Fundamental Groups of Schemes

Master thesis

under the supervision of Jilong Tong

Lei Yang

Universite Bordeaux 1

E-mail address: lyang0176@gmail.com

Contents

CHAPTER 1

Introduction

The topological fundamental group can be studied using the theory of covering spaces, since a fundamental group coincides with the group of deck transformations of the associated universal covering space. When the universal cover exists, the theory of covering spaces is entirely analogous to the Galois theory of fields, with the universal cover being the analogue of the algebraic closure of a field and the group of covering transformations being the analogue of the Galois group. The formalism of Galois categories and fundamental groups as introduced by A.Grothendieck in [Gro71] give a natural categorical generalization of the above two theories. Moreover, far beyond providing a uniform setting for the above two theories, this formalism gives rise to the construction and theory of étale fundamental group of schemes which classifies the finite étale covers of a connected scheme.

The aim of this master thesis is to understand the definition of étale fundamental groups of schemes. In the first chapter, we will give the formalism of Galois categories and fundamental groups. In the second chapter, we define the category of étale covers of a connected scheme and prove that it is a Galois category. The third chapter is devoted to giving some examples and properties of the étale fundamental group, while the fourth chapter is devoted to the study of the structure results concerning the geometric fundamental groups of smooth curves.

CHAPTER 2

Galois categories

In this chapter, we will give the formalism of Galois categories and fundamental groups. We first give an axiomatic characterization of categories that are equivalent to $\Pi-$sets, for some profinite group Π, then we give the proof of this equivalence, which is the main result of this chapter. In the last part, we establish some functoriality properties concerning such kind of categories.

1. Galois categories

§1. Definition and elementary properties. Let $\mathcal{C}$ be a category, X, Y two objects in $\mathcal{C}$. We will use the following notation:

- $Hom_{\mathcal{C}}(X, Y)$: Set of morphisms from X to Y in $\mathcal{C}$

- $Isom_{\mathcal{C}}(X, Y)$: Set of isomorphisms from X to Y in $\mathcal{C}$

- $Aut_{\mathcal{C}}(X) := Isom_{\mathcal{C}}(X, X)$

Recall moreover that a morphism $u : X \to Y$ of $\mathcal{C}$ is called a **strict epimorphism** if the fibre product $X \times_Y X$ exists in $\mathcal{C}$ and for any object Z in $\mathcal{C}$, the following sequence:

$$0 \longrightarrow Hom_{\mathcal{C}}(Y, Z) \xrightarrow{\ -\circ u\ } Hom_{\mathcal{C}}(X, Z) \underset{p_2}{\overset{p_1}{\rightrightarrows}} Hom_{\mathcal{C}}(X \times_Y X, Z)$$

is exact in the category of sets, i.e. the first map is injective, and for $\phi \in Hom_{\mathcal{C}}(X, Z)$ then ϕ lies in the image of the first map iff $\phi \circ p_1 = \phi \circ p_2$. For example, in the category of sets (or the category of finite sets), then a strict epimorphism in $\mathcal{C}$ is simply a surjective map of sets.

Let **FSets** denote the category of finite sets.

Definition 1.1. A **Galois category** is a category $\mathcal{C}$ such that there exists a covariant functor $F : \mathcal{C} \to FSets$ satisfying the following axioms:

(1) $\mathcal{C}$ has a final object $e_{\mathcal{C}}$ and finite fibre products exist in $\mathcal{C}$.

(2) Finite coproducts (and hence initial object) exist in $\mathcal{C}$ and categorical quotients by finite groups of automorphisms [1] exist in $\mathcal{C}$.

---

[1] Recall that a categorical quotient of $X \in \mathcal{C}$ by a finite group of automorphism G is an object $Z \in \mathcal{C}$ with $u : X \to Z$ a $G-$invariant morphism in $\mathcal{C}$, which satisfies the following universal property: any $G-$invariant $f : X \to Y$ in $\mathcal{C}$ factors through Z.

(3) Any morphism $u : Y \to X$ in $\mathcal{C}$ factors as $Y \xrightarrow{u'} X' \xrightarrow{u''} X$, where u' is a strict epimorphism and u'' is a monomorphism, which is an isomorphism onto a direct summand of X.

(4) F sends final objects to final objects and commutes with fibre products.

(5) F commutes with finite coproducts and categorical quotients by finite groups of automorphism and sends strict epimorphisms to strict epimorphisms.

(6) Let $u : Y \to X$ be a morphism in $\mathcal{C}$, then $F(u)$ is an isomorphism if and only if u is an isomorphism.

The functor F is called a **Fibre functor** for $\mathcal{C}$.

Remark 1.2.   (1) Initial object exists in $\mathcal{C}$, which we denote by $\emptyset_{\mathcal{C}}$. This is because the coproduct over emptyset $\emptyset$ is always an initial object. And by axiom(2), our claim follows.

(2) The decomposition $Y \xrightarrow{u'} X' \xrightarrow{u''} X$ in axiom (3) is unique in the sense that for any two such decompositions $Y \xrightarrow{u'_i} X' \xrightarrow{u''_i} X = X'_i \sqcup X''_i, i = 1, 2,$ there exists a (necessarily) unique isomorphism $\omega : X'_1 \xrightarrow{\sim} X'_2$ such that $\omega \circ u'_1 = u'_2$ and $u''_2 \circ \omega = u''_1$. See [Cad08], Lemma 2.3

(3) By combining axiom (3) (4) and (6), one also obtains that the functor $F : \mathcal{C} \to FSets$ preserves strict epimorphisms, monomorphisms, and initial objects. Conversely, for any $X_0 \in \mathcal{C}$, one has:

   (a) $F(X_0) = \emptyset$ if and only if X_0 is an initial object of $\mathcal{C}$;

   (b) $F(X_0) \cong \star$ if and only if X_0 is a final object of $\mathcal{C}$, where $\star$ denotes the final objects in $FSets$.

(4) A galois category is artinian. Indeed, apply the fibre functor to a decreasing sequence of monomorphism in $\mathcal{C}$, by (3) we get a decreasing sequence of monomorphism in $FSets$, which is stationary.

Definition 1.3. Let $\mathcal{C}$ be a Galois category,

- Let $F : \mathcal{C} \to FSets$ be a fibre functor, the **fundamental group** of $\mathcal{C}$ at F, which we denote by $\pi_1(\mathcal{C}; F)$, is the group of automorphisms of the functor F. (We recall that an automorphism of a functor $F : \mathcal{C} \to \mathcal{C}'$ is a compatible collection of isomorphisms $\{\sigma_C : F(C) \xrightarrow[\text{isom in } \mathcal{C}']{\cong} F(C) | \forall C \in \mathcal{C}\}$. Here, compatible means that $\forall$ morphism $f : C_1 \to C_2$ in $\mathcal{C}$, the diagram

$$
\begin{array}{ccc}
F(C_1) & \xrightarrow{\sigma_{C_1}} & F(C_1) \\
\downarrow{\scriptstyle F(f)} & & \downarrow{\scriptstyle F(f)} \\
F(C_2) & \xrightarrow{\sigma_{C_2}} & F(C_2)
\end{array}
$$

commutes.)

- Let $F_1, F_2 : \mathcal{C} \to FSets$ be two fibre functors, the **set of paths** from F_1 to F_2 in $\mathcal{C}$, which we denote by $\pi_1(\mathcal{C}; F_1, F_2) := Isom_{Fct}(F_1, F_2)$ is the set of isomorphisms of functors from $F_1 : \mathcal{C} \to FSets$ to $F_2 : \mathcal{C} \to FSets$.

§2. Examples and the main theorem.

§2.1. *The topological covers*. For any connected, locally arcwise connected and locally simply connected topological space B, let $\mathcal{C}_B^{top}$ denote the category of finite topological covers of B, and $\mathcal{C}_B$ the category of topological covers of B. For any $b \in B$, write $F_b : \mathcal{C}_B \to Sets$ for the functor sending $p : X \to B \in \mathcal{C}_B$ to $F_b(p) = p^{-1}(b)$. Then F_b naturally factors through the category $\mathcal{C}^{dist}(\pi_1^{top}(B, b))$ of discrete π_1^{top}−sets. The natural action of $\pi_1^{top}(B, b)$ on $F_b(p)$ is given by monodromy.

Lemma 1.4. monodromy *For any $p : Y \to B \in \mathcal{C}_B$, any path $c : [0,1] \to B$ and any $y \in F_{c(0)}(p)$, there exists a unique path $\tilde{c}_y : [0,1] \to Y$ such that $p \circ \tilde{c}_y = c$ and $\tilde{c}_y(0) = y$. Furthermore, if $c_1, c_2 : [0,1] \to B$ are two homotopic paths with fixed ends then $\tilde{c}_{1,y}(1) = \tilde{c}_{2,y}(1)$.*
In particular, one gets a well defined action $\rho_x(p) : \pi_1^{top}(B, b) \to Aut_{Sets}(F_b(p))$ sending $[\gamma] \in \pi_1^{top}(B, b)$ to $\rho_b(p)([\gamma]) : y \to \tilde{\gamma}(1)$ and ρ_b defines a group morphism $\rho_b : \pi_1^{top}(B, b) \to Aut_{Fct}(F_b)$, $[\gamma] \mapsto \rho_b(-)([\gamma])$.

For a proof of the above lemma, see [Sza09], Lemma 2.3.2. Actually, from the property of topological covers we have:

Proposition 1.5. *Assume that B is connected, locally arcwise connected and locally simply connected. Then $F_b : \mathcal{C}_B \to Sets$ induces an equivalence of categories*

$$F_b : \mathcal{C}_B \cong \mathcal{C}^{disc}(\pi_1^{top}(B, b))$$

.

See [Sza09] Theorem 2.3.4. Hence $\rho_x : \pi_1^{top}(B, b) \to Aut_{Fct}(F_b)$ is an isomorphism. And the category $\mathcal{C}_B^{top}$ of all finite topological covers of B is Galois with fibre functors $F_b|_{\mathcal{C}_B^{top}}, b \in B$

and $\pi_1(\mathcal{C}_B^{top}; F_b) = \widehat{\pi_1^{top}(B, b)}$ (where $\widehat{(-)}$ denotes the profinite completion).

§2.2. *The category $\mathcal{C}(\Pi)$ and the main theorem.*

Lemma 1.6. *Let Π be a profinite group , $\mathcal{C}(\Pi)$ the category of finite (discrete) sets with continuous left Π-action, $For : \mathcal{C}(\Pi) \to FSets$ the forgetful functor. Then $\mathcal{C}(\Pi)$ is a Galois category, and For is a fibre functor. Moreover, we have:*

$$\pi_1(\mathcal{C}(\Pi); For) \cong \Pi.$$

This lemma can be proved by checking that the axioms in the definition of Galois category are satisfied. Indeed $\mathcal{C}(\Pi)$ is the typical example of Galois categories. Let $\mathcal{C}$ be a Galois category equipped with a fibre functor F, we claim first that the fundamental group $\pi_1(\mathcal{C}; F)$ is equipped with a natural structure of profinite group. For this, set:

$$\pi := \prod_{X \in Ob(\mathcal{C})} Aut_{FSets}(F(X))$$

and endow π with the product topology of the discrete topologies, which gives it a structure of profinite group. Considering the monomorphism of groups:

$$\begin{array}{ccc} \pi_1(\mathcal{C}, F) & \hookrightarrow & \pi \\ \theta & \mapsto & (\theta(X))_{X \in Ob(\mathcal{C})} \end{array}$$

the group $\pi_1(\mathcal{C}; F)$ can be identified with the intersection of all subgroups of the form:

$$\mathcal{C}_\phi := \{(\sigma_X)_{X \in Ob(\mathcal{C})} \in \pi | \sigma_X \circ F(\phi) = F(\phi) \circ \sigma_Y\},$$

where $\phi : Y \to X$ describes the set of all morphisms in $\mathcal{C}$. By definition of the product topology, the $\mathcal{C}_\phi$'s are closed. So $\pi_1(\mathcal{C}, F)$ is closed as well and, equipped with the topology induced from the product topology on π, it becomes a profinite group.

By definition of this topology, the action of $\pi_1(\mathcal{C}; F)$ on $F(X)$ is continuous. This is because that we endow $F(X)$ with the discrete topology and that the subgroup of $\pi_1(\mathcal{C}, F)$ sending a certain element x in $F(X)$ to another element y in $F(X)$ is a subgroup of π which is the intersection of all $\mathcal{C}_\phi$ and $\{(\sigma_X)_{X \in Ob(\mathcal{C})} \in \pi | \sigma_X(x) = y\}$ hence it is a closed subgroup of $\pi_1(\mathcal{C}, F)$. And the group is of finite index in $\pi_1(\mathcal{C}, F)$, hence it is an open subgroup of $\pi_1(\mathcal{C}, F)$. Hence a fibre functor $F : \mathcal{C} \to FSets$ factors as:

$$\mathcal{C} \xrightarrow{\quad F \quad} FSets$$

$$\downarrow \qquad \nearrow {\scriptstyle For}$$

$$\mathcal{C}(\pi_1(\mathcal{C}; F))$$

By abuse of notation the induced functor $\mathcal{C} \to \mathcal{C}(\pi_1(\mathcal{C}; F))$ will be still denoted by F.

Theorem 1.7. (Main theorem) *Let $\mathcal{C}$ be a galois category. Then:*

(1) Any fibre funcor $F : \mathcal{C} \to FSets$ induces an equivalence of categories $F : \mathcal{C} \to \mathcal{C}(\pi_1(\mathcal{C}; F))$.

(2) For any two fibre functors $F_i : \mathcal{C} \to FSets, i = 1, 2$, the set of paths $\pi_1(\mathcal{C}; F_1, F_2)$ is non-empty. The profinite group $\pi_1(\mathcal{C}; F_1)$ is noncanonically isomorphic to $\pi_1(\mathcal{C}; F_2)$ with an isomorphism that is canonical up to inner automorphisms. In particular, the abelianization $\pi_1(\mathcal{C}; F)^{ab}$ of $\pi_1(\mathcal{C}; F)$ does not depend on F up to canonical isomorphisms.

In the following, after a discusion of the notion of connecte and Galois objects, we will give the proof in section 3. In the last section, we will discuss the functoriality of Galois categories.

2. Galois objects.

Given a category $\mathcal{C}$ and $X, Y \in \mathcal{C}$, we will say that X *dominates* Y in $\mathcal{C}$ if there exists at least one morphism from X to Y in $\mathcal{C}$, and writes $X \geq Y$.

We fix in this section a fibre functor $F : \mathcal{C} \to FSets$ of a galois category $\mathcal{C}$.

Definition 2.1. An object $X \in \mathcal{C}$ is called **connected** if it cannot be written as a coproduct $X = X_1 \sqcup X_2$ with $X_i \neq \emptyset_C, i = 1, 2$.

Proposition 2.2. (Minimality and connected components) *An object $X_0 \in \mathcal{C}$ is connected if and only if for any $X \in \mathcal{C}$, $X \neq \emptyset_C$ any monomorphism from X to X_0 in $\mathcal{C}$ is automatically an isomorphism.*
In particular, any object $X \in \mathcal{C}$ ($X \neq \emptyset_C$) can be written as: $X = \sqcup_{i=1}^r X_i$, with $X_i \in \mathcal{C}$ connected, $X_i \neq \emptyset_C, i = 1, ..., r$ and this decomposition is unique (up to permutation). We say that the $X_i, i = 1, ..., r$ are the connected components of X.

Proof. The "if" implication: write $X_0 = X_0' \sqcup X_0''$, we may assume that $X_0' \neq \emptyset_{\mathcal{C}}$. By 1.2 (3) the canonical morphism $i_{X_0'} : X_0' \to X_0$ is a monomorphism hence automatically an isomorphism, hence $F(X_0'') = \emptyset$ hence $X_0'' = \emptyset_{\mathcal{C}}$ by 1.2 (3).

The "only if" implication: assume that $X_0 \neq \emptyset_{\mathcal{C}}$. By axiom(3), any monomorphism $i : X \hookrightarrow X_0$ in $\mathcal{C}$ factors as $X \xrightarrow{i'} X_0' \xrightarrow{i''} X_0 = X_0' \sqcup X_0''$ with i' a strict epimorphism and i'' a monomorphism inducing an isomorphism onto X_0'. If $X_0' = \emptyset_{\mathcal{C}}$ then $F(X) = \emptyset$, which forces $X = \emptyset_{\mathcal{C}}$ and contradicts our assumption. So $X_0'' = \emptyset_{\mathcal{C}}$ and i'' is an isomorphism. Then, $i : X \hookrightarrow X_0$ is both a monomorphism and a strict epimorphism hence an isomorphism. As for the last assertion, since $\mathcal{C}$ is Artinian, for any $X \in \mathcal{C}, X \neq \emptyset$, there exists $X_1 \in \mathcal{C}$ connected, $X_1 \neq \emptyset_{\mathcal{C}}$ and a monomorphism $i_1 : X_1 \hookrightarrow X$. If i_1 is an isomorphism then X is connected. Otherwise, from axiom(3), i_1 factors as $X_1 \xrightarrow{i_1'} X' \xrightarrow{i_1''} X = X' \sqcup X''$ with i_1' a strict epimorphism and i_1'' a monomorphism inducing an isomorphism onto X'. Since i_1 and i_1'' are monomorphisms, i_1' is a monomorphism as well hence an isomorphism. We then iterate the argument on X''. By axiom(5), this process terminates after at most $|F(X)|$ steps. So we obtain a decomposition:

$$X = \sqcup_{i=1}^{r} X_i.$$

as a coproduct of finitely many non-initial connected objects, which proves the existence. For the unicity, assume that we have another such decomposition:

$$X = \sqcup_{i=1}^{s} Y_i.$$

For $1 \leq i \leq r$, let $1 \leq \sigma(i) \leq s$ such that $F(X_i) \cap F(Y_{\sigma(i)}) \neq \emptyset$. Then consider:

$$
\begin{array}{ccc}
X_i & \xrightarrow{\ i_{X_i}\ } & X \\[2pt]
{\scriptstyle p}\uparrow & & \uparrow{\scriptstyle i_{Y_{\sigma(i)}}} \\[2pt]
X_i \times_X Y_{\sigma(i)} & \xrightarrow{\ q\ } & Y_{\sigma(i)}
\end{array}
$$

Since i_{X_i} is a monomorphism, q is a monomorphism as well. Also, by axiom(4) one has $F(X_i \times_X Y_{\sigma(i)}) = F(X_i) \cap F(Y_{\sigma(i)}) \neq \emptyset$ and, since $Y_{\sigma(i)}$ is connected and q is a monomorphism, q is an isomorphism. Similarly, p is an isomorphism. $\qquad\square$

Proposition 2.3. (Morphisms from and to connected objects) *Let X_0, X be two objects of $\mathcal{C}$ with X_0 connected.*

(1) (Rigidity) *For any $\zeta_0 \in F(X_0)$, $\zeta \in F(X)$, there is at most one morphism from (X_0, ζ_0), to (X, ζ) in $\mathcal{C}^{pt}$;*

(2) (Domination by connected objects) *For any $(X_i, \zeta_i) \in \mathcal{C}^{pt}$, $i = 1, ..., r$ there exists $(X_0, \zeta_0) \in \mathcal{C}^{pt}$ with $X_0 \in \mathcal{C}$ connected such that $(X_0, \zeta_0) \geq (X_i, \zeta_i)$ in $\mathcal{C}^{pt}$, $i = 1, ..., r$.*
In particular, for any $X \in \mathcal{C}$, there exists $(X_0, \zeta_0) \in \mathcal{C}^{pt}$ with $X_0 \in \mathcal{C}$ connected such that the evaluation map:
$$
ev_{\zeta_0} : \begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(X_0, X) & \xrightarrow{\sim} & F(X) \\ u : X_0 \to X & \mapsto & F(u)(\zeta_0) \end{array}
$$
is bijective.

(3) (a) *If $X_0 \in \mathcal{C}$ is connected, then any morphism $u : X \to X_0$ in $\mathcal{C}$ is a strict epimorphism;*

(b) *If $u : X_0 \to X$ is a strict epimorphism in $\mathcal{C}$ and if X_0 is connected then X is also connected;*

(c) If $X_0 \in \mathcal{C}$ is connected, then any endomorphism $u : X_0 \to X_0$ in $\mathcal{C}$ is automatically an automorphism.

Proof.   (1) Let $u_i : (X_0, \zeta_0) \to (X, \zeta)$ be two morphisms in $\mathcal{C}^{pt}$, $i = 1, 2$. From axiom (1) the equilizer $ker(u_1, u_2) \xrightarrow{i} X$ exists in $\mathcal{C}$. From axiom (4), $F(ker(u_1, u_2)) \xrightarrow{F(i)} F(X_0)$ is the equalizer of $F(u_i) : F(X_0) \to F(X)$, $i = 1, 2$ in $FSets$. But by assumption, $\zeta_0 \in Ker(F(u_1), F(u_2)) = F(ker(u_1, u_2))$ so, in particular, $F(ker(u_1, u_2)) \neq \emptyset$ and it follows from 1.2 (3) that $ker(u_1, u_2) \neq \emptyset_{\mathcal{C}}$. Since an equalizer is always a monomorphism, it follows then from 2.2 that $i : ker(u_1, u_2) \xrightarrow{\simeq} X_0$ is an isomorphism, that is $u_1 = u_2$.

(2) Take $X = X_1 \times ... \times X_r$, $\zeta := (\zeta_1, ...\zeta_r) \in F(X_1) \times .. \times F(X_r) = F(X_1 \times ... \times X_2)$ by axiom (4). The ith projection $pr_i : X_0 \to X_i$ induces a morphism from (X, ζ) to (X_i, ζ_i) in $\mathcal{C}^{pt}$, $i = 1, ..., r$. So it is enough to prove that there exists $(X_0, \zeta_0) \in \mathcal{C}^{pt}$ with X_0 connected such that $(X_0, \zeta_0) \geq (X, \zeta)$ in $\mathcal{C}^{pt}$. If $X \in \mathcal{C}$ is connected then $Id : (X, \zeta) \to (X, \zeta)$ works. Otherwise, write

$$X = \sqcup_{i=1}^r X_i$$

as product of its connected components and let $i_{X_i} : X_i \hookrightarrow X$ denote the canonical monomorphism, $i = 1, ..., r$. Then, from axiom (2) one gets:

$$F(X) = \sqcup_{i=1}^r F(X_i)$$

hence, there exists a unique $1 \leq i \leq r$ such that $\zeta \in F(X_i)$ and $i_{X_i} : (X_i, \zeta) \hookrightarrow (X, \zeta)$ works.

(3) (a) It follows from axiom (3) that $u : X \to X_0$ factors as $X \xrightarrow{u'} X_0' \xrightarrow{u''} X_0' \sqcup X_0'' = X_0$, where u' is a strict epimorphism and u'' is a monomorphism inducing an isomorphism onto X_0'. Furthermore, $X \neq \emptyset_{\mathcal{C}}$ forces $X_0' \neq \emptyset_{\mathcal{C}}$ thus since X_0 is connected, $X_0'' = \emptyset_{\mathcal{C}}$ hence $u'' : X_0' \xrightarrow{\simeq} X_0$ is an isomorphism.

   (b) If $X_0 = \emptyset_{\mathcal{C}}$, the claim is obvious. Otherwise, write $X = X' \sqcup X''$ with $X' \neq \emptyset_{\mathcal{C}}$. Let $i_{X'} : X' \hookrightarrow X$ denote the canonical inclusion. Fix $\zeta' \in F(X')$, $\zeta_0 \in F(X_0)$ such that $F(u)(\zeta_0) = \zeta'$. From (2), there exists $(X_0', \zeta_0') \in \mathcal{C}^{pt}$, with X_0 connected and morphisms $p : (X_0', \zeta_0') \to (X_0, \zeta_0)$ and $q : (X_0', \zeta_0') \to (X', \zeta')$. From (3)(a) the morphism p is automatically a strict epimorphism, so $u \circ p$ is also a strict epimorphism. From (1) one has: $u \circ p = i_{X'} \circ q$ so $i_{X'} \circ q$ is a strict epimorphism, in particular, $F(X) = F(X')$, which forces $F(X'') = \emptyset$, therefore $X'' = \emptyset_{\mathcal{C}}$.

   (c) From axiom (6), we only need to prove $F(u) : F(X_0) \xrightarrow{\simeq} F(X)$ is an isomorphism. But $F(X_0)$ is finite, we only need to prove $F(u)$ is an epimorphism. By axiom (3), write $u : X_0 \to X_0$ as $X_0 \xrightarrow{u'} X_0' \xrightarrow{u''} X_0 = X_0' \sqcup X_0''$ with u' a strict epimorphism and u'' a monomorphism inducing an isomorphism onto X_0'. As X_0 is connected either $X_0' = \emptyset_{\mathcal{C}}$ or $X_0'' = \emptyset_{\mathcal{C}}$. Then either $X_0 = \emptyset_{\mathcal{C}}$ and thus the claim is straightforward, or $X_0 = X_0'$ thus u'' is an isomorphism and u is a strict epimorphism so by axiom (4) the conclusion follows.

$\square$

Corollary 2.4. *Let $X_0 \in \mathcal{C}$ be a connected object. $X_0 \neq \emptyset$, let $\zeta_0 \in F(X_0)$. Then the evaluation map:*

$$ev_{\zeta_0} : Aut_{\mathcal{C}}(X_0) \to F(X_0), u \mapsto F(u)(\zeta_0)$$

is injective. In particular,

$$|Aut_{\mathcal{C}}(X_0)| \leq |F(X_0)|$$

.

Definition 2.5. A connected object X_0 in $\mathcal{C}$ is an **Galois object** in $\mathcal{C}$ if for any $\zeta_0 \in F(X_0)$ the evaluation map $ev_{\zeta_0} : Aut_{\mathcal{C}}(X_0) \hookrightarrow F(X_0)$ is bijective.

Remark 2.6. Let X_0 be a connected object in $\mathcal{C}$, then X_0 is a Galois object iff one of the following conditions is satisfied:

(1) $Aut_{\mathcal{C}}(X_0)$ acts transitively on $F(X_0)$;

(2) $Aut_{\mathcal{C}}(X_0)$ acts simply transitively on $F(X_0)$;

(3) $|Aut_{\mathcal{C}}(X_0)| = |F(X_0)|$;

(4) $X_0/Aut_{\mathcal{C}}(X_0)$ is final in $\mathcal{C}$.

Since $Aut_{\mathcal{C}}(X_0)$ acts simply on $F(X_0)$, we get the equivalence of $(1), (2),$ and (3). By 1.2 $(3)(b)$, (4) is equivalent to $F(X_0/Aut_{\mathcal{C}}(X_0)) = *$. By axiom (5) this is also equivalent to $F(X_0)/Aut_{\mathcal{C}}(X_0) = *$, which is equivalent to (1).
By (4) the notion Galois object is independent of the fibre functor.

Proposition 2.7. (Galois closure) *Let $X \in \mathcal{C}$ be a connected object in $\mathcal{C}$, then there exists a Galois object $\hat{X} \in \mathcal{C}$ dominating X, which is minimal among all the Galois objects dominating X in $\mathcal{C}$.*

Proof. By 2.3(2) there exists $(X_0, \zeta_0) \in \mathcal{C}^{pt}$, such that X_0 is connected and that the evaluation map $ev_{\zeta_0} : Hom_{\mathcal{C}}(X_0, X) \xrightarrow{\simeq} F(X)$ is bijective. Write $Hom_{\mathcal{C}}(X_0, X) = \{u_1, ..., u_n\}$. For $i = 1, ..., n$, let $\zeta_i := F(u_i)(\zeta_0)$ and $pr_i : X^n \to X$ denote the ith projection. By the universal property of product, there is a unique morphism $\pi := (u_1, ..., u_n) : X_0 \to X^n$ such that $pr_i \circ \pi = u_i$.
By axiom (3), we can decompose $\pi : X_0 \to X^n$ as $X_0 \xrightarrow{\pi'} \hat{X} \xrightarrow{\pi''} X^n = \hat{X} \sqcup \hat{X}'$ with π' a strict epimorphism and π'' a monomorphism inducing an isomorphism onto $\hat{X}$. We claim that $\hat{X}$ is Galois and minimal among the Galois objects dominating X.
By 2.3(3)(b), $\hat{X}$ is connected. Set $\hat{\zeta}_0 := F(\pi')(\zeta_0) = (\zeta_1, ..., \zeta_n) \in F(\hat{X})$; we are to prove that $ev_{\hat{\zeta}_0} : Aut_{\mathcal{C}}(\hat{X}) \to F(\hat{X})$ is surjective that is, for any $\zeta \in F(\hat{X})$ there exists $\omega \in Aut_{\mathcal{C}}(\hat{X})$ such that $F(\omega)(\hat{\zeta}_0) = \zeta$. By 2.3 (2) there exists $(\tilde{X}_0, \tilde{\zeta}_0) \in \mathcal{C}^{pt}$ with $\tilde{X}_0$ connected and $(\tilde{X}_0, \tilde{\zeta}_0) \geq (X_0, \zeta_0)$ and $(\tilde{X}_0, \tilde{\zeta}_0) \geq (\hat{X}, \hat{\zeta}_0), \hat{\zeta}_0 \in F(\hat{X})$. So, on one hand, we can write $F(\omega)(\hat{\zeta}_0) = F(\omega \circ \pi')(\zeta_0)$, on the other hand, $\zeta = F(\rho_\zeta)(\zeta_0)$. But then by 2.3 (1), there exists $\omega \in Aut_{\mathcal{C}}(\hat{X})$ such that $F(\omega)(\hat{\zeta}_0) = \zeta$ if and only if there exists $\omega \in Aut_{\mathcal{C}}(\hat{X})$ such that $\omega \circ \pi' = \rho_\zeta$. To prove the existence of such an ω observe that:

(2.1) $$\{pr_1 \circ \pi'' \circ \rho_\zeta, ..., pr_n \circ \pi'' \circ \rho_\zeta\} = \{u_1, ..., u_n\}.$$

The inclusion $\subset$ is straightforward. To prove $\supset$, only need to prove that $pr_i \circ \pi'' \circ \rho_\zeta, 1 \leq i \leq n$ are all distinct. But since $pr_i \circ \pi'' \circ \pi' = u_i \neq u_j = pr_j \circ \pi'' \circ \pi', 1 \leq i \neq j \leq n$ and $\pi' : X_0 \to \hat{X}$ is a strict epimorphism, $pr_i \circ \pi'' \neq pr_j \circ \pi''$ as well. And, as X_0 is connected, $\rho_\zeta : X_0 \to \hat{X}$ is automatically a strict epimorphism hence $pr_i \circ \pi'' \circ \rho_\zeta \neq pr_h \circ \pi'' \circ \rho_\zeta$. From , there exists a permutation $\sigma \in \mathcal{S}_n$ such that $pr_{\sigma(i)} \circ \pi'' \circ \rho_\zeta = pr_i \circ \pi'' \circ \pi'$ and

from the universal property of product there exists a unique isomorphism $\sigma : X^n \simeq X^n$ such that $pr_i \circ \sigma = pr_{\sigma(i)}$. Hence $pr_i \circ \pi'' \circ \pi' = pr_i \circ \sigma \circ \pi'' \circ \rho_\zeta$, which forces $\pi'' \circ \pi' = \sigma \circ \pi'' \circ \rho_\zeta$. But then from the unicity of the decomposition in axiom(3), there exists an automorphism $\omega : \hat{X} \simeq \hat{X}$ satisfying $\sigma \circ \pi'' = \pi'' \circ \omega$ and $\omega \circ \pi' = \rho_\zeta$.

It remains to prove the minimality of $\hat{X}$. Let $Y \in \mathcal{C}$ be Galois and $q : Y \to X$ a morphism in $\mathcal{C}$. Fix $\eta_i \in F(Y)$ such that $F(q)(\eta_i) = \zeta_i, i = 1, ..., n$. Since Y is Galois, there exists $\omega_i \in Aut_\mathcal{C}(Y)$ such that $F(\omega_i)(\eta_1) = \eta_i$. This defines a unique morphism $\kappa := (q \circ \omega_1, ..., q \circ \omega_n) : Y \to X^n$ such that $pr_i \circ \kappa = q \circ \omega_i$. By axiom (3), $\kappa : Y \to X^n$ factors as $Y \xrightarrow{\kappa'} Z' \xrightarrow{\pi''} X^n = Z' \sqcup Z''$ with π' a strict epimorphism in $\mathcal{C}$ and π'' a monomorphism inducing an isomorphism onto Z'. It follows from 2.3(3) (b) that Z' is connected and $F(\kappa)(\eta_1) = (\zeta_1, ..., \zeta_n) = \hat{\zeta}_0$ hence Z' is the connected component of $\hat{\zeta}_0$ in X^n that is $\hat{X}$.

$\square$

Such an $\hat{X}$ is unique up to isomorphism; it is called the **Galois closure** of X.

3. Proof of the main theorem

Let $\mathcal{C}$ be a Galois category, and $F : \mathcal{C} \to Sets$ a fibre functor. Let $\mathcal{G}$ be a system of representatives of the isomorphism classes of Galois objects in $\mathcal{C}$. Let

$$\underline{\zeta} = (\zeta_X)_{X \in \mathcal{G}} \in \prod_{X \in \mathcal{G}} F(X).$$

For two pairs (X, ζ_X) and (Y, ζ_Y) with $X, Y \in \mathcal{G}$, we say that (X, ζ_X) dominates (Y, ζ_Y), denoted by $(X, \zeta_X) \geq (Y, \zeta_Y)$ if there exists a morphism $u_{X,Y} : X \to Y$ such that $F(u)(\zeta_X) = \zeta_Y$. Note that if this morphism exists, then it must be unique, by 2.3 (1), hence we have the following:

Lemma 3.1. *The set $\mathcal{G}^{\underline{\zeta}} := \{(X, \zeta_X)|X \in \mathcal{G}\}$ is a directed set with respect to the relation $\geq$ defined above.*

For any $(X, \zeta_X) \in \mathcal{G}^{\underline{\zeta}}$, we consider the evaluation map

$$ev_{X,\zeta_X} : \begin{array}{ccc} Hom_\mathcal{C}(X, -) & \to & F \\ (u : X \to Y) & \mapsto & F(u)(\zeta_X) \end{array}$$

Proposition 3.2. (Galois correspondence) *For any $X_0 \in \mathcal{G}$ let $\mathcal{C}^{X_0} \subset \mathcal{C}$ denote the full subcategory whose objects are the $X \in \mathcal{C}$ such that X_0 dominates any connected component of X in $\mathcal{C}$.*

(1) The evaluation map

$$ev_{X_0,\zeta_0} : Hom_\mathcal{C}(X_0, -)|_{\mathcal{C}^{X_0}} \xrightarrow{\simeq} F|_{\mathcal{C}^{X_0}}.$$

is a functor isomorphism. In particular, this induces an isomorphism of groups:

$$u^{\zeta_0} : Aut(F|_{\mathcal{C}^{X_0}}) \xrightarrow{\simeq} Aut(Hom_\mathcal{C}(X_0, -)|_{\mathcal{C}^{X_0}}) = Aut_\mathcal{C}(X_0)^{op}$$

(where the second equality is just Yoneda lemma)

(2) The functor $F|_{\mathcal{C}^{X_0}} : \mathcal{C}^{X_0} \to FSets$ factors through an equivalence of categories

$$
\begin{array}{ccc}
\mathcal{C}^{X_0} & \xrightarrow{\;\; F|_{\mathcal{C}^{X_0}} \;\;} & FSets \\
{\scriptstyle \cong}\Big\downarrow{\scriptstyle F|_{\mathcal{C}^{X_0}}} & \nearrow{\scriptstyle For} & \\
\mathcal{C}(Aut_{\mathcal{C}}(X_0)^{op}) & &
\end{array}
$$

Proof. (1) Since the diagram

$$
\begin{array}{ccc}
F(Y) & \xrightarrow{\;\; F(u) \;\;} & F(X) \\
{\scriptstyle ev_{(X_0,\zeta_0)}(Y)}\Big\uparrow & & \Big\uparrow{\scriptstyle ev_{(X_0,\zeta_0)}(X)} \\
Hom_{\mathcal{C}}(X_0, Y) & \xrightarrow{\;\; u\circ \;\;} & Hom_{\mathcal{C}}(X_0, X)
\end{array}
$$

commutes, $ev_{(X_0,\zeta_0)}$ is a morphism of functors. Also, since X_0 is connected, $ev_{(X_0,\zeta_0)}(X)$ is injective for $X \in \mathcal{C}^{X_0}$.

- If X is connected, it follows from 2.3 (3)(a) that any morphism $u : X_0 \to X$ in $\mathcal{C}$ is automatically a strict epimorphism. Write $F(X) = \{\zeta_1, ..., \zeta_n\}$ and let $\zeta_{0i} \in F(X_0)$ such that $F(u)(\zeta_{0i}) = \zeta_i$, $i = 1, ..., n$. Since $X_0 \in \mathcal{C}$ is Galois, there exists $\omega_i \in Aut_{\mathcal{C}}(X_0)$ such that $F(\omega)(\zeta_0) = \zeta_{0i}$, hence $ev_{(X_0,\zeta_0)}$ is surjective hence bijective.

- If X is not connected, the conclusion follows from 2.2 and axiom (5).

(2) We set $G := Aut_{\mathcal{C}}(X_0)$. From (1) and identify $F|_{X_0}$ with the functor $Hom_{\mathcal{C}}(X_0, -)|_{X_0}$ over which G^{op} acts naturally via composition on the right, whence a factoriza-

tion:
$$
\begin{array}{ccc}
\mathcal{C}^{X_0} & \xrightarrow{\;\; F^{X_0} \;\;} & FSets \\
{\scriptstyle F^{X_0}}\Big\downarrow & \nearrow{\scriptstyle For} & \\
\mathcal{C}(G^{op}) & &
\end{array}
$$
It remains to prove that $F|_{X_0} : \mathcal{C}^{X_0} \to \mathcal{C}(G^{op})$

is an equivalence of categories.

- $F|_{\mathcal{C}^{X_0}}$ is essentially surjective: Let $E \in \mathcal{C}(G^{op})$ and $e \in E$. By the same argument as in (1), one may assume that E is connected in $\mathcal{C}(G^{op})$ that is a transitive left $G^{op}-$set. Thus we get an epimorphism in $G^{op}-$sets:
$$
\begin{array}{rrcl}
p_e^0 : & G^{op} & \twoheadrightarrow & E \\
& \omega & \mapsto & \omega \cdot e
\end{array}
$$
Set $f_e := p_e^0 \circ ev_{\zeta_0}^{-1} : F(X_0) \twoheadrightarrow E$. Then for any $s \in S_e$ the stabilizer of e in G^{op}, and $\omega \in G$, one has: $f_e \circ F(s)(ev_{\zeta_0}(\omega)) = f_e(ev_{\zeta_0}(\omega))$. So, by the universal property of quotient, $f_e : F(X_0) \twoheadrightarrow E$ factors through $F(X_0)/S_e$. But if $p_e : X_0 \to X_0/S_e$ denotes the categorical quotient it follows from axiom(5) that $F(X_0) \twoheadrightarrow F(X_0)/S_e$ is $F(p_e) : F(X_0) \twoheadrightarrow F(X_0/S_e)$. As X_0 is connected, G acts simply on $F(X_0)$ hence:
$$|F(X_0)/S_e| = |F(X_0)|/|S_e| = [G : S_e] = |E|$$
So $\bar{f}_e : F(X_0)/S_e = F(X_0/S_e) \twoheadrightarrow E$ is actually an isomorphism in $G^{op}-$sets.

- $F|_{\mathcal{C}^{X_0}}$ is fully faithful: Let $X, Y \in \mathcal{C}^{X_0}$. We may again assume that X, Y are connected. The faithfulness follows directly from 2.3(1). For the fullness, for any morphism $u : F(X) \to F(Y)$ in $\mathcal{C}(G^{op})$, fix $e \in F(X)$.

Since $u : F(X) \to F(Y)$ is a morphism in $\mathcal{C}(G^{op})$ we have $S_e \subset S_{u(e)}$ hence $p_{u(e)} : X_0 \to X_0/S_{u(e)}$ factors through:

$$
\begin{array}{ccc}
X_0 & \xrightarrow{\ \ p_e\ \ } & X_0/S_e \\
\downarrow{\scriptstyle p_{u(e)}} & \swarrow{\scriptstyle p_{e,u(e)}} & \\
X_0/S_{u(e)} & &
\end{array}
$$

hence we get the following commutative diagram:

$$
\begin{array}{ccc}
F(X_0/S_e) & \xrightarrow{F(p_{e,u(e)})} & F(X_0/S_{u(e)}) \\
\downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle simeq} \\
F(X) & \xrightarrow{\ \ u\ \ } & F(Y)
\end{array}
$$

$\square$

Let $(X, \zeta_X), (Y, \zeta_Y) \in \mathcal{G}^{\subseteq}$ be two elements with $(X, \zeta_X) \geq (Y, \zeta_Y)$, let $u_{X,Y} : X \to Y$ be the unique morphism such that $F(u_{X,Y})(\zeta_X) = \zeta_Y$, then the following diagram is commutative:

$$
\begin{array}{ccc}
Hom_{\mathcal{C}}(Y, -) & \xrightarrow{ev_{Y,\zeta_Y}} & F(-) \\
\downarrow{\scriptstyle u^\star_{X,Y}} & \nearrow{\scriptstyle ev_{X,\zeta_X}} & \\
Hom_{\mathcal{C}}(X, -) & &
\end{array}
\qquad .
$$

Passing to limits, we obtain the following map

$$
ev_{\mathcal{G},\underline{\varsigma}} : \varinjlim_{(X,\zeta_X)\in\mathcal{G}^{\subseteq}} : Hom_{\mathcal{C}}(X, -) \to F(-)
$$

.

Proposition 3.3. *The morphism $ev_{\mathcal{G},\underline{\varsigma}}$ above is an isomorphism.*

Proof. By 2.3 (3)(a) the $u^\star_{X,Y}$ are automatically strict epimorphisms. Then the assertion follows from 3.2. $\square$

Corollary 3.4. $\pi_1(\mathcal{C}; F) \cong \varprojlim_{(X,\zeta_X)\in\mathcal{G}^{\subseteq}} Aut_{\mathcal{C}}(X)^{op}$.

Lemma 3.5. *Let $X, Y \in \mathcal{G}$ with $X \leq Y$, for any morphisms $\phi, \psi : Y \to X$ in $\mathcal{C}$ and for any $\omega_Y \in Aut_{\mathcal{C}}(Y)$ there is a unique automorphism $\omega_X := r_{\phi,\psi}(\omega_Y) : X \xrightarrow{\simeq} X$ in $\mathcal{C}$*

such that the following diagram commutes:
$$
\begin{array}{ccc}
Y & \xrightarrow{\ \omega_Y\ } & Y \\
\downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \phi} \\
X & \xrightarrow{\ \omega_X\ } & X.
\end{array}
$$

Proof. *(of lemma 3.5)* Since X is connected, $\psi : Y \to X$ is automatically a strict epimorphism hence the map

$$
\circ\psi : Aut_{\mathcal{C}}(X) \hookrightarrow Hom_{\mathcal{C}}(Y, X)
$$

is injective. But by 3.2 that $|Hom_{\mathcal{C}}(Y, X)| = |F(X)|$, from that X is Galois, $|F(X)| = |Aut_{\mathcal{C}}(X)|$. As a result the map $\circ\psi : Aut_{\mathcal{C}}(X) \xrightarrow{\simeq} Hom_{\mathcal{C}}(X, Y)$ is actually bijective. In

particular, there exists a unique $\omega_X : X \xrightarrow{\cong} X$ making the above diagram commutative.

$\square$

Proof. *(of 3.4)* By the above lemma we get a well defined surjective map:

$$r_{\phi,\psi} : Aut_{\mathcal{C}}(Y) \twoheadrightarrow Aut_{\mathcal{C}}(X)$$

which is a group epimorphism when $\phi = \psi$. As a result we get a projective system of finite groups. Denote by Π the projective limit of this system, then Π^{op} acts on $Hom_{\mathcal{C}}(X,-)$ by composition on the right, which induce a group monomorphism: $\Pi^{op} \hookrightarrow Aut(\varprojlim_{(X,\zeta_X)\in\mathcal{G}^{\underline{\zeta}}} Hom_{\mathcal{C}}(X,-)$ hence a group monomorphism $u^{\underline{\zeta}} : \pi_1(\mathcal{C};F) \hookrightarrow \Pi^{op}$

Let $\underline{\omega} := (\omega_X)_{X\in\mathcal{G}} \in \Pi$. For any $Z \in \mathcal{C}$ connected, let $\hat{Z}$ denote the Galois closure of Z in $\mathcal{C}$ and consider the bijective map:

$$\theta_{\underline{\omega}}(Z) : \xrightarrow[\simeq]{ev_{\zeta_{\hat{Z}}}^{-1}} Hom_{\mathcal{C}}(\hat{Z}, Z) \xrightarrow[\simeq]{\circ\omega_{\hat{Z}}} Hom_{\mathcal{C}}(\hat{Z}, Z) \xrightarrow[\simeq]{ev_{\zeta_{\hat{Z}}}} F(Z).$$

One checks that this defines a functor automorphism and that $u^{\underline{\zeta}}(\theta_{\underline{\omega}}) = \underline{\omega}$. By the definition of the topology on $\pi_1(\mathcal{C};F)$, $\pi_1(\mathcal{C};F) \to \Pi^{op} \to Aut_{\mathcal{C}}(X)^{op}$ is continuous for $X \in \mathcal{G}$, hence $u_{\underline{\zeta}} : \pi_1(\mathcal{C};F) \to \Pi^{op}$ is continuous. Since $\pi_1(\mathcal{C};F)$ is continuous, $u_{\underline{\zeta}^{-1}}$ is continuous as well.

$\square$

Proof. *(of the main theorem)*

(1) By 3.3 and 3.4 we only need to show: $ev_{\mathcal{G},\zeta}$ factors through an equivalence of categories: $ev_{\mathcal{G},\zeta} : \mathcal{C} \to \mathcal{C}(\Pi^{op})$ But this follows almost straightforwardly from 3.2. Indeed,

- $ev_{\mathcal{G},\zeta}$ is essentially surjective: For any $E \in \mathcal{C}(\Pi^{op})$ since E is equipped with the discrete topology, the action of Π^{op} on E factors through a finite quotient $Aut_{\mathcal{C}}(X)$ with $X \in \mathcal{G}$ and we apply 3.2 in $\mathcal{C}^X$.

- $ev_{\mathcal{G},\zeta}$ is fully faithful: For any $Z, Z' \in \mathcal{C}$, there exists $X \in \mathcal{G}$ such that $X \geq Z, X \geq Z'$ and, again, this allows us to apply 3.2 in $\mathcal{C}^X$.

(2) Let $F_i : \mathcal{C} \to FSets, i = 1, 2$ be fibre functors. Then any $\zeta^i \in \prod_{X\in\mathcal{G}} F_i(X)$ induces a functor isomorphism $ev_{\mathcal{G},\underline{\zeta}^i} : \varinjlim_{(X,\zeta_X)\in\mathcal{G}^{\underline{\zeta}^i}} Hom_{\mathcal{C}}(X,-) \to F(-)$, so we only need to prove that $\varinjlim_{(X,\zeta_X)\in\mathcal{G}^{\underline{\zeta}^1}} Hom_{\mathcal{C}}(X,-) \simeq \varinjlim_{(X,\zeta_X)\in\mathcal{G}^{\underline{\zeta}^2}} Hom_{\mathcal{C}}(X,-)$, but this follows from the universal property of the projective limit.

$\square$

4. Functoriality of Galois categories

Definition 4.1. *(fundamental functor)* Let $\mathcal{C}, \mathcal{C}'$ be two Galois categories. Then a covariant functor $H : \mathcal{C} \to \mathcal{C}'$ is a **fundamental functor** from $\mathcal{C}$ to $\mathcal{C}'$ if there exists a fibre functor $F' : \mathcal{C}' \to FSets$ such that $F' \circ H : \mathcal{C} \to FSets$ is again a fibre functor for $\mathcal{C}$[2]

---

[2]Or equivalently, if for any fibre functor $F' : \mathcal{C}' \to FSets$, the functor $F' \circ H : \mathcal{C} \to FSets$ is again a fibre functor, since two fibre functors of a Galois category are isomorphic. A fundamental functor is called an exact functor in [Gro71].

Let $\mathcal{C}$ (resp. $\mathcal{C}'$) be Galois categories with F(resp. F') a fibre functor on $\mathcal{C}$(resp. on $\mathcal{C}'$). Let $H : \mathcal{C} \to \mathcal{C}'$ be a fundamental functor such that $F' \circ H = F$. We can then define a morphism of profinite groups:

$$u_H : \pi_1(\mathcal{C}', F') \to \pi_1(C, F)$$

as follows: $\forall \Theta' \in \Pi'$, $\Theta' \in Aut(F')$, hence there exists a $\Theta \in Aut(F' \circ H)$ such that $\forall X \in \mathcal{C}, F(X) = F' \circ H(X)$, the following diagram is commutative:

$$
\begin{array}{ccc}
F' \circ H(X) & \xrightarrow{\Theta'_{H(X)}} & F' \circ H(X) \\
\Big\| & & \Big\| \\
F(X) & \xrightarrow{\quad \Theta \quad} & F(X)
\end{array}
\quad ,
$$

we define

$$u_H : \pi_1(\mathcal{C}', F') \to \pi_1(C, F) \quad \Theta' \mapsto \Theta.$$

Claim 4.2. The morphism u_H above is continuous.

Conversely, Let $u : \Pi' \to \Pi$ be a morphism of profinite groups, then any $X \in \mathcal{C}(\Pi)$ can be endowed with a continuous action of Π' via $u : \Pi' \to \Pi$, which defines a canonical fundamental functor:

$$H_u : \mathcal{C} \to \mathcal{C}(\Pi')$$

We obtain in this way the following commutative diagram of categories:

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\quad H \quad} & \mathcal{C}' \\
F \downarrow \cong & & F' \downarrow \cong \\
\mathcal{C}(\Pi)(H_{u_H}) & \longrightarrow & \mathcal{C}'
\end{array}
$$

Now we assume $\mathcal{C} = \mathcal{C}(\Pi), \mathcal{C}' = \mathcal{C}(\Pi'$ and $H = H_u$ for some morphism of profinite groups $u : \Pi' \to \Pi$. We have the following:

Proposition 4.3.   *(1) u is a trivial morphism if and only if for any object $X \in \mathcal{C}$, $H(X)$ is totally split in $\mathcal{C}'$.*

   (2) u is a monomorphism if and only if for any object $X' \in \mathcal{C}'$ there exists an object X in $\mathcal{C}$ and a connected component X'_0 of $H(X)$ which dominates X' in $\mathcal{C}'$.

   (3) u is an epimorphism iff one of the following two equivalent assertions are verified:

     (a) H sends connected objects to connnected objects;

     (b) H is fully faithful.

   (4) u is an isomorphism if and only if H is an equivalence of categories.

   (5) If $\mathcal{C} \xrightarrow{H} \mathcal{C}' \xrightarrow{H'} \mathcal{C}''$ is a sequence of fundamental functors of Galois categories with corresponding of profinite groups $\Pi \xleftarrow{u} \Pi' \xleftarrow{u'} \Pi''$. Then,

     (a) $ker(u) \supset im(u')$ if and only if $H'(H(X))$ is totally split in $\mathcal{C}''$, $X \in \mathcal{C}$;

(b) $ker(u) \subset im(u')$ if and only if for any connected object $X' \in \mathcal{C}'$ such that $H'(X')$ has a section in $\mathcal{C}''$, there exists $X \in \mathcal{C}$ and a connected component X'_0 of $H(X)$ which dominates X' in $\mathcal{C}'$.

Proof. We have the following lemma:

Lemma 4.4. *Given* $(X, \zeta) \in \mathcal{C}^{pt}$, *we will write* $(X, \zeta)_0 := (X_0, \zeta)$, *where* X_0 *denotes the connected component of* ζ *in* X. *We say that an object* $X \in \mathcal{C}$ *has a section in* $\mathcal{C}$ *if* $e_{\mathcal{C}} \geq X$ *and that an object* $X \in \mathcal{C}$ *is totally split in* $\mathcal{C}$ *if it is isomorphic to a finite coproduct of final objects.*

For any open subgroup $U \subset \Pi$,

(1) *$im(u) \subset U$ if and only if* $(e_{\mathcal{C}'}, \star) \geq (H(\Pi/U), 1))$ *in* $\mathcal{C}'^{pt}$;

(2) *Let:*
$$K_\Pi(im(u)) \lhd \Pi$$
denote the smallest normal subgroup in Π *containing* $im(u)$. *Then* $K_\Pi(im(u)) \subset U$ *if and only if* $H(\Pi/U)$ *is totally split in* $\mathcal{C}'$.
In particular, $u : \Pi' \to \Pi$ *is trivial if and only if for any object* $X \in \mathcal{C}$, $H(X)$ *is totally split in* $\mathcal{C}'$.

For any open subgroup $U' \subset \Pi'$,

(1) *$ker(u) \subset U'$ if and only if there exists an open subgroup* $U \subset \Pi$ *such that:* $(H(\Pi/U), 1)_0 \geq (\Pi'/U', 1)$ *in* $\mathcal{C}'^{pt}$.

(2) *if, furthermore,* $u : \Pi' \twoheadrightarrow \Pi$ *is an epimorphism, then* $Ker(u) \subset U'$ *if and only if there exists an open subgroup* $U \subset \Pi$ *and an isomorphism* $(H(\Pi/U), 1)_0 \to (\Pi'/U', 1)$ *in* $\mathcal{C}'^{pt}$.

In particular,

(1) *$u : \Pi' \hookrightarrow \Pi$ is a monomorphism if and only if for any connected object* $X' \in \mathcal{C}'$ *there exists a connected object* $X \in \mathcal{C}$ *and a connected component* $H(X)_0$ *of* $H(X)$ *in* $\mathcal{C}$ *such that* $H(X)_0 \geq X'$ *in* $\mathcal{C}'$.

(2) *if, furthermore,* $u : \Pi' \twoheadrightarrow \Pi$ *is an epimorphism, then* $u : \Pi' \twoheadrightarrow \Pi$ *is an isomorphism if and only if* $H : \mathcal{C} \to \mathcal{C}'$ *is essentially surjective.*

Proof. *(of lemma)* Since a closed subgroup S of a profinite group Π is the intersection of all the open subgroups of Π containing S, $\{1\}$ is the intersection of all open subgroups of Π. This yields the characterization of trivial morphisms and monomorphisms from the preceding assertions in (1) and (2).

(1) Note that $(e_{\mathcal{C}}, \star) \geq (H(\Pi/U), 1)$ in $\mathcal{C}'^{pt}$ if and only if the unique map $\phi : \star \to H(\Pi/U)$ sending $\star$ to U is a morphism in $\mathcal{C}'$ i.e., if and only if for any $\theta' \in H'$,
$$U = \phi(\star) = \phi(\theta' \cdot \star) = \theta' \cdot \phi(\star) = u(\theta)U.$$

For the second assertion of (1), note that $K_\Pi(Im(u)) \subset U$ if and only if for any $g \in \Pi/U$, the map $\phi_g : \star \to H(\Pi/U)$ sending $\star$ to gU is a morphism in $\mathcal{C}'$. This yields a surjective morphism $\bigsqcup_{g \in \Pi/U} \phi_g : \bigsqcup_{g \in \Pi/U} \star \to H(\Pi/U)$ in $\mathcal{C}'$, which is automatically injective by cardinality. Conversely, for any isomorphism $\bigsqcup_{i \in I} \phi_i : \bigsqcup_{i \in I} \star \to H(\Pi/U)$ in $\mathcal{C}'$, set $i_i : \star \to H(\Pi/U)$ for the morphism $\star \hookrightarrow \bigsqcup_{i \in I} \star \to H(\Pi/U)$ in $\mathcal{C}'$; by construction $i_i = \phi_{i_i}(\star)$.

(2) Since U' is closed of finite index in Π' and both Π and Π' are compact, $u(U')$ is closed of finite index in $im(u)$ hence open. So there exists an open subgroup $U \subset \Pi$ such that $U \cap im(u) \subset u(U')$. By definition, the connected component of 1 in $H(\Pi/U)$ in $\mathcal{C}'$ in $\mathcal{C}'$ is :

$$im(u)U/U \simeq im(u)/(U \cap im(u)) \simeq \Pi'/u^{-1}(U).$$

But $u^{-1}(U) = u^{-1}(U \cap im(u)) \subset U'$, whence a canonical epimorphism

$$(im(u)U/U, 1) \to (\Pi'/U', 1)$$

in $\mathcal{C}'^{pt}$. If furthermore, $im(u) = \Pi$, then one can take $U = u(U')$ and ϕ is nothing but the inverse of the canonical isomorphism $\Pi'/U' \to \Pi/U$. Conversely, assume that there exists an open subgroup $U \subset \Pi$ and a morphism $\phi : (im(u)U/U, 1) \to (\Pi'/U', 1)$ in $\mathcal{C}'^{pt}$. Then for any $g' \in \Pi'$, one has: $\phi(u(g')U) = g' \cdot \phi(1) = g'U'$. In particular, if $u(g') \in ker(u)$ then $g'U = \phi(u(g')U) = \phi(U) = U'$ whence $ker(u) \subset U'$. Eventually, note that since $ker(u)$ is normal in Π', the condition $ker(u) \subset U'$ does not depend on the choice of $\zeta \in F(X)$ defining the isomorphism $X' \to \Pi'/U'$.

$\square$

(1), (2) and (5) follows from 4.4 (2). (4) follows from 4.4 and (3). To prove assertion (3), we will show that u is an epimorphism $\Rightarrow$ (a) $\Rightarrow$ (b) $\Rightarrow$ u is an epimorphism. For the first implication, assume that u is an epimorphism. Then, for any connected object X in $\mathcal{C}(\Pi)$, the group Π acts transitively on X. But $H(X)$ is just X equipped with the Π' action $g' \cdot x = u(g') \cdot x$, $g' \in \Pi'$. Hence Π' acts transitively on $H(X)$ as well, i.e $H(X)$ is connected. For $(a) \Rightarrow u$ is an epimorhism, assume that if $X \in \mathcal{C}$ is connected then $H(X)$ is also connected in $\mathcal{C}'$. This holds, in particular, for any finite quotient Π/N of Π with N a normal open subgroup of Π, i.e. the canonical morphism $u_N : \Pi' \xrightarrow{\ u\ } \Pi \xrightarrow{\ pr_N\ } \Pi/N$ is a continuous epimorphism. Hence so is $u = \varprojlim u_N$. The implication $\Rightarrow (b)$ is straightforward. For $(b) \Rightarrow u$ is an epimorphism, observe that given an open subgroup $U \subset \Pi$, $U \neq \Pi$ there is no morphism from $\star$ to Π/U in $\mathcal{C}$. Hence if $H : \mathcal{C} \to \mathcal{C}'$ is fully faithful, there is no morphism as well from $\star$ to Π/U in $\mathcal{C}$. But from 4.4, this is equivalent to $im(u) \nsubseteq U$.

$\square$

Etale covers

Let X be a connected, locally noetherian scheme, and $\mathcal{C}_S$ the category of finite étale covers of X. The aim of this chapter is to prove that $\mathcal{C}_S$ is Galois.

For simplicity, we assume all the schemes are **locally noetherian** even though some results stated in the following sections remain valid without this assumption.

1. Some results in scheme theory.

Definition 1.1. Assume that $S = spec(A)$ is affine and let $P \in A[T]$ be a monic polynomial such that $P' \neq 0$. Set $B := A[T]/PA[T]$ and $C = B_b$ where $b \in B$ is such that $P'(t)$ becomes invertible in B_b (here t denotes the image of T in B). Then $spec(C) \to S$ is an étale morphism. We call such a morphism a **standard étale morphisms**.

Theorem 1.2. (Local structure of étale morphisms, cf. [Mil80], Thm. 3.14 and Rem. 3.15) *Let A be a noetherian local ring and set $S = spec(A)$. Let $\phi : X \to S$ an unramified (resp. étale) morphism. Then, for any $x \in X$, there exists an open neibourhood U of x such that one has a factorization:*

$$
\begin{array}{ccc}
U & \hookrightarrow & spec(C) \\
\downarrow{\scriptstyle \phi} & \swarrow & \\
S & &
\end{array}
$$

where $spec(C) \to S$ is a standard étale morphism and $U \hookrightarrow spec(C)$ is an immersion (resp. an open immersion).

Definition 1.3. An **étale cover** is a surjective finite étale morphism.

Lemma 1.4. (Stability) *Let P be a property of morphisms of shemes: We consider the following five conditions on P:*

(1) *P is stable under composition.*

(2) *P is stable under arbitrary base-change.*

(3) *closed immersions have P.*

(4) *P is stable by fibre products.*

(5) *For any $X \xrightarrow{f} Y \xrightarrow{g} Z$, if g is separated and $g \circ f$ has P then f has P.*

Then we have $(1) + (2) \Rightarrow (4)$, and $(1) + (2) + (3) \Rightarrow (5)$.

Example 1.5. The properties $P=$ surjective, flat, unramified, étale satisfy (1) and (2) hence (4). The properties $P=$separated, proper, finite satisfy (1), (2),(3) hence (4) and (5).

Lemma 1.6. (Topological properties of finite morphisms)

(1) A finite morphism is closed;

(2) A finite flat morphism is open.

Remark 1.7.   (1) Since being finite is stable under base-change, 1.6 (1) shows that a finite morphism is universally closed. Since finite morphisms are affine hence separated, this shows that finite morphisms are proper.

(2) 1.6 (2) also holds for flat morphisms which are locally of finite type.

Corollary 1.8. *Let S be a connected scheme. Then any finite étale morphism $\phi : X \to S$ is automatically an étale cover.*

2. The category of étale covers of a connected scheme

Let S be a connected scheme and Sch/S the category of $S-$schemes. Let $\mathcal{C}_S$ denote the full subcategory of Sch/S whose objects are étale covers of S.
Given a geometric point[1] $\bar{s} : spec(\Omega) \to S$, the underlying set associated with the scheme $X_{\bar{s}} := X \times_S spec(\Omega)$ will be denoted by $|X_{\bar{s}}|$. One thus obtain a functor:

$$F_{\bar{s}} : \quad \mathcal{C}_S \quad \to \quad FSets$$
$$\phi : X \to S \quad \to \quad |X_{\bar{s}}|$$

Theorem 2.1. *The category of étale covers of S is Galois. And for any geometric point $\bar{s} : spec(\Omega) \to S$, the functor $F_{\bar{s}} : \mathcal{C} \to FSets$ is a fibre functor for $\mathcal{C}_S$.*

By analogy with topology, for any geometric point $\bar{s} : spec(\Omega) \to S$, the profinite group:

$$\pi_1(S; \bar{s}) := \pi_1(\mathcal{C}_S; F_{\bar{s}})$$

is called the **étale fundamental group** of S with base point $\bar{s}$. Similarly, for any two geometric points $\bar{s}_i : spec(\Omega_i) \to S, i = 1, 2$, the set:

$$\pi_1(S; \bar{s}_1, \bar{s}_2) := \pi_1(\mathcal{C}; F_{\bar{s}_1}, F_{\bar{s}_2})$$

is called the set of étale paths from $\bar{s}_1$ to $\bar{s}_2$(Note that Ω_1 and Ω_2 may have different charateristics).

Proof. We check axioms (1) to (6) of the definition of a Galois category.
Axiom(1): The category of étale covers of S has a final object: $Id_S : S \to S$ and from 1.4, the fibre product (in the category of $S-$shemes) of any two étale covers of S over a third one is again an étale cover of S.
Axiom(2): The category of étale covers of S has an initial object: $\emptyset$ and the coproduct (in the category of $S-$schemes) of any two étale covers of S over a third one is again an étale cover of S. moreover we have the following:

Lemma 2.2. *Categorical quotients by finite groups of automorphisms exists in $\mathcal{C}_S$.*

Proof. Step 1: Assume first that $S = spec(A)$ is an affine scheme. Since étale cover are, in particular, finite hence affine morphisms, $\phi : X \to S$ is induced by a finite

---

[1]We recall that a geometric point of a scheme S is a morphism $\bar{s} : spec(\Omega) \to S$ where Ω is an algebraically closed field.

$A-$algebra $\phi^\# : A \to B$. But, then, if follows from the equivalence of category between the category of affine $S-$schemes and $(Alg/A)^{op}$ that the factorization

$$X \xrightarrow{\ p_G\ } spec(B^{G^{op}}) =: G\backslash X$$
$$\downarrow \phi \qquad \swarrow \phi_G$$
$$S$$

is the categorical quotient of $\phi : X \to S$ by G in the category of affine $S-$schemes. So, as $\mathcal{C}_S$ is a full subcategory of the category of affine $S-$schemes, to prove that $\phi_G : G\backslash X \to S$ is the categorical quotient of $\phi : X \to S$ by G in $\mathcal{C}_S$, it remains to prove that $\phi_G : G\backslash X \to S$ is in $\mathcal{C}_S$.

Lemma 2.3. (trivialization)*: An affine, surjective morphism $\phi : X \to S$ is an étale cover if and only if there exists a finite faithfully flat morphism $f : S' \to S$ such that the first projection $\phi' : X' := S' \times_S X \to S'$ is a totally split étale cover of S'.*

We apply the above lemma 2.2 to the quotient morphism $\phi_G : G\ X \to S$. For this, first apply 2.2 to the étale cover $\phi : X \to S$ to obtain a faithfully flat $A-$algebra $A \to A'$ such that $B \otimes_A A' = A'^n$ as $A'-$algebras. Tensoring the following exact sequence of $A-$algebras by the flat $A-$algebra A':

$$0 \to B^{G^{op}} \to B \xrightarrow{\sum_{g\in G^{op}}(Id_B-g)} \bigoplus_{g\in G^{op}} B$$

one gets the exact sequence of $A'-$algebras:

$$0 \to B^{G^{op}} \otimes_A A' \to B \otimes_A A' \xrightarrow{\sum_{g\in G^{op}}(Id_B-g)\otimes_A Id_{A'}} \bigoplus_{g\in G^{op}} B \otimes_A A'$$

whence:

(2.1) $$B^{G^{op}} \otimes_A A' = (B \otimes_A A')^{G^{op}} = (A'^n)^{G^{op}}.$$

But G^{op} is a subgroup of $Aut_{Alg/A'}(A'^n)$, which is nothing but the symmetric group $\mathcal{S}_n$ acting on the canonical coordinates $E := \{1,...n\}$ in A'^n. Hence:

$$(A'^E)^{G^{op}} = \otimes_{G\backslash E} A'.$$

In terms of schemes, if $f : S' \to S$ denotes the faithfully flat morphism corresponding to $A \hookrightarrow A'$ then $S' \times_S X$ is just the coproduct of n copies of S' over which G acts by permutation and 2.1 becomes

$$S' \times_S (G\backslash X) = G\backslash(\sqcup_E S') = \sqcup_{G\backslash E} S'.$$

Step 2, general case. We reduce to Step 1 by covering S with affine open subschemes(local existence) and using the unicity of categorical quotient up to canonical isomorphism (glueing) $\square$

Axiom (3): Before dealing with axiom (3), let us recall that, in the category of $S-$schemes, open immersions are monomorphisms and that:

Theorem 2.4. (Grothendieck-see[Mil80]) *In the category of $S-$schemes, faithfully flat morphisms of finite type are strict epimorphisms.*

Lemma 2.5. *Given a commutative diagram of schemes:*

$$
\begin{array}{ccc}
Y & \xrightarrow{\ u\ } & X. \\
\downarrow{\scriptstyle \psi} & \swarrow{\scriptstyle \phi} & \\
S & &
\end{array}
$$

If $\phi : X \to S$, $\psi : Y \to S$ are finite étale morphisms then $u : Y \to X$ is a finite étale morphism as well.

Axiom (4): For any étale covers $\phi : X \to S$ one has $F_{\bar{s}}(\phi) = \star$ if and only if $r(\phi) = 1$, which in turn, is equivalent to the fact that $\phi : X \to S$ is an isomorphism. Also, it follows straightforwardly from the universal property of fibre product and the definition of $F_{\bar{s}}$ that $F_{\bar{s}}$ commutes with fibre products.

Axiom (5): The fact that $F_{\bar{s}}$ commutes with finite coproducts and transforms strict epimorphisms into strict epimorphisms is straightforward. So it only remains to prove that $F_{\bar{s}}$ commutes with categorical quotients by finite groups of automorphisms. Let $\phi : X \to S$ be an étale cover and $G \subset Aut_{Sch/S}(\phi)$ a finite subgroup. Since the assertion is local on S, it follows from 2.2 that we may assume that $\phi : X \to S$ is totally split and that G acts on X by permuting the copies of S. But, then, the assertion is immediate since $G \backslash X = \sqcup_{G \backslash F_{\bar{s}}(\phi)} S$.

Axiom (6): For any two étale covers $\phi : X \to S$, $\psi : Y \to S$, let $u : Y \to X$ be a morphism over S such that $F_{\bar{s}}(u) : F_{\bar{s}}(\psi) \to F_{\bar{s}}(\phi)$ is bijective. It follows from 2.5 that $u : Y \to X$ is finite étale but, by assumption, it is also surjective hence $u : Y \to X$ is an étale cover. Moreover, still by assumption, it has rank 1 hence it is an isomorphism by 1.8. $\qquad\square$

Corollary 2.6.   *(1) $\pi_1(S; \bar{s})$ is a profinite group, and the action of $\pi_1(S; \bar{s})$ on $F_{\bar{s}}(X)$ is continuous.*

  (2) The induced functor

$$
\begin{array}{rccc}
For : & \mathcal{C}_S & \to & \mathcal{C}(\pi_1(S; \bar{s})) \\
 & X & \mapsto & X_{\bar{s}}
\end{array}
$$

  is an equivalence of categories.

3. Reformulation of functoriality

Let S and S' be connected schemes, equipped with geometric points $\bar{s} : Spec(\Omega) \to S$ and $\bar{s}' : Spec(\Omega) \to S'$, respectively. Assume given a morphism $\phi : S' \to S$ with $\phi \circ \bar{s}' = \bar{s}$. Then ϕ induces a base change functor

$$
\begin{array}{rccc}
H_\phi : & \mathcal{C}_S & \to & Fet_{S'} \\
 & X & \mapsto & X \times_S S'
\end{array}
$$

and a morphism $X \to Y$ to the induced morphism $X \times_S S' \to Y \times_S S'$. Moreover, the condition $\phi \circ \bar{s}' = \bar{s}$ implies that there is an equality of functors $F_{\bar{s}} = F_{\bar{s}'} \circ H_\phi$. So H_ϕ is a fundamental functor and we denote by $\phi_\star$ the corresponding morphism of fundamental groups $\phi_\star : \pi_1(S'; \bar{s}') \to \pi_1(S; s)$. Hence we can reformulate the functoriality properties as follows:

Proposition 3.1.   *(1) The map $\phi_\star$ is trivial if and only if for every connected finite étale cover $X \to S$ the base change $X \times_S S'$ is a trivial cover (i.e. isomorphic to a finite disjoint union of copies of S').*

(2) The map $\phi_\star$ is surjective if and only if for every connected finite étale cover $X \to S$ the base change $X \times_S S'$ is connected as well.

(3) The map $\phi_\star$ is injective if and only if for every connected finite étale cover $X' \to S'$ there exists a finite étale cover $X \to S$ and a morphism $X_i \to X'$ over S', where X_i is a connected component of $X \times_S S'$. In particular, if every connected finite étale cover $X' \to S'$ is of the form $X \times_S S' \to S'$ for a finite étale cover $X \to S$, then ϕ is injective.

(4) Let $S'' \xrightarrow{\psi} S' \xrightarrow{\phi} S$ be a sequence of morphisms of connected schemes, and let $\bar{s}, \bar{s}', \bar{s}''$ be geometric points of S, S', and S'' respectively, satisfying $\bar{s} = \phi \circ \bar{s}'$ and $\bar{s}' = \psi \circ \bar{s}''$. The sequence

$$\pi_1(S''; \bar{s}'') \xrightarrow{\psi_\star} \pi_1(S'; \bar{s}') \xrightarrow{\phi_\star} \pi_1(S; \bar{s})$$

is exact if and only if the following two conditions are satisfied.

(a) For every finite étale cover $X \to S$ the base change $X \times_S S'' \to S''$ is a trivial cover of S''.

(b) Given a connected finite étale cover $X' \to S'$ such that $X' \times_{S'} S''$ has a section over S'', there exists a connected finite étale cover $X \to S$ and an S'-morphism from a connected component of $X \times_S S'$ onto X'.

Properties and examples of the étale fundamental group

1. Spectrum of a field

Let k be a field and set $S := spec(k)$. Then:

Proposition 1.1. *For any geometric point $s : spec(\Omega) \to S$, there is a profinite group isomorphism:*

$$c_s : \pi_1(S, s) \simeq \Gamma_k.$$

Proof. Let $k \hookrightarrow \bar{k}$ be a fixed algebraic closure of k which defines a geometric point $\bar{s} : spec(\bar{s}) \to S$. Then by the main theorem of Galois categories, we have $\pi_1(S, s) \simeq \pi_1(S, \bar{s})$, canonically up to inner automorphisms. Hence we can assume that $\Omega = \bar{k}$. Let k_s be the separable closure of k in $\bar{k}$

We recall that X is a finite étale cover of S iff $X = spec(L)$ where L is a finite étale $k-$algebra. And X is a connected object in the category $\mathcal{C}_S$ iff L is a finite separable field extension of k.

$$
\begin{array}{rcl}
Fib_{\bar{s}}(X) & = & |X \times_S \bar{s}| \\
= |spec(L \otimes_k \Omega)| & \simeq & Hom_\Omega(L \otimes_k \Omega, \Omega) \\
\text{given by} \quad ker(\phi) & \leftarrow & \phi \\
= Hom_k(L, \Omega) & = & Hom_k(L, k_s)
\end{array}
.
$$

Hence we get $Fib_{\bar{s}} \cong Hom_k(-, k_s)$.Hence

$$
\begin{aligned}
\pi_1(S, s) &= Aut(Fib_{\bar{s}}) \\
&\cong Aut(Hom(-, k_s)) \\
&\cong Gal(k_s|k) = \Gamma_k
\end{aligned}
.
$$

Where the isomorphism $Gal(k_s|k) \cong Aut(Hom(-, k_s))$ is given sending $\sigma \in Gal(k_s|k)$ to

$$
\begin{array}{rccc}
\sigma_\star : & Hom(-, k_s) & \to & Hom(-, k_s) \\
& \phi & \mapsto & \sigma \circ \phi
\end{array}
.
$$

$\square$

2. The first homotopy sequence.

Let S be a connected scheme, $f : X \to S$ a proper morphism such that $f_*\mathcal{O}_X = \mathcal{O}_S$ and $s \in S$. Fix a geometric point $x_\Omega : spec(\Omega) \to X_{\bar{s}}$ with image again denoted by x_Ω in X and s_Ω in S.

Theorem 2.1. (First homotopy sequence) *Consider the canonical sequence of profinite groups induced by $(X_{\bar{s}}, x_\Omega) \to (X, x_\Omega) \to (S, s_\Omega)$:*

$$\pi_1(X_{\bar{s}}, x_\Omega) \xrightarrow{i} \pi_1(X, x_\Omega) \xrightarrow{p} \pi_1(S, s_\Omega) \to 1$$

Then $p : \pi_1(X, x_\Omega) \twoheadrightarrow \pi_1(S, s_\Omega)$ is an epimorphism and $p \circ i = 1$. If, furthermore, $f : X \to S$ is separable[1] then the sequence above is exact.

Proof. The main tool in the proof is the Stein factorization theorem. Let us recall that:

Theorem 2.2. (Stein factorization of a proper morphism) *Let $f : X \to S$ be a morphism such that $f_\star \mathcal{O}_X$ is a quasicoherent $\mathcal{O}_S-$algebra. Then $f_\star \mathcal{O}_X$ defines an $S-$scheme:*

$$p : S' = spec(f_\star \mathcal{O}_X) \to S$$

and $f : X \to S$ factors canonically as:

$$
\begin{array}{ccc}
S & \xleftarrow{\ p\ } & S' \\
{\scriptstyle f}\Big\uparrow & \nearrow{\scriptstyle f'} & \\
X & &
\end{array}
\quad .
$$

Furthermore,

(1) If $f : X \to S$ is proper then

(a) $p : S' \to S$ is finite and $f' : X \to S'$ is proper and with geometrically connected fibres;

(b) • *The set of connected components of X_s is one-to-one with $S_s'^{set}$, $s \in S$;*

   • *The set of connected components of $X_{\bar{s}}$ is one-to-one with $S_{\bar{s}}'^{set}$, $s \in S$.*

   In particular, if $f_\star \mathcal{O}_X = \mathcal{O}_S$ then $f : X \to S$ has geometrically connected fibres.

(2) If $f : X \to S$ is proper and separable then $p : S' \to S$ is an étale cover. In particular, $f_\star \mathcal{O}_X = \mathcal{O}_S$ if and only if $f : X \to S$ has geometrically connected fibres.

For a proof of this theorem, c.f. [Har77], Chapter III, Corollary 11.5.

Corollary 2.3. *Let $f : X \to S$ be a proper morphism such that $f_\star \mathcal{O}_X = \mathcal{O}_S$. Then, if S is connected, X is connected as well.*

Proof. *(of corollary)* It follows from (1)(b) of 2.2 that, if $f_\star \mathcal{O}_X = \mathcal{O}_S$ then f is geometrically connected and, in particular, has connected fibres. But, as $f : X \to S$ is proper, it is closed and $f_\star \mathcal{O}_X$ is coherent hence:

$$f(X) = supp(f_\star \mathcal{O}_X).$$

So $f_\star \mathcal{O}_X \cong \mathcal{O}_S$ also implies that $f : X \to S$ is surjective. As a result, if $f_\star \mathcal{O}_X = \mathcal{O}_S$ then f is closed, surjective, with connected fibres so, if S is connected, this forces X to be connected as well. □

---

[1]We recall that a scheme X over a field k is **separable** over k if for any field K of k the scheme $X \times_k K$ is reduced. This is equivalent to requiring that X be reduced and that, for any generic point η of X, the extension $k \hookrightarrow k(\eta)$ be separable. And a scheme X over a scheme S is separable over S if it is flat over S and for any $s \in S$ the scheme X_s is separable over $k(s)$. Separable morphisms satisfy the following elementary properties: (1) Any base change of a separable morphism is separable. (2) If $X \to S$ is separable and $X' \to X$ is étale then $X' \to S$ is separable.

For the proof of the theorem, we will apply the criterion of functoriality of fundamental functors. We first give a lemma stating under less restrictive hypothesis, $p \circ i = 1$ is true.

Lemma 2.4. *Let X, S be connected schemes, $f : X \to S$ a geometrically connected morphism and $s \in S$. Fix a geometric point $x_\Omega : spec(\Omega) \to X_{\bar{s}}$ with image again denoted by x_Ω in X and s_Ω in S and consider the canonical sequence of profinite groups induced by $(X_{\bar{s}}, x_\Omega) \to (X, x_\Omega) \to (S, s_\Omega)$:*

$$\pi_1(X_{\bar{s}}, x_\Omega) \xrightarrow{i} \pi_1(X, x_\Omega) \xrightarrow{p} \pi_1(S, s_\Omega).$$

Then, one always has $i \circ p = 1$.

Proof. Let $\phi : S' \to S$ be an étale cover and let $S'_{\bar{s}} := X_{\bar{s}} \times_S S'$, then

$$\begin{aligned} S'_{\bar{s}} = X_{\bar{s}} \times_S S' &= (X \times_S spec(k(\bar{s}))) \times_S S' \\ &= X \times_S (spec(k(\bar{s})) \times_S S') \\ &= X \times_S \sqcup_{S'_{\bar{s}}} spec(k(\bar{s})) \\ &= \sqcup_{S'_{\bar{s}}} X_{\bar{s}}. \end{aligned}$$

Hence $\bar{S}' \to X_{\bar{s}}$ is totally split. hence $p \circ i = 1$. $\qquad\square$

Now we return to the proof of 2.1.

- Exactness on the right: for any connected étale cover $\phi : S' \to S$ and let $X' := X \times_S S'$,with the notation for base change:

$$\begin{CD} X' @>{\phi'}>> X \\ @V{f'}VV \square @VV{f}V \quad\cdot \\ S' @>{\phi}>> S \end{CD}$$

Since $f'_\star(\mathcal{O}_{X'}) = f'_\star(\phi'^\star \mathcal{O}_X) = \phi^\star f_\star \mathcal{O}_X = \phi^\star \mathcal{O}_S = \mathcal{O}_{S'}$ and f' is proper, it follows from 2.2 (1)(b) that X' is connected.

- Exactness in the middle: by 2.4 we only need to show $ker(p) \subset im(i)$. Let $\phi : X' \to X$ be a connected étale cover, with the following notation:

$$\begin{CD} X' @>{\phi}>> X @>{f}>> S \\ @AAA \square @AAA \square @AA{\bar{s}}A \\ X'_{\bar{s}} @>{\bar{\phi}}>> X_{\bar{s}} @>>> k(\bar{s}) \end{CD}$$

Assume that $\bar{\phi}$ admits a section σ, now we prove that there is a model of ϕ which isa connected étale cover $S' \to S$.

As ϕ is finite étale and f is proper and separable, $g := f \circ \phi : X' \to S$ is also proper and separable. Consider its Stein factorization $X' \xrightarrow{g'} S' \xrightarrow{p} S$. By 2.2 (2), p is étale. Furthermore, as X' is connected and g' is surjective, S' is connected. The base change of $\phi : X' \to X$ has a section implies that the

base change of $X \times_S S'$ has a section as well. Hence the finite étale cover $X' \to X \times_S S'$ must be of degree 1, i.e. an isomorphism.

$\square$

Remark 2.5. In general, i is not injective.

Corollary 2.6. *Let X/k be a connected proper scheme over algebraically closed field k. Let $k \subset \Omega$ be an algebraically closed field extension, $x_\Omega \in X(\Omega)$ be a geometric point. Then the canonical morphism of profinite groups:*

$$\pi_1(X_\Omega; x_\Omega) \to \pi_1(X; x_\Omega)$$

induced by $(X_\Omega; x_\Omega) \to (X; x_\Omega)$ is an isomorphism.

The proof of 2.6 uses the following lemmas:

Lemma 2.7. (Künneth formula) *Let k be an algebraically closed field, X a connected, proper scheme over k and Y a connected scheme over k. For any $x : spec(k) \to X$ and $y : spec(k) \to Y$, the canonical morphism of profinite groups:*

$$\pi_1(X \times_k Y; (x, y)) \to \pi_1(X; x) \times \pi_1(Y; y)$$

induced by the projections $p_X : X \times_k Y \to X$ and $p_Y : X \times_k Y \to Y$ is an isomorphism.

Proof. From the following theorem:

Theorem 2.8. (c.f. [Gro71], Chap. VIII) *Let X be a scheme and $i : X^{red} \hookrightarrow X$ be the underlying reduced closed subscheme. Then the functor $i^* : \mathcal{C}_X \to \mathcal{C}_{X^{red}}$ is an equivalence of categories. In particular, if X is connected, it induces an isomorphism of profinite groups:*

$$\pi_1(i) : \pi_1(X^{red}) \to \pi_1(X)$$

we may assume that X is reduced hence, as k is algebraically closed, that X is separable over k. As X is proper, separable, geometrically connected and surjective over k, so is its base change $p_Y : X \times_k Y \to Y$. So, it follows from 2.2 (2) that $p_{Y*}\mathcal{O}_{X \times_k Y} = \mathcal{O}_Y$. Thus, one can apply 2.1 to $p_Y : X \times_k Y \to Y$ to get an exact sequence:

$$\pi_1((X \times_k Y)_y; x) \to \pi_1(X \times_k Y; (x, y)) \to \pi_1(Y; y) \to 1.$$

Furthermore, $X = (X \times_k Y)_y \to X \times_k Y \xrightarrow{p_X} X$ is the identity so $p_X : X \times_k Y \to X$ yields a section of $\pi_1(X; x) \to \pi_1(X \times_k Y; (x, y))$. $\square$

Proof. *(of corollary 2.6)* Surjectivity: Let $\phi : Y \to X$ be a connected étale. We are to prove that Y_Ω is again connected. But, as k is algebraically closed, if Y is connected then it is automatically geometrically connected over k and, in particular, Y_Ω is connected. Injectivity: One has to prove that for any connected étale cover $\phi : Y \to X_\Omega$, there exists an étale cover $\tilde\phi : \tilde Y \to X$ which is a model of ϕ over X. We begin with general lemma.

Lemma 2.9. *Let X be a connected scheme of finite type over a field k and let $k \hookrightarrow \Omega$ be a field extension of k. Then, for any étale cover $\phi : Y \to X_\Omega$, there exists a finitely generated $k-$algebra R contained in Ω and an affine morphism of finite type $\tilde\phi : \tilde Y \to X_R$ which is a model of $\phi : Y \to X_\Omega$ over X_R. Furthermore, if η denotes the generic point of $spec(R)$, then $\tilde\phi_{k(\eta)} : \tilde Y_{k(\eta)} \to X_{k(\eta)}$ is an étale cover.*

Proof. *(of lemma)* Since X is quasi-compact, there exists a finite covering of X by Zariski-open subschemes $X_i := spec(A_i) \hookrightarrow X$, $i = 1, ..., n$, where the A_i are finitely generated $k-$algebra. As $\phi : Y \to X_\Omega$ is affine, we can write $U_i := \phi^{-1}(X_{i\Omega}) = spec(B_i)$, where B_i is of the form:

$$B_i = A_i \otimes_k \Omega[\underline{T}]/\langle P_{i,1}, ..., P_{i,r_i}\rangle.$$

For each $1 \le j \le r_i$, the $\underline{\alpha}$th coefficient of $P_{i,j}$ is of the form:

$$\sum_k r_{i,j,\underline{\alpha},k} \otimes_k \lambda_{i,j,\underline{\alpha},k}$$

with $r_{i,j,\underline{\alpha},k} \in A_i$, $\lambda_{i,j,\underline{\alpha},k} \in \Omega$. So, let R_i denote the sub $k-$algebra of Ω generated by $\lambda_{i,j,\underline{\alpha},k}$ then B_i can also be written as:

$$B_i = A_i \otimes_k R_i[\underline{T}]/\langle P_{i,1}, ..., P_{i,r_i}\rangle \otimes_{R_i} \Omega.$$

Let R denote the sub-k-algebra of Ω generated by the R_i, then $k \hookrightarrow R$ is a finitely generated $k-$algebra and up to enlarging R, one may assume that the glueing data on the $U_i \cap U_j$ decend to R then one can construct $\tilde{\phi}$ by glueing the $spec(A_i \times_k R[\underline{T}]/\langle P_{i,1}, ..., P_{i,r_i}\rangle)$ along these descended gluing data. By construction $\tilde{\phi}$ is affine.

To conclude, since $k(\eta) \hookrightarrow \Omega$ is faithfully flat and $\phi : Y \to X_\Omega$ is finite and faithfully flat, the same is automatically true for $\tilde{\phi}_{k(\eta)} : \tilde{Y}_{k(\eta)} \to X_{k(\eta)}$, which is then étale since $\phi : Y \to X_\Omega$ is. $\qquad \square$

So, applying 2.9 to $\phi : Y \to X_\Omega$ and up to replacing R by R_r for some $r \in R\backslash\{0\}$, ome may assume that $\phi : Y \to X_\Omega$ is the base-change of some étale cover $\phi^0 : Y^0 \to X_R$. Note that, since $Y^0_\Omega = Y$ is connected, both Y^0_η and Y^0 are connected as well. Fix $s : spec(k) \to S$. Since the fundamental group does not depend on the fibre functor, one can assume that $k(x) = k$. Then from 2.7, one gets the canonical isomorphism of profinite groups:

$$\pi_1(X \times_k S; (x, s)) \xrightarrow{\cong} \pi_1(X; x) \times \pi_1(S; s).$$

Let $U \subset \pi_1(X \times_k S; (x, s))$ be the open subgroup corresponding to the étale cover $\phi^0 : Y^0 \to X \times_k S$ and let $U_X \subset \pi_1(X; x)$ and $U_S \subset \pi_1(S; s)$ be open subgroups such that $U_X \times U_S \subset U$. Then U_X and U_S correspond to connected étale covers $\psi_X : \tilde{X} \to X$ and $\psi_S : \tilde{S} \to S$ such that $\phi^0 : Y^0 \to X \times_k S$ is a quotient of $\psi_X \times_k \psi_S : \tilde{X} \times_k \tilde{S} \to X \times_k S$. Consider the following cartesian diagram:

Since $k(\eta) \subset \Omega$ and Ω is algebraically closed, one may assume that any point $\tilde{s} \in \tilde{S}$ above $s \in \tilde{S}$ has residue field contained in Ω and, in particular, one can consider the

associated $\Omega-$point $\tilde{s}_\Omega : spec(\Omega) \to \tilde{S}$. Then, one has the cartesian diagram:

$$
\begin{array}{ccc}
\tilde{Y}_S^0 & \longleftarrow & Y_\Omega \\
\downarrow & & \downarrow \\
X \times_k \tilde{S} & \xleftarrow{Id_X \times_k \tilde{s}_\Omega} & X_\Omega
\end{array}
$$

Again, since Y_Ω is connected, $\tilde{Y}^0$ is connected as well, from which it follows that $\tilde{Y}^0 \to X \times_k \tilde{S}$ corresponds to an open subgroup $V \subset \pi_1(X \times_k \tilde{S}) = \pi_1(X) \times U_S$ containing $\pi_1(\tilde{X} \times_k \tilde{S}) = U_X \times U_S$. Hence $V = U \times U_S$ for some open subgroup $U_X \subset U \subset \pi_1(X)$ hence $\tilde{Y}^0 \to X \times_k \tilde{S}$ is of the form $\tilde{Y} \times_k \tilde{S} \to X \times_k \tilde{S}$ for some étale cover $\tilde{\phi} : \tilde{Y} \to X$. $\qquad\square$

3. More examples

§1. Normal base scheme. Let S be a connected normal[2] scheme. Recall that we only consider locally noetherian scheme, in particular, S is irreducible.

Lemma 3.1. *Let $k(S) \hookrightarrow L$ be a finite separable field extension. Then the normalization of S in $k(S) \hookrightarrow L$ is finite over S.*

When S is normal, we can improve 1.2 as follows.

Lemma 3.2. *Let A be a noetherian integrally closed local ring with fraction field K and set $S = spec(A)$. Let $\phi : X \to S$ an unramified (resp. étale) morphism. Then, for any $x \in X$, there exists an open affine neighborhood U of x such that one has a factorization:*

$$
\begin{array}{ccc}
U & \longrightarrow & spec(C) \\
{\scriptstyle\phi}\downarrow & \swarrow & \\
S & &
\end{array}
$$

where $spec(C) \to S$ is a standard étale morphism, such that $C = A[T]/PA[T]$ can be chosen in such a way that the monic polynomial $P \in A[T]$ becomes irreducible in $K[T]$ and $U \hookrightarrow spec(C)$ is an immersion (resp. an open immersion).

Proof. We denote by $\mathfrak{m}$ the maximal ideal of A and s the corresponding closed point of S. By 1.2 we may assume that $\phi : X \to S$ is induced by an $A-$algebra of the form $A \to B_b$ with $B = A[T]/PA[T]$ and $b \in B$ such that $P'(t)$ is invertible in B_b. Since A is integrally closed, any monic factor of P in $K[T]$ is in $A[T]$. Let $x \in X_s$ and fix an irreducible monic factor Q of P mapping to 0 in $k(x)$. Write $P = QR$ in $A[T]$. As $\bar{P} \in k(s)[T]$ is separable, $\bar{Q}$ and $\bar{R}$ are coprime in $k(s)[T]$ or, equivalently:

$$\langle \bar{Q}, \bar{R} \rangle = k(s)[T].$$

But as Q is monic $M := A[T]/\langle Q, R \rangle$ is a finitely generated $A-$module so, from Nakayama, $A[T] = \langle Q, R \rangle$. By the Chinese remainder theorem:

$$A[T]/PA[T] = A[T]/QA[T] \times A[T]/RA[T].$$

---

[2]We recall that a scheme is called normal if every stalk is integrally closed.

Set $B_1 := A[T]/QA[T]$ and let b_1 denote the image of b in B_1. Then the open subscheme $U_1 := spec(B_{1b_1} \hookrightarrow X$ contains x and:

$$U_1 := spec(B_{1b_1}) \hookrightarrow X \to S$$

is a standard morphism of the required form. $\qquad\square$

Lemma 3.3. *Let $\phi : X \to S$ be an étale cover. Then X is also normal and, in particular, it can be written as the coproduct of its (finitely many) irreducible components. Furthermore, given a connected component of X_0 of X, the induced étale cover $X_0 \to S$ is the normalization of S in $k(S) \hookrightarrow k(X_0)$.*

Proof. We first prove the assertion when $S = spec(A)$ with A a noetherian integrally closed local ring and $\phi : X \to S$ is a standard morphism as in 3.2. Let $K(= k(S))$ denote the fraction field of A. By assumption, $L := C \otimes_A K = K[T]/PK[T]$ is a finite separable field extension of K. Let A^c denote the integral closure of A in $K \hookrightarrow L$. Since B is integral over A, one has $A \subset B \subset A^c \subset L$ hence $B_b \subset (A^c)_b = ((A^c)_b)^c \subset L$. So, to show that C is integrally closed in $K \hookrightarrow L$, it is enough to show that $A^c \subset B_b$. So let $\alpha \in A^c$ and write:

$$\alpha = \sum_{i=0}^{n-1} a_i t^i,$$

with $a_i \in K$, $i = 1, ...n$ and $n = deg(P)$. As $K \hookrightarrow L$ is separable of degree n, there are exactly n distinct morphisms of $K-$algebras:

$$\phi_i : L \hookrightarrow \bar{K}$$

Let $V_n(t) := V(\phi_1(t), ..., \phi_n(t))$ denote the Vandermonde matrix associated with $\phi_1(t), ..., \phi_n(t)$. Then one has:

$$|V_n(t)|(a_i)_{0 \le i \le n-1} =^t Com(V_n(t))(\phi_i(\alpha))_{1 \le i \le n}$$

(where $^t Com(-)$ denotes the transpose of the comatrix and $|-|$ the determinant). Hence, as the $\phi_i(t)$ and the $\phi_i(\alpha)$ are all integral over A, the $|V_n(t)|a_i$ are also all integral over A. By assumption, the a_i are in K and $|V_n(t)|$ is in K since it is symmetric in $\phi_i(t)$. So, as A is integrally closed, the $|V_n(t)|a_i$ are in A, from which the conclusion follows since $|V_n(t)|$ is a unit in C (recall that $P'(t)$ is invertible in C).

We now turn to the general case. From 3.2, the above already shows that X is normal and, in particular, it can be written as the coproduct of its (finitely many) irreducible components. So, without loss of generality we may assume that X is normal connected hence integral scheme. But then, for any open sub-scheme $U \subset S$, the ring $\mathcal{O}_X(\phi^{-1}(U))$ is integral ring and its local rings are all integrally closed so $\mathcal{O}_X(\phi^{-1}(U))$ is integrally closed as well and, since it is also integral over $\mathcal{O}_S(U)$, it is the integral closure of $\mathcal{O}_S(U)$ in $k(S) \hookrightarrow k(X)$. $\qquad\square$

The following lemma provides a converse to 3.3:

Lemma 3.4. *Let $k(S) \hookrightarrow L$ be a finite separable field extension which is unramified over S. Then the normalization $\phi : X \to S$ of S in $k(S) \hookrightarrow L$ is an étale cover.*

Proof. Since S is locally noetherian, $\phi : X \to S$ is finite by 3.1; it is also surjective [AM69] Thm.5.10. and, by construction it is unramified. So we are only to prove that

$\phi : X \to S$ is flat, namely that $\mathcal{O}_{S,\phi(x)} \hookrightarrow \mathcal{O}_{X,x}$ is a flat algebra, $x \in X$. One has a commutative diagram:

$$
\begin{array}{ccc}
\mathcal{O}_{X,x} & \twoheadleftarrow & C \\
\uparrow & \nearrow & \\
\mathcal{O}_{S,\phi(x)} & &
\end{array}
$$

where $\mathcal{O}_{S,\phi(x)} \to C$ is a standard algebra as in 3.2, $C \twoheadrightarrow \mathcal{O}_{X,x}$ is surjective and, as $\phi : X \to S$ is surjective, $\mathcal{O}_{S,\phi(x)} \hookrightarrow \mathcal{O}_{X,x}$. In particular,

$$\mathcal{O}_{S,\phi(x)} \otimes_{\mathcal{O}_{S,\phi(x)}} k(S) \hookrightarrow \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{X,x}} k(S)$$

is injective as well hence:

$$C \otimes_{\mathcal{O}_{S,\phi(x)}} k(S) \to \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{X,x}} k(S)$$

is non-zero. But, as $C \otimes_{\mathcal{O}_{S,\phi(x)}} k(S)$ is a field, the above morphism is actually injective and, as $\mathcal{O}_{S,\phi(x)} \to k(S)$ is faithfully flat, this implies that $C \twoheadrightarrow \mathcal{O}_{X,x}$ is injective hence bijective. $\qquad\square$

Denote by $FEAlg/k(S)$ the category of finite étale $k(S)-$algebras. 3.3 shows that there is a well-defined functor:

$$
\begin{array}{cccc}
R: & \mathcal{C}_S & \to & (FEAlg/k(S))^{op} \\
& X \to S & \mapsto & k(S) \hookrightarrow R(X) := \prod_{X_0 \in \pi_0(X)} k(X_0).
\end{array}
$$

Let $FEAlg/k(S)/S \subset FEAlg/k(S)$ denote the full subcategory of finite étale $k(S)$-algebras which are unramified over S. Then 3.3 and 3.4 imply the following theorem

Theorem 3.5. *The functor $R : \mathcal{C}_S \to FEAlg/k(S)$ is fully faithfull and induces an equivalence of categories $R : \mathcal{C}_S \to FEAlg/k(S)/S$ with pseudo-inverse the normalization functor.*

Corollary 3.6. *Let S be a connected, locally noetherian, normal scheme with generic point $\eta : k(S) \to S$. Let $k(S) \hookrightarrow \Omega$ be an algebraically closed field extension defining geometric points $\bar{s}_\eta : spec(\Omega) \to spec(k(S))$ and $\bar{s} : spec(\Omega) \to S$. Let $k(S) \hookrightarrow M_{k(S),S}$ denote the maximal algebraic field extension of $k(S)$ in Ω which is unramified over S. Then one has the canonical short exact sequence of profinite groups:*

$$
\begin{array}{ccc}
1 \longrightarrow \Gamma_{M_{k(S),S}} & \longrightarrow & \Gamma_{k(X)} \\
& & \| \\
& \pi_1(spec(k(S)); \bar{s}_\eta) & \longrightarrow \pi_1(S, \bar{s}) \longrightarrow 1
\end{array}
$$

(here $\Gamma\_$ denote the absolute Galois group) In particular, this defines a canonical profinite group isomorphism:

$$Gal(M_{k(S),S}|k(S)) \to \pi_1(X; x)$$

Proof. From 3.5 the base change functor $\eta^* : \mathcal{C}_S \to \mathcal{C}_{spec(k(\eta))}$ is fully faithful hence, from 4.3 (1), induce an epimorphism of profinite groups:

$$\pi_1(\eta) : \pi_1(spec(k(S)); \bar{s}_\eta) \twoheadrightarrow \pi_1(S, \bar{s})$$

whose kernel is $\Gamma_{M_{k(S),S}} by 4.4(2)$. $\qquad\square$

Example 3.7. Let S be a curve, smooth and geometrically connected over a field k and let $S \hookrightarrow \tilde{S}$ be the smooth compactification of S. Write $\tilde{S} \backslash S = P_1, ... P_r$. Then, with the notation of 3.6, the extension $k(S) \hookrightarrow M_{k(S),S}$ is just the maximal algebraic extension of $k(S)$ in Ω unramified outside the places $P_1, ..., P_r$.

§2. Abelian varieties.

§2.1. *Group schemes.* Let S be a scheme. A group scheme G over S is a group object in a category of S-schemes. That is, it is an S-scheme G equipped with one of the equivalent sets of data

(1) a triple of morphisms $\mu : G \times_S G \to G$, $e : S \to G$, and $\iota : G \to G$, satisfying the usual compatibilities of groups (namely associativity of μ, identity, and inverse axioms);

(2) a functor from schemes over S to the category of groups, such that composition with the forgetful functor to sets is equivalent to the presheaf corresponding to G under the Yoneda embedding.

Definition 3.8. *(Finite group schemes over a field k)* A finite group scheme k is a k-group scheme G, which is finite as a scheme over k.

Remark 3.9. An affine group scheme over k is of the form $spec(A)/k$ with μ, e, ι corresponding to

$$\mu^* : A \to A \otimes_k A;$$
$$e^* : A \to k;$$
$$i^* : A \to A$$

respectively. Any affine group scheme is the spectrum of a commutative Hopf k−algebra.

Example 3.10.   (1) $\mathbb{G}_{a,k} = spec\, k[x]$, with

$$\mu^* :\ k[x]\ \to\ k[x] \otimes_k k[x]$$
$$x\ \to\ x \otimes 1 + 1 \otimes x$$

(hence $\quad \mu :\ \begin{matrix} spec(k[x]) \otimes_k spec(k[x]) & \to & spec(k[x]) \\ (a,b) & \to & a+b \end{matrix}$)

(2) $\mathbb{G}_{m,k} = \mathbb{A}^1_k \backslash \{0\} = spec(k[x, x^{-1}])$ with

$$\mu^* :\ \begin{matrix} k[x, x^{-1}] & \to & k[x, x^{-1}] \otimes_k k[x, x^{-1}] \\ x & \to & x \otimes x \end{matrix} .$$

(3) $\mu_{n,k}$, where $n \in \mathbb{Z}_{\geq 1}$, $\mu_{n,k} = spec(\frac{k[x]}{x^n - 1})$ with

$$\mu^* :\ \begin{matrix} \frac{k[x]}{x^n-1} & \to & \frac{k[x]}{x^n-1} \otimes_k \frac{k[x]}{x^n-1} \\ x & \to & x \otimes x \end{matrix}$$

Let R be a k−algebra, $\mu_{n,k}(R) = Hom_k(spec(R), \mu_{n,k}) = Hom_{k-alg}(\frac{k[x]}{x^n-1}, R) = \{a \in R | a^n = 1\}$. We have the following:

Proposition 3.11. $\mu_{n,k}$ *is a finite group scheme over k. and*

- $(n; char(k)) = 1$ *implies that $\mu_{n,k}$ is smooth (=étale) over k;*

- *if $char(k) = p > 0$ and $n = p^m$, then $\mu_{n,k} = spec(\frac{k[x]}{(x^{p^m}-1)})$ is connected.*

(4) Suppose $n \in \mathbb{Z}_{\geq 1}$, $S = spec(k)$, let G be the abstract group $\mathbb{Z}/n\mathbb{Z}$, $S \times G :=$ $\coprod_{g \in G} S_g$, then we can define

$$\mu : \quad (S \times G) \times_S (S \times G) \longrightarrow S \times G$$
$$\|$$
$$S \times (G \times G)$$

which is induced by

$$\mu^* : \quad \prod_{g \in G} k_g \quad \to \quad \left(\prod_{g \in G} k_g\right) \otimes \left(\prod_{g \in G} k_g\right)$$
$$e_\lambda \quad \to \quad \sum_{(g,g') \in G \times G, g \cdot g' = b} e_{gg'}$$
.

Remark 3.12. Let k be a field, then the category of k-group schemes of finite type is an abelian category. In particular, for $f : G \to H$ a morphism of k-group schemes, its kernel and cokernel exist. More precisely, we have

(1) The kernel of f can be described by the following cartesian diagram

$$\begin{array}{ccc} G & \xrightarrow{f} & H \\ \uparrow & & \uparrow e \\ ker(f) & \longrightarrow & spec(k) \end{array}$$

where $e : spec(k) \to H$ is the neutral element of the group scheme H.

(2) We don't have an easy explicit description of the cokernel $coker(f)$. But we know the following universal property: for $g : H \to K$ a morphism of (finite type) k-group schemes such that $g \circ f = 0$, there exists then a unique morphism $\bar{g} : coker(f) \to K$ such that $\bar{g} \circ p = g$, where $p : H \to coker(f)$ is the canonical projection:

$$G \xrightarrow{f} H \xrightarrow{p} coker(f) \longrightarrow 0$$

with 0, g to K, and $\exists!$

Example 3.13. When $G = \mathbb{G}_{m,k}$,

$$f = [n] : \quad G \quad \to \quad G$$
$$a \quad \mapsto \quad a^n \quad ,$$

then the corresponding homomorphism of Hopf algebras is

$$[n]^\# : \quad k[x, x^{-1}] \quad \leftarrow \quad k[x, x^{-1}]$$
$$x^n \quad \leftarrow \quad x$$
.

Then by the cartesian diagram:

$$
\begin{array}{ccc}
k[x, x^{-1}] & \longleftarrow & k[x, x^{-1}] \\
\downarrow & \square & \downarrow \\
k[x]/(x^n - 1) & \longleftarrow & k
\end{array}
\quad ,
$$

we have $ker[n] = spec(k[x]/(x^n - 1))$.

Proposition 3.14. *Let G/k be a group scheme of finite type over a filed k. Let G^0 be the connected component of G containing 0 (called the **identity component** of G). Then*

- *G^0 has a natural structure of k-group scheme induced from G. In this way, G^0 becomes an open and closed subgroup scheme of G;*

- *The cokernel G/G^0 of $G^0 \hookrightarrow G$ is a finite étale group scheme over k.*

§2.2. *Abelian Varieties.* An **Abelian variety** over a field k is a smooth connected $k-$group scheme A/k such that A is a proper as k-schemes. For example, elliptic curves are $1-$dimensional abelian varieties.

Recall that an **isogeny** from an abelian variety A to an abelian variety B is a morphism of $k-$group schemes $A \xrightarrow{f} B$ such that one of the following equivalent conditions is satisfied:

- $dim(A) = dim(B)$, and $ker(f)$ is finite over k;

- $dim(A) = dim(B)$, and f is an epimorphism of k-group schemes (i.e., $coker(f) = 0$).

For A an abelian variety over k, we have the following facts:

- the group scheme structure on A/k is always commutative;

- $\forall n \in \mathbb{Z} \ n \geq 1$, the multiplication-by-n morphism

$$
\begin{array}{cccc}
[n]_A : & A & \to & A \\
& a & \to & a^n
\end{array}
$$

  is an isogeny of abelian varieties and $ker([n]_A)$ is a closed finite subgroup scheme of A.

In the following part of this section, k is always an algebraically closed field.
Let $p = char(k) \geq 0$. Set $A[n] := ker([n]_A)$.

Proposition 3.15. *Let A be an abelian variety of dimension $g \geq 0$.*

- *If $(n, p) = 1$, then $A[n] \to spec(k)$ is étale, hence can be thought as an abstract group. Moreover, $A[n] \cong (\mathbb{Z}/n\mathbb{Z})^{2g}$*

- *If $n = p^e$, then $A[n]$ is a finite group scheme over k, and $A[n](k) \cong (\mathbb{Z}/n\mathbb{Z})^r$, where r is an integer belonging to $\{0, 1, ..., g\}$, which is independent of the*

choice of the integer e. The multiplication-by-n morphism $[n]$ is no longer an étale isogeny, however by 3.14 we have the following exact sequence:

$$0 \to A[n]^0 \to A[n] \to A[n]/A[n]^0 \to 0$$

with $A[n]/A[n]^0$ finite étale over k. Hence we have the following diagram:

$$
\begin{array}{ccccccccc}
& & A[n] & & A[n]/A[n]^0 & & & & \\
& & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & A[n]^0 & \longrightarrow & A & \longrightarrow & A/A[n]^0 & \longrightarrow & 0 \\
& & & & \downarrow{\scriptstyle [n]} & \swarrow{\scriptstyle \textit{étale surjective}} & & & \\
& & & & A & & & &
\end{array}
$$

Since $A[n]/A[n]^0$ is étale over k, $A/A[n]^0$ is étale over k.

Remark 3.16. The integer r appearing in the previous proposition is called the p-rank of the abelian variety. For example, when $A = E$ is an elliptic curve, then its p-rank is equal to 1 if E is ordinary, otherwise, E is called supersingular.

Corollary 3.17. *Let l be a prime number, set $T_l(A) := \varprojlim A[l^n](k)$, the $l-$adic Tate module of A, then*

- *if $(l,p) = 1$, $T_l(A) \simeq \mathbb{Z}_l^{2g}$.*

- *if $l = p$, $T_p(A) \simeq \mathbb{Z}_p^r$, where r is the p-rank of A.*

A main reference for abelian varieties is [Mum70]. See also [Mil86] for a concise introduction.

Theorem 3.18. (Serre-Lang) *There is a canonical isomorphism of profinite groups:*

$$\pi_1(A; 0_A) \xrightarrow{\simeq} \prod_{l:prime} T_l(A)$$

Proof. Given a profinite group Π and a prime l, let $\Pi^{(l)}$ denote its maximal pro$-l$ quotient.
Claim 1: $\pi_1(A; 0_A)$ is abelian. In particular,

$$\pi_1(A, 0) = \prod_{l:prime} \pi_1(A, 0)^{(l)}.$$

Proof. *(of claim 1)*. From 2.7, the multiplication map $\mu : A \times_k A \to A$ induces a morphism of profinite groups $\pi_1(\mu) : \pi_1(A \times A) \to \pi_1(A)$. By Künneth formula, there exists an isomorphism $\pi_1(A \times A) \xrightarrow[(p_{1\star}, p_{2\star})]{\simeq} \pi_1(A) \times \pi_1(A)$, so we get a morphism of profinite groups

$$\pi_1(\mu) : \pi_1(A; 0_A) \times \pi_1(A; 0_A) \to \pi_1(A; 0_A).$$

The canonical section $\sigma_1 : A \to A \times_k A$ of the first projection $p_1 : A \times_k A \to A$ induce the morphism of profinite groups:

$$
\begin{array}{rccc}
\pi_1(\sigma_1) : & \pi_1(A; 0_A) & \to & \pi_1(A; 0_A) \times \pi_1(A; 0_A) \\
& \gamma & \mapsto & (\gamma, 1)
\end{array}
$$

and, by functoriality, $\pi_1(\mu) \circ \pi_1(\sigma_1) = Id$. The same holds for the second projection. As a result, the morphism $\pi_1(\mu)$ is given by $(\gamma_1, \gamma_2) \mapsto \gamma_1 \gamma_2$. As a result $\pi_1(A)$ is commutative. $\qquad\qquad\square$

Claim 2 (Serre-Lang): Let $\phi : X \to A$ be a connected étale cover, and x a point above $0 \in A$. Then X carries a unique structure of abelian variety such that $\phi : X \to A$ becomes a separable isogeny, and that $x \in X$ becomes the neutral element of x.

Proof. *(of claim 2)* The idea is to construct first the group structure on one fibre, and then extend it automatically by the formalism of Galois categories. Let $x : spec(k) \to X$ such that $\phi(x) = 0_A$. Then the pointed connected étale cover $\phi : (X; x) \to (A; 0_A)$ corresponds to a transitive $\pi_1(A; 0_A)-$set M together with a distinguished point $m \in M$. Since $\pi_1(A; 0_A)$ is abelian, the map:

$$\mu_M : \quad \begin{array}{ccc} M \times M & \to & M \\ (\gamma_1 m, \gamma_2 m) & \mapsto & \gamma_1 \gamma_1 m \end{array}$$

is well defined, maps (m, m) to m and is $\pi_1(A; 0_A) \times \pi_1(A; 0_A)-$equivariant if we endow M with the structure of $\pi_1(A; 0_A) \times \pi_1(A; 0_A)-$set induced by $\pi_1(\mu)$ (which corresponds to the étale cover $X \times_A (A \times_k A) \to A \times_k A$). Hence it corresponds to a morphism $\mu_X^0 : X \times_k X \to X \times_A (A \times_k A)$ above $A \times_k A$ or, equivalently, to a morphism $\mu_X : X \times_k X \to X$ fitting in:

$$
\begin{array}{ccc}
X \times_k X & \xrightarrow{\mu_X^0} X \times_A (A \times_k A) \longrightarrow & X \\
\downarrow{\scriptstyle \phi \times_k \phi} & \square & \downarrow{\scriptstyle \phi} \\
A \times_k A & \xrightarrow{\qquad \mu \qquad} & A
\end{array}
$$

and mapping (x, x) to x. By the same arguments, one constructs $i_X : X \to X$ above $[-1_A] : A \to A$ mapping x to x. With μ_X and i_X, one checks that this endows X with the structure of an algebraic group with unity x (hence, of an abelian variety since X is connected and $\phi : X \to A$ is proper) and such that $\phi : X \to A$ becomes a morphism of algebraic groups (hence a separable isogeny since $\phi : X \to S$ is an étale cover). $\qquad\square$

Now let $\phi : X \to A$ be a degree n isogeny (i.e $ker(\phi)$ is of exponent n). Then $ker(\phi) \subset ker([n]_X)$, hence one has a canonical commutative diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & ker(\phi) & \longrightarrow & X & \xrightarrow{\phi} & A & \longrightarrow & 0 \\
& & & \searrow^{0} & \downarrow{\scriptstyle [n]_X} & \swarrow_{\psi} & & & \\
& & & & X & & & &
\end{array}
$$

From the surjectivity of ϕ, one also has $\phi \circ \psi = [n]_A$. When l is a prime different from the characteristic p of k, combining this remark and claim 2, one gets that $([l^n] : A \to A)_{n \geq 0}$

is cofinal[3] among the finite étale covers of A with degree a power of l that is

$$\pi_1(A; 0_A)^{(l)} = \varprojlim A[l^n] = T_l(A).$$

When $l = p$, one has to be more careful. In this case, the isogeny

$$[p^n]_A : A \to A$$

is no longer étale. However, it factors as:

$$
\begin{array}{ccc}
A & \xrightarrow{\psi_n} & B_n \\
\downarrow{\scriptstyle [p^n]_A} & \swarrow{\scriptstyle \phi_n} & \\
A & &
\end{array}
\quad,
$$

where $B_n := A/A[p^n]^0$, ψ_n is the quotient morphism, and ϕ_n is the induced morphism (as $A[p^n]^0 \subset A[p^n]$). As a result, ψ_n is a purely inseparable isogeny, and ϕ_n is an étale isogeny with $ker(\phi_n) \cong A[p^n]/A[p^n]^0 \cong A[p^n](k)$ (as a constant group). Moreover, the translation map by an element of $ker(\phi_n)(k)$ in A induces an automorphism of B_n/A, we obtain an injective morphism of groups

$$ker(\phi_n)(k) \to Aut(B_n/A)$$

which must be an isomorphism for the reason of cardinality: Let now $\phi : X \to A$ be an étale isogeny of degree p^n. As before, there exists a morphism of group schemes $\psi : A \to X$ such that $\phi \circ \psi = [p^n]_A$.

$$
\begin{array}{ccc}
A & \xrightarrow{\psi} & X \\
 & \searrow{\scriptstyle [p^n]_A} & \downarrow{\scriptstyle \phi} \\
 & & A
\end{array}
$$

Since ϕ is étale, $\psi(A[p^n]^0) \subset ker(\phi) = 0$. Thus we obtain the following factorization:

$$
\begin{array}{ccccc}
 & & B_n & & \\
 & \nearrow{\scriptstyle \psi_n} & \downarrow{\scriptstyle \phi_n} & \searrow{\scriptstyle \exists} & \\
A & \xrightarrow{\hspace{1cm}\psi\hspace{1cm}} & & & X \\
 & \searrow{\scriptstyle [p^n]} & \downarrow & \swarrow{\scriptstyle \phi} & \\
 & & A & &
\end{array}
$$

In other words, the isogeny ϕ is dominated by ϕ_n. In this way, we show that the family $\{\phi_n : B_n \to A\}$ is cofinal among the étale isogeny of A of degree a power of p. Hence:

$$
\begin{aligned}
\pi_1^{(p)}(A) &\cong \varprojlim_n Aut(B_n/A)^{op} \\
&\cong \varprojlim_n (ker(\phi_n))(k) \\
&\cong \varprojlim_n A[p^n](k) \\
&= T_p(A)
\end{aligned}
$$

$\square$

---

[3]Let A be a set and let $\leq$ be a binary relation on A. Then a subset B of A is said to be **cofinal** if it satisfies the following condition: For every $a \in A$, there exists some $b \in B$ such that $a \leq b$.

§3. Geometrically connected schemes of finite type. Let S be a scheme geometrically connected and of finite type over a field k. Fix a geometric point $\bar{s} : spec(k(\bar{s}) \rightarrow S_{k^s}$ with image again denoted by $\bar{s}$ in S and $spec(k)$.

Proposition 3.19. *The morphisms* $(S_{k^s}, \bar{s}) \rightarrow (S, \bar{s}) \rightarrow (spec(k), \bar{s})$ *induce a canonical short exact sequence of profinite groups:*

$$1 \rightarrow \pi_1(S_{k^s}; \bar{s}) \xrightarrow{i} \pi_1(S; \bar{s}) \xrightarrow{p} \pi_1(spec(k); \bar{s}) \rightarrow 1.$$

Example 3.20. Assume furthermore that S is normal. Then the assumption that S is geometrically connected over k is equivalent to the assumption that $\bar{k} \cap k(S) = k$ and, with the notation of subsection §1, the short exact sequence above is just the one obtained from usual Galois theory:

$$1 \rightarrow Gal(M_{k(S),S}|k^s(S)) \rightarrow Gal(M_{k(S),S}|k(S)) \rightarrow \Gamma_k \rightarrow 1.$$

Proof. We use the criteria of 4.3. Exactness in the right: As S is geometrically connected over k, the scheme S_K is also connected for any finite separable field extension $k \hookrightarrow K$.

Exactness in the left: For any étale cover $f : X \rightarrow S_{k^s}$ we are to prove that there exists an étale cover $f : \tilde{X} \rightarrow S$ such that $f_{k(s))}$ dominates f. From 2.9, there exists a finite separable field extension $k \hookrightarrow K$ and an étale cover $\tilde{f} : \tilde{X} \rightarrow S_K$ which is a model of $f : X \rightarrow S_{k^s}$ over S_K. But then, the composite $f : \tilde{X} \rightarrow S_K \rightarrow S$ is again an étale cover whose base-change via $S_{k^s} \rightarrow S$ is the coproduct of $[K : k]$ copies of f hence in particular, dominates f.

Exactness in the middle: From 2.4, this amounts to show that $ker(p) \subset im(i)$. For any connected étale cover $\phi : X \rightarrow S$ such that $\phi_{k^s} : X_{k^s} \rightarrow S_{k^s}$ admits a section, say $\sigma : S_{k^s} \hookrightarrow X_{k^s}$, we are to prove that there exists a finite separable field extension $k \hookrightarrow K$ such that the base change of $spec(K) \rightarrow spec(k)$ via $S \rightarrow spec(k)$ dominates $\phi : X \rightarrow S$. So, let $k \hookrightarrow K$ be a finite separable field extension over which $\sigma : S_{k^s} \hookrightarrow X_{k^s}$ admits a model $\sigma_K : S_K \hookrightarrow X_K$. This defines a morphism from S_K to X over S by composing $\sigma_K : S_K \hookrightarrow X_K$ with $X_K \rightarrow X$. $\qquad \square$

4. G.A.G.A. theorems

We first define affine complex analytic spaces. Let $U \subset \mathbb{C}^n$ denote the polydisc of all $z = (z_1, ..., z_n) \in \mathbb{C}^n$ such that $|z_i| < 1$ and given analytic functions $f_1, ..., f_r : U \rightarrow \mathbb{C}$, let $\mathfrak{V}(f_1, ..., f_r)$ be the locally ringed spaces in $\mathbb{C}-$algebra whose underlying topological space the closed subset :

$$\cap_{i=1}^r f_i^{-1}(0) \subset U$$

endowed with the topology inherited from the transcendent topology on U and whose structural sheaf is:

$$\mathcal{O}_U/\langle f_1, ..., f_r \rangle,$$

where $\mathcal{O}_U$ is the sheaf of germs of analytic functions on U. As schemes over $\mathbb{C}$ are obtained by glueing affine schemes over $\mathbb{C}$ in the category $LR/\mathbb{C}$ of locally-ringed spaces in $\mathbb{C}-$algebras, complex analytic spaces are obtained by glueing "affine" complex analytic spaces in $LR/\mathbb{C}$. The category $AN/\mathbb{C}$ of complex analytic spaces is then the full subcategory of $LR/\mathbb{C}$ whose objects are locally isomorphic to affine complex analytic spaces.

Claim 4.1. Let X be a scheme of finite type over $\mathbb{C}$, then there exists $X^{an} \in AN/\mathbb{C}$ with a morphism of ringed spaces $\lambda_X : X^{an} \to X$ .Suppose $\phi : X \to Y$ is a morphism of schemes of locally finite type over $\mathbb{C}$. Then there exists a continuous map $\phi^{an} : X^{an} \to Y^{an}$ of analytic spaces such that $\lambda_Y \circ \phi^{an} = \phi \circ \lambda_X$. Moreover, if $X/\mathbb{C}$ is smooth, then X^{an} is a complex manifold. And if $\phi : X \to Y$ is a finite étale cover, then $\phi^{an} : X^{an} \to Y^{an}$ is a finite topological cover. Hence for X a scheme of finite type over $\mathbb{C}$, we get a functor $(-)^{an} : \mathcal{C}_X \to \mathcal{C}_X^{top}$. Furthermore, the map $\phi \mapsto \phi^{an}$ maps open immersions into open immersions.

The morphism $\lambda_X : X^{an} \to X$ is unique up to a unique $X-$isomorphism and we call X^{an} the analytification of X. There is a nice dictionary between the properties of X (resp. $\phi : X \to Y$) and those of X^{an} (resp. $\phi^{an} : X^{an} \to Y^{an}$). Morally, all those which are encoded in the completion of the local rings are preserved. For instance:

(1) Let P be the property of being connected, irreducible, regular, normal, reduced, of dimension d. Then X has P if and only if X^{an} has P;

(2) Let P be the property of being surjective, dominant, a closed immersion, finite, an isomorphism, a monomorphism, an open immersion, flat, unramified, étale, smooth. Then ϕ has P if and only if ϕ^{an} has P.

Let $LFT/\mathbb{C}$ be the category of schemes locally of finite type over $\mathbb{C}$.

Theorem 4.2. ([Gro71], VII, Thm.5.1) *Let X be a scheme locally of finite type over $\mathbb{C}$, the functor $(-)^{an} : LFT/\mathbb{C} \to AN/\mathbb{C}$ induce an equivalence of categories from the category of étale covers of X to the category of étale covers of X^{an}.*

Corollary 4.3. $\pi_1(X)$ *classifies the finite covers of X^{an}. π_1^{top} classifies all topological covers of X^{an} and*

$$\pi_1(X) \cong \widehat{\pi_1^{top}(X^{an})}$$

Example 4.4. Assume that $k = \mathbb{C}$ and A be an abelian variety of dimension g over k. Then A^{an} is a complex torus of dimension g: there exists a lattice $\Lambda \subset \mathbb{C}^g$ such that $A^{an} \cong \mathbb{C}^g/\Lambda$. Then, on the one hand, the universal covering of A is just the quotient map $\mathbb{C}^g \to A$ and has group $\pi_1^{top}(A(\mathbb{C}); 0_A) \simeq \Lambda$ whence

$$\pi_1(A; 0_A) \cong \widehat{\pi_1^{top}(A^{an}; 0)} \cong \widehat{Z}^g.$$

On the other hand, for any prime l:

$$T_l(A) = \varprojlim A[l^n]$$
$$= \varprojlim \frac{1}{l^n}\Lambda/\Lambda$$
$$= \varprojlim \Lambda/l^n\Lambda$$
$$= \Lambda^{(l)},$$

This recover the result in 3.18 when the field k is $\mathbb{C}$.

CHAPTER 5

Structure of geometric fundamental groups of smooth curves

1. Introduction

In this chapter, we discuss the structure results concerning the geometric fundamental groups of smooth curves. For simplicity, most of the time, we will mainly work in the compact case. As we are only mainly interested in the structure of fundamental group, we may and we will ignore the choice of the base point except section 2. Hence the fundamental group of a connected scheme at certain geometric point will be denoted simply by $\pi_1(X)$.

In the following, let k denote an algebraically closed field of arbitrary characteristic $p \geq 0$. Let X/k be a proper smooth connected curve of genus $g \geq 0$. We begin with the easiest case.

Proposition 1.1.
- *If $g = 0$, then $\pi_1(X, \bar{x}) = \{1\}$ is trivial. In other words, X is simply connected.*

- *If $g = 1$, then $\pi_1(X, \bar{x}) \cong TX$ the Tate module of the elliptic curve.*

Proof. (2) is the result of Theorem 3.18.

(1) is done by applying the Hurwitz's formular. Indeed, let $f : Y \to X$ be a finite étale cover of X, with Y is connected. In particular, Y is a smooth projective connected curve. Let g_Y be the genus of the curve Y. Then the Hurwitz formular tells us

$$2g_Y - 2 = deg(f)(2g_X - 2)$$

As $g_X = 0$, we obtain $2g_Y = 2 - 2degf$. As $g_X = 0$, we obtain

$$2g_Y = 2 - 2degf \geq 0,$$

as a result, we have $deg(f) = 1$, in other words, $f : Y \to X$ is a finite étale morphism of degree 1. As a result f is an isomorphism. From here, we obtain $\pi_1(X) = \{1\}$. □

Remark 1.2. When $g \geq 2$, the problem becomes more difficult. In the following, we will discuss some properties of the π_1 of such a curve. In particular, we will sketch the proof of the fact that $\pi_1(X)$ is topologically of finite type. At the end, we will also give an example of affine curve to see that π_1 in general doesn't satisfy the finiteness property above.

We begin with some useful properties about profinite groups which are topologically of finite type. Recall that a profinite group π is calle topologically of finite type if there is a finitely generated subgroup of π which is dense in π.

Proposition 1.3. ([FJ08],Proposition 16.10.6) *π is a profinite group, $f : \pi \to \pi$ is a group morphism from π to itself. If π is topologically of finite type, then f is surjective implies that f is an isomorphism.*

Definition 1.4. Let π be a profinite group. We define $\mathbf{Im}(\pi) = \{$ G finite group $\mid$ there exists a continuous surjective morphism of groups $f : \pi \twoheadrightarrow G\}/ \sim$, where $\sim$ is the isomorphism of finite groups.

Proposition 1.5. ([FJ08], Proposition 16.10.7) *G, H two profinite groups with G topologically of finite type, then:*

(1) If $Im(H) \subset Im(G)$, then H is a quotient of G.

(2) If $Im(H) = Im(G)$ then $H \cong G$

Combining 1.3 and 1.5, we have

Corollary 1.6. *Suppose $f : G \to H$ a surjective morphism of profinite groups, and H is topologically of finite type. Suppose that f induce an equality $Im(H) = Im(G)$, then f is an isomorphism.*

In order to apply this proposition, we need the following lemma.

Lemma 1.7. *Let k be an algebraically closed field, X/k a connected scheme of finite type. Let $k \subset K$ be an algebraically closed extension of fields, then the surjective morphism $\pi_1(X_K) \to \pi_1(X)$ induces an equality of sets $Im(\pi_1(X)) = Im(\pi_1(X_K))$.*

Proof. Let $\pi_1(X_K) \to G$ be a finite quotient which corresponds to covering $Z \to X_K$ of X_K, Galois of group G. Then it's sufficient to prove that there exists a finite Galois covering $Y \to X$ of Galois group G. As the scheme X_K/K is of finite presentation, and $Z \to X_K$ is finite, there exists an extension of finite type k' of k contained in K, and an étale Galois cover Z' of $X_{k'} = X \otimes_k k'$, such that the following diagram is cartesian:

$$
\begin{array}{ccc}
Z & \longrightarrow & Z' \\
\downarrow & & \downarrow \\
X_K & \longrightarrow & X_{k'}
\end{array}
$$

As k'/k is of finite type, there exist an integral $k-$scheme of finite type S which the finite Galois cover $Z' \to X_{k'}$ of Galois group G above the generic point of S can be extended to a finite Galois cover $\mathcal{Z}' \to X_S$ of Galois group G. Finally, let $s \in S(k)$ be a rational point of S (this is possible because S is of finite type over $k = \bar{k}$), hence we obtain a finite étale cover $Y = \mathcal{Z}_s \to X$, which is galois of group G. Hence the result. $\qquad\square$

Corollary 1.8. *Suppose k is an algebraically closed field, X/k a $k-$scheme connected of finite type. Denote $\pi_1(X)$ its étale fundamental group. Suppose $k \subset K$ an algebraically closed extension of fields, then if $\pi_1(X)$ is topologically of finite type, then the morphism $\pi_1(X_K) \to \pi_1(X)$ induced by base change (hence automatically surjective, by 1.3) is an isomorphism.*

Proof. Given the hypothesis, this results from 1.6 and 1.7. $\qquad\square$

2. Case of characteristic zero

In this section, let k be an algebraically closed field of characteristic 0, and X/k a smooth connected curve of genus g. The aim of this section is to determine the fundamental group of X. The main tool used here is Serre's GAGA principal, c.f. Section 4.4.

§1. The case $k = \mathbb{C}$**.** Denote $X(\mathbb{C})$ the set of $\mathbb{C}-$rational points of X, then $X(\mathbb{C})$ has naturally a structure of a connected Riemann surface, denote by X^{an}. Then X^{an} is the complement of n distinct points $y_1, y_2, ..., y_n$ of a compact connected Riemann surface of genus g. Concerning the topological fundamental group of such a Riemann surface, we have the following result.

Theorem 2.1. *Suppose that $\mathfrak{S}$ is a Riemann surface which is the complement of n distinct points $y_1, y_2, ..., y_n$ of a compact connected Riemann surface of genus g, then its topological fundamental group $\pi_1^{top}(\mathfrak{S})$ is a group of finite type, generated by $2g + n$ elements $\alpha_1, ...\alpha_g, \beta_1, ...\beta_g, \gamma_1, ...\gamma_n$ which satisfies the following relation:*

$$(\prod_{i=1}^{g}[\alpha_i, \beta_1]) \cdot \gamma_1 \cdot \gamma_2 \cdots \cdot \gamma_n = 1$$

In other words, $\pi_1^{top}(\mathfrak{S}) = \frac{\text{Free group generated by } \alpha_1, ...\alpha_g, \beta_1, ...\beta_g, \gamma_1, ...\gamma_n}{\langle(\prod_{i=1}^{g}[\alpha_i,\beta_1])\cdot\gamma_1\cdot\gamma_2\cdots\gamma_n\rangle}.$

Definition 2.2. Denote $\Pi_{g,n}$ the group generated by $2g+n$ elements $\alpha_1, ...\alpha_g, \beta_1, ...\beta_g, \gamma_1, ...\gamma_n$ with the only relation $(\prod_{i=1}^{g}[\alpha_i, \beta_1]) \cdot \gamma_1 \cdot \gamma_2 \cdots \cdot \gamma_n = 1$.

By the definition of $\pi_1(X)$, this profinite group classifies the finite étale covers of X. By the equivalence between the category of finite étale covers and the category of topological covers of finite degree of $X(\mathbb{C})$, and that the latter category is actually classified by the profinite completion of π_1^{top}, we have:

Corollary 2.3.

$$\pi_1(X) \simeq \widehat{\pi_1^{top}(X(\mathbb{C}))} \simeq \widehat{\Pi_{g,n}}.$$

In particular, $\pi_1(X)$ is topologically of finite type. More over, if $n \geq 1$(in other words, if the curve $X/\mathbb{C}$ is affine), $\pi_1(X)$ is free of rank $2g + n - 1$.

Example 2.4.   (1) $X = \mathbb{P}_{\mathbb{C}}^1$, the $\mathbb{P}_{\mathbb{C}}^1(\mathbb{C})^{an}$ is the complex projective line. As a topological space, it is homeomorphic to a sphere. As a result, $\mathbb{P}_{\mathbb{C}}^1(\mathbb{C})$ is contractible. Hence $\pi_1(\mathbb{P}_{\mathbb{C}}^1(\mathbb{C})) = \{1\}$. Hence we get $\pi_1^{top}(X^{an}) = 1$, hence $\pi_1(X) = \widehat{\pi_1^{top}(X^{an})} = 1$.

(2) $X = E$ an elliptic curve, $X^{an} = \mathbb{C}/\mathbb{Z} \oplus \mathbb{Z} \cdot \tau, \tau \in \mathbb{C}^*, Im(\tau) > 0$ (we can take $\tau \in$ the fundamental domain of modular group). Then we have:

$$\pi_1^{top}(X^{an}) = \mathbb{Z}^2,$$
$$\pi_1(X) = \widehat{\pi_1^{top}(X^{an})} = \widehat{\mathbb{Z}^2} = TE.$$

§2. General case. Here k is an algebraically closed field of characteristic 0, the idea is trying to use the conclusion in complex case.

Proposition 2.5. $k = \bar{k}, char(k) = 0$, X/k *a separated connected smooth curve, obtained as the complement of n rational points of a proper smooth curve of genus g over k. Then,*

(1) $\pi_1(X)$ is topologically of finite type, and isomorphic to $\widehat{\Pi_{g,n}}$.

(2) suppose $k \subset K$ an extension of algebraically closed fields, then the morphism induced by base change $\pi_1(X_K) \to \pi_1(X)$ is an isomorphism.

Proof. As the curve X/k is of finite presentation, there exists a subfield k_0 of k which is of finite type over $\mathbb{Q}$, and a k_0-scheme X_0, such that $X \cong X_0 \otimes_{k_0} k$. Denote $\overline{k_0} \subset k$ the algebraic closure of k_0 in k, and fix an embedding $\overline{k_0} \hookrightarrow \mathbb{C}$. We have the following commutative diagram:

$$
\begin{array}{ccccc}
X_{0,\mathbb{C}} & \longrightarrow & X_{0,\overline{k_0}} & \longleftarrow & X \\
\downarrow & \square & \downarrow & \square & \downarrow \\
spec(\mathbb{C}) & \longrightarrow & spec(\overline{k_0}) & \longleftarrow & spec(k)
\end{array}
$$

where $X_{0,\overline{k_0}}$ is obtained by the base change of X_0/k_0. From this we get the following two surjective morphisms:

$$\pi_1(X_{0,\mathbb{C}}) \twoheadrightarrow \pi_1(X_{0,\overline{k_0}}) \twoheadleftarrow \pi_1(X).$$

By 2.3, $\pi_1(X_{0,\mathbb{C}})$ is topologically of finite type, hence so is $\pi_1(X_{0,\overline{k_0}})$, by 1.8 the above surjective morphisms are isomorphisms. $\qquad\square$

Example 2.6. Let k be an algebraically closed field of characteristic 0

(1) $X = \mathbb{P}^1_k$, by 2.5, $\pi_1(X) = \widehat{\Pi_{0,0}} = \{1\}$.

(2) $X = E$ an elliptic curve over k, by 2.5 $\pi_1(X) = \widehat{\Pi_{1,0}} = \widehat{\mathbb{Z}^2} = TE$.

3. Case of positive characteristic

In this section, the letter k denotes an algebraically closed field of characteristic $p > 0$. Let as usual X/k be a proper smooth connected curve of genus g over k. Our aim here is to discuss some structure results of $\pi_1(X)$. In particular, we show that $\pi_1(X)$ is topologically of finite type.

§1. $\pi_1(X)^{(p')}$. In this section, we assume X/k to be proper. In the following, the notation p' always means "prime to p" and for G a profinite group, we denote by $G^{(p')}$ the maximal quotient of order prime to p. We want to determine the quotient $\pi_1(X, \bar{x})^{(p')}$ of $\pi_1(X, \bar{x})$. The strategy here is trying to use the corresponding results in characteristic zero case. The two tools used here are the deformation theory and Grothendieck's specialization theory of π_1.

Our aim is to show that $\pi_1^{(p')}(X)$ is isomorphic to $\widehat{\Pi_{g,0}}^{(p')}$. In particular, $\pi_1^{(p')}(X)$ is a profinite group of finite type, generated by $2g$ elements. Our strategy is try to use the facts corresponds to the characteristic 0 case by the specialization theory of Grothendieck.

§1.1. *Lifting of curves to characteristic 0.* The principal tool here is the deformation theory, as described in [Gro71]. We first cite the following result:

Theorem 3.1. ([Gro71], III, Thm 7.3 *Let A be a complete Noetherian local ring of residue field k. Denote $S = spec(A)$ the spectrum of A, and $s \in S$ it's closed point. Let X_0/s be a smooth projective scheme, such that*

$$H^2(X_0, T_{X_0/s}) = 0,$$

with $T_{X_0/k} := \mathcal{H}om_{\mathcal{O}_X}(\Omega^1_{X_0/s}, \mathcal{O}_{X_0})$ *its tangent bundle. Then there exists a proper smooth formal scheme $\mathfrak{X}$ over $\hat{S} = Spf(A)$ which lifts X_0/s. If moreover we have*

$$H^2(X_0, \mathcal{O}_{X_0}) = 0$$

we can even find a smooth projective $S-$scheme over S such that $X_s = X_0$.

Let $W = W(k)$ the witt ring of k. And $S = spec(W)$. According to 3.1 there exists a smooth projective $S-$scheme $\mathcal{X}/S$ such that $\mathcal{X}_s$ is the curve X.

$$
\begin{array}{ccc}
\mathcal{X}/S & \longleftarrow & X \\
\downarrow{\scriptstyle \text{smooth proper flat}} & \square & \downarrow \\
spec(W(k)) & \longleftarrow & spec(k)
\end{array}
$$

§1.2. *the specialization theory of Grothendieck.* Let $S = Spec(R)$ be the spectrum of a complete discrete valuation ring, with $s \in S$ its closed point and η its generic point. Just to simplify the presentation here, we assume that the residue field of R is algebraically closed (this assumption is unnecessary). Let $\bar{\eta}$ denote a geometric point of S above η. Let $\mathcal{X}/S$ denote a proper smooth curve with connected geometric fibers. Then we have the following commutative diagram

$$
\begin{array}{ccccc}
\mathcal{X}_{\bar{\eta}} & \xrightarrow{\ j\ } & \mathcal{X} & \xleftarrow{\ i\ } & \mathcal{X}_s \\
\downarrow & & \downarrow & & \downarrow \\
\bar{\eta} & \longrightarrow & S & \longleftarrow & s
\end{array}
$$

Then we have the following theorem of Grothendieck.

Theorem 3.2. *(1) The functor*

$$
i^\star : \begin{array}{ccc} \mathcal{C}_{\mathcal{X}} & \to & \mathcal{C}_{\mathcal{X}_s} \\ \mathcal{Y}/\mathcal{X} & \to & \mathcal{Y}_s/\mathcal{X}_s \end{array}
$$

is an equivalence of categories. In particular, $i_\star : \pi_1(\mathcal{X}_s) \xrightarrow{\simeq} \pi_1(\mathcal{X})$.

*(2) The composite $i_\star^{-1} \circ j_\star : \pi_1(\mathcal{X}_{\bar{\eta}}) \to \pi_1(\mathcal{X}_s)$ is call the **specialization morphism**, often denoted by sp. Then sp is always surjective. Moreover it induces an isomorphism between the maximal prime-to$-p$ quotients:*

$$sp^{(p')} : \pi_1(\mathcal{X}_{\bar{\eta}})^{(p')} \xrightarrow{\simeq} \pi_1(\mathcal{X}_s)^{(p')}.$$

§1.3. *Conclusion.* Now we use again the notation of §1.1. Let $S = Spec(W(k))$, with s its closed point and η its generic point. Let $\bar{\eta} = Spec(\Omega)$ be a geometric point of S above η, with Ω an algebraic closure of $Frac(W(k))$. Then the theorem of Grothendieck in §1.2 tells us that the specialization morphism

$$sp^{(p')} : \pi_1(\mathcal{X}_{\bar{\eta}}) \to \pi_1(\mathcal{X}_s)$$

is surjective. As Ω is an algebraically closed field of characteristic 0, $\pi_1(\mathcal{X}_{\bar{\eta}}) \cong \widehat{\Pi}_{g,0}$ topologically of finite type. In particular, so is $\pi_1(\mathcal{X}_s) \cong \pi_1(X)$. Moreover, sp induces an isomorphism between the maximal prime-to$-p$ quotients

$$\widehat{\Pi}_{g,0} \cong \pi_1(\mathcal{X}_{\bar{\eta}})^{(p')} \overset{\simeq}{\to} \pi_1(\mathcal{X}_s)^{(p')} \cong \pi_1(X)^{(p')}.$$

By combining 2.5, we obtain finally the following result.

Proposition 3.3. *Let X/k be a proper connected smooth curve over an algebraically closed field of characteristic $p \geq 0$. Then $\pi_1(X)$ is topologically of finite type. Moreover, its maximal prime-to-p quotient $\pi_1(X)^{(p')}$ is isomorphic to $\widehat{\Pi}_{g,0}^{(p')}$.*

§2. π_1^{ab}. Let k be an algebraically closed field of characteristic $p \geq 0$, and X/k a connected proper smooth curve of genus g. Recall that a **divisor** (or more precisely a **Weil divisor**) of X is a formal finite sum

$$\sum_{i=1}^{r} n_i x_i$$

with $x_i \in X$ closed points of X, and $n_i \in Z$. For D such a divisor, the sum $\sum_{i=1}^{r} n_i \in \mathbb{Z}$ is called the **degree** of D. For example, for $f \in k(X)$ a non-zero element of the function fields of X, its associated divisor is given by

$$div(f) := \sum_{x \in C} \nu_x(f) \cdot x,$$

where $\nu_x : k(X)^{\star} \to \mathbb{Z}$ is the (additive) valuation attached to the closed point $x \in X$ (so one has to show here that $\nu_x(f) = 0$ for all but finitely many closed points $x \in X$, we omit the details here). Such a divisor is called **principal**. As X/k is proper, the degree of a principal divisor is always zero. Two divisors D, D' are called **linearly equivalent** if their difference $D-D'$ is a principal divisor, namely of the form $div(f)$ for some $f \in k(X)^{\star}$. Now, the group of all divisors is denoted by $Div(X)$, and the quotient

$$Cl(X) := Div(X)/\{\text{principal divisors}\}$$

is the **(Weil) divisor class group** of X. Since the degree of a principal divisor is zero, we have a well-defined degree function on $Cl(X)$:

$$deg : Cl(X) \to \mathbb{Z}.$$

The subgroup of $Cl(X)$ of elements of degree 0 is denoted by $Cl^0(X)$.

Proposition 3.4. *Let k be an algebraically closed field, and X/k a proper smooth connected curve of genus $g \geq 0$.*

*(1) One can construct canonically an abelian variety $J(X)$ associated with X, called the **jacobian** of X, such that its group of $k-$rational points is canonically isomorphic to $Cl^0(X)$. Moreover, $J(X)$ is of dimension g.*

(2) Let $a \in X(k)$ be a rational point. Then there exits a closed immersion $\iota_a : X \hookrightarrow J(X)$ such that the associated map between the $k-$rational points is given by

$$\begin{array}{ccc} X(k) & \to & J(X)(k) = Cl^0(X) \\ x & \mapsto & \text{class of the divisor } x-a. \end{array}$$

*The closed immersion ι_a above is called the **Albanese morphism** associated with the rational point $a \in X(k)$.*

Consider now $\iota_a : X \hookrightarrow J(X)$ the Albanese morphism associated with a rational point $a \in X(k)$. It induces then a morphism of profinite groups

$$\iota_{a,\star} : \pi_1(X;a) \to \pi_1(J(X);0).$$

We have then the following result.

Proposition 3.5. *The morphism $\iota_{a,\star}$ is surjective, and induces an isomorphism*

$$\pi_1(X;a)^{ab} \xrightarrow{\cong} \pi_1(J(X);0).$$

§3. Some words about open curves. In 1957, S. Abhyankar stated the following conjecture, which is now a theorem:

Theorem 3.6. (Abhyankar's conjecture) *Let p be a prime number, and let k be an algebraically closed field of characteristic p. Let G be a finite group. Denote $p(G)$ the subgroup of G generated by its Sylow $p-$subgroups of G (i.e. $G/p(G)$ is the maximal quotient of G of order prime to p). Let X/k be a proper smooth connected curve of genus g over k, and $U \subset X$ the affine curve which is the complement of r points, with $r \geq 1$. Then there exists a connected Galois étale cover of U of Galois group G if and only if $G/p(G)$ is generated by $2g + r - 1$ elements.*

Remark 3.7.   (1) The necessity of this condition follows from the non-proper version of the specialization theory of Grothendieck 3.3.

    (2) The Abhyankar's conjecture is already proved by Raynaud ([Ray94]) in the affine line case, and by Harbater ([Har94]) in the general case. Pop ([Pop95]) has also given a proof in the general case, which is also based on the result of Raynaud in the affine line case.

    (3) The knowledge of the set $Im(\pi_1(X))$ is insufficient to determine the structure of this porfinite group $\pi_1(X)$, since $\pi_1(X)$ is not topologically of finite type. For example, we have seen that in the proof of 1.8 that the set $Im(\pi_1(X))$ is invariant by extensions of algebraically closed fields, but the structure of $\pi_1(X)$ really depends on the base field k. For example, when $U = \mathbb{A}^1_k$ is the affine line, then one can show that $\pi_1(U)^{(p)}$ is a free pro-p group, whose rank is equal to the cardinality of the base field k, c.f. [P.G00] Lemme 1.3 and the words between 1.3 and 1.4. In particular, the fundamental group of $\mathbb{A}^1_k$ is not topologically of finite type.

Bibliography

[AM69] M. F. Atiyah and I. G. MacDonald. *Introduction to commutative algebra*. Westview Press, 1969.

[Cad08] Anna Cadoret. ``Galois categories''. In: *Proceeding of the G.A.M.S.C. summer school*. 2008.

[FJ08] Michael D. Fried and M. Jarden. *Field arithmetic*. Vol. 11. Ergebnisse der Mathematik und ihrer Grenzgebiete. Springer, 2008.

[Gro71] A. Grothendieck. *Revêtements étales et groupe fondamental*. Springer-Verlag, 1971.

[Har77] Robin Hartshorne. *Algebraic Geometry*. Springer-Verlag, 1977.

[Har94] D. Harbater. ``Abhyankar's conjecture on Galois groups over curves''. In: *Invent. Math*. 117 (1994).

[Mil80] J. Milne. *Etale cohomology*. Princeton University Press, 1980.

[Mil86] J. Milne. ``Abelian varieties''. In: *Arithmetic Geometry*. Ed. by G. Cornell. Ed. by J.H. Silverman. Springer Verlag, 1986.

[Mum70] D. Mumford. *Abelian varieties*. Tata Institute of Fundamental Research, 1970.

[P.G00] P.Gille. ``Le groupe fondamental sauvage d'une courbe affine en caractéristique $p > 0$''. In: *Courbes semi-stables et groupe fondamental en géométrie algébrique*. Vol. 187. Progr. Math. Basel: Birkhäuser, 2000, pp. 217–231.

[Pop95] F. Pop. ``Étale covers of affine smooth curves. The geometric case of a conjecture of Shafarevich. On Abhyankar's conjecture''. In: *Invent. Math*. 119 (1995).

[Ray94] M. Raynaud. ``Revêtements de la droite affine en caractéristique $p > 0$ et conjecture d'Abhyankar''. In: *Invent. Math*. 116 (1994).

[Sza09] Tamás Szamuely. *Galois groups and fundamental groups*. Cambridge university press, 2009.