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Concurrent exceptional curves on del Pezzo surfaces of degree one

ALGANT Master's thesis

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" If your research adviser gives you a problem involving del Pezzo surfaces of degree 2 and 1, it means he really hates you."

Peter Swinnerton-Dyer.

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Contents

Introduction

A del Pezzo surface is a projective, non-singular, geometrically integral surface with ample anticanonical divisor. The degree of a del Pezzo surface is the self-intersection number of the canonical divisor, and this is at most 9. Over an algebraically closed field, del Pezzo surfaces of degree d are isomorphic to $\mathbb{P}^2$ blown up at $9-d$ points in general position for $d \neq 8$, and to $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{P}^2$ blown up in one point for $d = 8$. For degree at least three, del Pezzo surfaces can be embedded as surfaces of degree d in $\mathbb{P}^d$. A famous example is given by del Pezzo surfaces of degree three, which are exactly the smooth cubic surfaces in $\mathbb{P}^3$. For a del Pezzo surface of degree two, the linear system of the anticanonical divisor gives the surface the structure of a double cover of $\mathbb{P}^2$ ramified over a smooth curve of degree four, and for del Pezzo surfaces of degree one, the linear system of the bianticanonical divisor gives the surface the structure of a double cover of a cone Q in $\mathbb{P}^3$, ramified over a smooth curve that is cut out by a cubic surface.

Let X be a del Pezzo surface of degree d over an algebraically closed field k, and let K_X be the canonical divisor on X. An exceptional curve on X is an irreducible projective curve $C \subset X$ such that $C^2 = C \cdot K_X = -1$. For $d \geq 3$, the exceptional curves on X are exactly the lines on the model of degree d in $\mathbb{P}^d$. For $d = 3$ this gives a description of the 27 lines on a cubic surface. A lot is known about the exceptional curves on del Pezzo surfaces. For example, we know that there is a one-to-one correspondence between exceptional curves on X and their classes in Pic X, and we know what their images under the blow-up in $\mathbb{P}^2$ are, see Theorem 2.8. We also know how many exceptional curves there are.

d	1	2	3	4	5	6	7	8
exceptional curves on X	240	56	27	16	10	6	3	1

Now assume X is of degree one. Let φ be the morphism associated to $|-2K_X|$. In this thesis we prove the following two theorems.

Theorem 1. *Let P be a point on the ramification curve of φ. The number of exceptional curves that go through P is at most ten if char $k \neq 2$, and at most sixteen if char $k = 2$.*

Theorem 2. *Let R be a point outside the ramification curve of φ. The number of exceptional curves that go through R is at most twelve. If char $k = 0$, it is at most ten.*

In [SvL14], various examples of del Pezzo surfaces are given where ten exceptional curves go through one point outside the ramification curve, showing that the upper bound for char $k = 0$ in Theorem 2 is sharp. In Example 4.23 and Example 4.24, we show that the upper bounds given in Theorem 1 are sharp, too.

It is well known that on del Pezzo surfaces of degree three, the maximal number of exceptional curves through one point is three. The fact that three is an upper bound can be seen by looking at the maximal size of full subgraphs of the graph on the 27 exceptional curves. A geometrical argument can be found for instance in [Rei88], on page 102. A point on a del Pezzo surface of degree three that is contained in three exceptional curves is called an Eckardt point.

On a del Pezzo surface of degree two, the maximal number of exceptional curves through one point is four. As in the case of degree three, this upper bound is given by the graph on the 56 exceptional curves. A geometric argument why four is the upper bound is given in [TVAV09], Lemma 4.1. An example where this upper bound is reached is given in [STVAar], Example 7. A point on a del Pezzo surface of degree two that lies on four exceptional curves is called a generalized Eckardt point.

For del Pezzo surfaces of degree one, the situation is a little different. First of all, for char $k \neq 2$, the maximal size of full subgraphs of the graph on the 240 exceptional curves, which we will show is sixteen, is not equal to the maximal number of exceptional curves that can go through one point. Secondly, contrary to del Pezzo surfaces of degree two, where all generalized Eckardt points are outside the ramification curve, in the case of degree one we compute the maximum both for points on the ramification curve, as well as for points outside the ramification curve.

In Section 1, we define del Pezzo surfaces and study their main properties. We look more closely at del Pezzo surfaces of degree one in Subsection 1.1.

In sections 2,3 and 4 we work over an algebraically closed field.

In Section 2, we study the exceptional curves on del Pezzo surfaces. We look more closely at the exceptional curves on del Pezzo surfaces of degree one in Subsection 2.1, and show that they relate to hyperplanes in $\mathbb{P}^3$ that are tritangent to the branch curve of φ, and do not contain the vertex of the cone Q. This will later allow us to make the distinction between exceptional curves through one point on the ramification curve of φ, and exceptional curves through one point outside the ramification curve of φ.

In Section 3, we study the group G of permutations of the set E of exceptional classes in Pic X that preserve the intersection pairing. We prove various results about the action of G on E, that we will use in the fourth section.

In Section 4, we show that an upper bound for the number of exceptional curves through one point in X is sixteen. We show moreover that if the elements in a maximal set of exceptional curves that all intersect each other go through one point, then that point lies on the ramification curve of φ if and only if the set contains at least two curves that intersect with multiplicity three.

In Subsection 4.1 we focus on the number of exceptional curves through one point on the ramification curve. For char $k \neq 2$, we first show that this is at most twelve. Then we show that ten is a sharp upper bound. To this end, we define the following curves.

Let $Q_1, \ldots, Q_8$ be eight points in $\mathbb{P}^2$ such that no three of them lie on a line, and no six of them lie on a conic. For $i \in \{1, 2, 3, 4\}$, let L_i be the line through

Q_{2i} and Q_{2i-1}. For $i, j \in \{1, \dots, 8\}$, $i \neq j$, let $C_{i,j}$ the unique cubic through $Q_1, \dots, Q_{i-1}, Q_{i+1}, \dots, Q_8$ that is singular in Q_j.

We show that if the elements of a set of twelve exceptional curves go through one point on the ramification curve, we can reduce to a set containing the curves L_1, L_2, L_3, L_4, $C_{7,8}$, $C_{8,7}$, and $C_{6,5}$. The following proposition is therefore the key to the proof of Theorem 1.

Proposition 3. *Let char $k \neq 2$. Assume that the four lines L_1, L_2, L_3 and L_4 all intersect in one point P. Then the three cubics $C_{7,8}$, $C_{8,7}$, and $C_{6,5}$ do not all go through P.*

Finally we show that for char $k = 2$, sixteen is a sharp upper bound.

In Subsection 4.2 we focus on exceptional curves through one point outside the ramification curve. We first show that it is at most twelve, by showing that every set of exceptional curves of size bigger than twelve contains at least two curves intersecting with multiplicity three. To compute a sharp upper bound in the case char $k = 0$, we define the following.

Let $Q_1, \dots, Q_8$ be eight points in $\mathbb{P}^2$ such that no three of them lie on a line, and no six of them lie on a conic. Define the following curves.

L_1 is the line through Q_1 and Q_2;
L_2 is the line through Q_3 and Q_4;
C_1 is the conic through Q_1, Q_3, Q_5, Q_6 and Q_7;
C_2 is the conic through Q_1, Q_4, Q_5, Q_6 and Q_8;
C_3 is the conic through Q_2, Q_3, Q_5, Q_7 and Q_8;
C_4 is the conic through Q_2, Q_4, Q_6, Q_7 and Q_8;
D_1 is the quartic through all eight points with singular points in Q_1, Q_7 and Q_8;
D_2 is the quartic through all eight points with singular points in Q_2, Q_5 and Q_6;
D_3 is the quartic through all eight points with singular points in Q_3, Q_6 and Q_8;
D_4 is the quartic through all eight points with singular points in Q_4, Q_5 and Q_7.

As in the case of points on the ramification curve, we show that for a set of eleven or twelve exceptional curves going through one point outside the ramification curve, we can reduce to a set containing these ten curves. From the following proposition we can then deduce Theorem 2.

Proposition 4. *Assume that char $k = 0$. Then*

$$L_1,\ L_2,\ C_1, \dots C_4,\ D_1, \dots, D_4$$

do not all go through one point.

1 Del Pezzo surfaces

In this section we define del Pezzo surfaces and state their main properties. In Subsection 1.1 we will be more specific and focus on del Pezzo surfaces of degree one. We assume that the reader has a basic knowledge of algebraic geometry, and is familiar with concepts as variety, divisor, and Picard group. Most results in this section, as well as more information on del Pezzo surfaces, can be found in [Man74], Chapter IV, and [Kol96], Section III.3.

Definition 1.1. *Let k be a field, and X a variety over k. Then we say that X is nice if it is projective, smooth, and geometrically integral.*

Definition 1.2. *A del Pezzo surface is a nice surface X with ample anticanonical divisor $-K_X$.*

Let X be a del Pezzo surface with very ample anticanonical divisor $-K_X$. The linear system $|-K_X|$ determines an embedding $i : X \hookrightarrow \mathbb{P}^n$ for some n. If H is a hyperplane in $\mathbb{P}^n$, we have $i^*H \sim -K_X$. Therefore, the degree of $i(X)$ is equal to $(i^*H)^2 = (-K_X)^2 = K_X^2$. This leads to the following definition.

Definition 1.3. *The degree of a del Pezzo surface X is the self-intersection number K_X^2.*

Proposition 1.4. *The degree of a del Pezzo surface X is positive.*

Proof. Since $-K_X$ is ample, $-nK_X$ is very ample for some $n > 0$, hence determines an embedding of X into some projective space. Then $(-nK_X)^2$ is the degree of the image of X under this embedding, hence $n^2K_X^2 = (-nK_X)^2 > 0$. It follows that $K_X^2 > 0$. □

Definition 1.5. *Let $r \leq 8$, and let $P_1, \ldots, P_r$ be points in $\mathbb{P}^2$. Then we say that $P_1, \ldots, P_r$ are in general position if no three of them lie on a line, no six of them lie on a conic, and no eight of them lie on a singular cubic with one of these eight points at the singularity.*

Theorem 1.6. *For $r \leq 8$, let $P_1, \ldots, P_r$ be points in general position in $\mathbb{P}^2$. Let X be the blow-up of $\mathbb{P}^2$ in these points. Then $-K_X$ is ample, and very ample if $r \leq 6$.*

Proof. See [Man74], Theorem 24.5, and [Dem80], Theorem 1. □

Theorem 1.7. *Let k be an algebraically closed field, and let X be a del Pezzo surface over k. Then X is isomorphic to either $\mathbb{P}^1 \times \mathbb{P}^1$, in which case X is of degree 8, or to $\mathbb{P}^2$ blown up at $r \leq 8$ points in general position, in which case the degree of X is $9 - r$.*

Proof. See [Man74], Theorem 24.4, Theorem 26.2, and Remark 26.3. □

Remark 1.8. The previous two theorems give us an explicit description of all del Pezzo surfaces over algebraically closed fields; they are exactly those surfaces that are isomorphic to the blow-ups of $\mathbb{P}^2$ in $r \leq 8$ points in general position, and the surface $\mathbb{P}^1 \times \mathbb{P}^1$. Moreover, Theorem 1.7 implies that the degree of a del Pezzo surface over an algebraically closed field is at most 9, and a del Pezzo surface of degree 9 is just $\mathbb{P}^2$.

Since the anticanonical divisor of a del Pezzo surface is ample, a del Pezzo surface can be embedded in some projective space by a multiple of its anticanonical divisor $-K$. To study the various rational maps and morphisms given by multiples of $-K$, we need a couple of classical results.

Theorem 1.9. *(Nakai-Moishezon criterion). Let X be a nonsingular projective surface over an algebraically closed field. Then a divisor D on X is ample if and only if $D^2 > 0$ and $D \cdot C > 0$ for all irreducible curves C in X.*

Proof. See [Har77], Theorem V.1.10. □

Theorem 1.10. *(Riemann-Roch for surfaces). Let X be a nonsingular projective surface over an algebraically closed field k. Then for any divisor D on X we have*

$$l(D) - s(D) + l(K - D) = \frac{1}{2}D(D - K) + 1 + p_a,$$

where $l(D)$ is the dimension of the vectorspace $\mathcal{L}(D)$ of rational functions on X with poles at most at D, $s(D) = \dim H^1(X, \mathcal{L}(D))$, the superabundance of D, and p_a is the arithmetic genus of X.

Proof. See [Har77], Theorem V.1.6. □

Lemma 1.11. *Let X be a del Pezzo surface with canonical divisor K_X. Then we have $\dim H^1(X, \mathcal{L}(-mK_X)) = 0$ for all $m \geq 0$.*

Proof. See [Kol96], Corollary 3.2.5.1. □

The following lemma is well known, and can be found for instance in [Kol96], Corollary 3.2.5.2.

Lemma 1.12. *Let X be a del Pezzo surface of degree d over an algebraically closed field. Then for all positive integers m we have $l(-mK_X) = 1 + \frac{1}{2}m(m+1)d$.*

Proof. Let $m > 0$. Since X is geometrically rational, we have $p_a(X) = 0$ (see for example [Har77], Example II.8.20.2). Moreover, by the previous lemma we have $s(-mK_X) = 0$. Since $-K_X$ is ample we have $-K_X \cdot C > 0$ for all irreducible curves C in X by Nakai-Moishezon, so $(m+1)K_X \cdot C < 0$, hence $l((m+1)K_X) = 0$. From Riemann-Roch for surfaces it follows that

$$\begin{aligned} l(-mK_X) &= \frac{1}{2}((-mK_X)^2 - mK_X^2) + 1 \\ &= \frac{1}{2}(m^2 d - md) + 1 \\ &= 1 + \frac{1}{2}m(m+1)d. \end{aligned}$$

□

Remark 1.13. If X is a del Pezzo surface of degree $d \geq 3$, then $-K_X$ is very ample by Theorem 1.6. Therefore, the linear system $|-K_X|$ determines an embedding in $\mathbb{P}^n$, with $n = \frac{1}{2} \cdot 2 \cdot d = d$ by Lemma 1.12, and the image of X under this embedding has degree $(-K_X)^2 = d$. So for $d \geq 3$, a del Pezzo surface of degree d is isomorphic to a surface of degree d in $\mathbb{P}^d$.

Example 1.14. Let k be an algebraically closed field, and X a del Pezzo surface of degree 4 over k. Then X is isomorphic to $\mathbb{P}^2$ blown up in 5 points in general position. The anticanonical divisor $-K_X$ is very ample, and by Lemma 1.12 we have $l(-K_X) = 5$, so $-K_X$ determines an embedding $\varphi : X \hookrightarrow \mathbb{P}^4$. The image $\varphi(X)$ has degree 4, and it is the complete intersection of two quadric hypersurfaces in $\mathbb{P}^4$. To see this, let $\{v, w, x, y, z\}$ be a basis for $\mathcal{L}(-K_X)$. Let $V = \mathrm{Sym}^2(\mathcal{L}(-K_X))$ be the symmetric square of $\mathcal{L}(-K_X)$. Then V has dimension $\binom{6}{2} = 15$, and there is a canonical map $f : V \to \mathcal{L}(-2K_X)$. By Lemma 1.12 we have $l(-2K_X) = 13$, so the dimension of ker f is at least two, which means that there are two linearly independent quadratic forms vanishing on $\varphi(X)$. This means that $\varphi(X)$ is contained in the intersection of the two quadric hypersurfaces defined by these quadratic forms. Since their intersection has degree 4, which is the degree of $\varphi(X)$, we conclude that $\varphi(X)$ is in fact equal to this intersection.

Let k be an algebraically closed field, and let X be a del Pezzo surface of degree d over k. If X is not isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, then we know from Theorem 1.7 that X is isomorphic to $\mathbb{P}^2$ blown up in $9 - d$ points in general position. In this case we know a lot about the Picard group Pic X of X. For a divisor D on X, we denote its class in Pic X by $[D]$.

Proposition 1.15. *Let Y be a smooth surface over an algebraically closed field. Let $\widetilde{Y}$ be the blow-up of Y at a point P, with corresponding map $\pi : \widetilde{Y} \longrightarrow Y$. Let E be the exceptional curve above P. Then E is isomorphic to $\mathbb{P}^1$, and we have $E^2 = -1$. Moreover, we have an isomorphism Pic $Y \oplus \mathbb{Z} \longrightarrow$ Pic $\widetilde{Y}$ sending (D, n) to $\pi^*D + n[E]$. For all $C, D \in$ Pic Y we have $(\pi^*C) \cdot (\pi^*D) = C \cdot D$, and $(\pi^*C) \cdot [E] = 0$. Finally, we have $K_{\widetilde{Y}} \sim \pi^*K_Y + E$.*

Proof. See [Har77], Propositions V.3.1, V.3.2, and V.3.3. □

Proposition 1.16. *Let k be an algebraically closed field. For $1 \leq d \leq 8$, let Y be the blow-up of $\mathbb{P}^2$ in $r = 9 - d$ points $P_1, \ldots, P_r$ in general position. Let Pic Y be the Picard group of X, then we have Pic $Y \cong \mathbb{Z}^{10-d}$. More specifically, if E_i is the class of the exceptional curve corresponding to P_i, and L the class of the pullback of a line l in $\mathbb{P}^2$ not passing through any of the P_i, then $\{L, E_1, \ldots, E_r\}$ forms a basis for Pic Y.*

Proof. This follows from the previous proposition and the fact that Pic $\mathbb{P}^2 = \langle [l] \rangle$. □

Remark 1.17. Keeping the notation of the previous proposition, we have

$$\begin{aligned}&E_i^2 = -1 \text{ for all } i;\\ &E_i \cdot E_j = 0 \text{ for } i \neq j;\\ &L^2 = 1;\\ &L \cdot E_i = 0 \text{ for all } i.\end{aligned}$$

Since the canonical divisor $K_{\mathbb{P}^2}$ of $\mathbb{P}^2$ is linearly equivalent to $-3l$, we have $[-K_X] = 3L - \sum_{i=1}^{r} E_i$. It follows that $[-K_X] \cdot E_i = 1$ for all i.

1.1 Del Pezzo surfaces of degree one

Let X be a del Pezzo surface of degree one over an algebraically closed field k with anticanonical divisor $-K_X$. In this subsection we define the anticanonical model of X and see that this describes X as a smooth sextic surface in the weighted projective space $\mathbb{P}(2,3,1,1)$. Moreover, we will see that the linear system $|-2K_X|$ realizes X as a double cover of a quadric cone in $\mathbb{P}^3$. The linear system $|-K_X|$ defines a rational map that is not a morphism, but by blowing up X we can extend this map to an elliptic fibration. The results in this subsection can be found in [VA] and [CO99].

The anticanonical model of X

Definition 1.18. *The anticanonical ring of X is the graded ring*

$$R(X, -K_X) = \bigoplus_{m \geq 0} \mathcal{L}(-mK_X).$$

Definition 1.19. *The anticanonical model of X is the scheme Proj $R(X, -K_X)$.*

Since $-K_X$ is ample, X is isomorphic to its anticanonical model. We compute the anticanonical model of X as follows. By Lemma 1.12, we have $l(-K_X) = 2$. Let $\{z, w\}$ be a basis for $\mathcal{L}(-K_X)$. By Proposition 2.3 in [CO99], for all $m \geq 1$ the elements $z^m, z^{m-1}w, \ldots, zw^{m-1}, w^m$ are linearly independent in $\mathcal{L}(-mK_X)$. So z^2, zw, w^2 are linearly independent elements of $\mathcal{L}(-2K_X)$. Since $l(-2K_X) = 4$, we can choose an element $x \in \mathcal{L}(-2K_X)$ such that $\{z^2, zw, w^2, x\}$ forms a basis for $\mathcal{L}(-2K_X)$. Now $z^3, z^2w, zw^2, w^3, zx, wx$ are elements of $\mathcal{L}(-3K_X)$ and linearly independent by the arguments in [CO99], page 1200. Since $l(-3K_X) = 7$ we can therefore choose an element $y \in \mathcal{L}(-3K_X)$ to obtain a basis $\{z^3, z^2w, zw^2, w^3, zx, wx, y\}$ of $\mathcal{L}(-3K_X)$. We have $l(-4K_X) = 11$ and $l(-5K_X) = 16$, and together with the arguments in [CO99], page 1200 this implies that

$$\{z^4, z^3w, z^2w^2, zw^3, w^4, x^2, xz^2, xw^2, xzw, yz, yw\}$$

is a basis for $\mathcal{L}(-4K_X)$, and

$$\{z^5, z^4w, z^3w^2, z^2w^3, zw^4, w^5, x^2w, x^2z, xz^3, xw^3, xz^2w, xzw^2, xy, yz^2, yw^2, yzw\}$$

is a basis for $\mathcal{L}(-5K_X)$. Finally, since $l(-6K_X) = 22$, the 23 elements

$$\begin{aligned}&z^6, z^5w, z^4w^2, z^3w^3, z^2w^4, zw^5, w^6, x^3, x^2z^2, x^2w^2, x^2zw, xz^4, xz^3w,\\ &\qquad\qquad xz^2w^2, xzw^3, xw^4, xyz, xyw, y^2, yz^3, yz^2w, yzw^2, yw^3\end{aligned}$$

of $\mathcal{L}(-6K_X)$ are linearly dependent. Let $h(x, y, z, w) = 0$ be a dependence relation between them. If $\mathrm{char}(k) \neq 2, 3$ then x and y can be chosen such that h has the form

$$h = y^2 - x^3 - xf(z, w) - g(z, w),$$

where f and g are homogeneous polynomials in z and w of degree 4 and 6 respectively.

Let $k[x, y, z, w]$ be the graded $k-$algebra with grading defined by $\deg z = \deg w = 1$, $\deg x = 2$, and $\deg y = 3$. Then by Proposition 2.5 in [CO99] there exists a natural isomorphism between the anticanonical ring of X and $k[x, y, z, w]/(h)$. Therefore, X is isomorphic to the zero locus of h in the weighted projective space $\mathbb{P}(2, 3, 1, 1)$.

For the rest of this section we assume that $\mathrm{char}(k) \neq 2, 3$, and identify X with its anticanonical model inside $\mathbb{P}(2, 3, 1, 1)$.

The linear system $| - 2K_X|$

Let $p : \mathbb{P}(2, 3, 1, 1) \dashrightarrow \mathbb{P}(2, 1, 1)$ be the projection sending a point $(x : y : z : w)$ to $(x : z : w)$. This is a rational map that is well defined on X. The restriction to X is a morphism of degree 2. Let $i : \mathbb{P}(2, 1, 1) \hookrightarrow \mathbb{P}^3(a_0, a_1, a_2, a_3)$ be the $2-$uple embedding, sending $(x : z : w)$ to $(x : z^2 : zw : w^2)$. Then $i(\mathbb{P}(2, 1, 1))$ is a quadric cone Q given by $a_2^2 = a_1a_3$, with vertex $(1 : 0 : 0 : 0)$. The composition $\varphi = i \circ p : X \longrightarrow \mathbb{P}^3$ is the morphism defined by $| - 2K_X|$. It is a double covering of Q. The preimage of the vertex $(1 : 0 : 0 : 0)$ of Q under this morphism is the point $(1 : 1 : 0 : 0) = (1 : -1 : 0 : 0)$ in X. We define $\widetilde{X}$ to be the blow-up of X in this point with associated map $\pi : \widetilde{X} \longrightarrow X$. Moreover, we define $\widetilde{Q}$ to be the blow-up of Q in the vertex, with associated map $\rho : \widetilde{Q} \longrightarrow Q$. Then φ induces a morphism $\psi : \widetilde{X} \longrightarrow \widetilde{Q}$. The morphism ψ is ramified at the exceptional curve E in $\widetilde{X}$ above $(1 : 1 : 0 : 0)$, and at those points in $\mathbb{P}(2, 3, 1, 1)$ where $y = 0$, which are the points $(x : y : z : w)$ for which $x^3 + f(z, w)x + g(z, w) = 0$. The latter defines a surface in $\mathbb{P}(2, 3, 1, 1)$, whose image under ψ defines a cubic surface in $\mathbb{P}^3$. The branch curve of φ is therefore the union of the vertex V of Q and a curve B that is contained in the intersection of the cubic surface with Q. Since X is smooth it follows that B is too. Moreover, B is irreducible and reduced, so it is a smooth curve of degree six and genus four, see Proposition 3.1 in [CO99].

The linear system $| - K_X|$

The linear system $|-K_X|$ defines a rational map $\mu : X \dashrightarrow \mathbb{P}^1$, sending $(x : y : z : w)$ to $(z : w)$. This is not defined in the point $(1 : 1 : 0 : 0) \in X$, which is the unique basepoint of $| - K_X|$. As $\widetilde{X}$ is the blow-up of X in this point, the rational map μ induces a morphism $\nu : \widetilde{X} \longrightarrow \mathbb{P}^1$. The fiber under ν above a point $(z_0 : w_0) \in \mathbb{P}^1$ is isomorphic to the set of points $(x : y : z_0 : w_0) \in X$ with $y^2 = x^3 + xf(z_0, w_0) + g(z_0, w_0)$. This is an elliptic curve for almost all (z_0, w_0), so ν is an elliptic fibration.

The morphisms described above are shown in the following commutative diagram.

$$
\begin{tikzcd}
 & & & \mathbb{P}^1 \\
\widetilde{X} \arrow[r, "\pi"] \arrow[urrr, "\nu"] \arrow[d, "\psi"'] & X \arrow[r, hook] \arrow[urr, dashed, "\mu", "|-K_X|"'] \arrow[drr, "\varphi", "|-2K_X|"'] & \mathbb{P}(2,3,1,1) \arrow[r, dashed, "p"] & \mathbb{P}(2,1,1) \arrow[u, "p'"'] \arrow[d, "\cong"] \\
\widetilde{Q} \arrow[rrr, "\rho"] & & & Q \arrow[d, hook, "i"] \\
 & & & \mathbb{P}^3
\end{tikzcd}
$$

2 Exceptional curves

Let k be an algebraically closed field, and let X be a del Pezzo surface of degree d over k that is isomorphic to $\mathbb{P}^2$ blown up at $r = 9 - d$ points $\{P_1, \ldots, P_r\}$ in general position. Let $-K_X$ be the anticanonical divisor of X. Let $\pi : X \longrightarrow \mathbb{P}^2$ denote the blow-up. For all i, the inverse image $\pi^{-1}(P_i)$ of P_i is an exceptional curve on X. From Proposition 1.15 and Remark 1.17, we know that $\pi^{-1}(P_i)$ is isomorphic to $\mathbb{P}^1$, and $(\pi^{-1}(P_i))^2 = K_X \cdot \pi^{-1}(P_i) = -1$. As we will see, X contains more curves with these properties. In this section we define the general notion of an exceptional curve on a surface and describe the exceptional curves on a del Pezzo surface. In Subsection 2.1 we consider exceptional curves on del Pezzo surfaces of degree one, which have a very nice geometrical description. All results in this section can be found in [Man74], unless stated otherwise.

Definition 2.1. *Let Y be a nice surface. An exceptional curve on Y is an irreducible projective curve $C \subset Y$ such that $C^2 = C \cdot K_Y = -1$.*

The following proposition is a very classical result.

Proposition 2.2. *(Adjunction formula). Let Y be a nice surface over an algebraically closed field with canonical divisor K_Y, and C an irreducible projective curve on Y. Then*

$$2p_a(C) - 2 = C \cdot (C + K_Y),$$

where $p_a(C)$ is the arithmetic genus of C.

Proof. See [Har77], Proposition V.1.5. □

From the Adjunction formula it follows that for an exceptional curve C on X we have $2p_a(C) - 2 = -2$, hence $p_a(C) = 0$, so $C \cong \mathbb{P}^1$.

If X has degree $d \geq 3$, then X has very ample anticanonical divisor $-K_X$, which determines an embedding in $\mathbb{P}^n$ for some n. The image under this embedding of an exceptional curve C on X has degree $-K_X \cdot C = 1$, hence it is a line.

On a del Pezzo surface, every irreducible curve with negative self-intersection is in fact an exceptional curve. The following proposition can be found for instance in [Man74], Theorem 24.3.

Proposition 2.3. *Let Y be a del Pezzo surface over an algebraically closed field, and C an irreducible curve on Y with $C^2 < 0$. Then C is an exceptional curve.*

Proof. Since $-K_Y$ is ample and C is irreducible, we have $-K_Y \cdot C > 0$ by Theorem 1.9, so $K_Y \cdot C < 0$. Moreover, since C is irreducible we have $g_a(C) \geq 0$. From the adjunction formula it follows that

$$-2 \leq 2g_a(C) - 2 = C \cdot (C + K_Y) = C^2 + C \cdot K_Y \leq -2,$$

so equality must hold, hence $C^2 = K_Y \cdot C = -1$, so C is exceptional. □

We can now give the following condition for points in $\mathbb{P}^2$ to be in general position.

Proposition 2.4. *Let $Q_1, \dots, Q_8$ be eight points in $\mathbb{P}^2$ and let $\pi : Y \longrightarrow P^2$ be the blow-up in these points. Then $Q_1, \dots, Q_8$ are in general position if and only if Y is a del Pezzo surface.*

Proof. The fact that Y is a del Pezzo surface if $Q_1, \dots, Q_8$ are in general position is Theorem 1.6. For the converse, assume that three points $Q_j,\ Q_k$ and Q_l are on a line M in $\mathbb{P}^2$. Let M' be the strict transform of M on Y and let D_i be the exceptional curve above Q_i for all i. Then we have

$$\pi^* M = M' + D_j + D_k + D_l,$$

so

$$1 = M^2 = (\pi^* M)^2 = M'^2 + 2M' \cdot (D_j + D_k + D_l) + D_j^2 + D_k^2 + D_l^2 = M'^2 + 6 - 3,$$

hence $M'^2 = -2$, which contradicts Proposition 2.3. Analogously, a conic containing six of the Q_i and a singular cubic through seven of the Q_i with one of them at the singularity would have a strict transform on Y with self-intersection ≤ -2, contradicting Proposition 2.3. We conclude that $Q_1, \dots, Q_8$ are in general position. □

Exceptional curves can be 'blown down', as is described in the well-known theorem by Castelnuovo.

Theorem 2.5. *(Castelnuovo). If C is a curve on a nice surface Y over an algebraically closed field such that $C^2 = -1$ and $C \cong \mathbb{P}^1$, then there exists a morphism $f : Y \longrightarrow Y_0$ to a nonsingular projective surface Y_0, and a point $P \in Y_0$, such that Y is the blow-up of Y_0 at P, and C is the exceptional curve above P.*

Proof. See [Har77], Theorem V.5.7. □

After blowing down an exceptional curve on a del Pezzo surface, we obtain again a del Pezzo surface. Proposition 2.6 can be found in [Pie], Lemma 4.20.

Proposition 2.6. *Let Y be a del Pezzo surface of degree $d \leq 8$ over an algebraically closed field that is the blow-up of $r = 9 - d$ points in $\mathbb{P}^2$, and let C be an exceptional curve on Y. Let $f : Y \longrightarrow Y_0$ be a morphism to a nonsingular projective surface Y_0, such that Y is the blow-up of Y_0 in a point P, and such that C is the exceptional curve above P. Then Y_0 is a del Pezzo surface of degree $d + 1$.*

Proof. Let $K_Y,\ K_{Y_0}$ be the canonical divisors of $Y,\ Y_0$, respectively. By Proposition 1.15 we have $K_Y \sim f^* K_{Y_0} + C$, so, using Proposition 1.15, we have

$$K_{Y_0}^2 = (f^* K_{Y_0})^2 = (K_Y - C)^2 = K_Y^2 - 2K_Y \cdot C + C^2 = d + 2 - 1 = d + 1 > 0.$$

Let D be an irreducible curve on Y_0 containing P with multiplicity m, and let D' be its strict transform on Y. Then D' is an irreducible curve on Y, so $-K_Y \cdot D' > 0$

by Nakai-Moishezon. Therefore we have, using Proposition 1.15,

$$\begin{aligned}-K_{Y_0} \cdot D = f^*(-K_{Y_0}) \cdot f^*D = (-K_Y + C) \cdot f^*D &= -K_Y \cdot f^*D - C \cdot f^*D \\ &= -K_Y \cdot (D' + mC) \\ &= -K_Y \cdot D' + m > 0.\end{aligned}$$

From Nakai-Moishezon it follows that $-K_{Y_0}$ is ample, so Y_0 is a del Pezzo surface. Its degree is $K_{Y_0}^2 = d + 1$. $\square$

Let C be an exceptional curve in X. Then the class of C in Pic X satisfies

$$[C]^2 = [C] \cdot [K_X] = -1.$$

We call a class in Pic X satisfying these conditions *exceptional*. We will describe the exceptional classes in Pic X and show that there is a one-to-one correspondence between exceptional classes in Pic X and exceptional curves on X.

As we have seen, Pic X has a basis $\{L, E_1, \dots, E_r\}$, where E_i is the class of the exceptional curve above P_i, and L is the class of the strict transform of a line in $\mathbb{P}^2$ not going trough any of the P_i. If D is a class in Pic X given by $D = aL - \sum_{i=1}^r b_i E_i$, then D is an exceptional class if and only if $D^2 = D \cdot [K_X] = -1$, or, using the results in Remark 1.17,

$$a^2 - \sum_{i=1}^r b_i^2 = -1,$$

and

$$3a - \sum_{i=1}^r b_i = 1.$$

Using the fact that a and all b_i are integers, we can solve these two equations and find all exceptional classes in Pic X.

Proposition 2.7. *The exceptional classes in Pic X are the classes of the form $aL - \sum_{i=1}^r b_i E_i$ where $(a, b_1, \dots, b_r)$ is given by one of the rows of the following table, where all b_i can be permuted (we only consider the rows where $b_i = 0$ for all $i > r$).*

a	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8
0	−1	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0
2	1	1	1	1	1	0	0	0
3	2	1	1	1	1	1	1	0
4	2	2	2	1	1	1	1	1
5	2	2	2	2	2	2	1	1
6	3	2	2	2	2	2	2	2

Proof. See [Man74], Proposition 26.1. $\square$

Proposition 2.7 gives us a very explicit description of all exceptional classes in Pic X. The following theorem relates exceptional classes to exceptional curves on X.

THEOREM 2.8.

(i) There is a one-to-one correspondence between the set of exceptional curves on X and the set of exceptional classes in Pic X, given by the map sending an exceptional curve in X to its class in Pic X.

(ii) Let $f : X \longrightarrow \mathbb{P}^2$ be the blow-up of $\mathbb{P}^2$ in the points $P_1, \ldots, P_r$. Then the image $f(C)$ of an exceptional curve $C \subset X$ is one of the following types.

(a) One of the points P_i;

(b) a line passing through two of the points P_i;

(c) a conic passing through five of the points P_i;

(d) a cubic passing through seven of the points P_i such that one of them is a double point;

(e) a quartic passing through eight of the points P_i such that three of them are double points;

(f) a quintic passing through eight of the points P_i such that six of them are double points;

(g) a sextic passing trough eight of the points P_i such that seven of them are double points and one is a triple point.

(For $d = 2$, only $(a) - (d)$ hold; for $d = 3, 4$, only $(a) - (c)$ hold; for $d = 5, 6, 7$, only $(a) - (b)$ hold; for $d = 8$, only (a) holds.)

Proof. See [Man74], Theorem 26.2. □

REMARK 2.9. Theorem 2.8.(ii) gives a geometrical description of the table in Proposition 2.7. An exceptional class of the form $C = aL - \sum_{i=1}^{r} b_i E_i$, with $(a, b_1, \ldots, b_8)$ a solution given by Proposition 2.7, is either one of the E_i, or it is the class of the strict transform of a curve in $\mathbb{P}^2$ of degree a, going through P_i with multiplicity b_i for each i. Moreover, Theorem 2.8 tells us that these are in one-to-one correspondence with all exceptional curves on X. We can therefore count the exceptional curves on X using the table in Proposition 2.7, and obtain the following table.

d	1	2	3	4	5	6	7	8
exceptional curves on X	240	56	27	16	10	6	3	1

2.1 Exceptional curves on del Pezzo surfaces of degree one

Let X be a del Pezzo surface of degree one over an algebraically closed field k. Let E be the set of exceptional curves on X. We have $|E| = 240$. As in Subsection 1.1, let $\varphi : X \longrightarrow \mathbb{P}^3(a_0, a_1, a_2, a_3)$ be the morphism corresponding to the linear system $|-2K_X|$. We have seen that this is a double covering of a quadric cone Q given by $a_2^2 = a_1 a_3$ in $\mathbb{P}^3$, that branches over a sextic curve B and an isolated branch point at the vertex of Q. In this subsection we show that the exceptional curves on X are related to hyperplane sections of Q that do not pass through the vertex of Q, and are tritangent to B. We start by studying the elements in $|-K_X|$.

Proposition 2.10 and Proposition 2.12 can both be found in [CO99].

Proposition 2.10. *For every element $D \in |-K_X|$, its image $\varphi(D)$ is a line in Q passing trough the vertex of Q. Conversely, a line through the vertex of Q pulls back under φ to an element of $|-K_X|$.*

Proof. As we saw in Subsection 1.1, $\mathcal{L}(-K_X)$ is generated by two elements z and w. Let $D \in |-K_X|$, then D is of the form $\alpha z + \beta w = 0$, with $\alpha, \beta \in k$. Without loss of generality we can assume that $\alpha \neq 0$. Then $z = -\frac{\beta}{\alpha} w$, and $\varphi(D)$ is contained in the two hyperplanes $a_1 = \frac{\beta^2}{\alpha^2} a_3$ and $a_1 = -\frac{\beta}{\alpha} a_2$ in $\mathbb{P}^3$, both containing the vertex of Q. Since φ is finite, $\varphi(D)$ is equal to their intersection.
Conversely, let M be a line in Q through the vertex of Q. Then M is the intersection of two hyperplanes $\gamma a_1 + \delta a_2 + \varepsilon a_3 = 0$ and $\lambda a_1 + \mu a_2 + \nu a_3 = 0$ in $\mathbb{P}^3$. Keeping the notation of Subsection 1.1, we identify Q with $\mathbb{P}(2,1,1)$. Under this identification, M is given by a linear relation in z and w. Therefore M projects under the map $p' : \mathbb{P}(2,1,1) \longrightarrow \mathbb{P}(1,1)$ to a point in $\mathbb{P}^1$. The fiber of ν above a point in $\mathbb{P}^1$ is an element of $|-K_X|$, so $\varphi^* M$ is an element of $|-K_X|$. □

To prove the following proposition, we first need a Lemma.

Lemma 2.11. *Let Y, Z be two normal projective varieties, and $f : Y \longrightarrow Z$ a finite morphism of degree d. Let D, D' be two divisors on Z. Then $f^*D \cdot f^*D' = d(D \cdot D')$, and for a divisor C on Y we have $f^*D \cdot C = D \cdot f_*C$.*

Proof. See [HS00], Theorem A.2.3.2, and [Kol96], Proposition VI.2.11. □

Proposition 2.12.

(i) If e is an exceptional curve on X, then $\varphi(e)$ is a smooth conic in Q not containing the vertex of Q. Moreover $\varphi|_e : e \longrightarrow \varphi(e)$ is one-to-one.

(ii) If H is a hyperplane in $\mathbb{P}^3$ not containing the vertex of Q, then $\varphi^* H$ has an exceptional curve as component if and only if it has at least three (maybe infinitely near) singular points. If this is the case, then $\varphi^* H = e_1 + e_2$ with e_1, e_2 exceptional curves, and $e_1 \cdot e_2 = 3$.

Proof.

(i) Let H be a hyperplane in $\mathbb{P}^3$, then we have $\deg \varphi(e) = H \cdot \varphi(e)$ and $\varphi^* H \sim -2K_X$. Let $[k(e) : k(\varphi(e))] = n$, then $\varphi_*(e) = n\varphi(e)$, so by Lemma 2.11 we have

$$H \cdot n\varphi(e) = H \cdot \varphi_*(e) = \varphi^* H \cdot e = -2K_X \cdot e = 2,$$

hence $\deg \varphi(e) = \frac{2}{n}$. Therefore, n is either 1 or 2. If $n = 2$, then $\deg \varphi(e) = 1$, so $\varphi(e)$ is a line M in Q and $\varphi|_e : e \longrightarrow M$ is $2:1$. Then $\varphi^* M = e$. But $\varphi^* M$ is an element in $|-K_X|$ by Proposition 2.10, which gives a contradiction. Therefore we have $n = 1$, so $\varphi|_e : e \longrightarrow \varphi(e)$ is one-to-one and $\deg \varphi(e) = 2$. Since $\varphi(e)$ is irreducible, it is a smooth conic in Q.

(ii) Let H be a hyperplane in $\mathbb{P}^3$ not containing the vertex of Q, so that $C = H \cap Q$ is a smooth conic section of Q. First assume that $\varphi^* H$ has

an exceptional curve e_1 as component. If $\varphi^*H = me_1$ for some $m \geq 1$, then $2 = \varphi^*H \cdot e_1 = -m$, which is a contradiction. Therefore, φ^*H is not irreducible. Since deg $\varphi = 2$ and φ^*H is not in the ramification divisor of φ, it follows that we have $\varphi^*H = e_1 + e_2$, where e_2 is irreducible and distinct form e_1. But then we have $e_1 \cdot e_2 = e_1 \cdot \varphi^*H - e_1^2 = e_1 \cdot -2K_X - e_1^2 = 3$. Therefore, φ^*H has three (maybe infinitely near) singular points.
Conversely, assume that φ^*H has at least three (maybe infinitely near) singular points. We have $(\varphi^*H)^2 = 4$ and $\varphi^*H \cdot K_X = -2K_X^2 = -2$. If φ^*H were irreducible, then, by the adjunction formula, we would have

$$2p_a(\varphi^*H) - 2 = \varphi^*H(\varphi^*H + K_X) = 4 - 2 = 2,$$

so $p_a(\varphi^*H) = 2$. Since φ^*H has at least three (maybe infinitely near) singularities, this would imply that it has genus at most $g(\varphi^*H) \leq 2 - 3 < 0$, which is impossible. We conclude that φ^*H is not irreducible. Therefore, since deg $\varphi = 2$ and φ^*H is not the ramification divisor, we have $\varphi^*H = D_1 + D_2$, where D_1 and D_2 are irreducible and D_1 is distinct from D_2. Since C is smooth, the singular points of φ^*H are the intersections between D_1 and D_2, so $D_2 \cdot D_2 \geq 3$. Since $\varphi(D_1) = \varphi(D_2)$, the automorphism of X sending a point $(x : y : z : w)$ to $(x : -y : z : w)$ is an involution that interchanges D_1 and D_2, so $D_1 \cdot K_X = D_2 \cdot K_X$ and $D_1^2 = D_2^2$. Hence from

$$-2 = \varphi^*H \cdot K_X = D_1 \cdot K_X + D_2 \cdot K_X$$

it follows that $D_1 \cdot K_X = D_2 \cdot K_X = -1$. Finally, we have

$$4 = (-2K_X)^2 = D_1^2 + D_2^2 + 2D_1 \cdot D_2 = 2D_1^2 + 2D_1 \cdot D_2,$$

so $2D_1^2 = 4 - 2D_1 \cdot D_2 \leq -2$. Therefore $D_1^2 < 0$, hence from Proposition 2.3 it follows that $D_1^2 = D_2^2 = -1$ and so $D_1 \cdot D_2 = 3$. We conclude that D_1 and D_2 are exceptional curves with intersection multiplicity three.

□

Remark 2.13. From the previous proposition we can conclude that if e_1, e_2 are exceptional curves on X such that $e_1 \cdot e_2 = 3$, the points in the intersection $e_1 \cap e_2$ are exactly the points in the intersection of e_i with the ramification curve of φ, for $i = 1, 2$. We conclude that there is a bijection between the sets

$$\{\{e_1, e_2\} \mid e_1, e_2 \in E;\ e_1 \cdot e_2 = 3\}$$

and

$$\{H \mid H \subset \mathbb{P}^3 \text{ hyperplane tritangent to } B;\ (1:0:0:0) \notin H\}.$$

3 The Weyl group acting on exceptional curves

To count the maximal number of exceptional curves through one point, we will make a lot of use of the group that permutes the exceptional classes in the Picard group while preserving the intersection pairing. In this section we describe this group and study its action on the exceptional classes on a del Pezzo surface of degree one. All results in this section about root systems and the Weyl group can be found in [Man74].

Let X be a del Pezzo surface of degree d over an algebraically closed field k, such that X is isomorphic to $\mathbb{P}^2$ blown up in $r = 9 - d$ points $P_1, \ldots, P_r$. Let $E_i \in \text{Pic } X$ be the class of the exceptional curve above P_i for all i, and let L be the class of the strict transform of a line not going through any of the P_i. Let K_X be the class of the canonical divisor on X. As we have seen, Pic X is a free abelian group of rank $r+1$. Consider the $\mathbb{R}$-vectorspace $\mathbb{R} \otimes_{\mathbb{Z}} \text{Pic } X$. Since $\{L, E_1, \ldots, E_r\}$ is a basis for the Picard group, the set $\{1 \otimes L, 1 \otimes E_1 \ldots, 1 \otimes E_r\}$ is a basis for $\mathbb{R} \otimes_{\mathbb{Z}} \text{Pic } X$.

Lemma 3.1. *For $0 < r \leq 8$, the intersection number $(\cdot, \cdot)$ is negative-definite on the orthogonal complement $K_X^{\perp}$ of K_X in $\mathbb{R} \otimes_{\mathbb{Z}}$ Pic X.*

Proof. Let $D = aL - \sum_{i=1}^{r} b_i E_i \in \text{Pic } X$. Then we have

$$K_X \cdot D = \left(-3L + \sum_{i=1}^{r} E_i\right) \cdot \left(aL - \sum_{i=1}^{r} b_i E_i\right) = -3a + \sum_{i=1}^{r} b_i,$$

so $K_X \cdot D = 0$ if and only if $3a = \sum_{i=1}^{r} b_i$. Now assume $D \in K_X^{\perp}$. Note that D has self-intersection $a^2 - \sum_{i=1}^{r} b_i^2$. By Cauchy-Schwarz we have

$$\sum_{i=1}^{r} b_i^2 = \frac{1}{r} \sum_{i=1}^{r} b_i^2 \sum_{i=1}^{r} 1^2 \geq \frac{1}{r} \left(\sum_{i=1}^{r} b_i\right)^2,$$

so

$$a^2 - \sum_{i=1}^{r} b_i^2 \leq a^2 - \frac{1}{r} \left(\sum_{i=1}^{r} b_i\right)^2 = a^2 - \frac{9}{r} a^2 < 0.$$

We conclude that $D^2 < 0$, so the intersection number is negative definite on $K_X^{\perp}$. $\square$

Definition 3.2. *We define $\left(K_X^{\perp}, \langle \cdot, \cdot \rangle\right)$ to be the vector space in $\mathbb{R} \otimes_{\mathbb{Z}}$ Pic X with inner product $\langle \cdot, \cdot \rangle = -(\cdot, \cdot)$. Note that this inner product is positive-definite by Lemma 3.1.*

We now give the definition of a root system.

Definition 3.3. *Let V be a finite-dimensional vector space over a field $l \subseteq \mathbb{R}$ with a positive-definite inner product $\langle \cdot, \cdot \rangle$. A root system in V is a finite set R of non-zero vectors, called roots, that satisfy the following conditions:*

(i) the roots span V;

(ii) for all $r \in R$, we have $\lambda r \in R \Longrightarrow \lambda = \pm 1$;

(iii) for all $r, s \in R$, we have $s - 2r\frac{\langle r,s\rangle}{\langle r,r\rangle} \in R$;

(iv) for all $r, s \in R$, the number $2\frac{\langle r,s\rangle}{\langle r,r\rangle}$ is an integer.

Define the set

$$R_r = \{D \in \text{Pic } X \mid D^2 = -2;\ D \cdot K_X = 0\}.$$

Proposition 3.4. *The set R_r is a root system of rank r in $\left(K_X^{\perp}, \langle \cdot, \cdot \rangle\right)$.*

Proof. See [Man74], Proposition 25.2. □

From now on we assume that X is a del Pezzo surface of degree one, so $r = 8$.

Proposition 3.5. *The root system R_8 is isomorphic to the classical rootsystem E_8. Moreover, a basis for R_8 is given by the elements $r_1, \ldots, r_8$, given by*

$$E_1 - E_2,\ E_2 - E_3, \ldots, E_7 - E_8,\ L - E_1 - E_2 - E_3.$$

Proof. See [Man74], Theorem 25.4 and Proposition 25.5.6. □

Definition 3.6. *The Weyl group $W(R_8)$ is the group of permutations of the roots of R_8 generated by the reflections with respect to $r_1, \ldots, r_8$ (the reflection with respect to r_i is given by $s \mapsto s - 2r_i\frac{\langle s,r_i\rangle}{\langle r_i,r_i\rangle}$ for all $s \in R_8$).*

Theorem 3.7. *The following groups are isomorphic:*

(i) the group of automorphisms of Pic X preserving K_X and the intersection pairing;

(ii) the group of permutations of the exceptional classes in Pic X preserving their pairwise intersection multiplicities;

(iii) the Weyl group $W(R_8)$.

The order of the group $W(R_8)$ is $2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$.

Proof. See [Man74], Theorem 23.9 and 26.6. □

Let E be the set of exceptional classes in Pic X. Recall that E is in one-to-one correspondence with the set of exceptional curves on X. For the rest of this thesis we refer to $W(R_8)$, the group of permutations of E preserving the intersection pairing, as G. Since G preserves the intersection pairing on E, we can use results about the action of G on E when computing the maximal number of exceptional curves that go through one point. The following proposition will be used a lot.

Proposition 3.8. *Let E and G be as above. Then we have:*

(i) the group G acts transitively on E;

(ii) for all $r \leq 8$ such that $r \neq 7$, the group G acts transitively on the set

$$\{(e_1, \ldots, e_r) \in E^r \mid \forall i \neq j : e_i \cdot e_j = 0\}.$$

Proof. See [Man74], Corollaries 26.7 and 26.8. □

Let U be the set

$$\{(e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8) \in E^8 \mid \forall i \neq j : e_i \cdot e_j = 0\}.$$

The group G acts transitively on U by Proposition 3.8. We show some results about U that will be useful later.

Lemma 3.9. *For $u = (e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8) \in U$, there exists a morphism $f : X \longrightarrow \mathbb{P}^2$, and points $Q_1, \ldots, Q_8 \in \mathbb{P}^2$ that are in general position, such that X is the blow-up of $\mathbb{P}^2$ at $Q_1, \ldots, Q_8$, and for all i, the element e_i is the class in Pic X of the exceptional curve above Q_i.*

Proof. Set $Y_0 = X$. Since all Q_i are disjoint, by Theorem 2.5, for each $i \in \{1, \ldots, 8\}$ there is a nonsingular projective surface Y_i, and a morphism $f_i : Y_{i-1} \longrightarrow Y_i$ that is the blow-up of Y_i in Q_i, where e_i is the class in Pic X of the exceptional curve above Q_i. Since k is algebraically closed, by Proposition 2.6, the surface Y_i is a del Pezzo surface of degree $i + 1$ for all i. It follows that $Y_8 = \mathbb{P}^2$, and the composition of the f_i is a morphism $f : X \longrightarrow \mathbb{P}^2$ that is the blow-up in $Q_1, \ldots, Q_8$. Since X is a del Pezzo surface, from Proposition 2.4 it follows that $Q_1, \ldots, Q_8$ are in general position. □

Corollary 3.10. *For $u = (e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8) \in U$, there is a unique element $l \in$ Pic X such that $K_X = -3l + \sum_{i=1}^{8} e_i$. Moreover, the set $\{l, e_1, \ldots, e_8\}$ forms a basis for Pic X.*

Proof. Let $u = (e_1, \ldots, e_8) \in U$. By the previous lemma there exists a morphism $f : X \longrightarrow \mathbb{P}^2$, and points $Q_1, \ldots, Q_8 \in \mathbb{P}^2$ that are in general position, such that X is the blow-up of $\mathbb{P}^2$ at $Q_1, \ldots, Q_8$, and e_i is the class of the exceptional curve above Q_i for all i. By Remark 1.17 we have $K_X = -3l + \sum_{i=1}^{8} e_i$, where l is the class of the strict transform of a line in $\mathbb{P}^2$ not containing any of the Q_i. By Proposition 1.16 we know that $\{l, e_1, \ldots, e_8\}$ forms a basis for Pic X. □

Remark 3.11. Let $u = (e_1, \ldots, e_8) \in U$, and let l be the unique element such that $K_X = -3l + \sum_{i=1}^{8} e_i$, which exists by Corollary 3.10. Let $g_1, g_2 \in G$ be such that $g_1(u) = g_2(u)$. This implies $g_1(l) = g_2(l)$, since g_1, g_2 fix K_X. Therefore, g_1 and g_2 act the same on the basis $\{l, e_1, \ldots, e_8\}$ of Pic X, hence they act the same on Pic X. Since G acts faithfully on Pic X we conclude that $g_1 = g_2$. Therefore, the action of G on U is free. Since G acts transitively on U and U is not empty, this implies that $|U| = |G|$.

Remark 3.12. Let A be the set of 240 vectors $(a, b_1, \ldots, b_8)$ that are in the table in Proposition 2.7 (where the b_i can be permuted). We have a map

$$f : U \longrightarrow \mathrm{Hom}_{\mathrm{Set}}(E, A)$$

as follows. Given $u = (e_1, \ldots, e_8) \in U$, let l be the unique element such that $K_X = -3l + \sum_{i=1}^{8} e_i$, which exists by Corollary 3.10. Then we define $f(u)$ as follows.

$$f(u) : E \longrightarrow A, \; e \longmapsto (e \cdot l, e \cdot e_1, \ldots, e \cdot e_8).$$

For $\alpha = (a, b_1, \ldots, b_8) \in A$ and $e = al - \sum_{i=1}^{8} b_i e_i \in E$ we have $f(u)(e) = \alpha$, hence $f(u)$ is surjective. Since E and A have the same cardinality, it follows that $f(u)$ is a bijection.

To study G and its action on E further, we first look at the intersection multiplicities on E. These results will be useful later.

Lemma 3.13. *For every $e \in E$ there is a unique $c \in E$ such that $e \cdot c = 3$.*

Proof. Since G acts transitively on E, it is enough to check this for $e = E_1$. Let $c = aL - \sum_{i=1}^{8} b_i E_i \in E$. Then c intersects e with multiplicity three if and only if $b_1 = 3$. By looking at the table in Proposition 2.7, we find that there is one solution for c with $b_1 = 3$, given by $c = 6L - 3E_1 - \sum_{i=2}^{8} 2E_i$. □

Remark 3.14. Since for every element e in E there is a unique element intersecting e with multiplicity three, and G acts transitively on E, the group G acts transitively on the set

$$\{(e_1, e_2) \in E^2 \mid e_1 \cdot e_2 = 3\}.$$

Lemma 3.15. *There is a bijection between the sets*

$$Z = \{(e_0, e_1) \in E^2 \mid e_0 \cdot e_1 = 0\} \quad \text{and} \quad Z' = \{(e_1, e_2) \in E^2 \mid e_1 \cdot e_2 = 2\},$$

given by

$$f : Z \longrightarrow Z', \; (e_0, e_1) \longmapsto (e_1, e_2), \; \text{where } e_2 \text{ is such that } e_0 \cdot e_2 = 3.$$

Proof. Let $e_0 = E_1$ and $e_1 = E_2$. Then (e_0, e_1) is an element in Z. By looking at the table in Proposition 2.7, we see that there is exactly one exceptional class intersecting e_0 with multiplicity three, which is $e_2 = 6L - 3E_1 - \sum_{i=2}^{8} 2E_i$. We have $e_1 \cdot e_2 = 2$. Since G acts transitively on Z by Proposition 3.8, it follows that for all elements $(c_0, c_1) \in Z$, the unique element c_2 such that $c_0 \cdot c_2 = 3$ has the property that $c_1 \cdot c_2 = 2$. Therefore, f is well defined. Let $(c_1, c_2) \in Z'$. Then $f^{-1}((c_1, c_2))$ consists of all elements $(c, c_1) \in Z$ such that $c \cdot c_2 = 3$. From Lemma 3.13 we know that there is one such c, say c_0. We conclude that $f^{-1}((c_1, c_2)) = \{(c_0, c_1)\}$, so all fibers of f have cardinality one. Therefore, f is a bijection. □

Corollary 3.16. *G acts transitively on the set*

$$Z = \{(e_1, e_2) \in E^2 \mid e_1 \cdot e_2 = 2\}.$$

Proof. This follows from Lemma 3.15 and Proposition 3.8. □

Proposition 3.17. *Any two elements in E intersect each other with multiplicity at most three. Let $e \in E$ be an exceptional class in Pic X. Then there are exactly 56 exceptional classes disjoint from e, there are 126 exceptional classes intersecting*

e with multiplicity one, and there are 56 exceptional classes intersecting e with multiplicity two.

Proof. Since G acts transitively on E, it is enough to check this for $e = E_1$. Let $c = aL - \sum_{i=1}^{8} b_i E_i \in E$. Then $e \cdot c = b_1$. We can now easily compute the results case by case.
We have $e \cdot c = 0$ if and only if $b_1 = 0$. Looking at the table in Proposition 2.7, we find the following possibilities.

a	0	1	2	3
number of possibilities for c	7	21	21	7

This gives a total of 56 exceptional classes that are disjoint from e. By the bijection in Lemma 3.15, this gives also 56 exceptional classes intersecting e with multiplicity two.
Similarly, c intersects e with multiplicity one if and only if $b_1 = 1$. This gives the following possibilities.

a	1	2	3	4	5
number of possibilities for c	7	35	42	35	7

Therefore we find a total of 126 exceptional classes intersecting e with multiplicity one.
From Lemma 3.13 we know that there is one c such that $e \cdot c = 3$. Since we have a total of 240 exceptional classes, we conclude that these are all the possibilities. $\square$

Lemma 3.18. *Let $e_1,\ e_2 \in E$ such that $e_1 \cdot e_2 = 0$. Then there are exactly 72 elements of E intersecting both e_1 and e_2 with multiplicity one.*

Proof. We know that G acts transitively on the set $\{(e_1, e_2) \in E^2 \mid e_1 \cdot e_2 = 0\}$ from Proposition 3.8, so it is enough to check this for $e_1 = E_1$ and $e_2 = E_2$. Let $e = aL - \sum_{i=1}^{8} b_i E_i \in E$. Then $E_1 \cdot e = b_1$ and $E_1 \cdot e = b_2$, so $E_1 \cdot e = E_2 \cdot e = 1$ if and only if $b_1 = b_2 = 1$. Looking at the table in Proposition 2.7, we find the following possibilities.

a	1	2	3	4	5
number of possibilities for e	1	20	30	20	1

This gives a total of 72 exceptional classes that intersect both E_1 and E_2 with multiplicity one. $\square$

We are now able to prove a couple of propositions about the action of G on various subsets of E. These propositions are very useful for our purpose in the next section. First, we need some lemmas.

Lemma 3.19. *Let V be the set*

$$V = \{(e_0, e_1, e_2) \in E^3 \mid e_0 \cdot e_1 = e_0 \cdot e_2 = 1;\ e_1 \cdot e_2 = 0\}.$$

Then $|V| = 967680$, and G acts transitively on V.

Proof. Since G preserves intersection multiplicities, it acts on V. Fix $e_1 \in E$. By Proposition 3.17 there are exactly 56 exceptional classes disjoint from e, and by Lemma 3.18, for each e_2 of those 56 there are exactly 72 exceptional classes intersecting both e_1 and e_2 with multiplicity one. Therefore we have

$$|V| = 240 \cdot 56 \cdot 72 = 967680.$$

Let $e_0 = L - E_1 - E_2$, $e_1 = E_1$, and $e_2 = E_2$. Then $v = (e_0, e_1, e_2)$ is an element in V. Let G_v be the stabilizer of v in G and Gv the orbit of v in V, then we have $[G : G_v] = |Gv| \leq |V|$. We want to show that the latter is an equality. Let W_v be the set

$$W_v = \{e \in E \mid e \cdot e_0 = e \cdot e_1 = e \cdot e_2 = 0\}.$$

For $e = aL - \sum_{i=1}^{r} b_i E_i \in W_v$, the condition $e \cdot e_0 = e \cdot e_1 = e \cdot e_2 = 0$ is equivalent to $a = b_1 = b_2 = 0$. Looking at the table in Proposition 2.7, we see that there are only 6 possibilities for e, which are $E_3, \ldots, E_8$. So we have $W_v = \{E_3, \ldots, E_8\}$. Since G preserves intersection multiplicities, G_v acts on W_v. Let $g \in G_v$. If $gE_i = E_i$ for $3 \leq i \leq 8$, then g fixes $E_1, \ldots, E_8$. But then g fixes every element in E and since G acts faithfully on E this implies that g is the identity. Therefore, G_v acts faithfully on W_v, hence $G_v \subseteq S_6$, so $|G_v| \leq 720$. We now have

$$967680 = \frac{2^{14} \cdot 3^5 \cdot 5^2 \cdot 7}{720} \leq \frac{|G|}{|G_v|} = |Gv| \leq |V| = 967680,$$

so we have equality everywhere and so $Gv = V$. We conclude that G acts transitively on V. □

Lemma 3.20. *Let H be a group, let A, B be H-sets, and $f : A \longrightarrow B$ a morphism of H-sets. Then the following hold.*

(i) if H acts transitively on A, then H acts transitively on $f(A)$;

(ii) if H acts transitively on A and A is finite, then all non-empty fibers of f have the same cardinality, say n, and $|f(A)| = \frac{|A|}{n}$;

(iii) if H acts transitively on B, then all fibers of f have the same cardinality.

Proof.

(i) Let $f(a), f(a') \in f(A)$ with $a, a' \in A$. Assume that H acts transitively on A, then there is an $h \in H$ such that $ha = a'$. Since f is a morphism of H-sets, we have $hf(a) = f(ha) = f(a')$, so H acts transitively on $f(A)$.

(ii) Let $b, b' \in B$ such that $f^{-1}(b)$ and $f^{-1}(b')$ are non-empty. Then we have $b, b' \in f(A)$, so there is an element $h \in H$ such that $hb = b'$ by (i). But then $|f^{-1}(b')| = |f^{-1}(hb)| = |hf^{-1}(b)| = |f^{-1}(b)|$. We conclude that all non-empty fibers have the same cardinality, say n. It is now immediate that $|A| = |f^{-1}(B)| = \sum_{b \in f(A)} n = n|f(A)|$, so $|f(A)| = \frac{|A|}{n}$.

(iii) Let $b, b' \in B$. Since H acts transitively on B, there is an $h \in H$ such that $hb = b'$, so $|f^{-1}(b')| = |f^{-1}(hb)| = |hf^{-1}(b)| = |f^{-1}(b)|$.

□

Lemma 3.21. *Let $e_1 = E_1$ and $e_2 = L - E_1 - E_2$. Then there are 32 elements e in E such that $e_1 \cdot e = 1$ and $e_2 \cdot e = 0$.*

Proof. Let $e = aL - \sum_{i=1}^{8} b_i E_i \in E$, then $e_1 \cdot e = 1$ and $e_2 \cdot e = 0$ if and only if $b_1 = 1$ and $a - b_1 - b_2 = 0$. Looking at the table in Proposition 2.7, we find the following possibilities.

a	1	2	3
number of possibilities for e	6	20	6

This gives a total of 32 possibilities for e. □

Proposition 3.22. *G acts transitively on the set*

$$W = \{(e_1, e_2) \in E^2 \mid e_1 \cdot e_2 = 1\}.$$

Proof. Consider the set $V = \{(e_1, e_2, e_3) \in E^3 \mid e_1 \cdot e_2 = e_1 \cdot e_3 = 1;\ e_2 \cdot e_3 = 0\}$. We have a projection $f : V \longrightarrow W$ on the first two coordinates. Consider the elements $e_1 = E_1$ and $e_2 = L - E_1 - E_2$. Then $w = (e_1, e_2)$ is an element of W. Let $e \in E$, then $(e_1, e_2, e) \in V$ if and only if $e_1 \cdot e = 1$ and $e_2 \cdot e = 0$. By Lemma 3.21 this gives 32 possibilities for e, so $|f^{-1}((e_1, e_2))| = 32$. Since G acts transitively on V by Lemma 3.19, it follows from Lemma 3.20 that all non-empty fibers of f have cardinality 32, and $|f(V)| = \frac{|V|}{32} = 30240$. By Proposition 3.17 we have $|W| = 240 \cdot 126 = 30240$. We conclude that $f(V) = W$, hence f is surjective. Therefore, G acts transitively on W by Lemma 3.20. □

Now that we know that G acts transitively on all pairs in E that intersect with multiplicity one, we can easily get more results on the intersection multiplicities in E.

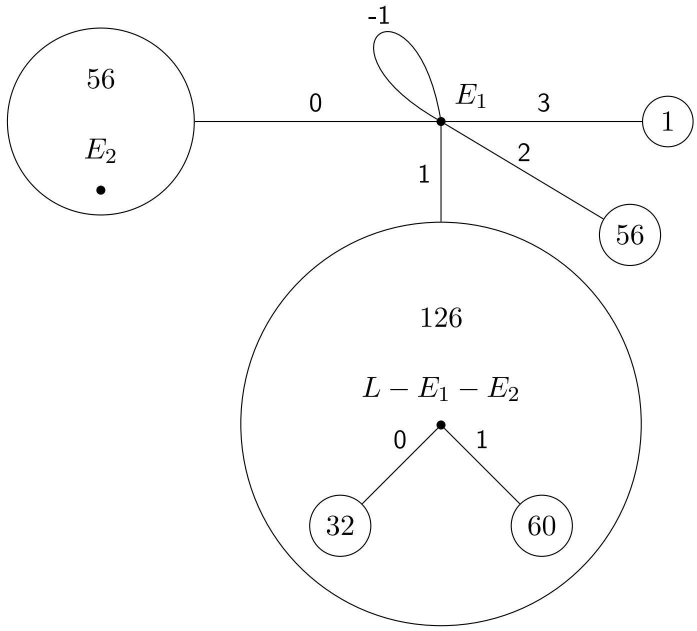
Lemma 3.23. *For each pair $(e_1, e_2) \in E^2$ such that $e_1 \cdot e_2 = 1$ there are exactly 60 exceptional classes $e \in E$ such that $e_1 \cdot e = e_2 \cdot e = 1$.*

Proof. By Proposition 3.22 it is enough to check this for one pair. Let $e_1 = E_1$ and $e_2 = L - E_1 - E_2$. Then $e_1 \cdot e_2 = 1$. Now let $e = aL - \sum_{i=1}^{r} b_i E_i \in E$, then $e_1 \cdot e = 1$ if and only if $b_1 = 1$, and $e_2 \cdot e = 1$ if and only if $a - b_1 - b_2 = 1$. Combining this we have $e_1 \cdot e = e_2 \cdot e = 1$ if and only if $b_1 = 1$ and $a - b_2 = 2$. Looking at the table in Proposition 2.7, we find all following possibilities.

a	2	3	4
number of possibilities for e	15	30	15

This gives a total of 60 exceptional classes intersecting both e_1 and e_2 with multiplicity one. □

The following graph shows some of the information we obtained so far about the intersection multiplicities in E. The vertexes are subsets of E and the number in a vertex is the cardinality of the subset. The numbers on the edges between the vertexes are the intersection multiplicities between the elements in the two subsets. By Lemma 3.19, replacing the elements $L - E_1 - E_2,\ E_1,\ E_2$ by any other triple $e_0,\ e_1,\ e_2$ with $e_0 \cdot e_2 = e_0 \cdot e_2 = 1$ and $e_1 \cdot e_2 = 0$ will give the same graph.



We conclude this section by looking at the action of G on the sets

$$V = \{(e_1, e_2, e_3) \in E^3 \mid \forall i \neq j : e_i \cdot e_j = 1\}$$

and

$$W = \{(e_1, e_2, e_3, e_4) \in E^4 \mid \forall i \neq j : e_i \cdot e_j = 1\}.$$

G acts on both V and W, since G preserves the intersection pairing on E.

We start by introducing some notation. Let Z be the set

$$Z = \{(\{e_1, e_2\}, \{e_3, e_4\}, \{e_5, e_6\}, \{e_7, e_8\}) \mid \forall i : e_i \in E;\ \forall i \neq j : e_i \cdot e_j = 0\}.$$

Recall the set U defined by

$$U = \{(e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8) \in E^8 \mid \forall i \neq j : e_i \cdot e_j = 0\}.$$

We have $|Z| = \frac{|U|}{2^4}$, so from Remark 3.11 it follows that $|Z| = \frac{|G|}{2^4} = 2^{10} \cdot 3^5 \cdot 5^2 \cdot 7$. Moreover, from Proposition 3.8, it follows that G acts transitively on Z.

Let $f : W \longrightarrow V$ be the projection on the first three coordinates. We define a map $g : Z \longrightarrow W$ as follows. For $z = (\{e_1, e_2\}, \{e_3, e_4\}, \{e_5, e_6\}, \{e_7, e_8\}) \in Z$, let l be the unique element in E such that $K_X = -3l + \sum_{i=1}^{8} e_i$. Then we set

$$g(z) = (l - e_1 - e_2, l - e_3 - e_4, l - e_5 - e_6, l - e_7 - e_8).$$

Let $h \in G$, then $K_X = hK_X = -3hl + \sum_{i=1}^{8} he_e$, so

$$g(hz) = (hl - he_1 - he_2, hl - he_3 - he_4, hl - he_5 - he_6, hl - he_7 - he_8) = hg(z).$$

Therefore, the map g is a morphism of G-sets.

Let Y be the image of g. The following commutative diagram shows the maps and sets that are defined.

$$\begin{tikzcd}
U \arrow[d, two heads] & & \\
Z \arrow[r, "g"] \arrow[dr] & W \arrow[r, "f"] & V \\
& Y \arrow[u, hook] \arrow[ur] &
\end{tikzcd}$$

Lemma 3.24. *The map g is injective.*

Proof. Consider the elements $e_1 = L - E_1 - E_2$, $e_2 = L - E_3 - E_4$, $e_3 = L - E_5 - E_6$ and $e_4 = L - E_7 - E_8$. Then $w = (e_1, e_2, e_3, e_4)$ is an element in W. The fiber of g above w consists of the elements $(\{c_1, c_2\}, \{c_3, c_4\}, \{c_5, c_6\}, \{c_7, c_8\}) \in Z$ such that

$$\begin{aligned}
c_1 \cdot e_1 = c_2 \cdot e_1 = 1 &\text{ and } c_1 \cdot e_i = c_2 \cdot e_i = 0 \text{ for all } i \neq 1; \qquad (1)\\
c_3 \cdot e_2 = c_4 \cdot e_2 = 1 &\text{ and } c_3 \cdot e_i = c_4 \cdot e_i = 0 \text{ for all } i \neq 2;\\
c_5 \cdot e_3 = c_6 \cdot e_3 = 1 &\text{ and } c_5 \cdot e_i = c_6 \cdot e_i = 0 \text{ for all } i \neq 3;\\
c_7 \cdot e_4 = c_8 \cdot e_4 = 1 &\text{ and } c_7 \cdot e_i = c_8 \cdot e_i = 0 \text{ for all } i \neq 4.
\end{aligned}$$

Clearly, c_i and c_{i+1} are interchangeable for $i \in \{1, 3, 5, 7\}$. Let $c_1 = aL - \sum_{i=1}^{8} b_i E_i$. Then (1) implies $a - b_1 - b_2 = 1$ and $a = b_3 + b_4 = b_5 + b_6 = b_7 + b_8$. Looking at the table in Proposition 2.7, the only possibilities for c_1 and c_2 are E_1 and E_2. Analogously we find that the only possibilities for c_3 and c_4 are E_3 and E_4, the only possibilities for c_5 and c_6 are E_5 and E_6, and the only possibilities for c_7 and c_8 are E_7 and E_8. Therefore we have $g^{-1}(w) = \{(\{E_1, E_2\}, \{E_3, E_4\}, \{E_5, E_6\}, \{E_7, E_8\})\}$, hence the fiber above w has cardinality one. Since G acts transitively on Z, we conclude from Lemma 3.20 that all non-empty fibers of g have cardinality one, so g is injective. $\square$

Remark 3.25. By the previous proposition, the map $g : Z \longrightarrow Y$ is a bijection. Since g is a G-map, it follows that Y is a G-set, and that G acts transitively on Y.

Lemma 3.26. *Consider the elements of E given by*

$$e_1 = L - E_1 - E_2; \qquad c_1 = 3L - \sum_{i=1}^{6} E_i - 2E_7$$

$$e_2 = L - E_3 - E_4; \qquad c_2 = 3L - \sum_{i=1}^{6} E_i - 2E_8.$$

$$e_3 = L - E_5 - E_6;$$

Then $w_1 = (e_1, e_2, e_3, c_1)$ and $w_2 = (e_1, e_2, e_3, c_2)$ are elements in W that are not in Y.

Proof. It is easy to check that w_1 and w_2 are in W. We want to show that the fibers of g above w_1 and w_2 are empty. Let $z = (\{d_1, d_2\}, \{d_3, d_4\}, \{d_5, d_6\}, \{d_7, d_8\}) \in Z$, and write $d_1 = rL - \sum_{i=1}^{8} s_i E_i$. Then $z \in g^{-1}(w_1)$ implies that $d_1 \cdot e_1 = 1$ and $d_1 \cdot e_2 = d_1 \cdot e_3 = d_1 \cdot c_1 = 0$, which is equivalent to $r - s_1 - s_2 = 1$, $r = s_3 + s_4 = s_5 + s_6$, and $3r - \sum_{i=1}^{6} s_i - 2s_7 = 0$. But this implies

$$0 = 3r - \sum_{i=1}^{6} s_i - 2s_7 = 3r - (s_1 + s_2) - 2r - 2s_7 = r - (s_1 + s_2) - 2s_7 = 1 - 2s_7,$$

and since s_7 is an integer this has no solutions.We conclude that the fiber of g above w_1 is empty and analogously the fiber of g above w_2 is empty. This proves that w_1 and w_2 are not in Y. □

Proposition 3.27. *Let $v = (e_1, e_2, e_3)$ be an element of V. The following hold.*

(i) The group G acts transitively on V.

(ii) We have $|f^{-1}(v)| = 26$, and $|f^{-1}(v) \cap Y| = 24$.

(iii) For $e \in f^{-1}(v) \cap Y$ and $\{c_1, c_2\} = f^{-1}(v) \setminus Y$, we have $e \cdot c_1 = e \cdot c_2 = 1$, and $c_1 \cdot c_2 = 3$.

Proof.

(i) Consider the map $\lambda = f \circ g : Z \to V$. Note that λ is a G-map, since both f and g are. We want to show that λ is surjective. Let

$$e_1 = L - E_1 - E_2, \; e_2 = L - E_3 - E_4, \; e_3 = L - E_5 - E_6.$$

Then $v = (e_1, e_2, e_3) \in V$. Note that

$$\lambda\left((\{E_1, E_2\}, \{E_3, E_4\}, \{E_5, E_6\}, \{E_7, E_8\})\right) \;=\; v,$$

so the fiber of λ above v is not empty. To compute this fiber, we first compute the fiber of f above v. Let $e = aL - \sum_{i=1}^{8} E_i \in E$. The conditions $e \cdot e_1 = 1$, $e \cdot e_2 = 1$ and $e \cdot e_3 = 1$ are equivalent to $a - b_1 - b_2 = a - b_3 - b_4 = a - b_5 - b_6 = 1$. By looking at the table in Proposition 2.7 we find all possibilities.

a	1	2	3	4	5
number of possibilities for e	1	8	8	8	1

We find a total of 26 possibilities for e, so $|f^{-1}(v)| = 26$. Since $\lambda^{-1}(v)$ is not empty, by Lemma 3.20, we have $|\lambda(Z)| = \frac{|Z|}{\lambda^{-1}(v)}$. Since we have

$$\lambda(Z) \leq |V| = 240 \cdot 126 \cdot 60 = 1814400,$$

this implies

$$\lambda^{-1}(v) \geq \frac{|Z|}{1814400} = 24. \tag{2}$$

Since $g : Z \longrightarrow Y$ is a bijection we have $\lambda^{-1}(v) \leq 26$. Consider the elements $c_1 = 3L - \sum_{i=1}^{6} E_i - 2E_7$ and $c_2 = 3L - \sum_{i=1}^{6} E_i - 2E_8$, then $w_1 = (e_1, e_2, e_3, c_1)$ and $w_2 = (e_1, e_2, e_3, c_2)$ are both elements in $f^{-1}(v)$. By Lemma 3.26, we know that the fibers of g above w_1 and w_2 are empty. It follows that $\lambda^{-1}(v) \leq 24$, which together with (2) implies $\lambda^{-1}(v) = 24$. This means that

$$|\lambda(Z)| = \frac{|Z|}{24} = |V|,$$

so λ is surjective. Since G acts transitively on Z, we conclude that G acts transitively on V, too.

(ii) In part (i) we showed that $|f^{-1}(v)| = 26$ and $|\lambda^{-1}(v)| = 24$. Since g is a bijection, we have $|f^{-1}(v) \cap Y| = |\lambda^{-1}(v)| = 24$. Since f is a G-map, and G acts transitively on V, the result holds for all elements in V.

(iii) This is an easy check, after writing down the 26 elements we found in part (i).

□

Proposition 3.28. *The set W has two orbits under the action of G.*

Proof. From Remark 3.25 it follows that Y is an orbit under the action of G on W. Therefore $O = W - Y$ is also a G-set. Consider the restriction of f to O,

$$f|_O : O \longrightarrow V.$$

Let e_1, e_2, e_3, c_1, c_2 be as in Lemma 3.26, and let $v = (e_1, e_2, e_3)$, $w_1 = (e_1, e_2, e_3, c_1)$, and $w_2 = (e_1, e_2, e_3, c_2)$. Then we have $v \in V$, and $w_1, w_2 \in f|_O^{-1}(v)$ by Lemma 3.26. From Proposition 3.27 we know that $|f^{-1}(v) \cap Y| = 24$, so $\left| f|_O^{-1}(v) \right| = 2$. This implies $f|_O^{-1}(v) = \{w_1, w_2\}$. For $r_7 = E_7 - E_8 \in R_8$, the reflection with respect to r_7 is an element in G, say h. Since $e_1 \cdot r_7 = e_1 \cdot r_7 = e_3 \cdot r_7 = 0$, for $i \in \{1, 2, 3\}$ we have

$$he_i = e_i - 2r_7 \frac{e_i \cdot r_7}{r_7 \cdot r_7} = e_i,$$

so h is contained in the stabilizer G_v of v in G. We have

$$hc_1 = c_1 - 2r_7 \frac{c_1 \cdot r_7}{r_7 \cdot r_7} = c_1 + 2r_7 = 3L - \sum_{i=1}^{6} E_i - 2E_8 = c_2,$$

so h interchanges w_1 and w_2, hence G_v acts transitively on $f|_O^{-1}(v)$. Since G acts transitively on V, this holds for all elements in V. Now let $o, o' \in O$. Let $a = f(o)$

and $b = f(o')$. Since G acts transitively on V there is an element $h_1 \in G$ such that $b = h_1 a$. Then $f(o') = h_1 f(o) = f(h_1 o)$, so o' and $h_1 o$ are in the same fiber of f. Since G_b acts transitively on this fiber, there is al element $h_2 \in G_b$ such that $o' = h_2 h_1 o$. We conclude that G acts transitively on O. It follows that W consists of the orbits O and Y. $\square$

4 Maximal cliques and the maximum

In this section we prove Theorem 1 and Theorem 2. Recall that we defined X to be a del Pezzo surface of degree one over an algebraically closed field k, and E the set of exceptional classes in Pic X. We want to compute the maximal number of exceptional curves on X that go through one point P. Let Q be the quadratic cone such that X is a double cover of Q under the map φ, branched over a smooth curve B of degree 6. We distinguish two cases. In Subsection 4.1, we will consider the case when $\varphi(P)$ lies on B, and prove Theorem 1. In Subsection 4.2, we consider the case when $\varphi(P)$ does not lie on B, and prove Theorem 2.

Since there is a one-to-one correspondence between exceptional curves on X and exceptional classes in Pic X, a set of exceptional curves can go through one point only if the corresponding set of exceptional classes pairwise intersect with multiplicity greater than zero. Therefore, we start by looking at sets of exceptional curves that pairwise intersect positively, and we do this by studying the graph on E.

Definition 4.1. *By $\mathcal{G}$ we denote the weighted graph whose vertices are the elements of E, and where two vertices are connected by an edge of weight n if and only if the corresponding elements of E intersect with multiplicity $n > 0$.*

Remark 4.2. Since G is the group of automorphisms of E that preserve intersection multiplicities, it follows that G is the automorphism group of $\mathcal{G}$.

Definition 4.3. *A clique in $\mathcal{G}$ is a weighted subgraph of $\mathcal{G}$ in which every two vertices are connected by an edge. The size of a clique is the number of vertices contained in it.*

As we mentioned before, a number of exceptional curves go through one point only if the corresponding classes form a clique in $\mathcal{G}$. Of course, the converse is not always true. However, the maximal size of the cliques in $\mathcal{G}$ does give us a first upper bound for the maximal number of exceptional curves that go through one point. To compute this, we first need a couple of lemmas.

Definition 4.4. *A maximal clique in $\mathcal{G}$ is a clique that is maximal with respect to inclusion.*

Lemma 4.5. *The size of a clique in $\mathcal{G}$ that contains no edges of weight one is at most three.*

Proof. Let K be a maximal clique in $\mathcal{G}$ without edges of weight one. We distinguish two cases.
First assume that K contains an edge of weight two. Then by Proposition 3.16, we can without loss of generality assume that K contains the vertices corresponding to $e_1 = E_1$ and $e_2 = 3L - 2E_1 - \sum_{i=2}^{7} E_i$. Let $e = aL - \sum_{i=1}^{8} b_i E_i \in E$, then $e \cdot e_1 > 1$ and $e \cdot e_2 > 1$ if and only if $b_1 > 1$ and $3a - 2b_1 - \sum_{i=2}^{7} b_i > 1$. By looking at the table in Proposition 2.7 we find only one possibility for e, which is $e = 6L - 2\sum_{i=1}^{7} E_i - 3E_8$. We conclude that K consists of the edges corresponding

to e_1, e_2 and e and thus has size three.
Now assume that K contains an edge of weight three. Then by Remark 3.14 we can without loss of generality assume that K contains the vertices corresponding to $c_1 = E_1$ and $c_2 = 6L - 3E_1 - 2\sum_{i=2}^{2} E_i$. Let $c = aL - \sum_{i=1}^{8} b_i E_i \in E$, then $c \cdot c_1 > 1$ and $c \cdot c_2 > 1$ if and only if $b_1 > 1$ and $6a - 3b_1 - \sum_{i=2}^{8} b_i > 1$. This has no solutions in the table in Proposition 2.7, so K consists only of the vertices corresponding to c_1 and c_2. □

Lemma 4.6. *Let e_1, e_2 be elements in E that are not disjoint, and let c_1, c_2 be such that $e_1 \cdot c_1 = e_2 \cdot c_2 = 3$. Then the following are equivalent.*

(i) $e_1 \cdot e_2 = 1$;

(ii) $e_1 \cdot c_2 > 0$ *and* $e_2 \cdot c_1 > 0$;

(iii) $e_1 \cdot c_2 = e_2 \cdot c_1 = 1$.

Proof. First assume that $e_1 \cdot e_2 = 1$. Since G acts transitively on the set of pairs of exceptional classes intersecting with multiplicity one, we can assume without loss of generality that $e_1 = E_1$ and $e_2 = L - E_1 - E_2$. Then $c_1 = 6L - 3E_1 - 2\sum_{i=2}^{2} E_i$ and $c_2 = 5L - E_1 - E_2 - 2\sum_{i=3}^{8} E_i$. It is straightforward to check that $e_1 \cdot c_2 = e_2 \cdot c_1 = 1$, so (i) implies (iii).
Since (iii) obviously implies (ii), it is now enough to prove that (ii) implies (i). To this end, assume that $e_1 \cdot c_2 > 0$ and $e_2 \cdot c_1 > 0$. If $e_1 \cdot c_2 = 2$ then from the bijection in Lemma 3.15 we have $e_1 \cdot e_2 = 0$, which is a contradiction. Therefore we have $e_1 \cdot c_2 = 1$. Without loss of generality we take $e_1 = E_1$ and $c_2 = L - E_1 - E_2$. Then $e_2 = 5L - E_1 - E_2 - 2\sum_{i=3}^{8} E_i$, so $e_1 \cdot e_2 = 1$. □

The previous lemma states that if we have two pairs (e_1, c_1) and (e_2, c_2) of exceptional classes intersecting with multiplicity three, the four classes together form a clique in $\mathcal{G}$ if and only if $e_1 \cdot e_2 = e_1 \cdot c_2 = e_2 \cdot c_1 = c_1 \cdot c_2 = 1$. We call the pair $(\{e_1, c_1\}, \{e_2, c_2\})$ an *intersecting pair*.

Corollary 4.7. *G acts transitively on the set*

$$S = \{(e_1, e_2, c_1, c_2) \in E^4 \mid (\{e_1, c_1\}, \{e_2, c_2\}) \text{ is an intersecting pair.}\}$$

Proof. Consider the set $T = \{(e_1, e_2) \in E^2 \mid e_1 \cdot e_2 = 1\}$ and the map

$$\lambda : T \longrightarrow S,\ (e_1, e_2) \longmapsto (e_1, e_2, c_1, c_2),$$

where $c_1 \cdot e_1 = c_2 \cdot e_2 = 3$. Note that λ is well defined by Lemma 4.6. Let (e_1, e_2, c_1, c_2) be an element in S. Then $(\{e_1, c_1\}, \{e_2, c_2\})$ is an intersecting pair, so $e_1 \cdot e_2 = 1$, and $\lambda((e_1, e_2)) = (e_1, e_2, c_1, c_2)$. We conclude that λ is surjective. The statement now follows from the fact that G acts transitively on T. □

Corollary 4.8. *Every maximal clique K in $\mathcal{G}$ of size bigger than two that does not contain any edge of weight two is of the form*

$$K = \{e_1, \dots, e_n, c_1, \dots, c_n \mid \forall i \neq j : (\{e_i, c_i\}, \{e_j, c_j\}) \text{ is an intersecting pair}\}.$$

Proof. Let K be a maximal clique not containing any edge of weight two. If K would not contain any edge of weight one, it would consist of two vertexes connected by an edge of weight three, hence it would have size two. Therefore we can assume that K contains at least one edge of weight one. Let $e_1, \ldots, e_n$ be a subclique of maximal size in K that only contains edges of weight one. By Lemma 4.6, for all $i \neq j$, the unique elements $c_i, c_j \in E$ such that $e_i \cdot c_i = e_j \cdot c_j = 3$ satisfy $e_i \cdot c_j = e_j \cdot c_i = c_i \cdot c_j = 1$. Since K is maximal, it follows that K also contains the n elements $c_1, \ldots, c_n$. If there would be another element $d \in K$, then, since there is only one element intersecting d with multiplicity three, either $d \cdot e_i = 1$ for all i, or $d \cdot c_i = 1$ for all i. But this contradicts the fact that the set $\{e_1, \ldots, e_n\}$ is maximal. We conclude that $K = \{e_1, \ldots, e_n, c_1, \ldots, c_n\}$. $\square$

Lemma 4.9. *The maximal size of the cliques in $\mathcal{G}$ that contain an edge of weight two is thirteen.*

Proof. Let K be a clique of maximal size in $\mathcal{G}$ that contains an edge of weight two. By Proposition 3.16 we can without loss of generality assume that K contains $e_1 = E_1$ and $e_2 = 3L - 2E_1 - \sum_{i=2}^{7} E_i$. Let $e = aL - \sum_{i=1}^{8} E_i \in E$. The conditions $e \cdot e_1 \geq 1$ and $e \cdot e_2 \geq 1$ are equivalent to $b_1 \geq 1$ and $3a - 2b_1 - \sum_{i=2}^{7} b_i \geq 1$. By looking at the table in Proposition 2.7 we find all possibilities.

a	1	2	3	4	5	6
number of possibilities for e	1	20	36	41	22	7

We find a total of 127 possibilities. As it is too tedious to compute all pairwise intersection multiplicities by hand, we do this with MAGMA and find that the maximal size of a clique in the graph on these 127 exceptional curves is eleven. This gives a total of 13 elements in K. We conclude that the maximal size of a clique that contains an edge of weight two is thirteen. $\square$

Let V, W, Z, f, g and Y be as in the diagram on page 25. Let $\mathcal{S}$ be the set of all cliques of size sixteen. The following proposition gives us a first upper bound for the maximal number of exceptional curves going through one point.

Proposition 4.10. *The following hold.*

(i) The maximal size of a clique in $\mathcal{G}$ is sixteen.

(ii) Every clique of size sixteen is of the form

$$\{e_1, \ldots, e_8, c_1, \ldots, c_8 \mid \forall i \neq j : (\{e_i, c_i\}, \{e_j, c_j\}) \text{ is an intersecting pair}\},$$

and every clique that has no edges of weight two is contained in a clique of size sixteen.

(iii) For $y = (e_1, \ldots, e_4) \in Y$, there is a unique element in $\mathcal{S}$ containing $e_1, \ldots, e_4$. This gives rise to a map $s : Y \longrightarrow \mathcal{S}$, which is surjective.

Proof.

(i) Let K be a maximal clique in $\mathcal{G}$. We consider two cases. First assume that K contains an edge of weight two. Then K has size at most thirteen by Lemma 4.9.

Now assume that K contains no edges of weight two. From Corollary 4.8 it follows that K is of the form

$$K = \{e_1, \ldots, e_n, c_1, \ldots, c_n \mid \forall i \neq j : (\{e_i, c_i\}, \{e_j, c_j\}) \text{ is an intersecting pair}\}.$$

By Corollary 4.7 we can without loss of generality assume that K contains the four exceptional classes

$$e_1 = L - E_1 - E_2; \qquad e_2 = L - E_3 - E_4;$$

$$c_1 = 5L - E_1 - E_2 - 2\sum_{i=3}^{8} E_i; \quad c_2 = 5L - 2E_1 - 2E_2 - E_3 - E_4 - 2\sum_{i=5}^{8} E_i.$$

Let e_3 be a different element in K. Then $e_1 \cdot e_3 = e_2 \cdot e_3 = 1$, so by Proposition 3.27 we can take without loss of generality $e_3 = L - E_5 - E_6$. Then K also contains the unique exceptional curve intersecting e_3 with multiplicity three, which is $c_3 = 5L - 2\sum_{i=1}^{4} E_i - E_5 - E_6 - 2E_7 - 2E_8$.

Let e_4 be a different element in K. Then $e_4 \cdot e_1 = e_4 \cdot e_2 = e_4 \cdot e_3 = 1$, so by Proposition 3.27 there are 26 possibilities for e_4. They are

$$L - E_7 - E_8;$$
$$2L - E_i - E_j - E_k - E_7 - E_8 \text{ for } i \in \{1, 2\}, j \in \{3, 4\}, k \in \{5, 6\};$$
$$3L - 2E_i - \sum_{j \in \{1,\ldots,i-1,i+2,\ldots,8\}} E_j \text{ for } i \in \{1, 3, 5, 7\};$$
$$3L - 2E_i - \sum_{j \in \{1,\ldots,i-2,i+1,\ldots,8\}} E_j \text{ for } i \in \{2, 4, 6, 8\};$$
$$4L - 2E_i - 2E_j - 2E_k - \sum_{l \notin \{i,j,k\}} E_l \text{ for } i \in \{1, 2\}, j \in \{3, 4\}, k \in \{5, 6\};$$
$$5L - 2\sum_{i=1}^{6} E_i - E_7 - E_8.$$

The elements $e_4 = 3L - \sum_{i=1}^{6} E_i - 2E_7$ and $c_4 = 3L - \sum_{i=1}^{6} E_i - 2E_8$ intersect all other 24 elements with multiplicity one and satisfy $e_4 \cdot c_4 = 3$, so they are both in K. As we have seen in Proposition 3.27, all other 24 elements e have the property that (e_1, e_2, e_3, e) is in Y, so by Proposition 3.28, without loss of generality we can assume that $e_5 = L - E_7 - E_8$ is in K. Then K also contains $c_5 = 5L - 2\sum_{i=1}^{6} E_i - E_7 - E_8$, since $e_5 \cdot c_5 = 3$. Of the remaining 22 elements, the only elements intersecting e_5 with multiplicity one are

$$3L - 2E_1 - \sum_{i=3}^{8} E_i; \qquad 3L - E_1 - E_2 - 2E_3 - \sum_{i=5}^{8} E_i;$$

$$3L - 2E_2 - \sum_{i=3}^{8} E_i; \qquad 3L - \sum_{i=1}^{4} E_1 - 2E_5 - E_7 - E_8;$$

$$3L - E_1 - E_2 - 2E_3 - \sum_{i=5}^{8} E_i; \qquad 3L - \sum_{i=1}^{4} E_1 - 2E_6 - E_7 - E_8.$$

These intersect pairwise with multiplicity three or one, so they are all contained in K. This gives sixteen elements in K.

(ii)-(iii) These points follow from the construction of the clique in part (i). □

Corollary 4.11. *The number of exceptional curves on X that go through one point is at most sixteen.*

Proof. This follows directly from Proposition 4.10. □

Corollary 4.12. *G acts transitively on $\mathcal{S}$, and $|\mathcal{S}| = 2025$.*

Proof. By Proposition 4.10, there is a surjective map $s : Y \longrightarrow \mathcal{G}$. Therefore, G acts transitively on $\mathcal{S}$ by Lemma 3.20. Let K be a clique of size sixteen. From Proposition 4.10 (ii), it follows that there are $16 \cdot 14 \cdot 12$ tuples $(c_1, c_2, c_3) \in K^3$ with $c_1 \cdot c_2 = c_1 \cdot c_3 = c_2 \cdot c_3 = 1$. Moreover, for a fixed tuple $(e_1, e_2, e_3) \in K^3$ with $e_1 \cdot e_2 = e_1 \cdot e_3 = e_2 \cdot e_3 = 1$ there are ten exceptional classes in K intersecting e_1, e_2 and e_3 with multiplicity one. From Proposition 3.27 (iii) and the fact that all cliques of size sixteen are maximal, it follows that two of those ten, say d_1 and d_2, are such that $(e_1, e_2, e_3, d_i) \notin Y$. The other eight e are such that (e_1, e_2, e_3, e) is an element of Y. We conclude that $|s^{-1}(K)| = 16 \cdot 14 \cdot 12 \cdot 8 = 21504$. Since s is surjective, we have $|\mathcal{S}| = |s(Y)| = \frac{|Y|}{21504} = 2025$. □

4.1 Points on the ramification curve

By Proposition 2.12, a hyperplane section H of Q that intersects B with multiplicity two in three (possibly infinitely near) points and does not contain the vertex of Q pulls back under φ to the sum of two exceptional curves that intersect in three (possibly infinitely near) points. From now on, we call two exceptional curves or exceptional classes intersecting with multiplicity three a *pair*.

Proposition 4.14 will give an upper bound for the maximal number of exceptional curves going through one point on the ramification curve of φ. The proof uses the following well-known proposition.

Proposition 4.13. *(Hurwitz). Let $f : C \longrightarrow D$ be a finite separable morphism of complete, nonsingular curves over an algebraically closed field k. Let $n = deg\ f$. Then*

$$2g(C) - 2 = n \cdot (2g(D) - 2) + deg\ R,$$

where R is the ramification divisor of f. We have $deg\ R \geq \sum_{P \in C}(e_P - 1)$, with equality if f only has tame ramification.

Proof. See [Har77], Proposition IV.2.2 and Corollary IV.2.4. □

I got the idea for Proposition 4.14 from Niels Lubbes, who was so kind to share this with us. Later I found that part of it was also done in [TVAV09].

Proposition 4.14. *Assume that char $k \neq 2$. Then the number of exceptional curves that go through one point on the ramification curve of φ is at most twelve.*

Proof. Fix a point $p \in B$, and let M be the tangent line to B at p. The set of planes trough M is a pencil P of planes in $\mathbb{P}^3$, hence can be parametrized by $\mathbb{P}^1$. Let $\lambda : B \dashrightarrow \mathbb{P}^1$ be the rational map sending every point $x \notin M$ to the unique plane trough M containing x. Then since B is smooth, this extends to a morphism $\lambda : B \longrightarrow \mathbb{P}^1$. As is shown in Lemma 4.5 (1) in [TVAV09], the morphism λ is separable, and deg $\lambda = 4$. Therefore, by Proposition 4.13 we have

$$2g(B) - 2 = (\deg\,\lambda)(2g(\mathbb{P}^1) - 2) + \deg\,R,$$

where R is the ramification divisor of λ. We have $g(B) = 4$, so this gives

$$\sum_{x \in B} (e_x - 1) \leq \deg\,R = 6 - (4 \cdot -2) = 14.$$

Let H be a plane through p that is tritangent to B. Then H contains two points where λ ramifies with ramification degree 2, or one point where λ ramifies with ramification degree four, hence H contributes 2 or 3 to the degree of R. Therefore, there are at most 7 tritangent planes going through p, which is Lemma 4.5 (1) in [TVAV09]. Note that P contains exactly one plane H' containing the vertex of Q. The intersection of H' with Q is a double line, each component intersecting B with multiplicity three. Therefore, the morphism λ branches over H', hence we counted H' as one of the 7 tritangent planes through p. We conclude that there are at most 6 planes tritangent to B and not going through the vertex of Q. By the bijection in Remark 2.13, this gives an upper bound of twelve exceptional curves going through $\varphi^{-1}(p)$. □

We will later give a sharper upper bound for the number of exceptional curves through one point on B.

Remark 4.15. Let K be a set of exceptional curves all going through a point P on the ramification curve of φ. Let e be an exceptional curve in K. There is a unique exceptional curve c intersecting e with multiplicity three, and by Remark 2.13, their intersection $c \cap e$ consists exactly of those points in e that are on the ramification curve of φ. We conclude that P is also contained in c.

From the previous remark we conclude that a maximal clique that corresponds to a set of exceptional curves all going through one point P on the ramification curve of φ is a union of pairs. Therefore, by Proposition 4.10 and Lemma 4.6, such a maximal clique is contained in a clique of size sixteen.

Let S be a clique of size sixteen in $\mathcal{G}$, and G_S the stabilizer of S in G. Consider the sets

$$I = \{(e_1, e_2, e_3) \in S^3 \mid e_1 \cdot e_2 = e_1 \cdot e_3 = e_2 \cdot e_3 = 1\},$$

and

$$J = \{(e_1, e_2) \in S^2 \mid e_1 \cdot e_2 = 1\}.$$

Proposition 4.16. *The group G_S acts transitively on I.*

Proof. Since S consists of eight pairs, we have $|I| = 16 \cdot 14 \cdot 12 = 2688$. Fix an element $\iota = (e_1, e_2, e_3)$ in I. We want to show that the orbit $G_S\iota$ has size 2688. Let $G_{S,\iota}$ be the stabilizer of ι in G_S. We have $G_S\iota = [G_S : G_{S,\iota}]$, and

$$[G : G_{S,\iota}] = [G : G_S][G_S : G_{S,\iota}].$$

By Corollary 4.12 we have $[G : G_S] = GS = 2025$. Moreover, we have

$$[G : G_{S,\iota}] = [G : G_\iota][G_\iota : G_{\iota,S}].$$

By Proposition 3.27 we have $[G : G_\iota] = G\iota = 240 \cdot 126 \cdot 60 = 1814400$. We now compute $[G_\iota : G_{\iota,S}] = G_\iota S$. Since G_ι acts transitively on the 24 exceptional curves e such that (e_1, e_2, e_3, e) is an element in Y, and every element in y is contained in a unique clique of size sixteen, the orbit $G_\iota S$ contains all different cliques that are the images under s of these 24 elements in Y, where s is the map in Proposition 4.10. Now fix e such that $y = (e_1, e_2, e_3, e)$ is in Y, and let $K = s(y)$. By Proposition 3.27, the clique K contains the two exceptional classes d_1, d_2 such that (e_1, e_2, e_3, d_1) and (e_1, e_2, e_3, d_2) are in $W \setminus Y$, and we have $d_1 \cdot d_2 = 3$. By Proposition 4.10 (ii), we know that K also contains the unique $c_1, c_2, c_3 \in E$ such that for $i \in \{1, 2, 3\}$ we have $e_i \cdot c_i = 3$. We conclude that the seven other elements in K are among the 24 exceptional classes f such that (e_1, e_2, e_3, f) is an element in Y. Therefore, they determine the same unique clique of size sixteen as e. We conclude that there are $\frac{24}{8} = 3$ different cliques containing ι. So we have $|G_\iota S| \geq 3$, and we conclude that

$$[G : G_{S,\iota}] \geq 240 \cdot 126 \cdot 60 \cdot 3 = 5443200$$

It follows that $[G_S : G_{S,\iota}] \geq \frac{5443200}{2025} = 2688$. Since $[G_S : G_{S,\iota}] = |G_S\iota| \leq 2688$, this finishes the proof. $\square$

Corollary 4.17. *The group G_S acts transitively on J.*

Proof. We have a projection map $\lambda : I \longrightarrow J$ on the first two coordinates. Since S consists of eight pairs, if we fix two elements e_1, e_2 such that $(e_1, e_2) \in J$, there are $16 - 4 = 12$ elements $e \in S$ such that $(e_1, e_2, e) \in I$. Therefore, λ is surjective. From Proposition 4.16 it follows that G_S acts transitively on J. $\square$

Corollary 4.18. *The group G_S acts transitively on S.*

Proof. We have a projection map $\lambda : J \longrightarrow S$ on the first coordinate. For every element e in S there are 14 elements c such that $(e, c) \in J$, so λ is surjective. From Corollary 4.17 it follows that G_S acts transitively on S. $\square$

The following proposition allows us to say something about all cliques of a certain type by considering only one of them, which is very useful.

Proposition 4.19. *For $n \in \{2, 3, 5, 6, 7, 8\}$, G acts transitively on the set*

$$D_n = \left\{ \{e_1, \ldots, e_n, c_1, \ldots, c_n\} \;\middle|\; \begin{array}{c} \forall i : e_i, c_i \in E; \\ \forall i \neq j : (\{e_i, c_i\}, \{e_j, c_j\}) \text{ is an intersecting pair} \end{array} \right\}.$$

Proof. The case $n = 2$ follows from Corollary 4.7, and $n = 3$ follows from Proposition 3.27(i) and Lemma 4.6. By Proposition 4.16 and Lemma 4.6, the stabilizer G_S of S in G acts transitively on the set

$$\{(e_1, e_2, e_3, c_1, c_2, c_3) \in S^6 \mid \forall i \neq j : (\{e_i, c_i\}, \{e_j, c_j\}) \text{ is an intersecting pair.}\}$$

Since S consists of eight pairs, the cliques of five pairs in S are the complements of the cliques of three pairs in S, so this implies that G_S acts transitively on the set of cliques of five pairs in S. By Corollary 4.12 this implies the statement for $n = 5$. The cases $n = 6$ and $n = 7$ are proved analogously since we showed that G_S acts transitively on J and on S. Finally, $n = 8$ is Corollary 4.12. □

Proposition 4.20. *There are two orbits under the action of G on the set*

$$\{(e_1, \ldots, e_4, c_1, \ldots, c_4) \in E^8 \mid \forall i \neq j : (\{e_i, c_i\}, \{e_j, c_j\}) \text{ is an intersecting pair.}\}$$

Proof. This follows from Proposition 3.28 and Lemma 4.6. □

From Proposition 4.22 we will deduce a sharp upper bound for the number of exceptional curves going through one point on the ramification curve of φ if char $k \neq 2$. We need a lemma first. This lemma will also be used in the next subsection.

Let $\mathbb{P}^2$ be the projective plane over k with coordinates x, y, z. Let $R_1, \ldots, R_9$ be nine points in $\mathbb{P}^2$, with $R_i = (x_i : y_i : z_i)$ for $i \in \{1, \ldots, 9\}$. We define the following lists of polynomials in x, y, z.

$$\begin{aligned}
\mathrm{Mon}_1 &= [x, y, z];\\
\mathrm{Mon}_2 &= [x^2, xy, xz, y^2, yz, z^2];\\
\mathrm{Mon}_3 &= [x^3, x^2y, x^2z, xy^2, xyz, xz^2, y^3, y^2z, yz^2, z^3];\\
\mathrm{Mon}_4 &= [x^4, x^3y, x^3z, x^2y^2, x^2yz, x^2z^2, xy^3, xy^2z, xyz^2, xz^3, y^4, y^3z, y^2z^2, yz^3, z^4].
\end{aligned}$$

For $i = 3, 4$, let Mon_i^1 be the list of derivatives of Mon_i with respect to x, let Mon_i^2 be the list of derivatives of Mon_i with respect to y, and let Mon_i^3 be the list of derivatives of Mon_i with respect to z. Define the following matrices, where each row is defined up to scaling.

$$\begin{aligned}
&M = (a_{i,j})_{i,j\in\{1,2,3\}} && \text{with } a_{i,j} = \mathrm{Mon}_1[j](R_i);\\
&N = (b_{i,j})_{i,j\in\{1,\ldots,6\}} && \text{with } b_{i,j} = \mathrm{Mon}_2[j](R_i);\\
&L = (c_{i,j})_{i,j\in\{1,\ldots,10\}} && \text{with } c_{i,j} = \begin{cases} \mathrm{Mon}_3[j](R_i) & \text{for } i \leq 8\\ \mathrm{Mon}_3^1[j](R_8) & \text{for } i = 9\\ \mathrm{Mon}_3^3[j](R_8) & \text{for } i = 10 \end{cases};\\
&H = (d_{i,j})_{i,j\in\{1,\ldots,15\}} && \text{with } d_{i,j} = \begin{cases} \mathrm{Mon}_4[j](R_i) & \text{for } i \leq 6\\ \mathrm{Mon}_4^{i-6}[j](R_7) & \text{for } i \in \{7,8,9\}\\ \mathrm{Mon}_4^{i-9}[j](R_8) & \text{for } i \in \{10,11,12\}\\ \mathrm{Mon}_4^{i-12}[j](R_9) & \text{for } i \in \{13,14,15\} \end{cases}.
\end{aligned}$$

Lemma 4.21. *The following hold.*

(i) The points R_1, R_2, and R_3 are collinear if and only if det$(M) = 0$.

(ii) The points $R_1, \ldots, R_6$ are on a conic if and only if det$(N) = 0$.

(iii) If $y_8 \neq 0$, then the points $R_1, \ldots, R_8$ are on a cubic with a singular point at R_8 if and only if det$(L) = 0$.

(iv) If char $k = 0$, then the points $R_1, \ldots, R_9$ are on a quartic that is singular at R_7, R_8 and R_9 if and only if det$(H) = 0$.

Proof.

(i) The determinant of M is zero if and only if there is a non-zero element in the nullspace of M, that is, there is a non-zero vector (m_1, m_2, m_3) such that for all $i \in \{1, 2, 3\}$, we have $m_1 a_{i,1} + m_2 a_{i,2} + m_3 a_{i,3} = 0$. But this vector exists if and only if the line defined by $m_1 x + m_2 y + m_3 z$ contains all three points.

(ii) This proof goes analogously to the proof of (i).

(iii) The determinant of L is zero if and only if there is a non-zero vector $(l_1, \ldots, l_{10})$ such that for all $i \in \{1, \ldots, 10\}$, we have $l_1 c_{i,1} + \cdots + l_{10} c_{i,10} = 0$. This is the case if and only if the cubic C defined by $\lambda : \sum_{i=1}^{10} l_i \mathrm{Mon}_3[i]$ contains all eight points, and moreover, the derivatives λ_x, λ_z of λ with respect to x and z vanish in R_8. Since we have $x\lambda_x + y\lambda_y + z\lambda_z = 3\lambda$ and $y_8 \neq 0$, this implies that also the derivative λ_y of λ with respect to y vanishes in R_8, hence C is singular in R_8.

(iv) The determinant of H is zero if and only if there is a non-zero vector $(h_1, \ldots, h_{15})$ such that for all $i \in \{1, \ldots, 15\}$, we have $h_1 d_{i,1} + \cdots + h_{15} d_{i,15} = 0$. This is the case if and only if the quartic K defined by $\lambda : \sum_{i=1}^{15} h_i \mathrm{Mon}_4[i]$ contains $R_1, \ldots, R_6$, and moreover, the derivatives $\lambda_x, \lambda_y, \lambda_z$ of λ with respect to x, y, and z vanish in R_7, R_8 and R_9. Since we have $x\lambda_x + y\lambda_y + z\lambda_z = 4\lambda$ and char $k = 0$, this implies that also R_7, R_8, and R_9 are in contained in λ.

□

Proposition 4.22. *Assume that char $k \neq 2$. Let $Q_1, \ldots, Q_8$ be eight points in $\mathbb{P}^2$ in general position. Let L_i be the line through Q_{2i} and Q_{2i-1} for $i \in \{1, 2, 3, 4\}$, and $C_{i,j}$ the unique cubic through $Q_1, \ldots, Q_{i-1}, Q_{i+1}, \ldots, Q_8$ that is singular in Q_j. Assume that the four lines L_1, L_2, L_3 and L_4 all intersect in one point P. Then the three cubics $C_{7,8}$, $C_{8,7}$, and $C_{6,5}$ do not all go through P.*

Proof. Assume that $C_{7,8}$, $C_{8,7}$, and $C_{6,5}$ go through P. First note that if P were equal to one of the Q_i, then three of the eight Q_i would be on a line, which would contradict the fact that $Q_1, \ldots, Q_8$ are in general position. We conclude that P is not equal to one of the Q_i.
Let $(x : y : z)$ be the coordinates in $\mathbb{P}^2$. Without loss of generality we can choose four points in general position in $\mathbb{P}^2$, and we set

$$Q_1 = (0 : 1 : 1); \qquad Q_3 = (1 : 0 : 1)$$
$$Q_5 = (1 : 1 : 1); \qquad P = (0 : 0 : 1).$$

Then we have the following.

$$\begin{aligned} &L_1 \text{ is the line given by } x = 0; \\ &L_2 \text{ is the line given by } y = 0; \\ &L_3 \text{ is the line given by } x = y. \end{aligned}$$

Since L_4 contains P, and is unequal to L_1 and L_2, there is an $m \in k^*$ such that L_4 is the line $my = x$. Since Q_2, Q_7 and Q_8 are not in L_2, and Q_4 is not in L_1, there are $a, b, c, u, v \in k$ such that

$$\begin{aligned} Q_2 &= (0 : 1 : a); & Q_7 &= (m : 1 : v); \\ Q_4 &= (1 : 0 : b); & Q_8 &= (m : 1 : c). \\ Q_6 &= (1 : 1 : u); \end{aligned}$$

We define $\mathbb{A}^6$ to be the affine space with coordinate ring $T_6 = k[a, b, c, m, u, v]$. Points in $\mathbb{A}^6$ correspond to configurations of the points $Q_1, \ldots, Q_8$. The fact that $C_{6,5}$, $C_{7,8}$ and $C_{8,7}$ go through P gives polynomial equations in these six variables, hence defines an algebraic set A_0 in $\mathbb{A}^6$. We define S_0 to be the algebraic set of all points in $\mathbb{A}^6$ that correspond to the configurations where three of the points $Q_1, \ldots, Q_8$ lie on a line, or six of the points lie on a conic. We want to show that A_0 is contained in S_0.

Claim 4.22.1: There is a unique cubic through $Q_1, Q_2, Q_3, Q_4, Q_5, Q_8$ and P that is singular in Q_5.
Proof: The vector space spanned by all monomials in x, y, z of degree three has dimension ten, so cubics in $\mathbb{P}^2$ correspond to points in $\mathbb{P}^9$. Requiring that a point lies on a cubic defines a hyperplane in $\mathbb{P}^9$. Requiring that a cubic contains and is singular in a point gives three linear conditions. We conclude that all cubics through these seven points with a singularity at one of them are in the intersection of 9 hyperplanes of $\mathbb{P}^9$, which gives at least one point, so at least one cubic. Assume that this cubic is not unique. Then there are two linearly independent cubics D_1 and D_2 that go through these seven points with a singularity at Q_5. Let l_i be the tangent line to D_i at P for $i = 1, 2$.
If the equations defining l_1 and l_2 are not linearly independent, then there is a linear combination F of D_1 and D_2 that is singular in P. But then the line L_3 through P, Q_5 and Q_6 intersects F in at least four points counted with multiplicity, which implies that F has L_3 as a component, hence is reducible. If F splits in three lines, then, since the five points Q_1, Q_2, Q_3, Q_4 and Q_8 are not on L_3, they are contained in the other two lines. But then there would be at least three points on a line, contradicting the fact that the points are in general position. On the other hand, if F splits in a line and a smooth conic C, then, since Q_5 is a singular point of F, it is in the intersection of L_3 and C. The five points Q_1, Q_2, Q_3, Q_4 and Q_8 are not contained in L_3, hence they are in C, too. But then there are six points in general position on the smooth conic C, which gives a contradiction. We conclude that l_1 and l_2 must be linearly independent.
Since the equations defining l_1 and l_2 are linearly independent, the two lines span the whole plane, so there is a linear combination G of D_1 and D_2 such that L_1 is the tangent line to G at P. But then L_1 intersects G in four points counted

with multiplicity, so it is contained in G. Therefore G is reducible. If G splits in three lines, then, since the points Q_3, Q_4, Q_5, Q_8 are not contained in L_1, each of the other two lines contains two of these five points. But since Q_5 is a singular point of G, it is contained in the intersection of two lines, so there is a line that contains three of the Q_i. On the other hand, if G splits in L_1 and a smooth conic, then, since Q_5 is a singular point of G, it lies in the intersection of the conic and L_1, hence L_1 contains Q_1, Q_2 and Q_5. In both cases, the points $Q_1, \dots, Q_8$ are not in general position, leading to a contradiction. We conclude that there is a unique cubic through $Q_1, Q_2, Q_3, Q_4, Q_5, Q_8$ and P that is singular in Q_5. (□)

Let D be the unique cubic through $Q_1, Q_2, Q_3, Q_4, Q_5, Q_8$ and P that is singular in Q_5. By uniqueness, it must be equal to $C_{6,5}$. Note that D intersects L_4 in Q_8 and in P. If L_4 were contained in D, then by the same reasoning as used in Claim 4.22.1, there would be either three of the Q_i on L_4, or six of the Q_i on a smooth conic, which is not possible. Therefore L_4 is not contained in D, so Q_7 is the third point of intersection of L_4 with D. By Lemma 4.21, the equation expressing that Q_7 is in D is given by $\det(L) = 0$, where L is the matrix associated to $(R_1, \dots, R_8) = (Q_1, Q_2, Q_3, Q_4, Q_7, Q_8, P, Q_5)$. We have

$$\det(L) = m(m-1)(c-v)(b-1)(a-1)f,$$

where

$$f = (a - ac - bc + bm)v + b(a-1)m^2 + b(c-2a)m + a(b+c-1).$$

The first five factors of $\det(L)$ define subsets of S_0, hence do not correspond to configurations where $Q_1, \dots, Q_8$ are in general position. Therefore, $C_{6,5}$ goes through P if and only if $f = 0$. Define $U = Z(a - ac - bc + bm)$.

Claim 4.22.2: $U \cap A_0$ is contained in S_0.
Proof: Let $(a_0, b_0, c_0, m_0, u_0, v_0) \in A_0$ be such that $a_0 - a_0c_0 - b_0c_0 + b_0m_0 = 0$. Then, since $f(a_0, b_0, c_0, m_0, u_0, v_0) = 0$, we have also

$$b_0(a_0 - 1)m_0^2 + b_0(c_0 - 2a_0)m_0 + a_0(b_0 + c_0 - 1) = 0.$$

But then $f(a, b, c, m, u, v) = 0$ for every v, so the whole line L_4 is contained in D. As we have seen before, this implies that the points $Q_1, \dots, Q_8$ are not in general position. (□)

Analogously, the fact that $C_{7,8}$ goes through P is expressed by $\det(L')$, where L' is the matrix L associated to $(R_1, \dots, R_8) = (Q_1, Q_2, Q_3, Q_4, Q_5, Q_6, P, Q_8)$ in Lemma 4.21. We have

$$\det(L') = m(u-1)(m-1)(b-1)(a-1)g,$$

where $g = \beta u + \gamma$ with

$$\beta = bm^3 + (1 - bc - c)m^2 + (c^2 - 2c + 1)m + a(1-c) + c^2 - c,$$

and

$$\gamma = -abm^3 + (abc + ab + ac - a + b - 2bc)m^2 + (ab - 2abc + a + 2bc^2 - b - ac^2 + 2c^2 - 2c)m \\ + a(bc - b + 2c^2 - 2c) - bc^2 + bc - 2c^3 + 2c^2.$$

The first five factors of $\det(L')$ correspond to configurations where the eight points are not in general position, so $C_{7,8}$ contains P if and only if $g = 0$. Define $V = Z(\beta)$. By the same reasoning as in Claim 4.22.2, we have $V \cap A_0 \subseteq S_0$.

Set
$$v' = \frac{-b(a-1)m^2 + b(c-2a)m + a(b+c-1)}{a - ac - bc + bm} \quad \text{and} \quad u' = \frac{-\gamma}{\beta}.$$

Define $\mathbb{A}^4$ to be the affine space with coordinate ring $T_4 = k[m, a, b, c]$, and let $K_4 = \mathrm{Frac}(T_4)$ be the field of rational fractions of elements in T_4. Consider the ring homomorphism $T_6 \longrightarrow K_4$ defined by

$$(m, a, b, c, u, v) \longmapsto (m, a, b, c, u', v').$$

This defines an injective rational map $i : \mathbb{A}^4 \dashrightarrow \mathbb{A}^6$, which is a section of the projection $\mathbb{A}^6 \longrightarrow \mathbb{A}^4$ on the first four coordinates. Let $A_0' = A_0 \setminus ((A_0 \cap U) \cup (A_0 \cap V))$. Showing that $A_0 \subseteq S_0$ is equivalent to showing that $A_0' \subseteq S_0$. Note that, since i is defined outside the subvarieties of $\mathbb{A}^4$ defined by $a - ac - bc + bm$ and β, we have $i^{-1}(A_0') \cong A_0'$. Let $A_1 = \overline{i^{-1}(A_0')}$ and $S_1 = i^{-1}(S_0)$, then $A_0' \subseteq S_0$ is equivalent to $A_1 \subseteq S_1$.

Let L'' be the matrix L associated to $(R_1, \ldots, R_8) = (Q_1, Q_2, Q_3, Q_4, Q_5, Q_6, P, Q_7)$ in Lemma 4.21. Similarly to $C_{7,8}$, the fact that $C_{8,7}$ contains P is expressed by $\det(L'')$. We have

$$\det(L'') = -2abm(m-1)^2(b-1)(a-1)(a+b-1) f_1 \cdot f_2 \cdot f_3,$$

with
$$f_1 = ac - a + bcm - bm^2 - c^2 + cm + c - m,$$

$$f_2 = abm^2 - 2abm + ab - ac^2 + 2ac - a - bc^2 + 2bcm - bm^2,$$

and

$$\begin{aligned} f_3 = abcm^2 - 2abcm + abc - abm^3 + abm^2 + abm - ab - ac^2m + 2ac^2 \\ + acm^2 - 3ac - am^2 + am + a + 2bc^2m - bc^2 - 3bcm^2 + bc + bm^3 \\ + bm^2 - bm - 2c^3 + 3c^2m + 3c^2 - cm^2 - 4cm - c + m^2 + m. \end{aligned}$$

Since $\operatorname{char} k \neq 2$, the determinant of L'' equals zero if and only if at least one of the non-constant factors equals zero. We can show that all non-constant factors of $\det(L'')$ define subvarieties of S_1. If $a = 0$, then Q_2, Q_3 and Q_5 are contained in the line $x - z = 0$. Similarly, $b = 0$ implies that Q_1, Q_4 and Q_5 are on the line $y - z = 0$, and $a + b - 1 = 0$ implies that Q_2, Q_4, and Q_5 are on the line $x(a-1) - ay + z = 0$. If $m = 0$ then $L_4 = L_2$, and $m = 1$ implies $L_4 = L_3$, so in both cases there are four points on a line. If $a = 1$ or $b = 1$, then two points would be the same. Let $(R_1, \ldots, R_6) = (Q_3, \ldots, Q_8)$, and let N be the corresponding matrix from Lemma 4.21. We compute the determinant of N and find that $f_1 f_2 f_3$ divides $\det(N)$. This means that f_1, f_2, as well as f_3 define subsets of S_1. We conclude that all irreducible components of A_1 are contained in S_1, which finishes the proof. □

We can now prove Theorem 1.

Proof of Theorem 1. First note that by Corollary 4.11, the number of exceptional curves through any point in X is at most sixteen in all characteristics.
Now assume char $k \neq 2$. Consider the clique $K = \{e_1, \ldots, e_6, c_1, \ldots, c_6\}$, where

$$\begin{aligned}
e_1 &= L - E_1 - E_2; \\
e_2 &= L - E_3 - E_4; \\
e_3 &= L - E_5 - E_6; \\
e_4 &= L - E_7 - E_8; \\
e_5 &= 3L - E_1 - E_2 - E_3 - E_4 - E_5 - E_6 - 2E_8; \\
e_6 &= 3L - E_1 - E_2 - E_3 - E_4 - 2E_5 - E_7 - E_8,
\end{aligned}$$

and c_i is the unique class in E such that $e_1 \cdot c_i = 3$, for all $i \in \{1, \ldots, 6\}$. By Remark 2.9, the classes $e_1, \ldots, e_6, c_5$ are the strict transforms of the four lines through P_i and P_{i+1} for $i \in \{1, 3, 5, 7\}$, and the unique cubics through $P_1, \ldots, P_6$ and P_8 respectively $P_1, \ldots, P_5$, P_7 and P_8 respectively $P_1, \ldots, P_6$ and P_7, that are singular in P_8 respectively P_5 respectively P_7.
Now let K' be a clique in $\mathcal{G}$ consisting of at least six pairs, and let

$$\{(f_1, d_1), \ldots, (f_6, d_6)\}$$

be a set of six pairs in K'. Since G acts transitively on the set of cliques of six pairs in E by Proposition 4.19, after changing the indexes and interchanging f_i's and d_j's if necessary, there is an element $g \in G$ such that $f_i = g(e_i)$ and $d_i = g(c_i)$ for $i \in \{1, \ldots, 6\}$. Let $E_i' = g(E_i)$. Then, since the E_i' are pairwise disjoint, by Lemma 3.9 we can blow down $E_1', \ldots, E_8'$ to points $Q_1, \ldots, Q_8 \in \mathbb{P}^2$ that are in general position, such that X is the blow-up of $\mathbb{P}^2$ at $Q_1, \ldots, Q_8$, and E_i' is the class in Pic X of the exceptional curve above Q_i for all i. By the bijection in Remark 3.12, the element f_i is the class of the strict transform of the line through Q_{2i-1} and Q_i for $i \in \{1, \ldots, 4\}$, the elements f_5 and f_6 are the classes of the strict transforms of the unique cubics through $Q_1, \ldots, Q_6$ and Q_8 respectively $Q_1, \ldots, Q_5$, Q_7 and Q_8, that are singular in Q_8 respectively Q_5, and d_i is the unique class in E intersecting f_i with multiplicity three for all i. From Proposition 4.22 it follows that the curves corresponding to $f_1, \ldots, f_6, d_5$ and d_6 can not all go through one point.
We conclude that the curves corresponding to the classes in a clique containing at least six pairs can not go through one point. Since any maximal number of exceptional curves going through one point on the ramification curve forms a clique consisting of only pairs, hence of even size, we conclude that this number is at most ten. □

The following examples show that the upper bounds in Theorem 1 can be reached.

Example 4.23. Define the following eight points in $\mathbb{P}^2_{\mathbb{Q}}$.

$$
\begin{aligned}
Q_1 &= (0:1:1); & Q_5 &= (1:1:1);\\
Q_2 &= (0:5:3); & Q_6 &= (4:4:5);\\
Q_3 &= (1:0:1); & Q_7 &= (-2:2:1);\\
Q_4 &= (-1:0:1); & Q_8 &= (2:-2:1).
\end{aligned}
$$

They are in general position, as can be checked by asserting that the determinants of the appropriate matrices in Lemma 4.21 are all nonzero. Therefore, the blow-up of $\mathbb{P}^2$ in $(Q_1, \dots, Q_8)$ is a del Pezzo surface S. We have the following four lines in $\mathbb{P}^2$.

$$
\begin{aligned}
&\text{The line } L_1 \text{ through } Q_1 \text{ and } Q_2, && \text{which is given by } x = 0;\\
&\text{the line } L_2 \text{ through } Q_3 \text{ and } Q_4, && \text{which is given by } y = 0;\\
&\text{the line } L_3 \text{ through } Q_5 \text{ and } Q_6, && \text{which is given by } x = y;\\
&\text{the line } L_4 \text{ through } Q_7 \text{ and } Q_8, && \text{which is given by } x = -y.
\end{aligned}
$$

On S, we define the four exceptional curves $e_1, \dots, e_4$ to be the strict transforms of $L_1, \dots, L_4$. Let $C_{7,8}$ be the unique cubic through $Q_1, \dots, Q_6, Q_8$ that is singular in Q_8. Let $(R_1, \dots, R_8) = (Q_1, Q_2, Q_3, Q_4, Q_5, Q_6, Q_7, Q_8)$, and let L be the corresponding matrix from Lemma 4.21. Then the equation defining $C_{7,8}$ is the determinant of L', where L' is equal to L after replacing the first row by Mon_3. We compute this determinant and find

$$C_{7,8} : x^3 - \tfrac{3}{4}x^2y - \tfrac{31}{12}xy^2 + \tfrac{10}{3}xyz - xz^2 - y^3 + \tfrac{8}{3}y^2z - \tfrac{5}{3}yz^2 = 0.$$

We define the singular cubic $C_{8,7}$ analogously and compute its defining equation, which is

$$C_{8,7} : x^3 + \tfrac{13}{4}x^2y + \tfrac{43}{4}xy^2 - 14xyz - xz^2 + 15y^3 - 40y^2z + 25yz^2 = 0.$$

Let the exceptional curves e_5, c_5 on S be the strict transforms of $C_{7,8}$ and $C_{8,7}$, respectively. Since $L_1, \dots, L_4, C_{7,8}, C_{8,7}$ all go through the point $(0:0:1)$, the six exceptional curves $e_1, \dots, e_5, c_5$ go through one point P in S. Moreover, since $e_5 \cdot c_5 = 3$, this point P lies on the ramification curve of φ. Therefore, by Remark 4.15, the unique exceptional curves $c_1, \dots, c_4$ such that $e_i \cdot c_i = 3$ for $i \in \{1, \dots, 4\}$ go through P, too. We conclude that the ten exceptional curves $e_1, \dots, e_5, c_1, \dots, c_5$ all go through P.

Example 4.24. Let $f = x^5 + x^2 + 1 \in \mathbb{F}_2[x]$, and let $F \cong \mathbb{F}_2[x]/(f)$ be the finite field of 32 elements defined by adjoining a root α of f to F_2. Define the following eight points in $\mathbb{P}^2_F$.

$$
\begin{aligned}
Q_1 &= (0:1:1); & Q_5 &= (1:1:1);\\
Q_2 &= (0:1:\alpha^{19}); & Q_6 &= (\alpha^{20}:\alpha^{20}:\alpha^{16});\\
Q_3 &= (1:0:1); & Q_7 &= (\alpha^{24}:\alpha^{25}:1);\\
Q_4 &= (1:0:\alpha^5); & Q_8 &= (\alpha^{30}:1:\alpha^5).
\end{aligned}
$$

Again, we can easily check that the determinants of the appropriate matrices in Lemma 4.21 are all nonzero, such that these points are in general position. Therefore, the blow-up of $\mathbb{P}^2$ in $(Q_1,\dots,Q_8)$ is a del Pezzo surface S. We have the following four lines in $\mathbb{P}^2$.

$$
\begin{aligned}
&\text{The line } L_1 \text{ through } Q_1 \text{ and } Q_2, \text{ which is given by } x=0;\\
&\text{the line } L_2 \text{ through } Q_3 \text{ and } Q_4, \text{ which is given by } y=0;\\
&\text{the line } L_3 \text{ through } Q_5 \text{ and } Q_6, \text{ which is given by } x=y;\\
&\text{the line } L_4 \text{ through } Q_7 \text{ and } Q_8, \text{ which is given by } x=\alpha^{30}y.
\end{aligned}
$$

Let $C_{i,j}$ the unique cubic through $Q_1,\dots,Q_{i-1},Q_{i+1},\dots,Q_8$ that is singular in Q_j. We compute the defining equations of $C_{1,2},\ C_{3,4},\ C_{5,6},\ C_{7,8}$ and $C_{8,7}$ and obtain

$$
\begin{aligned}
C_{1,2} :& x^3+\alpha^{24}x^2y+\alpha^{28}x^2z+\alpha^{30}xy^2+\alpha^9xyz+\alpha^{26}xz^2+\alpha^{13}y^3+\alpha^6yz^2=0;\\
C_{3,4} :& x^3+\alpha^{12}x^2y+\alpha^4xy^2+\alpha^{11}xyz+\alpha^{21}xz^2+y^3+\alpha^{23}y^2z+\alpha^{12}yz^2=0;\\
C_{5,6} :& x^3+\alpha^4x^2y+\alpha^{28}x^2z+\alpha^{25}xy^2+\alpha^{20}xyz+\alpha^{26}xz^2+\alpha^{17}y^3+\alpha^9y^2z+\alpha^{29}yz^2=0;\\
C_{7,8} :& x^3+\alpha x^2y+\alpha^{28}x^2z+\alpha^{17}xy^2+\alpha^{10}xyz+\alpha^{26}xz^2+\alpha^{16}y^3+\alpha^8y^2z+\alpha^{28}yz^2=0;\\
C_{8,7} :& x^3+\alpha^{26}x^2y+\alpha^{28}x^2z+\alpha^{19}xy^2+\alpha^{10}xyz+\alpha^{26}xz^2+\alpha^{16}y^3+\alpha^8y^2z+\alpha^{28}yz^2=0.
\end{aligned}
$$

Let the exceptional curves $e_1,\dots,e_8$ be the strict transforms of the curves

$$L_1,\dots,L_4,C_{1,2},C_{3,4},C_{5,6},C_{7,8},$$

and let c_8 be the strict transform of $C_{8,7}$. Since $L_1,\dots,L_4,C_{1,2},C_{3,4},C_{5,6},C_{7,8},C_{8,7}$ all go through the point $(0:0:1)$, the exceptional curves $e_1,\dots,e_8,c_8$ all go through one point P on S. Moreover, since $e_8\cdot c_8=3$, this point P lies on the ramification curve of φ. Therefore, by Remark 4.15, the unique exceptional curves $c_1,\dots,c_7$ such that $e_i\cdot c_i=3$ for $i\in\{1,\dots,7\}$ go through P, too. We conclude that the sixteen exceptional curves $e_1,\dots,e_8,c_1,\dots,c_8$ all go through P.

4.2 Points outside the ramification curve

In this subsection we prove Theorem 2.

Lemma 4.25. *For $e_1,\ e_2\in E$ with $e_1\cdot e_2=1$, there are 138 elements e in E such that $e\cdot e_1\geq 1$ and $e\cdot e_2\geq 1$.*

Proof. From Proposition 3.22 it follows that it is enough to show this for $e_1 = E_1$ and $e_2 = L - E_1 - E_2$. Let $e = aL - \sum_{i=1}^{8} E_i b_i \in E$. Then the conditions $e \cdot e_1 \geq 1$, $e \cdot e_2 \geq 1$ are equivalent to $b_1 \geq 1$ and $a - b_1 - b_2 \geq 1$. By looking at the table in Proposition 2.7 we find all possibilities.

a	2	3	4	5	6
number of possibilities for e	15	37	50	28	8

We find a total of 138 possibilities for e such that e intersects both e_1 and e_2. □

Lemma 4.26. *The maximal size of a clique in $\mathcal{G}$ without any pairs is twelve. Moreover, for $e_1, e_2 \in E$ with $e_1 \cdot e_2 = 1$, there are 640 cliques of size twelve without pairs that contain e_1 and e_2.*

Proof. As

$$\{L - E_1 - E_2,\ L - E_3 - E_4,\ L - E_5 - E_6,\ L - E_7 - E_8\}$$

is a clique of size four without pairs, the maximal size of a clique in $\mathcal{G}$ without pairs is bigger than three. So by Proposition 4.5, such a clique contains two exceptional classes intersecting with multiplicity one. Let K be a maximal clique without pairs. By Proposition 3.22 we can assume that K contains $e_1 = E_1$ and $e_2 = L - E_1 - E_2$. By Lemma 4.25, there are 138 exceptional classes that intersect both e_1 and e_2 positively. But since K contains no pairs, the two exceptional classes that intersect e_1 or e_2 with multiplicity three are not in K. This leaves us with 136 possibilities for elements in K. Since it is too tedious to compute all pairwise intersection multiplicities by hand, we compute with MAGMA the maximal size of a clique in the graph on these 136 exceptional curves that does not contain any edges of weight three. This maximum is ten, hence K has size twelve, and there are 640 such cliques. □

Lemma 4.26 gives an upper bound for the number of exceptional curves going through one point outside the ramification curve of φ, which is twelve. We will compute a sharp upper bound.

Let T be the clique consisting of the following twelve elements.

$$t_1 = L - E_1 - E_2; \qquad t_7 = 4L - 2\sum_{i\in\{1,6,8\}} E_i - \sum_{j\in\{2,3,4,5,7\}} E_j;$$

$$t_2 = L - E_3 - E_4; \qquad t_8 = 4L - 2\sum_{i\in\{2,3,8\}} E_i - \sum_{j\in\{1,4,5,6,7\}} E_j;$$

$$t_3 = L - E_5 - E_6; \qquad t_9 = 4L - 2\sum_{i\in\{3,6,7\}} E_i - \sum_{j\in\{1,2,4,5,8\}} E_j;$$

$$t_4 = L - E_7 - E_8; \qquad t_{10} = 4L - 2\sum_{i\in\{4,5,8\}} E_i - \sum_{j\in\{1,2,3,6,7\}} E_j;$$

$$t_5 = 4L - 2\sum_{i\in\{1,3,5\}} E_i - \sum_{j\in\{2,4,6,7,8\}} E_j; \qquad t_{11} = 4L - 2\sum_{i\in\{2,4,6\}} E_i - \sum_{j\in\{1,3,5,7,8\}} E_j;$$

$$t_6 = 4L - 2\sum_{i\in\{1,4,7\}} E_i - \sum_{j\in\{2,3,5,6,8\}} E_j; \qquad t_{12} = 4L - 2\sum_{i\in\{2,5,7\}} E_i - \sum_{j\in\{1,3,4,6,8\}} E_j.$$

Let G_T be the stabilizer of T in G.

Proposition 4.27. *Let $\mathcal{T}$ be the set of all cliques in $\mathcal{G}$ of size twelve that do not contain any pair. The following hold.*

(i) The group G acts transitively on $\mathcal{T}$;

(ii) we have $T^4 \cap W = T^4 \cap Y$, and G_T acts transitively on $T^4 \cap Y$;

(iii) we have $|\mathcal{T}| = 179200$.

Proof.

(i) Consider the two sets

$$A = \{\{e_1, e_2\} \mid e_1, e_2 \in E;\ e_1 \cdot e_2 = 1\},$$

and

$$C = \{(a, K) \in A \times \mathcal{T} \mid a \subset K\}.$$

We have $|A| = \frac{240 \cdot 126}{2} = 15120$. To compute the cardinality of C, note that by Lemma 4.26, for every element $a \in A$ there are 640 elements $K \in \mathcal{T}$ such that $(a, K) \in C$. We conclude that $|C| = |A| \cdot 640 = 9676800$. We will show that G acts transitively on C. Define the set

$$D = \{((e_1, e_2, e_3, e_4), K) \in Y \times \mathcal{T} \mid e_1, \ldots, e_4 \in K\}.$$

It is an easy check that the clique T is an element in $\mathcal{T}$. Let $F = G(y, T)$ be the orbit of (y, T) under the action of G on D. We have a map

$$\lambda : F \longrightarrow C,\ ((e_1, e_2, e_3, e_4), K) \longmapsto (\{e_1, e_2\}, K).$$

The group G acts transitively on F, and we will show that λ is surjective, which implies that G acts transitively on C.
First we compute the cardinality of F. We have a projection $\gamma : F \longrightarrow Y$ on the first coordinate. Let $y = (t_1, t_2, t_3, t_4) \in Y$. Since G acts transitively on F, the stabilizer G_y of y in G acts transitively on the fiber $\gamma^{-1}(y)$. Therefore, we have $|\gamma^{-1}(y)| = |G_y(y, T)| = \frac{|G_y|}{|G_{y,K}|}$. The stabilizer G_y of y in G has cardinality $|G_y| = \frac{|G|}{|Gy|} = \frac{|G|}{|Y|} = 16$, since G acts transitively on Y. Note that the stabilizer of y in G contains all permutations of $(E_1, \ldots, E_8)$ that are generated by the permutations of E_i and E_{i+1} for $i \in \{1, 3, 5, 7\}$. There are 16 of these, so that means that G_y consists exactly of these permutations. For every permutation g in G_y, we have $gT \neq T$, except for the identity and the permutation permuting all E_i, E_{i+1} for $i \in \{1, 3, 5, 7\}$. Therefore we have $\frac{|G_y|}{|G_{y,K}|} = \frac{16}{2} = 8$, so $\gamma^{-1}(y)$ has size eight. Since G acts transitively on Y, all fibers of γ have size eight, and $|F| = 8 \cdot |Y| = 348364800$.
Now consider the element $c = (\{L - E_1 - E_2, L - E_3 - E_4\}, T)$ in C. We compute the cardinality of $\lambda^{-1}(c)$. By looking at T we see that for the elements $e_1 = L - E_1 - E_2$ and $e_2 = L - E_3 - E_4$ there are six elements e_3 in T such that $e_1 \cdot e_3 = e_2 \cdot e_3 = 1$, and for each of those six e_3 there are three elements e_4 in T such that $(e_1, \ldots, e_4) \in W$. Since we can interchange e_1 and e_2, it follows that the fiber above c has cardinality at most $2 \cdot 6 \cdot 3 = 36$. Since G acts

transitively on F, it follows that all non empty fibers of λ have cardinality at most 36. But then we have $|\lambda(F)| \geq \frac{|F|}{36} = 9676800 = |C|$, so we conclude that there is equality everywhere, and λ is surjective.
Finally, we have a projection $\delta : C \longrightarrow \mathcal{T}$ on the second coordinate, which is surjective since every element in $\mathcal{T}$ contains an element in A by Lemma 4.5. Therefore, G acts transitively on $\mathcal{T}$.

(ii) From the fact that the fibers of λ have cardinality 36 it follows that all elements in $T^4 \cap W$ are elements in Y, so $T^4 \cap W = T^4 \cap Y$. Let $\kappa : F \longrightarrow \mathcal{T}$ be the composition $\lambda \circ \delta$, then κ is surjective since both λ and δ are. Since G acts transitively on F, the stabilizer G_T of T acts transitively on the fiber $\kappa^{-1}(T)$ above T. We have a projection $\kappa^{-1}(T) \longrightarrow T^4 \cap Y$ on the first coordinate, which is injective. By surjectivity of κ, we have

$$|\kappa^{-1}(T)| = \frac{|F|}{|\mathcal{T}|} = \frac{348364800}{179200} = 1944.$$

By looking at the intersection multiplicities in T we have

$$|T^4 \cap Y| = |T^4 \cap W| = 12 \cdot 9 \cdot 6 \cdot 3 = 1944.$$

We conclude that the projection is a bijection, hence G_T acts transitively on $T^4 \cap Y$.

(iii) There are 640 elements in $\mathcal{T}$ containing e_1 and e_2 by Lemma 4.26. By looking at the elements in T we see that for a fixed element $c_1 \in T$ there are 9 elements c_2 in T such that $c_1 \cdot c_2 = 1$. Since G acts transitively on $\mathcal{T}$ this holds for all elements in $\mathcal{T}$, so we have $|\mathcal{T}| = \frac{640 \cdot 240 \cdot 126}{12 \cdot 9} = 179200$.

$\square$

Corollary 4.28. *G_T acts transitively on T.*

Proof. We have a surjective map $Y^4 \cap Y \longrightarrow T$ projecting on the first coordinate, so this follows from the previous proposition. $\square$

From the following result, which is again purely geometrical, we will deduce Theorem 2.

Let $Q_1, \ldots, Q_8$ be eight points in general position in $\mathbb{P}^2$. Define the following curves:

L_1 is the line through Q_1 and Q_2;
L_2 is the line through Q_3 and Q_4;
C_1 is the conic through Q_1, Q_3, Q_5, Q_6 and Q_7;
C_2 is the conic through Q_1, Q_4, Q_5, Q_6 and Q_8;
C_3 is the conic through Q_2, Q_3, Q_5, Q_7 and Q_8;
C_4 is the conic through Q_2, Q_4, Q_6, Q_7 and Q_8;
D_1 is the quartic through all eight points with singular points in Q_1, Q_7 and Q_8;
D_2 is the quartic through all eight points with singular points in Q_2, Q_5 and Q_6;
D_3 is the quartic through all eight points with singular points in Q_3, Q_6 and Q_8;
D_4 is the quartic through all eight points with singular points in Q_4, Q_5 and Q_7.

Proposition 4.29. *Assume that char $k = 0$. Then $L_1,\ L_2,\ C_1, \ldots C_4,\ D_1, \ldots, D_4$ do not all go through one point.*

Proof. We assume that these ten curves go through a common point P. First note that if P were equal to one of the Q_i, then one of the conics would contain six of the eight Q_i, which would contradict the fact that $Q_1, \ldots, Q_8$ are in general position. We conclude that P is not equal to one of the Q_i.
Let $(x : y : z)$ be the coordinates in $\mathbb{P}^2$. Without loss of generality we can choose four points in general position in $\mathbb{P}^2$, and we set

$$Q_1 = (1 : 0 : 1); \qquad Q_6 = (0 : -1 : 1);$$
$$Q_5 = (0 : 1 : 1); \qquad P = (-1 : 0 : 1).$$

Since L_1 contains Q_1 and P, it is given by $y = 0$, so we can set $Q_2 = (a_1 : 0 : a_3)$ for some $a_1,\ a_3 \in k$. Let $b_1, b_2, b_3, c_1, c_2, c_3, d_1, d_2, d_3, e_1, e_2, e_3 \in k$ be such that

$$Q_3 = (b_1 : b_2 : b_3); \qquad Q_7 = (d_1 : d_2 : d_3);$$
$$Q_4 = (c_1 : c_2 : c_3); \qquad Q_8 = (e_1 : e_2 : e_3).$$

Since the four points $Q_1,\ Q_5,\ Q_6$ and P are in general position, the linear system of quadrics through $Q_1,\ Q_5,\ Q_6$ and P is two-dimensional. Therefore it is generated by two linearly independent quadrics, and we take these to be $x^2 + y^2 - z^2$ and xy. Let $l, m \in k$ be such that

C_1 is given by $x^2 + y^2 - z^2 = 2lxy$;
C_2 is given by $x^2 + y^2 - z^2 = 2mxy$.

Since Q_3, Q_4, Q_7, and Q_8 are not contained in L_1, there are $s, t, u \in k$ such that

L_2 is given by $sy = x + z$;
The line L_3 through P and Q_7 is given by $ty = x + z$;
The line L_4 through P and Q_8 is given by $uy = x + z$.

We define $\mathbb{A}^{19}$ to be the affine space with coordinate ring

$$T_{19} = k[l, m, s, t, u, a_1, a_3, b_1, b_2, b_3, c_1, c_2, c_3, d_1, d_2, d_3, e_1, e_2, e_3].$$

Points in $\mathbb{A}^{19}$ correspond to configurations of the points $Q_1, \ldots, Q_8$. The fact that all ten curves go through P gives polynomial equations in these 19 variables, hence defines an algebraic set A_0 in $\mathbb{A}^{19}$. We define S_0 to be the algebraic set of all points in $\mathbb{A}^{19}$ that correspond to the configurations where three of the points $Q_1, \ldots, Q_8$ lie on a line, or six of the points lie on a conic. We want to show that A_0 is contained in S_0, which would mean that all possibilities for the ten curves to go through P imply that $Q_1, \ldots, Q_8$ are not in general position, giving a contradiction and thus proving our statement. Since the equations defining A_0 are very big, we do a couple of reduction steps to obtain something that we can actually compute.

Step 1
Let $b_1' = sb_2 - b_3,\ c_1' = sc_2 - c_3,\ d_1' = td_2 - d_3$ and $e_1' = ue_2 - e_3$. The fact that the

points Q_3 and Q_4 lie on the line L_2, the point Q_7 lies on the line L_3, and Q_8 lies on L_4 implies $b_1 = b_1'$, $c_1 = c_1'$, $d_1 = d_2'$, and $e_1 = e_1'$. Define $\mathbb{A}^{15}$ to be the affine space with coordinate ring

$$T_{15} = k[l, m, s, t, u, a_1, a_3, b_2, b_3, c_2, c_3, d_2, d_3, e_2, e_3],$$

and consider the ring homomorphism $T_{19} \longrightarrow T_{15}$ defined by

$$\begin{aligned}&(l, m, s, t, u, a_1, a_3, b_1, b_2, b_3, c_1, c_2, c_3, d_1, d_2, d_3, e_1, e_2, e_3)\\ &\qquad\longmapsto (l, m, s, t, u, a_1, a_3, b_1', b_2, b_3, c_1', c_2, c_3, d_1', d_2, d_3, e_1', e_2, e_3).\end{aligned}$$

This corresponds to an embedding $i_1 : \mathbb{A}^{15} \hookrightarrow \mathbb{A}^{19}$, and A_0 lies in $i_1(\mathbb{A}^{15})$. Let A_1 be $i_1^{-1}(A_0)$ and $S_1 = i_1^{-1}(S_0)$, then $A_0 \subseteq S_0$ is equivalent to $A_1 \subseteq S_1$.

Step 2
Since Q_3 and Q_4 are in the intersection of L_2 with C_1 and C_2, respectively, we have

$$(sb_2 - b_3)^2 + b_2^2 - b_3^2 - 2l(sb_2 - b_3)b_2 = 0; \tag{1}$$

$$(sc_2 - c_3)^2 + c_2^2 - c_3^2 - 2m(sc_2 - c_3)c_2 = 0. \tag{2}$$

Since P is also in the intersection of L_2 with C_1, we can divide (1) by b_2 and obtain the equation

$$(s^2 - 2ls + 1)b_2 + 2(l - s)b_3 = 0. \tag{3}$$

Similarly, we can divide (2) by c_2 and obtain

$$(s^2 - 2ms + 1)c_2 + 2(l - s)c_3 = 0. \tag{4}$$

Let $V_1,\ V_2$ be the subvarieties of $\mathbb{A}^{15}$ defined by $s^2 - 2ls + 1 = 0$ and $s^2 - 2ms + 1 = 0$, respectively.

<u>Claim 4.29.1:</u> $V_1 \cap A_1$ and $V_2 \cap A_1$ lie in S_1.
<u>Proof:</u> Let $(l, m, s, t, u, a_1, a_3, b_1, b_3, c_1, c_3, d_2, d_3, e_2, e_3)$ be a point in A_1 and assume that $s^2 - 2ls + 1 = 0$. Then, by (3), we have $2(l - s)b_3 = 0$, which implies that C_1 contains the whole line L_2. But then $Q_4 \in C_1$, which means that the six points $Q_1,\ Q_3,\ Q_4,\ Q_5,\ Q_6$ and Q_7 lie on a conic. Since S_1 consists of all points in $\mathbb{A}^{15}$ that correspond to configurations of the points $Q_1, \ldots, Q_8$ where three of the points lie on a line or six of the points lie on a conic, we conclude that $V_1 \cap A_1$ lies in S_1. The proof for $V_2 \cap A_1$ goes analogously. $(\square)$

Let

$$b_1' = \frac{-2(l - s)b_3}{s^2 - 2ls + 1} \quad \text{and} \quad c_1' = \frac{-2(l - s)c_3}{s^2 - 2ms + 1}.$$

Define $\mathbb{A}^{13}$ to be the affine space with coordinate ring

$$T_{13} = k[l, m, s, t, u, a_1, a_3, b_3, c_3, d_2, d_3, e_2, e_3],$$

and let $K_{13} = \mathrm{Frac}(T_{13})$ be the field of rational fractions of elements in T_{13}. Consider the ring homomorphism $T_{15} \longrightarrow K_{13}$ defined by

$$\begin{aligned}&(l, m, s, t, u, a_1, a_3, b_1, b_3, c_1, c_3, d_2, d_3, e_2, e_3)\\ &\qquad\longmapsto (l, m, s, t, u, a_1, a_3, b_1', b_3, c_1', c_3, d_2, d_3, e_2, e_3).\end{aligned}$$

This defines an injective rational map $i_2 : \mathbb{A}^{13} \dashrightarrow \mathbb{A}^{15}$. Let

$$A_1' = A_1 \setminus ((A_1 \cap V_1) \cup (A_1 \cap V_2)).$$

By Claim 4.29.1, showing that $A_1 \subseteq S_1$ is equivalent to showing that $A_1' \subseteq S_1$.

Note that, since i_2 is defined outside the subvarieties of $\mathbb{A}^{13}$ defined by $s^2 - 2ls + 1 = 0$ and $s^2 - 2ms + 1 = 0$, we have $i_2^{-1}(A_1') \cong A_1'$. Let $A_2 = \overline{i_2^{-1}(A_1')}$ and $S_2 = i_2^{-1}(S_1)$, then $A_1' \subseteq S_1$ is equivalent to $A_2 \subseteq S_2$.

Step 3

Since Q_7 and Q_8 are in the intersection of L_3 with C_1 and L_4 with C_2, respectively, we have

$$(td_2 - d_3)^2 + d_2^2 - d_3^2 - 2ld_2(td_2 - d_3) = 0; \tag{5}$$

$$(ue_2 - e_3)^2 + e_2^2 - e_3^2 - 2me_2(ue_2 - e_3) = 0. \tag{6}$$

Since P is also in the intersection of L_3 with C_1 and in the intersection of L_4 with C_2, we can divide (5) by d_2 and obtain the equation

$$(t^2 - 2lt + 1)d_2 + 2ld_3 - 2td_3 = 0. \tag{7}$$

Similarly, we can divide (6) by e_2 and obtain

$$(u^2 - 2mu + 1)e_2 + 2me_3 - 2ue_3 = 0. \tag{8}$$

Let $U_1,\ U_2$ be the subvarieties of $\mathbb{A}^{13}$ defined by $t^2 - 2lt + 1 = 0$ and $u^2 - 2mu + 1 = 0$, respectively.

<u>Claim 4.29.2:</u> $U_1 \cap A_2$ and $U_2 \cap A_2$ lie in S_2.
<u>Proof:</u> Analogously to the proof of Claim 4.29.1, $U_1 \cap A_2$ and $U_2 \cap A_2$ consist of points in $\mathbb{A}^{13}$ corresponding to configurations where L_3 is contained in C_1, and L_4 is contained in C_2, respectively. If L_3 is contained in C_1 then C_1 is reducible, so three of the points $Q_1,\ Q_3,\ Q_5,\ Q_6$ and Q_7 are on a line. Equivalently, $L_4 \subset C_2$ implies that three of the Q_i are on a line. Since S_2 contains all points in $\mathbb{A}^{13}$ corresponding to configurations of the Q_i where three of them lie on a line, we conclude that $U_1 \cap A_2$ and $U_2 \cap A_2$ are both in S_2. ($\square$)

Let

$$d_2' = \frac{2td_3 - 2ld_3}{t^2 - 2lt + 1} \quad \text{and} \quad e_2' = \frac{2ue_3 - 2me_3}{u^2 - 2mu + 1}.$$

Define $\mathbb{A}^{11}$ to be the affine space with coordinate ring

$$T_{11} = k[l, m, s, t, u, a_1, a_3, b_3, c_3, d_3, e_3],$$

and let K_{11} be the function field of A_{11}, consisting of rational fractions of elements in T_{11}. Consider the ring homomorphism $T_{13} \longrightarrow K_{11}$ defined by

$$(l, m, s, t, u, a_1, a_3, b_3, c_3, d_2, d_3, e_2, e_3) \longmapsto (l, m, s, t, u, a_1, a_3, b_3, c_3, d_2', d_3, e_2', e_3).$$

This defines an injective rational map $i_3 : \mathbb{A}^{11} \dashrightarrow \mathbb{A}^{13}$. Let

$$A_2' = A_2 \setminus ((A_2 \cap U_1) \cup (A_2 \cap U_2)).$$

By Claim 4.29.2, showing that $A_2 \subseteq S_2$ is equivalent to showing that $A_2' \subseteq S_2$.

Since i_3 is defined outside the subvarieties of $\mathbb{A}^{11}$ defined by $t^2 - 2lt + 1 = 0$ and $u^2 - 2mu + 1 = 0$, we have $i_3^{-1}(A_2') \cong A_2'$. Let $A_3 = \overline{i_3^{-1}(A_2')}$ and $S_3 = i_3^{-1}(S_2)$, then $A_2' \subseteq S_2$ is equivalent to $A_3 \subseteq S_3$.

Step 4
Since no four of the points Q_3, Q_5, Q_7, Q_8 and P are on a line, there is a unique conic C through these five points. Note that C intersects L_1 in P, and L_1 is not contained in C since then C would contain six of the Q_i. Since C_3 contains all of the five points, we conclude that $C = C_3$, and the second intersection point of C and L_1 is Q_2. Let $(R_1, \dots, R_6) = (Q_2, Q_3, Q_5, Q_7, Q_8, P)$, and let N be the corresponding matrix from Lemma 4.21. We have

$$\det(N) = \tfrac{1}{2}e^2d^2b^2(a_1 + a_3)(u-1)(t-1)(s-1)(s-t)(m-u)(l-t)(l-s)(\alpha a_1 + \beta),$$

with

$$\begin{aligned}\alpha = l^2su + l^2s - lmsu - lms - \tfrac{1}{2}lstu - \tfrac{1}{2}lst + \tfrac{1}{2}lsu^2 + lsu + \tfrac{1}{2}ls - \tfrac{1}{2}lu - \tfrac{1}{2}l \\ + \tfrac{1}{2}mst - msu - \tfrac{1}{2}ms + \tfrac{1}{2}su^2 + \tfrac{1}{2}t + \tfrac{1}{2}mt + \tfrac{1}{2}m - \tfrac{1}{2}u - \tfrac{1}{2}stu,\end{aligned}$$

and

$$\beta = \tfrac{1}{2}a_3\left(lstu + lst - lsu^2 + ls + lu + l - mst - ms - mt - m + stu - su^2 - t + u\right).$$

All factors of $\det(N)$ except for $\alpha a_1 + \beta$ define a subset of S_3, hence correspond to configurations where the points are not in general position. Therefore, C_3 contains P if and only $\alpha a_1 + \beta = 0$. By the same reasoning as before, we have $Z(\alpha) \cap A_3 \subset S_3$. Define $\mathbb{A}^{10}$ to be the affine space with coordinate ring

$$T_{10} = k[l, m, s, t, u, a_3, b_3, c_3, d_3, e_3],$$

and let $K_{10} = \mathrm{Frac}(T_{10})$. Consider the ring homomorphism $T_{11} \longrightarrow K_{10}$ defined by

$$(l, m, s, t, u, a_1, a_3, b_3, c_3, d_3, e_3) \longmapsto \left(l, m, s, t, u, \frac{-\beta}{\alpha}, a_3, b_3, c_3, d_3, e_3\right).$$

This defines an injective rational map $i_4 : \mathbb{A}^{10} \dashrightarrow \mathbb{A}^{11}$. Let $A_3' = A_3 \setminus (Z(\alpha) \cap A_3)$. Showing that $A_3 \subseteq S_3$ is equivalent to showing that $A_3' \subseteq S_3$.

Since i_4 is defined everywhere outside the subvariety of $\mathbb{A}^{10}$ defined by $\alpha = 0$, we have $i_4^{-1}(A_3') \cong A_3'$. Let $A_4 = \overline{i_4^{-1}(A_3')}$ and $S_4 = i_4^{-1}(S_3)$, then $A_3' \subseteq S_3$ is equivalent to $A_4 \subseteq S_4$.

Step 5
The equation expressing that P is contained in C_4, is the determinant of the matrix N in Lemma 4.21 corresponding to $(R_1, \dots, R_6) = (Q_2, Q_4, Q_6, Q_7, Q_8, P)$. This determinant is given by

$$\tfrac{1}{4}e_3^2d_3^2c_3^2a_3^2(u+1)(t+1)(s+1)(s-u)(m-u)(m-s)(l-t)(l-m)f,$$

where

$$f = (l^2-1)(s-1)u^2 + (m^2-1)(s+1)t^2 - 2s(m-1)(l+1)(tu+t-u) - l^2s + l^2 \\ + 2lms - 2ls - m^2s - m^2 + 2ms.$$

All factors except for f define subsets of S_4, so P is contained in C_4 if and only if $f=0$. Since f is quadratic in t and u and does not depend on a_3, b_3, c_3, d_3, e_3, it defines a conic D in the affine plane over $k(l,m,s)$ with coordinates t and u. One point on this conic is given by $(t,u)=(l,m)$. Let l_1, l_2 be the tangent lines at P to C_1 and C_2, respectively. Note that $\frac{d(x/z)}{d(y/z)}l_1 = l$ and $\frac{d(x/z)}{d(y/z)}L_3 = t$, and both lines contain P. Therefore $t=l$ implies $l_1 = L_3$, and this means that P is equal to Q_7. Similarly, $u=m$ implies $P=Q_8$. Therefore the point $(t,u)=(l,m)$ corresponds to a subvariety of S_4, so we can assume $t-l\neq 0$ and $u-m\neq 0$. So we can parametrize D by intersecting D with a line M through (l,m), where M is given by $v(t-l)=(u-m)$ for a parameter v. Intersecting D with M gives

$$(av^2+cv+b)t^2 + (2mv - 2v^2l - cvl + m - cv + c)t \\ + am^2 - 2mvl + v^2l^2 - cm + cvl - (a+b+c) = 0.$$

Since $t=l$ is a solution, we can divide by $t-l$ and obtain

$$(av^2+cv-b)t - l(av^2+cv-b) + 2((l^2-1)(ms-1)v + m^2-1) = 0. \tag{9}$$

Let $\gamma = av^2+cv-b$. As before, $\gamma=0$ implies that all t satisfy (9), which means that M is contained in D, so D is reducible. But then, by the same reasoning as before, the points $Q_1,\dots,Q_8$ would not be in general position, giving a contradiction. We conclude that $\gamma\neq 0$.

Let $\mathbb{A}^9$ be the affine space with coordinate ring $T_9 = k[l,m,s,a_3,b_3,c_3,d_3,e_3,v]$, and let $K_9 = \mathrm{Frac}(T_9)$. Let

$$t' = -2\frac{(l^2-1)(ms-1)v+m^2-1}{av^2+cv-b} + l,$$

and consider the ring homomorphism $T_{10} \longrightarrow K_9$ defined by

$$(l,m,s,t,u,a_3,b_3,c_3,d_3,e_3) \longmapsto (l,m,s,t',t'v+m-vl,a_3,b_3,c_3,d_3,e_3)\,.$$

This defines an injective rational map $i_5 : \mathbb{A}^9 \dashrightarrow \mathbb{A}^{10}$. Let

$$A_4' \;=\; A_4 \;\setminus\; (A_4 \;\cap\; Z(\gamma)\;).$$

Showing that $A_4\subseteq S_4$ is equivalent to showing that $A_4'\subseteq S_4$.

We have $i_5^{-1}(A_4')\cong A_4'$. Let $A_5 = \overline{i_5^{-1}(A_4')}$ and $S_5 = i_5^{-1}(S_4)$, then $A_4'\subseteq S_4$ is equivalent to $A_5\subseteq S_5$.

Finishing the proof

For i in $\{1,2,3,4\}$, the expression stating that P is contained in D_i is given by $\det(H_i)=0$, where H_i is the matrix H Lemma 4.21 with

H_1 is the matrix associated to $(R_1,\dots,R_9) = (Q_2,Q_3,Q_4,Q_5,Q_6,Q_1,Q_7,Q_8)$;

H_2 is the matrix associated to $(R_1,\dots,R_9) = (Q_1,Q_3,Q_4,Q_7,Q_8,Q_2,Q_5,Q_6)$;

H_3 is the matrix associated to $(R_1,\dots,R_9) = (Q_1,Q_2,Q_4,Q_5,Q_7,Q_3,Q_6,Q_8)$;

H_4 is the matrix associated to $(R_1,\dots,R_9) = (Q_1,Q_2,Q_3,Q_6,Q_8,Q_4,Q_5,Q_7)$.

With MAGMA, we compute these determinants. For $i \in \{1, 2, 3, 4\}$, we have $\det(H_i) = \lambda_i g_i$, where λ_i defines a subset of S_5, and g_i is irreducible. The algebraic set A_5 is the zero set of the radical of the ideal $(g_1, \ldots, g_4) \subset T_9$. Let $\delta = v^2(ls - l - ms - m + 2s)^2(l - m)(l + 1)(m - 1)(l - 1)(m + 1) \in T_9$. With MAGMA we can compute the Gröbner basis of I, and then it is a relatively easy check that δ is contained in I. Therefore, A_5 is contained in the union of the varieties defined by the factors of δ. As these are all subsets of S_5, we conclude that A_5 is contained in S_5. This finishes the proof. □

Lemma 4.30. *All maximal cliques of size eleven contain a pair.*

Proof. By Lemma 4.5 it is sufficient to prove this statement for all cliques of size eleven containing the two elements $e_1 = L - E_1 - E_2$ and $e_2 = L - E_3 - E_4$. From Lemma 4.25 we know that there are 138 exceptional curves intersecting both e_1 and e_2 positively. Since it is too tedious to compute all maximal cliques of size nine in the graph on these 138 exceptional curves by hand, we use MAGMA to compute them, and check that they all contain at least one pair. □

Lemma 4.31. *Let K be a clique of size eleven without pairs. Then K is contained in a clique of size twelve without pairs.*

Proof. The clique K is not maximal by the previous lemma, so it is contained in a clique of size twelve. Let $e_1 = L - E_1 - E_2$, and $e_2 = L - E_1 - E_2$. With MAGMA, we compute all (not necessarily maximal) cliques of size twelve in the graph on these 138 curves. There are no cliques of size twelve with only one pair. Therefore, if K were contained in a clique of size twelve that has a pair, K would contain a pair too, which is a contradiction. □

Proposition 4.32. *Assume that char $k = 0$. Then the number of exceptional curves that go through one point outside the ramification curve of φ is at most ten.*

Proof. By Lemma 4.26 we know that it is at most twelve. Consider the twelve classes

$$\begin{aligned}
e_1 &= L - E_1 - E_2;\\
e_2 &= L - E_3 - E_4;\\
e_3 &= 2L - E_1 - E_3 - E_5 - E_6 - E_7;\\
e_4 &= 2L - E_1 - E_4 - E_5 - E_6 - E_8;\\
e_5 &= 2L - E_2 - E_3 - E_5 - E_7 - E_8;\\
e_6 &= 2L - E_2 - E_4 - E_6 - E_7 - E_8;\\
e_7 &= 4L - 2E_1 - E_2 - E_3 - E_4 - E_5 - E_6 - 2E_7 - 2E_8;\\
e_8 &= 4L - E_1 - 2E_2 - E_3 - E_4 - 2E_5 - 2E_6 - E_7 - E_8;\\
e_9 &= 4L - E_1 - E_2 - 2E_3 - E_4 - E_5 - 2E_6 - E_7 - 2E_8;\\
e_{10} &= 4L - E_1 - E_2 - E_3 - 2E_4 - 2E_5 - E_6 - 2E_7 - E_8;\\
e_{11} &= 5L - 2E_1 - 2E_2 - 2E_3 - 2E_4 - 2E_5 - E_6 - E_7 - 2E_8;\\
e_{12} &= 5L - 2E_1 - 2E_2 - 2E_3 - 2E_4 - E_5 - 2E_6 - 2E_7 - E_8.
\end{aligned}$$

It is straightforward to check that they form a clique without pairs. By Remark 2.9

we know that $e_1, \ldots, e_{10}$ are the classes in Pic X of the strict transforms of the curves $L_1, L_2, C_1, \ldots, C_4, D_1, \ldots, D_4$, defined with respect to $Q_i = P_i$ for $i \in \{1, \ldots, 8\}$.
Let $T = \{c_1, \ldots, c_{12}\}$ be a clique of size twelve that does not contain any pairs. By Proposition 4.27, after changing the indexes if necessary, there is an element $g \in G$ such that $c_i = g(e_i)$ for $i \in \{1, \ldots, 12\}$. Let $E'_i = g(E_i)$. Then, since the E'_i are pairwise disjoint, by Lemma 3.9 we can blow down $E'_1, \ldots, E'_8$ to points $R_1, \ldots, R_8 \in \mathbb{P}^2$ that are in general position, such that X is the blow-up of $\mathbb{P}^2$ at $R_1, \ldots, R_8$, and E'_i is the class in Pic X of the exceptional curve above R_i for all i. By the bijection in Remark 3.12, the elements $c_1, \ldots, c_{10}$ are the classes of the strict transforms of $L_1, L_2, C_1, \ldots, C_4, D_1, \ldots, D_4$ defined with respect to $Q_i = R_i$ for $i \in \{1, \ldots, 8\}$. From Proposition 4.29 it follows that the curves corresponding to $c_1, \ldots, c_{10}$ can not all go through one point.
Since every set of twelve exceptional curves without pairs corresponds to a clique of size twelve without pairs, we conclude that the number of exceptional curves going through one point outside the ramification curve of φ is less than twelve.
Let K be a clique of size eleven without pairs. By Lemma 4.31, K is contained in a clique of size twelve without pairs, say H. As we just showed, the clique H contains a set of ten classes $\{d_1, \ldots, d_{10}\}$ such that the corresponding curves do not all go trough one point. By Corollary 4.28, there is an element g in G_H such that K contains $g(d_1), \ldots, g(d_{10})$. But then, as we did above, we can blow down $g(E_1), \ldots, g(E_8)$, such that $g(d_1), \ldots, g(d_{10})$ are the classes of the strict transforms of $L_1, L_2, C_1, \ldots, C_4, D_1, \ldots, D_4$ defined with respect to $Q_i = g(E_i)$ for $i \in \{1, \ldots, 8\}$, which can not all go through one point. We conclude that the elements in K can not all go through one point, so the maximum is less than eleven. □

Proof of Theorem 2.
By Lemma 4.26, the number of exceptional curves that go through one point outside the branch curve of X is at most twelve. If char $k = 0$, it is at most ten by Proposition 4.32. □

The following example is in [SvL14].

Example 4.33. Assume that the characteristic of k is unequal to 2,3 and 5. Let $\beta, \delta \in k^*$, and let S be the surface in $\mathbb{P}(2,3,1,1)$ given by

$$y^2 = x^3 + f(z,w)x + g(z,w),$$

where

$$f = -27(\beta^4 + 12\beta^3 + 14\beta^2 - 12\beta + 1)w^4,$$

and

$$g = \delta z^5 w + 54(\beta^2 + 1)(\beta^4 + 18\beta^3 + 74\beta^2 - 18\beta + 1)w^6.$$

Assume that S is smooth, so it is a del Pezzo surface of degree one. Consider the point $Q = (x_0 : y_0 : 0 : 1) \in S$ with $x_0 = 3(\beta^2 + 6\beta + 1)$, and $y_0 = 108\beta$. Note that Q is outside that ramification curve of φ, since $y_0 \neq 0$.
Let α, ε be in a field extension of k such that $\alpha^2 = \alpha + 1$ and $\delta = -6(\beta + \alpha^5)\varepsilon^5$. Since char $k \neq 2,3,5$, there are ten such pairs (α, ε). Now consider the curve $C_{\alpha,\varepsilon}$ in $\mathbb{P}(2,3,1,1)$ defined by

$$x = \varepsilon^2 z^2 + 6\alpha\varepsilon z w + x_0 w^2,$$

$$y = -\varepsilon^3 z^3 + 3(\beta + 2\alpha + 3)\varepsilon^2 z^2 w + 18\alpha(\beta + 1)\varepsilon z w^2 + y_0 w^3.$$

Let μ be the restriction to $U = \mathbb{P}(2,3,1,1) - \{z = w = 0\}$ of the projection $\mathbb{P}(2,3,1,1) \dashrightarrow \mathbb{P}^1_{k(\alpha,\varepsilon)}$ on the last two coordinates. By Lemma 2.1 in [SvL14], the curve $C_{\alpha,\varepsilon}$ is a section of μ. Moreover, it is a quick check that $C_{\alpha,\varepsilon}$ is contained in S, so from Lemma 2.2 in [SvL14] it follows that $C_{\alpha,\varepsilon}$ is an exceptional curve in S over $k(\alpha,\varepsilon)$. It is easy to see that Q is contained in $C_{\alpha,\varepsilon}$. Since there are ten pairs (α,ε), we conclude there are ten exceptional curves through Q over a field extension of k.

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