebraicity and separability). The preminent example in this context is the one of group schemes over k. In fact given such a functor $M : Sch/k \to Grp$, we have that M verifies the three last conditions above and if we assume in addition the first condition we are done. Algebraic groups in particular form a class of group schemes (over k). Two very frequent examples of group schemes are

$$\mathbb{G}_a : (Sch/k)^{op} \longrightarrow Ab$$
$$X \longmapsto (\mathcal{O}_X(X), +)$$

and

$$\mathbb{G}_m : (Sch/k)^{op} \longrightarrow Ab$$
$$X \longmapsto (\mathcal{O}_X(X)^*, \cdot)$$

where $\mathcal{O}_X(X)^*$ denotes the group of units of the global-sections ring of the structural sheaf of the k-scheme X.

Here below we state two relevant results in Galois cohomology.

Proposition 4.11. *Let $k \in Fld$ and $l \in Fld_k$. Assume l/k is Galois. We have that:*

- $H^1(l/k, \mathbb{G}_m) = 0$ ("Hilbert 90")
- $H^q(l/k, \mathbb{G}_a) = 0, \quad \forall q \geq 1.$

In particular we have that $H^1(k, \mathbb{G}_m) = 0$ (particular case of "Hilbert 90").

Remark 4.12. For $q > 1$ it can be that $H^q(l/k, \mathbb{G}_m) \neq 0$. As an example consider the case $q = 2$: we have that $H^2(k, \mathbb{G}_m) \cong Br(k)$, where $Br(k)$ denotes the *Brauer group of the field k*. For $l \in Fld_k$ we have that $H^2(l/k, \mathbb{G}_m) \cong Br(l/k) \hookrightarrow Br(k) \cong H^2(k, \mathbb{G}_m)$.

Chapter 5

Hilbert 90. An étale view

> You cannot always wait for the perfect time.
> Sometimes you [must] dare to jump.
>
> (*anonymous*)

In this last chapter we present a application of Thm 3.21 to one of the main results in classical Galois Cohomology, namely *Hilbert Theorem 90*. We will first rephrase this theorem in terms of the étale cohomology and then outline a path that eventually brings to a proof of this result in the "étale setting".

First of all we will consider the classical formulation of Hilbert 90 (as stated in Prop. 4.11) for the case of the Galois extension k_s/k, where the notation is the same as in Thm 3.21. Namely the result in this case can be stated as

$$H^1(G_k, k_s^*) = 0.$$

with $G_k = \mathcal{G}al(k_s/k)$. Now by Thm 3.21 we have that for every $\mathscr{F} \in Ab(Spec(k)_{ét})$ there is isomorphism

$$H^q_{ét}(Spec(k), \mathscr{F}) \cong H^q(G_k, \varinjlim_{l \in \Sigma} \mathscr{F}(Spec(l)))$$

of abelian groups, where Σ is as in the statement of the theorem.

Remark 5.1. Observe that for every such $\mathscr{F}$ we can identify $\varinjlim_{l \in \Sigma} \mathscr{F}(Spec(l))$ with the stalk $\mathscr{F}_{\bar{s}}$ at the geometric point $\bar{s} = Spec(k_s) \to Spec(k)$ (for a proof see [Tam, p.116-118]).

Given $S \in Sch$ let's consider the étale abelian presheaf

$$\mathbb{G}_m : (\acute{E}t/S)^{op} \longrightarrow Ab$$
$$X \longmapsto \mathbb{G}_m(X) := \mathcal{O}_X(X)^*.$$

This presheaf is co-represented by the scheme $S \times_{Spec(\mathbb{Z})} Spec(\mathbb{Z}[t, t^{-1}])$, namely

$$\mathbb{G}_m(-) \simeq \mathrm{Hom}_S(-, S \times_{Spec(\mathbb{Z})} Spec(\mathbb{Z}[t, t^{-1}])).$$

By a theorem of Grothendieck (see for example [Zom, Thm 1.1, p.1]) we have that $\mathbb{G}_m \in Ab(S_{\acute{e}t})$. Let's take in exam the case when $S = Spec(k)$. We have that the stalk of the sheaf $\mathbb{G}_m$ on at the point $\bar{s}$ is

$$(\mathbb{G}_m)_{\bar{s}} = \varinjlim_{l \in \Sigma} \mathbb{G}_m(Spec(l)) = \varinjlim_{l \in \Sigma} \mathcal{O}_{Spec(l)}(Spec(l))^* = \varinjlim_{l \in \Sigma} l^* = k_s^*.$$

So by Thm 3.21 applied to the case when $F = \mathbb{G}_m$ and Rmk 5.1 we have that

$$H^1(G_k, k_s^*) = H^1(G_k, (\mathbb{G}_m)_{\bar{s}}) \cong H^1_{\acute{e}t}(Spec(k), \mathbb{G}_m).$$

Next step will be defining the notion of Picard group of a *ringed site* $(T_{\mathcal{C}}, \mathcal{O}_T)$, which in our case will the *small Zariski* or *small étale* site. Given $S \in Sch$ we will denote them respectively with S_{Zar} and $S_{\acute{e}t}$. The former is defined as the site $T_{\mathcal{C}}$, with $\mathcal{C} = SubOp_S$ the category whose objects are the open subschemes of S and $Cov(T_{\mathcal{C}})$ consisting of the jointly surjective families of morphisms in $SubOp_S$. In order to do this let's introduce some further notions about sheaves on sites.

Definition 5.2. We define a **ringed site** to be a pair $(T_{\mathcal{C}}, \mathcal{O}_T)$ where $T = T_{\mathcal{C}}$ is a site and $\mathcal{O}_T \in Ring(T)$.

Definition 5.3. Given $(T_{\mathcal{C}}, \mathcal{O}_T)$ a ringed site we define a **presheaf of $\mathcal{O}_T$-modules** as a pair $(\mathscr{F}, \eta)$ where

- $\mathscr{F} \in PAb(T)$
- $(\mathcal{O} \times \mathscr{F} \xrightarrow{\eta} \mathscr{F}) \in PSh(T)$

such that for every $U \in \mathcal{C}$, the map $\mathcal{O}(U) \times \mathscr{F}(U) \xrightarrow{\eta_U} \mathscr{F}(U)$ endows $\mathscr{F}(U)$ with a $\mathcal{O}(U)$-module structure.

Definition 5.4. Given $(\mathscr{F}, \eta)$ and $(\mathscr{G}, \theta)$ two presheaves of $\mathcal{O}_T$-modules we define a **morphism of presheaves of $\mathcal{O}_T$-modules** as a morphism $(\mathscr{F} \xrightarrow{\varphi} \mathscr{G}) \in PAb(T)$ such that the diagram

$$\begin{array}{ccc} \mathcal{O} \times \mathscr{F} & \xrightarrow{\eta} & \mathscr{F} \\ \downarrow{\scriptstyle id_{\mathcal{O}} \times \varphi} & & \downarrow{\scriptstyle \varphi} \\ \mathcal{O} \times \mathscr{G} & \xrightarrow{\theta} & \mathscr{G} \end{array}$$

commutes. The category of presheaves of $\mathcal{O}_T$-modules and associated morphisms will be denoted by $PMod(\mathcal{O}_T)$.

Definition 5.5. Given $(\mathscr{F}, \eta) \in PMod(\mathcal{O}_T)$ we say that $(\mathscr{F}, \eta)$ is a **sheaf of $\mathcal{O}_T$-modules**, if $\mathscr{F} \in Ab(T)$.

Definition 5.6. Given $(\mathscr{F}, \eta)$ and $(\mathscr{G}, \theta)$ two sheaves of $\mathcal{O}_T$-modules we say that $(\mathscr{F} \xrightarrow{\varphi} \mathscr{G}) \in PAb(T)$ is a **morphism of sheaves of $\mathcal{O}_T$-modules**, if $\varphi \in PMod(\mathcal{O}_T)$. The category of sheaves of $\mathcal{O}_T$-modules and associated morphisms will be denoted by $Mod(\mathcal{O}_T)$.

Definition 5.7. Let $(T_{\mathcal{C}}, \mathcal{O}_T)$ be a ringed site and let $\mathscr{F} = (\mathscr{F}, \eta) \in Mod(\mathcal{O}_T)$. We say that $\mathscr{F}$ is **quasi-coherent**, if for every $U \in \mathcal{C}$ there exists a covering $\{U_i \to U\}_{i \in I} \in Cov(\mathcal{C})$ such that for every $i \in I$ there exists a sequence of $\mathcal{O}_T|_{U_i}$-modules

$$\bigoplus_{K(i)} \mathcal{O}_T|_{U_i} \to \bigoplus_{J(i)} \mathcal{O}_T|_{U_i} \to \mathscr{F}|_{U_i} \to 0$$

which is exact. In this case we say that $\mathscr{F}$ admits a **local presentation**. Quasi-coherent sheaves of $\mathcal{O}_T$-modules form a category which we denote with $QCoh(\mathcal{O}_T)$.

Definition 5.8. Let $(T_{\mathcal{C}}, \mathcal{O}_T)$ be a ringed site and $\mathscr{F}, \mathscr{G} \in Mod(\mathcal{O}_T)$. We define the **tensor product of $\mathscr{F}$ and $\mathscr{G}$ over $\mathcal{O}_T$** as the sheaf of $\mathcal{O}_T$-modules $\mathscr{F} \otimes_{\mathcal{O}_T} \mathscr{G}$ obtained by sheafifying the presheaf defined by setting $U \mapsto \mathscr{F}(U) \otimes_{\mathcal{O}_T(U)} \mathscr{G}(U)$ for every $U \in \mathcal{C}$.

Definition 5.9. Given $\mathscr{F} \in Mod(\mathcal{O}_T)$ we say that $\mathscr{F}$ is **invertible**, if there exists $\mathscr{G} \in Mod(\mathcal{O}_T)$ such that $\mathscr{F} \otimes_{\mathcal{O}_T} \mathscr{G} \cong \mathcal{O}_T$.

Definition 5.10. Given $\mathscr{F}, \mathscr{G} \in Mod(\mathcal{O}_T)$ and denote respectively with F and G their isomorphism classes. We set $F \bullet_{\mathcal{O}_T} G := [\mathscr{F} \otimes_{\mathcal{O}_T} \mathscr{G}]_{\sim}$. We define the **Picard group** $Pic(T)$ **of the ringed site** $(T_{\mathcal{C}}, \mathcal{O}_T)$ to be the set of isomorphism classes of invertible sheaves of $\mathcal{O}_T$-modules. We have that $Pic(T) \in Ab$.

Definition 5.11. Let $T = T_{\mathcal{C}}$ be a site. If $\mathcal{C}$ has a terminal object E we define the **global-sections functor of the site** $T_{\mathcal{C}}$ as the evaluation functor $\Gamma_E : Ab(T) \to Ab$ that sends $\mathscr{F} \mapsto \mathscr{F}(E)$. If $\mathcal{C}$ does not have such kind of object we define the above functor by setting $\Gamma_{T_{\mathcal{C}}}(\mathscr{F}) := \mathrm{Hom}_{PSh(T_{\mathcal{C}})}(e, \mathscr{F})$, where e is a terminal object in $PSh(T_{\mathcal{C}})$. In this fashion we set respectively $H^q(T_{\mathcal{C}}, \mathscr{F}) := (R^q\Gamma_E)(\mathscr{F})$ and $H^q(T_{\mathcal{C}}, \mathscr{F}) := (R^q\Gamma_{T_{\mathcal{C}}})(\mathscr{F})$.

Proposition 5.12. *Let $(T_{\mathcal{C}}, \mathcal{O}_T)$ be a ringed site. There is a canonical isomorphism*

$$H^1(T_{\mathcal{C}}, \mathcal{O}_T^*) \cong Pic(T)$$

Proof. The main ingredient here is the canonical identification between $H^1(T_{\mathcal{C}}, \mathcal{O}_T^*)$ and the set of isomorphism classes of $\mathcal{O}_T^*$-torsors. For details see for instance [S.P., Tag 040E]. □

In our case we deal with the category $\acute{E}t/Spec(k)$ and with the site $T = Spec(k)_{\acute{e}t}$ and so the above result reads

$$H^1_{\acute{e}t}(Spec(k), \mathbb{G}_m)) = H^1_{\acute{e}t}(Spec(k), \mathcal{O}^*_{Spec(k)}) \cong Pic(Spec(k)_{\acute{e}t}).$$

The next step will be proving that $Pic(Spec(k)_{\acute{e}t}) \cong Pic(Spec(k)_{Zar})$.

Definition 5.13. Let $S \in Sch$ and $\mathscr{F} \in QCoh(\mathcal{O}_S)$. We define the associated sheaf of $\mathcal{O}_{S_{\acute{e}t}}$-modules (resp. of $\mathcal{O}_{S_{Zar}}$-modules) $\mathscr{F}^a$ by setting

$$\mathscr{F}^a : (T \xrightarrow{f} S) \mapsto (f^*\mathscr{F})(T)$$

where $f^*\mathscr{F}$ denotes the pullback sheaf of $\mathscr{F}$ via f.

Proposition 5.14. *Given $S \in Sch$ and $\mathscr{F} \in QCoh(\mathcal{O}_S)$ we have that the functor*

$$\begin{aligned} QCoh(\mathcal{O}_S) &\longrightarrow QCoh(\mathcal{O}_{S_{\acute{e}t}}) \\ \mathscr{F} &\longmapsto \mathscr{F}^a \end{aligned}$$

is an equivalence of categories. (The same holds for S_{Zar}.).

Proof. See for instance [S.P., Tag 03DX]. The proof essentially develops following a "descent theory argument". □

From this we get that invertible sheaves on the site S_τ with $\tau \in \{\acute{e}t, Zar\}$ descend to invertible sheaves on the scheme S and isomorphism classes "on one side" correspond to isomorphism classes "on the other side". So what we get in our case is that

$$Pic(Spec(k)_{\acute{e}t}) \cong Pic(Spec(k)_{Zar}) \cong Pic(Spec(k)).$$

To conclude it just suffices to observe that

$$Pic(Spec(k)) = 0.$$

This follows from this argument: let $\mathscr{F}$ an invertible sheaf on S and let $\mathscr{G} \in Mod(\mathcal{O}_S)$ such that $\mathscr{F} \otimes_{\mathcal{O}_S} \mathscr{G} \simeq \mathcal{O}_S$. We want to prove that $\mathscr{F} \simeq \mathcal{O}_S$. It is enough to check things on the stalks. In our case $S = Spec(k)$ so for the unique point $x \in Spec(K)$ we have that

$$(\mathscr{F} \otimes_{\mathcal{O}_S} \mathscr{G})_x \cong \mathscr{F}_x \otimes_{\mathcal{O}_{S,x}} \mathscr{G}_x \cong \mathcal{O}_{Spec(k),x} \cong k$$

hence $\mathscr{F}_x \cong k$ and we are done.

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