

CONSTRUCTION OF COMPACT COMPLEX SURFACES UNIFORMIZED BY THE UNIT BALL

Alessandro ROSSETTI
rossetti.math@gmail.com

Advised by Prof. Vincent KOZIARZ

ALGANT MASTER THESIS - JULY 2013
UNIVERSITY OF STELLENBOSCH AND UNIVERSITÉ BORDEAUX 1



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Alessandro Rossetti

July 1, 2013

Contents

Introduction	i
1 The Universal Elliptic Curve $E(N)$	1
1.1 The topological space $\Gamma \backslash \mathbb{H}^*$	3
1.2 $\Gamma \backslash \mathbb{H}^*$ as a Riemann surface	5
1.3 The Riemann Surfaces $Y(N)$ and $X(N)$	8
1.4 $\tilde{E}(N)$ and its compactification $E(N)$	12
2 Numerical Invariants, Special Curves	20
3 The Cyclic Cover $A(N, n)$	30
3.1 Ramified and Cyclic Coverings	30
3.2 The Construction of $A(N, n)$	34
Bibliography	38

Introduction

In this thesis we analyze a certain class of surfaces of general type which are obtained by ramified covering of the universal elliptic surface. We are going to follow the work of R. Livné [Liv81] and M. Inoue [Ino94]. In the first chapter we construct the moduli space of elliptic curves $Y(N)$ with full level N structure and its compactification $X(N)$.

If $N \geq 3$, $Y(N)$ is a fine moduli space and we can construct the universal family and extend it to an elliptic surface $p: \tilde{E}(N) \rightarrow Y(N)$. After the compactification $E(N)$ of the elliptic surface, as the last part of the chapter, we see the fibration p admits N^2 sections

$$\Gamma(i, j), \quad i, j = 0, \dots, N-1,$$

which intersect each fiber F at its N -torsion points.

In the second chapter we calculate numerical invariants and special curves of the elliptic curve $E(N)$. We use the approach of Kodaira [Kod60] to find the canonical bundle $K_{E(N)}$ of $E(N)$ and its degree.

In the last part of the chapter we prove some facts about the algebraic equivalence between divisors, in particular we prove an important Lemma of T. Shioda [Shi72].

It turns out that

$$\Gamma = \sum_{i,j=0}^{N-1} \Gamma(i, j)$$

is divisible by N (resp. $N/2$) if N is odd (resp. if N is even). Let n be an integer such that

$$\begin{aligned} n|N & \quad \text{if } N \text{ is odd,} \\ n|(N/2) & \quad \text{if } N \text{ is even.} \end{aligned}$$

and let $[\Gamma] = nL$ for some line bundle $L \in H^1(E(N), \mathcal{O}^*)$ be a n -th root of Γ .

By the general theory of cyclic covers, explained in the first part of the last chapter, there is a branched cyclic cover $A(N, n)$ of $E(N)$, totally branched over the $\Gamma(i, j)$'s and nowhere else.

For $N \geq 6$ these turn out to be of general type.

One of the interesting characteristics of these surfaces is the high ratio of c_1^2/c_2 . Surface with $c_1^2 > 2c_2$ are rather infrequent, indeed, Miyaoka proved that always $c_1^2 \leq 3c_2$, and Yau has shown that if $c_1^2 = 3c_2$ (over \mathbb{C}) the surface is actually uniformized by the ball, or is \mathbb{P}^2 or \mathbb{C}^2 . Most

of the construction of surfaces of general type with $c_1^2 = 3c_2$ are done by defining a discrete subgroup $\Gamma \subset \text{PU}(2, 1)$.

Among the surfaces $A(N, n)$, it turns out that for

$$(N, n) = (7, 7), (8, 4), (9, 3) \text{ or } (12, 2),$$

one has $c_1^2 = 3c_2$.

In addition, if $(N, n) = (5, 5)$, $A(N, n)$ is non minimal and the minimal model has $c_1^2 = 3c_2$.

Chapter 1

The Universal Elliptic Curve

$E(N)$

Although our main focus is the congruence subgroups of $SL(2, \mathbb{Z})$ and the action on the upper half plane, let us first consider a more general theory. Let $\mathbb{C} \cup \{\infty\}$ and consider the group $GL(2, \mathbb{C})$ acts on $\mathbb{C} \cup \{\infty\}$ via

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \mapsto \frac{az + b}{cz + d}.$$

Suppose that this is not the identity transformation. From the theory of the Jordan canonical form, we see that the matrix γ is conjugate to one of the following two forms:

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}; \quad \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \lambda \neq \mu.$$

Therefore, our transformation is of the following types:

$$z \mapsto z + \lambda^{-1}; \quad z \mapsto cz, \quad c \neq 1.$$

In the first case, we call γ **parabolic**. In the second case, we call γ **elliptic** if $|c| = 1$, **hyperbolic** if c is real and positive and **loxodromic** otherwise. The classification can be done by means of the trace of γ ($\text{tr}(\gamma)$)(for the proof see [Shi71]):

Proposition 1.1. *Let $\gamma \in SL(2, \mathbb{C})$, $\gamma \neq \pm 1_2$. Then*

$$\gamma \text{ is parabolic} \Leftrightarrow \text{tr}(\gamma) = \pm 2,$$

$$\gamma \text{ is elliptic} \Leftrightarrow |\text{tr}(\gamma)| \text{ is real and } |\text{tr}(\gamma)| < 2,$$

$$\gamma \text{ hyperbolic} \Leftrightarrow |\text{tr}(\gamma)| \text{ is real and } |\text{tr}(\gamma)| > 2,$$

$$\gamma \text{ loxodromic} \Leftrightarrow |\text{tr}(\gamma)| \text{ is not real.}$$

If $\gamma \in SL(2, \mathbb{R})$, then γ maps the upper half plane \mathbb{H} (defined as $\mathbb{H} := \{z \in \mathbb{C} | \text{Im}(z) > 0\}$) onto itself. It is also known that every holomorphic automorphism of \mathbb{H} is obtained from an element of $SL(2, \mathbb{R})$.

Note that γ induces the identity map if and only if it is a scalar matrix. Therefore the group of all holomorphic automorphisms of \mathbb{H} is isomorphic to $\mathrm{SL}(2, \mathbb{R})/\{\pm 1_2\}$.

We are about to see that the number of fixed point of γ is one or two, according as γ is parabolic or not and now we want to classify the transformation on terms of the fixed points.

If $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R})$ and $i = \sqrt{-1}$, we have $\gamma(i) = i$ if and only if $a = d$, $b = -c$, $a^2 + b^2 = 1$. Therefore, the special orthogonal group

$$\mathrm{SO}(2) = \{\gamma \in \mathrm{SL}(2, \mathbb{R}) \mid {}^t\gamma\gamma = 1_2\}$$

is the isotropy subgroup of $\mathrm{SL}(2, \mathbb{R})$ at i . The action of $\mathrm{SL}(2, \mathbb{R})$ on \mathbb{H} is transitive and \mathbb{H} is homeomorphic to $\mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(2)$, through the map $\gamma \mapsto \gamma(i)$.

For every $z \in \mathbb{H}$, we can find an element σ of $\mathrm{SL}(2, \mathbb{R})$ so that $\sigma(i) = z$. Then

$$\sigma \cdot \mathrm{SO}(2) \cdot \sigma^{-1} = \{\gamma \in \mathrm{SL}(2, \mathbb{R}) \mid \gamma(z) = z\}.$$

Since every element of $\mathrm{SO}(2)$ has characteristic roots of absolute value 1, this shows that an element of $\mathrm{SL}(2, \mathbb{R})$ with at least one fixed point in \mathbb{H} must be either $\pm 1_2$ or elliptic.

For every $s \in \mathbb{R} \cup \{\infty\}$, put

$$\begin{aligned} F(s) &= \{\gamma \in \mathrm{SL}(2, \mathbb{R}) \mid \gamma(s) = s\}, \\ P(s) &= \{\gamma \in \mathrm{SL}(2, \mathbb{R}) \mid \gamma \text{ parabolic or } = \pm 1_2\}. \end{aligned}$$

Since $\mathrm{SL}(2, \mathbb{R})$ acts transitively on $\mathbb{R} \cup \{\infty\}$, we can find an element of σ of $\mathrm{SL}(2, \mathbb{R})$ so that $\sigma(\infty) = s$. Then $F(s) = \sigma F(\infty)\sigma^{-1}$, $P(s) = \sigma P(\infty)\sigma^{-1}$. Now we see easily that

$$\begin{aligned} F(\infty) &= \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid a \in \mathbb{R}^*, b \in \mathbb{R} \right\}, \\ P(\infty) &= \left\{ \pm \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \mid h \in \mathbb{R} \right\}. \end{aligned}$$

This shows that if an element γ of $\mathrm{SL}(2, \mathbb{R})$, $\neq \pm 1_2$, has at least one fixed point on $\mathbb{R} \cup \{\infty\}$, then γ is either parabolic or hyperbolic. To summarize we get the following

Proposition 1.2. *Let $\gamma \in \mathrm{SL}(2, \mathbb{R})$, $\gamma \neq \pm 1_2$. Then*

γ is parabolic $\Leftrightarrow \gamma$ has only one fixed point on $\mathbb{R} \cup \{\infty\}$,

γ is elliptic $\Leftrightarrow \gamma$ has one fixed point z in \mathbb{H} , and the other \bar{z} ,

γ hyperbolic $\Leftrightarrow \gamma$ has two fixed points on $\mathbb{R} \cup \{\infty\}$.

Let us now fix a discrete subgroup Γ of $\mathrm{SL}(2, \mathbb{R})$. A point z of \mathbb{H} is called an **elliptic point** of Γ if there exists an elliptic element σ of Γ such that $\sigma(z) = z$. Similarly, a point s of $\mathbb{R} \cup \{\infty\}$ is called a **cusps** of Γ if there exists a parabolic element of τ of Γ such that $\tau(s) = s$. If w is a cusp (resp. an elliptic point) of Γ and $\gamma \in \Gamma$, then we see easily that $\gamma(w)$ is also a cusp (resp. an elliptic point) of Γ .

1.1 The topological space $\Gamma \backslash \mathbb{H}^*$

Hereafter till the end of this section, we denote by Γ any discrete subgroup of $SL(2, \mathbb{R})$, and by \mathbb{H}^* the union of \mathbb{H} and the cusps of Γ . We observe that Γ acts on \mathbb{H}^* , hence the quotient space $\Gamma \backslash \mathbb{H}^*$ is meaningful. Before defining a topology on the quotient space, we will define a topology of \mathbb{H}^* . For every $z \in \mathbb{H}$, as a fundamental system of open neighbourhoods of z , we take the usual one. For a fundamental system of open neighbourhoods of cusp $s \neq \infty$, we take all sets of the form:

$$\{s\} \cup \{\text{the interior of a circle in } \mathbb{H} \text{ tangent to the real axis at } s\}.$$

If ∞ is a cusp, we take the sets

$$\{\infty\} \cup \{z \in \mathbb{H} \mid \text{Im}(z) > c\},$$

for all positive numbers c , as a fundamental system of open neighbourhoods of ∞ . It can easily be seen that this defines a Hausdorff topology on \mathbb{H}^* , and every element Γ acts on \mathbb{H}^* as homeomorphism.

For a cusp $s \leq \infty$ of Γ , put

$$\begin{aligned} P(s) &= \{\gamma \in SL(2, \mathbb{R}) \mid \gamma(s) = s, \gamma \text{ parabolic or } = \pm 1_2\}, \\ \Gamma_s &= P(s) \cap \Gamma = \{\gamma \in \Gamma \mid \gamma(s) = s\}. \end{aligned}$$

The neighbourhoods of s of above type are obviously stable under $P(s)$. To study the structure of $\Gamma \backslash \mathbb{H}^*$, let us assume that ∞ is a cusp of Γ . For every $\sigma \in \Gamma$, we let c_σ denote the lower left entry of the matrix σ . Then

$$\Gamma_\infty = \{\sigma \in \Gamma \mid c_\sigma = 0\}$$

and a generator of Γ_∞ modulo $\pm 1_2$ is $\pm \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$. Now we will write some important Lemmas to construct the topology of $\Gamma \backslash \mathbb{H}^*$ (for the proofs see [Shi71]):

Lemma 1.1. $|c_\sigma|$ depends only on the double coset $\Gamma_\infty \sigma \Gamma_\infty$.

Lemma 1.2. Given $M > 0$, there are only finitely many double cosets $\Gamma_\infty \sigma \Gamma_\infty$ such that $\sigma \in \Gamma$ and $|c_\sigma| \leq M$.

Lemma 1.3. There exists a positive number r , depending only on Γ , such that $|c_\sigma| \geq r$ for all $\sigma \in \Gamma - \Gamma_\infty$. Moreover, for such an r , one has $\text{Im}(z) \cdot \text{Im}(\sigma(z)) \leq 1/r^2$ for all $z \in \mathbb{H}$ and all $\sigma \in \Gamma - \Gamma_\infty$.

Lemma 1.4. For every cusp s of Γ , there exists a neighbourhood U of s in \mathbb{H}^* such that $\Gamma_s = \{\sigma \in \Gamma \mid \sigma(U) \cap U \neq \emptyset\}$.

Observe that two points of the set U are equivalent under Γ only if they are so under Γ_s , and hence $\Gamma_s \backslash U$ may be identified with a subset of $\Gamma \backslash \mathbb{H}^*$; moreover U contains no elliptic point of Γ .

Lemma 1.5. *For every cusp of Γ and for every compact subset K of \mathbb{H} , there exists a neighborhood U of s such that $U \cap \gamma(K) = \emptyset$ for every $\gamma \in \Gamma$.*

Let us now consider the quotient topology of $\Gamma \backslash \mathbb{H}^*$, namely we take

$$\{X \subset \Gamma \backslash \mathbb{H}^* \mid \pi^{-1}(X) \text{ is open in } \mathbb{H}^*\}$$

to be the class of all open sets in $\Gamma \backslash \mathbb{H}^*$, where π is the natural projection of \mathbb{H}^* to $\Gamma \backslash \mathbb{H}^*$. If U is as in Lemma 1.4, then $\pi(U)$ can be identified with $\Gamma_s \backslash U$, and is a neighbourhood of $\pi(s)$.

Theorem 1.1. *The quotient space $\Gamma \backslash \mathbb{H}^*$, with the above topology, is a Hausdorff space.*

Proof. It is well known that $\Gamma \backslash \mathbb{H}$ is a Hausdorff space. Since $\Gamma \backslash \mathbb{H}^*$ is the union of $\Gamma \backslash \mathbb{H}$ and the equivalence classes of cusps, it remains to show that an equivalence class of cusps can be separated from an equivalence class of cusps. Lemma 1.5 takes care of the former case. Therefore let us consider two cusps s and t which are not Γ -equivalent. Without loss of generality, we may assume $t = \infty$. Let Γ_∞ and $\pm \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$ be as before. Define three sets L , K , and V as follows:

$$\begin{aligned} L &= \{z \in \mathbb{C} \mid \text{Im}(z) = u\}, \\ K &= \{z \in L \mid 0 \leq \text{Re}(z) \leq |h|\}, \\ V &= \{z \in \mathbb{H}^* \mid \text{Im} > u\}, \end{aligned}$$

where u is a positive number. Since K is compact, we can find by Lemma 1.5, a neighbourhood U of s so that $K \cap \Gamma U = \emptyset$. We may assume that the boundary of U is a circle tangent to the real line \mathbb{R} . Let us show that $V \cap \Gamma U = \emptyset$. Assume, on the contrary, that $\gamma(U) \cap V \neq \emptyset$ for some $\gamma \in \Gamma$. Since $\gamma(s) \neq \infty$, the boundary of $\gamma(U)$ is a circle tangent to \mathbb{R} . Therefore, if $\gamma(U) \cap V \neq \emptyset$, then $\gamma(U) \cap L \neq \emptyset$, hence $\gamma(U)$ intersect some translation of K by an element of Γ_∞ , i.e., there exists an element δ of Γ_∞ such that $\gamma(U) \cap \delta(K) \neq \emptyset$. Then $\delta^{-1}\gamma(U) \cap K \neq \emptyset$, a contradiction. This completes the proof. \square

The example of the modular group $\text{SL}(2, \mathbb{Z})$

Now we shall studying the modular group $\text{SL}(2, \mathbb{Z})$. It is clear that $\text{SL}(2, \mathbb{Z})$ is a discrete subgroup of $\text{SL}(2, \mathbb{R})$. Let us determine its cusps and elliptic points. First let us show that the cusps of $\Gamma = \text{SL}(2, \mathbb{Z})$ are exactly the points in $\mathbb{Q} \cup \{\infty\}$. It is clear that ∞ is a fixed point under the parabolic element $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ of Γ . If $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a parabolic element of Γ , it has only a fixed point s . If s is finite, it satisfies

$$cs^2 + (d - a)s - b = 0, \quad c \neq 0.$$

Since the discriminant of this equation vanishes, s must be contained in \mathbb{Q} . Conversely, for $p/q \in \mathbb{Q}$ with $p \in \mathbb{Z}$, $q \in \mathbb{Z}$, $(p, q) = 1$, take integers t and u so that $pt - qu = 1$. Then $\sigma = \begin{pmatrix} p & u \\ q & t \end{pmatrix} \in \Gamma$, and $\sigma(\infty) = p/q$. Since the image of a cusp under any element of Γ is a cusp, this shows that all points of $\mathbb{Q} \cup \{\infty\}$ are cusps of Γ . Moreover we have shown that all cusps are equivalent to the cusp at ∞ . Thus $\Gamma \backslash \mathbb{H}^* = (\Gamma \backslash \mathbb{H}) \cup \{\infty\}$.

Next let us determine the elliptic points of $SL(2, \mathbb{Z})$. If σ is an elliptic element of $SL(2, \mathbb{Z})$, $|\text{tr}(\sigma)|$ is an integer and < 2 by Proposition 1.1. Therefore the characteristic polynomial of σ is either $x^2 + 1$ or $x^2 \pm x + 1$ so that $\sigma^4 = 1$ or $\sigma^6 = 1$, and $\sigma^2 \neq 1$. Thus for the determination of elliptic elements, it is sufficient to consider the cases $\sigma^4 = 1$ and $\sigma^3 = 1$. With a bit of calculation we find (for details see [Shi71])

$\sigma^4 = 1$: Every elliptic element σ in $SL(2, \mathbb{Z})$ of order 4 is conjugate to $\pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ in $SL(2, \mathbb{Z})$. Therefore every elliptic point of order 2 is equivalent to the fixed point of $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, that is i .

$\sigma^3 = 1$: Every elliptic element σ in $SL(2, \mathbb{Z})$ of order 3 is conjugate to either $\tau = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ or $\tau^2 = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$ in $SL(2, \mathbb{Z})$. Thus every elliptic point of order 3 is equivalent to the point $e^{2\pi i/3}$.

It can be shown that every subgroup Γ of $SL(2, \mathbb{R})$ has a **fundamental domain**.

We call F a fundamental domain for $\Gamma \backslash \mathbb{H}$, if

1. F is a connected open subset of \mathbb{H} ,
2. no two points of F are equivalent under Γ ,
3. every point of \mathbb{H} is equivalent to some point of the closure of F under Γ .

It can be verified that the interior F of the set

$$\{w \in \mathbb{C} \mid -1/2 \leq \text{Re}(w) \leq 1/2, |w| \geq 1\}$$

is a fundamental domain for $SL(2, \mathbb{Z})$.

1.2 $\Gamma \backslash \mathbb{H}^*$ as a Riemann surface

By a **Riemann surface**, we shall mean a one-dimensional connected Hausdorff space M on which there is defined a complex structure S with the following properties:

1. S is a collection of pairs (U_α, p_α) with α in a set A of indices, where $\{U_\alpha\}_{\alpha \in A}$ is an open covering of M , and p_α is a homeomorphism of U_α onto an open subset of \mathbb{C} .
2. If $U_\alpha \cap U_\beta \neq \emptyset$, the map

$$p_\beta \circ p_\alpha^{-1}: p_\alpha(U_\alpha \cap U_\beta) \rightarrow p_\beta(U_\alpha \cap U_\beta)$$

is holomorphic.

Let us now define a complex structure on $\Gamma \backslash \mathbb{H}^*$. Denote by ϕ the natural projection map of \mathbb{H}^* to $\Gamma \backslash \mathbb{H}^*$. For each $v \in \mathbb{H}^*$, put

$$\Gamma_v = \{\gamma \in \Gamma \mid \gamma(v) = v\}.$$

By Lemma 1.4, there exists an open neighbourhood U of v such that

$$\Gamma_v = \{\gamma \in \Gamma \mid \gamma(U) \cap U \neq \emptyset\}.$$

Then we have a natural injection $\Gamma_v \backslash U \rightarrow \Gamma \backslash \mathbb{H}^*$, and $\Gamma_v \backslash U$ is an open neighbourhood of $\phi(v)$ in $\Gamma \backslash \mathbb{H}^*$. If v is neither an elliptic point nor a cusp, Γ_v contains only 1 and possibly -1, so that the map $\phi: U \rightarrow \Gamma_v \backslash U$ is a homeomorphism. We take $(\Gamma_v \backslash U, \phi^{-1})$ as a member of the complex structure of $\Gamma \backslash \mathbb{H}^*$.

Next assume that v is an elliptic point, and denote by $\bar{\Gamma}_v$ the transformation group $\Gamma_v \cdot \{\pm 1\} / \{\pm 1\}$. Let λ be a holomorphic isomorphism of \mathbb{H} onto the unit disc D such that $\gamma(v) = 0$. If $\bar{\Gamma}_v$ is of order n , then $\lambda \bar{\Gamma}_v \lambda^{-1}$ consists of the transformations

$$w \mapsto \zeta^k w, \quad k = 0, 1, \dots, n-1, \quad \zeta = e^{2\pi i/n}.$$

Then we can define a map $p: \Gamma_v \backslash U \rightarrow \mathbb{C}$ by $p(\phi(z)) = \lambda(z)^n$. We see that p is a homeomorphism onto an open subset of \mathbb{C} . Thus we include $(\Gamma_v \backslash U, p)$ in our complex structure.

Let s be a cusp of Γ , and let ρ be an element of $\text{SL}(2, \mathbb{R})$ such that $\rho(s) = \infty$.

Then

$$\rho \Gamma_s \rho^{-1} \cdot \{\pm 1\} = \left\{ \pm \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}^m \mid m \in \mathbb{Z} \right\}$$

with a positive number h . Then we can define a homeomorphism p of $\Gamma_s \backslash U$ into an open subset of \mathbb{C} by $p(\phi(z)) = \exp[2\pi i \rho(z)/h]$, and include $\Gamma_s \backslash U$ in our complex structure. It is now easy to check the condition (2) for our complex structure.

Let us recall some basic properties of a compact Riemann surface M . If $H_i(M, \mathbb{Z})$ denotes the i -dimensional homology group of M with coefficient in \mathbb{Z} , then we have:

$$\begin{aligned} H_0(M, \mathbb{Z}) &\cong \mathbb{Z}, \\ H_1(M, \mathbb{Z}) &\cong \mathbb{Z}^{2g}, \\ H_2(M, \mathbb{Z}) &\cong \mathbb{Z}, \\ H_p(M, \mathbb{Z}) &= 0 \quad \text{for } p > 2. \end{aligned}$$

The non-negative integer g is called the **genus** of M . The **Euler characteristic** χ of M is defined by

$$\chi = \sum_{p=0}^2 (-1)^p \dim H_p(M, \mathbb{Z}) = 2 - 2g.$$

If we take a triangulation of M and let c_p denote the number of p -simplexes then $\chi = c_0 - c_1 + c_2$.

Let M and M' be two compact Riemann surfaces and $f: M' \rightarrow M$ a holomorphic mapping. Then f is either constant or surjective. Suppose f is surjective, then (M', f) is called a **covering** of M . If $z_0 \in M'$, $w_0 = f(z_0)$, and if u and t are local parameters at z_0 and w_0 , respectively, which map z_0 and w_0 to the origin, then we can express f in the form

$$t(f(z)) = a_e u(z)^e + a_{e+1} u(z)^{e+1} \dots, \quad a_e \neq 0$$

in a neighbourhood of z_0 , with a positive integer e . The integer e is independent of the choice of u and t , and called the **ramification index** of the covering (M', f) at z_0 . There are only finitely many, say h , inverse images of w_0 by f . If e_1, \dots, e_h are their respective ramification indices, the number

$$n = e_1 + \dots + e_h$$

depends only on M , M' and f , and is independent of w_0 . We call n the **degree** of the covering. If g and g' are the genera of M and M' , respectively, then these integers are connected by **Hurwitz formula**

$$2g' - 2 = n(2g - 2) + \sum_{z \in M'} (e_z - 1),$$

where e_z is the ramification index at z .

Endowed with the complex structure defined above, $\Gamma \backslash \mathbb{H}^*$ becomes a compact Riemann surface. If Γ' is a subgroup of Γ of finite index, the natural map $\Gamma' \backslash \mathbb{H}^* \rightarrow \Gamma \backslash \mathbb{H}^*$ defines a covering in the above sense. Let $\bar{\Gamma}$ and $\bar{\Gamma}'$ denote the images of Γ and Γ' by the natural map

$$\mathrm{SL}(2, \mathbb{R}) \longrightarrow \mathrm{SL}(2, \mathbb{R}) / \{\pm 1\}.$$

Then the degree of the covering is exactly $[\bar{\Gamma} : \bar{\Gamma}']$.

For every $z \in \mathbb{H}^*$, put

$$\bar{\Gamma}_z = \{\gamma \in \bar{\Gamma} \mid \gamma(z) = z\}, \quad \bar{\Gamma}'_z = \bar{\Gamma}_z \cap \bar{\Gamma}'.$$

Consider a commutative diagram

$$\begin{array}{ccc} \mathbb{H}^* & \xrightarrow{\text{identity}} & \mathbb{H}^* \\ \phi' \downarrow & & \downarrow \phi \\ \Gamma' \backslash \mathbb{H}^* & \xrightarrow{f} & \Gamma \backslash \mathbb{H}^* \end{array}$$

where each map is a natural projection. Let $z \in \mathbb{H}^*$, $p = \phi(z)$, and $f^{-1}(p) = (q_1, \dots, q_h)$. Choose points w_k of \mathbb{H}^* so that $q_k = \phi'(w_k)$.

Proposition 1.3. *The ramification index e_k of f at q_k is $[\bar{\Gamma}_{w_k} : \bar{\Gamma}'_{w_k}]$. Moreover, if $w_k = \sigma_k(z)$ with $\sigma_k \in \bar{\Gamma}$, then $e_k = [\bar{\Gamma}_z : \sigma_k^{-1} \bar{\Gamma}' \sigma_k \cap \bar{\Gamma}_z]$, and $\bar{\Gamma} = \bigcup_{k=1}^h \bar{\Gamma}' \sigma_k \bar{\Gamma}_z$. Especially if $\bar{\Gamma}'$ is a normal subgroup of $\bar{\Gamma}$, then $e_1 = \cdots = e_h$ and $[\bar{\Gamma} : \bar{\Gamma}'] = e_1 h$.*

1.3 The Riemann Surfaces $Y(N)$ and $X(N)$

For every positive integer N , put

$$\begin{aligned} \Gamma(N) &= \{\gamma \in \mathrm{SL}(2, \mathbb{Z}) \mid \gamma \equiv 1_2 \pmod{(N)}\} \\ &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}) \mid a \equiv d \equiv 1, b \equiv c \equiv 0 \pmod{N\mathbb{Z}} \right\}. \end{aligned}$$

Then $\Gamma(N)$ is a normal subgroup of $\mathrm{SL}(2, \mathbb{Z})$ and called **the principal congruence subgroup of level N** . From now on we make the assumption $N \geq 3$.

Lemma 1.6. *If $f: \mathrm{SL}(2, \mathbb{Z}) \rightarrow \mathrm{SL}(2, \mathbb{Z}/N\mathbb{Z})$ is defined by $f(\alpha) = \alpha \pmod{(N)}$, then the sequence*

$$1 \longrightarrow \Gamma(N) \longrightarrow \mathrm{SL}(2, \mathbb{Z}) \longrightarrow \mathrm{SL}(2, \mathbb{Z}/N\mathbb{Z}) \longrightarrow 1$$

is exact.

Proof. The only non-trivial point is the surjectivity of f . We shall prove more generally that the map $\mathrm{SL}(m, \mathbb{Z}) \rightarrow \mathrm{SL}(m, \mathbb{Z}/N\mathbb{Z})$ is surjective for any positive integer m , i.e., if $A \in \mathrm{GL}(m, \mathbb{Z})$ and $\det(A) \equiv 1 \pmod{(N)}$, then $A \equiv B \pmod{(N)}$ for some $B \in \mathrm{SL}(m, \mathbb{Z})$. If $m = 1$, this is obvious. Therefore assume the assertion to be true for $m - 1$, and $m > 1$. Now for such an A , by elementary division theory, we can find two elements U and V of $\mathrm{SL}(m, \mathbb{Z})$ such that UAV is a diagonal matrix. Let a_1, \dots, a_m be the diagonal elements of UAV , and $b = a_2 \cdots a_m$. Put

$$W = \begin{pmatrix} b & 1 & & & \\ b-1 & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}, \quad X = \begin{pmatrix} 1 & -a_2 & & & \\ 0 & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}, \quad \text{and}$$

$$A' = \begin{pmatrix} 1 & 0 & & & \\ 1-a_1 & a_1 a_2 & & & \\ & & a_3 & & \\ & & & \ddots & \\ & & & & a_m \end{pmatrix}$$

Since $a_1b = \det(A) \equiv 1 \pmod{N}$, we see that $WUAVX \equiv A' \pmod{N}$. By the induction assumption, there exists an element C of $\mathrm{SL}(m-1, \mathbb{Z})$ such that

$$C \equiv \begin{pmatrix} a_2a_2 & & & \\ & a_3 & & \\ & & \ddots & \\ & & & a_m \end{pmatrix} \pmod{N}.$$

Put

$$B = U^{-1}W^{-1} \begin{pmatrix} 1 & 0 \\ 1 - a_1 & C \\ 0 & \end{pmatrix} X^{-1}V^{-1}.$$

Then B has the required property. \square

If $N = \prod_p p^e$ is the decomposition on N into the product of power of distinct primes p , we see that

$$\begin{aligned} \mathbb{Z}/N\mathbb{Z} &\cong \prod_p (\mathbb{Z}/p^e\mathbb{Z}), \\ \mathrm{GL}(2, \mathbb{Z}/N\mathbb{Z}) &\cong \prod_p \mathrm{GL}(2, \mathbb{Z}/p^e\mathbb{Z}), \\ \mathrm{SL}(2, \mathbb{Z}/N\mathbb{Z}) &\cong \prod_p \mathrm{SL}(2, \mathbb{Z}/p^e\mathbb{Z}). \end{aligned}$$

Now consider an exact sequence

$$1 \longrightarrow X \longrightarrow \mathrm{GL}(2, \mathbb{Z}/p^e\mathbb{Z}) \longrightarrow \mathrm{GL}(2, \mathbb{Z}/p\mathbb{Z}) \longrightarrow 1.$$

Since X consists of the elements of $\mathrm{GL}(2, \mathbb{Z}/p^e\mathbb{Z})$ which are congruent to 1_2 modulo (p) , the order of X is $p^{4(e-1)}$. It is well known that the order of $\mathrm{GL}(2, \mathbb{Z}/p\mathbb{Z})$ is $(p^2 - 1)(p^2 - p)$. Therefore,

$$\begin{aligned} \text{the order of } \mathrm{GL}(2, \mathbb{Z}/p^e\mathbb{Z}) &= p^{4(e-1)}(p^2 - 1)(p^2 - p) \\ &= p^{4e}(1 - p^{-1})(1 - p^{-2}), \\ \text{the order of } \mathrm{SL}(2, \mathbb{Z}/p^e\mathbb{Z}) &= p^{3e}(1 - p^{-2}). \end{aligned}$$

By Lemma 1.6, we obtain

$$[\Gamma(1) : \Gamma(N)] = N^3 \cdot \prod_{p|N} (1 - p^{-2}).$$

Since $-1_2 \in \Gamma(2)$ and $-1 \notin \Gamma(N)$ for $N > 2$, we find

$$[\bar{\Gamma}(1) : \bar{\Gamma}(N)] = \begin{cases} (N^3/2) \cdot \prod_{p|N} (1 - p^{-2}) & \text{if } N > 2 \\ 6 & \text{if } N = 2. \end{cases}$$

Proposition 1.4. *If $N > 1$, $\Gamma(N)$ has no elliptic element.*

Proof. We have seen that every elliptic element of $\Gamma(1) = \mathrm{SL}(2, \mathbb{Z})$ is conjugate to one of the following elements:

$$\pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \pm \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \quad \pm \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}.$$

None of these is congruent to 1_2 modulo (N) if $N > 1$. Since $\Gamma(N)$ is normal subgroup of $\Gamma(1)$, we obtain our proposition. \square

Let us now denote the topological quotient space $\Gamma \backslash \mathbb{H} = Y(N)$. $Y(N)$ is a non-compact Riemann surface, which can be compactified by adding a finite number of points, the cusps to get $X(N) = \Gamma \backslash \mathbb{H}^*$. Let us have a look more deeply about the compactification of $Y(N)$. The usual procedure for $Y(N)$ is to note that, since it is one-dimensional, there is a unique non-singular complete algebraic curve $X(N)$ such that $Y(N) = X(N) \setminus \{\text{finite set}\}$. Then, from the theory of algebraic surfaces, one can also find a canonical $E(N)$: the unique so-called non singular relatively minima model over $X(N)$. We seek to describe $X(N)$ and $E(N)$ by more "scissors and glue" construction involving torus embeddings, to do this we are going to follow [AS10].

We deal first with the cusp ∞ . Consider the subgroup

$$\Gamma(N)_\infty = \left\{ \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \mid h \equiv 0 \pmod{(N)} \right\}$$

and factor $\pi: \mathbb{H} \rightarrow \Gamma \backslash \mathbb{H} = Y(N)$ via

$$\begin{array}{ccc} \mathbb{H} & \xrightarrow{\exp} & \mathring{\Delta} \\ & \searrow \pi & \swarrow \pi' \\ & Y(N) & \end{array}$$

where q is the coordinate on $\mathring{\Delta}$ and \exp is defined by

$$q = e^{2\pi iw/N}.$$

This makes $\mathring{\Delta}$ isomorphic to $\Gamma(N)_\infty \backslash \mathbb{H}$, hence π factors via \exp . Moreover, define

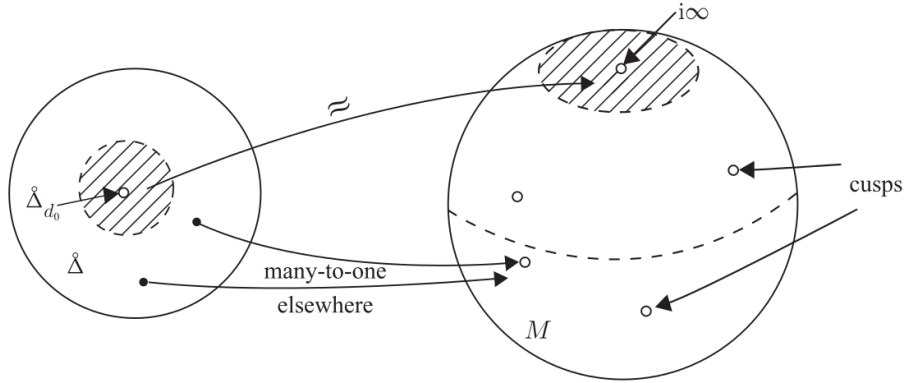
$$\begin{aligned} \mathbb{H}_d &= \{w \mid \mathrm{Im}(w) \geq d\}, \\ \mathring{\Delta}_d &= \{q \mid 0 < |q| \leq e^{-2\pi d/N}\}. \end{aligned}$$

Then $\mathbb{H}_d = \exp^{-1}(\mathring{\Delta}_d)$ and $\mathring{\Delta}_d \cong \Gamma(N)_\infty \backslash \mathbb{H}$. The following lemma is easy to check.

Lemma 1.7. *There exists d_0 such that, for all $w \in \mathbb{H}$, $\gamma \in \Gamma(N)$,*

$$w \text{ and } \gamma(w) \in \mathbb{H}_{d_0} \Rightarrow \gamma \in \Gamma(N)_\infty.$$

Therefore $\text{res}\pi'$ maps $\mathring{\Delta}_{d_0}$ injectively to $Y(N)$:



Moreover, as $d \rightarrow \infty$, it is well known that the sets $\pi(\mathbb{H}_d) \subset Y(N)$ are a fundamental system of neighborhoods of the cusp ∞ . Therefore we find that we can glue via this map by taking $Y(N)$ plus

$$\Delta_{d_0} = \{q \mid |q| \leq e^{-2\pi d_0/N}\}$$

and identifying them via $\text{res}\pi'$ on $\mathring{\Delta}_{d_0}$.

Next, every rational point p/q also defines a cusp of $Y(N)$, except that p/q and $\gamma(p/q) \in \Gamma(N)$ define the same cusp, now we can use the fact that $SL(2, \mathbb{Z})$ acts transitively on the set of rational points plus ∞ , hence $\Gamma(N) \backslash SL(2, \mathbb{Z})$ acts on $Y(N)$ and permutes transitively all its cusps. Thus, if we know how to fill in one, we can fill in the others by acting by $\Gamma(N) \backslash SL(2, \mathbb{Z})$.

Let us now find the ramification indices of the covering

$$X(N) \longrightarrow X(1).$$

Let ϕ_N denote the projection map of \mathbb{H}^* to $\Gamma(N) \backslash \mathbb{H}^*$. The ramification index at $\phi_N(z)$, for $z \in \mathbb{H}^*$, is $[\bar{\Gamma}(1)_z : \bar{\Gamma}(N)_z]$. Therefore the ramification index at $\phi_N(z)$ is 2 or 3 accordingly. Furthermore, putting

$$\mu_N = [\bar{\Gamma}(1) : \bar{\Gamma}(N)],$$

we see that the number of points on $X(N)$ lying above $\phi_1(z)$ is $\mu_N/2$ or $\mu_N/3$ accordingly.

If s is a cusp, s is $\Gamma(1)$ -equivalent to ∞ . Now we have

$$\begin{aligned} \bar{\Gamma}(1)_\infty &= \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^m \mid m \in \mathbb{Z} \right\}, \\ \bar{\Gamma}(N)_\infty &= \bar{\Gamma}(N) \cap \bar{\Gamma}(1)_\infty = \left\{ \begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix}^m \mid m \in \mathbb{Z} \right\}, \end{aligned}$$

so that $[\bar{\Gamma}(1)_\infty : \bar{\Gamma}(N)_\infty] = N$. Therefore $\Gamma(N)$ has exactly μ_N/N inequivalent cusps.

Proposition 1.5. *Let $\bar{\Gamma}'$ be a subgroup of $\bar{\Gamma}(1)$ of index μ , and ν_2, ν_3 the numbers of $\bar{\Gamma}'$ -inequivalent elliptic points of order 2, 3, respectively. Further let ν_∞ be the number of $\bar{\Gamma}'$ -inequivalent cusps. Then the genus of $\bar{\Gamma}' \backslash \mathbb{H}^*$ is given by*

$$g = 1 + \frac{\mu}{12} - \frac{\nu_2}{4} - \frac{\nu_3}{3} - \frac{\nu_\infty}{2}.$$

Proof. Consider the covering $\bar{\Gamma}' \backslash \mathbb{H}^* \rightarrow \bar{\Gamma}(1) \backslash \mathbb{H}^*$. Let e_1, \dots, e_t be the ramification indices at the points of $\bar{\Gamma}' \backslash \mathbb{H}^*$ lying above $\phi_1(e^{2i\pi/3})$. Then $\mu = e_1 + \dots + e_t$, and e_i is 1 or 3. The number of i for which $e_i = 1$ is ν_3 . If $t = \nu_3 + \nu'_3$, we have $\mu = \nu_3 + 3\nu'_3$, so that $\sum_{i=1}^t e_i = \mu - t = 2\nu'_3 = 2(\mu - \nu_3)/3$. Similarly, if e_P is the ramification index at a point P of $\bar{\Gamma}' \backslash \mathbb{H}^*$, we have

$$\begin{aligned} \sum (e_P - 1) &= (\mu - \nu_2)/2 \quad (P \text{ lying above } \phi_1(i)), \\ \sum (e_P - 1) &= \mu - \nu_\infty \quad (P \text{ lying above } \phi_1(\infty)). \end{aligned}$$

We obtain our assertion from Hurwitz formula. □

In the case $\bar{\Gamma}' = \bar{\Gamma}(N)$, we have $\nu_2 = \nu_3 = 0$ if $N > 1$, and $\nu_\infty = \mu_N/N$. Thus we obtain the formula for the genus g_N of $\Gamma(N) \backslash \mathbb{H}^*$:

$$g_N = 1 + \mu_N \cdot (N - 6)/12N \quad (N > 1).$$

1.4 $\tilde{E}(N)$ and its compactification $E(N)$

The assumption that we took $N \geq 3$, guarantees that $Y(N)$ is a fine moduli space. Thus, there exists a universal family, $\tilde{E}(N) \rightarrow Y(N)$, whose fibers are elliptic curves. To construct $\tilde{E}(N)$, let $A\Gamma(N) = \Gamma(N) \ltimes \mathbb{Z}^2$, where $\Gamma(N)$ acts on \mathbb{Z}^2 in the obvious way:

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \underline{l} = \begin{pmatrix} m \\ n \end{pmatrix} \mapsto \begin{pmatrix} am + bn \\ cm + dn \end{pmatrix}.$$

$A\Gamma(N)$ acts discretely on $\mathbb{C} \times \mathbb{H}$ via

$$(\gamma, \underline{l}) : (z, w) \longrightarrow \left(\frac{z + mw + n}{cw + d}, \frac{aw + b}{cw + d} \right)$$

and $\tilde{E}(N) = A\Gamma(N) \backslash \mathbb{C} \times \mathbb{H}$.

The elliptic surface $\tilde{E}(N)$ can be completed to a non-singular algebraic surface $E(N)$. Now we shall look deeply to the compactification of the universal elliptic curve, to complete the diagram

$$\begin{array}{ccc} \tilde{E}(N) & \hookrightarrow & E(N) \\ \tilde{p} \downarrow & & p \downarrow \\ Y(N) & \hookrightarrow & X(N) \end{array}$$

Now look upstairs at $\mathbb{C} \times \mathbb{H}$. Define

$$A\Gamma(N)_1 = \left(\begin{array}{l} \text{subgroup of } A\Gamma(N) \text{ generated by} \\ (z, w) \mapsto (z + 1, w) \text{ and} \\ (z, w) \mapsto (z, w + k) \end{array} \right) \cong \mathbb{Z}^2;$$

$$A\Gamma(N)_2 = \left(\begin{array}{l} \text{subgroup of } A\Gamma(N) \text{ generated by } A\Gamma(N)_1 \\ \alpha: (z, w) \mapsto (z + w, w) \end{array} \right).$$

Factor $\pi: \mathbb{C} \times \mathbb{H} \rightarrow A\Gamma(N) \backslash (\mathbb{C} \times \mathbb{H}) = \tilde{E}(N)$ via:

$$\begin{array}{ccc} \mathbb{C} \times \mathbb{H} & \xrightarrow{\text{exp}} & \mathbb{C}^* \times \mathring{\Delta}, \\ & \searrow \pi & \swarrow \pi' \\ & & \tilde{E}(N) \end{array}$$

where x is the coordinate on \mathbb{C}^* , and q that on $\mathring{\Delta}$, and where exp is defined by

$$\begin{aligned} x &= e^{2\pi iz}, \\ q &= \exp^{2i\pi w/N}. \end{aligned}$$

This makes $\mathbb{C}^* \times \mathring{\Delta}$ isomorphic to $A\Gamma(N)_1 \backslash (\mathbb{C} \times \mathbb{H})$. Now, $A\Gamma(N)_1$ is a normal subgroup of $A\Gamma(N)_2$ and $A\Gamma(N)_1 \backslash A\Gamma(N)_2 \cong \mathbb{Z}$, with generator α , and $A\Gamma(N)_1 \backslash A\Gamma(N)_2 \cong \mathbb{Z}$ acts on $\mathbb{C}^* \times \mathring{\Delta}$. The previous lemma now gives us:

Lemma 1.8. *There exists d_0 such that, for all $(z, w) \in \mathbb{C} \times \mathbb{H}$ and $\gamma \in A\Gamma$,*

$$(z, w) \text{ and } \gamma(z, w) \in \mathbb{C} \times \mathbb{H}_{d_0} \Rightarrow \gamma \in A\Gamma(N)_2.$$

Therefore,

$$\text{res } \pi': (\mathbb{C}^* \times \mathring{\Delta}_{d_0}) / \{\alpha^n\} \rightarrow \tilde{E}(N)$$

is injective. To compactify $\tilde{E}(N)$ over $\infty \in X(N)$, it suffices to enlarge $\mathbb{C}^* \times \mathring{\Delta}_{d_0}$ to an analytic manifold M over Δ_{d_0} , equivalently with respect to the action of α and so that, $\text{mod } \alpha$, we get a manifold proper over Δ_{d_0} :

$$\begin{array}{ccc} \mathbb{C}^* \times \mathring{\Delta}_{d_0} \hookrightarrow & M & \\ \downarrow & \downarrow & \searrow \alpha \text{ acting} \\ & & M / \{\alpha^n\} \\ & & \swarrow \text{proper} \\ \mathring{\Delta}_{d_0} \hookrightarrow & \Delta_{d_0} & \end{array}$$

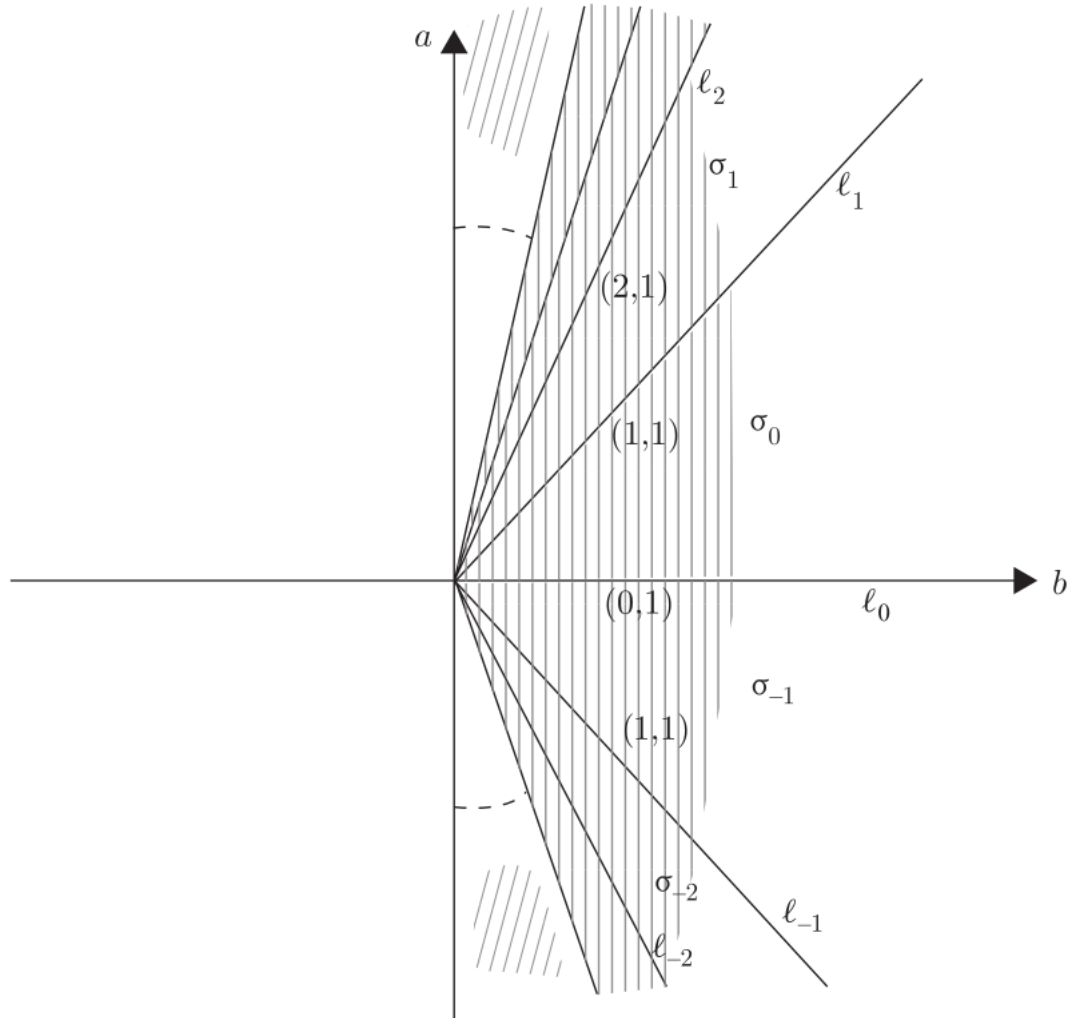
We can think of $\mathbb{C}^* \times \mathring{\Delta}$ as an open subset of the two-dimensional torus $\mathbb{C}^* \times \mathbb{C}^*$ (with coordinates x, q). Thus α acts on whole torus by

$$(x, q) \mapsto (q^N x, q).$$

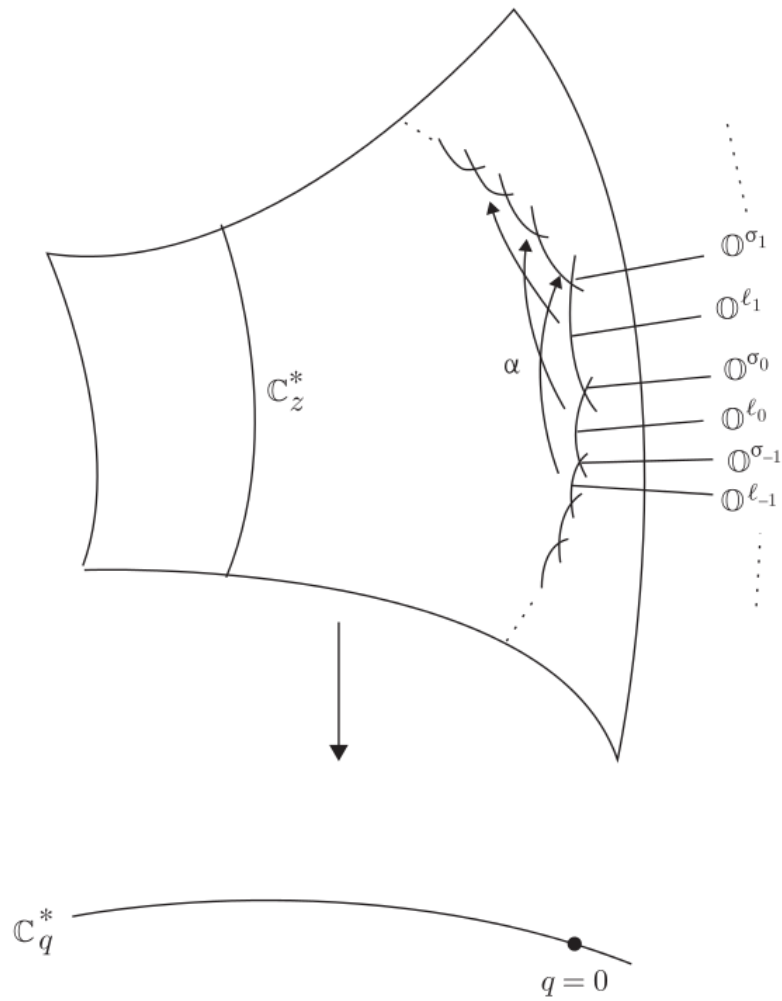
We shall construct a torus embedding $\mathbb{C}^* \times \mathbb{C}^* \subset X_{\{\sigma_\alpha\}}$. In fact, we can identify $N(\mathbb{C}^* \times \mathbb{C}^*)$ with $\mathbb{Z} \times \mathbb{Z}$ and note that α acts on $N(\mathbb{C}^* \times \mathbb{C}^*)$ by

$$(a, b) \mapsto (a + kb, b).$$

We choose $\{\sigma_\alpha\}$ be the following infinite chain $\sigma_n, n \in \mathbb{Z}$:



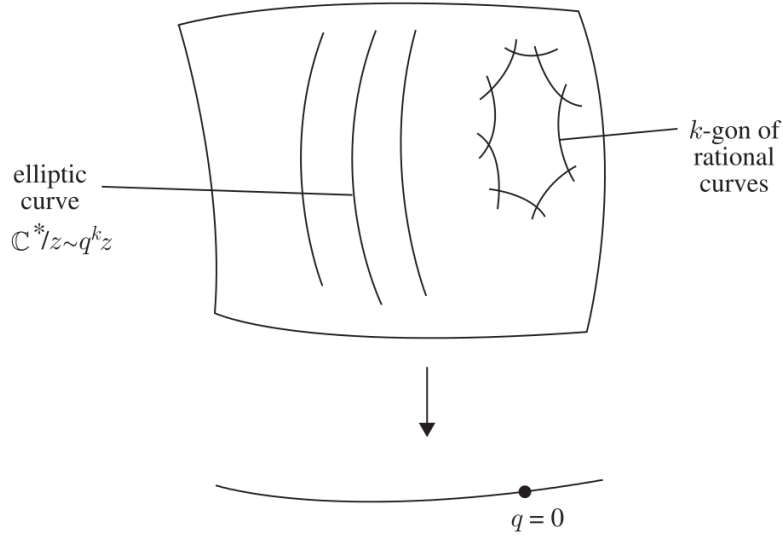
Note that α carries σ_n to σ_{n+k} , so that, $\text{mod } \alpha$, there are only finitely many σ . The corresponding $E(N)_{\{\sigma_n\}}$ may be pictured as follows:



Clearly α acts on $\tilde{E}(N)_{\{\sigma_n\}}$. Since each σ_i is generated by a basis of $\mathbb{Z} \times \mathbb{Z}$, it follows that $E(N)_{\{\sigma_n\}}$ is a manifold, i.e., smooth. Moreover, a whole neighbourhood of the boundary $E(N)_{\{\sigma_n\}}/(\mathbb{C}^* \times \mathbb{C}^*)$ is contained in $\mathbb{C}^* \times \mathring{\Delta}_{d_0}$, so define M to be

$$\begin{aligned} M &= \text{interior of closure of } \mathbb{C}^* \times \mathring{\Delta}_{d_0} \text{ in } \tilde{E}(N)_{\{\sigma_n\}} \\ &= (\mathbb{C}^* \times \mathring{\Delta}_{d_0}) \cup (\tilde{E}(N)_{\{\sigma_n\}} \setminus (\mathbb{C}^* \times \mathbb{C}^*)). \end{aligned}$$

What happens when we divide by α ? Clearly α does not act discontinuously on the whole of $\mathbb{C}^* \times \mathbb{C}^*$ so we cannot form $(\mathbb{C}^* \times \mathbb{C}^*)/\{\alpha^n\}$. However, it can be checked to act discontinuously on M , and the quotient looks.



We can now define that part of $E(N)$ that lies over $\infty \in X(N)$ by glueing $\tilde{E}(N)$ and $M/\{\alpha^n\}$ together on the common open set $(\mathbb{C}^* \times \Delta_{d_0})/\{\alpha^n\}$. As before, we can take care of the other cusps by pushing this boundary around by $\Gamma(N) \backslash SL(2, \mathbb{Z})$. This gives us a compact non-singular surface $E(N)$, proper over $X(N)$, with its fibers elliptic curves over $Y(N)$ and rational N -gons over the cusps. In this way, we find an analytic construction not only of $Y(N)$ and $\tilde{E}(N)$, but also of their natural completions $X(N)$ and $E(N)$.

As the last topic of this chapter, it is to look more in details about the **group of sections**. Before the main statement, we have to do a bit of work (for more details see §9 [Kod63a].

As we have seen before

$$\tilde{E}(N) = p^{-1}(Y(N)) = A\Gamma(N) \backslash \mathbb{H} \times \mathbb{C}$$

It is clear that the formula

$$((w, z_1)) - ((w, z_2)) = ((w, z_1 - z_2))$$

defines on $\tilde{E}(N)$ a structure of analytic fibre system of abelian groups over $Y(N)$.

We define $E^\sharp(N)$ to be the open subset of $E(N)$ consisting of all points z satisfying

$$|\partial\tau_a(p(z))/\partial z_1| + |\partial\tau_a(p(z))/\partial z_2| > 0, \quad (1.1)$$

where we denote by τ_a the local uniformization variable on $X(N)$ with the center $a \in X(N)$, and $\tau_a(u)$ the value of τ_a at a point u in a neighbourhood of a , finally we denote by (z_1, z_2) a local coordinate of a point z in $E(N)$.

We write the fibre of $E(N)$ over a cusp a_ρ in the form

$$C_{a_\rho} = \sum_s \Theta_{\rho s}$$

and let

$$C_{a_\rho}^\sharp = \bigcup_s \Theta_{\rho s}^\sharp, \quad \Theta_{\rho s}^\sharp = E^\sharp(N) \cap \Theta_{\rho s}.$$

Obviously we have

$$E^\sharp(N) = \bigcup_{\rho} C_{a_\rho}^\sharp \cup \tilde{E}(N).$$

Theorem 1.2. *There exists on $E^\sharp(N)$ a unique structure of analytic fibre system of abelian groups over $X(N)$ which is an extension of the structure on $\tilde{E}(N)$ of analytic fibre system of abelian groups over $Y(N)$ define by the formula 1.1. The analytic group structures of the fibres $C_{a_\rho}^\sharp$ of $E^\sharp(N)$ is $\mathbb{C}^* \times \mathbb{Z}_N$.*

Following §5 [Shi72], let C_v be a singular fibre of type I_N , then $\Theta_{v,0}^\sharp$ is a multiplicative group an the quotient group $C^\sharp/\Theta_{v,0}^\sharp$ is a cyclic group of order N .

Proposition 1.6. *Let us consider the fibre $F = p^{-1}(u)$ of $E(N)$ over the generic point u of $X(N)$ and let $F(\mathbb{C})$ be the group of \mathbb{C} -rational points on F . Finally let consider $F(\mathbb{C})_0$ the subgroup of $F(\mathbb{C})$ consisting of s such that $s(v) \in \Theta_{v,0}^\sharp$ for all cusps v . Then $F(\mathbb{C})_0$ is a torsion-free subgroup of finite index in $F(\mathbb{C})$.*

Proof. Suppose s is an element of $F(\mathbb{C})_0$ of finite order $n > 1$. Applying Lemma 2.2 to the divisor $D = n[(s) - (o)]$, we get

$$n[(s) - (o)] \approx n([(s) - (o)](o))C_{u_0},$$

since D does not meet $\Theta_{v,i}$, for $i \geq 1$. By taking the intersection number of both side with the divisor (s) , we have

$$((s)(s)) + ((o)(o)) = 2((s)(o)) \geq 0.$$

This contradicts the fact that $((s)(s)) = ((o)(o)) = -(p_a + 1) < 0$ (Proposition 2.2).

Hence $F(\mathbb{C})_0$ is torsion-free. It is clear that $F(\mathbb{C})_0$ is a subgroup of finite index in $F(\mathbb{C})$. \square

We can give now a more precise result. For brevity, we denote by $\mathcal{S}(E(N))$ the group of global holomorphic sections of $E(N)$ over the base curve $X(N)$ ($\mathcal{S}(E(N)) = H^0(X(N), \Omega)$).

Theorem 1.3. *Let $E(N)$ be the universal elliptic curve attached to $\Gamma(N)$.*

If $\Gamma(N)$ is torsion-free and all cusps are of the first kind, then the group of sections $\mathcal{S}(E(N))$ is isomorphic to a subgroup of $\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$.

Proof. This Theorem is a immediate consequence of Proposition 1.6, i.e. the injectivity of the homomorphism

$$\mathcal{S}(E(N)) \longrightarrow \prod_v C_v^\sharp/\Theta_{v,0}^\sharp \cong \prod \mathbb{Z}/N\mathbb{Z}.$$

This completes the proof. \square

Theorem 1.4. *For the universal elliptic curve $E(N)$ of level N , the group of sections $\mathcal{S}(E(N))$ of $E(N)$ over the base curve $X(N)$ consists of N^2 sections of order N .*

Proof. By Theorem 1.3, $\mathcal{S}(E(N))$ is isomorphic to a subgroup of $(\mathbb{Z}/N\mathbb{Z})^2$, since all singular fibres of $E(N)$ are of type I_N . Hence we have only to prove that $E(N)$ admits (at least) N^2 sections. As we have seen above, $\tilde{E}(N)$ is the quotient $\mathbb{H} \times \mathbb{C}$ by the group of automorphisms of the form:

$$(\gamma, \underline{m}): (z, w) \longrightarrow \left(\frac{z + mw + n}{cw + d}, \frac{aw + b}{cw + d} \right).$$

We denote by $((w, z))$ the image of $(w, z) \in \mathbb{H} \times \mathbb{C}$ in $\tilde{E}(N)$. Note that $\mathbb{H} \times \mathbb{C}$ (\mathbb{H}) is the universal covering of $\tilde{E}(N)$ (or $Y(N)$).

Now if s' is a holomorphic section of $\tilde{E}(N)$ over $Y(N)$, then s' is induced by a holomorphic map

$$f: \mathbb{H} \longrightarrow \mathbb{H} \times \mathbb{C}, \quad f(w) = (w, z), \quad (1.2)$$

such that, for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(N)$, we have

$$z(\gamma \cdot w) = \frac{z + mw + n}{cw + d}. \quad (1.3)$$

Two function $z(w)$ and $z'(w)$ satisfying 1.3 induce the same s' if and only if

$$z'(w) = z(w) + m_1 w + m_2, \quad m_1, m_2 \in \mathbb{Z}$$

In particular s' is a section of finite order if and only if we have

$$z(w) = a_1 w + a_2 \quad (1.4)$$

with rational numbers a_1, a_2 with the property:

$$(a_1, a_2)(\gamma - 1) \in \mathbb{Z} \oplus \mathbb{Z} \quad \text{for all } \gamma \in \Gamma(N). \quad (1.5)$$

In our case we can see that the condition 1.5 is equivalent to

$$a_1 = \frac{m_1}{N}, \quad a_2 = \frac{m_2}{N}$$

with integers m_1 and m_2 . Hence we get N^2 section s'_m of $\tilde{E}(N)$ over $Y(N)$:

$$s'_m: Y(N) = \Gamma(N) \backslash \mathbb{H} \ni (w) \longmapsto \left(\left(z, \frac{m_1 w + m_2}{N} \right) \right), \quad (1.6)$$

where $m = (m_1, m_2)$ runs over pairs of integers mod N . We shall show that each s'_m can be extended to a holomorphic section

s_m of $E(N)$ over $X(N)$. To examine the behaviour of s'_m at the cusp v of $X(N)$, we may assume that v is the cusp at the infinity v_0 , because any cusp can be transformed to v_0 by a modular transformation. We put $v = v_0$ and

$$q = e^{2\pi iw/N}, \quad x = e^{2\pi iz}.$$

Let R be a small neighbourhood of v with the local parameter q . The part C_v^\sharp of the singular fibre C_v (of type I_N) is covered by N open sets W_i ($0 \leq i \leq N-1$) of $E(N)$ with coordinates $((q, x))_i$.

The section s'_m on $R - \{v\}$ can be expressed as

$$q \longmapsto ((q, e^{2\pi i(m_1 w + m_2)/N}))_0.$$

Since we have

$$\begin{aligned} ((q, e^{2\pi i(m_1 w + m_2)/N}))_0 &= ((q, e^{2\pi i m_2/N} q^{m_1}))_0 \\ &= ((q, e^{2\pi i m_2/N}))_{-m_1}, \end{aligned}$$

it is clear that s'_m can be extended to a holomorphic section over R ; in particular we have

$$s_m(v) = ((0, e^{2\pi i m_2/N}))_{-m_1} \in W_{-m_1}.$$

Thus we have proved the existence of N^2 sections of order N of $E(N)$ over $X(N)$. This completes the proof of the Theorem. \square

Chapter 2

Numerical Invariants, Special Curves

In this section we summarize the facts we will use about the universal elliptic curve $E(N)$.

In the elliptic fibration $p: E(N) \rightarrow X(N)$ the singular fibers, as we saw in Chapter 1, are all of Kodaira type I_N (see §6[Kod63a])— an N -gon of -2 curves. The first thing that we want to do, is to construct the canonical bundle $K_{E(N)}$ of the universal elliptic curve. Before that we have to define the numerical invariants (for more details see [BH96] and [Bea96]) that we will use:

1. As we have seen before the number $h^{1,0}(S)$ of holomorphic 1-forms on a Riemann surface S is its **genus** $g(S)$. In general, the number $h^{n,0}(M)$ of holomorphic forms of top degree on a compact complex n -manifold M is called the **geometric genus** of M and denoted p_g .
2. An alternative generalization of the notion of the genus is the number

$$p_a = h^{n,0}(M) - h^{n-1,0}(M) + \cdots + (-1)^{n-1}h^{1,0}(M),$$

called the **arithmetic genus** of M . Using $h^{a,0}(M) = h^{0,a}(M)$ we can also write

$$p_a = (-1)^n(\chi(\mathcal{O}_M) - 1).$$

3. The number $h^{1,0}(M)$ of holomorphic 1-forms on a compact complex manifold M is often denoted $q(M)$ and called the **irregularity** of M .
4. For an arbitrary curve C on a surface M we define the **virtual genus** $\pi(C)$ as

$$\pi(C) = \frac{K \cdot C + C \cdot C}{2} + 1$$

Theorem 2.1. *The canonical bundle $K_{E(N)}$ of $E(N)$ is induced from the complex line bundle $\mathfrak{k} - \mathfrak{f}$ over $X(N)$ by the elliptic fibration p of $E(N)$ onto $X(N)$:*

$$K_{E(N)} = p^*(\mathfrak{k} - \mathfrak{f}), \quad (2.1)$$

where

$$\begin{aligned} \mathfrak{k} &= \text{the canonical bundle of } X(N), \\ \mathfrak{f} &= \text{a line bundle on } X(N). \end{aligned}$$

Proof. We take a sequence of distinct points $u_1, u_2, \dots, u_\nu, \dots$ on $Y(N)$ and consider the divisor

$$C^{(n)} = \sum_{\nu=1}^n C_{u_\nu}$$

on $E(N)$. The virtual genus $\pi'(C^{(n)})$ of $C^{(n)}$ is equal to 1. Therefore using the formula (for the proof see Th.(2.2)[Kod60])

$$\dim |K + C| = \pi'(C) + p_g - q + k + m - 2,$$

where p_g is the **geometric genus**, q is the **irregularity** of $E(N)$, m is the number of connected components of C and k is the dimension of the kernel of the homomorphism

$$r_C^*: H^1(E(N), \Omega) \longrightarrow H^1(C, \Omega_C).$$

In our case we obtain

$$\dim |K_{E(N)} + C^{(n)}| = p_a + k + n - 1, \quad k \geq 0, \quad (2.2)$$

where p_a is the **arithmetic genus** of $E(N)$.

This implies that the complete linear system $|K_{E(N)} + C^{(n)}|$ contains an effective divisor D , provided that $n > 1 - p_a$. The intersection multiplicity $D \cdot C_u$ of D and any fibre C_u of $E(N)$ vanishes. Hence D can be written in the form

$$D = \sum_{\rho} \sum_s m_{\rho s} \Theta_{\rho s} + \sum_u m_u C_u,$$

where the $\Theta_{\rho s}$ denote irreducible components of the singular fibres C_{a_ρ} of $E(N)$ and the coefficients m_u vanish except for a finite number of points u on $Y(N)$. Since

$$K_{E(N)} \cdot \Theta_{\rho s} = C_u \cdot \Theta_{\rho s} = 0,$$

we have

$$\Theta_{\rho t} \cdot \sum_s m_{\rho s} \Theta_{\rho s} = 0$$

Hence, by the Lemma below, the divisor $\sum_s m_{\rho_s} \Theta_{\rho_s}$ is a multiple of C_{a_ρ} . The canonical divisor $D - C^{(n)}$ can be written in the form

$$\sum k_u C_u,$$

where the coefficients k_u vanish except for a finite number of points u on $X(N)$. Thus we see that

$$K_{E(N)} = \left[\sum k_u C_u \right]. \quad (2.3)$$

We consider the holomorphic section $o: u \rightarrow o(u)$ of $E(N)$ which maps each point $u \in X(N)$ onto the unit $o(u)$ of the fibre C_u of $E(N)$. For the sake of brevity we identify $X(N)$ with the curve $o(X(N))$ on $E(N)$ by means of the biholomorphic map $u \rightarrow o(u)$ of $X(N)$ onto $o(X(N))$. Denoting by $K_{X(N)}$ the restriction of the curve $X(N) = o(X(N))$, we infer from (2.3) that

$$K_{E(N)} = p^*(K_{X(N)}). \quad (2.4)$$

The complex line bundle \mathfrak{f} over $X(N)$ coincides with the normal bundle of $X(N)$ in $E(N)$, i.e., the restriction $[X(N)]_{X(N)}$ of the complex line bundle $[X(N)]$ over $E(N)$ to $X(N)$.

The canonical bundle \mathfrak{k} of $X(N)$ is given by the adjunction formula

$$\mathfrak{k} = K_{X(N)} + [X(N)]_{X(N)}.$$

Hence we get

$$K_{X(N)} = \mathfrak{k} - \mathfrak{f}.$$

Combining this with (2.4) we obtain (2.1). \square

Lemma 2.1. *Let $C_{a_\rho} = \sum_s n_{\rho_s} \Theta_{\rho_s}$ be a simple singular fibre of an analytic fibre space of elliptic curves over a curve and let $D = \sum_s m_{\rho_s} \Theta_{\rho_s}$ be a divisor composed of the components Θ_{ρ_s} of C_{a_ρ} . If the intersection multiplicity $(D \cdot \Theta_{\rho_s})$ vanishes for all components Θ_{ρ_s} , then D is a multiple of C_{a_ρ} .*

Now we want to prove that:

$$q(E(N)) = g_N = 1 + \frac{(N-6)\mu_N}{12N}.$$

To see that, we will use the **Albanese variety** $\text{Alb}(E(N))$ and **Albanese mapping**

$$\text{alb}: E(N) \longrightarrow \text{Alb}(E(N)).$$

We recall here some basic facts (for more details see [GH11]) the Albanese variety is defined as the quotient $\text{Alb}(E(N)) = H^0(E(N), \Omega^1)^* / \Lambda$ where Λ are the linear functions obtained by integrating over cycles in $H_1(E(N), \mathbb{Z})$. Explicitly, if (η_1, \dots, η_q) is a basis for the holomorphic

one-forms on $E(N)$, then $H^0(E(N), \Omega^1)^* \cong \mathbb{C}^g$ and Λ is the lattice of vectors

$$\left(\int_{\gamma} \eta_1, \dots, \int_{\gamma} \eta_g \right), \quad \gamma \in H_1(E(N), \mathbb{Z}).$$

Thus the mapping alb is given by choosing a base point p_0 and then for $p \in E(N)$ setting

$$\text{alb}(p) = \left(\int_{p_0}^p \eta_1, \dots, \int_{p_0}^p \eta_g \right).$$

We are ready for the following (for details see [Tom11])

Proposition 2.1. *If $p: E(N) \rightarrow X(N)$ is an elliptic fibration and for some fiber F the reduced curve F_r is singular, then*

$$q(E(N)) = g_N.$$

Proof. The hypothesis implies that under the Albanese map $\text{alb}: E(N) \rightarrow \text{Alb}(E(N))$ the fiber F_r goes to a point (since all of its components are rational curves). This shows in the canonical diagram

$$\begin{array}{ccc} E(N) & \xrightarrow{\text{alb}} & \text{Alb}(E(N)) \\ p \downarrow & & \Psi \downarrow \\ X(N) & \xrightarrow{\text{jac}} & J(X(N)) \end{array}$$

where J is the Jacobian variety. Ψ is a finite surjective map and hence $\dim \text{Alb}(E(N)) = \dim J(X(N)) = g_N$. \square

Another invariant, that we will go to calculate, will be the **Euler number**, denoted by c_2 , of the surface $E(N)$.

The formula (2.3) implies that $(K_{E(N)}^2)$ vanishes. Hence, by the Noether formula:

$$\chi(\mathcal{O}_{E(N)}) = \frac{1}{12}(K_{E(N)}^2 + c_2),$$

we have

$$12(p_a + 1) = c_2.$$

By the Euler number $e(C_{a_\rho})$ of a singular fibre $C_{a_\rho} = \sum_s n_{\rho s} \Theta_{\rho s}$ we shall mean the Euler number of the polyhedron $\bigcup_s \Theta_{\rho s}$.

Theorem 2.2. *The arithmetic genus p_a and the Euler characteristic of the surface $E(N)$ is given by the formula*

$$c_2 = 12(p_a + 1) = \sum_{\rho} e(C_{a_\rho}) = \mu_N.$$

Proof. The value of $e(C_{a_\rho})$, in our case, is just equal to N . since the Euler number of any general fibre C_u of $E(N)$ vanishes, the proof follows immediately. \square

So we have

$$\begin{aligned} c_2 &= \mu_N, \\ p_a + 1 &= p_g - q + 1 = \frac{(K_{E(N)}^2 + c_2)}{12} = \frac{\mu_N}{12}, \\ p_g &= \frac{(N-3)\mu_N}{6N}. \end{aligned}$$

Now we want to calculate the degree of the line bundles \mathfrak{k} and \mathfrak{f} and the canonical bundle $K_{E(N)}$ as due to Kodaira in [Kod63b].

Theorem 2.3. *We have*

$$\deg \mathfrak{f} = -p_a - 1. \quad (2.5)$$

Proof. It follows from (2.1) that $\dim |K_{E(N)} + C^{(n)}|$ is equal to the dimension of the complete linear system $|\mathfrak{k} - \mathfrak{f} + \sum_{\nu=1}^n u_\nu|$ on the curve $X(N)$. Hence, for all sufficiently large values of n , we have

$$\dim |K_{E(N)} + C^{(n)}| = g_N - \deg \mathfrak{f} + n - 2.$$

Combining this with (2.2), we get

$$g_N - \deg \mathfrak{f} = p_a + k + 1,$$

while for all sufficiently large values of n the integer k is equal to g_N (for details see pag. 124 [I.K.]). Consequently we obtain (2.5). \square

To calculate the degree of \mathfrak{k} , it is sufficient to see that by the classical Gauss-Bonnet theorem, then,

$$\deg \mathfrak{k} = -\chi(X(N)) = 2g_N - 2.$$

So to summarize we have

$$\begin{aligned} \deg \mathfrak{f} &= -\frac{\mu_N}{12}, \\ \deg \mathfrak{k} &= 2g - 2 = \frac{(N-6)\mu_N}{6N}, \\ \deg(\mathfrak{k} - \mathfrak{f}) &= \frac{N-4}{4N}\mu_N. \end{aligned}$$

On the fibres of p , we know

$$p^{-1}(v) = \begin{cases} \text{a non-singular elliptic curve} & \text{if } v \neq \text{cusp}, \\ \sum_{i=0}^{N-1} \Theta_{v,i} & \text{if } v = \text{cusp}, \end{cases}$$

where $\Theta_{v,i}$ is a non-singular rational curve and from the fact that $0 = \Theta_{v,i} \cdot F = \Theta_{v,i} \cdot \sum_t \Theta_{t,i}$ and

$$\Theta_{v,i}^2 + \sum_{t \neq v} \Theta_{v,i} \cdot \Theta_{t,i} = 0,$$

we get $\Theta_{v,i}^2 = -2$. As we have seen in Chapter 1 $E(N)$ has N^2 sections

$$\Gamma(i, j), \quad i, j = 0, \dots, N-1$$

where $\Gamma(0, 0) =$ the zero-section, $\Gamma(i, j)$'s are mutually disjoint and

$$\{\Gamma(i, j) \cap F\} = \{N - \text{division points on } F\}$$

where F is a generic fibre of the fibration p .

The intersection number of the canonical divisor and $\Gamma(i, j)$ is

$$K_{E(N)} \cdot \Gamma(i, j) = \deg(\mathfrak{k} - \mathfrak{f}) = \frac{N-4}{4N} \mu_N.$$

It is well know that the genus of $\Gamma(i, j)$ is equal to g_N , so we can calculate the self-intersection of $\Gamma(i, j)$ from the adjunction formula:

$$g_N = \frac{K_{E(N)} \cdot \Gamma(i, j) + \Gamma(i, j)^2}{2} + 1,$$

and get

$$\Gamma(i, j)^2 = \deg \mathfrak{f} = -\frac{\mu_N}{12}.$$

Let

$$\Gamma = \sum_{i,j} \Gamma(i, j).$$

Then Γ is a non-singular curve on $E(N)$,

$$\Gamma \cap F = \{N - \text{division points on } F\} \sim N^2[0_F]$$

where \sim is the linear equivalence relation and by the Abel's Theorem on elliptic curves we get

$$\Gamma \cap \Theta_{v,i} = \{N\text{-th roots of unity } \mathbb{C}^* = \mathbb{P}^1 - \{0, \infty\}\}$$

where $\Theta_{v,i} = \mathbb{P}^1$, $\Theta_{v,i} \cap \Theta_{v,i-1} = 0$ and $\Theta_{v,i} \cup \Theta_{v,i+1} = \infty$.

Now we are ready to prove some facts about the algebraic equivalence between divisors and a remark on the Nèron-Severi group of $E(N)$.

Lemma 2.2 (Lemma of T.Shioda). *Let F be a general fibre of the fibration $p: E(N) \rightarrow X(N)$ and let D be divisor on $E(N)$ such that $D|_F \sim 0$.*

$$D \approx (D \cdot \Gamma(0, 0)) \cdot F + \sum_{v: \text{ cusp}} (\Theta_{v,1}, \dots, \Theta_{v,N-1}) \cdot A_N^{-1} \begin{pmatrix} D \cdot \Theta_{v,1} \\ \vdots \\ D \cdot \Theta_{v,N-1} \end{pmatrix}$$

where \approx is the algebraic equivalence relation and

$$A_N = [\Theta_{v,i} \cdot \Theta_{v,j}]_{1 \leq i,j \leq N-1} = \begin{pmatrix} -2 & 1 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ 1 & -2 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 1 & -2 & 1 & 0 & \cdot & \cdot & 0 \\ \cdot & & & \cdot & & & & \cdot \\ \cdot & & & & \cdot & & & \cdot \\ \cdot & & & & & \cdot & & 0 \\ 0 & \cdot & \cdot & \cdot & 0 & 1 & -2 & 1 \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 & 1 & -2 \end{pmatrix}$$

and the components of $A_N^{-1} \begin{pmatrix} D \cdot \Theta_{v,1} \\ \cdot \\ \cdot \\ D \cdot \Theta_{v,N-1} \end{pmatrix}$ are integers.

Proof. Take h general fibres F_1, F_2, \dots, F_h where $F_i \neq F_j$ ($i \neq j$). Then

$$\begin{aligned} &\rightarrow H^1\left(E(N), \mathcal{O}\left(D - \sum_{i=1}^h F_i\right)\right) \rightarrow H^1(E(N), \mathcal{O}(D)) \\ &\rightarrow H^1(F_1, \mathcal{O}) \oplus H^1(F_2, \mathcal{O}) \oplus \dots \oplus H^1(F_h, \mathcal{O}) \rightarrow H^2\left(E(N), \mathcal{O}\left(D - \sum_{i=1}^h F_i\right)\right) \rightarrow \end{aligned}$$

where $H^1(F_i, \mathcal{O}) \cong \mathbb{C}$. Hence, if h is sufficiently large

$$H^2\left(E(N), \mathcal{O}\left(D - \sum_{i=1}^h F_i\right)\right) \neq 0.$$

Since $H^2\left(E(N), \mathcal{O}\left(D - \sum_{i=1}^h F_i\right)\right) \cong H^1\left(E(N), \mathcal{O}\left(K - \sum_{i=1}^h F_i - D\right)\right)$, there exists an effective divisor $D' \in |K_{E(N)} + \sum_{i=1}^h F_i - D|$. Namely

$$K_{E(N)} + \sum_{i=1}^h F_i - D \sim D'.$$

For any fibre F of $p: E(N) \rightarrow X(N)$,

$$D' \cdot F = K_{E(N)} \cdot F + \sum_{i=1}^h F_i \cdot F - D \cdot F = 0.$$

Hence $D' = \sum_{\alpha} m_{\alpha} D_{\alpha}$ where D_{α} 's are irreducible curves contained in the fibres of p . Since

$$K \approx \frac{N-4}{4N} \mu_N \cdot F, \quad F_i \approx F, \quad \sum_{i=1}^{N-1} \Theta_{v,i} \approx F,$$

we obtain

$$D \approx pF + \sum_{i=1, v: \text{ cusp}}^{N-1} q_{v,i} \Theta_{v,i}$$

for some $p, q_{v,i} \in \mathbb{Z}$. Since $\Gamma(0,0) \cdot \Theta_{v,i} = 1$ if $i = 0$ and $\Gamma(0,0) \cdot \Theta_{v,i} = 0$ if $i = 1, \dots, N-1$, we obtain

$$D \cdot \Gamma(0,0) = pF \cdot \Gamma(0,0) + \sum_{i=1}^{N-1} q_{v,i} \Theta_{v,i} \cdot \Gamma(0,0) = p.$$

Since $F \cdot \Theta_{v,j} = 0$,

$$D \cdot \Theta_{v,j} \sum_{i=1}^{N-1} q_{v,i} \Theta_{v,i} \cdot \Theta_{v,j}$$

for $j = 1, 2, \dots, N-1$. Let $A_N = [\Theta_{v,i} \cdot \Theta_{v,j}]_{1 \leq i, j \leq N-1}$. Then A_N is non-singular and

$$\begin{pmatrix} q_{v,1} \\ \vdots \\ q_{v,N-1} \end{pmatrix} = A_N^{-1} \begin{pmatrix} D \cdot \Theta_{v,1} \\ \vdots \\ D \cdot \Theta_{v,N-1} \end{pmatrix}.$$

Thus

$$D \approx (D \cdot \Gamma(0,0)) \cdot F + \sum_{v: \text{ cusp}} (\Theta_{v,1}, \dots, \Theta_{v,N-1}) \cdot A_N^{-1} \begin{pmatrix} D \cdot \Theta_{v,1} \\ \vdots \\ D \cdot \Theta_{v,N-1} \end{pmatrix}$$

□

Lemma 2.3. *Let $A_N^{-1} = [x_{ik}]$. Then*

(i)

$$x_{jk} = \begin{cases} \frac{-j(n-k)}{N}, & j \leq k \\ \frac{-k(N-j)}{N}, & j > k. \end{cases}$$

(ii)

$$\sum_{k=1}^{N-1} x_{jk} = \frac{-j(N-j)}{2} = \begin{cases} -mj + \frac{j(j+1)}{2} - \frac{j}{2}, & N = 2m, \\ -mj + \frac{j(j-1)}{2}, & N = 2m + 1. \end{cases}$$

Lemma 2.4. $\mu_N/12$ is divisible by N if $N \geq 5$.

Proof. Let $\sigma(N) = N^2 \prod_{p|N, p: \text{ prime}} (1 - p^{-2})$. It is sufficient to prove that $24|\sigma(N)$.

- (i) Assume $p \geq 5$ and p prime. Then (a) $p = 3h + 1$, $h = 2m$, $m \geq 1$, or (b) $p = 3h + 2$, $h = 2m + 1$, $m \geq 0$.

(a) $\sigma(p) = p^2(1 - p^{-2}) = (p + 1)(p - 1) = 12(3m + 1)m$. Since $3m + 1$ or m is even, $24|\sigma(p)$.

(b) $\sigma(p) = (p - 1)(p + 1) = 12(m + 1)(3m + 2)$. Since $m + 1$ or $3m + 2$ is even, $24|\sigma(p)$.

- (ii) Assume $N = p_1^{h_1} \cdots p_r^{h_r}$ ($i \neq j$), p_i :prime, $h_i \geq 1$ where some $p_i \neq 2, 3$. Then $p_i \geq 5$ and, hence by (i), $24|\sigma(p_i)$. Since

$$\begin{aligned}\sigma(N) &= \sigma(p_1^{h_1}) \cdots \sigma(p_r^{h_r}), \\ \sigma(p_i^{h_i}) &= p_i^{2h_i}(1 - p_i^{-2}) = p_i^{2h_i-2} \cdot \sigma(p_i),\end{aligned}$$

we get $24|\sigma(N)$.

- (iii) Finally we assume $N = 2^{h_1} \cdot 3^{h_2} \geq 5$.

- (a) In case $h_1, h_2 \geq 1$,

$$\begin{aligned}\sigma(N) &= \sigma(2^{h_1}) \cdot \sigma(3^{h_2}) = 2^{2h_1}(1 - 2^{-2})3^{2h_2}(1 - 3^{-2}) \\ &= 2^{2h_1-2} \cdot 3^{2h_2-2}(2^2 - 1)(3^2 - 1) = 24 \cdot 2^{2h_1-2} \cdot 3^{2h_2-2}.\end{aligned}$$

Hence $24|\sigma(N)$.

- (b) In case $h_1 = 0$ and $h_2 \geq 2$,

$$\sigma(N) = \sigma(3^{h_2}) = 3^{2h_2-2}(3^2 - 1) = 3^{2h_2-2} \cdot 8 = 24 \cdot 3^{2h_2-3},$$

where $2h_2 - 3 > 0$. Hence $24|\sigma(N)$.

- (c) In case $h_1 \geq 3$ and $h_2 = 0$,

$$\sigma(N) = \sigma(2^{h_1}) = 2^{2h_1-2}(2^2 - 1) = 2^{2h_1-2} \cdot 3 = 24 \cdot 2^{2h_1-5}$$

where $2h_1 - 5 > 0$. Hence $24|\sigma(N)$.

□

Lemma 2.5.

$$\Gamma \approx N^2 \cdot \Gamma(0, 0) - (1 - N^2) \frac{\mu_N}{12} F - \sum_{v: \text{cusp}, j=1, \dots, N-1} \frac{j(N-j)}{2} N \cdot \Theta_{v,j}.$$

Proof. Let $D = \Gamma - N^2 \cdot \Gamma(0, 0)$. Then $D|F = 0$. By the Lemma of Shioda

$$D \approx (D \cdot \Gamma(0, 0)) \cdot F + \sum_{v: \text{cusp}} (\Theta_{v,1}, \dots, \Theta_{v,N-1}) \cdot A_N^{-1} \begin{pmatrix} D \cdot \Theta_{v,1} \\ \vdots \\ D \cdot \Theta_{v,N-1} \end{pmatrix}$$

where

$$\begin{aligned}D \cdot \Gamma(0, 0) &= (1 - N^2) \cdot \Gamma(0, 0)^2 = -(1 - N^2) \cdot \mu_N / 12, \\ D \cdot \Theta_{v,i} &= \Gamma \cdot \Theta_{v,i} N \quad \text{for } i = 1, \dots, N - 1.\end{aligned}$$

Thus Lemma 2.3, 2.4 imply Lemma 2.5. □

Lemmas 2.3,2.4,2.5 imply

Proposition 2.2. *Assume $N \geq 4$. Then*

$$\Gamma = \sum_{i,j=0}^{N-1} \Gamma(i,j) \text{ is divisible } \begin{cases} \text{by } N & \text{if } N \text{ is odd} \\ \text{by } N/2 & \text{if } N \text{ is even.} \end{cases}$$

Remark 2.1. In the first section of the paper [Shi72], Shioda remarked that the Néron-Severi group $NS(E(N))$ is torsion-free. This fact can be proved as follows:

Proof of the Remark. Let D be a divisor on $E(N)$ such that $D \neq 0$ and $nD \approx 0$ for some positive integer n . Then

$$\sum_{\nu=0}^2 (-1)^\nu \dim H^\nu(E(N), \mathcal{O}(D)) = p_g - q + 1 = \frac{\mu_N}{12} \geq 1$$

where $H^0(E(N), \mathcal{O}(D)) = 0$ and $H^2(E(N), \mathcal{O}(D)) \cong H^0(E(N), \mathcal{O}(K_{E(N)} - D)) = 0$. Hence there exists an effective divisor $D' \in |K_{E(N)} - D|$. Since D' is effective and $D' \cdot F = K_{E(N)} \cdot F - D \cdot F = 0$, we obtain that $D'|F \sim 0$. By the Lemma of Shioda and by the fact that $D' \cdot \Theta_{v,i} = 0$, we get

$$D' \approx (D' \cdot \Gamma(0,0))F.$$

Since $K_{E(N)} = p^*(\mathfrak{k} - \mathfrak{f})$, we obtain that $D \approx hF$ for some integer h . Since $nD \approx 0$, $0 = D \cdot \Gamma(0,0) = h \cdot F \cdot \Gamma(0,0) = h$. Thus $h = 0$ and $D \approx 0 \cdot F = 0$. □

Chapter 3

The Cyclic Cover $A(N, n)$

3.1 Ramified and Cyclic Coverings

Before go deeply to the construction of the cyclic cover $S_d(N)$ we need some basic facts about cyclic covers (for more details see [Per77] and [BPVdV84]). We first use the word "*covering*" in two ways.

Firstly in the sense of **analytic covering space**. This is a triple (X, Y, π) where X and Y are complex manifold and $\pi: X \rightarrow Y$ a surjective holomorphic map such that all points $y \in Y$ have a connected neighbourhood V_y with the property that $\pi^{-1}(V_y)$ consists of the union of disjoint open subsets of X , each of which is mapped isomorphically onto V_y by π .

If we use the corresponding topological concept we shall speak of **topological covering**.

Secondly, we shall use the word "*covering*" for triples (X, Y, π) where X and Y are connected normal complex spaces and π a finite, surjective proper holomorphic map.

In this last case there exists a proper analytic subset of X , outside of which π is a topological covering. Indeed, on $X' = X \setminus \pi^{-1}(\pi(\text{Sing}X) \cup \text{Sing}Y)$ the map π is a covering between manifolds, so it is a topological covering outside of $\pi^{-1}(\pi(S))$ where $S = \{x \in X'; \text{rank}(f\pi)_x \leq \dim X - 1\}$. By definition the degree of π is the one of $\pi|_{X' \setminus \pi^{-1}(\pi(S))}$. Properness of π implies that for every $x \in X$ there exists at least one connected open neighbourhood V of $\pi(x) \in Y$ such that $\pi^{-1}(V)$ is a union of disjoint connected open neighbourhood U_i of x_i , ($i = 1, \dots, n$), where $\pi^{-1}\pi(x) = \{x = x_1, x_2, \dots, x_n\}$. If $V' \subset V$ is another connected neighbourhood of $\pi(x)$, we claim that $\pi^{-1}(V') \cap U_1 = U'_1$ is connected. The degree is called the **local degree** e_x of π at x or the **branching order** of π at x . If $e_x \geq 2$ we say that π is **ramified** at x , and x is called a **ramification point**. The images of ramification points are called **branch points**.

Now if X and Y are manifolds, then the set of ramification points is the zero divisor R of the canonical section in $\text{Hom}(p^*(K_Y), K_X)$, i.e.

$$K_X = \pi^*(K_Y) \otimes \mathcal{O}_X(R). \quad (3.1)$$

The divisor R is called the **ramification divisor** of π . Formula 3.1, together with the specification of R , given by Lemma 3.1 below, is called the **Hurwitz-formula**.

We observe that the properness of π implies that $\pi: X \setminus \pi^{-1}(\pi(R)) \rightarrow Y \setminus \pi(R)$ is a covering in the first sense; in particular, a covering in the second sense if $R = 0$ is one in the first sense too.

To empathise the difference we shall frequently call a covering in the first sense **unbranched** or **unramified** and one in the second sense a **branched** or **ramified** covering, as soon as $R \neq 0$.

Lemma 3.1. *If $R = \sum r_j R_j$, where R is the ramification divisor of some branched covering and the R_j 's its irreducible components, then $r_j = e_j - 1$, where e_j is the branching order at any point $x \in R_j$ which is smooth on R_{red} , and for which $y = \pi(x)$ is smooth on $B_j = \pi(R_j)$.*

Proof. Let (t_1, \dots, t_n) be the local coordinates on Y , centred at y , such that B_j is given by $t_1 = 0$. If $s = 0$ is local equation for R_j at x , then we have $\pi^*(t_1) = \epsilon \cdot s^{e_j}$, where ϵ does not vanish around x , and in fact can be taken to be 1 if s is suitably chosen. If we set $\omega = dt_1 \wedge \dots \wedge dt_n$, then $\pi^*(\omega) = s^{e_j-1} ds \wedge d\pi^*(t_2) \wedge \dots \wedge d\pi^*(t_n)$. This not only show that $(s, \pi^*(t_2), \dots, \pi^*(t_n))$ is local coordinate system at x (so $e_j = e_x$), but also that the zero divisor of $\pi^*(\omega)$ is $(e_j - 1)R_j$. Hence $r_j = e_j - 1$. \square

Lemma 3.2. *Let X and Y be compact connected complex manifolds and $f: X \rightarrow Y$ a covering of degree d . If \mathcal{L} is a line bundle on Y with $f^*\mathcal{L} = \mathcal{O}_X$, then $\mathcal{L}^{\otimes d} = \mathcal{O}_Y$.*

Proof. Since $f_*\mathcal{O}_X$ is locally free of rank d this is an immediate consequence of $f_*\mathcal{O}_X = f_*f^*\mathcal{L} = \mathcal{L} \otimes f_*\mathcal{O}_X$. \square

Let Y be a connected complex manifold and B a divisor on Y which is either effective or zero. Suppose we have a line bundle \mathcal{L} on Y such that

$$\mathcal{O}_Y(B) = \mathcal{L}^{\otimes n},$$

and a section $s \in \Gamma(Y, \mathcal{O}_Y(B))$ vanishing exactly along B . We denote by L the total space of \mathcal{L} and we let $p: L \rightarrow Y$ be the bundle projection. If $t \in \Gamma(L, p^*\mathcal{L})$ is the tautological section then the zero divisor of $p^*s - t^*$ defines an analytic subspace X in L .

If $B \neq 0$ and reduced, X is an irreducible normal analytic subspace of L , and $\pi = p|_X$ exhibits X as an n -fold ramified covering of Y with branch-locus B . We call (X, Y, π) the n -cyclic covering of Y branched along B , determined by \mathcal{L} .

If $\text{Pic}(Y)$ has no torsion, then B uniquely determines \mathcal{L} and we may speak of the n -cyclic covering of Y , branched B .

It is clear from the above description that X has at most singularities over singular points of B . In particular if B is reduced and smooth, then also X is smooth.

Lemma 3.3. *Let $\pi: X \rightarrow Y$ be the n -cyclic covering of Y branched along smooth divisor B and determined by \mathcal{L} , where $\mathcal{L}^{\otimes n} = \mathcal{O}_Y(B)$. Let B_1 be the reduced divisor $\pi^{-1}(B)$ on X . Then*

$$(i) \quad \mathcal{O}_X(B_1) = \pi^*\mathcal{L}$$

$$(ii) \quad \pi^*B = nB_1 \text{ (in particular } n \text{ is the branching order along } B_1)$$

$$(iii) \quad \mathcal{K}_X = \pi^*(\mathcal{K}_Y \otimes \mathcal{L}^{n-1})$$

Proof. If we embed Y as the zero-section in L , then the section $t \in \Gamma(L, p^*\mathcal{L})$ has divisor Y , so $\mathcal{O}_L(Y) = p^*\mathcal{L}$. By construction Y and $X \subset L$ intersect transversally in B_1 , so $\mathcal{O}_X(B_1) = \mathcal{O}_L(Y)|_X = \pi^*\mathcal{L}$. The identity $\pi^*B = nB_1$ follows from the equation $p^*s - t^n = 0$ for X in L . The formula \mathcal{K}_X is an application of Lemma 3.1 \square

Lemma 3.4. *Let $\pi: X \rightarrow Y$ be as in Lemma 3.3. Then $\pi_*\mathcal{O}_X \cong \bigoplus_{j=0}^{n-1} \mathcal{L}^{-j}$.*

Proof. For an open set $V \subset Y$, any holomorphic function f on $p^{-1}(V)$ has a unique power series expansion $f = \sum_{k=0}^{\infty} a_k t^k$, $a_k \in \Gamma(V, \mathcal{L}^{-k})$. Every function $\pi^{-1}(V) \subset p^{-1}(V)$ is the restriction of such an f . Using the equation $t^n = \pi^*s$, we obtain a unique expansion $\sum_{k=0}^{n-1} b_k t^k$, $b_k \in \Gamma(V, \mathcal{L}^{-k})$ for holomorphic functions on $\pi^{-1}(V)$. \square

We can summarize some important facts about the cyclic cover

Theorem 3.1. *Let V be a smooth algebraic variety, $d \geq 2$ and integer, D a reduced effective divisor on V divisible by d in $\text{Pic}(V)$. Then*

1. *There exist cyclic d -sheeted covers $\pi_\Delta: W_\Delta \rightarrow V$, totally branched over D and nowhere else.*
2. *The covers $\pi_\Delta: W_\Delta \rightarrow V$ are one-to-one correspondence with d^{th} -roots of D , i.e., divisor classes Δ satisfying $d_\Delta \equiv D$ (linear equivalence), and hence there are d^{2g} of them where $g = h^{1,0}(V)$. Now fix Δ satisfying $d_\Delta \equiv D$ and suppose, in fact, that $f \in \mathbb{C}(V)$, the rational function on V gives this equivalence:*

$$(f) = d\Delta - D.$$

Then

3. $\mathbb{C}(W_\Delta) = \mathbb{C}(V)(\sqrt[d]{f})$ and W_Δ is irreducible unless $k\Delta = 0 \in \text{Pic}(V)$ for some k/d , $k \neq d$.
we assume $\Delta \neq 0 \in \text{Pic}(V)$. Let e denote the Euler characteristic. Then

4.

$$e(W_\Delta) = d \cdot e(V) - (d-1)e(D).$$

5. *An automorphism $\phi: V \rightarrow V$ lifts to W_Δ iff*

- (i) $\phi(D) = D$
(ii) $\phi(\Delta) \equiv \Delta$,

and if ϕ lifts, it lifts in d different ways,

6. W_Δ is smooth iff D is.

7. Assume V is a surface, and D_1, D_2 two irreducible curves on V . Let \tilde{D}_1, \tilde{D}_2 be the (reduced) inverse image in W_Δ of D_1 and D_2 , respectively. Then

$$\begin{aligned} \tilde{D}_1 \cdot \tilde{D}_2 &= dD_1 \cdot D_2 && \text{if } D_1, D_2 \not\subset D, \\ &D_1 \cdot D_2 && \text{if exactly one of } D_1, D_2 \subset D, \\ &\frac{1}{d}D_1 \cdot D_2 && \text{if both } D_1, D_2 \subset D. \end{aligned}$$

Sketch of the proof. Given $f \in \mathbb{C}(V)$ with $(f) = d\Delta - D$, define

$$\phi_{ij}: \mathcal{O}(-j\Delta) \otimes \mathcal{O}(-k\Delta) \longrightarrow \mathcal{O}((d-k-j)\Delta)$$

by setting, for s and t sections of $\mathcal{O}(-j\Delta), \mathcal{O}(-k\Delta)$, respectively.

$$\phi_{jk}(s \otimes t) = i(f \cdot s \cdot t). \quad (3.2)$$

Here $f \cdot s \cdot t$ is a section of $\mathcal{O}(-D + (d-k-j)\Delta)$.

$\mathcal{O}(-D + (d-k-j)\Delta)$ is contained in $\mathcal{O}((d-k-j)\Delta)$, since D is effective, and we let i_0 denote this inclusion.

Using the ϕ_{jk} we define a ring structure on the sheaf $S = \bigoplus_{j=0}^{d-1} \mathcal{O}(-j\Delta)$ by setting, for s and t sections of $\mathcal{O}(-j\Delta)$ and $\mathcal{O}(-k\Delta)$, respectively, with $0 \leq j, k \leq d-1$.

$$s \cdot t = \begin{cases} s \otimes t \in \mathcal{O}((-j-k)\Delta) & \text{if } j+k \leq d-1 \\ \phi_{jk}(s \otimes t) & \text{if } j+k \geq d. \end{cases} \quad (3.3)$$

And we let $W_\Delta = \text{Spec} S$.

Conversely, given a cyclic d -sheeted cover $\pi: W \rightarrow V$, with branch locus D , let $S = \pi_* \mathcal{O}_W$. The Galois group of the cover μ_d acts, and S breaks into a sum of character eigenspaces:

$$S = \bigoplus_{i=0}^{d-1} S_i.$$

S is a locally free, rank d sheaf. The S_i are invertible sheaves, with $S_0 \cong \mathcal{O}_V$.

Let Δ be the divisor class with $\mathcal{O}(-\Delta) \cong S_1$. Then the ring structure on \mathcal{O}_W gives a map

$$\phi: \bigotimes_{i=0}^d \mathcal{O}_V(-\Delta) \longrightarrow \mathcal{O}_V.$$

ϕ fails to be surjective exactly over D , and hence ϕ gives an isomorphism

$$\phi: \bigotimes^d \mathcal{O}_V(-\Delta) \sim \mathcal{O}_V(-D) \longrightarrow \mathcal{O}_V \quad (3.4)$$

which proves $d\Delta \cong D$, and we get (1) and (2). The first part of (3) is obvious. If $k\Delta \equiv 0$ for some $k/d, k < d$, then $D \cong 0$ and since D is effective, $D = \emptyset$. Hence W_Δ is unramified. Since f is k/d -th power, the equation $x^d = f$ is reducible and so is W_Δ .

Conversely, if W_Δ is reducible, the equation $x^d = f$ is reducible. If $D \neq \emptyset$, let v be the valuation $\mathbb{C}(V)$ associated to some irreducible component of D . Since D is reduced, $v(f) = 1$, hence $x^d = f$ is irreducible. If, on the other hand, $D = \emptyset$ and the cover is reducible, $f = g^a$, $a > 1$ and $\frac{d}{a}\Delta = (g)$, which proves the second part of (3).

(4) follows from the additivity of the Euler characteristic. The condition in (5) are clearly necessary, and they are sufficient since if $(g) = \Delta - \phi(\Delta)$, g gives an isomorphism of ringed spaces

$$g: \bigoplus_{k=0}^{d-1} (-k\Delta) \longrightarrow \bigoplus_{k=0}^{j-1} (-k\phi(\Delta)),$$

since it is compatible with the ring structure defined above.

Locally D is given by a principal ideal (h) , and W_Δ is given by the equation

$$x^d = h. \quad (3.5)$$

At a point $p \in D$, D is non-singular iff $h \notin m_{p,V}^2$ the maximal ideal of V at p , but this is equivalent to $x^d - h \in m_{p,W_\Delta}^2$, proving (6).

(7) is a consequence of the functoriality of homology coupled with the fact that for $D_0 \subset D$, $\pi_\Delta^* D_0 = d\tilde{D}_0$. \square

3.2 The Construction of $A(N, n)$

In this part we combine the general results of above with the structure of $E(N)$ to construct a class of covers of $E(N)$, where N is now fixed and we assume $N \geq 4$.

By Proposition 2.2, $\Gamma = \sum_{i,j=0}^{N-1} \Gamma(i, j)$ is divisible by N if N is odd, and by $N/2$ if N is even. Let n be an integer such that $n \geq 2$ and

$$\begin{aligned} n|N & \quad \text{if } N \text{ is odd,} \\ n|(N/2) & \quad \text{if } N \text{ is even.} \end{aligned}$$

Then $[\Gamma] = nL$ for some line bundle $L \in H^1(E(N), \mathcal{O}^*)$. Hence we can construct, in the bundle space of L , an n -fold branched covering

$$\phi: A(N, n) \longrightarrow E(N)$$

along a non-singular branch locus $\Gamma(\subset E(N))$.

Let

- K_S = the canonical line bundle of a compact complex surface S ,
- $e(X)$ = the topological Euler number of a space X ,
- $\tau(S)$ = the topological index of $S = (K_S^2 - 2e(S))/3$,
- $p_g(S)$ = the geometric genus of S , $q(S)$ = the irregularity of S .

In the following, we shall calculate some numerical invariants of $A(N, n)$.

Proposition 3.1.

$$K_{A(N,n)}^2 = \frac{N(n-1)\{(5n+1)N - 24n\}}{12n} \cdot \mu(N),$$

$$e(A(N, n)) = \frac{6n + (n-1)N(N-6)}{6} \cdot \mu(N).$$

Proof. By Theorem 3.1,

$$K_{A(N,n)}^2 = nK_{E(N)}^2 + 2(n-1)K_{E(N)} \cdot \Gamma + \frac{(n-1)^2}{n}\Gamma^2$$

where

$$K_{E(N)}^2 = 0,$$

$$K_{E(N)} \cdot \Gamma = \sum_{i,j} K_{E(N)} \cdot \Gamma(i, j) = N^2 \cdot \frac{N-4}{4N} \mu(N) = \frac{N(N-4)}{4} \mu(N),$$

$$\Gamma^2 = \sum_{i,j} \Gamma(i, j)^2 = N^2 \frac{-\mu(N)}{12} = -\frac{N^2}{12} \mu(N).$$

Hence

$$\begin{aligned} K_{A(N,n)}^2 &= 2(n-1) \cdot \frac{N(N-4)}{4} \cdot \mu(N) - \frac{(n-1)^2}{n} \frac{N^2}{12} \mu(N) \\ &= \frac{N(n-1)\{(5n+1)N - 24n\}}{12n} \cdot \mu(N). \end{aligned}$$

By Theorem 3.1,

$$e(A(N, n)) = ne(E(N)) - (n-1)e(\Gamma)$$

where

$$e(E(N)) = \mu(N),$$

$$e(\Gamma) = \sum_{i,j} e(\Gamma(i, j)) = N^2 \cdot e(X(N)) = N^2(2 - 2g(X(N)))$$

$$= N^2 \cdot 2 \left(-\frac{N-6}{12N} \cdot \mu(N) \right) = -\frac{N(N-6)}{6} \cdot \mu(N).$$

Hence

$$\begin{aligned} e(A(N, n)) &= n \cdot \mu(N) + (n-1) \frac{N(N-6)}{6} \cdot \mu(N) \\ &= \frac{6n + (n-1)N(N-6)}{6} \mu(N). \end{aligned}$$

□

Theorem 3.2. *Assume $N \geq 5$. Then*

- (i) $3e(A(N, n)) \geq K_{A(N, n)}^2 \geq 2e(A(N, n))$,
- (ii) $3e(A(N, n)) = K_{A(N, n)}^2$ if and only if $(N, n) = (7, 7), (8, 4), (9, 3), (12, 2)$,
- (iii) $K_{A(N, n)}^2 = 2e(A(N, n))$ if and only if $(N, n) = (5, 5)$.

Proof. By Proposition 3.1

$$3e(A(N, N)) - K_{A(N, n)}^2 = \frac{\mu(N)}{12n} \{(n-1)N - 6n\}^2 \geq 0.$$

The equality holds if and only if $N = 6n/(n-1)$. Since $n \geq 2$, this is equivalent to $(N, n) = (7, 7), (8, 4), (9, 3), (12, 2)$. By Proposition 3.1

$$K_{A(N, n)}^2 - 2e(A(N, n)) = \frac{\mu(N)}{12n} \{(n^2 - 1)N^2 - 24n^2\}.$$

If $N = 5$ (and hence $n = 5$), then

$$K_{A(N, n)}^2 - 2e(A(N, n)) = \frac{\mu(5)}{12 \cdot 5} \{(5^2 - 1) \cdot 5^2 - 24 \cdot 5^2\} = 0.$$

If $N \geq 6$, then, since $n \geq 2$,

$$\begin{aligned} K_{A(N, n)}^2 - 2e(A(N, n)) &\geq \frac{\mu(N)}{12n} \{(n^2 - 1)36 - 24n^2\} \\ &= \frac{\mu(N)}{12n} (12n^2 - 36) = \frac{\mu(N)}{n} (n^2 - 3) > 0. \end{aligned}$$

□

Proposition 3.2. (i) *If $N \geq 6$, then $A(N, n)$ is a minimal surface of general type with positive topological index.*

- (ii) *$A(5, 5)$ is a surface of general type with $K_{A(5, 5)}^2 = 200$ and $e(A(5, 5)) = 100$. Let*

$$\Gamma^*(i, j) = \phi^{-1}(\Gamma(i, j)) \quad (\text{hence } 5\Gamma^*(i, j) = \phi^*\Gamma(i, j)).$$

Then $\Gamma^(i, j)$'s are exceptional curves of the first kind. Let A_0 be the surface obtained by blowing down $\Gamma^*(i, j)$'s. Then A_0 is a minimal surface of general type with $K_{A_0}^1 = 225$ and $e(A_0) = 75$ (and hence $K_{A_0}^2 = 3e(A_0)$).*

Proof. By Theorem 3.1

$$K_{A(N,n)} = \phi^* K_{E(N)} + (n-1)\Gamma^* = \phi^* \Phi^*(\mathfrak{k} - \mathfrak{f}) + (n-1) \sum_{i,j} \Gamma^*(i, j),$$

$$\dim |\mathfrak{f}| - \dim |\mathfrak{k} - \mathfrak{f}| = \deg \mathfrak{f} + 1 - g(X(N)) = \frac{3-N}{6N} \mu(N).$$

Since $\deg \mathfrak{f} = -\mu(N)/12 < 0$, $\dim |\mathfrak{f}| = -1$. Hence

$$\dim |\mathfrak{k} - \mathfrak{f}| = \frac{N-3}{6N} \mu(N) - 1 > 0 \quad N \geq 5.$$

In particular, $p_g(A(N, n)) > 0$. If there exists an exceptional curve of the first kind on $A(N, n)$, then it is contained in the divisor $K_{A(N,n)}$ and, hence is one of $\Gamma^*(i, j)$'s, while

$$g(\Gamma^*(i, j)) = g(\Gamma(i, j)) = 1 + \frac{(N-6)\mu(N)}{12N},$$

$$\Gamma^*(i, j)^2 = \frac{\Gamma(i, j)^2}{n} = -\frac{\mu(N)}{12n}.$$

If $N \geq 6$, then $g(\Gamma^*(i, j)) \geq 1$ and hence $A(N, n)$ is minimal. If $N = 5$, then $\mu(5) = \frac{1}{2}5^3 \cdot (1 - 5^{-2}) = 60$. Hence $g(\Gamma^*(i, j)) = 0$ and $\Gamma^*(i, j)^2 = -1$, namely, $\Gamma^*(i, j)$'s are exceptional curves of the first kind. Since $\mu(5) = 60$, $g(X(5)) = 1 + (5-6)\mu(5)/(12 \cdot 5) = 0$ and $\deg(\mathfrak{k} - \mathfrak{f}) = ((5-4)/(4 \cdot 5))\mu(5) = (1/20)60 = 3$,

$$K_{A(5,5)} = 3\phi^* F + 4 \sum_{i,j=0}^4 \Gamma^*(i, j)$$

where F is a general fibre of $p: E(5) \rightarrow X(5)$ and $F \cdot \Gamma^*(i, j) = 1$. Hence $K_{A_0} = 3F_*$ where F_* is a non-singular curve with $g(F_*) = 11$. In particular A_0 is minimal. Since $K_{A(5,5)}^2 = 200$ and $e(A(5, 5)) = 100$ by Proposition 3.1, A_0 is a minimal surface of general type with $K_{A_0}^2 = 225$ and $e(A_0) = 75$. By Proposition 3.1,

$$\begin{aligned} K_{A(N,n)}^2 &= \frac{N(n-1)\{(5n+1)N-24n\}}{12n} \cdot \mu(N) \\ &\geq \frac{N(n-1)\{(5n+1)5-24n\}}{12n} \cdot \mu(N) \\ &= \frac{N(n-1)(n+5)}{12n} \cdot \mu(N) > 0 \end{aligned}$$

for $N \geq 5$. Thus $A(N, n)$ is of general type. The topological index $\tau(A(N, n))$ is positive if $N \geq 6$ by Theorem 3.2 \square

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