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On the unirationality of Del Pezzo surfaces over an arbitrary field

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Introduction

Given a variety over a field k that is rational over an algebraic closure of k one can wonder whether it is rational also over k. This topic, that we will address as the question about descent of rationality, can be included in the more general context of descent for field extensions, i.e. the plan of establishing under which conditions properties that are valid over an extension of a field k hold also over k.

To approach the descent of rationality we need some knowledge in descent theory, that is discussed, together with the main properties of rational points, in Chapter 2.

Given a rational variety over an infinite field k, the Zariski density of its set of rational points is a consequence of its rationality. Then we see that, in the case of infinite fields, the density of the set of rational points and, in particular, the existence of a rational point, is a necessary condition for rationality. In this thesis we will study some families of varieties for which the existence of a rational point is also a sufficient condition to rationality.

The simplest example of such varieties is given by the so called Severi-Brauer varieties, i.e. any variety of dimension n that becomes isomorphic to the n-dimensional projective space over an algebraic closure of the ground field. Indeed a Severi-Brauer variety of dimension n is rational over the ground field k if and only if it is isomorphic to $\mathbb{P}^n_k$ if and only if it has a rational point. This result is a theorem of F. Châtelet (1944), in Section 3.3 we present the proof given in [Gil], 5, §5.1, Theorem 5.1.3.

The isomorphism classes of Severi-Brauer varieties are strictly connected to the Brauer group of the ground field, as we will see in Section 3.3, this connection helps in some cases to prove the existence of a rational point, indeed if the Brauer group of the ground field is trivial, every Severi-Brauer variety has a rational point.

In Chapter 3 we give the classical definition of the Brauer group via central simple algebras, then following [Se1], X, §5 and §6 we prove that it is isomorphic to a certain cohomology group and we describe its connection with Severi-Brauer varieties.

The second example we consider is given by Del Pezzo surfaces, which are surfaces that become isomorphic, over an algebraic closure of the ground field, either to $\mathbb{P}^1 \times \mathbb{P}^1$ or to a blowing-up of the projective plane with center

$r \leq 8$ rational points satisfying some geometric conditions. In particular, Severi-Brauer surfaces are just a particular case of Del Pezzo surfaces.

Del Pezzo surfaces are characterized by the fact that their anticanonical divisor $-K$ is ample, indeed they are, by definition, all the nonsingular surfaces satisfying this property. They are classified by the self intersection of the anticanonical divisor, which is is an integer between 1 and 9, called the degree of the surface.

A Del Pezzo surface that over an algebraic closure of the ground field becomes isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ has degree 8, while a Del Pezzo surface that over an algebraic closure of the ground field becomes isomorphic to a blowing-up of the projective plane with center r points has degree $9 - r$. In particular Severi-Brauer surfaces are the Del Pezzo surfaces of degree 9. So the degree classifies somehow how far is the surface from being a Severi-Brauer surface.

In Chapter 4 we study the main properties of Del Pezzo surfaces: after the definition and some basic examples, we study the classification over an algebraically closed field following [Ko1], III, §3, and we give a detailed description of the structure of the (-1)-curves. In Section 4.4 we prove that the same classification remains valid also over a separably closed field, i.e. that the descent from an algebraic closure to a separable closure of a field works well in the case of Del Pezzo surfaces. While in Section 4.3 we see that over an arbitrary field the situation is more complicated.

In Chapter 5 we investigate the descent of rationality for Del Pezzo surfaces over an arbitrary field. Following [Man], IV, §29, Theorem 29.4 we prove that a Del Pezzo surfaces of degree ≥ 5 is rational if and only if it has a rational point. Following [Ko2], we prove that a Del Pezzo surface of degree 3 is unirational if and only if it has a rational point. Then, combining these two results, we prove that unirationality, provided the existence of a rational point, holds also in degree 4.

In the last Section we give a brief account of the last results of research in degrees 1 and 2. C. Salgado, D. Testa and A. Várilly-Alvarado have proven unirationality for Del Pezzo surfaces of degree 2 that contain a rational point satisfying some geometric conditions, and there is hope that the same result holds also without the additional conditions on the point. R. van Luijk and C. Salgado have proven that, under some special conditions on the surface, the set of rational points of a Del Pezzo surface of degree 1 over a number field is Zariski dense. While unirationality remains an open problem.

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I thank my family and my friends for their support and for having endured me during this year.

I apologize for the slight on English this paper could be. Mathematics is the only language, besides my mother tongue, I feel comfortable with.

Chapter 1

Background

In this chapter we recall the main definitions and results in Algebraic Geometry and Group Cohomology that are used in the rest of the text.

Section 1.1 contains general facts in Algebraic Geometry. The main references are: [Har] for the geometry of algebraic varieties, [Dol] for weighted spaces and Section 7.6 in [Bos] for Weil restriction.

Section 1.2 contains specific results about the geometry of nonsingular surfaces and their birational classification, the main reference is Chapter V in [Har].

Section 1.3 contains some facts in Group Cohomology that are used in Chapter 3, the main reference is [Se2].

1.1 Generalities

Definition 1.1 (Variety)**.** A variety over a field k is an integral, separated scheme of finite type over $\mathrm{Spec}(k)$.

Let X be a variety over a field, we denote by $\mathcal{O}_X$ its structural sheaf. For any point $x \in X$, we denote by $\mathcal{O}_{X,x}$ the stalk of $\mathcal{O}_X$ at x. $\mathcal{O}_{X,x}$ is a local ring, we denote by m_x its maximal ideal and by $k(x)$ its residue field.

Definition 1.2 (Divisors)**.** Let X be a nonsingular variety over a field k, we define the group of divisors of X to be the free abelian group generated by the closed integral sub-schemes of codimension 1 in X.

A divisor D is principal if there exists a nonzero rational function f on X such that $D = \sum_Y v_Y(f)Y$, where Y runs over the closed integral sub-schemes of codimension 1 in X and v_Y is the valuation of the discrete valuation ring $\mathcal{O}_{X,\eta}$, where η is the generic point of Y.

The set of principal divisors is a subgroup of the group of divisors of X and the quotient is called the group of classes of divisors of X. Two divisors C, D said to be linearly equivalent, $C \sim D$, if they belong to the same class.

A divisor $\sum_Y n_Y Y$ is effective if $n_Y \geq 0$ for all Y. If $D = \sum_Y n_Y Y$ is an effective divisor we define the support of D to be the closed subscheme union of the Y's such that $n_Y \neq 0$.

Let D be a divisor of X, the linear system $|D|$ associated to D is the set of effective divisors of X linearly equivalent to D. The linear system $|D|$ can be identified to the projective space associated to the k-vector space $H^0(X, \mathcal{O}(D))$ (see [Har], II, 7, Proposition 7.7).

Definition 1.3 (Picard group)**.** Let X be a ringed space, an invertible sheaf on X is a locally free sheaf of $\mathcal{O}_X$-modules of rank one. Let $\mathrm{Pic}(X)$ be the set of isomorphism classes of invertible sheaves on X.

Proposition 1.4. *Let X be a ringed space, then* $\mathrm{Pic}(X)$ *is an abelian group, called the Picard group of X, and* $\mathrm{Pic}(X) \cong H^1(X, \mathcal{O}_X^\times)$*.*

Proof. See [Har], II,§6, Proposition 6.12, and III, §4, Exercise 4.5. □

Proposition 1.5. *Let X be a nonsingular variety, then to every divisor D of X can be associated an invertible sheaf $\mathcal{O}(D)$ such that: D is principal if and only if $\mathcal{O}(D) \cong \mathcal{O}_X$; if D is effective then $\mathcal{O}(-D)$ is the ideal sheaf of the support of D; the map that associate to each divisor its invertible sheaf induces an isomorphism between the group of classes of divisors of X and* $\mathrm{Pic}(X)$*.*

Proof. See [Har], II, §6, Propositions 6.11, 6.15, 6.18. □

Throughout this paper $\mathrm{Pic}(X)$ will denote both the Picard group of X and the group of classes of divisors of X, under the identification provided by Proposition 1.5. With an abuse of notation we will identify also a divisor with its class, writing $D \in \mathrm{Pic}(X)$, where D is a representative for its class.

Definition 1.6 (Canonical sheaf)**.** Let X be a nonsingular variety of dimension n over an algebraically closed field k. Let $\Omega_{X/k}$ be the sheaf of relative differentials of X over k (see [Har], II, §8, p. 175 for the definition), then $\Omega_{X/k}$ is a locally free sheaf of $\mathcal{O}_X$-modules of rank 2 by Proposition 8.15 in [Har], II, §8. We define the canonical sheaf of X to be the invertible sheaf $\omega_X := \Lambda^n \Omega_{X/k}$. The divisor class on X whose associated sheaf is ω_X is called the canonical divisor of X and it is denoted by K_X.

Theorem 1.7 (Serre's duality)**.** *Let X be a projective nonsingular variety of dimension n over an algebraically closed field, let $\mathscr{L}$ be an invertible sheaf on X, then*

$$H^i(X, \mathscr{L}) \cong H^{n-i}(X, \mathscr{L}^{-1} \otimes_{\mathcal{O}_X} \omega_X), \quad i = 0, 1, 2.$$

Proof. See [Har], III, 7, Corollary 7.12. □

Definition 1.8. Let X be a projective variety over a field k and $\mathscr{F}$ a coherent sheaf of $\mathcal{O}_X$-modules on X, we denote by $H^i(X, \mathscr{F})$, $i \geq 0$, the Cech cohomology groups of $\mathscr{F}$ on X.

For all $i \geq 0$, let $h^i(X, \mathscr{F}) := \dim_k H^i(X, \mathscr{F})$. If D is a divisor on X, let $h^i(X, D) := h^1(X, \mathcal{O}(D))$ for all $i \geq 0$.

Let $\chi(\mathscr{F}) := \sum_{i \geq 0} (-1)^i h^i(X, \mathscr{F})$ be the Euler characteristic of $\mathscr{F}$. Theorem 5.2 in [Har], III, §5 says that $h^i(X, \mathscr{F})$ and $\chi(\mathscr{F})$ are integers. In particular $\chi(\mathscr{F}) = \sum_{0 \leq i \leq \dim X} (-1)^i h^i(X, \mathscr{F})$.

The arithmetic genus of X is $p_a(X) := (-1)^{\dim X}(\chi(\mathcal{O}_X) - 1)$.

Definition 1.9. Let X be a variety over a field k. A sheaf $\mathscr{F}$ of $\mathcal{O}_X$-modules is generated by global sections on X if for every $x \in X$ the image of $H^0(X, \mathscr{F})$ in the stalk $\mathscr{F}_x$ at x generates $\mathscr{F}_x$ as $\mathcal{O}_{X,x}$-module.

An invertible sheaf $\mathscr{L}$ on X is ample if for every coherent sheaf $\mathscr{F}$ on X there is a positive integer $n_{\mathscr{F}}$ such that $\mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{L}^{\otimes n}$ is generated by global sections on X, for all $n \geq n_{\mathscr{F}}$.

An invertible sheaf $\mathscr{L}$ on X is very ample if there is a morphism $\phi : X \to \mathbb{P}^n_k$ for some n, such that ϕ gives an isomorphism of X with an opens subscheme of a closed subscheme of $\mathbb{P}^n_k$ and $\phi^*(\mathcal{O}_{\mathbb{P}^n_k}(1)) \cong \mathscr{L}$.

We say that a divisor D on X is generated by global section, or ample, or very ample, if $\mathcal{O}(D)$ is.

Proposition 1.10. *Let X be a variety over a field k, let $\mathscr{L}$ be an invertible sheaf generated by global sections $s_0, \ldots, s_n \in H^0(X, \mathscr{L})$, then there exists a unique morphism of k-varieties $\phi : X \to \mathbb{P}^n_k$ such that $\phi^*(\mathcal{O}_{\mathbb{P}^n_k}(1)) \cong \mathscr{L}$ and $s_i = \phi^*(x_i)$, $\forall i = 0, \ldots, n$, under this isomorphism. Moreover every morphism $X \to \mathbb{P}^n_k$ arises in this way.*

Proof. See [Har], II,§7, Theorem 7.1. □

Proposition 1.11. *Let X be a variety over a field k, let $\mathscr{L}$ be a very ample invertible sheaf on X, then $\mathscr{L}$ is ample and generated by global sections.*

Proof. $\mathscr{L}$ is ample by Theorem 5.17 in [Har], II, §5, while it is generated by global sections by Proposition 1.10. □

Proposition 1.12. *Let X be a proper variety over a field k, let $\mathscr{L}$ be a very ample invertible sheaf on X, then the morphism $\phi : X \to \mathbb{P}^n_k$ associated to $\mathscr{L}$ in Definition 1.9 is a closed immersion.*

Proof. See [Har], II, §4, Exercise 4.4. □

Proposition 1.13. *Let X be a proper variety over a field k, let $\mathscr{L}$ be a very ample invertible sheaf on X, then $H^0(X, \mathscr{L})$ generates the graded ring $\oplus_{m \geq 0} H^0(X, \mathscr{L}^{\otimes m})$.*

Proof. Let $\phi : X \to \mathbb{P}^n_k$ be the morphism associated to $\mathscr{L}$ in Definition 1.9, then ϕ is a closed immersion by Proposition 1.12 and $\phi^*(\mathcal{O}_{\mathbb{P}^n_k}(1)) \cong \mathscr{L}$, then $\mathscr{L}^{\otimes m} \cong \phi^*(\mathcal{O}_{\mathbb{P}^n_k}(m))$ for all $m \geq 0$ and

$$H^0(X, \mathscr{L}^{\otimes m}) = H^0(\mathbb{P}^n_k, \phi_*(\mathscr{L}^{\otimes m})) \cong H^0(\mathbb{P}^n_k, \mathcal{O}_{\phi(X)}(m))$$

Since $H^0(\mathbb{P}^n_k, \mathcal{O}_{\phi(X)}(1))$ generates $\oplus_{m\geq 0} H^0(\mathbb{P}^n_k, \mathcal{O}_{\phi(X)}(m))$, then also $H^0(X, \mathscr{L})$ generates $\oplus_{m\geq 0} H^0(X, \mathscr{L}^{\otimes m})$. □

Proposition 1.14. *Let X be a variety over a field k, let $\mathscr{L}$ be an invertible sheaf on X, then $\mathscr{L}$ is ample if and only if there is a positive integer n such that $\mathscr{L}^{\otimes n}$ is ample if and only if there is a positive integer m such that $\mathscr{L}^{\otimes m}$ is very ample.*

Proof. See [Har], II, 7, Proposition 7.5 and Theorem 7.6. □

Proposition 1.15. *Let $\phi : Y \to X$ be a closed immersion of proper varieties over a field k, let $\mathscr{L}$ be an ample invertible sheaf on X. Then $\phi^*(\mathscr{L})$ is an ample invertible sheaf on Y.*

Proof. See [Har], III, §5, Exercise 5.7. □

Proposition 1.16. *Let X be a projective variety over a field k, let $\mathscr{L}$ be an ample invertible sheaf on X, then $H^0(X, \mathscr{L}^{-1}) = 0$.*

Proof. See [Har], III, 7, Exercise 7.1. □

Proposition 1.17. *Let $\varphi : S \to R$ be a surjective morphism of graded rings preserving degrees, then φ induces a closed immersion* $\mathrm{Proj}(R) \to \mathrm{Proj}(S)$.

Proof. See [Har], II, §3, Exercise 3.12. □

Proposition 1.18. *Let k be a field and $S = \oplus_{m\geq 0} S_m$ a graded ring which is a finitely generated k-algebra, then* $\mathrm{Proj}(S) \cong \mathrm{Proj}(\oplus_{m\geq 0} S_{md})$ *for all $d > 0$.*

Proof. Let $d > 0$ and $S^d := \oplus_{m\geq 0} S_{md}$. For any $f \in S_m$, $m > 0$, let $S\{\frac{1}{f}\}$ be the ring of degree 0 elements of $S[\frac{1}{f}]$, then $S\{\frac{1}{f}\}$ and $S^d\{\frac{1}{f^d}\}$ are isomorphic and the isomorphisms are compatible on the open coverings of basic open sets $\mathrm{Spec}(S\{\frac{1}{f}\})$ and $\mathrm{Spec}(S^d\{\frac{1}{f^d}\})$ of $\mathrm{Proj}(S)$ and $\mathrm{Proj}(S^d)$ respectively. Glueing these isomorphisms we conclude that $\mathrm{Proj}(S) \cong \mathrm{Proj}(S^d)$. □

Proposition 1.19. *Let X be a projective variety over a field k, let $\mathscr{L}$ be an ample invertible sheaf on X, then $X \cong \mathrm{Proj}(\oplus_{m\geq 0} H^0(X, \mathscr{L}^{\otimes m}))$.*

Proof. By Proposition 1.14 there is a positive integer d such that $\mathscr{L}^{\otimes d}$ is very ample on X, then by Proposition 1.12 it induces a closed immersion $\phi : X \to \mathbb{P}^n_k$ for some n. With the same argument used in Proposition 1.13 we see that $H^0(X, \mathscr{L}^{\otimes md}) \cong H^0(\phi(X), \mathcal{O}_{\phi(X)}(m))$ for all $m \geq 0$, then

$$X \cong \phi(X) \cong \mathrm{Proj}(\oplus_{m\geq 0} H^0(\phi(X), \mathcal{O}_{\phi(X)}(m))) \cong \mathrm{Proj}(\oplus_{m\geq 0} H^0(X, \mathscr{L}^{\otimes md}))$$

and by Proposition 1.18 we get $X \cong \mathrm{Proj}(\oplus_{m\geq 0} H^0(X, \mathscr{L}^{\otimes m}))$. □

Definition 1.20 (Weighted projective space)**.** Let k be a field and $n \geq 0$, the polynomial ring $S = k[x_0, \ldots, x_n]$ is a graded ring generated by the degree 1 elements $x_0, \ldots, x_n$. Let $S_{(a_0,\ldots,a_n)} = k[x_0^{a_0}, \ldots, x_n^{a_n}]$ be the graded subring of S generated by the elements $x_0^{a_0}, \ldots, x_n^{a_n}$. We define the weighted projective space associated to $(a_0, \ldots, a_n)$ to be $\mathbb{P}_k(a_0, \ldots, a_n) = \mathrm{Proj}(S_{(a_0,\ldots,a_n)})$, it is a normal, projective variety of dimension n over k (see [Dol], §1.2 and 1.3, Proposition 1.3.3).

For all $m \in \mathbb{Z}$, denote by $\mathcal{O}_{\mathbb{P}_k(a_0,\ldots,a_n)}(m)$ the coherent sheaf of $\mathcal{O}_{\mathbb{P}_k(a_0,\ldots,a_n)}$-modules on $\mathbb{P}_k(a_0, \ldots, a_n)$ associated to the graded $S_{(a_0,\ldots,a_n)}$-module $S_{(a_0,\ldots,a_n)}(m)$.

Proposition 1.21. *Let k be a field, then for every positive integer m such that $a_i|m$ for $i = 0, \ldots, n$ we have that $\mathcal{O}_{\mathbb{P}_k(a_0,\ldots,a_n)}(m)$ is an ample invertible sheaf on $\mathbb{P}_k(a_0, \ldots, a_n)$.*

Proof. See [Del], Corollaire 1.6 and Proposition 2.3. □

Proposition 1.22. *Let k be a field. Every irreducible hypersurface X in $\mathbb{P}_k(a_0, \ldots, a_n)$ is given by an irreducible homogeneous polynomial $g \in S_{(a_0,\ldots,a_n)}$. In that case we denote by $\deg X$ the degree of the polynomial g.*

Proof. Let $g, h \in S_{(a_0,\ldots,a_n)}$ be two homogeneous polynomials, let X, Y be the subvariety of $\mathbb{P}_k(a_0, \ldots, a_n)$ associated to g, h respectively. Since $\mathbb{P}_k(a_0, \ldots, a_n)$ is projective, we can assume that X, Y are two varieties in $\mathbb{P}_k^N$ for some N, then if $X \neq Y$ we have that $X \cap Y$ has dimension $n - 2$ because X and Y are irreducible (see [Har], I, §7, Theorem 7.2). Thus we have proved that if X is an irreducible hypersurface in $\mathbb{P}_k(a_0, \ldots, a_n)$ then X is defined by only one homogeneous polynomial in $S_{(a_0,\ldots,a_n)}$. □

Definition 1.23. Let k be a field and X an irreducible nonsingular hypersurface in $\mathbb{P}_k(a_0, \ldots, a_n)$, for all $m \in \mathbb{Z}$ we set $\mathcal{O}_X(m) := \mathcal{O}_X \otimes_{\mathcal{O}_{\mathbb{P}_k(a_0,\ldots,a_n)}} \mathcal{O}_{\mathbb{P}_k(a_0,\ldots,a_n)}(m)$.

Proposition 1.24. *Let k be a field and X an irreducible nonsingular hypersurface in $\mathbb{P}_k(a_0, \ldots, a_n)$, then $\omega_X^{-1} \cong \mathcal{O}_X(\sum_{i=0}^n a_i - \deg X)$. If $\sum_{i=0}^n a_i - \deg X > 0$ then ω_X^{-1} is ample.*

Proof. By Theorem 3.3.4 in [Dol], §3.3, we have that $\omega_X \cong \mathcal{O}_X(\deg X - \sum_{i=0}^n a_i)$. Since ω_X is an invertible sheaf, we have that $\omega_X^{-1} \cong \mathcal{O}_X(\sum_{i=0}^n a_i - \deg X)$. Let $m := \sum_{i=0}^n a_i - \deg X$, if $m > 0$, then $\mathcal{O}_X(ma_0 \cdot \cdots \cdot a_n)$ is ample by Propositions 1.21 and 1.15, then also $\omega_X^{-1} \cong \mathcal{O}_X(m)$ is ample by Proposition 1.14. □

Definition 1.25. Let $f : X \to X'$ be a morphism of varieties over k, f is birational if there are nonempty open subsets $U \subset X$ and $V \subset Y$ such that $f|_U : U \to V$ is an isomorphism.

If f is birational then then X and X' have the same dimension.

If f is a blowing-up (see [Har], II, §7 for the definition), then f is a birational morphism (see [Har], II, §7, Proposition 7.16).

We say that $f : X \to X'$ is a monoidal transformation if it is a blowing-up with center a closed point of X'.

Definition 1.26. Let X, Y be two varieties over a field k, a rational map $X \dashrightarrow Y$ is an equivalence class of pairs (U, f) where U is an open subset of X and $f : U \to Y$ is a morphism, and where $(U, f), (V, g)$ are equivalent if $f|_{U \cap V} = g|_{U \cap V}$.

A variety X over a field k is said to be unirational if there exists a dominant rational map $\mathbb{P}^n_k \dashrightarrow X$ defined over k.

Definition 1.27. Let $f : X \dashrightarrow X'$ be a rational map of varieties over a field k, f is a birational map if there are nonempty open subsets $U \subset X$ and $V \subset Y$ such that $f|_U : U \to V$ is an isomorphism.

We say that X and X' are k-birationally equivalent if there exists a birational map $X \dashrightarrow X'$ defined over k, i.e. if there are nonempty open subsets $U \subset X$ and $V \subset X'$ with $U \cong V$ as k-schemes.

A variety X over a field k is k-rational if it is k-birationally equivalent to $\mathbb{P}^n_k$, where $n = \dim X$.

Remark 1.28. Birational equivalence is an equivalence relation, in particular we have the following results. Let X, Y be two varieties over a field k.

If X and Y are k-birationally equivalent, then X is rational over k if and only if Y is rational over k, and X is unirational over k if and only if Y is unirational over k.

If there is a dominant rational map $X \dashrightarrow Y$ over k and X is k-rational, then Y is unirational over k.

Proposition 1.29. *Let X be a variety of dimension n over a field k, suppose that there is a dominant rational map $f : \mathbb{A}^m_k \dashrightarrow X$ for some $m > n$, then X is unirational over k.*

Proof. See [Ko2], Lemma 11. □

Definition 1.30 (Weil restriction)**.** Let L/k be a finite Galois extension and Y a scheme over L, then we denote by $\mathfrak{R}_{L/k}(Y)$, if it exists, the Weil restriction of Y with respect to the field extension L/k (see [Bos], 7, §7.6 for the definition). If it exists, $\mathfrak{R}_{L/k}(Y)$ is a scheme over k and there is a bijection $Hom_k(X, \mathfrak{R}_{L/k}(Y)) \xrightarrow{\sim} Hom_L(X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(L), Y)$.

Proposition 1.31. *Let L/k be a finite Galois extension, then for all $n \geq 0$ we have that $\mathfrak{R}_{L/k}(\mathbb{A}^n_L)$ and $\mathfrak{R}_{L/k}(\mathbb{P}^n_L)$ exist, $\mathfrak{R}_{L/k}(\mathbb{A}^n_L) \cong \mathbb{A}^{[L:k]n}_k$ and $\mathfrak{R}_{L/k}(\mathbb{P}^n_L)$ is a proper variety over k birationally equivalent to $\mathbb{P}^{[L:k]n}_k$.*

Proof. The existence comes from Theorem 4 in [Bos], 7, §7.6. The isomorphism $\mathfrak{R}_{L/k}(\mathbb{A}^n_L) \cong \mathbb{A}^{[L:k]n}_k$ is shown in the proof of Theorem 4 in [Bos], 7, §7.6. The Weil restriction $\mathfrak{R}_{L/k}(\mathbb{P}^n_L)$ is a proper variety over k by Proposition 5 in [Bos], 7, §7.6, moreover open immersions are invariant under Weil restriction (see [Bos], 7, §7.6, Proposition 2), then if we take an open immersion $\mathbb{A}^n_L \to \mathbb{P}^n_L$, which exists by the definition of the projective space, we have an open immersion $\mathfrak{R}_{L/k}(\mathbb{A}^n_L) \to \mathfrak{R}_{L/k}(\mathbb{P}^n_L)$ and in particular an open immersion $\mathbb{A}^{[L:k]n}_k \to \mathfrak{R}_{L/k}(\mathbb{P}^n_L)$, then $\mathfrak{R}_{L/k}(\mathbb{P}^n_L)$ is a variety of dimension $[L:k]n$ which is birationally equivalent to $\mathbb{P}^{[L:k]n}_k$. □

1.2 Geometry of surfaces

Definition 1.32 (Surface)**.** A surface over a field k is a geometrically integral, nonsingular, projective variety of dimension 2 over k.

Definition 1.33 (Curve)**.** Let X be a surface, a curve in X is a closed subscheme of codimension 1 in X, in particular a curve in X is an effective divisor of X.

Proposition 1.34. *Let C be an integral curve over an algebraically closed field, if $p_a(C) = 0$ then $C \cong \mathbb{P}^1_k$.*

Proof. See [Liu], §7.4.1, Proposition 4.1. □

Definition 1.35. Let X be a surface over an algebraically closed field and C, D two curves in X having no common irreducible component. Let $P \in C \cap D$ and f, g local equations of C, D in $\mathcal{O}_{X,P}$. We define $(C.D)_P$ to be the dimension of $\mathcal{O}_{X,P}/(f,g)$ as k-vector space. If $(C.D)_P = 1$ we say that C, D meet transversally at P. We say that C and D meet transversally if they meet transversally at P for every $P \in C \cap D$.

Theorem 1.36 (Intersection pairing)**.** *Let X be a surface over an algebraically closed field. There is a unique pairing $\mathrm{Pic}(X) \times \mathrm{Pic}(X) \to \mathbb{Z}$, called the intersection pairing and denoted by $C.D$ for any two divisors C, D, such that:*

i) if C, D are two curves in X having no common irreducible component, then $C.D = \sum_{P \in C \cap D} (C.D)_P$;

ii) if C, D are nonsingular curves in X meeting transversally then $C.D = \#(C \cap D)$;

iii) $C.D = D.C$ for all divisors $C, D \in \mathrm{Pic}(X)$;

iv) $C.(D + D') = C.D + C.D'$ for all divisors $C, D, D' \in \mathrm{Pic}(X)$;

v) $C.D = \chi(\mathcal{O}_X) - \chi(\mathcal{O}(C)) - \chi(\mathcal{O}(D)) + \chi(\mathcal{O}(C+D))$ *for all divisors* $C, D \in \mathrm{Pic}(X)$.

Proof. See [Har], V, §1, Theorem 1.1, Proposition 1.4 and Exercise 1.1. □

Proposition 1.37 (Adjunction formula)**.** *Let X be a surface over an algebraically closed field, let C be an irreducible curve in X, then*

$$2p_a(C) - 2 = C.(C + K_X) \tag{1.1}$$

Proof. See [Se3], IV, §8, Proposition 5. □

Theorem 1.38 (Riemann-Roch)**.** *Let X be a surface over an algebraically closed field, let D be a divisor on X, then*

$$h^0(X, D) - h^1(X, D) + h^0(X, K_X - D) = \frac{1}{2}D.(D - K_X) + 1 + p_a(X) \tag{1.2}$$

i.e., in other words, $\chi(\mathcal{O}(D)) = \frac{1}{2}D.(D - K_X) + \chi(\mathcal{O}_X)$.

Proof. See [Har], V, §1, Theorem 1.6. □

Theorem 1.39 (Nakay-Moishezon criterion)**.** *Let X be a surface over an algebraically closed field, a divisor D on X is ample if and only if $D^2 > 0$ and $D.C > 0$ for all irreducible curves C in X.*

Proof. See [Har], V, §1, Theorem1.10. □

Definition 1.40 (Numerical equivalence)**.** Let X be a surface over an algebraically closed field, we say that two divisors $C_1, C_2 \in \mathrm{Pic}(X)$ are numerically equivalent, write $C_1 \equiv C_2$, if $C_1.D = C_2.D$ for all divisors $D \in \mathrm{Pic}(X)$. Numerical equivalence is an equivalence relation in $\mathrm{Pic}(X)$ and the subset of divisors numerically equivalent to 0 is a subgroup of $\mathrm{Pic}(X)$, we denote the quotient by $\mathrm{Num}(X)$. $\mathrm{Num}(X)$ is a finitely generated free abelian group (see [Har], V, §1, Exercise 1.8), we define $\rho(X)$ to be the rank of $\mathrm{Num}(X)$.

Theorem 1.41 (Hodge index theorem)**.** *Let X be a surface over an algebraically closed field, let $H \in \mathrm{Pic}(X)$ be an ample divisor and $D \in \mathrm{Pic}(X)$ such that $D.H = 0$ and $D \not\equiv 0$, then $D^2 = 0$. In particular if $D_1, \ldots, D_n$ is a basis of $\mathrm{Num}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ that diagonalize the intersection pairing, and D_1 is ample, then the intersection pairing restricted to $\mathbb{R}D_2 \oplus \cdots \oplus \mathbb{R}D_n$ is negative defined.*

Proof. See [Har], V, §1, Theorem 1.9 and Remark 1.9.1. □

Proposition 1.42. *Let X be a surface over a field k, let H be an ample divisor generated by global sections on X, then H induces a finite morphism $\phi : X \to \mathbb{P}^n_k$ such that $\phi(X)$ spans $\mathbb{P}^n_k$, where $n = h^0(X, H) - 1$. If k is algebraically closed we have also $H^2 = \deg \phi \cdot \deg \phi(X)$.*

Proof. Choose a basis $s_0, \dots, s_n$ of the k-vector space $H^0(X, \mathcal{O}(H))$, then H is generated by the global sections $s_1, \dots, s_n$ and we ca apply Proposition 1.10. Let $\phi : X \to \mathbb{P}^n_k$ be the morphism induced by H and the choice of $s_1 \dots, s_n$. Suppose that there is an irreducible curve C in X such that $\phi(C)$ is a point in $\mathbb{P}^n_k$, take a hyperplane in $\mathbb{P}^n_k$ which misses that point, then its inverse image under ϕ is an effective divisor $D \sim H$ such that $D \cap C = \emptyset$, then $D.C = 0$, which contradicts the fact that H is ample, as $D.C = H.C$ by Theorem 1.36 and $H.C > 0$ by Proposition 1.39. Then ϕ is a projective morphism with finite fibers and by Stein factorization (see [Har], III, 11, Exercise 11.1) we conclude that ϕ is a finite morphism.

Let use the notation introduced in the proof of Theorem 7.1 in [Har], II, §7. Suppose that $\phi(X)$ is contained in a hyperplane of equation $a_0x_0 + \cdots + a_nx_n = 0$ in $\mathbb{P}^n_k$, then the image of $a_0x_0 + \cdots + a_nx_n$ has to be 0 under the maps $A[y_0, \dots, y_n] \to H^0(X_i, \mathcal{O}_{X_i})$ for $i = 0, \dots, n$, that gives $a_0 = \cdots = a_n = 0$, by how those maps are defined. Then $\phi(X)$ cannot be contained in a hyperplane of $\mathbb{P}^n_k$, hence it spans $\mathbb{P}^n_k$.

Let assume that k is algebraically closed. Since ϕ is finite and X is nonsingular, then $\phi(X)$ is a nonsingular closed subvariety of dimension 2 in $\mathbb{P}^n_k$. By Bertini's theorem (see [Har], II, §8, Theorem 8.18) we can find two hyperplanes H_1, H_2 in $\mathbb{P}^n_k$, not containing $\phi(X)$, such that $H_1 \cap H_2 \cap \phi(X)$ is a nonsingular subvariety of dimension 0, i.e. a finite set of points, and then $\deg \phi(X) = \#(H_1 \cap H_2 \cap \phi(X))$. Moreover, since ϕ is finite, we can choose H_1, H_2 such that $\#\phi^{-1}(H_1 \cap H_2 \cap \phi(X)) = \deg \phi \cdot \#(H_1 \cap H_2 \cap \phi(X))$, but $\phi^{-1}(H_1 \cap H_2 \cap \phi(X)) = \phi^{-1}(H_1) \cap \phi^{-1}(H_2)$ and we can choose H_1, H_2 such that $\phi^{-1}(H_i)$ is a nonsingular curve in X, for $i = 1, 2$, moreover $\phi^{-1}(H_1), \phi^{-1}(H_2)$ are linearly equivalent to H and meet transversally, then $\#\phi^{-1}(H_1 \cap H_2 \cap \phi(X)) = H^2$ by Theorem 1.36 and we get that $H^2 = \deg \phi \cdot \deg \phi(X)$. □

Proposition 1.43. *Let X be a surface over an algebraically closed field k and H a very ample divisor on X, let $\phi : X \to \mathbb{P}^n_k$ be the closed immersion induced by H, then if C is a curve in X we have that* $\deg \phi(C) = H.C$.

Proof. See [Har], V, §1, Exercise 1.2. □

Definition 1.44. A (-1) curve of a surface X is a curve E in X such that $E \cong \mathbb{P}^1$ and $E^2 = -1$.

Proposition 1.45. *Let X be a surface over an algebraically closed field, let P be a closed point of X and $f : \tilde{X} \to X$ be the monoidal transformation of X with center P, let E be the exceptional divisor of f. Then we have the following properties:*

i) E is a (-1)-curve;

ii) $\operatorname{Pic} \tilde{X} \cong \operatorname{Pic} X \oplus \mathbb{Z}$;

iii) if $C, D \in \operatorname{Pic} X$, then $(f^*C).(f^*D) = C.D$, $(f^*C).E = 0$;

iii) $K_{\tilde{X}} = f^*K_X + E$ and $K_{\tilde{X}}^2 = K_X^2 - 1$;

iv) if $C \in \operatorname{Pic}(X)$ is an effective divisor and P has multiplicity r on C, then $f^*C = \tilde{C} + rE$, where $\tilde{C}$ is the strict transform of C under f.

Proof. See [Har], V, §3, Propositions 3.1, 3.2, 3.3 and 3.6. □

Proposition 1.46. *Let $f : X \to X'$ be a birational morphism of surfaces over an algebraically closed field, let $n(f)$ be the number of irreducible curves C in X contracted by f (i.e. such that $f(C)$ is a point). Then $n(f)$ is finite, f can be factored into a composition of exactly $n(f)$ monoidal transformations.*

Proof. See [Har], V, §5, Corollary 5.4. □

The motivation of Definition 1.44 is the fact that the exceptional divisor of a monoidal transformation of surfaces is a (-1)-curve. The following theorem says that in fact every (-1)-curve on a surface X is the exceptional divisor of some monoidal transformation of surfaces $X \to X'$.

Theorem 1.47 (Castelnuovo's contraction criterion)**.** *Let X be a surface over an algebraically closed field, if E is a (-1)-curve on X, then there exists a surface X' and a point P on X' such that X is isomorphic to the monoidal transformation of X' with center P, and E corresponds to the exceptional divisor.*

Proof. See [Har], V, §5, Theorem 5.7. □

Definition 1.48 (Minimal surface)**.** A surface X over a field k is minimal over k if any birational morphism of surfaces $f : X \to X'$ over k is an isomorphism.

Proposition 1.49. *Let X be a surface over an algebraically closed field, then X is minimal if and only if X contains no (-1)-curves.*

Proof. It is a consequence of Theorem 1.47. □

Theorem 1.50. *Every surface over an algebraically closed field k admits a birational morphism to a minimal surface over k.*

Proof. See [Har], V, §5, Theorem 5.8. □

Definition 1.51. Let X be a surface over a field k, a divisor D on X is nef if $D.C \geq 0$ for every curve C in X.

Theorem 1.52. *Let X be a minimal surface over an algebraically closed field, then X satisfies exactly one of the following conditions:*

i) K_X *is nef;*

ii) $\rho(X) = 2$ *and* X *is a* $\mathbb{P}^1$*-bundle over a projective nonsingular irreducible curve* C*;*

iii) $\rho(X) = 1$ *and* $-K_X$ *is ample.*

Proof. See [Ko1], III, §2, Theorem 2.3, or [Has], Corollary 2.20. □

Theorem 1.53 (Castelnuovo's rationality criterion)**.** *Let* X *be a surface over an algebraically closed field, then* X *is rational if and only if* $h^1(X, \mathcal{O}_X) = h^2(X, \mathcal{O}(2K_X)) = 0$.

Proof. See [Ko1], III, §2, Theorem 2.4. □

1.3 Group cohomology

Definition 1.54. Let G be a group. A G-group is a group A endowed with an action of G compatible with the group structure of A, let denote the action by $g : A \to A$ for all $g \in G$. An abelian G-group is called a G-module.

Let A be a G-group. We define $H^0(G, A) := \{a \in A : g(a) = a, \forall g \in G\}$ to be the maximal subgroup of A on which the action of G is trivial.

Let use the notation $A = (A, \cdot, 1)$ for the group structure of A. We define $H^1(G, A)$ to be the set of maps $\varphi : G \to A$ such that $\varphi(gg') = \varphi(g) \cdot g(\varphi(g'))$ for all $g, g' \in G$, modulo the equivalence relation $\varphi \sim \psi$ if and only if there is an element $a \in A$ such that $\varphi(g) = a^{-1} \cdot \psi(a) \cdot g(a)$ for all $g \in G$. $H^1(G, A)$ is a pointed set with the constant map 1 as special element.

If A is a G-module we can define the cohomology groups $H^i(G, A)$ for $i \geq 0$, as the right derived functors of the left-exact covariant functor $A \mapsto H^0(G, A)$ from the category of G-modules to the category of abelian groups. In particular the first cohomology group coincides with the $H^1(G, A)$ defined above.

Proposition 1.55. *Let* k *be a field,* L/k *a Galois extension and* $\overline{k}$ *a separable closure of* k *containing* L*. Then there is an exact sequence*

$$0 \to H^2(\mathrm{Gal}(L/k), L^\times) \to H^2(\mathrm{Gal}(\overline{k}/k), \overline{k}^\times) \to H^2(\mathrm{Gal}(\overline{k}/L), \overline{k}^\times)$$

which gives the transition maps of the direct limit in the following equality

$$H^2(\mathrm{Gal}(\overline{k}/k), \overline{k}^\times) = \varinjlim_{\substack{L/k \text{ finite} \\ \text{Galois}}} H^2(\mathrm{Gal}(L/k), L^\times)$$

Proof. See [Se1], X, §4, Proposition 6 and compare [Se1], X, §3 with [Se2], I, §2.2, Corollary 1. □

Proposition 1.56. *Let L/k be a finite Galois extension and $G = \mathrm{Gal}(L/k)$, then $H^1(G, GL_n(L)) = \{1\}$ for all $n \geq 1$.*

Proof. See [Se1], X, §1, Proposition 3. $\square$

Proposition 1.57. *Let G be a group and $0 \to A \to B \to C \to 0$ an exact sequence of G-groups such that A is contained in the center of B, where the center of B is the set $\{x \in B : xb = bx, \forall b \in B\}$. Then there is an exact sequence of pointed sets*

$$\begin{aligned} 0 \to H^0(G,A) \to H^0(G,B) \to H^0(G,C) & \\ \to H^1(G,A) \to H^1(G,B) \to H^1(G,C) \to H^2(G,A) & \end{aligned}$$

Proof. See [Se2], I, §5.7, Proposition 43. $\square$

Corollary 1.58. *Let L/k be a Galois field extension and $G = \mathrm{Gal}(L/k)$. Then for any positive integer n there is an exact sequence of pointed sets*

$$0 \to H^1(G, PGL_n(L)) \to H^2(G, L^\times)$$

Proof. The action of G on the coordinates of matrices gives to $GL_n(L)$ and $PGL_n(L)$ a natural structure of G-groups, then we have an exact sequence of G-groups

$$0 \to L^\times \to GL_n(L) \to PGL_n(L) \to 0$$

where $L^\times$ maps into the center of $GL_n(L)$, then by Proposition 1.57 we have a long exact sequence of pointed sets

$$\cdots \to H^1(G, GL_n(L)) \to H^1(G, PGL_n(L)) \to H^2(G, L^\times)$$

but $H^1(G, GL_n(L)) = 0$ by Proposition 1.56, then we have the desired exact sequence: $0 \to H^1(G, PGL_n(L)) \to H^2(G, L^\times)$. $\square$

Chapter 2

Field extensions and rational points

This chapter introduces some notions and results that are necessary to deal with the questions discussed in this paper.

In Section 2.1 we present some properties of varieties that are stable under field extension.

In Section 2.2 we define the set of rational points of a variety and we prove its main properties.

In Section 2.3 we introduce the language of Galois descent.

Notation. For any variety X over a field k and any field extension K/k we set $X_K := X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(K)$, in particular if $x \in X$ is a point then $x_K = x \times_{\mathrm{Spec}(k)} \mathrm{Spec}(K)$ is the inverse image of x under the projection $X_K \to X$.

2.1 Under field extension

Let k be a field and K an extension of k.

Proposition 2.1. *Let X be a nonsingular, projective, geometrically integral variety over k, then X_K is a nonsingular, projective variety over K.*

Proof. X_K is a separated, projective scheme of finite type over K because separated morphisms, projective morphisms and morphisms of finite type are stable under base extension, moreover X_K is integral as X is geometrically integral, and $\dim X_K = \dim X$, thus X_K is a projective variety of dimension 2 over K. X is nonsingular if and only if the base extension to the algebraic closure is nonsingular if and only if X_K is nonsingular. □

Proposition 2.2. *Let X be a projective variety over k, $\mathscr{F}$ be a coherent sheaf on X, then $\mathscr{F} \otimes_k K$ is a coherent sheaf over X_K and $H^i(X_K, \mathscr{F} \otimes_k K) \cong$*

$H^i(X, \mathscr{F}) \otimes_k K$, $\forall i \geq 0$.
In particular $h^i(X_K, \mathscr{F} \otimes_k K) = h^i(X, \mathscr{F})$, $\forall i \geq 0$, and $\chi(\mathscr{F} \otimes_k K) = \chi(\mathscr{F})$.

Proof. Since k is a field, $\mathscr{F}$ is flat over $\operatorname{Spec}(k)$ and also K is a flat k-algebra, moreover X is projective, hence proper, over $\operatorname{Spec}(k)$, then by Corollary 5 in [Mum], II, §5, p. 53, we have $H^i(X_K, \mathscr{F} \otimes_k K) \cong H^i(X, \mathscr{F}) \otimes_k K$, $\forall i \geq 0$. □

Proposition 2.3. *Let X be a projective variety over k and $\mathscr{L}$ be an invertible sheaf on X. Then $\mathscr{L} \otimes_k K$ is an invertible sheaf on X_K, and $\mathscr{L}$ is very ample on X if and only if $\mathscr{L} \otimes_k K$ is very ample on X_K. In particular $\mathscr{L}$ is ample on X if and only if $\mathscr{L} \otimes_k K$ is ample on X_K.*

Proof. The cohomological criterion for ampleness (see [Har], III, §5, Proposition 5.3) says that $\mathscr{L}$ is ample if and only if for any coherent sheaf $\mathscr{F}$ on X there exists a positive integer n_0 such that $H^i(X, \mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{L}^n) = 0$ $\forall i > 0$ and $\forall n \geq n_0$. Let suppose that $\mathscr{L} \otimes_k K$ is ample on X_K, let $\mathscr{F}$ be a coherent sheaf on X, then $\mathscr{F} \otimes_k K$ is a coherent sheaf on X_K, since $\mathscr{L} \otimes_k K$ is ample there exists a positive integer n_0 such that for all $n \geq 0$ and all $i > 0$ we have:

$$H^i(X_K, (\mathscr{F} \otimes_k K) \otimes_{\mathcal{O}_{X_K}} (\mathscr{L} \otimes_k K)^n) = 0$$

but $(\mathscr{F} \otimes_k K) \otimes_{\mathcal{O}_{X_K}} (\mathscr{L} \otimes_k K)^n \cong (\mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{L}^n) \otimes_k K$, so by Proposition 2.2 we conclude that $H^i(X, \mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{L}^n) \otimes_k K = 0, \forall i > 0$ and $\forall n \geq n_0$, and in particular $H^i(X, \mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{L}^n) = 0, \forall i > 0$ and $\forall n \geq n_0$. Thus $\mathscr{L}$ is ample on X.

Suppose now that $\mathscr{L}$ is ample on X, to prove that $\mathscr{L} \otimes_k K$ is ample on X_K it is enough to prove that one of its positive powers is very ample. Without loss of generality we can assume that $\mathscr{L}$ is very ample, let $j : X \to \mathbb{P}^N_k$ be the closed immersion associated to $\mathscr{L}$ (it is closed because X is projective over k), since closed immersions are stable under base extension, we have a closed immersion $j_K : X_K \cong X \times_{\mathbb{P}^N_k} \mathbb{P}^N_K \to \mathbb{P}^N_K$. In particular $j_K^*(\mathcal{O}_{\mathbb{P}^N_K}(1)) \cong (j^*(\mathcal{O}_{\mathbb{P}^N_k}(1))) \otimes_k K \cong \mathscr{L} \otimes_k K$ (see [Har], II, §5, Exercise 5.11), thus $\mathscr{L} \otimes_k K$ is very ample on X_K. □

Proposition 2.4. *Let X be a surface over k, let $\mathscr{L}, \mathscr{M}$ be two invertible sheaves on X, then the formula*

$$\mathscr{L}.\mathscr{M} = \chi(\mathcal{O}_X) - \chi(\mathscr{L}) - \chi(\mathscr{M}) + \chi(\mathscr{L} \otimes_{\mathcal{O}_X} \mathscr{M}) \tag{2.1}$$

gives a symmetric bilinear form $\operatorname{Pic}(X) \times \operatorname{Pic}(X) \to \mathbb{Z}$ and for every algebraic extension K of k we have $(\mathscr{L} \otimes_k K).(\mathscr{M} \otimes_k K) = \mathscr{L}.\mathscr{M}$.

Proof. Let K be an algebraic extension of k, then $\mathscr{L} \otimes_k K$ and $\mathscr{M} \otimes_k K$ are invertible sheaves on X_K by Proposition 2.3. We have also

$$(\mathscr{L} \otimes_k K) \otimes_{\mathcal{O}_{X_K}} (\mathscr{M} \otimes_k K) \cong (\mathscr{L} \otimes_{\mathcal{O}_X} \mathscr{M}) \otimes_k K$$

then, using Proposition 2.2, we obtain

$$(\mathscr{L} \otimes_k K).(\mathscr{M} \otimes_k K) = \mathscr{L}.\mathscr{M} \tag{2.2}$$

Now, let K be an algebraic closure of k, then from the discussion above and Theorem 1.36 we have that the map $\mathrm{Pic}(X) \times \mathrm{Pic}(X) \to \mathbb{Z}$ induced by the formula (2.1) is a symmetric bilinear form. $\square$

Proposition 2.5. *Let X be a nonsingular, projective, geometrically integral variety, then $\mathcal{O}_{X_K} \cong \mathcal{O}_X \otimes_k K$, $\omega_{X_K} \cong \omega_X \otimes_k K$. If X is a surface we have also $K^2_{X_K} = K^2_X$.*

Proof. Let $\{U_i = \mathrm{Spec}(A_i)\}_{i\in I}$ be an open affine covering of X, then $\mathcal{V} = \{V_i := U_i \times_{\mathrm{Spec}(k)} \mathrm{Spec}(K) = \mathrm{Spec}(A_i \otimes_k K)\}_{i\in I}$ is an open affine covering of X_K, so $\mathcal{O}_{X_K}$ is the glueing of $\{\mathcal{O}_{V_i}\}_{i\in I}$ over the open covering $\mathcal{V}$, which is isomorphic to $\mathcal{O}_X \otimes_k K$.

Let $\Omega_{X/k}$ be the sheaf of relative differentials of X over k. Let U be an open affine subset of X and $V = U \times_{\mathrm{Spec}(k)} \mathrm{Spec}\, K$, by Proposition 8.2 A in [Har], §8, and what we proved above, we have that:

$$\Omega_{V/K} \cong \Omega_{U/k} \otimes_{\mathcal{O}_U} \mathcal{O}_V \cong \Omega_{U/k} \otimes_{\mathcal{O}_U} \mathcal{O}_U \otimes_k K \cong \Omega_{U/k} \otimes_k K$$

Moreover, we have $\omega_X = \bigwedge^2 \Omega_{X/k}$, and:

$$\omega_V \cong \bigwedge^2 \Omega_{V/K} \cong \bigwedge^2 (\Omega_{U/k} \otimes_k K) \cong (\bigwedge^2 \Omega_{U/k}) \otimes_k K \cong \omega_U \otimes_k K$$

Then, with a glueing argument as above, we can conclude that $\Omega_{X_K/K} \cong \Omega_{X/k} \otimes_k K$ and $\omega_{X_K} \cong \omega_X \otimes_k K$.

Moreover, using the formulas (2.1) and (2.2), we get that $K^2_{X_K} = K^2_X$. $\square$

2.2 Rational points

Definition 2.6. Let X be a variety over a field k, the set of k-rational points of X is the set $X(k) := \mathit{Hom}_k(\mathrm{Spec}(k), X)$ of morphisms from $\mathrm{Spec}(k)$ to X as k-schemes.

Notation. We recall the notation introduced in Definition 1.1. For any variety X over a field and any point $x \in X$, we denote by $k(x)$ the residue field of $\mathcal{O}_X$ at x.

Proposition 2.7. *Let X be a variety over a field k, to give a morphism $\mathrm{Spec}(k) \to X$ is equivalent to give a point $x \in X$ and an inclusion $k(x) \to k$. So $X(k)$ is the set of points $x \in X$ such that $k(x) = k$.*

Proof. See [Har], II, §2, Exercise 2.7. $\square$

Corollary 2.8. *Let X be a variety over a field k and let K/k be any field extension, then we have an inclusion $X(k) \subset X_K(K)$.*

Proof. Let $x \in X(k)$, then $k(x) = k$ by Proposition 2.7. We have $x \cong \mathrm{Spec}(k(x)) \cong \mathrm{Spec}(k)$ and $\mathrm{Spec}(k(X_K)) \cong x_K \cong \mathrm{Spec}(k \otimes K) \cong \mathrm{Spec}(K)$, then $k(x_K) = \mathrm{Spec}(K)$ and $x_K \in X_K(K)$ by Proposition 2.7. Moreover, if $\pi : X_K \to X$ is the projection, we have that $x_K \cong \pi^{-1}(x)$, thus the map $X(k) \to X_K(K)$ sending x to x_K is injective and we get an inclusion $X(k) \subset X(K)$. $\square$

Proposition 2.9. *Let X be a variety over a field k, $x \in X$ is a closed point if and only if $k(x)$ is a finite extension of k. So $X(k)$ is a subset of the set of closed points of X.*

Proof. Without loss of generality we ca assume that X is affine. Let A be a k-algebra of finite type, let $X = \mathrm{Spec}(A)$, take a point $x \in X$ and let $\mathfrak{p}$ be the prime ideal of A corresponding to the point x. Then $k(x) = Frac(A/\mathfrak{p})$ is a finite extension of k if and only if $Frac(A/\mathfrak{p}) = A/\mathfrak{p}$ (see [Lan], IX, §1, Corollary 1.2), if and only if $\mathfrak{p}$ is a maximal ideal, if and only if x is a closed point in X. $\square$

Proposition 2.10. *Let X, Y be two varieties over a field k, then*

$$(X \times_{\mathrm{Spec}(k)} Y)(k) = X(k) \times Y(k)$$

Proof. Let π_X, π_Y be the projections of $X \times_{\mathrm{Spec}(k)} Y$ with respect to X, Y respectively. An element in $(X \times_{\mathrm{Spec}(k)} Y)(k)$ is a morphism of k-varieties $f : \mathrm{Spec}(k) \to X \times_{\mathrm{Spec}(k)} Y$, and we have that the pair $(\pi_X \circ f, \pi_Y \circ f)$ is an element in $X(k) \times Y(k)$. Moreover if $(g_X, g_Y) \in X(k) \times Y(k)$ then g_X, g_Y are morphisms that make commutative the following diagram

$$\begin{tikzcd}
\mathrm{Spec}(k) \arrow[drr, "g_Y"] \arrow[ddr, "g_X"'] & & \\
& X \times_{\mathrm{Spec}(k)} Y \arrow[r, "\pi_Y"'] \arrow[d, "\pi_X"] & Y \arrow[d] \\
& X \arrow[r] & \mathrm{Spec}(k)
\end{tikzcd}$$

then, by the universal property of fibred product, there exists a unique morphism of k-varieties $f : \mathrm{Spec}(k) \to X \times_{\mathrm{Spec}(k)} Y$ such that $\pi_X \circ f = g_X$ and $\pi_Y \circ f = g_Y$, then $f \in (X \times_{\mathrm{Spec}(k)} Y)(k)$ and $(\pi_X \circ f, \pi_Y \circ f) = (g_X, g_Y)$. Thus we have a natural bijection $(X \times_{\mathrm{Spec}(k)} Y)(k) \xrightarrow{\sim} X(k) \times Y(k)$. $\square$

Example 2.11. *Let k be a field. Let $\mathbb{A}^n_k$ be the n-dimensional affine space over k, then $\mathbb{A}^n_k(k)$ can be identified with the set k^n of coordinates on $\mathbb{A}^n_k$.*

If X is an affine variety over k, we can write $X = \operatorname{Spec}(k[x_1, \dots, x_n]/I)$ and think of X as a closed subvariety of $\mathbb{A}^n_k$. Then $X(k)$ can be identified with the subset $\{v \in k^n : f(v) = 0, \forall f \in I\}$ of k^n.

Example 2.12. *Let k be a field. Let $\mathbb{P}^n_k$ be the n-dimensional projective space k, then $\mathbb{P}^n_k(k)$ can be identified with the set $(k^{n+1} \setminus \{0\})/k^\times = \{(v_0 : \cdots : v_n) \in \mathbb{P}(k^{n+1})\}$ of homogeneous coordinates on $\mathbb{P}^n_k$.*

If X is a projective variety over k, we can think of X as a closed subvariety of $\mathbb{P}^n_k$ for some n, let I be the homogenous ideal of $k[x_0, \dots, x_n]$ corresponding to X, then $X(k)$ can be identified with the subset

$$\{(v_0 : \cdots : v_n) \in \mathbb{P}(k^{n+1}) : f(v_0, \dots, v_n) = 0, \forall f \in I\} = \\ = (\{v \in k^{n+1} : f(v) = 0, \forall f \in I\} \setminus 0)/k^\times$$

of the set of homogeneous coordinates of $\mathbb{P}^n_k$.

Remark 2.13. Let X be a variety over a field k and K/k any field extension. After Examples 2.11 and 2.12 we see that the inclusion $X(k) \subset X_K(K)$ given in Corollary 2.8 is not just an injective map, but a real inclusion on local coordinates (over an affine open covering), compatible with the inclusion $k \subset K$.

Remark 2.14. From Examples 2.11 and 2.12 we get that $\mathbb{A}^n_k(k)$ and $\mathbb{P}^n_k(k)$ are nonempty over any field k and for all $n \geq 0$. The property of having rational points holds for all varieties over a separably closed field, as stated in Proposition 2.20. We will see later examples of varieties without rational points over some non separably closed fields.

Example 2.15. *Let k be a field, L/k a finite Galois extension and Y a variety over L. Then, if the Weil restriction $\mathfrak{R}_{L/k}(Y)$ exists, we can identify the set of L-rational points of X with the set of k-rational points of $\mathfrak{R}_{L/k}(Y)$. Indeed, $Y(L) = Hom_L(\operatorname{Spec}(L), Y)$ and $(\mathfrak{R}_{L/k}(Y))(k) = Hom_k(\operatorname{Spec}(k), \mathfrak{R}_{L/k}(Y))$ by Definition 2.6, and there is a bijection*

$$Hom_k(\operatorname{Spec}(k), \mathfrak{R}_{L/k}(Y)) \xrightarrow{\sim} Hom_L(\operatorname{Spec}(L), Y)$$

by Definition 1.30. Moreover this identification is functorial in Y (see [Bos], 7, §7.6 for more details).

Proposition 2.16. *Let X be a variety over a field k. If k is algebraically closed, then $X(k)$ is the set of closed points of X.*

Proof. Without loss of generality we can assume that X is affine, then Proposition 2.16 is a consequence of Hilbert's Nullstellensatz in its weak form (see [A-M], 5, Exercise 17). $\square$

Proposition 2.17. *Let X be a variety over a field k. If k is algebraically closed, then $X(k)$ is dense in X.*

Proof. Without loss of generality we can assume that X is affine. From Proposition 2.16 we have that $X(k)$ is the set of closed points of X. Let A be a finitely generated k-algebra such that $X \cong \mathrm{Spec}(A)$, since the Jacobson ideal of A, i.e. the intersection of all the maximal ideals of A, is zero (see [A-M], 1, Exercises 2, 3 and 4), then for all $a \in A$ there is a maximal ideal m of A such that $a \notin m$, then the closed point corresponding to m belongs to the basic open set $\mathrm{Spec}(A[\frac{1}{a}])$. Thus the set of closed points of X is dense in X and also $X(k)$ is dense in X. □

Definition 2.18. Let k be a field and $X \subset \mathbb{A}^n_k$, $Y \subset \mathbb{A}^m_k$ two affine subvarieties, let $(\alpha_1, \dots, \alpha_n), (\beta_1, \dots, \beta_m)$ be the coordinates induced by $\mathbb{A}^n_k, \mathbb{A}^m_k$ on X, Y respectively. We say that a map $f : X(k) \dashrightarrow Y(k)$ is defined by rational functions on the coordinates if there are m rational functions $f_1, \dots, f_m \in k(x_1, \dots, x_n)$ and a nonempty open subset $U \subset X$ such that $f(\underline{\alpha}) = (f_1(\underline{\alpha}), \dots, f_m(\underline{\alpha}))$ for all $\underline{\alpha} = (\alpha_1, \dots, \alpha_n) \in U(k)$.

Proposition 2.19. *Let X, Y be two affine varieties over an algebraically closed field k, let $f : X(k) \dashrightarrow Y(k)$ be a map which can be defined by rational functions on the coordinates, then f induces a rational map of k-varieties $X \dashrightarrow Y$ whose restriction to $X(k)$ is f.*

Proof. Without loss of generality we can assume that $X = \mathrm{Spec}(k[x_1, \dots, x_n]/I)$ and $Y = \mathrm{Spec}(k[y_1, \dots, y_m]/J)$, where I, J are the ideals defining $X \subset \mathbb{A}^n_k$ and $Y \subset \mathbb{A}^m_k$ respectively. Under the identification introduced in Example 2.11, choose a system of coordinates on X and Y:

$$X(k) = \{\underline{\alpha} = (\alpha_1, \dots, \alpha_n) \in k^n : g(\alpha_1, \dots, \alpha_n) = 0, \forall g \in I\}$$
$$Y(k) = \{\underline{\beta} = (\beta_1, \dots, \beta_m) \in k^n : h(\beta_1, \dots, \beta_m) = 0, \forall h \in J\}$$

Let $f_1, \dots, f_m \in k(x_1, \dots, x_n)$ be the rational functions defining f, say $f_i = g_i/h_i$ with $g_i, h_i \in k[x_1, \dots, x_n]$ such that $gcd(g_i, h_i) = 1$ for $i = 1, \dots, m$. Let $d = \prod_{i=1}^m h_i \in k[x_1, \dots, x_n]$ then $df_i \in k[x_1, \dots, x_n]$ for all $i = 1, \dots, n$ and $U = \mathrm{Spec}((k[x_1, \dots, x_n)[\frac{1}{d}])/I)$ is the largest open subset of X such that $f(\underline{\alpha}) = (f_1(\underline{\alpha}), \dots, f_m(\underline{\alpha}))$ is well defined for all $\underline{\alpha} \in U(k)$. Thus U is nonempty by Definition 2.18.

Let define a morphism of rings $\varphi : k[y_1, \dots, y_m]/J \to (k[x_1, \dots, x_n][\frac{1}{d}])/I$ by $\varphi(y_i) = f_i(x_1, \dots, x_n)$ for $i = 1, \dots, m$. We note that φ is well defined, indeed if $h \in J$, then $\varphi(h) = h(\varphi(y_1), \dots, \varphi(y_m)) \in I$, as $h(f_1(\underline{\alpha}), \dots, f_m(\underline{\alpha})) = 0$ for all $\underline{\alpha} \in U(k)$ by the definition of f. Let $F : U \to Y$ be the morphism induced by φ. For any $\underline{\alpha} \in U(k)$ let $m_{\underline{\alpha}}$ be the ideal defining $\underline{\alpha}$ as closed point of X, $m_{\underline{\alpha}}$ is generated by $\{x_i - \alpha_i : i = 1, \dots, n\}$. One can easily verify that $\varphi(y_j - f_j(\alpha)) = f_j(\underline{x}) - f_j(\underline{\alpha}) \in m_{\underline{\alpha}}$ for all $j = 1, \dots, m$. Let $m_{f(\underline{\alpha})}$ be the ideal of $k[y_1, \dots, y_m]/J$ generated by $y_j - f_j(\underline{\alpha}), j = 1, \dots, m$. Since $m_{f(\underline{\alpha})}$

is a maximal ideal contained in $\varphi^{-1}(m_\alpha)$, we have that $\varphi^{-1}(m_\alpha) = m_{f(\alpha)}$ is the ideal defining $f(\alpha)$ as closed point in Y. Thus $F|_{U(k)} = f|_{U(k)}$. The rational map $X \dashrightarrow Y$ represented by F is the desired map. □

Proposition 2.20. *Let X be a variety over a field k. If k is separably closed, then $X(k)$ is dense in X.*

Proof. See [Bor], AG, §13, Corollary 13.3. □

Proposition 2.21. *If k is an infinite field, then $\mathbb{P}^n_k(k)$ is dense in $\mathbb{P}^n_k$, $\forall n > 0$.*

Proof. Let U be a nonempty open subset of $\mathbb{P}^n_k$, since $\mathbb{P}^n_k$ is irreducible we have that U is dense in $\mathbb{P}^n_k$, then it is enough to prove that $U(k)$ is dense in U. Let $k[x_0, \ldots, x_n]$ be the polynomial ring defining $\mathbb{P}^n_k$, let U be the complement in $\mathbb{P}^n_k$ of the closed subvariety defined by the homogeneous ideal generated by x_0 in $k[x_0, \ldots, x_n]$, then $U \cong \mathrm{Spec}(k[y_1, \ldots, y_n]) = \mathbb{A}^n_k$. So it is enough to prove that $\mathbb{A}^n_k(k)$ is dense in $\mathbb{A}^n_k$. Let $f \in k[y_1, \ldots, y_n]$. Since k is infinite there is an element $v \in k^n$ such that $f(v) \neq 0$, let $x \in X(k)$ be the rational point corresponding to v under the identification stated in Example 2.11, then x belong to the basic open set $\mathrm{Spec}(k[y_1, \ldots, y_n][\frac{1}{f}])$. □

Proposition 2.22. *Let X, Y be irreducible varieties over a field k, let $f : X \dashrightarrow Y$ be a dominant rational map defined over k. If $X(k)$ is dense in X, then also $Y(k)$ is dense in Y.*

Proof. Let $U \subset X$ be an open subset on which f is defined, then $U(k) = U \cap X(k)$ is dense in U. Since f is dominant, $f(U(k))$ is dense in the image of f, which is dense in Y, so $f(U(k))$, which is a subset of $Y(k)$, is dense in Y. Then $Y(k)$ is dense in Y. □

Corollary 2.23. *Let X be a variety over an infinite field k, if X is unirational over k, then $X(k)$ is dense in X.*

Proof. Apply Propositions 2.21 and 2.22. □

Remark 2.24. In particular we see that, in the case of a variety X over an infinite field k, $X(k) \neq \emptyset$ is a necessary condition for rationality and unirationality. In Chapters 3 and 5 we will study some families of varieties for which this condition is equivalent to unirationality.

2.3 Galois descent

Let k be a field and $\overline{k}$ a separable closure of k.

Notation. We recall the notation introduced at the beginning of Chapter 2. Let X be a variety over k and K/k a field extension, then $X_K = X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(K)$, moreover we set $\overline{X} := X_{\overline{k}}$. If $x \in X(k)$, then $x_K = x \times_{\mathrm{Spec}(k)} \mathrm{Spec}(K) \in X_K(K)$ (see Corollary 2.8) and in particular $\overline{x} = x_{\overline{k}} \in \overline{X}(\overline{k})$.

Definition 2.25. Let K/k be any field extension, let Y be a variety over K, we say that Y is defined over k if there is a variety X over k such that $X_K \cong Y$ as K-varieties.

Let X, X' be varieties over k, let $f : X_K \to X'_K$ be a morphism of K-varieties, we say that f is defined over k if there is a morphism $g : X \to X'$ such that $f = g \times \mathrm{Id}_{\mathrm{Spec}(K)}$.

Proposition 2.26. *Let K/k be an algebraic field extension and Y a quasi-projective variety defined over K, then there is a finite extension k'/k, with $k' \subset K$, such that Y is defined over k.*

Proof. Since Y is quasi-projective, then it can be covered by finitely many open affine subsets $\{U_i\}_{i=1,\dots,m}$. Fix $i \in \{1, \dots, m\}$, then $U_i = \mathrm{Spec}(K[x_1, \dots, x_n]/I)$ for some n and some ideal I of $K[x_1, \dots, x_n]$. Since $K[x_1, \dots, x_n]$ is a noetherian ring, we have that I is generated by finitely many polynomials $f_1, \dots f_r \in K[x_1, \dots, x_r]$. Let k_i be the extension of k generated by the coefficients of $f_1, \dots, f_r$, then k_i/k is a finite extension and

$$U_i = \mathrm{Spec}(k_i[x_1, \dots, x_n]/(f_1, \dots, f_r)) \times_{\mathrm{Spec}(k_i)} \mathrm{Spec}(K)$$

i.e. U_i is defined over k_i. Let k' be the subfield of K generated by $k_1, \dots, k_m$, then k'/k is a finite extension, as k' is generated over k by finitely many algebraic elements. For $i = 1, \dots, m$ we have that U_i is defined over k', let U'_i be the variety over k' such that $U'_{i,K} = U_i$. It is easy to see that we can glue $U'_1, \dots, U'_m$ to obtain a variety X over k' such that $X_K = Y$. $\square$

Definition 2.27. Let k be a field, L/k a Galois extension and $G = \mathrm{Gal}(L/k)$. For all $g \in G$ let denote by $g : \mathrm{Spec}(L) \to \mathrm{Spec}(L)$ the k-automorphism of $\mathrm{Spec}(L)$ given by the automorphism $g^{-1} : L \to L$.

Let X be a variety over k. For all $g \in G$ and all affine open subset $U = \mathrm{Spec}(A)$ of X, let denote by $g_U : U_L \to U_L$ the automorphism given by the automorphism $g^{-1} : A \otimes_k L \to A \otimes_k L$, which is induced by the natural action of G on L. We denote by $g : X_L \to X_L$ the automorphism obtained glueing the automorphisms g_U over an open affine covering of X.

Remark 2.28. The above definition gives an action of G over X_L, that we call the natural action of G over X_L. Moreover, for all $g \in G$ we have a commutative diagram

$$\begin{tikzcd} X_L \arrow[r, "g"] \arrow[d] & X_L \arrow[d] \\ \mathrm{Spec}(L) \arrow[r, "g"] & \mathrm{Spec}(L) \end{tikzcd}$$

where the vertical arrows are the morphism that define the L-scheme structure on X_L.

Remark 2.29. In the situation of Definition 2.27, if we identify $U_L(L)$ to the set of coordinates on U_L as in Examples 2.11, we have that the natural action of G over U_L restricted to $U_L(L)$ coincides with the action of G on the coordinates of the L-rational points on U_L.

Moreover the natural action of G over $X_L = X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(L)$ coincides with the natural action of G over the right factor, i.e. for all $g \in G$ there is a commutative diagram

$$\begin{tikzcd} X_L \arrow[r, "g"] \arrow[d] & X_L \arrow[d] \\ X \arrow[r, "\mathrm{Id}"] & X \end{tikzcd}$$

where the vertical arrows are the projections on the left factor. And $X(k)$ is the set of points in $X_L(L)$ that are stable under the natural action of G on X_L.

Proposition 2.30. *Let L/k be a finite Galois extension of fields and $G = \mathrm{Gal}(L/k)$, let Y be a quasi-projective scheme over L and suppose that there is a collection of endomorphisms $\{\alpha_g, g \in G\}$ of Y as scheme such that $\alpha_{g_1 g_2} = \alpha_{g_1} \circ \alpha_{g_2}$ for all $g_1, g_2 \in G$ and the following diagram*

$$\begin{tikzcd} Y \arrow[r, "\alpha_g"] \arrow[d] & Y \arrow[d] \\ \mathrm{Spec}(L) \arrow[r, "g"] & \mathrm{Spec}(L) \end{tikzcd}$$

commutes for all $g \in G$, where $g : \mathrm{Spec}(L) \to \mathrm{Spec}(L)$ is the automorphism associated to $g \in G$ induced by the natural action of G over $\mathrm{Spec}(L)$. Then there exists a quasi- projective scheme X over k such that there is an *isomorphism of L-schemes $f : Y \to X_L$ and the following diagram*

$$\begin{tikzcd} Y \arrow[r, "\alpha_g"] \arrow[d, "f"'] & Y \arrow[d, "f"] \\ X_L \arrow[r, "g"] & X_L \end{tikzcd}$$

commutes for all $g \in G$, where $g : X_L \to X_L$ is the automorphism associated to $g \in G$ induced by the natural action of G over X_L.

Proof. See [Bos], 6, §6.2, Example B. □

Proposition 2.31. *Let X be a quasi-projective variety over a field k and L/k a Galois extension. If Z is a closed subvariety of X_K such that Z is invariant under the natural action of* $\mathrm{Gal}(L/k)$*, then Z is defined over k.*

Proof. By Proposition 2.26 we can assume that L/k is finite, then apply Proposition 2.30 to Z with the family of endomorphism $\{g : Z \to Z, g \in G\}$ given by the natural action of G. The hypothesis of Proposition 2.30 are satisfied by Remark 2.28. □

Definition 2.32. Let X, Y be two varieties over a field k, L/k a Galois extension and $G = \mathrm{Gal}(L/k)$. Let $Hom_L(X_L, Y_L)$ be the set of morphisms from X_L to Y_L as L-schemes.

For all $f \in Hom_L(X_L, Y_L)$ and all $g \in G$ we set $g.f := g \circ f \circ g^{-1}$. This defines a action of G over $Hom_L(X_L, Y_L)$ called the natural action of G over $Hom_L(X_L, Y_L)$.

Proposition 2.33. *Let X, Y be two varieties over a field k, L/k a Galois extension and $G =$ $\mathrm{Gal}(L/k)$. Let $f \in Hom_L(X_L, Y_L)$, then f is defined over k if and only if f is fixed by the natural action of G over $Hom_L(X_L, Y_L)$.*

Proof. It is an easy consequence of Definition 2.32 and Remark 2.28. □

Remark 2.34. Let X be a variety over a field k, L/k a Galois extension and $G = \mathrm{Gal}(L/k)$. Let $\mathscr{F}$ be a quasi coherent sheaf of $\mathcal{O}_{X_L}$-modules on X_L. The natural action of G over X_L gives for any $g \in G$ an automorphism of X_L as scheme, so in particular an automorphism of the structural sheaf $\mathcal{O}_{X_L}$, that induces an automorphism $g : \mathscr{F} \to \mathscr{F}$. We can conclude that we have a well defined action of G over $\mathscr{F}$.

Proposition 2.35. *Let X be a surface over a field k, L/k a Galois extension and $G =$ $\mathrm{Gal}(L/k)$. Then the natural action of G over X_L induces an action of G over* $\mathrm{Pic}(X_L)$ *and the intersection pairing* $\mathrm{Pic}(X_L) \times \mathrm{Pic}(X_L) \to \mathbb{Z}$ *defined by the formula (2.1) is invariant under the action of G, i.e. for any $\mathscr{L}, \mathscr{M} \in$* $\mathrm{Pic}(X)$ *and any $g \in G$ we have $(g(\mathscr{L})).(g(\mathscr{M})) = \mathscr{L}.\mathscr{M}$.*

Proof. Let $g \in G$, since $g : X_L \to X_L$ is an automorphism, we have that if C is an integral curve in X_L then $g(C)$ is again an integral curve in X_L. So G induces an action on the group of divisors of X.

Let f be a rational function on X_L, C an integral curve in X_L and η its generic point, then $\mathcal{O}_{X_L,\eta}$ is a discrete valuation ring whose field of fractions is the field of functions of C, let denote by v_C the valuation of $\mathcal{O}_{X_L,\eta}$. $\mathcal{O}_{X_L,\eta}$ is a subring of its completion which is isomorphic to $L[[t]]$, the formal power series ring in one variable over L (see [A-M], Theorem 11.22 and Lemma 10.23). For every $g \in G$, let $g(\eta)$ be the generic point of $g(C)$, then $\mathcal{O}_{X_L,g(\eta)}$ is isomorphic over L to the subring $g^{-1}(\mathcal{O}_{X_L,\eta})$ of $L[[t]]$. Under these identifications, f is an element in the field of fractions of $L[[t]]$, which is the ring of formal Laurent series $L((t))$. Then f ca be written uniquely as $f = \sum_{n \geq n_0} a_n t^n$, where $n_0 \in \mathbb{Z}$, $a_n \in L$ for all $n \geq n_0$ and $a_{n_0} \neq 0$. The action of G over $L((t))$ gives $g^{-1}(f) = \sum_{n \geq n_0} g^{-1}(a_n) t^n$ for all $g \in G$, since $g^{-1} : L \to L$ is an automorphism for all $g \in G$, we have that $v_{g(C)}(g^{-1}(f)) = n_0 = v_C(f)$ for all $g \in G$.

Let D be the principal divisor associated to f, then $D = \sum_C v_C(f)C$, where C runs over the integral curves in X_L, by definition (see Definition 1.22). If $g \in G$ we have that $g(D) = \sum v_C(f)g(C) = \sum v_{g(C)}(g^{-1}(f))g(C)$, then $g(D)$ is again a principal divisor of X_L. Thus the action of G on the group of divisors of X_L induces an action of G over $\mathrm{Pic}(X_L)$.

Now we want to prove that the intersection pairing defined by (2.1) is invariant under the action of G over $\mathrm{Pic}(X_L)$. Since the above defined action of G over $\mathrm{Pic}(X_L)$ is compatible with the group structure of $\mathrm{Pic}(X_L)$, and since every divisor of X_L can be written as formal sum of integral curves, it is enough to prove that the intersection pairing is invariant under the action of G in the case of two integral curves. Let C, D be two integral curves in X, and consider $\mathscr{L} = \mathcal{O}_{X_L}(C)$ and $\mathscr{M} = \mathcal{O}_{X_L}(D)$.

From Proposition 2.4 we know that the intersection pairing is invariant under field extension. Let K be an algebraic closure of L, then combining Proposition 2.4 and Theorem 1.36 we get $\mathscr{L}.\mathscr{M} = C_K.D_K$. The action of G over L extends to an action of G over K, which is the action induced on K as L-vector space. So, without loss of generality, we can assume that C_K, D_K are irreducible, without common irreducible components and meet transversally (see [Har], V, §1, Lemma 1.2), then $C_K.D_K = \sum_{P \in C_K \cap D_K} (C_K.D_K)_P = \#(C_K \cap D_K)$, again by Theorem 1.36.

Let $g \in G$, since $g : X_K \to X_K$ is an automorphism, we have that $g(C_K)$ and $g(D_K)$ are irreducible and $g(C_K) \neq g(D_K)$, then $g(C_K)$ and $g(D_K)$ have no common irreducible components.

Let P be a closed point in X_K, we have that $P \in X_K(K)$ by Proposition 2.16, and $k(P) = K$ by Proposition 2.7. Then $\mathcal{O}_{X_K,P}$ is a subring of its completion which is isomorphic to $K[[u, v]]$, the formal power series ring in two variables over K (see [A-M], 11, Remark 2 after Proposition 11.24). Let

f_C, f_D be local equations of C_K, D_K in $\mathcal{O}_{X_K,P}$, they can be seen as formal power series in $K[[u,v]]$. The action of G over K induces an action of G over $K[[u,v]]$. Then for any $g \in G$, we have that the local equations of $g(C), g(D)$ in $\mathcal{O}_{X_K,P}$ correspond to $g^{-1}(f_C), g^{-1}(f_D)$ in $K[[u,v]]$. Moreover we have that $g^{-1}(\mathcal{O}_{X_K,P})$ is the local ring $\mathcal{O}_{X_K,g(P)}$ in $K[[u,v]]$.

Fix $g \in G$. $\mathcal{O}_{X_K,P}/(f_C, f_D)$ has dimension 1 as K-vector space if and only if m_P is generated by f_C, f_D, where m_P is the maximal ideal of $\mathcal{O}_{X_K,P}$, if and only if $g^{-1}(f_C), g^{-1}(f_D)$ generate $g^{-1}(m_P)$, but $g^{-1}(m_P) = m_{g(P)}$ is the maximal ideal of $g^{-1}(\mathcal{O}_{X_K,P})$, then $P \in C_K \cap D_K$ if and only if $g(P) \in g(C_K) \cap g(D_K)$, and $g(C_K), g(D_K)$ meet transversally. Thus we have

$$\begin{aligned} g(C_K).g(D_K) &= \sum_{P \in g(C_K) \cap g(D_K)} (g(C_K).g(D_K))_P = \\ &= \#(g(C_K) \cap g(D_K)) = \#(C_K \cap D_K) = C_K.D_K \end{aligned}$$

Since $g(C_K) = g(C)_K$, as they are defined by the same equation $g^{-1}(f_C)$ in $K[[u,v]]$, and similarly $g(D_K) = g(D)_K$; using the invariance of the intersection pairing under field extension (proved in Proposition 2.4), we conclude that

$$g(\mathscr{L}).g(\mathscr{M}) = g(C)_K.g(D)_K = C_K.D_K = \mathscr{L}.\mathscr{M}$$

□

Corollary 2.36. *Let X be a surface over a field k, L/k a Galois extension and $G = \mathrm{Gal}(L/k)$. The natural action of G over X_L induces an action over the set of (-1)-curves of X_L.*

Proof. Let $g \in G$ and let E be a (-1)-curve of X_L. By Proposition 2.35 we have $g(E)^2 = E^2 = -1$, then it is enough to prove that $g(E)$ is L-isomorphic to $\mathbb{P}^1_L$. Let take an isomorphism $f : C \to \mathbb{P}^1_L$ defined over L. Since $\mathbb{P}^1_L = \mathbb{P}^1_k \times_{\mathrm{Spec}(k)} \mathrm{Spec}(L)$, we have a well defined natural action of G over $\mathbb{P}^1_L$ as in Definition 2.27, then we have isomorphism $g \circ f \circ g^{-1} : g(E) \to \mathbb{P}^1_L$ defined over L. Indeed from the commutative diagram in Remark 2.28 applied to X_L and $\mathbb{P}^1_L$, we get a commutative diagram

$$\begin{tikzcd} g(C) \arrow[d, "g \circ f \circ g^{-1}"'] & C \arrow[l, "g"'] \arrow[r] \arrow[d, "f"'] & \mathrm{Spec}(L) \arrow[d, "\mathrm{Id}"'] \\ \mathbb{P}^1_L & \mathbb{P}^1_L \arrow[l, "g"] \arrow[r] & \mathrm{Spec}(L) \end{tikzcd}$$

□

Definition 2.37. Let X be a variety over k, a sub-variety of $\overline{X}$ is said Galois invariant if it is invariant under the natural action of $\Gamma_k := \mathrm{Gal}(\overline{k}/k)$ over $\overline{X}$.

Remark 2.38. In general, if X is a variety over k, L/k is a Galois extension and Z is a sub-variety of X_L, we say that Z is Galois invariant if $\overline{Z}$ is invariant under the natural action of Γ_k over $\overline{X}$, i.e. if Z is invariant under the natural action of $\mathrm{Gal}(L/k)$ over X_L.

Proposition 2.39. *Let X be a surface over a field k. If $\{E_1, \dots, E_r\}$ is a Galois invariant collection of pairwise disjoint (-1)-curves in $\overline{X}$, then there exists a surface X' over k and a birational morphism $X \to X'$ defined over k, such that its extension $\overline{X} \to \overline{X'}$ is a birational morphism contracting exactly $E_1, \dots, E_r$.*

Proof. Let H be a very ample divisor on X, such that $\overline{H}$ is very ample on $\overline{X}$, we have that $\overline{H}$ is invariant under the action of Γ_k, then $\overline{H}.E_i = \overline{H}.E_j$ for all $i, j \in \{1, \dots, r\}$ by Proposition 2.35, and $H' = \overline{H} + \sum_{i=1}^{r} (\overline{H}.E_i)E_i$ is Γ_k-invariant, hence defined over k by Proposition 2.31. Since $E_i^2 = -1$ and $E_i.E_j = 0$ for all $i, j \in \{1, \dots, r\}, i \neq j$, we have that $H'|_{E_i} \cong \mathcal{O}_{E_i}$ for all $i = 1, \dots, r$. Following the proof of Theorem 1.47 (see [Har], V, §5, Theorem 5.10), we can show that H' is generated by global sections on $\overline{X}$, so we get a birational surjective morphism $\varphi : \overline{X} \to X' := \mathrm{Proj}(\sum_{n \geq 0} H^0(\overline{X}, \mathcal{O}(nH')))$, where X' is a surface. We note that since H' is defined over k, also X' is defined over k, moreover, since φ is a k-isomorphism of $\overline{X} \setminus \{E_1, \dots, E_r\}$ onto its image and it contracts every E_i, for $i = 1, \dots, r$, and the collection $\{E_1, \dots, E_r\}$ is Γ_k-invariant, we can conclude that φ is fixed by the action of Γ_k and then defined over k by Proposition 2.33. □

Proposition 2.40. *Let X be a surface over a field k, then X is minimal if and only if $\overline{X}$ admits no Galois invariant collection of pairwise disjoint (-1)-curves.*

Proof. Suppose that X is not minimal and let $\varphi : X \to X'$ be a birational morphism to a surface X' non isomorphic to X. By Proposition 1.46, $\overline{X}$ admits a (-1)-curve E_1 contracted by φ and contains only finitely many such curves, let $\{E_1, \dots, E_r\}$ be the orbit of E_1 under the action of $\Gamma_k = \mathrm{Gal}(\overline{k}/k)$. Then, by Corollary 2.36, $\{E_1, \dots, E_r\}$ is a Galois invariant collection of (-1)-curves on $\overline{X}$. By Hodge index theorem (Theorem 1.41), we have that the intersection form on $\mathbb{Z}E_1 + \dots + \mathbb{Z}E_r$ is negative defined, so, for all $i, j \in \{1, \dots, r\}, i \neq j$, the determinant of the matrix

$$\begin{pmatrix} E_i^2 & E_i.E_j \\ E_j.E_i & E_j^2 \end{pmatrix}$$

is positive, that means $1 - (E_i.E_j)^2 > 0$, and in particular $E_i.E_j = 0$ for all $i, j \in \{1, \dots, r\}, i \neq j$, since $E_i.E_j \in \mathbb{Z}, \forall i, j \in \{1, \dots, r\}$. Thus $E_i \cap E_j = \emptyset$

for all $i, j \in \{1, \ldots, r\}, i \neq j$, and finally $\{E_1, \ldots, E_r\}$ is also a set of pairwise disjoint curves.

Conversely, suppose that $\{E_1, \ldots, E_r\}$ is a Galois invariant collection of pairwise disjoint (-1)-curves, by Proposition 2.39 there is a surface X' over k and a birational morphism $X \to X'$ which is not an isomorphism, thus X is not minimal. $\square$

Proposition 2.41. *Let X be a variety over a field k, $x \in X$ a closed point such that $k(x)$ is a separable extension of k and L/k a Galois extension such that $k(x) \subset L$. Then $x_L \subset X_L$ is the disjoint union of $[k(x) : k]$ closed points and the blowing-up of X_L with center x_L over L is a succession of monoidal transformations with these points as centers, composed in any order.*

Proof. Without loss of generality we can assume that X is affine. Let $X = \mathrm{Spec}(A)$ where A is a k-algebra of finite type, let m be the maximal ideal of A defining x, then $k(x) \cong A/m$, then $x \cong \mathrm{Spec}(A/m) \cong \mathrm{Spec}(k(x))$ as subscheme of X. Since the extension $k(x)/k$ is finite (see Proposition 2.9) and separable we can apply the primitive element theorem, so $k(x) \cong k[t]/(P)$ for a suitable monic polynomial $P \in k[t]$ of degree $[k(x) : k]$. Since $k(x) \subset L$, the polynomial P splits in $[k(x) : k]$ distinct linear factors over L. Then we have $k(x) \otimes_k L \cong \prod_{i=1}^{[k(x):k]} L$ and $x_L \cong \mathrm{Spec}(k(x) \otimes_k L)$ is a union of $n = [k(x) : k]$ distinct closed points $x_1, \ldots, x_n$ of X_L.

Let $f : X' \to X_L$ be the blowing up of X_L with center x_L, then by the universal property of blowing up, we can factor f through successive monoidal transformations f_i of center x_i, for $i = 1, \ldots, n$. So we get a diagram:

$$\begin{tikzcd} X'' \arrow[rr, "f_1 \circ \cdots \circ f_n"] & & X_L \\ & X' \arrow[ul, "g"] \arrow[ur, "f"'] & \end{tikzcd}$$

where g is a birational morphism which does not contract any curve, hence an isomorphism. Thus we have that f is the same as $f_n \circ \cdots \circ f_1$. $\square$

Chapter 3

Severi-Brauer varieties

Severi-Brauer varieties are the first example of varieties over a field k that are rational over an algebraic closure of k and for which we know a criterion for their rationality over k. In Section 3.3 we will prove that a Severi-Brauer variety is rational over the ground field k if and only if it is isomorphic to some $\mathbb{P}^n_k$ if and only if it contains a k-rational point. Moreover we will give a condition on the field k which assures that every Severi-Brauer variety over k is rational over k.

The first two sections of this chapter are devoted to the classical definition of the Brauer group of a field k as the group of equivalence classes of central simple algebras over k and to its identification with a certain cohomological group. Even though that identification is a very classical result, we will give a sketch of the proof because it similar to the proof, given in details in the third section, of the identification between the set of isomorphism classes of Severi-Brauer varieties and a subset of the Brauer group, which is a more relevant result in our context. Actually, there is a closed connection between central simple algebras and Severi-Brauer varieties, but it will not be developed in this paper.

3.1 Central simple algebras

Let k be a field, a k-algebra is a vector space over k equipped with a bilinear binary operation compatible with scalars that makes it a ring. The dimension of a k-algebra is the dimension of the underlying vector space. Throughout this chapter a k-algebra will be a finite dimensional k-algebra.

Definition 3.1. A ring is simple if it is non zero and it has no nontrivial two-sided ideals.

Definition 3.2. A k-algebra A is a division algebra if every non zero element of A is a unit.

Remark. In particular a division algebra is simple.

Definition 3.3. Let S be a subset of a ring R, the centralizer of S in R is the set $C_R(S) := \{x \in R : xs = sx, \forall s \in S\}$. In particular $C(R) := C_R(R)$ is called the center of R.

We note that every k-algebra contains k in its center.

Definition 3.4. A k-algebra is central if its center is k.

Proposition 3.5. *Let D be a division algebra, then the algebra $M_n(D)$ of $n \times n$ matrices over D is simple and the center of $M_n(D)$ is identified to the center of D under the canonical embedding $D \to M_n(D)$ given by scalar matrices.*

Proof. For $1 \leq i, j \leq n$ let $e_{i,j}$ be the $n \times n$ matrix whose entries are all zero except the (i, j)-th which is 1, then $\{e_{i,j}\}_{1 \leq i,j \leq n}$ is a basis of $M_n(D)$. Let I be a nonzero two-sided ideal of $M_n(D)$, let $a = (a_{i,j})_{1 \leq i,j \leq n} \in I$ be a nonzero element, then for $e_{i,j} \in \{e_{i,j}\}_{1 \leq i,j \leq n}$ there exists a matrix $b \in I$ such that $e_{i,j}a = be_{i,j}$, which gives that $a_{j,k} = 0$ for all $k \neq j$, so we conclude that a is a nonzero scalar matrix. Since D is a division algebra we have that a is also invertible in $M_n(D)$, so $I = M_n(D)$.

We know that the center of $M_n(D)$ is contained in the subring of scalar matrices, and in fact in its center. Scalar matrices can be identified with the elements of D under the canonical embedding $D \to M_n(D)$, so the center of $M_n(D)$ is contained in the center of D, while the other inclusion is trivial. □

Definition 3.6. Let A be a ring, a simple A-module is a nonzero A-module with no nontrivial submodules.

Proposition 3.7 (Schur's Lemma)**.** *Let A be a k-algebra and S, S' two simple A-modules, then any nonzero morphism of A-modules $S \to S'$ is an isomorphism. Moreover $\mathrm{End}_A(S)$, the group of endomorphisms of S as A-module, is a division k-algebra.*

Proof. See [Lan], XVII, §1, Proposition 1.1. □

Proposition 3.8 (Double Centralizer Theorem)**.** *Let E be a central simple algebra, let A be a simple sub-algebra, then $C_E(C_E(A)) = A$.*

Proof. See [Jac], §4.6, Theorem 4.10. □

Theorem 3.9. *Let A be a simple k-algebra, then A is isomorphic to $M_n(D)$ for some n and some division k-algebra D.*

Proof. Let S be a simple A-module (for example a minimal left ideal of A), let $E := \mathrm{End}_A(S)$ be the group of endomorphisms of S as A-module. By left-multiplication we get a homomorphism $A \to E$ which is injective because it is nonzero and its kernel is reduced to 0, as the image of 1 is

Id_S and A is simple. So A is a simple subalgebra of E and $C_E(A) = E$. By Proposition 3.7 we have that E is a division algebra, so $S \cong E^n$ as E-algebra for some n. By Proposition 3.8 we have $A \cong C_E(C_E(A)) = \mathrm{End}_E(S) \cong \mathrm{End}_E(E^n) \cong M_n(D)$, where $D = E^{opp}$ is the division algebra E with reversed multiplication. □

Proposition 3.10. *A k-algebra A is central simple if and only if it is isomorphic to $M_n(D)$ for some n and some central division k-algebra D.*

Proof. Let A be a central simple algebra, by Theorem 3.9 A is isomorphic to a finite dimensional matrix algebra over a division algebra D, from Proposition 3.5 we get that the center of D is the same as the center of A, namely k. The converse comes from Proposition 3.5. □

Theorem 3.11. *Let A be a central simple k-algebra and B any k-algebra. If J is a two sided ideal of $A \otimes_k B$, then $J = A \otimes_k I$ for some two sided ideal I of B.*

Proof. See [Jac], §4.6, Corollary 1. □

Proposition 3.12. *The tensor product of two central simple k-algebras is a central simple k-algebra.*

Proof. Let A, B be two central simple k-algebras, from Theorem 3.11 we have that any two-sided ideal of $A \otimes_k B$ is of the form $A \otimes_k I$ where I is a two sided ideal of B, since B is simple we get that also $A \otimes_k B$ is simple. Moreover $C(A \otimes_k B) = C(A) \otimes_k C(B) \cong k$ since A and B are both central. □

Proposition 3.13. *Let A be a central simple k-algebra and L/k a field extension, then $A \otimes_k L$ is a central simple L-algebra.*

Proof. L is a field, hence simple, then as in the proof of Proposition 3.12 we get that $A \otimes_k L$ is simple, moreover $C(A \otimes_k L) = C(A) \otimes_k C(L) = k \otimes_k L \cong L$. □

Proposition 3.14. *Let A be a k-algebra and L/k a field extension such that $A \otimes_k L \cong M_n(L)$ for some n, then A is a central simple k-algebra.*

Proof. Suppose that A is not simple, let I be a nontrivial two-sided ideal of A, then $I \otimes_k L$ is a nontrivial two-sided ideal of $A \otimes_k L$ which contradicts the fact that $M_n(L)$ is simple by Proposition 3.5. Thus A is simple, moreover $C(A)$ is a field extension of k, i.e. a k-vector space, but $C(A) \otimes_k L \cong C(M_n(L)) = L$, so $C(A) = k$. □

Proposition 3.15. *Let A be a k-algebra, then A is central and simple if and only if there exists a finite Galois extension L/k such that $A \otimes_k L \cong M_n(L)$ for some n.*

Proof. For the direct implication see [Bou], VIII, §10, n°5, Corollaire 3. The converse comes from Proposition 3.14. □

3.2 Brauer group

Definition 3.16. We say that two central simple k-algebras A and B are similar if $A \otimes_k M_n(k) \cong B \otimes_k M_m(k)$ for some n and m, and in that case we write $A \sim B$. It is easy to see that similarity is an equivalence relation, we define $\mathrm{Br}(k)$ to be the set of central simple k-algebras modulo similarity.

Proposition 3.17. $\mathrm{Br}(k)$ *is a group with respect to* $\otimes_k$.

Proof. Since $M_n(k) \otimes_k M_m(k) \cong M_{nm}(k)$ for all n, m, if $A, A', B, B' \in \mathrm{Br}(k)$ such that $A \sim A'$ and $B \sim B'$, then $A \otimes_k B \sim A' \otimes_k B'$. Since $A \otimes_k M_n(k) \sim A$ for all n, then the class of k is a neutral element. Let A^{opp} be the algebra A with reversed multiplication, then $A \otimes_k A^{opp} \cong M_{n^2}(k)$ for some n (see [Jac], §4.6, Theorem 4.6), so the class of A^{opp} is an inverse to the class of A. □

Let L/k be a field extension, we have that $(A \otimes_k M_n(k)) \otimes_k L \cong (A \otimes_k L) \otimes_L M_n(L)$ and $(A \otimes_k B) \otimes_k L \cong (A \otimes_k L) \otimes_L (B \otimes_k L)$, for all central simple k-algebras A, B and all n. Then the extension of scalars $A \mapsto A \otimes_k L$ induces a well defined group homomorphism $\mathrm{Br}(k) \to \mathrm{Br}(L)$.

Definition 3.18. Let L/k be a field extension, we denote by $\mathrm{Br}(L/k)$ the kernel of the morphism $\mathrm{Br}(k) \to \mathrm{Br}(L)$ and we say that an element $A \in \mathrm{Br}(k)$ is split by L if $A \otimes_k L$ is a matrix algebra over L, i.e. $A \in \mathrm{Br}(L/k)$.

From Proposition 3.15 and the above definition we get that

$$\mathrm{Br}(k) = \varinjlim_{\substack{L/k \text{ finite} \\ \text{Galois}}} \mathrm{Br}(L/k) \tag{3.1}$$

For every Galois extension L/k let fix the notation $H^2(L/k) := H^2(\mathrm{Gal}(L/k), L^\times)$. Let $\overline{k}$ be a separable closure of k, from Proposition 1.55 we know that for every finite Galois extension L/k we have an exact sequence

$$0 \to H^2(L/k) \to H^2(\overline{k}/k) \to H^2(\overline{L}/L)$$

and that

$$H^2(\overline{k}/k) = \varinjlim_{\substack{L/k \text{ finite} \\ \text{Galois}}} H^2(L/k) \tag{3.2}$$

Proposition 3.19. *Let L be a field and ϕ an automorphism of $M_n(L)$ as L-algebra. Then ϕ is an inner automorphism, i.e. there exists $T \in GL_n(L)$ such that $\phi(A) = TAT^{-1}$ for all $A \in M_n(L)$.*

Proof. By linearity ϕ is completely determined by its action on rank one matrices in $M_n(L)$, moreover any rank one $n \times n$ matrix can be represented as xy^t where $x, y \in L^n$ are considered as $n \times 1$ matrices over L, and every product xy^t is a rank one $n \times n$ matrix.

Let fix two nonzero elements $u, y \in L^n$, then uy^t is a nonzero matrix in $M_n(L)$, then also $\phi(uy^t)$ is nonzero in $M_n(L)$ as ϕ is injective, let $z \in L^n$ such that $\phi(uy^t)z$ is nonzero. Let define $Tx := \phi(xy^t)z$ for all $x \in L^n$, then the linearity of ϕ give that T is a linear endomorphism of L^n, moreover $Tu \neq 0$ by the choice of z, so T is nonzero. Let $A \in M_n(L)$ and $x \in L^n$, we have

$$TAx = \phi(Axy^t)z = \phi(A)\phi(xy^t)z = \phi(A)Tx$$

and we get that $TA = \phi(A)T$. If $v \in L^n$, since $Tu \neq 0$ and ϕ is surjective, there exists $A \in M_n(L)$ such that $\phi(A)Tu = v$ but we have also $TAu = \phi(A)Tu = v$, so we get that T is surjective and hence invertible. So we conclude that $\phi(A) = TAT^{-1}$ for all $A \in M_n(L)$. □

Proposition 3.20. *Let L be a field, then* $\mathrm{Aut}_L(M_n(L)) \cong PGL_n(L)$ *for all* n.

Proof. The morphism of groups $GL_n(L) \to \mathrm{Aut}_L(M_n(L))$ that sends an invertible matrix T to the induced inner automorphism of $M_n(L)$ is surjective by Proposition 3.19. Moreover, if $T_1T_2 \in GL_n(L)$ induce the same automorphism of $M_n(L)$, we have $T_1AT_1^{-1} = T_2AT_2^{-1}$ for all $A \in M_n(L)$, which is equivalent to $T_2^{-1}T_1A = AT_2^{-1}T_1$ for all $A \in M_n(L)$, then $T_2^{-1}T_1$ has to be a scalar matrix, i.e. T_2 is a nonzero scalar multiple of T_1. Thus we have proved that in fact $\mathrm{Aut}_L(M_n(L)) \cong PGL_n(L)$. □

For every finite Galois extension L/k let $\mathrm{Br}_n(L/k)$ be the set of $A \in \mathrm{Br}(L/k)$ such that $A \otimes_k L \cong M_n(L)$. For every n we have $\mathrm{Br}_n(L/k) \subset \mathrm{Br}_{n+1}(L/k)$, then $\mathrm{Br}(L/k) = \varinjlim_n \mathrm{Br}_n(L/k)$.

Proposition 3.21. *Let L/k be a finite Galois extension and $G = \mathrm{Gal}(L/k)$, then for all positive integers n there is a bijection between $\mathrm{Br}_n(L/k)$ and $H^1(G, \mathrm{Aut}_L(M_n(L)))$, where $\mathrm{Aut}_L(M_n(L))$ is the group of automorphisms of $M_n(L)$ as L-algebra.*

Proof. For a definition of $H^1(G, \mathrm{Aut}_L(M_n(L)))$ see Section 1.3. Let define a map $\theta : \mathrm{Br}_n(L/k) \to H^1(G, \mathrm{Aut}_L(M_n(L)))$ in the following way: to an element $A \in \mathrm{Br}_n(L/k)$, which is a division k-algebra of dimension n^2, we associate an L-isomorphism $M_n(L) \to A \otimes_k L$ and we set $\theta(A)$ to be the map $\theta_f : G \to \mathrm{Aut}_L(M_n(L))$ that sends $g \in G$ to $\theta_f(g) := f^{-1} \circ g \circ f \circ g^{-1}$. An easy computation shows that each θ_f and θ are well defined, see [Der], §3 for the details. Another easy computation shows that θ is bijective, see [Der], §3, Proposition 3.1 or [Se1], X, §2, Proposition 4. □

Proposition 3.22. *Let L/k be a finite Galois extension, then there is an isomorphism* $\mathrm{Br}(L/k) \to H^2(L/k)$.

Proof. Let $G = \mathrm{Gal}(L/k)$. For every positive integer n there is an injective morphism of pointed sets $H^1(G, PGL_n(L)) \to H^2(L/k)$ by Proposition 1.58, while Proposition 3.21 and Proposition 3.20 give a bijection $\mathrm{Br}_n(L/k) \to H^1(G, PGL_n(L))$, so we have an injection $\delta_n : \mathrm{Br}_n(L/k) \to H^2(L/k)$. One can easily show that the δ_n are compatible with the inclusions $\mathrm{Br}_n(L/k) \subset \mathrm{Br}_{n+1}(L/k)$, so we get an injective map $\delta_L : \mathrm{Br}(L/k) = \varinjlim_n \mathrm{Br}_n(L/k) \to H^2(L/k)$ that is in fact a morphism of groups. Moreover one can show that $\delta_{[L:k]}$ is surjective (see [Der], §3, Theorem 3.2), then δ_L is also surjective and hence an isomorphism. □

Proposition 3.23. $\mathrm{Br}(k) \cong H^2(\overline{k}/k)$.

Proof. The isomorphisms δ_L for L/k finite Galois extension, are compatible with the inclusions $\mathrm{Br}(L/k) \to \mathrm{Br}(L'/k)$ and $H^2(L/k) \to H^2(L'/k)$ for $L \subset L'$ finite Galois extensions of k. So by the equalities (3.1), (3.2) and the properties of directs limits, we have an isomorphism $\mathrm{Br}(k) \to H^2(\overline{k}/k)$. □

Proposition 3.24. *i) If k is algebraically closed, then* $\mathrm{Br}(k) = 0$.

ii) If k is separably closed, then $\mathrm{Br}(k) = 0$.

iii) If k is a finite field, then $\mathrm{Br}(k) = 0$.

iv) If k is an extension of transcendence degree 1 over an algebraically closed field, then $\mathrm{Br}(k) = 0$.

v) If k is the maximal unramified extension of a p-adic field, then $\mathrm{Br}(k) = 0$.

vi) If k is an algebraic extension of $\mathbb{Q}$ containing all the roots of 1, then $\mathrm{Br}(k) = 0$.

Proof. i) and ii) come from Proposition 3.23, using the fact that if k is a separably closed field, then $\Gamma_k = 0$ and in particular $H^2(\overline{k}/k) = 0$.

For the rest, see [Se1], X, §7, Exemples de corps à groupe de Brauer nul. □

3.3 Severi-Brauer varieties

Let k be a field and K an algebraic closure of k.

Definition 3.25. A variety X over k is a Severi-Brauer variety if $X_K \cong \mathbb{P}^n_K$, where n is the dimension of X.

If X is a Severi-Brauer variety of dimension n over k and k'/k is an algebraic extension such that $X_{k'} \cong \mathbb{P}^n_{k'}$ we say that X splits over k', or, alternatively, that k' is a splitting field for X.

Let see two examples of Severi-Brauer varieties.

Example 3.26. *For all $n \geq 0$ and for all field extensions k'/k, we have that $\mathbb{P}^n_k \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k') \cong \mathbb{P}^n_{k'}$. Then $\mathbb{P}^n_k$ is a Severi-Brauer variety of dimension n over k that splits over any algebraic extension k'/k.*

Remark 3.27. From Example 3.26 we get that if X is a Severi-Brauer over k that splits over an algebraic extension k'/k, then every algebraic extension k''/k' is a splitting field for X.

Example 3.28. *Let X be an irreducible conic in $\mathbb{P}^2_k$, if $X(k) \neq \emptyset$ then the parametrization of X with the lines passing through a k-rational point of X gives an isomorphism $X \to \mathbb{P}^1_k$. Since X is defined by a polynomial of degree 2 then there is a quadratic extension k'/k such that $X_{k'}$ has a k'-rational point, then X is a Severi-Brauer variety of dimension 1 that splits either over k or over a quadratic extension of k.*

Theorem 3.29. *Let X be a Severi-Brauer variety over k, then X splits over k if and only if $X(k) \neq \emptyset$.*

Proof. If X has dimension 0, then the result is trivial. Let suppose then that X has dimension $n \geq 1$. If $X \cong \mathbb{P}^n_k$ then $X(k) \neq \emptyset$ by Example 2.12.

Conversely, if $X(k) \neq \emptyset$, let $x \in X(k)$ and $f : X' \to X$ be the monoidal transformation of center x, let E be the exceptional divisor associated to f, then $E \cong \mathbb{P}^{n-1}_k$ (see [Sha], II, §4.3). We have that $X_K \cong \mathbb{P}^n_K$, let consider x_K as a point in $\mathbb{P}^n_K(K)$. Let $f_K : X'_K \to \mathbb{P}^n_K$ be the extension of f to K, then f_K is the monoidal transformation with center x_K, so we can consider X'_K as a subvariety of $\mathbb{P}^n_K \times \mathbb{P}^{n-1}_K$. Let denote by $x_0, \dots, x_n$ be a system of homogeneous coordinates on $\mathbb{P}^n_K$ and $y_0, \dots, y_n$ coordinates on $\mathbb{P}^{n-1}_K$, without loss of generality we can assume that X'_K is defined by the equations $x_i y_j = x_j y_i$ for $i, j \in \{0, \dots, n-1\}$. Let $\pi : X' \to \mathbb{P}^{n-1}_K$ be the morphism induced by the projection on the second factor, we have that $\pi|_{E_K} : E_K \to \mathbb{P}^{n-1}_K$ is an isomorphism, indeed it is the extension to K of the isomorphism $E \cong \mathbb{P}^{n-1}_k$, moreover, from the equations of X, it easy to see that the fibers of π are lines of $\mathbb{P}^n_K$ contained in X'. Let L be a hyperplane in $E \cong \mathbb{P}^{n-1}_k$, then $\pi(L_K)$ is a hyperplane in $\mathbb{P}^{n-1}_K$. Since f_K is the morphism induced by the first projection of the fibred product $\mathbb{P}^n_K \times \mathbb{P}^{n-1}_K$ and it is an isomorphism outside E_K, then $f_K(\pi^{-1}(\pi(L_K)))$ is a hyperplane in $\mathbb{P}^n_K$.

Let H be an ample divisor on X, let d be the degree of H_K in $\mathbb{P}^n_K$ and $H' := f^*H - dE$, then H'_K is a divisor on X'_K.

If l is a fiber of π, then H' induces a divisor of degree 0 on l, then $\mathcal{O}_{X'_K}(H')|_l \cong \mathcal{O}_l$ and in particular $\mathcal{O}_{X'_K}(H')$ is generated by global sections on l. Thus H'_K is generated by global sections on X_K and it induces a morphism $\psi_K : X'_K \to \mathbb{P}^N_K$. which is the extension to K of the rational map $\psi : X' \to \mathbb{P}^N_k$ induced by H'. Then ψ is indeed a morphism.

We have also $\mathcal{O}_{X'_K}(H')|_{E_K} \cong \mathcal{O}_{E_K}(H'_K.E_K) \cong \mathcal{O}_{E_K}(d)$, then ψ factors as composition of π followed by the d-uple embedding of $\mathbb{P}^{n-1}_K$ in $\mathbb{P}^N_K$. Let $D = f(\psi^{-1}(\psi(L)))$, then $D_K = f_K(\pi^{-1}(\pi(L_K)))$ is a hyperplane in $X_K \cong \mathbb{P}^n_K$, hence a very ample divisor.

By Proposition 2.3 we have that D is very ample on X, let $\phi : X \to \mathbb{P}^m_k$ be the closed immersion induced by D (by Proposition 1.12), then its extension $\phi_K : X_K \to \mathbb{P}^m_K$ is the closed immersion induced by D_K. Since D_K is a hyperplane in $X_K \cong \mathbb{P}^n_K$, we get that ϕ_K is an isomorphism, then $m = n$ and also ϕ is an isomorphism. □

Corollary 3.30. *Let X be a Severi-Brauer variety over k, then there is a finite Galois extension L/k such that X splits over L.*

Proof. Let $\overline{k}$ be a separable closure of k, we use the notation introduced at the beginning of Section 2.3. By Proposition 2.20 we have $\overline{X}(\overline{k}) \neq \emptyset$, let take a point $z \in \overline{X}(\overline{k})$. By Proposition 2.26 there exists a finite extension k'/k, $k' \subset \overline{k}$, such that z is defined over k'. Since $k' \subset \overline{k}$ we have that k' is separable over k. Let L be the normal closure of k' over k, then L is a finite Galois extension of k and z is defined over L. Let $x \in X_L$ be the point such that $\overline{x} = z$, then $k(x) = L$ by Proposition 2.41 and $x \in X_L(L)$ by Proposition 2.7. Thus $X_L(L) \neq \emptyset$ and X splits over L by Theorem 3.29. □

Remark 3.31. After Corollary 3.30 and Remark 3.27 we have that a variety X over k is a Severi-Brauer variety if and only if there exists a finite Galois extension L/k such that $X_L \cong \mathbb{P}^n_L$, where n is the dimension of X.

For all positive integers n, let SB_n be the set of isomorphism classes of $n-1$-dimensional Severi-Brauer varieties over k. For every finite Galois extension L/k and every positive integer n, let $SB_n(L)$ be the set of Severi-Brauer varieties $X \in SB_n$ such that $X_L \cong \mathbb{P}^{n-1}_L$, then

$$SB_n = \varinjlim_{\substack{L/k \text{ finite} \\ \text{Galois}}} SB_n(L) \tag{3.3}$$

We note that for all finite Galois extension L/k we have a special element $\mathbb{P}^{n-1}_k \in SB_n(L)$ that makes $SB_n(L)$ a pointed set, and by Example 3.26 the transition maps in the direct limit (3.3) are morphisms of pointed sets.

Proposition 3.32. *Let L be a field, then the group of automorphisms of $\mathbb{P}^{n-1}_L$ as scheme over L is isomorphic to $PGL_n(L)$, for all positive integers n.*

Proof. See [Har], II, §7, Example 7.1.1, the proof given there works also if L is not algebraically closed. □

Proposition 3.33. *Let L/k be a finite Galois extension and $G = \mathrm{Gal}(L/k)$, for all positive integers n there is an isomorphism of pointed sets*

$$\theta_{L,n} : SB_n(L) \to H^1(G, PGL_n(L))$$

Proof. Let fix a finite Galois extension L/k and a positive integer n, let define $\theta_{L,n} : SB_n(L) \to H^1(G, PGL_n(L))$ in the following way: to each $X \in SB_n(L)$ we associate an isomorphism of L-schemes $f : \mathbb{P}_L^{n-1} \to X_L$ and set $\theta_{L,n}$ to be the map $\theta_f : G \to PGL_n(L)$ that sends $g \in G$ to $\theta_f(g) := f^{-1} \circ g \circ f \circ g^{-1}$.

1. We prove that the map θ_f is well defined: from Definition 2.32 we have that $g \circ f \circ g^{-1} = g.f$ is a morphism of L-schemes from $\mathbb{P}_L^{n-1}$ to X_L, and in fact an isomorphism, as g, g^{-1} are bijective maps. Then, for all $g \in G$, $\theta_f(g)$ is an automorphism of $\mathbb{P}_L^{n-1}$ as L-scheme, i.e. an element of $PGL_n(L)$, as stated in Proposition 3.32.

2. We prove that $\theta_{L,n}$ is well defined: let $X \in SB_n(L)$ and $f : \mathbb{P}_L^{n-1} \to X_L$ an isomorphism associated to X, then for all $g_1, g_2 \in G$ we have

$$\theta_f(g_1) \circ g_1.\theta_f(g_2) = f^{-1} \circ g_1.f \circ g_1.(f^{-1} \circ g_2.f) = \theta_f(g_1 g_2)$$

so $\theta_f \in H^1(G, PGL_n(L))$.
Let $f_i : \mathbb{P}_L^{n-1} \to X_L$, i=1,2, be two isomorphisms associated to X, then $f_2^{-1} \circ f_1$ is an automorphism of $\mathbb{P}_L^{n-1}$, i.e. an element of $PGL_n(L)$, by Proposition 3.32. We have that for all $g \in G$

$$(f_2^{-1} \circ f_1)^{-1} \circ \theta_{f_2}(g) \circ g.(f_2^{-1} f_1) = f_1^{-1} \circ f_2 \circ (f_2^{-1} \circ g \circ f_2 \circ g^{-1}) \circ g \circ f_2^{-1} \circ f_1 \circ g^{-1} = \theta_{f_1}(g)$$

then $\theta_{f_1} \sim \theta_{f_2}$ and they are the same element in $H^1(G, PGL_n(L))$. Let choose the identity map Id of $\mathbb{P}_L^{n-1}$ as isomorphism associated to $\mathbb{P}_k^{n-1} \in SB_n(L)$, then $\theta_{\mathrm{Id}}(g) = \mathrm{Id}$ for all $g \in G$, so θ_{Id} is the special element of $H^1(G, PGL_n(L))$ and $\theta_{L,n}$ is a well defined morphism of pointed sets.

3. We prove that $\theta_{L,n}$ is injective: let $X_1, X_2 \in SB_n(L)$ such that $\theta_{L,n}(X_1) = \theta_{L,n}(X_2)$, by part 2. we can choose the associated isomorphisms $f_i : \mathbb{P}_L^{n-1} \to X_{iL}$, i=1,2, such that $\theta_{f_1} = \theta_{f_2}$, i.e.

$$\begin{gathered} \theta_{f_1}(g) = \theta_{f_2}(g), \quad \forall g \in G \quad \Longleftrightarrow \\ f_1^{-1} \circ g \circ f_1 \circ g^{-1} = f_2^{-1} \circ g \circ f_2 \circ g^{-1}, \quad \forall g \in G \quad \Longleftrightarrow \\ f_2 \circ f_1^{-1} = g \circ f_2 \circ f_1^{-1} \circ g^{-1} = g.(f_2 \circ f_1^{-1}), \quad \forall g \in G \end{gathered}$$

then $f_2 \circ f_1^{-1} : X_{1L} \to X_{2L}$ is an isomorphism defined over k, then X_1 and X_2 are the same element in $SB_n(L)$ and $\theta_{L,n}$ is injective.

4. We prove that $\theta_{L,n}$ is surjective: let $\varphi \in H^1(G, PGL_n(L))$, for $g \in G$ let $g : \mathbb{P}_L^{n-1} \to \mathbb{P}_L^{n-1}$ be the morphism of schemes induced by $g^{-1} : L \to L$. Since $\varphi(g) : \mathbb{P}_L^{n-1} \to \mathbb{P}_L^{n-1}$ is a morphism of L-schemes for all $g \in G$, we have that the following diagram

$$\begin{tikzcd} \mathbb{P}_L^{n-1} \arrow[r, "\varphi(g) \circ g"] \arrow[d] & \mathbb{P}_L^{n-1} \arrow[d] \\ \mathrm{Spec}(L) \arrow[r, "g"] & \mathrm{Spec}(L) \end{tikzcd}$$

commutes for all $g \in G$. Moreover, if $g_1, g_2 \in G$ we have

$$\varphi(g_1 g_2) \circ (g_1 g_2) = \varphi(g_1) \circ (g_1 . \varphi(g_2)) \circ (g_1 \circ g_2) = (\varphi(g_1) \circ g_1) \circ (\varphi(g_2) \circ g_2)$$

then $\mathbb{P}_L^{n-1}$ with the collection of endomorphisms $\{\varphi(g) \circ g, g \in G\}$ verify the hypothesis of Proposition 2.30, so there exists a variety X over k and an isomorphism $f : \mathbb{P}_L^{n-1} \to X_L$ such that $f \circ \varphi(g) \circ g = g \circ f$, then $X \in SB_n(L)$ and $\theta_f = f^{-1} \circ g \circ f \circ g^{-1} = f^{-1} \circ f \circ \varphi(g) \circ g \circ g^{-1} = \varphi(g)$ for all $g \in G$, so $\theta_{n,L}(X) = \varphi$.

□

Proposition 3.34. *For any positive integer n there is an injective morphism of pointed sets $SB_n \to \mathrm{Br}(k)$.*

Proof. Let fix a positive integer n. By Proposition 1.58 we have an injective morphism of pointed sets $\delta_{n,L} : H^1(G, PGL_n(L)) \to H^2(L/k)$. Then $\delta_{n,L} \circ \theta_{n,L} : SB_n(L) \to H^2(L/k)$ is an injective morphism of pointed sets. For all $L \subset L'$ finite Galois extensions of k, the morphisms $\delta_{n,L} \circ \theta_{n,L}, \delta_{n,L'} \circ \theta_{n,L'}$ are compatible with the inclusions $SB_n(L) \subset SB_n(L')$ and $H^2(L/k) \subset H^2(L'/k)$, then, taking the direct limit in the equations 3.3 and 3.2, we get a well defined injective morphism of pointed sets $SB_n \to H^2(\overline{k}/k)$, where $\overline{k}$ is a separable closure of k. Moreover $H^2(\overline{k}/k) \cong \mathrm{Br}(k)$ by Proposition 3.23, then we obtain an injective morphism $SB_n \to \mathrm{Br}(k)$. □

Corollary 3.35. *If $\mathrm{Br}(k) = 0$, then every Severi-Brauer variety over k splits over k.*

Proof. Indeed if $\mathrm{Br}(k) = 0$, then also $SB_n = 0$ for all $n \geq 1$ by Proposition 3.34. Then for all $n \geq 1$ there is only one isomorphism class of Severi-Brauer varieties of dimension n, i.e. every Severi-Brauer of dimension n is isomorphic to $\mathbb{P}_k^n$. □

Remark 3.36. For each of the fields k listed in Proposition 3.24 and for every $n \geq 0$, we have that, up to isomorphism, $\mathbb{P}_k^n$ is the only Severi-Brauer variety of dimension n over k.

Chapter 4

Del Pezzo surfaces

Del Pezzo surfaces are the second example of varieties over a field k that are rational over an algebraic closure of k and for which we know some sufficient conditions for their unirationality over k.

This chapter is devoted to the classification of Del Pezzo surfaces by degree. We will prove that over an algebraically or separably closed field they are rational, in particular they are, up to isomorphism, either $\mathbb{P}^1_k \times \mathbb{P}^1_k$ or a blowing-up of $\mathbb{P}^2_k$ in at most 8 k-rational points that satisfy some special conditions, but over an arbitrary field this is not always the case, in Section 4.3 we give some counter examples. The properties of Del Pezzo surfaces and their (-1)-curves over an algebraically or separably closed field are studied in detail in Sections 4.2 and 4.4, while the question about their rationality or unirationality over an arbitrary field is investigated in the next chapter.

4.1 Definition and examples

Let k be a field.

Definition 4.1. A surface X over k is a Del Pezzo surface if its anticanonical divisor $-K_X$ is ample.

In this section we see some examples of Del Pezzo surfaces. By Proposition 2.3 we can assume, without loss of generality, that k is algebraically closed. This fact will be better explained in Section 4.3.

Example 4.2. *The projective plane $\mathbb{P}^2_k$ is a Del Pezzo surface. Indeed, we have that $\mathrm{Pic}(\mathbb{P}^2_k) \cong \mathbb{Z}$ and any invertible sheaf on $\mathbb{P}^2_k$ is isomorphic to $\mathcal{O}_{\mathbb{P}^2_k}(n)$ for some $n \in \mathbb{Z}$ (see [Har], II, §6, Proposition 6.4), moreover $\mathcal{O}_{\mathbb{P}^2_k}(n)$ is ample if and only if $n > 0$ (see [Har], II, §7, Example 7.6.1). Now, $\omega_{\mathbb{P}^2_k} \cong \mathcal{O}_{\mathbb{P}^2_k}(-3)$ (see [Har], II, §8, Example 8.20.1), then $\mathcal{O}(-K_{\mathbb{P}^2_k}) \cong \mathcal{O}_{\mathbb{P}^2_k}(3)$ is ample and $\mathbb{P}^2_k$ is a Del Pezzo surface.*

Let H be a line on $\mathbb{P}^2_k$, then its associated sheaf is $\mathcal{O}_{\mathbb{P}^2_k}(1)$, which is a generator of $\mathrm{Pic}(\mathbb{P}^2_k)$ and $H^2 = 1$ (see [Har], V, §1, Example 1.4.2), we have seen that $-K_{\mathbb{P}^2_k}$ is linearly equivalent to $3H$, then $K^2_{\mathbb{P}^2_k} = 9H^2 = 9$.

Example 4.3. *Let X be a Del Pezzo surface, let $f : \tilde{X} \to X$ be a monoidal transformation with center a closed point $P \in X$. Let use Proposition 1.45 and Theorem 1.39 to understand in which cases $\tilde{X}$ is again a Del Pezzo surface. We will see in Lemma 4.20 that the converse always works.*

Since $K^2_{\tilde{X}} = K^2_X - 1$, a necessary condition is that $K^2_X \geq 2$. Let E be the exceptional divisor associated to f in $\tilde{X}$, let C be an irerducible curve in $\tilde{X}$, let m be the multiplicity of $f(C)$ at P and $f^*f(C) = C + rE$ for some $r \leq m$, then

$$-K_{\tilde{X}}.C = (f^*(-K_X)-E).(f^*f(C)-rE) = -K_X.f(C)-r \geq -K_X.f(C)-m$$

which is positive if $-K_X.f(C) > m$. So we see that $\tilde{X}$ is a Del Pezzo surface if and only if $-K_X.C > m$ for every irreducible curve C in X with multiplicity m at P.

If $X = \mathbb{P}^2_k$, then the condition $-K_X.C > m$ for every curve C in X with multiplicity m at P is verified, indeed if C is an irreducible curve of degree $d \geq 1$ in $\mathbb{P}^2_k$, we have that $m \leq d$ and then $-K_{\mathbb{P}^2_k}.C = 3d > m$.

Similarly, one can prove that if X is a blowing-up of $\mathbb{P}^2_k$ with center a set of closed points $P_1, \ldots, P_r \in \mathbb{P}^2_k$, with $0 \leq r \leq 8$, that verify the conditions in Definition 4.21, then X is a Del Pezzo surface with $K^2_X = 9-r$. In Theorem 4.22 we will see that over an algebraically closed field all non minimal Del Pezzo surfaces are of this type.

Example 4.4. *$\mathbb{P}^1_k \times \mathbb{P}^1_k$ is a Del Pezzo surface. Indeed, $\mathbb{P}^1_k \times \mathbb{P}^1_k$ is isomorphic to a quadric surface in $\mathbb{P}^3_k$ via the Segre embedding (see [Har], I, §2, Exercise 2.15), so $\omega_{\mathbb{P}^1_k \times \mathbb{P}^1_k} \cong \mathcal{O}_{\mathbb{P}^1_k \times \mathbb{P}^1_k}(-2)$ (see [Har], II, §8, Example 8.20.3) and $\mathcal{O}(-K_{\mathbb{P}^1_k \times \mathbb{P}^1_k}) \cong \mathcal{O}_{\mathbb{P}^1_k \times \mathbb{P}^1_k}(2)$ is ample by Proposition 1.15 as $\mathcal{O}_{\mathbb{P}^3_k}(2)$ is ample on $\mathbb{P}^3_k$ (see [Har], II, §7, Example 7.6.1). Thus $\mathbb{P}^1_k \times \mathbb{P}^1_k$ is a Del Pezzo surface. We have also $\mathrm{Pic}(\mathbb{P}^1_k \times \mathbb{P}^1_k) \cong \mathbb{Z} \oplus \mathbb{Z}$ (see [Har], II, §6, Example 6.6.1) and $K^2_{\mathbb{P}^1_k \times \mathbb{P}^1_k} = 8$ (see [Har], V, §2, Corollary 2.11).*

Example 4.5. *A nonsingular intersection of two quadric hypersurfaces in $\mathbb{P}^4_k$ is a Del Pezzo surface. Let Q_1, Q_2 be two quadric hypersurfaces in $\mathbb{P}^4_k$ such that $X = Q_1 \cap Q_2$ is a surface, without loss of generality we ca assume that Q_1 and Q_2 are nonsingular. We have that $\omega_{Q_1} \cong \mathcal{O}_{Q_1}(-3)$, $\mathcal{O}_{Q_1}(X) \cong \mathcal{O}_{Q_1}(2)$ and so $\omega_X \cong \omega_{Q_1} \otimes_{\mathcal{O}_{Q_1}} \mathcal{O}_{Q_1}(X) \otimes_{\mathcal{O}_{Q_1}} \mathcal{O}_X \cong \mathcal{O}_X(-1)$ (see [Har], II, §8, Example 8.20.1 and Proposition 8.20). Then $\mathcal{O}_X(-K_X) \cong \mathcal{O}_X(1)$ is ample by Proposition 1.15 and X is a Del Pezzo surface.*

Example 4.6. *A cubic surface is a Del Pezzo surface. Let X be a cubic surface over k, i.e. a nonsingular projective variety of dimension 2 and* *degree 3, then X is a hypersurface in $\mathbb{P}^3_k$, indeed if we take a closed immersion $X \to \mathbb{P}^n_k$ with $n > 3$, than there are two hypersurfaces S_1, S_2, of degree d_1, d_2 respectively, in $\mathbb{P}^n_k$ such that $X \subset S_1 \cap S_2$, then $\deg X = 3$ is a multiple of $d_1 d_2$, then there is $i \in \{1, 2\}$ such that $d_i = 1$, then $S_i \cong \mathbb{P}^{n-1}_k$ and $X \subset S_i$. Thus by induction on n we can prove that there exists a closed immersion $X \to \mathbb{P}^3_k$. Then X is a cubic surface in $\mathbb{P}^3_k$. We have $\omega_X \cong \mathcal{O}_X(-1)$ (see [Har], II, §8, Example 8.20.3), then $\mathcal{O}(-K_X) \cong \mathcal{O}_X(1)$ is ample by Proposition 1.15 and X is a Del Pezzo surface.*

In the next examples we need the notions of weighted projective spaces and degree of a hypersurface in a weighted projective space, see Definition 1.20 and Proposition 1.22 for the definitions.

Example 4.7. *A nonsingular irreducible hypersurface of degree 4 in $\mathbb{P}_k(1, 1, 1, 2)$ is a Del Pezzo surface. Indeed if X is a nonsingular irreducible hypersurface of degree 4 in $\mathbb{P}_k(1, 1, 1, 2)$, we have that $\omega_X^{-1} \cong \mathcal{O}_X(1)$ is ample by Proposition 1.24.*

Example 4.8. *A nonsingular irreducible hypersurface of degree 6 in $\mathbb{P}_k(1, 1, 2, 3)$ is a Del Pezzo surface. Indeed if X is a nonsingular irreducible hypersurface of degree 6 in $\mathbb{P}_k(1, 1, 2, 3)$, we have that $\omega_X^{-1} \cong \mathcal{O}_X(1)$ is ample by Proposition 1.24.*

4.2 Classification over an algebraically closed field

Let X be a Del Pezzo surface over an algebraically closed field k, and let $K := K_X$ be its canonical divisor.

Proposition 4.9. *$H^0(X, \mathcal{O}_X) = k$, $H^1(X, \mathcal{O}_X) = 0$, $H^2(X, \mathcal{O}_X) = 0$, in particular $\chi(\mathcal{O}_X) = 1$.*

Proof. Since it is a surface, X is irreducible and projective over k, hence connected, thus $H^0(X, \mathcal{O}_X) = k$.

Since $-K$ is an ample divisor, Proposition 1.16 gives $H^0(X, K) = 0$, then $H^2(X, \mathcal{O}_X) \cong H^0(X, K) = 0$, by Serre's duality (Theorem 1.7).

For $H^1(X, \mathcal{O}_X) = 0$, see [Ko1], III, §3, Lemma 3.2.1. □

Corollary 4.10. *X is a rational surface.*

Proof. Since $-K$ is ample, then also $-2K$ is ample by Proposition 1.14, then $h^0(X, \mathcal{O}(2K)) = 0$ by Proposition 1.16. Moreover, from Proposition 4.9 we have that $h^1(X, \mathcal{O}_X) = 0$, then we can apply Theorem 1.53. □

Lemma 4.11. *We have $h^2(X, -mK) = 0$ for all $m \geq 0$, and $h^0(X, -mK) > K^2$ for all $m \geq 1$.*

Proof. By Serre's duality (Theorem 1.7) and Propositions 1.14 and 1.16 we have $h^2(X, -mK) = h^0(X, (m+1)K) = 0$ for all $m \geq 0$. Then by the Riemann-Roch formula (1.38) we get $h^0(X, -K) \geq \frac{1}{2}m(m+1)K^2 + 1 > K^2$ for all $m \geq 1$. $\square$

Lemma 4.12. *If C is a curve in X such that $\mathcal{O}(-C) \cong \mathcal{O}(K)$, then C is connected.*

Proof. We have an exact sequence:

$$0 \to \mathcal{O}(K) \to \mathcal{O}_X \to \mathcal{O}_C \to 0$$

which gives a long exact sequence of cohomology groups:

$$0 \to H^0(X, \mathcal{O}(K)) \to H^0(X, \mathcal{O}_X) \to H^0(C, \mathcal{O}_C) \to H^1(X, \mathcal{O}(K))$$

Since $H^0(X, \mathcal{O}(K)) = 0$ and, by Serre's duality (Theorem 1.7), $H^1(X, \mathcal{O}(K)) = H^1(X, \mathcal{O}_X) = 0$, then $H^0(C, \mathcal{O}_C) = H^0(X, \mathcal{O}_X) = k$ and we conclude that C is connected. $\square$

Lemma 4.13. *A general member of $|-K|$ is irreducible and reduced.*

Proof. Let $D = \sum_{i=1}^s a_i C_i \in |-K|$ be an effective divisor not irreducible and reduced, with $n_i > 0$ and C_i integral curve for all $i = 1, \ldots, s$. Then by the adjunction formula (1.1) we have

$$\begin{aligned} 2p_a(C_i) - 2 = C_i.(C_i + K) = C_i.(C_i - \sum_{j=1}^s \frac{a_j}{a_i} C_j + \frac{a_i - 1}{a_i} K) = \\ = -\sum_{j \neq i} \frac{a_j}{a_i} C_i.C_j - \frac{a_i - 1}{a_i} C_i.(-K) < 0 \end{aligned}$$

for all $i = 1, \ldots, s$, indeed since $\mathcal{O}(K)$ is the ideal sheaf of $\sum_{i=1}^s C_i$, we have that $\sum_{i=1}^s C_i$ is connected by Lemma 4.12, then if D is not irreducible we have $\sum_{j \neq i} \frac{a_j}{a_i} C_i.C_j > 0$. Moreover $C_i.(-K) > 0$ by Theorem 1.39, then if D is irreducible but not reduced we have $\frac{a_i - 1}{a_i} C_i.(-K) > 0$. So if D is not irreducible and reduced $p_a(C_i) = 0$ for all $i = 1, \ldots, s$ and Proposition 1.34 gives $C_i \cong \mathbb{P}^1_k$ for all $i = 1, \ldots, s$.

Let fix $i \in \{1, \ldots, s\}$, from the long exact sequence of cohomology associated to the exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}(C_i) \to \mathcal{O}_{C_i}(C_i) \to 0$$

we get that $h^0(X, C_i) \leq h^0(C_i, \mathcal{O}_{C_i}(C_i)) + h^0(X, \mathcal{O}_X)$. Remark 1.3.2 in [Har], IV, 1, gives $h^0(C_i, \mathcal{O}_{C_i}(C_i)) = C_i^2 + 1$, then using also Proposition 4.9 and the adjunction formula (1.1) we conclude that $h^0(X, C_i) \leq C_i^2 + 2 = -K.C_i$.

Since there is a closed immersion $C_i \to \sum_{j=1}^s a_jC_j$, we have an injection of sheaves $\mathcal{O}(K) \to \mathcal{O}(-C_i)$ and then $\mathcal{O}(C_i) \to \mathcal{O}(-K)$, which induces an injection $\varphi_i : H^0(X, C_i) \to H^0(X, -K)$. The injection φ_i corresponds to the map that sends an effective divisor C_i' linearly equivalent to C_i to the effective divisor $D + a_i(C_i' - C_i) \in |-K|$. Let $D' \in |-K|$, if $gcd(a_i, \ldots, a_s) = 1$ we can find integers $b_1, \ldots, b_s$ such that $\sum_{i=1}^s a_ib_i = 1$. Let $C_i' := C_i + b_i(D - D')$, then

$$\oplus_{i=1}^s \phi_i : \oplus_{i=1}^s H^0(X, C_i) \to H^0(X, -K)$$

sends $\oplus_{i=1}^s C_i'$ to $D + \sum_{i=1}^s a_ib_i(D' - D) = D'$, thus $\oplus_{i=1}^s \phi_i$ is surjective and we have

$$h^0(X, -K) \leq \sum_{i=1}^s h^0(X, C_i) \leq -K.\sum_{i=1}^s C_i \leq (-K)^2 = K^2$$

that contradicts Lemma 4.11.

Let Y be the fixed component of $|-K|$ and $U = X \setminus Y$, then $\mathcal{O}(-K)$ is generated by global sections on U (see [Har], II, §7, Lemma 7.8) and, by Proposition 1.10, it induces a morphism $\phi : U \to \mathbb{P}_k^N$ for some N, such that $\phi^*(\mathcal{O}_{\mathbb{P}_k^N}(1) \cong \mathcal{O}_U(-K)$. Since $-K$ is ample we have that $K^2 \geq 1$ (see Theorem 1.39), then Lemma 4.11 gives $h^0(X, -K) > K^2 \geq 1$, i.e. $h^0(X, -K) \geq 2$. Thus for a general $D \in |-K|$, Y is strictly contained in the support of D. Write $D = D_0 + D_1$ where D_0, D_1 are effective divisors on X such that: if Y has dimension 1, the support of D_0 is Y and the support of D_1 does not contain any irreducible component of Y; if Y has dimension ≤ 0, $D_0 = 0$ and $D_1 = D$. Since Y is strictly contained in the support of D we have that $D_1 \neq 0$ for a general $D \in |-K|$, and in particular $D_1 \cap U \neq \emptyset$ for a general $D \in |-K|$. Thus $\mathcal{O}_U(-K) \ncong \mathcal{O}_U$ and $\phi(U)$ has dimension ≥ 1. Let Z be the closure of $\phi(U)$ in $\mathbb{P}_k^N$. If Z is a curve, by Bertini's theorem (see [Har], II, §8, Remark 8.18.1) a general hyperplane of $\mathbb{P}_k^N$ meets Z, and in particular $\phi(U)$, in a nonsingular finite set of points, then for a general divisor $D \in |-K|$ we have that D_1 is reduced. If Z is a surface, a general hyperplane of $\mathbb{P}_k^N$ cuts on Z, and in particular on $\phi(U)$, a reduced divisor, then then for a general divisor $D \in |-K|$ we have that D_1 is reduced.

Thus for a general $D = \sum_{i=1}^s a_iC_i \in |-K|$ we have that $D_1 = \sum_{i=1}^r a_iC_i$, $r \leq s$, is reduced, then $gcd(a_1, \ldots, a_r) = 1$ and in particular $gcd(a_1, \ldots, a_s) = 1$. Thus we can apply the above reduction ad absurdum to D and conclude that a general $D \in |-K|$ is irreducible and reduced. □

Proposition 4.14. *For all $m \geq 0$ we have $h^0(X, -mK) = \frac{1}{2}m(m+1)K^2 + 1$, $h^1(X, -mK) = 0$ and $h^2(X, -mK) = 0$.*

Proof. Let $m \geq 0$. Lemma 4.11 gives $h^2(X, -mK) = 0$.

Let $C \in |-K|$, by Lemma 4.13 we can assume that C is an integral curve. We have an exact sequence

$$0 \to \mathcal{O}(-mK) \to \mathcal{O}(-(m+1)K) \to \mathcal{O}_C(-(m+1)K) \to 0 \qquad (4.1)$$

that gives an exact sequence of cohomology groups

$$H^1(X,-mK) \to H^1(X,-(m+1)K) \to H^1(C,-(m+1)K|_C)$$

From the adjunction formula (1.1) and Theorem 1.36 we have $2p_a(C)-2 = C.(C+K) = -K.(-K+K) = 0$, then C is an integral curve of arithmetic genus 1. Let D be a divisor on C such that $\mathcal{O}_C(-(m+1)K) = \mathcal{O}_C(D)$. Since $\deg D = (m+1)C^2 > 0$ we have $D \neq 0$, then, by Exercises 1.5 and 1.9 in [Har], IV, 1, we get $h^0(C,-(m+1)K|_C) < \deg D + 1$ and $h^1(C,-(m+1)K|_C) = h^0(C,-(m+1)K|_C) - \deg D < 1$, thus $H^1(C,-(m+1)K|_C) = 0$ for all $m \geq 0$.

For $m = 0$ we apply Proposition 4.9, then, by induction, we can conclude that $h^1(X,-mK) = 0$ for all $m \geq 0$.

The Riemann-Roch formula (1.2) and Proposition 4.9 give now

$$h^0(X,-mK) = \frac{1}{2}(-mK).(-mK-K) + \chi(\mathcal{O}_X) = \frac{1}{2}m(m+1)K^2 + 1$$

for all $m \geq 0$ □

Proposition 4.15. *Let* $R = \oplus_{m\geq 0} H^0(X,-mK)$*,*

if $K^2 = 1$*, then* $\oplus_{m\leq 3} H^0(X,-mK)$ *generates* R*;*

if $K^2 = 2$*, then* $\oplus_{m\leq 2} H^0(X,-mK)$ *generates* R *and* $-K$ *is generated by global sections;*

if $K^2 \geq 3$*, then* $H^0(X,-K)$ *generates* R *and* $-K$ *is very ample.*

Proof. Let define $\alpha(1) = 3$, $\alpha(2) = 2$ and $\alpha(n) = 1$ for $n \geq 3$. Let $C \in |-K|$ be an integral curve of arithmetic genus 1 as in the proof of Proposition 4.14. Let $m \geq 0$, from the sequence (4.1) and Proposition 4.14 we get an exact sequence

$$0 \to H^0(X,-mK) \to H^0(X,-(m+1)K) \to H^0(C,-(m+1)K|_C) \to 0 \tag{4.2}$$

Since $h^0(X,\mathcal{O}_X) = 1$ and $H^0(X,-K) \geq 2$ by Propositions 4.9 and 4.14, we have that $H^0(C,-K|_C) \neq 0$. Let D be an effective divisor on C such that $\mathcal{O}_C(-K) = \mathcal{O}_C(D)$, let $Q = \oplus_{m\leq\alpha(K^2)} H^0(X,-mK)$, then $Q|_C = \oplus_{m\leq\alpha(K^2)} H^0(C,-mK|_C) = \oplus_{m\leq\alpha(K^2)} H^0(C,mD)$.

Since $\deg mD = mK^2$ for all $m \geq 0$ and according to Corollary 3.2 in [Har], IV, §3, we have that mD is very ample for $mK^2 \geq 3$, so we see that $\alpha(K^2)D$ is very ample on C, then by Proposition 1.13 we have that $H^0(C,\alpha(K^2)D)$ generates $H^0(C,d\alpha(K^2)D)$ for all $d \geq 0$. Let D' be the support of D, then D' is a finite set of closed points of C, the exact sequence

$$0 \to \mathcal{O}_C(mD) \to \mathcal{O}_C((m+1)D) \to \mathcal{O}_{D'}((m+1)D) \to 0$$

gives an exact sequence

$$0 \to H^0(C, mD) \to H^0(C, (m+1)D) \to H^0(D', (m+1)D) \to 0 \qquad (4.3)$$

for all $m \geq 1$, because $H^1(C, mD) = 0$ for all $m \geq 1$ (via [Har], IV, §1, Theorem 1.3 and Proposition 1.5) as in the proof of Proposition 4.14.

Since D' is a finite set of points we have $H^0(D', (m+1)D)$ is the direct sum of the stalks of $\mathcal{O}_{D'}((m+1)D)$ at the points of D'. By Corollary 3.2 in [Har], IV, §3 we have that mD is generated by global sections on C if $mK^2 \geq 2$, then $\mathcal{O}_{D'}((m+1)D)$ is generated by the image of $H^0(C, 2D) \oplus H^0(C, 3D)$ for all $m \geq 1$. So we can conclude that $Q|_C$ generates $H^0(C, mD)$ for all $m \geq 0$, by induction on m in the sequence (4.3). Then, by induction on m in the sequence 4.2, we conclude that Q generates R.

If $K^2 = 2$, then $-K|_C$ is generated by global sections (see [Har], IV, §3, Corollary 3.2), then from the sequence (4.2) with $m = 0$ we get that also $-K$ is generated by global sections.

If $K^2 = 3$, we have that $-mK$ is very ample for some positive integer m, by Proposition 1.14, but $Q = H^0(X, -K)$ generates $H^0(X, -mK)$, as we have proved, then also $-K$ is very ample. □

Proposition 4.16. *If $K^2 \leq 4$, then $X \cong \mathrm{Proj}(\oplus_{m \geq 0} H^0(X, -mK))$ is a non minimal surface. In particular*

if $K^2 = 4$, then X is a complete intersection of two quadric hypersurfaces in $\mathbb{P}^4_k$ and a line of $\mathbb{P}^4_k$ is contained in X if and only if it is a (-1)-curve of X;

if $K^2 = 3$, then X is a cubic surface in $\mathbb{P}^3_k$ and a line of $\mathbb{P}^3_k$ is contained in X if and only if it is a (-1)-curve of X;

if $K^2 = 2$, then X is a hypersurface of degree 4 in $\mathbb{P}_k(1,1,1,2)$, moreover there is a finite morphism $X \to \mathbb{P}^2_k$ of degree 2 and ramified on a quartic curve in $\mathbb{P}^2_k$;

if $K^2 = 1$, then X is a hypersurface of degree 6 in $\mathbb{P}_k(1,1,2,3)$.

Proof. Since $-K$ is an ample invertible sheaf on X, we have that $X \cong \mathrm{Proj}(\oplus_{m \geq 0} H^0(X, -mK))$ by Proposition 1.19.

Proposition 4.15 says that if $K^2 \geq 3$ then $-K$ is very ample, then, applying Propositions 1.12 and 1.42, $-K$ induces a closed immersion $\phi : X \to \mathbb{P}^n_K$, where $n = h^0(X, -K) - 1 = K^2$ by Proposition 4.14, $\phi(X)$ spans $\mathbb{P}^n_k$ and $K^2 = \deg \phi \cdot \deg \phi(X) = \deg \phi(X)$ as ϕ is injective.

If $K^2 = 4$, then $\phi(X)$ is a surface of degree 4 in $\mathbb{P}^4_k$ not contained in any hyperplane of $\mathbb{P}^4_k$, then there are two hypersurfaces Q_1, Q_2 of degrees $d_1, d_2 > 1$ in $\mathbb{P}^4_k$ such that $\phi(X) \subset Q_1 \cap Q_2$, then $\deg \phi(X)$ is a multiple of $d_1 d_2$, but $\deg \phi(X) = 4$, then the only possible choice is $d_1 = d_2 = 2$. So

Q_1, Q_2 are two quadric hypersurfaces of $\mathbb{P}^4_k$ and by Exercise 6.5 in [Har], II, §6 we have that $\phi(X) = Q_1 \cap Q_2$.
Let C be an integral curve over X, by Proposition 1.43 we have $\deg \phi(C) = -K.C$. Then $\phi(C)$ is a line in $\mathbb{P}^4_k$ if and only if $\deg \phi(C) = 1$ and $\phi(C) \cong \mathbb{P}^1_k$, if and only if $K.C = -1$ and $C \cong \mathbb{P}^1_k$. By adjunction formula (1.1) this is equivalent to $C^2 = -1$ and $C \cong \mathbb{P}^1_k$, thus $\phi(C)$ is a line in $\mathbb{P}^4_k$ if and only if C is a (-1)-curve.

If $K^2 = 3$, $\phi(X)$ is a surface of degree 3 in $\mathbb{P}^3_k$, then it is a cubic surface. Let C be an integral curve over X, by Proposition 1.43 we have $\deg \phi(C) = -K.C$. Then $\phi(C)$ is a line in $\mathbb{P}^3_k$ if and only if $\deg \phi(C) = 1$ and $\phi(C) \cong \mathbb{P}^1_k$, if and only if $K.C = -1$ and $C \cong \mathbb{P}^1_k$. By adjunction formula (1.1) this is equivalent to $C^2 = -1$ and $C \cong \mathbb{P}^1_k$, thus $\phi(C)$ is a line in $\mathbb{P}^3_k$ if and only if C is a (-1)-curve.

If $K^2 = 2$, by Proposition 4.14 we have that $h^0(X, -K) = 3$ and $h^0(X, -2K) = 7$, then $H^0(X, -K)$ generates a subspace of dimension 6 of $H^0(X, -2K)$. Let $s_0, s_1, s_2 \in H^0(X, -K)$ be a basis of $H^0(X, -K)$, since $R := \oplus_{m \geq 0} H^0(X, -mK))$ is generated by $\oplus_{m \leq 2} H^0(X, -mK)$ by Proposition 4.15, there is an element $t \in H^0(X, -2K)$ such that s_0, s_1, s_2, t generates R as k-algebra. Using the notation introduced in Definition 1.20 we can define a surjective morphism of graded rings preserving degrees $\varphi : S_{(1,1,1,2)} = k[x_0, x_1, x_2, x_3^2] \to R$ by $\varphi(x_i) = s_i$ for $i = 0, 1, 2$ and $\varphi(x_3^2) = t$. By Proposition 1.17 φ induces a closed immersion $f : X \to \mathbb{P}_k(1, 1, 1, 2)$ such that $\omega_X^{-1} \cong f^*\mathcal{O}_{\mathbb{P}_k(1,1,1,2)}(1)$. By Proposition 1.24 we have that $\omega_X^{-1} \cong f^*\mathcal{O}_{\mathbb{P}_k(1,1,1,2)}(5 - \deg f(X))$, then $\deg f(X) = 4$, i.e. $f(X)$ is given by a homogeneous polynomial $g \in S_{(1,1,1,2)}$ of degree 4. Without loss of generality we can write $g(x_0, x_1, x_2, x_3^2) = (x_3^2)^2 - h(x_0, x_1, x_2)$ where $h \in k[x_0, x_1, x_2]$ is a polynomial of degree 4.

Proposition 4.15 says that $-K$ is generated by global sections, then we can apply Proposition 1.42, so $-K$ induces a finite morphism $\phi : X \to \mathbb{P}^n_k$, where $n = K^2$ as before and $\phi(X)$ is a nonsingular surface in $\mathbb{P}^2_k$, then ϕ is surjective and of degree 2. Since ϕ is the restriction to X of the projection $\mathbb{P}_k(1, 1, 1, 2) \dashrightarrow \mathbb{P}^2_k$ from the point $(0, 0, 0, 1)$, then we easily see that the ramification locus of ϕ is the quartic curve in $\mathbb{P}^2_k$ defined by the homogeneous polynomial $h(x_0, x_1, x_2)$.

If $K^2 = 1$, by Proposition 4.14 we have that $h^0(X, -K) = 2, h^0(X, -2K) = 4$ and $h^0(X, -K) = 7$, then $H^0(X, -K)$ generates a subspace of dimension 3 in $H^0(X, -2K)$, and $\oplus_{m \leq 2} H^0(X, -mK)$ generates a subspace of dimension 6 in $H^0(X, -3K)$. Since $\oplus_{m \leq 3} H^0(X, -mK)$ generates $R = \oplus_{m \geq 0} H^0(X, -mK)$ by Proposition 4.15, we can choose $s_0, s_1 \in H^0(X, -K)$, $u \in H^0(X, -2K)$ and $v \in H^0(X, -3K)$ such that s_0, s_1, u, v generate R as k-algebra. Using the notation introduced in Definition 1.20 we can define a surjective morphism of graded rings preserving degrees $\varphi : S_{(1,1,2,3)} = k[x_0, x_1, x_2^2, x_3^3] \to R$ by $\varphi(x_i) = s_i$ for $i = 0, 1$, $\phi(x_2^2) = u$ and $\phi(x_3^3) = v$.

By Proposition 1.17 φ induces a closed immersion $f : X \to \mathbb{P}_k(1,1,2,3)$ such that $\omega_X^{-1} \cong f^*\mathcal{O}_{\mathbb{P}_k(1,1,2,3)}(1)$. By Proposition 1.24 we have that $\omega_X^{-1} \cong f^*\mathcal{O}_{\mathbb{P}_k(1,1,2,3)}(7 - \deg f(X))$, then $\deg f(X) = 6$.

For the non minimality see [Ko1], III, §3, Corollary 3.6. $\square$

Proposition 4.17. *If $\rho(X) = 1$ then $X \cong \mathbb{P}_k^2$.*

Proof. Let H be a generator of $\mathrm{Num}(X)$ such that $-K = rH$ with $r > 0$. Since $-K$ is ample, then by Propositions 2.3 and 1.39 also H is ample and $H^2 > 0$, and since H is a generator, every curve $C \in |H|$ is irreducible and reduced. Let C be a curve in $|H|$, by the adjunction formula (1.1), we have

$$2p_a(C) - 2 = C.(C + K) = (1 - r)H^2 \leq 0 \tag{4.4}$$

which gives $1 \leq r \leq 3$.

If $r > 1$, then $p_a(C) = 0$, and $C \cong \mathbb{P}_k^1$. We have an exact sequence:

$$0 \to \mathcal{O}_X \to \mathcal{O}(C) \to \mathcal{O}_C(C) \to 0 \tag{4.5}$$

The invertible sheaf $\mathcal{O}(C)$ is generated by global sections on $X \setminus C$, moreover $\mathcal{O}_C(C) \cong \mathcal{O}_{\mathbb{P}_k^1}(H^2)$ is very ample as $H^2 > 0$, hence generated by global sections on C, and $H^1(X, \mathcal{O}_X) = 0$ by Proposition 4.9, then $\mathcal{O}(C)$ is generated by global sections also on C. So H is generated by global sections on X and by Proposition 1.42 it induces a finite morphism $\phi : X \to \mathbb{P}_k^n$, where $n = h^0(X, H) - 1$, then $\phi(X)$ is a surface that spans $\mathbb{P}_k^n$ and $H^2 = \deg \phi \cdot \deg \phi(X)$.

We have $h^1(X, \mathcal{O}_X) = 0$ by Proposition 4.9, and $h^1(C, \mathcal{O}_C(C)) = 0$ by Theorem 5.1 in [Har], III, §5, as $\mathcal{O}_C(C) \cong \mathcal{O}_{\mathbb{P}_k^1}(H^2)$. Then by the long exact sequence of cohomology groups associated to the sequence (4.5) we get $H^1(X, \mathcal{O}(C)) = 0$, and in particular $h^1(X, H) = 0$. We have $h^0(X, K - H) = h^0(X, -(r+1)H) = 0$ by Proposition 1.16, then the Riemann-Roch formula (1.2) gives

$$h^0(X, H) = \frac{1}{2}H.(H - K) + \chi(\mathcal{O}_X) = \frac{1}{2}(r+1)H^2 + 1$$

If $r = 3$, we have $H^2 = 1$, $n = 2$ and $\deg \phi = 1$, so $\phi : X \to \mathbb{P}_k^2$ is an isomorphism.

If $r = 2$, we have $H^2 = 2$, $n = 3$, $\deg \phi(X) = 2$ since $\phi(X)$ spans $\mathbb{P}_k^3$, and $\deg \phi = 1$. Then $\phi(X) : X \to \phi(X)$ is an isomorphism and $\phi(X)$ is a nonsingular quadric in $\mathbb{P}_k^3$, then $\rho(\phi(X)) = 2$ (see [Har], II, 6, Example 6.6.1), which contradicts the fact that $\rho(X) = 1$.

If $r = 1$, then $-K$ is a generator of $\mathrm{Num}(X)$. By Theorem 5.14 in [Ko1], II, §5, there is a rational curve C in X such that $-K.C \leq 3$, since $-K$ is ample and it is a generator of $\mathrm{Num}(X)$, we have that $C \equiv -mK_X$ for some $m > 0$, then $K^2 \leq mK = -K.C \leq 3$, then X contains a (-1)-curve by Proposition 4.16, which contradicts the fact that $\rho(X) = 1$.

Thus the only possibility is $r = 3$ and $X \cong \mathbb{P}_k^n$. $\square$

Lemma 4.18. *Every irreducible curve E in X with $E^2 < 0$ is a (-1)-curve and $-K.E = 1$.*

Proof. The irreducibility of E gives $p_a(E) \geq 0$ with equality if and only if $E \cong \mathbb{P}^1$. Since $-K$ is an ample divisor, from Theorem 1.39 we have that $K.E < 0$, then the adjunction formula says

$$-2 \leq 2p_a(E) - 2 = E.(E+K) \leq -2$$

so $E^2 = -1$, $p_a(E) = 0$ and $K.E = -1$, which implies that $E \cong \mathbb{P}^1_k$ and $-K.E = 1$. □

Proposition 4.19. *Let X be a minimal Del Pezzo surface over k, then either $\rho(X) = 1$, $X \cong \mathbb{P}^2_k$ and $K^2 = 9$, or $\rho(X) = 2$, $X \cong \mathbb{P}^1_k \times \mathbb{P}^1_k$ and $K^2 = 8$.*

Proof. Let C be an irreducible curve in X, since $-K_X$ is ample, by Theorem 1.39 (Nakay-Moishezon criterion) we have $K_X.C < 0$, so K_X is not nef. Then condition *i)* in Theorem 1.52 does not hold for minimal Del Pezzo surfaces.

After Theorem 1.52, Proposition 4.17 and the above discussion, we have that either $\rho(X) = 1$ and $X \cong \mathbb{P}^2_k$ or $\rho(X) = 2$ and X is a $\mathbb{P}^1_k$ bundle over a projective nonsingular irreducible curve C.

In the first case, $\mathrm{Pic}(\mathbb{P}^2_k) \cong \mathbb{Z}$ and $K^2_{\mathbb{P}^2_k} = 9$ from Example 4.2.

In the second case, X is a minimal rational ruled surface, i.e. a Hirzebruch surface $F_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1_k} \oplus \mathcal{O}_{\mathbb{P}^1_k}(n))$, for some $n \geq 0, n \neq 1$. According to Lemma 4.18 and to [Bea], IV, Propositions IV.1 we have that $n = 0$ and $X \cong \mathbb{P}^1_k \times \mathbb{P}^1_k$. Then $\mathrm{Pic}(\mathbb{P}^1_k \times \mathbb{P}^1_k) \cong \mathbb{Z} \oplus \mathbb{Z}$ and $K^2_{\mathbb{P}^1_k \times \mathbb{P}^1_k} = 8$ from Example 4.4. □

Lemma 4.20. *Let $f : X \to X'$ be a birational morphism of surfaces, if X is a Del Pezzo surface, then also X' is a Del Pezzo surface.*

Proof. By Proposition 1.46 f can be factored into a finite number of monoidal transformations, then, without loss of generality, we can assume that f is a monoidal transformation. Using Proposition 1.45 and the fact that $K^2_X > 0$, as $-K_X$ is ample, we have $(-K_{X'})^2 = K^2_{X'} = K^2_X + 1 > 0$.

Let E in X be the exceptional curve of f, from the adjunction formula we get $-K_X.E = E^2 - 2g(E) + 2 = 1$, as E is a (-1)-curve. Let C be any irreducible curve in X', using again Proposition 1.45 we have:

$$\begin{aligned} -K_{X'}.C = (-f^*K_{X'}).(f^*C) = (-K_X + E).(f^*C) = -K_X.(f^*C) = \\ = -K_X.(\tilde{C} + rE) = -K_X.\tilde{C} + r(-K_X.E) \end{aligned}$$

where $\tilde{C}$ is an irreducible curve in X and $r \geq 0$. Thus $-K_{X'}.C \geq -K_X.\tilde{C} > 0$, because of the ampleness of $-K_X$. By Theorem 1.39 (Nakai-Moishezon criterion) we get that $-K_{X'}$ is ample and so X' is a Del Pezzo surface. □

Definition 4.21. Let $1 \le r \le 8$, r closed points $P_1, \dots, P_r$ in $\mathbb{P}^2_k$ are in general position if they satisfy the following conditions:

i) $P_i \neq P_j$ for all $i, j \in \{1, \dots, r\}, i \neq j$;

ii) if $r \geq 3$, no three of them lie on a line;

iii) if $r \geq 6$, no six of them lie on a conic;

iv) if $r = 8$, there is no cubic which contains all $P_1, \dots, P_8$ and is singular at one of them.

Theorem 4.22. *Either $X \cong \mathbb{P}^1_k \times \mathbb{P}^1_k$ or X arises as a blowing-up of $\mathbb{P}^2_k$ in $r \leq 8$ points in general position. In the last case $K^2 = 9 - r$.*

Proof. Let X be a minimal Del Pezzo surface, by Proposition 4.19 we have that either $X \cong \mathbb{P}^2_k$ or $X \cong \mathbb{P}^1_k \times \mathbb{P}^1_k$. Let now X be a Del Pezzo surface which is not minimal, by Theorem 1.50 there exists a birational morphism $f : X \to X'$ to a minimal model, X' is a minimal Del Pezzo surface by Lemma 4.20, so X' is isomorphic either to $\mathbb{P}^2_k$ or to $\mathbb{P}^1_k \times \mathbb{P}^1_k$.

If X is the blowing-up of $\mathbb{P}^1_k \times \mathbb{P}^1_k$ with center a closed point P, let E be the exceptional divisor and E_1, E_2 the strict transforms of two distinct lines L_1, L_2 passing through P. From Proposition 2.3 in [Har], V, §2 we have $L_i^2 = 0$ in $\mathbb{P}^1_k \times \mathbb{P}^1_k$, while Proposition 1.45 gives $E_i^2 = L_i^2 + E^2 = -1$ and $E_1.E_2 = L_1.L_2 + E^2 = 0$, then E_1, E_2 are two disjoint (-1)-lines in X. So, by Theorem 1.47, we can contract them with a birational morphism $g : X \to X''$ which is the composition of two successive monoidal transformations. Proposition 4.19 gives $\rho(X') = 2$, then, using Proposition 1.45, we have $\rho(X) = \rho(X') + 1 = 3$ and $\rho(X'') = \rho(X') - 2 = 1$, thus X'' is minimal and $X'' \cong \mathbb{P}^2_k$, again by Proposition 4.19.

Then, without loss of generality, we can suppose that there is a birational morphism $f : X \to \mathbb{P}^2_k$. By Proposition 1.46, f can be factored into composition of $r \geq 1$ monoidal transformations $f_i : X_i \to X_{i-1}$ with center $Q_i \in X_{i-1}$, for $i = 1, \dots, r$, where r is the number of curves contracted by f, $X_0 = \mathbb{P}^2_k$, $X_r \cong X$ and we can identify f with $f_1 \circ \dots \circ f_r$. Applying Proposition 1.45 r times, we get $K_X^2 = K_{\mathbb{P}^2_k}^2 - r = 9 - r$, but $K_X^2 > 0$ because $-K_X$ is ample, so we conclude that $r \leq 8$.

Let $P_1 = Q_1$ and for $i = 2, \dots, r$ let $P_i \in \mathbb{P}^2_k$ be the image of Q_i under the map $f_1 \circ \dots \circ f_{i-1}$. Then $\{P_1, \dots, P_r\}$ is a collection of r closed points of $\mathbb{P}^2_k$, $1 \leq r \leq 8$, we want to prove that they are in general position.

We note that, according to the universal property of blowing-up, f is independent of the order of composition of the f_i, $i = 1, \dots, r$. So it is enough to verify the cases below. Moreover we note that by Lemma 4.20 X_i is a Del Pezzo surface for all $i = 0, \dots, r$. For $i = 1, \dots, r$ let $E_i \subset X_i$ be the exceptional divisor of f_i.

Let suppose $r \geq 2$ and $P_1 = P_2$, this means that $Q_2 \in E_1$, let $m \geq 1$ be the multiplicity of Q_2 on E_1, then by Proposition 1.45 the strict transform of E_1 under f_2 is $\tilde{E_1} = f_2^* E_1 - mE_2$, a curve with self-intersection $\tilde{E_1}^2 = E_1^2 + m^2 E_2^2 = -(1+m^2) \leq -2$, which contradicts Lemma 4.18 for the Del Pezzo surface X_2.

Let suppose that $r \geq 3$ and that P_1, P_2, P_3 lie on a line L in $\mathbb{P}^2_k$, from [Har], V, §1, Example 1.4.2, we have that $L^2 = 1$. The strict transform $\tilde{L}$ of L under $f := f_1 \circ f_2 \circ f_3$ is a curve in X_3, using Proposition 1.45, we have $\tilde{L} = f^* L - (f_2 \circ f_3)^* E_1 - f_3^* E_2 - E_3$ and $\tilde{L}^2 = L^2 + E_1^2 + E_2^2 + E_3^2 = -2$, which contradicts Lemma 4.18 for the Del Pezzo surface X_3.

Let suppose that $r \geq 6$ and that $P_1, \dots, P_6$ lie on a conic C in $\mathbb{P}^2_k$, from [Har], V, §1, Example 1.4.2, we have that $C^2 = 4$. As above let $\tilde{C}$ be the strict transform of C under $f_1 \circ \cdots \circ f_6$, $\tilde{C}$ is a curve in X_6 with $\tilde{C}^2 = C^2 + E_1^2 + \cdots + E_6^2 = -2$, which contradicts Lemma 4.18 for the Del Pezzo surface X_6.

Let suppose that $r = 8$ and that $P_1, \dots, P_8$ lie on a cubic C in $\mathbb{P}^2_k$, such that C has multiplicity $m \geq 2$ at P_8. From [Har], V, §1, Example 1.4.2, we have that $C^2 = 9$. As above let $\tilde{C}$ be the strict transform of C under $f_1 \circ \cdots \circ f_8$, $\tilde{C}$ is a curve in X_8 with $\tilde{C}^2 = C^2 + E_1^2 + \cdots + E_7^2 + m^2 E_8^2 \leq -2$, which contradicts Lemma 4.18 for the Del Pezzo surface X_8. □

Definition 4.23. Let X be a Del Pezzo surface, we define the degree of X to be K^2.

Remark 4.24. After Theorem 4.22, we see that non minimal Del Pezzo surfaces can be classified by their degree and that there are only two non isomorphic minimal Del Pezzo surfaces.

Proposition 4.25. *Let $f : X \to \mathbb{P}^2_k$ be a blowing-up with center r closed points in general position $P_1, \dots, P_r \in \mathbb{P}^2_k$, $1 \leq r \leq 8$. For $i = 1, \dots, r$ let E_i be the inverse image of P_i under f and L be the inverse image of a line in $\mathbb{P}^2_k$ which does not contain any of the P_i, $i = 1, \dots, r$. Then:*

i) $\mathrm{Pic}(X) \cong \mathbb{Z}L \oplus \mathbb{Z}E_1 \oplus \cdots \oplus \mathbb{Z}E_r$;

ii) $L^2 = 1$, $E_i^2 = -1$, $L.E_i = 0$, $E_i.E_j = 0$, $\forall i, j \in \{1, \dots, r\}, i \neq j$;

iii) $K_X = -3L + E_1 + \cdots + E_r$.

Proof. Since the image of L is a line in $\mathbb{P}^2_k$, then it is a generator of $\mathrm{Pic}(\mathbb{P}^2_k) \cong \mathbb{Z}$, so we get i) applying Proposition 1.45. Moreover L does not meet any E_i, $i = 1, \dots, r$, because its image in $\mathbb{P}^2_k$ does not contain any P_i, $i = 1, \dots, r$, and E_i does not meet E_j for all $i, j \in \{1, \dots, r\}$, $i \neq j$ because the points $P_1, \dots, P_r$ are pairwise distinct. Applying again Proposition 1.45 and the previous observation, we have ii). For iii) we have that $\omega_{\mathbb{P}^2_k} \cong \mathcal{O}_{\mathbb{P}^2_k}(-3)$ by Example 4.2, and $\mathcal{O}(f_* L) \cong \mathcal{O}_{\mathbb{P}^2_k}(1)$, so $K_{\mathbb{P}^2_k} = -3L$ and, applying Proposition 1.45, $K_X = -3L + E_1 + \cdots + E_r$. □

Proposition 4.26. *Let X be a blowing-up of $\mathbb{P}^2_k$ in r points in general position, $1 \leq r \leq 8$. Then the (-1)-curves in X are exactly of the following types:*

type a: the inverse image of one the r points;

type b: if $r \geq 2$, the strict transform of a line containing two of the r points;

type c: if $r \geq 5$, the strict transform of a conic containing five of the r points;

type d: if $r \geq 7$, the strict transform of a cubic containing seven of the r points, with multiplicity 2 at one of them;

type e: if $r = 8$, the strict transform of a quartic containing all the 8 points, with multiplicity 2 at three of them;

type f: if $r = 8$, the strict transform of a quintic containing all the 8 points, with multiplicity 2 at six of them;

type g: if $r = 8$, the strict transform of a sextic containing all the 8 points, with multiplicity 2 at seven of them and multiplicity 3 at the remaining one.

In particular a non minimal Del Pezzo surface X contains only finitely many (-1)-curves, as listed below:

degree of X	*8*	*7*	*6*	*5*	*4*	*3*	*2*	*1*
number of (-1)-curves	*1*	*3*	*6*	*10*	*16*	*27*	*56*	*240*

Proof. Let $f : X \to \mathbb{P}^2_k$ be a blowing-up with center r closed points in general position $P_1, \ldots, P_r \in \mathbb{P}^2_k$, $1 \leq r \leq 8$. With an argument as in the proof Theorem 4.22 we can easily see that *type a*, *type b*, *type c* and *type d* curves are all (-1)-curves in X.

Let E be a (-1)-curve in X, according to Proposition 4.25 we ca write, up to linear equivalence, $E = dL - \sum_{i=1}^r m_i E_i$, where $d \geq 0$ is the degree of the image f_*E of E in $\mathbb{P}^2_k$ and $m_i \geq 0$ is the multiplicity of P_i on f_*E, for $i = 1, \ldots, r$. From Lemma 4.18 we have $-K_X.E = 1$, so

$$E^2 = d^2 - \sum_{i=1}^r m_i^2 = -1 \qquad \text{and} \qquad -K_X.E = 3d - \sum_{i=1}^r m_i = 1$$

which gives

$$\sum_{i=1}^r m_i^2 = d^2 + 1 \qquad \text{and} \qquad \sum_{i=1}^r m_i = 3d - 1 \tag{4.6}$$

Using the Cauchy-Schwartz's inequality we have:

$$\left(\sum_{i=1}^{r} m_i\right)^2 \leq \left(\sum_{i=1}^{r} 1\right)\left(\sum_{i=1}^{r} m_i^2\right) = r\left(\sum_{i=1}^{r} m_i^2\right)$$

and in particular $(3d-1)^2 \leq r(d^2+1)$, so $(9-r)d^2 - 6d + 1 - r \leq 0$, which gives

$$d \leq \frac{3+\sqrt{r(10-r)}}{9-r}$$

and the following upper bounds for d:

r	1	2	3	4	5	6	7	8
$d \leq$	0	1	1	1	2	2	3	7

We can interpret the above list as follows: if $K_X^2 = 8$ then $r = 1$ and X contains only (-1)-curves of *type a*, if $K_X \leq 7$ then $r \geq 2$ and X may contain (-1)-curves not of *type a*. Let suppose $K_X \leq 7$ and let E be a (-1)-curve not of *type a*, then E is the strict transform of a rational curve f_*E of degree d in $\mathbb{P}_k^2$.

If $d = 1$ then (4.6) gives $m_1^2 + \cdots + m_r^2 = 2$ and $m_1 + \cdots + m_r = 2$, so E is of *type b*.

If $d = 2$ then (4.6) gives $m_1^2 + \cdots + m_r^2 = 5$ and $m_1 + \cdots + m_r = 5$, so E is of *type c*.

If $d = 3$ then (4.6) gives $m_1^2 + \cdots + m_r^2 = 10$ and $m_1 + \cdots + m_r = 8$, so E is of *type d*.

If $d = 4$ then (4.6) gives $m_1^2 + \cdots + m_r^2 = 17$ and $m_1 + \cdots + m_r = 11$, so E is of *type e*.

If $d = 5$ then (4.6) gives $m_1^2 + \cdots + m_r^2 = 26$ and $m_1 + \cdots + m_r = 14$, so E is of *type f*.

If $d = 6$ then (4.6) gives $m_1^2 + \cdots + m_r^2 = 37$ and $m_1 + \cdots + m_r = 17$, so E is of *type g*.

If $d = 7$ then (4.6) gives $m_1^2 + \cdots + m_r^2 = 50$ and $m_1 + \cdots + m_r = 20$, but the two equations have no common solution $(m_1, \ldots, m_r)$ with $r = 8$ and $m_i \geq 0$ for all $i = 1, \ldots, r$, so there are no (-1)-curves with $d = 7$.

Moreover, the number of (-1)-curves in X, can be calculated by the following formulas:

$$r + \binom{r}{2} + \binom{r}{5} + (r-6)\binom{r}{6} \qquad \text{if } 1 \leq r \leq 7$$

$$r + \binom{r}{2} + \binom{r}{5} + (r-6)\binom{r}{6} + \binom{r}{3} + \binom{r}{6} + \binom{r}{7} \qquad \text{if } r = 8$$

□

Proposition 4.27. *Let X be a Del Pezzo surface of degree $K^2 \geq 3$, let $f : X' \to X$ be a monoidal tranformation with center a closed point $x \in X$. Then X' is a Del Pezzo surface if and only if x does not lie on any exceptional curve of X.*

Proof. If $K^2 = 9$ it is trivial, see also Example 4.17.

If $X \cong \mathbb{P}^1_k \times \mathbb{P}^1_k$ see the second paragraph of the proof of Theorem 4.22.

If X is a non minimal Del Pezzo surface and x lies on a (-1)-curve E of X, let $\tilde{E}$ be the strict transform of E under f and E' be the exceptional divisor on X' corresponding to f. Then by Proposition 1.45 we have that $\tilde{E}^2 = E^2 + E'^2 = -2$ and by Lemma 4.18 we conclude that X' is not a Del Pezzo surface.

Conversely, if X is a non minimal Del Pezzo surface and X' is not a Del Pezzo surface. By Theorem 4.22 we have that X is, up to isomorphism, a blowing-up of $\mathbb{P}^2_k$ in $r \leq 7$ points $P_1, \ldots, P_r$ in general position, say $g : X \to \mathbb{P}^2_k$. Let $P_{r+1} = g(x)$, then $g \circ f : X' \to \mathbb{P}^2_k$ is a blowing-up with center $P_1, \ldots, P_{r+1}$. Again by Theorem 4.22 we have that $P_1, \ldots, P_{r+1}$ are not in general position. We have three possible cases:

i) if $P_{r+1} = P_i$ for some $i \in \{1, \ldots, r\}$, then x belongs to the inverse image of P_i under g, which is a (-1)-curve of X by Proposition 4.26;

ii) if $P_{r+1} \neq P_i$ for all $i = 1, \ldots, r$ and P_{r+1} lies on a line $L_{i,j}$ containing two distinct points P_i, P_j for some $i, j \in \{1, \ldots, r\}$, then x belongs to the strict transform of $L_{i,j}$ under g, which is a (-1)-curve of X by Proposition 4.26;

iii) if $P_{r+1} \neq P_i$ for all $i = 1, \ldots, r$ and P_{r+1} lies on a conic C passing through five of the $P_1, \ldots, P_r$, then x belongs to the strict transform of C under g, which is a (-1)-curve of X by Proposition 4.26.

□

Remark 4.28. From Theorem 4.22 and Proposition 4.27 we get that X is a Del Pezzo surface over k if and only if $X \cong \mathbb{P}^1_k \times \mathbb{P}^1_k$ or X arises as blowing-up of $\mathbb{P}^2_k$ in $r \leq 8$ points in general position.

Proposition 4.29. *Let $f : X \to \mathbb{P}^2_k$ be a blowing-up with center r closed points in general position $P_1, \ldots, P_r \in \mathbb{P}^2_k, 1 \leq r \leq 8$. For $i = 1, \ldots, r$, let E_i be the inverse image of P_i under f. For $i, j \in \{1, \ldots, r\}, i < j$, let $L_{i,j}$ be the strict transform under f of the line containing P_i and P_j. If $r = 5$ let C be the strict transform under f of the conic containing $P_1, \ldots, P_5$. We have that*

i) $E_1, \ldots, E_n$ are pairwise disjoint;

ii) if $r \geq 2$, then $L_{i,j}$ and E_s are disjoint if and only if $i \neq s \neq j$, otherwise they meet in exactly one point;

iii) if $r \geq 2$ and $L_{i,j}$, $L_{s,t}$ are two distinct curves, then $L_{i,j}$ and $L_{s,t}$ are disjoint if and only if $\{i, j\} \cap \{s, t\} \neq \emptyset$, otherwise they meet in exactly one point;

iv) if $r = 5$, then C meets E_i in exactly one point for all $i = 1, \dots, 5$, while C and $L_{i,j}$ are disjoint for all $i, j \in \{1, \dots, 5\}, i < j$.

Proof. i) If $i, j \in 1, \dots, r, i \neq j$, Proposition 4.25 says that $E_i.E_j = 0$, then E_j and E_j are disjoint.

ii) If $i, j, s \in \{1, \dots, r\}, i \neq j$, since $P_i, \dots, P_r$ are in general position and $f(L_{i,j})$ is the line containing P_i, P_j, then $f(L_{i,j})$ contains P_s if and only if $s = i$ or $s = j$, and in that case the multiplicity of $f(L_{i,j})$ at P_s is 1 as $L_{i,j}$ is nonsingular. Then by Proposition 1.45 we have that if $i \neq s \neq j$ then $L_{i,j}.E_s = 0$, i.e. $L_{i,j}$ and E_s are disjoint, otherwise $L_{i,j}.E_s = 1$, i.e. $L_{i,j}$ and E_s meet in exactly one point, as they are two distinct curves in X.

iii) Suppose that $i, j, s, t \in \{1, \dots, r\}, i < j, s < t, (i, j) \neq (s, t)$. By definition we have that $f(L_{i,j})$ and $f(L_{s,t})$ are two distinct lines in $\mathbb{P}^2_k$, then they meet in exactly one point, say P. Since $P_i, \dots, P_r$ are in general position, we have that $P \in \{P_1, \dots, P_r\}$ if and only if $\{i, j\} \cap \{s, t\} \neq \emptyset$. Then by Proposition 1.45 we have that if $\{i, j\} \cap \{s, t\} \neq \emptyset$, then $L_{i,j}.L_{s,t} = 0$, i.e. $L_{i,j}$ and $L_{s,t}$ are disjoint, otherwise $L_{i,j}.L_{s,t} = 1$, i.e. $L_{i,j}$ and $L_{s,t}$ meet in exactly one point, as they are two distinct curves in X.

iv) If $r = 5$, let $i \in \{1, \dots, 5\}$, since $f(C)$ contains P_i with multiplicity 1 by definition of C, then by Proposition 1.45 we have that $C.E_i = 1$, i.e. C and E_i meet in exactly one point, as they are two distinct curves in X.

If $i, j \in \{1, \dots, 5\}, i < j$, from the definition of C and $L_{i,j}$ we have that $f(C)$ and $f(L_{i,j})$ meet in exactly two points, counted with multiplicity, which are P_i, P_j. Then by Proposition 1.45 we have that $C.L_{i,j} = 0$, i.e. C and $L_{i,j}$ are disjoint, as they are two distinct curves in X.

□

4.3 Del Pezzo surfaces over an arbitrary field

In this section we give a list of properties, already proven for Del Pezzo surfaces over an algebraically closed field in Section 4.2, that hold true for Del Pezzo surfaces over any field. Then we consider properties that do not remain valid over an arbitrary field and we give some counter examples. We start with a very useful remark.

Remark 4.30. Let X be a surface over a field k. According to Propositions 2.3 and 2.5 we have that the following statements are equivalent:

i) X is a Del Pezzo surface of degree d;

ii) there exists an extension K/k such that X_K is a Del Pezzo surface of degree d;

iii) X_K is a Del Pezzo surface of degree d for all field extensions K over k.

After Remark 4.30 it is clear why the surfaces described in Examples 4.2, 4.3, 4.4, 4.5, 4.6, 4.7 and 4.8 are Del Pezzo surfaces independently of the choice of the ground field k.

Let k be a field and K an algebraic closure of k.

Proposition 4.31. *Let X be a Del Pezzo surface over k. Then*

i) $h^0(X, \mathcal{O}_X) = 1, h^1(X, \mathcal{O}_X) = 0, h^2(X, \mathcal{O}_X) = 0$ *and* $\chi(\mathcal{O}_X) = 1$;

ii) $h^0(X, -mK_X) = \frac{1}{2}m(m+1)K_X^2+1, h^1(X, -mK_X) = 0, h^2(X, -mK_X) = 0$ *for all* $m \geq 0$;

Proof. After Remark 4.30 we have that X_K is Del Pezzo surface, then combining Propositions 2.5, 4.9, 4.14 and 2.2 we obtain the result. □

Proposition 4.32. *Let X be a Del Pezzo surface over k,*

if $K_X^2 = 9$, *then X is a Severi-Brauer surface;*

if $K_X^2 = 4$, *then X is a complete intersection of two quadric hypersurfaces in $\mathbb{P}^4_k$ and every line of $\mathbb{P}^4_k$ contained in X is a (-1)-curve of X;*

if $K_X^2 = 3$, *then X is a cubic surface in $\mathbb{P}^3_k$ and every line of $\mathbb{P}^3_k$ contained in X is a (-1)-curve of X;*

if $K_X^2 = 2$, *then X is a hypersurface of degree 4 in $\mathbb{P}_k(1, 1, 1, 2)$, moreover there is a finite morphism $X \to \mathbb{P}^2_k$ of degree 2 and ramified on a quartic curve in $\mathbb{P}^2_k$;*

if $K_X^2 = 1$, *then X is a hypersurface of degree 6 in $\mathbb{P}_k(1, 1, 2, 3)$.*

Proof. If $K_X^2 = 9$, by Remark 4.30 and Theorem 4.22 we have that $X_K \cong \mathbb{P}^2_K$, then X is a Severi-Brauer surface.

By Remark 4.30 and Proposition 2.2 we see that Proposition 4.15 holds over any field, then the embeddings that are given in Proposition 4.16 are defined over k.

If $K_X^2 = 4$, then X is a surface in $\mathbb{P}^4_k$, let Q_1, Q_2 be two hypersurfaces in $\mathbb{P}^4_k$ such that $X \subset Q_1 \cap Q_2$, then $Q_{i,K}$ is a hypersurface of $\mathbb{P}^4_K$ for $i = 1, 2$ and $X_K \subset Q_{1,K} \cap Q_{2,K}$. By Proposition 4.16 we have that $Q_{1,K}, Q_{2,K}$ are quadric hypersurfaces and $X_K = Q_{1,K} \cap Q_{2,K}$, then Q_1, Q_2 are quadric hypersurfaces in $\mathbb{P}^4_k$ and $X = Q_1 \cap Q_2$.

Let $Y^3 = \mathbb{P}^3_k$, $Y^2 = \mathbb{P}_k(1, 1, 1, 2)$, $Y^1 = \mathbb{P}_k(1, 1, 2, 3)$. If $K_X^2 \leq 3$, since X is a geometrically integral hypersurface in $Y^{K_X^2}$, by Proposition 4.16 we have that X_K is defined by an irreducible homogeneous polynomial of the desired degree with coefficients in k, then X is defined by the same polynomial and we can conclude.

The assertions about the lines come from Proposition 4.16 and the following remark: if E is a k-rational curve in X such that E_K is a (-1)-curve in X_K, then E is a (-1)-curve in X (it is an immediate consequence of Proposition 2.4 and Theorem 3.29).

If $K_X^2 = 2$, the finite morphism $\phi_K : X_K \to \mathbb{P}^2_K$ of Proposition 4.16 is induced by $-K_{X_K}$, then it is the extension of the finite morphism $\phi : X \to \mathbb{P}^2_k$ indced by $-K$. Since ϕ_K has degree 2, then also ϕ has degree 2, moreover ϕ_K is ramified over a curve of degree 4 in $\mathbb{P}^2_K$ which is the ramification locus of ϕ extended to K. Thus ϕ is ramified over a quartic curve of degree 4 in $\mathbb{P}^2_k$. □

Proposition 4.33. *Let $f : X \to X'$ be a birational morphism of surfaces over k. If X is a Del Pezzo surface then also X' is a Del Pezzo surface.*

Proof. Let $f_K : X_K \to X'_K$ be the extension of f to K, X_K is a Del Pezzo surface by Remark 4.30, then X'_K is a Del Pezzo surface by Lemma 4.20. So X' is a Del Pezzo surface over k, again by Remark 4.30. □

Proposition 4.34. *If X is a blowing-up of $\mathbb{P}^2_k$ in $r \leq 8$ k-rational points in general position, then X is a Del Pezzo surface of degree $9 - r$ over k.*

Proof. We have that X_K is a blowing-up of $\mathbb{P}^2_K$ in $r \leq 8$ points in general position, so X_K is a Del Pezzo surface of degree $9-r$ over K by Remark 4.28, then X is a Del Pezzo surface of degree $9 - r$ over k by Remark 4.30. □

In general Theorem 4.22 does not hold over an arbitrary field. In Section 4.4 we will prove that it remains valid if the field is separably closed, while here we give some counter examples.

We start with two examples of Del Pezzo surfaces over $\mathbb{Q}$ that are not rational over $\mathbb{Q}$.

Example 4.35. *Let X be the quadric surface in $\mathbb{P}^3_{\mathbb{Q}}$ defined by the homogeneous polynomial $x_0^2 + x_1^2 + x_2^2 + x_3^2 \in \mathbb{Q}[x_0, x_1, x_2, x_3]$. We have that X is a surface and $X_{\mathbb{C}} \cong \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ (see [Har], I, Exercises 5.12 and 2.15), then X is a Del Pezzo surface by Remark 4.30. Moreover X is minimal over $\mathbb{Q}$ as $X_{\mathbb{C}}$ is minimal over $\mathbb{C}$. It is clear that X has no $\mathbb{Q}$-rational points, then X is not rational over $\mathbb{Q}$ by Remark 2.24. In particular X is a minimal Del Pezzo surface of degree 8 which is not isomorphic to $\mathbb{P}^1_{\mathbb{Q}} \times \mathbb{P}^1_{\mathbb{Q}}$.*

Example 4.36. *Let X be the hypersurface of $\mathbb{P}_{\mathbb{Q}}(1, 1, 1, 2)$ defined by the polynomial $x_0^4 + x_1^4 + x_2^4 + (x_3^2)^2 \in \mathbb{Q}[x_0, x_1, x_2, x_3^2]$. X is a Del Pezzo surface of degree 2 over $\mathbb{Q}$ (by Example 4.7 and Remark 4.30) without $\mathbb{Q}$-rational points, then X is not rational over $\mathbb{Q}$ by Remark 2.24.*

The next example shows a Del Pezzo surface of degree 7 over $\mathbb{Q}$ which is rational over $\mathbb{Q}$, even a blowing-up of $\mathbb{P}^2_{\mathbb{Q}}$, but cannot be represented as a blowing-up of $\mathbb{P}^2_{\mathbb{Q}}$ in two $\mathbb{Q}$-rational points.

Example 4.37. *Let $P_1 = (1 : i : 0 : 0), P_2 = (1 : -i : 0 : 0) \in \mathbb{P}^2_{\mathbb{C}}$, then P_i is not defined over $\mathbb{Q}$ for $i = 1, 2$. We have that $\{P_1, P_2\}$ is $\Gamma_{\mathbb{Q}}$-invariant, hence defined over $\mathbb{Q}$ by Proposition 2.31. Let Z be the closed subvariety of $\mathbb{P}^2_{\mathbb{Q}}$ such that $Z_{\mathbb{C}} = P_1 \cup P_2$, then Z is a closed point in $\mathbb{P}^2_{\mathbb{Q}}$, as it is an integral subvariety of dimension 0. Let X be the blowing-up of $\mathbb{P}^2_{\mathbb{Q}}$ with center Z. By Proposition 2.41 we have that $X_{\mathbb{C}}$ is a Del Pezzo surface of degree 7 over $\mathbb{C}$, then by Remark 4.30 X is a Del Pezzo surface of degree 7 over $\mathbb{Q}$, but it is not a blowing-up of $\mathbb{P}^2_{\mathbb{Q}}$ in two $\mathbb{Q}$-rational points, because P_1, P_2 are not defined over $\mathbb{Q}$.*

4.4 Classification over a separably closed field

In this section we prove that the classification of Del Pezzo surfaces given in Section 4.2, for an algebraically closed field, holds also over a separably closed field.

Let k be a separably closed field, if k is perfect then k is algebraically closed and there is nothing to prove. Then we can assume that k is a non perfect separably closed field of positive characteristic p. Let denote by K an algebraic closure of k, then K/k is a purely inseparable extension.

Proposition 4.38. *Let X be a projective variety over k and k'/k an algebraic extension, then the projection $X_{k'} \to X$ is a homeomorphism at the level of topological spaces.*

Proof. Since X is projective, then there is a positive integer n and a closed immersion $\phi : X \to \mathbb{P}^n_k$ which extends to a closed immersion $\phi_{k'} : X_{k'} \to \mathbb{P}^n_{k'}$ and we have a commutative diagram

$$\begin{tikzcd} X_{k'} \arrow[r, "\phi_{k'}"] \arrow[d] & \mathbb{P}^n_{k'} \arrow[d] \\ X \arrow[r, "\phi"] & \mathbb{P}^n_k \end{tikzcd}$$

where the vertical arrows are the projections. Thus we see that it is enough to prove that the projection $\mathbb{P}^n_{k'} \to \mathbb{P}^n_k$ is a homeomorphism at the level of topological spaces.

Let prove it first for $k' = K$. The projection $\mathbb{P}^n_K \to \mathbb{P}^n_k$ is surjective and closed, let $\pi : \mathbb{P}^n_K \to \mathbb{P}^n_k$ be its associated continuous map of topological spaces, then π is surjective, closed and sends closed points to closed points.

Now we prove that if π were not surjective, than π would be not surjective at the level of closed points. If $y \in \mathbb{P}^n_K$ is any point, its closure $\overline{\{y\}}$ in $\mathbb{P}^n_K$ is a subvariety of $\mathbb{P}^n_K$, then is contains at least one closed point. If $y, z \in \mathbb{P}^n_K$ such that $\pi(y) = \pi(z) = x \in \mathbb{P}^n_k$, then $\pi(\overline{\{y\}}) = \overline{\{x\}} = \pi(\overline{\{z\}})$ because π is a closed morphism of topological spaces, but π sends closed points to closed points, then there are closed points $y' \in \overline{\{y\}}$ and $z' \in \overline{\{z\}}$ such that $\pi(y') = \pi(z')$. Thus it is enough to prove that π is injective at the level of closed points. By Proposition 2.16 we have that the set of closed points of $\mathbb{P}^n_K$ is the set of its K-rational points. Let take a closed point $x = (\alpha_0 : \cdots : \alpha_n) \in \mathbb{P}^n_K$, without loss of generality we can assume that $\alpha_0 \neq 0$ and in particular $\alpha_0 = 1$, then x corresponds to the maximal ideal $m' = (x_1 - \alpha_1 x_0, \ldots, x_n - \alpha_n x_0)$ of $K[x_0, \ldots, x_n]$. The projection $\mathbb{P}^n_K \to \mathbb{P}^n_k$ is induced by the natural inclusion of graded rings $\varphi : k[x_0, \ldots, x_n] \to K[x_0, \ldots, x_n]$. Since K/k is a purely inseparable extension in characteristic p, for $i = 1, \ldots, n$ there are positive integers s_i such that $\alpha_i^{p^{s_i}} \in k$, let take $s_1, \ldots, s_n$ to be minimal with that property. For $i = 1, \ldots, n$, we have that $(x_i - \alpha_i x_0)^{p^{s_i}} = x_i^{p^{s_i}} - \alpha_i^{p^{s_i}} x_0^{p^{s_i}} \in k[x_0, \ldots, x_n]$ and that s_i is the minimal positive integer such that $(x_i - \alpha_i x_0)^{p^{s_i}} \in k[x_0, \ldots, x_n]$. Then $m_x = (x_1^{p^{s_1}} - \alpha_1^{p^{s_1}} x_0^{p^{s_1}}, \ldots, x_n^{p^{s_n}} - \alpha_1^{p^{s_n}} x_0^{p^{s_n}})$ is an ideal of $k[x_0, \ldots, x_n]$ such that $\varphi(m_x) \subset m'$. Since K/k is purely inseparable, then the closed subset of $\mathbb{P}^n_K$ defined by the ideal of $K[x_0, \ldots, x_m]$ generated by $\varphi(m_x)$ is the closed point x, then m_x is a homogeneous ideal of $k[x_0, \ldots, x_n]$ such that its radical $\mathrm{Rad}(m_x)$ is the maximal ideal that defines the closed point $\pi(x)$. Since the extension K/k is purely inseparable, then the map that associates to a closed point $x \in \mathbb{P}^n_K(K)$ the homogeneous ideal m_x of $k[x_0, \ldots, x_n]$ defined as above, is injective. Now, if $x, y \in \mathbb{P}^n_K(K)$ are two points such that $\pi(x) = \pi(y)$, then $\mathrm{Rad}(m_x) = \mathrm{Rad}(m_y)$. Since φ is an inclusion we have that $\varphi(\mathrm{Rad}(m_x))$ defines the same closed subset of $\mathbb{P}^n_K$ as $\varphi(m_x)$, i.e. the closed point x as we have seen above. The same holds for $\varphi(\mathrm{Rad}(m_y))$, then we conclude that $x = y$ and that π is injective. Thus π is a bijective closed continuous map of topological spaces, then it is a homeomorphism.

If k'/k is an algebraic extension, let K be an algebraic closure of k containing k', then we have a commutative diagram of projections

$$\begin{tikzcd} \mathbb{P}^n_K \arrow[rr, "\pi_k"] \arrow[dr, "\pi_{k'}"'] & & \mathbb{P}^n_k \\ & \mathbb{P}^n_{k'} \arrow[ur, "\pi"'] & \end{tikzcd}$$

We have that π_k and $\pi_{k'}$ are homeomorphisms at the level of topological spaces, because of what we have proves above. Then at the level of topological spaces we have that $\pi = \pi_k \circ \pi_{k'}^{-1}$ is a homeomorphism. $\square$

Proposition 4.39. *Let X be a surface over k such that X_K contains a (-1)-curve E', then X contains a (-1)-curve E such that $E_K \cong E'$.*

Proof. By Proposition 4.38 the projection $\pi : X_K \to X$ is a homeomorphism at the level of topological spaces, then then $E := \pi(E')$ is an irreducible curve in X

Since closed immersions are stable under base change, we have that E_K is a curve in X_K and, by the universal property of the fibred product, there is a closed immersion $E' \to E_K$. Since π is a homeomorphism at the level of topological spaces, then we can conclude that E_K is an irreducible curve in X_K and that E' is the curve E_K with the reduced induced structure. Then E_K is linearly equivalent, as divisor on X_K, to nE' for some positive integer n. The adjunction formula (1.1) gives $p_a(E_K) = 1 - \frac{n^2+n}{2} \leq 0$, but $p_a(E_K) \geq 0$ as E_K is an irreducible over an algebraically closed field. Then $n = 1$ and $E_K \cong E'$ is a (-1)-curve in X_K, in particular $E_K^2 = -1$ and $E_K \cong \mathbb{P}^1_K$. By Proposition 2.4 we have that $E^2 = E_K^2 = -1$, by Proposition 2.20 we have that $E(k) \neq \emptyset$ and then Theorem 3.29 says that $E \cong \mathbb{P}^1_k$, thus E is a (-1)-curve in X. □

Theorem 4.40. *Let X be Del Pezzo surface over k, then either $X \cong \mathbb{P}^1_k \times \mathbb{P}^1_k$ and $K_X^2 = 8$, or X arises as a blowing-up of $\mathbb{P}^2_k$ in $r \leq 8$ k-rational points in general position and $K_X^2 = 9 - r$.*

Proof. By Remark 4.30 we have that X_K is a Del Pezzo surface over K, then Theorem 4.22 says that X_K is either $\mathbb{P}^1_K \times \mathbb{P}^1_K$ or a blowing up of $\mathbb{P}^2_K$ in $r \leq 8$ closed points.

If $X_K \cong \mathbb{P}^2_K$, then X is a Severi-Brauer surface, by Proposition 2.20 we have that $X(k) \neq \emptyset$, then Theorem 3.29 says that X splits over k, i.e. $X \cong \mathbb{P}^2_k$.

If X_K is not a minimal surface, then it is a blowing-up $f : X_K \to \mathbb{P}^2_K$ with center $r \leq 8$ closed points in general position, we prove by induction on r that these r points are k-rational and f is defined over k. Let E' be a (-1)-curve in X_K which is contracted by f, by Proposition 4.39 there is a (-1)-curve E in X such that $E_K \cong E'$. Let H be a very ample divisor on X, then by Proposition 2.3 H_K is very ample on X_K, let $H' = H + (H.E)E$. Following the proof of Theorem 1.47 (see [Har], V, §5, Theorem 5.10) we can show that H'_K is generated by global sections on X, then it induces a birational surjective morphism $\phi : X_K \to X' := \mathrm{Proj}(\oplus_{m\leq 0})H^0(X_K, mH'_K)$, where X' is a surface, $\phi(E')$ is a point and ϕ is an isomorphism of $X_K \setminus E_K$ onto its image. Since H'_K is defined over k we have that X' and ϕ are defined over k, then since E' is defined over k and it is contracted by ϕ the closed point $\phi(E')$ is defined over k, i.e. it is k-rational. By Lemma 4.20 we have that X' is a Del Pezzo surface of degree $K_{X'}^2 = K_X^2 + 1$, then it is a blowing-up $g : X' \to \mathbb{P}^2_K$ with center $r - 1$ points in general position. By the inductive hypothesis we have that these $r - 1$ points are k-rational and g is defined over k, then also $g(\phi(E'))$ is a k-rational point of $\mathbb{P}^2_K$ and $f = g \circ \phi$ is defined over k. Thus X is a blowing-up of $\mathbb{P}^2_k$ in $r \leq 8$ k-rational points in general position and $K_X^2 = 9 - r$ by Theorem 4.22 and Remark 4.30

Suppose now that $X_K \cong \mathbb{P}^1_K \times \mathbb{P}^1_K$, then $K_X^2 = 8$ by Proposition 4.19 and Remark 4.30. By Proposition 2.20 we have that $X(k) \neq \emptyset$, take $x \in X(k)$ and let $f : X' \to X$ be the monoidal transformation with center x, let denote by E the inverse image of x by f, then E is a (-1)-curve in X. Since f extends to a monoidal transformation $f_K : X'_K \to X_K$, by Proposition 4.27 and Remark 4.30 we have that X' is a Del Pezzo surface of degree 7, then by what we proved above X' is isomorphic to a blowing-up of $\mathbb{P}^2_k$ in two points and by Propositions 4.26 and 4.39 X' contains exactly three (-1)-curves: E, E_1, E_2. Let $D = f(E_1) + f(E_2)$, then D is an effective divisor on X and the associated linear system $|D|$ induces a rational map $\phi : X \dashrightarrow \mathbb{P}^n_k$ for some n, let $\phi_K : \mathbb{P}^1_K \times \mathbb{P}^1_K \dashrightarrow \mathbb{P}^n_K$ be its extension to K, then ϕ_K is the rational map induced by the linear system $|D_K|$. Let $p_i : \mathbb{P}^1_K \times \mathbb{P}^1_K \to \mathbb{P}^1_K$, $i = 1, 2$ be the two projections, as in the second paragraph of the proof of Theorem 4.22 we have that $f_K(E_{i,K}) = p_i^{-1}(P_i(x_K)), i = 1, 2$ are two lines passing through the point x_K, then ϕ_K is the Segre embedding $\mathbb{P}^1_K \times \mathbb{P}^1_K \to \mathbb{P}^3_K$ (see [Har], II, §7, Example 7.6.2). So also ϕ is an embedding and $\phi(X)$ is the image of the Segre embedding $\psi : \mathbb{P}^1_k \times \mathbb{P}^1_k \to \mathbb{P}^3_k$, then X is isomorphic to $\mathbb{P}^1_k \times \mathbb{P}^1_k$ via the isomorphism $\psi^{-1} \circ \phi$. □

Remark 4.41. After Theorem 4.40 and Proposition 4.39 it is easy to check that Propositions 4.16, 4.19, 4.25, 4.26, 4.27 and 4.29 hold also over a separably closed field.

Chapter 5

Unirationality

This chapter is devoted to study the rationality and unirationality of Del Pezzo surfaces over an arbitrary field k. We will prove that a Del Pezzo surface of degree ≥ 5 with a k-rational point is rational, while Del Pezzo surfaces of degree 3 and 4 with a k-rational point are unirational. In both cases, if k is infinite, we can conclude that the set of k-rational points is dense.

In Section 5.5 we will mention the last developments of research for Del Pezzo surfaces of degree 1 and 2. In particular some conditions for unirationality of Del Pezzo surfaces of degree 2 and some conditions for the density of the set of k-rational points for Del Pezzo surfaces of degree 1.

Let k be a field, $\overline{k}$ a separable closure of k and $\Gamma_k = \mathrm{Gal}(\overline{k}/k)$. Let X be a Del Pezzo surface over k. For any point $x \in X$ we denote $\overline{x} = x \times_{\mathrm{Spec}\, k} \mathrm{Spec}(\overline{k})$. If $x \in X(k)$, then $\overline{x} \in \overline{X}(\overline{k})$.

Remarks 1.28 and 4.41 will be tacitly used throughout this chapter.

5.1 Points on the (-1)-curves

Definition 5.1. Let X be a Del Pezzo surface over k, let L/k be a Galois extension and $x \in X_L(L)$. We say that x is a point

of type 0: if $\overline{x}$ does not lie on any (-1)-curve of $\overline{X}$;

of type 1: if $\overline{x}$ lies on exactly one (-1)-curve of $\overline{X}$;

of type 2: if $\overline{x}$ lies on the intersection of exactly two (-1)-curves of $\overline{X}$;

Proposition 5.2. *If $X(k)$ contains a point* of type 1*, then there exists a birational morphism $X \to X'$ over k, where X' is a Del Pezzo surface over k, $K_{X'}^2 = K_X^2 + 1$ and $X'(k) \neq \emptyset$.*

Proof. Let $x \in X(k)$ be a point *of type 1* and E the (-1)-curve of $\overline{X}$ containing $\overline{x}$. By Remark 2.29 we have that $\overline{x}$ is fixed by the natural action of Γ_k over $\overline{X}$. Moreover E is the only (-1)-curve of $\overline{X}$ that contains $\overline{x}$ and the natural action of Γ_k over $\overline{X}$ permutes the set of (-1)-curves of $\overline{X}$ by Corollary 2.36, then E is Galois invariant, and hence defined over k by Proposition 2.31. By Proposition 2.39 there exists a surface X' over k and a birational morphism $f : X \to X'$ over k, which is a monoidal trasformation contracting E over $\overline{k}$. X' is a Del Pezzo surface by Lemma 4.20 and $K_{X'}^2 = K_X^2 + 1$ by Proposition 1.45. Moreover $f(E)$ is a k-rational point, because both f and E are defined over k. Then $X'(k) \neq \emptyset$. □

Proposition 5.3. *If $K_X^2 \geq 4$, any closed point of X is either* of type 0, *or* of type 1, *or* of type 2.

Proof. By Theorem 4.40 we have that $\overline{X}$ is a blowing-up of $\mathbb{P}^2_{\overline{k}}$ in $r \leq 5$ points in general position. By Proposition 4.26 we have that $\overline{X}$ contains only (-1)-curves of *type a, b* or *c*, and no more than one curve of *type c.*

By Proposition 4.29 we have that: if E_1, E_2, E_3 are three distinct (-1)-curves of *type a*, then $E_1 \cap E_2, \cap E_3 = \emptyset$; if E_1, E_2 are two distinct (-1)-curves of *type a* and L is a (-1)-curve of *type b* or of *type c* in $\overline{X}$, then $E_1 \cap E_2 \cap L = \emptyset$; if E is a (-1)-curve of *type a* and L_1, L_2 are two distinct (-1)-curves of *type b* in $\overline{X}$, then $E \cap L_1 \cap L_3 = \emptyset$; if E is a (-1)-curve of *type a*, L is a (-1)-curves of *type b* and C is a (-1)-curve of *type c* in $\overline{X}$, then $E \cap L \cap C = \emptyset$; if L_1, L_2 are two distinct (-1)-curves of *type b* and C is a (-1)-curve of *type c* in $\overline{X}$, then $L_1 \cap L_2 \cap C = \emptyset$. Since there are no more possible combinations of three distinct (-1)-curves on $\overline{X}$, then we conclude that $X(k)$ contains only points *of type 0, 1* or *2.* □

5.2 Degree ≥ 5

Proposition 5.4. *If $K_X^2 = 9$ and $X(k) \neq \emptyset$ then $X \cong \mathbb{P}^2_k$.*

Proof. $\overline{X}$ is a Del Pezzo surface of degree 9 over $\overline{k}$ by Remark 4.30, then $\overline{X} \cong \mathbb{P}^2_{\overline{k}}$ by Theorem 4.40, then X is a Severi-Brauer surface over k and we conclude by Proposition 3.29. □

Proposition 5.5. *If $K_X^2 = 8$ and $\overline{X}$ is a not a minimal surface, then X is k-rational.*

Proof. $\overline{X}$ is a Del Pezzo surface of degree 8 by Remark 4.30, then by Theorem 4.40 and Proposition 4.26 it contains a unique (-1)-curve E, which is Galois invariant by Corollary 2.36 and defined over k by Proposition 2.31. By the same argument used in the proof of Propostition 5.2, E can be contracted by a birational morphism $f : X \to X'$, where X' is a Del Pezzo surface over k, $K_{X'}^2 = 9$, f is a monoidal transformation defined over k and $f(E) \in X'(k)$. Then $X' \cong \mathbb{P}^2_k$ by Proposition 5.4 and we conclude that X is k-rational. □

Remark 5.6. We have proved that there are no minimal Del Pezzo surfaces X over k such that $\overline{X}$ is non minimal. So a Del Pezzo surface X over k of degree 8 is minimal if and only if $\overline{X} \cong \mathbb{P}^1_{\overline{k}} \times \mathbb{P}^1_{\overline{k}}$.

Moreover, we have proved that, up to isomorphism, there is only one non minimal Del Pezzo surface of degree 8, which is the blowing-up of $\mathbb{P}^2_k$ with center a point.

Proposition 5.7. *If $K_X^2 = 7$, then X is k-rational.*

Proof. From Theorem 4.40 and Proposition 4.26 we know that, up to isomorphism, $\overline{X}$ is a blowing-up of $\mathbb{P}^2_{\overline{k}}$, say $f : \overline{X} \to \mathbb{P}^2_{\overline{k}}$, with center two distinct points $P_1, P_2 \in \mathbb{P}^2_{\overline{k}}$, and $\overline{X}$ contains exactly three (-1)-curves: two *type a* (-1)-curves E_1, E_2, which are the inverse images under f of P_1, P_2 respectively, and one *type b* (-1)-curve E, which is the strict transform under f of the line L containing P_1, P_2 in $\mathbb{P}^2_{\overline{k}}$. By Proposition 4.29 we have that E_1, E_2 are disjoint, while E meets both E_1 and E_2. Since the natural action of Γ_k over $\overline{X}$ induces an action on the set of (-1)-curves and respects the intersection pairing (see Corollary 2.36 and Proposition 2.35), we have that $\{E_1, E_2\}$ is a Galois invariant pair of disjoint (-1)-curves, then f is defined over k by Proposition 2.39 and is induced by a birational morphism $g : X \to X'$ where X' is a Del Pezzo surface of degree 9.

Since $\{E_1, E_2\}$ is Galois invariant and f is defined over k, we have that $\{P_1, P_2\}$ is a Galois invariant pair of points, then also the line L is Galois invariant, hence defined over k by Proposition 2.31. So X' contains a k-rational curve C such that $\overline{C} \cong L$ and in particular $X'(k) \neq \emptyset$. Then $X' \cong \mathbb{P}^2_k$ by Proposition 5.4 and X is k-rational as it is k-birationally equivalent to X' through g. □

Remark 5.8. We have proved that there are no minimal Del Pezzo surfaces of degree 7 over k, for any field k.

Proposition 5.9. *If $\overline{X} \cong \mathbb{P}^1_{\overline{k}} \times \mathbb{P}^1_{\overline{k}}$ and $X(k) \neq \emptyset$, then X is k-rational.*

Proof. Let fix a point $x \in X(k)$ and let $f : X' \to X$ be the monoidal transformation with center x. Then f extends to a monoidal transformation $\overline{X'} \to \mathbb{P}^1_{\overline{k}} \times \mathbb{P}^1_{\overline{k}}$ with center $\overline{x}$. By Proposition 4.27 we have that $\overline{X'}$ is a Del Pezzo surface of degree 7, then X' is a Del Pezzo surface of degree 7 over k, by Remark 4.30 and it is k-rational by Proposition 5.7, then also X is k-rational because f is a birational morphism over k. □

Proposition 5.10. *If $K_X^2 = 6$ and $X(k) \neq \emptyset$, then X is k-rational.*

Proof. By Theorem 4.40 we have that $\overline{X}$ is, up to isomorphism, a blowing up of $\mathbb{P}^2_{\overline{k}}$, say $f : \overline{X} \to \mathbb{P}^2_{\overline{k}}$, with center three distinct, not aligned closed points $P_1, P_2, P_3 \in \mathbb{P}^2_{\overline{k}}$. For $i = 1, 2, 3$ let E_i be the inverse image of P_i under f, for $i, j \in \{1, 2, 3\}$, $i < j$, let $L_{i,j}$ be the strict transform under f of the line passing through P_i, P_j in $\mathbb{P}^2_{\overline{k}}$. By Proposition 4.26 we have that the (-1)-curves of $\overline{X}$ are exactly $E_1, E_2, E_3, L_{1,2}, L_{1,3}, L_{2,3}$.

Let $x \in X(k)$, by Proposition 5.3 we have that x is either a point *of type 0* or *of type 1* or *of type 2*.

If x is a point *of type 1*, then Proposition 5.2 says that X is not a minimal surface over k.

If x is a point *of type 2*, without loss of generality we can assume that $\overline{x} = E_1 \cap L_{1,2}$. Since $\overline{x}$ is fixed by the natural action of Γ_k over $\overline{X}$, then $\{E_1, L_{1,2}\}$ is Galois invariant, and we easily see (using Proposition 2.35, Corollary 2.36 and Proposition 4.29) that the action of Γ_k on the set of (-1)-curves of $\overline{X}$ is given by exactly two permutations: the identity Id and the permutation σ such that $\sigma(E_1) = L_{1,2}$, $\sigma(E_2) = L_{1,3}$, $\sigma(E_3) = L_{2,3}$ and $\sigma^2 = \mathrm{Id}$. Then $\{E_2, L_{1,3}\}$ is a Galois invariant pair of disjoint (-1)-curves (again by Proposition 4.29), and X is not a minimal surface over k, by Proposition 2.40.

If X is not a minimal surface, then it is k-birationally equivalent to a Del Pezzo surface X' of degree ≥ 7 with a k-rational point, X' is k-rational by one of Propositions 5.4, 5.5, 5.7, 5.9, then also X is k-rational.

If x is a point *of type 0*, let $X' \to X$ be the monoidal transformation with center x and $g : \overline{X'} \to \overline{X}$ its extension to $\overline{k}$. Let $P_4 = f(\overline{x}) \in \mathbb{P}^2_{\overline{k}}$, then $f \circ g : \overline{X}' \to \mathbb{P}^2_{\overline{k}}$ is a blowing-up of $\mathbb{P}^2_{\overline{k}}$ in P_1, P_2, P_3, P_4 and $\overline{X'}$ is a Del Pezzo surface of degree 5 by Proposition 4.27, then X' is a Del Pezzo surface of degree 5 by Remark 4.30. For $i, j \in \{1, 2, 3, 4\}, i < j$, define E_i and $L_{i,j}$ as above, replacing $\overline{X}$ by $\overline{X'}$ and f by $f \circ g$. By Proposition 4.29 we have that $L_{1,4}, L_{2,4}, L_{3,4}$ are pairwise disjoint and $\{L_{1,4}, L_{2,4}, L_{3,4}\}$ is the set of (-1)-curves of $\overline{X'}$ that meet E_4. Since g is defined over k and $x \in X(k)$, we have that E_4 is Galois invariant and k-rational, then $\{L_{1,4}, L_{2,4}, L_{3,4}\}$ is Galois invariant and (by Proposition 2.39 and Lemma 4.20) there exists a Del Pezzo surface X'' over k of degree 8 and a birational morphism $X' \to X''$. Since E_4 is k-rational and contained in X', we have that $X'(k) \neq \emptyset$ and also $X''(k) \neq \emptyset$, then we get that X'' is k-rational by Proposition 5.5 or 5.9. But X and X'' are k-birationally equivalent, then also X is k-rational. □

Remark 5.11. In particular we have proved that if X is a Del Pezzo surface of degree 6 over k with a k-rational point $x \in X(k)$ such that $\overline{x}$ lies on a (-1)-curve of $\overline{X}$, then X is not minimal over k.

Proposition 5.12. *If $K_X^2 = 5$ and $X(k) \neq \emptyset$, then X is k-rational.*

Proof. By Theorem 4.40 we have that $\overline{X}$ is, up to isomorphism, a blowing up of $\mathbb{P}^2_{\overline{k}}$, say $f : \overline{X} \to \mathbb{P}^2_{\overline{k}}$, with center four distinct, closed points in general position $P_1, P_2, P_3, P_4 \in \mathbb{P}^2_{\overline{k}}$. For $i = 1, 2, 3, 4$ let E_i be the inverse image of P_i under f, for $i, j \in \{1, 2, 3, 4\}$, $i < j$, let $L_{i,j}$ be the strict transform under f of the line passing through P_i, P_j in $\mathbb{P}^2_{\overline{k}}$.

By Proposition 4.26 we have that $\overline{X}$ contains exactly ten (-1)-curves: four *type a* (-1)-curves E_1, E_2, E_3, E_4 and six *type b* (-1)-curves $L_{i,j}, i, j \in \{1, 2, 3, 4\}, i < j$.

Let $x \in X(k)$, by Proposition 5.3 we have that x is either a point *of type 0* or *of type 1* or *of type 2*.

If x is a point *of type 1* then Proposition 5.2 says that X is not minimal over k.

If x is a point *of type 2*, without loss of generality we can reduce to two cases: $\overline{x} = E_1 \cap L_{1,2}$ or $\overline{x} = L_{1,2} \cap L_{3,4}$.

If $\overline{x} = E_1 \cap L_{1,2}$, then $\{E_1, L_{1,2}\}$ is Galois invariant and also $\{E_2, L_{1,3}, L_{1,4}, L_{3,4}\}$ is Galois invariant, as it is the set of (-1)-curves of $\overline{X}$ that meet E_1 or $L_{1,2}$ but are not E_1 nor $L_{1,2}$ (see Proposition 4.29). Moreover $\{E_2, L_{1,3}, L_{1,4}, L_{3,4}\}$ is a set of pairwise disjoint curves by Proposition 4.29, then Proposition 2.40 says that X is not minimal over k.

If $\overline{x} = L_{1,2} \cap L_{3,4}$, then $\{L_{1,2}, L_{3,4}\}$ is Galois invariant and also $\{E_1, E_2, E_3, E_4\}$ is Galois invariant, as it is the set of (-1)-curves of $\overline{X}$ that meet $L_{1,2}$ or $L_{3,4}$ but are not $L_{1,2}$ nor $L_{3,4}$ (see Proposition 4.29). Moreover $\{E_1, E_2, E_3, E_4\}$ is a set of pairwise disjoint curves by Proposition 4.29, then Proposition 2.40 says that X is not minimal over k.

If X is not a minimal surface, then it is k-birationally equivalent to a Del Pezzo surface X' of degree ≥ 6 with a k-rational point, X' is k-rational by one of Propositions 5.4, 5.5, 5.7, 5.9, 5.10, then also X is k-rational.

If x is a point *of type 0*, let $X' \to X$ be the monoidal transformation with center x and $g : \overline{X'} \to \overline{X}$ its extension to $\overline{k}$, X' is a Del Pezzo surface of degree 4 over k by Proposition 4.27 and Remark 4.30. Let $P_5 = f(\overline{x}) \in \mathbb{P}^2_{\overline{k}}$, then $f \circ g : \overline{X'} \to \mathbb{P}^2_{\overline{k}}$ is a blowing up of $\mathbb{P}^2_{\overline{k}}$ in P_1, P_2, P_3, P_4, P_5. For $i, j \in \{1, 2, 3, 4, 5\}, i < j$, define E_i and $L_{i,j}$ as above, replacing $\overline{X}$ by $\overline{X'}$ and f by $f \circ g$, moreover let C be the strict transform under $f \circ g$ of the conic containing P_1, P_2, P_3, P_4, P_5. Since $x \in X(k)$ we have that E_5 is defined over k and k-rational, then $X'(k) \neq \emptyset$. Moreover E_5 is Galois invariant, then also then set $\{L_{1,5}, L_{2,5}, L_{3,5}, L_{4,5}, C\}$, of the (-1)-curves of $\overline{X}'$ that meet E_5 and are not E_5, is Galois invariant. Moreover $L_{1,5}, L_{2,5}, L_{3,5}, L_{4,5}, C$ are pairwise disjoint by Proposition 4.29, then by Proposition 2.39 there is a Del Pezzo surface X'' over k of degree 9 and a birational morphism $X' \to X''$. Since $X'(k) \neq \emptyset$ then also $X''(k) \neq \emptyset$, then $X'' \cong \mathbb{P}^2_k$ by Proposition 5.4 and X is k-rational as it is k-birationally equivalent to X''. $\square$

Remark 5.13. In particular we have proved that if X is a Del Pezzo surface of degree 5 over k with a k-rational point $x \in X(k)$ such that $\overline{x}$ lies on a (-1)-curve of $\overline{X}$, then X is not minimal over k.

5.3 Degree 3

Let X be a Del Pezzo surface of degree 3 over k. By Proposition 4.32 X is a cubic surface in $\mathbb{P}^3_k$. Throughout this section we consider X as hypersurface in $\mathbb{P}^3_k$ defined by a polynomial of degree 3.

Definition 5.14. Let L/k be a Galois extension, for any point $y \in X_L(L)$ let C_y be the intersection of X_L with its tangent plane at y.

Lemma 5.15. *Let L/k be a Galois extension and $y \in X_L(L)$ a point* of type 0*, then C_y is an integral plane cubic curve with a double point at y and C_y is rational over L.*

Proof. Let H be the tangent plane of X_L at y, we can choose a system of homogeneous coordinates x_0, x_1, x_2, x_3 on $\mathbb{P}^3_L$ such that $y = (1 : 0 : 0 : 0)$ and H is defined by the equation $x_3 = 0$. Then x_0, x_1, x_2 are homogeneous coordinates on $H \cong \mathbb{P}^2_L$. Since X is defined by an irreducible homogeneous polynomial of degree 3 in x_0, x_1, x_2, x_3, then C_y is defined, as hypersurface in H, by a homogeneous polynomial of degree 3 in x_0, x_1, x_2, thus C_y is a cubic plane curve, with multiplicity 2 at y as C_y is contained in the tangent plane of X at y. Since y is a point *of type 0* and every line of $\mathbb{P}^3_{\overline{k}}$ contained in $\overline{X}$ is a (-1)-curves by Proposition 4.16, then $\overline{C_y}$ is irreducible and reduced. Thus C_y is an integral plane cubic curve with a double point at y.

The parametrization of C_y with the lines of H that contain y gives a birational equivalence between C_y and $\mathbb{P}^1_L$. Thus C_y is rational over L. □

Lemma 5.16. *If L/k is a Galois extension and $y, y' \in X_L(L)$ are two points of type 0 such that $y \notin C_{y'}$ and $y' \notin C_y$, then there is a dominant rational map*

$$\phi : C_y \times C_{y'} \dashrightarrow X_L$$

Proof. By Propositions 4.16 and 4.26, X_L contains only finitely many lines of $\mathbb{P}^3_L$, C_y and $C_{y'}$ are integral cubic curves by Lemma 5.15, then only finitely many points in $C_y(L)$ and only finitely many points in $C_{y'}(L)$ lie on a line of $\mathbb{P}^3_L$ contained in X_L. Moreover $C_y \cap C_{y'}$ is a finite set of closed points as $C_y \neq C_{y'}$ by hypothesis. Then, for a general $u \in C_y(L)$ and a general $u' \in C_{y'}(L)$ we have that $u \neq u'$ and the line $l_{(u,u')}$ of $\mathbb{P}^3_L$ passing through u and u' is not contained in X_L. The line $l_{(u,u')}$ intersects X_L in exactly one more closed point $f(u, u') \in X_L(L)$.

The map $f : C_y(L) \times C_{y'}(L) \dashrightarrow X_L(L)$ that sends a general couple $(u, u') \in C_y(L) \times C_{y'}(L)$ to the point $f(u, u')$ is defined by rational functions on the coordinates on an open affine subset of $C_y \times C_{y'}$, then f induces a rational map $\phi : C_y \times C_{y'} \dashrightarrow X_L$ by Proposition 2.19.

To prove that ϕ is dominant, without loss of generality we can assume that L is separably closed. Take $z \in C_y(L)$, $z \neq y$, such that f is well defined at (z, y') and let $x = f(z, y') \in X_L(L)$.

Let H be the tangent plane of X_L at y, let $\pi : \mathbb{P}^3_L \dashrightarrow H$ be the projection from x. Then $f^{-1}(x)$ is the set of pairs $(u, u') \in C_y(L) \times C_{y'}(L)$ such that $\pi(u') = u$, and $f^{-1}(x)$ has the same dimension as $C_y \cap \pi(C_{y'})$. Since both C_y and $\pi(C_{y'})$ contain z but with different multiplicities, then $C_y \neq \pi(C_{y'})$, and their intersection has dimension 0. Thus $f^{-1}(x)$ has dimension 0. Since L is separably closed, we have that $f^{-1}(x)$ is dense in $\phi^{-1}(x)$ by Proposition 2.20, then also $\phi^{-1}(x)$ has dimension 0 and we conclude that ϕ is dominant by Exercise 3.22 in [Har], II, §3. $\square$

Proposition 5.17. *If $X(k) \neq 0$, then X is unirational over k.*

Proof. Let $x \in X(k)$.

Let suppose first that k is an infinite field. Then a general line l containing x in $\mathbb{P}^2_k$ is not contained in X and intersect $\overline{X}$ in three distinct closed points $\overline{x}, z, z'$ such that z, z' are *of type 0*. Then $z \notin C_{z'}$ and $z' \notin C_z$, indeed: if, for example, $z \in C_{z'}$, then l is contained in the tangent plane of $\overline{X}$ at z' and intersects X in z' with multiplicity ≥ 2, which contradicts the fact that l meets X in three distinct points.

If z, z' are defined over k, there are $y, y' \in X(k)$ such that $\overline{y} = z, \overline{y'} = z'$. C_y and C'_y are k-rational by Lemma 5.15, then $C_y \times C_{y'}$ is k-birationally equivalent to $\mathbb{P}^1_k \times \mathbb{P}^1_k$ and hence k-rational. Lemma 5.16 gives a dominant rational map $C_y \times C_{y'} \dashrightarrow X$, then X is unirational over k.

If z, z' are not defined over k, then there is a quadratic extension L/k such that z, z' are defined over L and conjugate under the natural action of Γ_k over $\overline{X}$. Let $y, y' \in X_L(L)$ such that $\overline{y} = z, \overline{y'} = z'$, then y, y' are conjugate under the natural action of $G = \mathrm{Gal}(L/k) \cong \mathbb{Z}/2\mathbb{Z}$ over X_L and also C_y and C'_y are conjugate under the natural action of G over X_L. For any $u \in C_y(L)$, let $u' \in C_{y'}(L)$ be its conjugate under the action of G over X_L. Looking at the definition of the map f in the proof of Lemma 5.16, we see that, since u and u' are conjugate under the action of G, $\phi(u, u')$ is a closed point defined over k. Moreover C_y is rational over L by Lemma 5.15, then we get a map $g : \mathbb{P}^1_L(L) \dashrightarrow X(k)$ that sends $u \in C_y$ to $f(u, u')$ and is defined by rational functions on the coordinates on an open affine subset of $\mathbb{P}^1_L$. In particular we get that $X(k)$ contains infinitely many k-rational points and then there is a point in the image of G such that its inverse image under g has dimension 0 (see the proof of Lemma 5.16). By Example 2.15, we have a functorial identification $\mathbb{P}^1_L(L) = (\mathfrak{R}_{L/k}(\mathbb{P}^1_L))(k)$, then g induces a map $(\mathfrak{R}_{L/k}(\mathbb{P}^1_L))(k) \dashrightarrow X(k)$ defined by rational functions on the coordinates on an open affine subset of $\mathfrak{R}_{L/k}(\mathbb{P}^1_L)$. Then, by Proposition 2.19 we obtain a rational map $\mathfrak{R}_{L/k}(\mathbb{P}^1_L) \dashrightarrow X$ defined over k which is dominant as there is a point in X whose inverse image is nonempty and has dimension 0. Since $\mathfrak{R}_{L/k}(\mathbb{P}^1_L)$ is birationally equivalent to $\mathbb{P}^2_k$ over k by Proposition 1.31, we can conclude that X is unirational over k.

If k is a finite field, we could not find a line l that intersect $\overline{X}$ in two points that satisfy the hypothesis of Lemma 5.16 (see Example 5.18), then we work with all the lines passing through x at the same time.

Without loss of generality we can choose a system of homogeneous coordinates x_0, x_1, x_2, x_3 on $\mathbb{P}^3_L$ such that $x = (1:0:0:0)$ and that the tangent plane H of X at x is defined by the equation $x_3 = 0$. We can work on the open affine subset $\mathbb{P}^3_k \setminus \{x_0 = 0\}$ which we denote by $\mathbb{A}^3_k$ with coordinates x_1, x_2, x_3, then x is the point $(0,0,0)$ in $\mathbb{A}^3_k$, H is the plane of equation $x_3 = 0$ and X is defined by the irreducible polynomial

$$f(x_1, x_2, x_3) = f_1(x_1, x_2, x_3) + f_2(x_1, x_2, x_3) + f_3(x_1, x_2, x_3)$$

where $f_i \in k[x_1, x_2, x_3]$ is a homogeneous polynomial of degree i, for $i = 1, 2, 3$, and $f_1(x_1, x_2, x_3) = x_3$. The lines through x in $\mathbb{A}^3_k \setminus H$ can be parametrized by $(u_1, u_2) \in \mathbb{A}^2_k(k)$ as follows: let $l_{(u_1,u_2)}$ be the line containing x and the point of coordinates $(u_1, u_2, 1)$, then the points of $l_{(u_1,u_2)}$ have coordinates $(\lambda u_1, \lambda u_2, \lambda)$ for $\lambda \in k$. Let deonte by $k(u_1, u_2)$ the field of rational funtions in u_1, u_2 with coefficients in k.

$l_{(u_1,u_2)}$ intersects $\overline{X}$ in two more points $z_{(u_1,u_2),1}, z_{(u_1,u_2),2}$, besides x, of coordinates $z_{(u_1,u_2),i} = (\lambda_i u_1, \lambda_i u_2, \lambda_i)$ for $i = 1, 2$, where $\lambda_1, \lambda_2 \in \overline{k(u_1, u_2)}$ are the zeros of the following quadratic polynomial in λ

$$f_1(u_1, u_2, 1) + \lambda f_2(u_1, u_2, 1) + \lambda^2 f_3(u_1, u_2, 1)$$

which is irreducible as $f(x_1, x_2, x_3)$ is. Then λ_1, λ_2 are in a quadratic extension $L_{(u_1,u_2)}$ of $k(u_1, u_2)$ and they are conjugate over $k(u_1, u_2)$, in particular we have $z_{(u_1,u_2),1}$ and $z_{(u_1,u_2),2}$ are conjugate over $k(u_1, u_2)$.

The tangent plane H_z of $X_{L_{(u_1,u_2)}}$ at a point $z = (z_1, z_2, z_3) \in X_{L_{(u_1,u_2)}}(L_{(u_1,u_2)})$ in $\mathbb{A}^3_{L_{(u_1,u_2)}}$ has equation $\sum_{i=1}^3 \left(\frac{\partial f}{\partial x_i}(z)\right)(x_i - z_i) = 0$, then the points in H_z have coordinates $(v_1, v_2, v_3(v_1, v_2))$, where

$$v_3(v_1, v_2) := z_3 + \left(\frac{\partial f}{\partial x_3}(z)\right)^{-1} \sum_{i=1}^{2} \left(\frac{\partial f}{\partial x_i}(z)\right)(v_i - z_i)$$

The general line through z in H_z can be parametrized by $v \in L_{(u_1,u_2)}$ as follows: the points of the line $h_{z,v}$ passing through z and $(v, 1, v_3(v, 1))$ have coordinates

$$(z_1 + \mu(v - z_1), z_2 + \mu(1 - z_2), z_3 + \mu(v_3(v, 1) - z_3)) \tag{5.1}$$

For $i = 1, 2$, $v_1, v_2 \in k(u_1, u_2)$, let $h_{(u_1,u_2,v_1,v_2),i} := h_{z_{(u_1,u_2),i}, v_1 + \lambda_i v_2}$ be the general line through $z_{(u_1,u_2),i}$ in the tangent plane of $X_{L_{(u_1,u_2)}}$ at $z_{(u_1,u_2),i}$. To obtain a parametrization of $C_{(u_1,u_2),i} := C_{z_{(u_1,u_2),i}}$ with the lines through $z_{(u_1,u_2),i}$ in the tangent plane of $X_{L_{(u_1,u_2)}}$ at $z_{(u_1,u_2),i}$, we intersect $X_{L_{(u_1,u_2)}}$ with the general line $h_{(u_1,u_2,v_1,v_2),i}$ as follows: we take $z = z_{(u_1,u_2),i}$ and we substitute the parametrization (5.1) in f, so we get a polynomial of degree

3 in μ, with coefficients in $L_{(u_1,u_2)}$, which has a double zero $\mu = 0$ and the third one can be written as rational function in the variables u_1, u_2, v_1, v_2, with coefficients in k.

Since $z_{(u_1,u_2),1}$ and $z_{(u_1,u_2),2}$ are conjugate over $k(u_1, u_2)$, then also $C_{(u_1,u_2),1}$ and $C_{(u_1,u_2),2}$ are conjugate over $k(u_1, u_2)$.

For any point $y_1 \in C_{(u_1,u_2),1}(L_{(u_1,u_2)})$ let $y_2 \in C_{(u_1,u_2),2}(L_{(u_1,u_2)})$ be its conjugate, then the third intersection point with $X_{L_{(u_1,u_2)}}$ of the line passing through y_1 and y_2 is defined over $k(u_1, u_2)$ and its coordinates are rational functions in u_1, u_2, v_1, v_2 with coefficients in k. So we get a map $g : \mathrm{Spec}(k[u_1, u_2, v_1, v_2])(k) \dashrightarrow X(k)$ which is defined by rational functions on the coordinates, by Proposition 2.19 it induces a rational map

$$\psi : \mathbb{A}^4_k = \mathrm{Spec}(k[u_1, u_2, v_1, v_2]) \dashrightarrow X$$

defined over k. The map g comes from the map defined over the infinite field $k(u_1, u_2)$

$$\left(\mathrm{Spec}(k[v_1, v_2])_{k(u_1,u_2)}\right)(k(u_1, u_2)) \dashrightarrow X_{k(u_1,u_2)}(k(u_1, u_2)) \tag{5.2}$$

which is dominant from the above argument for infinite fields. Then ψ is dominant and, by Proposition 1.29, we get that X is unirational over k. $\square$

The next example shows that over a finite field k we can find a Del Pezzo surface that has no *type 0* k-rational points and such that the construction of the morphism that gives unirationality over infinite fields in the proof of Proposition 5.17 does not apply.

Example 5.18. *Let $k = \mathbb{F}_2$ be the field with two elements, let X be the cubic surface in $\mathbb{P}^3_{\mathbb{F}_2}$ defined by the homogeneous polynomial $x_0^3 + x_1^3 + x_2^3 + x_3^3$, then*

$$\begin{aligned} X(\mathbb{F}_2) = \{&(1:1:0:0), (1:0:1:0), (1:0:0:1), (0:1:1:0),\\ &(0:1:0:1), (0:0:1,1), (1,1,1,1)\} \end{aligned}$$

it is immediate to see that each point in $X(\mathbb{F}_2)$ lies on one of the following lines of $\mathbb{P}^3_{\mathbb{F}_2}$

$$\begin{cases} x_0 + x_1 = 0 \\ x_2 + x_3 = 0 \end{cases} \qquad \begin{cases} x_0 + x_2 = 0 \\ x_1 + x_3 = 0 \end{cases} \qquad \begin{cases} x_0 + x_3 = 0 \\ x_1 + x_2 = 0 \end{cases}$$

it is easy also to verify that the three lines of $\mathbb{P}^3_{\mathbb{F}_2}$ listed above are contained in X, then they are three (-1)-curves on X by Proposition 4.32. Thus X is an example of a Del Pezzo surfaces over a finite field k such that $X(k) \neq \emptyset$ and all its k-rational points lie on a (-1)-curve.

Let L/k be a quadratic extension, then $L \cong \mathbb{F}_4 \cong \mathbb{F}_2[t]/(t^2 + t + 1)$. Let α be a root of $t^2 + t + 1$ in L, then, up to a permutation of the coordinates of $\mathbb{P}^3_L$, the points of $X_L(L)$ are the following five:

$$(1:1:1:1), (1:1:0:0), (1:1:\alpha:\alpha), (\alpha:\alpha:0:0), (\alpha:\alpha:\alpha:\alpha)$$

and they lie on the line of $\mathbb{P}^3_L$ of equations $x_0 + x_1 = 0$ and $x_2 + x_3 = 0$, which is contained in X_L and is a (-1)-line of X_L. Thus we can conclude that all the L-rational points in X_L are contained in a (-1)-curve and hence all the lines in $\mathbb{P}^3_{\mathbb{F}_2}$ passing through a $\mathbb{F}_2$-rational point of X intersect X in two more points that are not of type 0.

5.4 Degree 4

Proposition 5.19. *Let X be a Del Pezzo surface of degree 4 over k. If $X(k) \neq \emptyset$, then X is unirational over k.*

Proof. Let $x \in X(k)$, by Proposition 5.3 we have that x is either a point *of type 0* or *of type 1* or *of type 2*.

If x is a point *of type 0*, let $X' \to X$ be the monoidal transformation with center x, and E the associated exceptional divisor in X'. We have that X' is a Del Pezzo surface of degree 3 over k by Proposition 4.27, moreover E is k-rational, then $X'(k) \neq \emptyset$ and X' is unirational over k by Proposition 5.17.

If x is a point *of type 1*, then Proposition 5.2 says that there is a Del Pezzo surface X' of degree 5 over k such that $X'(k) \neq \emptyset$ and a birational morphism $X \to X'$. Since X' is k-rational by Proposition 5.12 we have that also X is rational (and then unirational) over k.

If x is a point *of type 2*, without loss of generality we can assume that $\overline{x} = E' \cap E$ where E' is a (-1)-curve of *type a* and E is a (-1)-curve of *type b* or of *type c*

From Remark 4.30 and Theorem 4.40 we have that $\overline{X}$ is, up to isomorphism, a blowing-up of $\mathbb{P}^2_{\overline{k}}$, say $f : \overline{X} \to \mathbb{P}^2_{\overline{k}}$, with center five closed points in general position $P_1, P_2, P_3, P_4, P_5 \in \mathbb{P}^2_{\overline{k}}$. For $i = 1, 2, 3, 4, 5$ let E_i be the inverse image of P_i under f and L the inverse image of a line not containing any of the $P_i, i = 1, 2, 3, 4, 5$. Without loss of generality we can assume that $E' = E_1$ and then $f(\overline{x}) = P_1$.

If E is a (-1)-curve of *type b*, without loss of generality we can assume that $E = L_{1,5}$ is the strict transform under f of the line $l_{1,5}$ containing P_1 and P_5 in $\mathbb{P}^2_{\overline{k}}$. Let C be the conic in $\mathbb{P}^2_{\overline{k}}$ containing P_1, P_2, P_3, P_4 and tangent to $l_{1,5}$ in P_1. Let $\tilde{C}$ be the strict transform of C under f, then $\tilde{C} = 2L - E_1 - E_2 - E_3 - E_4$ in $\mathrm{Pic}(\overline{X})$. By Proposition 4.25 we have that $L_{1,5} = L - E_1 - E_5$ and $-K_{\overline{X}} = 3L - E_1 - E_2 - E_3 - E_4 - E_5$, so we see that $\tilde{C} = -K_{\overline{X}} - L_{1,5} - E_1$ in $\mathrm{Pic}(\overline{X})$. Since $x \in X(k)$ we have that $\{E_1, L_{1,5}\}$ is Galois invariant, since $\omega_{\overline{X}} \cong \omega_X \otimes_k \overline{k}$ we have that also $K_{\overline{X}}$ is Galois invariant, then $\mathcal{O}_{\overline{X}}(\tilde{C})$ is Galois invarian by Proposition 2.35, in particular the linear system $|\tilde{C}|$ associated to $\tilde{C}$ on $\overline{X}$ is Galois invariant.

Without loss of generality we can identify $\overline{X}$ with its image under the closed immersion $\overline{X} \to \mathbb{P}^4_{\overline{k}}$ induced by $-K_{\overline{X}}$ as in Proposition 4.16. By

Propositions 1.43 and 4.25 we have that $\deg \tilde{C} = -K_{\overline{X}}.\tilde{C} = 2$, then $\tilde{C}$ is a conic in $\mathbb{P}^4_{\overline{k}}$, irreducible as it is the strict transform under f of the irreducible conic C.

Moreover $\tilde{C}.L_{1,5} = 1$ by Proposition 4.25, then $\tilde{C}$ and $L_{1,5}$ meet in a point, since C is tangent to $l_{1,5}$ in P_1 we have that $\tilde{C}$ intersect $L_{1,5}$ in a point of E_1, which can be only $\overline{x}$ by Proposition 4.29.

Let $g \in \Gamma_k$, since $|\tilde{C}|$ is Galois invariant, we have that $g(\tilde{C})$ is linearly equivalent to $\tilde{C}$. Suppose that $g(\tilde{C}) \neq \tilde{C}$, since $g(\tilde{C}).\tilde{C} = \tilde{C}^2 = 0$, we have that $g(\tilde{C})$ and $\tilde{C}$ are disjoint by Theorem 1.36, but $\tilde{C}$ contains the point $\overline{x}$ which is defined over k, thus we get a contradiction. Then $g(\tilde{C}) = \tilde{C}$ for all $g \in \Gamma_k$, i.e. $\tilde{C}$ is Galois invariant, hence defined over k by Proposition 2.31. Let D be the curve in X such that $\overline{D} = \tilde{C}$. We have that D is an irreducible conic over k with a k-rational point, then $D \cong \mathbb{P}^1_k$ by Example 3.28.

Let η be the generic point of D. The field extension $k(\eta)/k$ is purely transcendental of degree 1, and we have a morphism $j : \mathrm{Spec}(k(\eta)) \to X$ corresponding to η, let $j \times \mathrm{Id}_{\mathrm{Spec}(\eta)} : \mathrm{Spec}(k(\eta)) \to X_{k(\eta)}$ be the induced morphism, its image y is a closed rational point in $X_{k(\eta)}(k(\eta))$ (by Proposition 2.7), whose image under the projection $X_{k(\eta)} \to X$ is η.

$X_{k(\eta)}$ is a Del Pezzo surface of degree 4 over $k(\eta)$ by Remark 4.30. Since $\tilde{C}$ is not a (-1)-curve in $\overline{X}$ and y is not a closed point defined over k we have that y is a point *of type 0* on $X_{k(\eta)}$. Then $X_{k(\eta)}$ is unirational over $k(\eta)$ by what we have proved above, i.e. there is a dominant rational map $\mathbb{P}^2_{k(\eta)} \dashrightarrow X_{k(\eta)}$, we compose it with the projection $X_{k(\eta_1)} \to X$, which is surjective, and we get a dominant rational map of k-schemes $\mathbb{P}^2_{k(\eta)} \dashrightarrow X$. Let denote by $K(X), K(\mathbb{P}^2_{k(\eta)})$ the function fields of $X, \mathbb{P}^2_{k(\eta)}$ respectively. Since the above rational map is dominant, we have a an induced morphism $\mathrm{Spec}(K(\mathbb{P}^2_{k(\eta)})) \to \mathrm{Spec}(K(X))$ defined over k, which corresponds to a morphism of k-algebras $K(X) \to K(\mathbb{P}^2_{k(\eta)})$. Since $k(\eta)/k$ is an extension purely transcendental of degree 1, we get that the function field $K(\mathbb{P}^2_{k(\eta)})$ of $\mathbb{P}^2_{k(\eta)}$ is a purely transcendental extension of degree 3 over k. Then by Theorem 4.4 in [Har], I, §4, we obtain a dominant rational map $\mathbb{P}^3_k \dashrightarrow X$ and by Proposition 1.29 we have that X is unirational over k.

If E is a (-1)-curve of *type c* of $\overline{X}$, then it is the strict transform under f of the conic C containing P_1, P_2, P_3, P_4, P_5 in $\mathbb{P}^2_{\overline{k}}$. Let l be the tangent line of C at P_1 in $\mathbb{P}^2_{\overline{k}}$ and let $\tilde{l}$ be the strict transform of L under f. Since P_1, P_2, P_3, P_4, P_5 are in general position, then C is an irreducible conic and l does not contain any of the $P_i, i = 2, 3, 4, 5$, then $\tilde{l} = L - E_1$. By Proposition 4.25 we have that $E = 2L - E_1 - E_2 - E_3 - E_4 - E_5$ and $-K_{\overline{X}} = 3L - E_1 - E_2 - E_3 - E_4 - E_5$, then $\tilde{l} = -K_{\overline{X}} - E_1 - E$ in $\mathrm{Pic}(\overline{X})$. Since $x \in X(k)$ we have that $\{E_1, E1\}$ is Galois invariant, moreover also $K_{\overline{X}}$ is Galois invariant, as $\omega_{\overline{X}} \cong \omega_x \otimes_k \overline{k}$, then $\mathcal{O}_{\overline{X}}(\tilde{l})$ is Galois invarian by Proposition 2.35, in particular the linear system $|\tilde{l}|$ associated to $\tilde{l}$ on $\overline{X}$ is Galois invariant.

Without loss of generality we can identify $\overline{X}$ with its image under the closed immersion $\overline{X} \to \mathbb{P}^4_{\overline{k}}$ induced by $-K_{\overline{X}}$ as in Proposition 4.16. By Propositions 1.43 and 4.25 we have that $\deg \tilde{l} = -K_{\overline{X}}.\tilde{l} = 2$, then $\tilde{l}$ is a conic in $\mathbb{P}^4_{\overline{k}}$, irreducible as it is the strict transform of a line under f.

Moreover $\tilde{l}.E = 1$ by Proposition 4.25, then $\tilde{l}$ and E meet in one point, since l is tangent to C in P_1 we have that $\tilde{l}$ intersects E in a point of E_1, which can be only $\overline{x}$ by Proposition 4.29. Since $\tilde{l}^2 = 0$, the same argument used in the previous case gives that $\tilde{l}$ is defined over k. Let D be the curve in X such that $\overline{D} = \tilde{l}$. We have that D is an irreducible conic over k with a k-rational point, then $D \cong \mathbb{P}^1_k$ by Example 3.28.

Then we can proceed as in the previous case to conclude that X is unirational over k. □

5.5 About degrees 1 and 2

This section gives a short presentation of the latest results for Del Pezzo surfaces of degree 1 and 2. For proofs and details we refer to the people who are working on these topics.

Concerning unirationality of Del Pezzo surfaces of degree 2, we report the newest results of Cecília Salgado (Leiden University), Damiano Testa (University of Warwick) and Anthony Várilly-Alvarado (Rice University). Their work consists in correcting and improving Manin's Theorem 29.4 in [Man].

Let X be a Del Pezzo surface of degree 2 over k. By Proposition 4.32 there is a finite morphism $\phi : X \to \mathbb{P}^2_k$ of degree 2 and ramified on a quartic curve C in $\mathbb{P}^2_k$.

Theorem 5.20. *If X contains a k-rational point P which does not lie on $\phi^{-1}(C)$, nor $\overline{P}$ lie on the intersection of four (-1)-curves of $\overline{X}$, then X is unirational over k.*

For finite fields, they are working on a lower bound on the size of k, which assures that X has a k-rational point that satisfies the hypothesis of Theorem 5.20.

Concerning Del Pezzo surfaces of degree 1, Cecília Salgado and Ronald van Luijk (Leiden University) have approached the problem of density of rational points, producing the following result.

Let X be a Del Pezzo surface of degree 1 over a number field k. By Proposition 4.32 X is a hypersurface of degree 6 in $\mathbb{P}_k(1,1,2,3)$. Let denote by w_0, w_1, x, y the coordinates on $\mathbb{P}_k(1,1,2,3)$, then X has a model in $\mathbb{P}_k(1,1,2,3)$ given by the equation $y^2 = x^3 + f(w_0,w_1)x + g(w_0,w_1)$ where $f, g \in k[w_0,w_1]$ are homogeneous polynomials of degree 4 and 6 respectively.

Let $\pi : X \dashrightarrow \mathbb{P}^1_k$ be the restriction to X of the projection $\mathbb{P}_k(1,1,2,3) \dashrightarrow \mathbb{P}^1_k$ on the first two coordinates. Then $(0:0:1:1)$ is the only point on X on which π is not defined, let $\mathcal{E} \to X$ be the monoidal transformation with center $(0:0:1:1)$ and let $\tilde{\pi} : \mathcal{E} \to \mathbb{P}^1_k$ be the induced morphism.

Theorem 5.21. *There is an explicit elliptic threefold $T \dashrightarrow \mathcal{E}$, such that if the fiber T_Q has infinitely many k-rational points for some point $Q \in \mathcal{E}(k)$ of infinite order on a smooth fiber of $\tilde{\pi}$, then $X(k)$ is Zariski dense in X.*

Corollary 5.22. *Let $\mathcal{M}$ be the moduli space of Del Pezzo surfaces of degree 1. Then the set $\{X \in \mathcal{M}(\mathbb{Q}) : X(\mathbb{Q}) \text{ is Zariski dense in } X\}$ is dense in $\mathcal{M}(\mathbb{R})$.*

The converse of Theorem 5.21 is conjectured to be true, but not known yet.

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