Algebra, Geometry and Number Theory
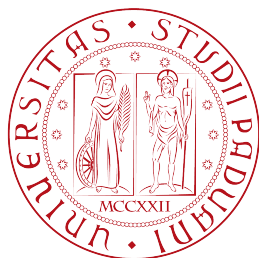
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ALGANT Erasmus Mundus

Representations
of SL$_2$ and GL$_2$
in defining characteristic

Andrea Pasquali

andrea.pasquali91@gmail.com

Advised by Dr. Lenny Taelman

Università degli Studi
di Padova

Universiteit
Leiden

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Contents

Introduction

Let p be a prime number, and let $q = p^r$ with $r \in \mathbb{N}_{\geq 1}$. We consider algebraic representations of the affine group schemes SL_2 and GL_2 of 2×2 matrices of determinant 1 (resp. invertible), defined over $\overline{\mathbb{F}}_p$. We compare them to the linear representations over $\overline{\mathbb{F}}_p$ of the finite groups $\mathrm{SL}_2(\mathbb{F}_q)$ and $\mathrm{GL}_2(\mathbb{F}_q)$.

Let Std be the standard representation of SL_2 and GL_2 acting on $\overline{\mathbb{F}}_p^2$, and let $V_j = \mathrm{Sym}^j \mathrm{Std}$ for $j \in \mathbb{N}$. For $n \in \mathbb{Z}$, let D^n be the 1-dimensional representation of GL_2 given by the n-th power of the determinant. Let F be the Frobenius endomorphism of $\overline{\mathbb{F}}_p$. This can be extended to an endomorphism F of SL_2 and GL_2 that raises all matrix entries to the p. For a representation V of SL_2 or GL_2, define the representation $V^{[i]}$ to be the same vector space where a matrix M acts as $F^i(M)$ would act on V. An algebraic representation of SL_2 (resp. GL_2) induces a representation of the finite group $\mathrm{SL}_2(\mathbb{F}_q)$ (resp. $\mathrm{GL}_2(\mathbb{F}_q)$) over $\overline{\mathbb{F}}_p$.

We use without proving it the following theorem:

Theorem ([5], §2.8). *The irreducible algebraic representations of the affine group scheme SL_2 over $\overline{\mathbb{F}}_p$ are the following (up to isomorphism):*

$$\bigotimes_{i=0}^{s} V_{j_i}^{[i]}$$

for $s \in \mathbb{N}$ and $0 \leq j_i < p$ for every i.

In this thesis we give proofs of the following results:

Theorem. *The irreducible representations of the finite group $\mathrm{SL}_2(\mathbb{F}_q)$ over $\overline{\mathbb{F}}_p$ are the following (up to isomorphism):*

$$\bigotimes_{i=0}^{r-1} V_{j_i}^{[i]}$$

for $0 \leq j_i < p$ for every i.

Theorem. *The irreducible algebraic representations of the affine group scheme GL_2 over $\overline{\mathbb{F}}_p$ are the following (up to isomorphism):*

$$D^n \otimes \bigotimes_{i=0}^{s} V_{j_i}^{[i]}$$

for $n \in \mathbb{Z}$, $s \in \mathbb{N}$ and $0 \leq j_i < p$ for every i. The irreducible representations of the finite group $\mathrm{GL}_2(\mathbb{F}_q)$ over $\overline{\mathbb{F}}_p$ are the following (up to isomorphism):

$$D^n \otimes \bigotimes_{i=0}^{r-1} V_{j_i}^{[i]}$$

for $0 \leq n < q - 1$ and $0 \leq j_i < p$ for every i.

The description of the irreducible representations of $\mathrm{SL}_2(\mathbb{F}_q)$ can be found in [1], §30. We give a more detailed proof in a more modern language. The analogous result for $\mathrm{GL}_2(\mathbb{F}_q)$ is also stated there, without proof. The description of the irreducible representations of the affine group scheme GL_2 can be deduced from the general theory in [6], but we give a direct proof instead.

It should be noted that representations over $\overline{\mathbb{F}}_p$ of both the affine group schemes SL_n and GL_n and the finite groups $\mathrm{SL}_n(\mathbb{F}_q)$ and $\mathrm{GL}_n(\mathbb{F}_q)$ are not semisimple for $n > 1$. Hence describing the irreducible representations is not enough to describe all the representations.

Notice that the above results imply that every irreducible representation of the finite groups SL_2 and GL_2 over $\overline{\overline{\mathbb{F}}}_p$ is induced by an irreducible algebraic representation of the corresponding group scheme. There is in fact a general theorem, proved by Steinberg, that states:

Theorem (Steinberg, [9]). *Let G be a reductive algebraic group defined over $\mathbb{F}_p$. Denote by G_k its base change to $k = \overline{\mathbb{F}}_p$. Then an algebraic representation of G_k induces a representation of G over k. If G_k is simply connected, then every irreducible representation of G over k is the restriction of an irreducible algebraic representation of G_k.*

We refer to [9] and [6] for the general theory leading to this theorem. The proof relies on the classification by dominant weights of the irreducuble representations of G_k. The affine group scheme SL_n is simply connected for all n, while GL_n is not. We showed that the conclusion still holds for GL_2, but we were not able to find a proof or a counterexample for any GL_n with $n \geq 3$.

We also prove the following result:

Theorem. *Let $\rho : \mathrm{SL}_n \to \mathrm{GL}(V)$ be an irreducible algebraic representation of the affine group scheme SL_n over an algebraically closed field. Let $i : \mathrm{SL}_n \to \mathrm{GL}_n$ be the canonical map. Then there is an irreducible algebraic representation $\tilde{\rho} : \mathrm{GL}_n \to \mathrm{GL}(V)$ such that the diagram*

$$\begin{array}{ccc} \mathrm{SL}_n & \xrightarrow{\ \rho\ } & \mathrm{GL}(V) \\ {\scriptstyle i}\downarrow & \nearrow{\scriptstyle \tilde{\rho}} & \\ \mathrm{GL}_n & & \end{array}$$

commutes.

This gives a tool to pass from representations of SL_n to representations of GL_n for any n. However, it does not seem to be enough to deduce an analog of Steinberg's theorem for GL_n.

1 Background

1.1 Notation and conventions

The set $\mathbb{N}$ consists of the non-negative integers. The category of sets is denoted by **Sets**. Algebras over a field are assumed to be commutative. Given a field k, the category of k-algebras is denoted by $k - \mathbf{Alg}$. The category of (abstract) groups is denoted by **Grp**, the category of (abstract) abelian groups is denoted by **Ab**, and the category of (abstract) finite groups is denoted by **FGrp**. Given a field k and a group G, we denote by $k[G]$ the group algebra of G. The category of affine schemes over k is denoted by **AffSch**$/k$. The symmetric group acting on the set $\{1,\ldots,n\}$ is denoted by S_n. If $\varphi : G \to \operatorname{Aut} S$ is an action of a group G on a set S, we denote the element $\varphi(g)(s) \in S$ by $g.s$, for any $g \in G$ and $s \in S$. Given two topological spaces X and Y, we denote by $\operatorname{Cont}(X,Y)$ the set of continuous maps $X \to Y$. Given a ring R, we denote by $\operatorname{Mat}_m(R)$ the ring of $m \times m$ matrices with entries in R. Given a ring R and two elements $a, b \in R$, we denote by $[a,b]$ the element $ab - ba \in R$. Throughout this thesis, p denotes a prime number.

1.2 Affine group schemes

For sections 1.2 to 1.8 we refer to [6], [10] and [7].

Definition 1.1. Let k be a field. An *affine group scheme over k G* is a representable functor $G : k - \mathbf{Alg} \to \mathbf{Grp}$, *i.e.* G is naturally isomorphic to $\operatorname{Hom}(A, -)$ for some k-algebra A. A morphism of affine group schemes is a natural transformation. The category of affine group schemes over k will be denoted by **AffGrSch**$/k$.

Equivalently, an affine group scheme can be defined as a representable functor $G : (\mathbf{AffSch}/k)^{op} \to \mathbf{Grp}$. Notice that if G is represented by A, then A is unique up to unique isomorphism by Yoneda's lemma. Giving a group scheme structure on a scheme is giving compatible group structures on its R-points for every k-algebra R.

Example 1.1. 1. The *additive group* $\mathbb{G}_{a,k}$ is the functor assigning to a k-algebra R its additive group $(R, +)$. It is represented by $k[X]$, since giving a morphism $k[X] \to R$ is the same as giving an element $x \in R$.

2. The *multiplicative group* $\mathbb{G}_{m,k}$ is the functor assigning to a k-algebra R its group of units, *i.e.* $\mathbb{G}_{m,k}(R) = (R^\times, \cdot)$. It is represented by $k[X, X^{-1}]$ (which is just alternative notation for what should more precisely be written $k[X, Y]/(XY - 1)$). Indeed, if we have a morphism $\varphi : k[X, X^{-1}] \to R$, this identifies an element $x = \varphi(X) \in R$, and we have that $x \in R^\times$ because X is invertible in $k[X, X^{-1}]$. Conversely,

such a φ is determined by $x = \varphi(X)$. This group scheme will be denoted $\mathbb{G}_\mathrm{m}$ if no confusion is likely.

3. The *general linear group* $\mathrm{GL}_{n,k}$ is the functor assigning to a k-algebra R the multiplicative group of invertible $n \times n$ matrices with entries in R. To give such a matrix is the same as to give n^2 elements of R, with the condition that the determinant must be invertible. It is then easy to check that the representing algebra is

$$k[X_{ij}, Y]_{i,j=1,\ldots,n}/(Y \det(X_{ij}) - 1)$$

where $\det(X_{ij})$ is the determinant formula in the variables X_{ij}. We will denote this algebra by $k[X_{ij}, \det^{-1}]$, and the group scheme by GL_n. Notice that $\mathrm{GL}_1 = \mathbb{G}_\mathrm{m}$ (they really have the same definition).

4. The *special linear group* $\mathrm{SL}_{n,k}$ is the functor assigning to a k-algebra R the group of $n \times n$ matrices with entries in R and determinant equal to 1. It should now be clear that it is represented by the algebra

$$k[X_{ij}]_{i,j=1,\ldots,n}/(\det(X_{ij}) - 1).$$

This group scheme will usually be denoted SL_n.

5. The *group of n-th roots of unity* $\mu_{n,k}$ is the functor assigning to a k-algebra R the multiplicative group $(\{x \in R \mid x^n = 1\}, \cdot)$. It is represented by the algebra $k[X]/(X^n - 1)$, and we will denote it by μ_n.

1.3 Morphisms and constructions

Recall that a morphism of affine group schemes is a natural transformation of functors $k - \mathbf{Alg} \to \mathbf{Grp}$.

Definition 1.2. A *closed immersion* $H \to G$ is an affine group scheme morphism such that the corresponding algebra map is surjective. In this case H is a *closed subgroup* of G. We will denote closed immersions by $H \rightarrowtail G$.

Notice that in this case H is represented by a quotient of A. Notice also that the composition of two closed immersions is again a closed immersion.

Example 1.2. 1. For every $n \in \mathbb{N}$, there is a closed immersion $z : \mu_n \rightarrowtail \mathbb{G}_\mathrm{m}$, which is defined as follows. Let R be a k-algebra, we can define on R-points the map

$$z(R) : \mu_n(R) \to \mathbb{G}_\mathrm{m}(R)$$

$$\zeta \mapsto \zeta$$

If $\alpha : R \to S$ is a k-algebra map, then the diagram

$$
\begin{array}{ccc}
\mu_n(R) & \xrightarrow{z(R)} & \mathbb{G}_{\mathrm{m}}(R) \\
{\scriptstyle \mu_n(\alpha)} \downarrow & & \downarrow {\scriptstyle \mathbb{G}_{\mathrm{m}}(\alpha)} \\
\mu_n(S) & \xrightarrow[z(S)]{} & \mathbb{G}_{\mathrm{m}}(S)
\end{array}
$$

is clearly commutative, hence z is a morphism of affine group schemes. The corresponding algebra map is

$$k[X, X^{-1}] \to k[X]/(X^n - 1)$$

$$X \mapsto X$$

which is a surjection.

2. For every $n \in \mathbb{N}$, there is a closed immersion $\mathbb{G}_{\mathrm{m}} \rightarrowtail \mathrm{GL}_n$, given on R-points by

$$\mathbb{G}_{\mathrm{m}}(R) \to \mathrm{GL}_n(R)$$

$$g \mapsto \begin{pmatrix} g & & \\ & \ddots & \\ & & g \end{pmatrix}$$

This corresponds to the algebra map

$$k[X_{ij}, \det^{-1}] \to k[X, X^{-1}]$$

$$X_{ij} \mapsto \delta_{ij} X$$

which is surjective.

3. For every $n \in \mathbb{N}$, there is a closed immersion $\mathrm{SL}_n \rightarrowtail \mathrm{GL}_n$, given on R-points by

$$\mathrm{SL}_n(R) \to \mathrm{GL}_n(R)$$

$$M \mapsto M$$

which corresponds to the surjective algebra map

$$k[X_{ij}, \det^{-1}] \to k[X_{ij}]/(\det(X_{ij}) - 1)$$

$$X_{ij} \mapsto X_{ij}.$$

4. For every $n \in \mathbb{N}$, there is a closed immersion $\mu_n \rightarrowtail \mathrm{SL}_n$, given on R-points by

$$\mu_n(R) \to \mathrm{SL}_n(R)$$

$$\zeta \mapsto \begin{pmatrix} \zeta & & \\ & \ddots & \\ & & \zeta \end{pmatrix}.$$

It corresponds to the algebra map

$$k[X_{ij}]/(\det(X_{ij}) - 1) \to k[X]/(X^n - 1)$$

$$X_{ij} \mapsto \delta_{ij} X$$

which is surjective.

Definition 1.3. An affine group scheme represented by A is called *of finite type* if A is a finitely generated k-algebra. A *linear algebraic group over k* is an affine group scheme G such that its representing algebra is reduced and there exists a closed immersion $G \rightarrowtail \mathrm{GL}_{n,k}$ for some $n \in \mathbb{N}$.

Affine group schemes of finite type always admit a closed immersion into some GL_n, so they are linear algebraic groups under our definition (see [10], §3.4). Conversely, it is obvious that if G admits a closed immersion in GL_n then it is of finite type.

Consider now a morphism $\Phi : G \to H$ of affine group schemes. We can define an affine group scheme $\ker \Phi$ in the natural way

$$(\ker \Phi)(R) = \ker(\Phi(R))$$

for every k-algebra R. This turns out to be a representable functor hence an affine group scheme, and it is true that monomorphisms in **AffGrSch**/k are the morphisms that have trivial kernel. Epimorphisms and surjective maps are more complicated, and we refer to [7], §VII.

Let k' be a k-algebra. Then every k'-algebra is in a natural way a k-algebra, which allows us to define base changes.

Definition 1.4. Let G be an affine group scheme over k. We define its *base change* to k' to be the affine group scheme

$$G_{k'} : k' - \mathbf{Alg} \to \mathbf{Grp}$$

$$R \mapsto G(R).$$

If G is represented by the k-algebra A, then $G_{k'}$ is represented by the k'-algebra $A \otimes_k k'$.

1.4 Diagonalisable group schemes

Let M be an abelian group, and let R be a k-algebra. There is a canonical bijection

$$\mathrm{Hom}_{k-\mathbf{Alg}}(k[M], R) \cong \mathrm{Hom}_{\mathbf{Grp}}(M, R^\times)$$

hence the functor $k - \mathbf{Alg} \to \mathbf{Grp}$ sending R to $\mathrm{Hom}(M, R^\times)$ is an affine group scheme represented by $k[M]$.

Definition 1.5. A group scheme G represented by $k[M]$ for an abelian group M is called *diagonalisable*. Diagonalisable group schemes form a full subcategory of affine group schemes, which will be denoted by **DiagGrSch**$/k$.

Example 1.3. Suppose $M = \mathbb{Z}$. Then $k[M]$ has basis $\{e_n \mid n \in \mathbb{Z}\}$, with $e_n \cdot e_m = e_{n+m}$. So $k[M]$ is isomorphic as a k-algebra to $k[X, X^{-1}]$ by $e_1 \mapsto X$, hence this algebra represents the affine group scheme $\mathbb{G}_\mathrm{m}$. The group scheme $\mathbb{G}_\mathrm{m}$ is then diagonalisable, corresponding to the abelian group $\mathbb{Z}$.

Suppose now $M = \mathbb{Z}/n\mathbb{Z}$. Then $k[M]$ has basis $\{e_0, \ldots, e_{n-1}\}$ with $e_i = e_1^i$ for $i = 0, \ldots, n-1$. So $k[M]$ is isomorphic as a k-algebra to $k[X]/(X^n - 1)$. Hence the group scheme μ_n is also diagonalisable for all $n \in \mathbb{N}$, and it corresponds to the abelian group $\mathbb{Z}/n\mathbb{Z}$.

Theorem 1.1. *Let k be a field. The functors* $\mathrm{Hom}(-, \mathbb{G}_\mathrm{m}) : \textbf{DiagGrSch}/k \to \textbf{Ab}$ *and* $F : \textbf{Ab} \to \textbf{DiagGrSch}/k$ *defined by*

$$\mathrm{Hom}(-, \mathbb{G}_\mathrm{m}) : G \mapsto \mathrm{Hom}(G, \mathbb{G}_\mathrm{m})$$

and

$$F : M \mapsto \mathrm{Hom}(k[M], -)$$

are quasi-inverses of one another, so they define an equivalence of categories.

Proof. Omitted, see [10], §2.2. $\qquad\square$

Corollary 1.2. *The maps defined by* $N \mapsto (x \mapsto x^N)$ *give isomorphisms* $\mathrm{Hom}(\mathbb{G}_\mathrm{m}, \mathbb{G}_\mathrm{m}) \cong \mathbb{Z}$ *and* $\mathrm{Hom}(\mu_n, \mathbb{G}_\mathrm{m}) \cong \mathbb{Z}/n\mathbb{Z}$.

Proof. We have seen in Example 1.3 that $\mathbb{G}_\mathrm{m}$ and μ_n are diagonalisable corresponding to $\mathbb{Z}$ and $\mathbb{Z}/n\mathbb{Z}$ respectively. Using Theorem 1.1 it follows that $\mathrm{Hom}(\mathbb{G}_\mathrm{m}, \mathbb{G}_\mathrm{m}) \cong \mathbb{Z}$ and $\mathrm{Hom}(\mu_n, \mathbb{G}_\mathrm{m}) \cong \mathbb{Z}/n\mathbb{Z}$, and the fact that the maps giving the isomorphisms are $N \mapsto (x \mapsto x^N)$ can be checked by the explicit constructions of the examples and of the equivalence of categories. $\qquad\square$

1.5 Constant group schemes

Let Γ be a finite group. Define a functor $k - \textbf{Alg} \to \textbf{Grp}$ by

$$R \mapsto \mathrm{Cont}(\mathrm{Spec}\, R, \Gamma)$$

where we put on Γ the discrete topology. This is an affine group scheme represented by the algebra k^Γ, because as a topological space Γ is the same as $\coprod_\Gamma \mathrm{Spec}\, k = \mathrm{Spec}\, k^\Gamma$.

Definition 1.6. Let Γ be a finite group. We will call the affine group scheme $\mathrm{Hom}(k^\Gamma, -)$ the *constant group scheme* Γ_k, and we will denote it by Γ if no confusion is likely.

Notice that the constant group scheme construction defines in fact a functor $(-)_k : \mathbf{FGrp} \to \mathbf{AffGrSch}/k$. We have two basic properties of constant group schemes:

Lemma 1.3. *Let Γ be a finite group, and let X be a connected affine scheme over k. Then $\mathrm{Hom}_{\mathbf{AffSch}/k}(X, \Gamma_k) = \Gamma$.*

Proof. Notice that

$$\mathrm{Hom}_{\mathbf{AffSch}/k}(X, \Gamma_k) = \mathrm{Hom}_{\mathbf{AffSch}/k}(X, \coprod_\Gamma \mathrm{Spec}\, k).$$

Then the conclusion follows from the fact that X is connected. $\square$

Lemma 1.4. *Let Γ be a finite group, and let H be an affine group scheme over k. Then the map given by taking k-points*

$$\alpha : \mathrm{Hom}_{\mathbf{AffGrSch}/k}(\Gamma_k, H) \to \mathrm{Hom}_{\mathbf{Grp}}(\Gamma, H(k))$$

is a bijection.

Proof. Let us define the inverse map $\beta : \mathrm{Hom}(\Gamma, H(k)) \to \mathrm{Hom}(\Gamma_k, H)$ of α. Take $\varphi \in \mathrm{Hom}(\Gamma, H(k))$. For a k-algebra R, we have that $\Gamma_k(R)$ is isomorphic to a sum of copies of Γ indexed by the connected components of $\mathrm{Spec}\, R$ (this is just a slight generalisation of Lemma 1.4, the proof is similar). Recall that affine schemes, being quasi-compact, have a finite number of connected components. If $H \cong \mathrm{Hom}(A, -)$, then write $R = \bigoplus e_i R$ as an R-module, where the e_i's are orthogonal idempotents corresponding to the n connected components of $\mathrm{Spec}\, R$, and define the map $\beta(\varphi)(R) : \Gamma_k(R) \to H(R)$ by

$$\beta(\varphi)(R) : (g_1, \ldots, g_n) \mapsto \sum_{i=1}^n \varphi(g_i) e_i.$$

Every $\varphi(g_i)$ is a map from A to k, so this defines a map from A to R, *i.e.* an element of $H(R)$. This defines a map of affines group schemes $\beta(\varphi)$, and it is easy to check that β is the inverse of α. $\square$

Example 1.4. Consider the affine group scheme μ_n over an algebraically closed field k, with $n \neq 0$ in k. Then we have a (non canonical) isomorphism $\mu_n \cong (\mathbb{Z}/n\mathbb{Z})_k$. To prove it, let us first fix an isomorphism $\mu_n(k) \cong \mathbb{Z}/n\mathbb{Z}$. Let us define a map

$$\varphi(R) : \mu_n(R) \to (\mathbb{Z}/n\mathbb{Z})_k (R) = \mathrm{Cont}(\mathrm{Spec}\, R, \mathbb{Z}/n\mathbb{Z})$$

for every k-algebra R. To do so, fix an $x \in R$ such that $x^n = 1$. For a prime ideal $\mathfrak{p} \in \mathrm{Spec}\, R$, consider the image of x in $k' = \mathrm{Frac}(R/\mathfrak{p})$. Notice that since $Rx = R$, we have that $x \notin \mathfrak{p}$ hence $x \neq 0$ in k'. Then $x^n = 1$ in k'. The

field k' being an algebraic extension of k, it follows that $x \in \mu_n(k)$, and we define $\varphi(R)(x)(\mathfrak{p})$ to be the corresponding element in $\mathbb{Z}/n\mathbb{Z} \cong \mu_n(k)$. The map $\operatorname{Spec} R \to \mathbb{Z}/n\mathbb{Z}$ sending $\mathfrak{p}$ to $\varphi(R)(x)(\mathfrak{p})$ is continuous (write $\operatorname{Spec} R$ as the disjoint union of its connected components, then this map is easily locally constant). This defines an affine group scheme morphism

$$\varphi : \mu_n \to (\mathbb{Z}/n\mathbb{Z})_k$$

so now to conclude it is enough to prove that the algebras $k[X]/(X^n - 1)$ and $k^{\mathbb{Z}/n\mathbb{Z}}$ are isomorphic. Indeed, since n is invertible in k, the equation $X^n = 1$ has n distinct solutions in k, so that if we choose a primitive n-th root of unity ζ, we have $X^n - 1 = \prod_{m=0}^{n-1}(X - \zeta^m)$. Now define the map $k[X]/(X^n - 1) \to \prod_{m=0}^{n-1} k \cong k^{\mathbb{Z}/n\mathbb{Z}}$ by $X \mapsto (\zeta^m)_{m=0,\dots,n-1}$. By the Chinese Remainder Theorem this is an isomorphism and we are done.

Notice that $\mu_n \not\cong (\mathbb{Z}/n\mathbb{Z})_k$ if $n = 0$ in k. For instance, if $\operatorname{char} k = p > 0$, then the affine group scheme μ_p is connected, so if it were constant it would be the group with one element. Notice also that in this case μ_p is not reduced (it consists of a single point "of multiplicity p").

1.6   Representations

We assume that the reader knows the basic definitions and results about linear representations of finite groups, at the level of [3], §1. Throughout this thesis, all representations of groups are understood to be finite-dimensional.

Consider a finite-dimensional vector space V over a field k. The functor mapping a k-algebra R to the group $\operatorname{Aut}_R(R \otimes_k V)$ is representable. We will call $\operatorname{GL}(V)$ the corresponding affine group scheme. The choice of a basis of V induces an isomorphism $\operatorname{GL}(V) \cong \operatorname{GL}_{n,k}$, where $n = \dim V$. The maps $R^\times \to \operatorname{Aut}_R(R \otimes V)$ given by

$$r \mapsto (v \mapsto r \cdot v)$$

for any k-algebra R, for any $r \in R$ and $v \in V$ define a closed immersion $\mathbb{G}_\mathrm{m} \rightarrowtail \operatorname{GL}(V)$.

Definition 1.7. A (algebraic) *representation* of an affine group scheme G defined over a field k is a pair (V, ρ), where V is a finite dimensional k-vector space and ρ is a group scheme morphism $\rho : G \to \operatorname{GL}(V)$.

By abuse of notation, we will also refer to a representation (V, ρ) simply by V or ρ. A *subrepresentation* of V is a vector subspace $U \subset V$ such that $R \otimes U$ is closed under the action of $G(R)$ for every k-algebra R. A representation is called *irreducible* if it has exactly two subrepresentations, namely itself and the zero representation.

If $(V, \rho), (U, \pi)$ are representations of an affine group scheme G over a field k, we define their *tensor product* (over k) to be the representation

$(V \otimes_k U, \rho \otimes_k \pi)$, where the map $\rho \otimes_k \pi$ is defined by

$$(\rho \otimes_k \pi)(R) : g \mapsto (v \otimes u \mapsto \rho(R)(g)(v) \otimes \pi(R)(g)(u))$$

for all pure tensors $v \otimes u$ of $V \otimes_k U$, for every k-algebra R. Tensoring is exact in both arguments. We will write $V^{\otimes j}$ for the tensor product of j copies of a representation V. Given a representation V of an affine group scheme and a natural number $j \in \mathbb{N}$, we define its *j-th symmetric power* to be the representation

$$\mathrm{Sym}^j V = V^{\otimes j} / \langle \{v_1 \otimes \cdots \otimes v_j - v_{\sigma 1} \otimes \cdots \otimes v_{\sigma j} \mid v_1, \ldots, v_j \in V, \ \sigma \in S_j\} \rangle.$$

If V has basis $\{e_1, \ldots, e_n\}$, then the map

$$\mathrm{Sym}^j V \to k[X_1, \ldots, X_n]_j$$

given by

$$\overline{e}_{i_1} \otimes \cdots \otimes \overline{e}_{i_j} \mapsto X_{i_1} \cdots X_{i_j}$$

is an isomorphism.

Example 1.5. Let $G = \mathrm{GL}_{n,k}$ for some field k, and fix an integer $j \in \mathbb{Z}$. Let $D^j = k$ as a k-vector space. We can define a 1-dimensional representation $\eta_j : G \to \mathrm{GL}(D^j)$ by setting, for every k-algebra R, and for every $g \in G(R)$,

$$\eta_j(R)(g) = (\det g)^j.$$

This is a representation for every $j \in \mathbb{Z}$ (it is the trivial one for $j = 0$). Notice that if $h : \mathrm{SL}_{n,k} \to \mathrm{GL}_{n,k}$ is the canonical immersion, we have that $\eta_j h$ is the trivial representation of $\mathrm{SL}_{n,k}$ for every $j \in \mathbb{Z}$. Notice also that for every $j, l \in \mathbb{Z}$ we have

$$D^j \otimes D^l \cong D^{j+l}.$$

1.7   Jordan-Hölder decomposition

In the following sections we will need some tools to handle representations that are not irreducible but cannot be written as a direct sum of irreducibles. One such tool is Jordan-Hölder theory, of which we will state the essential results only for the case in which we are interested.

Definition 1.8. Let G be a finite group, k a field, and V be a representation of G. A *composition series* of V is a finite descending chain of subrepresentations

$$V = V_1 \supset V_2 \supset \cdots \supset V_n \supset V_{n+1} = 0$$

such that all the quotients V_i/V_{i+1} are irreducible, for $i = 1, \ldots, n$. The irreducible representations V_i/V_{i+1} are called the *factors* of the series. If V and W are k-representations of a group G, $(V_i)_i$ is a composition series for V and $(W_j)_j$ is a composition series for W, then $(V_i)_i$ and $(W_j)_j$ are called *equivalent* if they have the same factors, counted with multiplicities, up to isomorphism.

The main result is the following:

Theorem 1.5. *Let G be a finite group, k a field, and V a representation of G. Then a composition series of V exists, and any two such series are equivalent.*

Proof. Omitted, see [2], §13. $\qquad\square$

In particular, the factors of a composition series of V are well defined up to isomorphism, and they are called the *composition factors* of V.

Definition 1.9. Let V be a representation of a finite group G. A *subquotient* of V is a representation of G that is isomorphic to the quotient of two subrepresentations of V.

Every irreducible subquotient of a representation V is isomorphic to a composition factor of V as it follows from this lemma:

Lemma 1.6. *Let G be a finite group, k a field, and let V, W, U be representations of G over k. If there is an exact sequence*

$$0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0$$

then V and $U \oplus W$ have equivalent composition series.

Proof. Omitted. $\qquad\square$

2   Motivating result

In this section we start focusing our attention to the case in which our affine group schemes are defined over a field k of positive characteristic p. In this case we can relate the representation theory of an affine group scheme to the representation theory over k of a class of finite groups. In the case we are considering we can use the relationship with the affine group scheme to deduce information about the finite groups. Let us start by recalling a basic result in representation theory:

Theorem 2.1 (Maschke)**.** *Let G be a finite group and let k be a field. Let $\rho : G \to \operatorname{Aut} V$ be a representation of G, with V a k-vector space. Let $U \subset V$ be a subrepresentation. If*

$$\operatorname{char} k \nmid |G|$$

then there exists a subrepresentation $W \subset V$ such that $V = U \oplus W$.

Proof. Omitted, see [3], Proposition 1.5. □

Notice that this immediately implies that every representation of G can be written as a direct sum of irreducible representations. The proof of Maschke's Theorem heavily relies on being able to divide by the order of G, and in fact the conclusion does not hold if $|G| = 0$ in k.

Example 2.1. Let $G = \mathbb{Z}/p\mathbb{Z}$, and let $k = \overline{\mathbb{F}}_p$. Let $V = k^2$ as a vector space over k, and define ρ by

$$\rho : 1 \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

It is easy to check that (V, ρ) is a representation of G. The vector space V has the invariant subspace $U \subset V$ generated by the first basis vector in the basis we have chosen. However, suppose that U has a complement W that is a subrepresentation. Then $\dim W = 1$, and there is a basis for which all the elements in the image of ρ are diagonal matrices. But $\rho(1)$ is not diagonalisable, since its Jordan normal form is not diagonal, contradiction.

2.1   The Frobenius endomorphism

Let $k = \overline{\mathbb{F}}_p$, and let $q = p^r$ for $r \in \mathbb{N}_{\geq 1}$. Let us call F the Frobenius endomorphism of k, given by $F : x \mapsto x^p$ for every $x \in k$. We define the finite field $\mathbb{F}_q$ as a subfield of k by

$$\mathbb{F}_q = \{x \in k \mid F^r(x) = x\}.$$

Let $G \cong \operatorname{Hom}(A, -)$ be an affine group scheme of finite type defined over $\mathbb{F}_p$. Then $G(\mathbb{F}_q) \cong \operatorname{Hom}(A, \mathbb{F}_q)$ is finite since A is finitely generated. Consider now the base change G_k of G to k. Notice that the inclusion $\mathbb{F}_q \to k$

induces an inclusion $G(\mathbb{F}_q) \to G_k(k)$. Let V be a k-vector space, and let $\rho : G_k \to \mathrm{GL}(V)$ be a representation of G_k. Then ρ induces a representation $\rho_q : G(\mathbb{F}_q) \to \mathrm{Aut}\, V$ of the finite group $G(\mathbb{F}_q)$ over k, by means of

$$\rho_q = \rho(k)|_{G(\mathbb{F}_q)}.$$

We will abuse the notation, and sometimes write ρ instead of ρ_q.

If G is an affine group scheme of finite type defined over $\mathbb{F}_p$, then F defines an endomorphism $G \to G$ (seeing G as a closed subgroup of GL_n, this is raising matrix entries to the p).

Definition 2.1. Let G be an affine group scheme of finite type over $\mathbb{F}_p$. Let G_k be the base change of G to k. Let V be a k-vector space, and let $\rho : G_k \to \mathrm{GL}(V)$ be a representation of G_k. For every $r \in \mathbb{N}$, let $V^{[r]} = V$ as a vector space, and we define the *r-th twisted representation*

$$\rho^{[r]} : G_k \to \mathrm{GL}(V^{[r]})$$

of ρ by

$$\rho^{[r]} = \rho \circ F^r.$$

If $G_k \rightarrowtail \mathrm{GL}_{n,k}$, this means that for a k-algebra R, if

$$g = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \in G_k(R)$$

then

$$F^r(R)(g) = \begin{pmatrix} a_{11}^{p^r} & \dots & a_{1n}^{p^r} \\ \vdots & \ddots & \vdots \\ a_{n1}^{p^r} & \dots & a_{nn}^{p^r} \end{pmatrix}$$

and

$$\rho^{[r]}(R)(g) = \rho(R)(F^r(R)(g)).$$

We will apply these constructions to the group schemes SL_n and GL_n, which can be defined over $\mathbb{F}_p$. We will usually write $\mathrm{SL}_n(\mathbb{F}_q)$ (resp. $\mathrm{GL}_n(\mathbb{F}_q)$) instead of $\mathrm{SL}_{n,\mathbb{F}_p}(\mathbb{F}_q)$ (resp. $\mathrm{GL}_{n,\mathbb{F}_p}(\mathbb{F}_q)$), and SL_n (resp. GL_n) instead of $\mathrm{SL}_{n,k}$ (resp. $\mathrm{GL}_{n,k}$).

Notice that $|\mathrm{SL}_n(\mathbb{F}_q)|$ and $|\mathrm{GL}_n(\mathbb{F}_q)|$ are both divisible by p for every $n > 1$. In particular, Maschke's Theorem does not apply to their representations in characteristic p.

2.2 Steinberg's Theorem

The classification of the irreducible algebraic representations of SL_n and GL_n in characteristic p is known in terms of highest weights thanks to Chevalley, for more on this see for instance [6], §II.2. For the group scheme SL_2, highest weights are in bijection with natural numbers, and the description of the irreducible representations can be made very explicit.

We will now present a particular case of the results in [9], which relates the representations of the group scheme G to those of the finite groups $G(\mathbb{F}_q)$. We will only state it for $G = SL_{n,\mathbb{F}_p}$, but it holds for any simply connected reductive group G defined over $\mathbb{F}_p$. For the definitions of these terms, and the proof of the theorem, see [9] or [6].

Theorem 2.2 (Steinberg, [9])**.** *Let $\rho : SL_n \to GL(V)$ be an irreducible representation over $\overline{\mathbb{F}}_p$ of the affine group scheme SL_n. Let $q = p^r$ for some $r \in \mathbb{N}_{\geq 1}$. Then the representation ρ_q of the finite group $SL_n(\mathbb{F}_q)$ over $\overline{\mathbb{F}}_p$ is irreducible. Moreover, every irreducible representation of $SL_n(\mathbb{F}_q)$ over $\overline{\mathbb{F}}_p$ is isomorphic to ρ_q for some irreducible representation ρ of the affine group scheme SL_n.*

Proof. See [9] or [6]. $\qquad\square$

The motivation of this thesis was to investigate whether this theorem holds in more generality, or to find examples where it fails if we make weaker assumptions. We have first restricted our attention to the groups SL_n and GL_n, and we were able to find an answer in the case $n = 2$ (the case $n = 1$ is trivial). The main reason behind this choice is that while SL_n and GL_n are very closely related, the affine group scheme GL_n does not satisfy the hypotheses of the general statement of Theorem 2.2 (it is not simply connected). So the main question is: does a similar result hold for $G = GL_n$? In other words, does every irreducible representation of $GL_n(\mathbb{F}_q)$ over $\overline{\mathbb{F}}_p$ arise as the restriction of an irreducible representation of the affine group scheme GL_n? We were not able to find an answer nor a counterexample for any $n \geq 3$. However, there is some that can be said.

2.3 Extension of representations from SL_n to GL_n

A first important result that shows how the group schemes SL_n and GL_n are related is the following:

Theorem 2.3. *Let $SL_n = SL_{n,k}$ and $GL_n = GL_{n,k}$, with k an algebraically closed field, and let $i : SL_n \rightarrowtail GL_n$ be the canonical map. Let $\rho : SL_n \to GL(V)$ be an irreducible algebraic representation over k. Then there exists an*

irreducible algebraic representation $\tilde{\rho} : \mathrm{GL}_n \to \mathrm{GL}(V)$ *such that the diagram*

$$
\begin{array}{ccc}
\mathrm{SL}_n & \xrightarrow{\ \rho\ } & \mathrm{GL}(V) \\
{\scriptstyle i}\big\downarrow & \nearrow{\scriptstyle \tilde{\rho}} & \\
\mathrm{GL}_n & &
\end{array}
$$

commutes.

In other words, given an irreducible representation of SL_n, we can extend it to an irreducible representation of GL_n. Notice that Theorem 2.3 is true in any characteristic.

To prove this theorem, we will need a lemma.

Lemma 2.4. *Let* $n' \in \mathbb{N}$ *be a divisor of* n, *and let* k *be an algebraically closed field. Denote by* j *the natural map* $j : \mu_{n'} \rightarrowtail \mathrm{SL}_n$, *and by* h *the natural map* $h : \mathbb{G}_m \rightarrowtail \mathrm{GL}(V)$. *Let* $\rho : \mathrm{SL}_n \to \mathrm{GL}(V)$ *be an irreducible algebraic representation over* k. *Then there exists a map* $\tau : \mu_{n'} \to \mathbb{G}_m$ *such that the diagram*

$$
\begin{array}{ccc}
\mu_{n'} & \xrightarrowtail{\ j\ } & \mathrm{SL}_n \\
{\scriptstyle \tau}\big\downarrow & & \big\downarrow{\scriptstyle \rho} \\
\mathbb{G}_m & \xrightarrowtail[\ h\ ]{} & \mathrm{GL}(V)
\end{array}
$$

commutes.

Proof. This is the key lemma in the proof of Theorem 2.3, and it requires different approaches for different values of n'.

Step 1. Suppose $n' \neq 0$ in k. In this case, by Example 1.4 there is a (non canonical) isomorphism of group schemes $\mathbb{Z}/n'\mathbb{Z} \cong \mu_{n'}$.

Consider now k-points. We know that $\mu_{n'}(k) = <\zeta_{n'}>$ is a cyclic group of order n'. Hence the endomorphism $\rho j(k)(\zeta_{n'})$ of V is diagonalisable since $n \neq 0$. Fix a basis such that $V \cong k^m$ and

$$
\rho j(k)(\zeta_{n'}) = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_{n'} \end{pmatrix}
$$

with respect to that basis, where $\lambda_i^{n'} = 1$ for all i's. Consider the eigenspace V_{λ_1} relative to λ_1. Then we claim that V_{λ_1} is stable under the action of SL_n. Indeed, let R be a k-algebra. We have that $\mu_{n'}(R) \subset \mu_n(R) \subset \mathrm{SL}_n(R)$ is in the center of the group, hence $R \otimes V_{\lambda_1}$ is closed under the action of $\mathrm{SL}_n(R)$. Then V_{λ_1} gives a nonzero subrepresentation of ρ, and since ρ is irreducible we conclude that $V_{\lambda_1} = V$, hence $\lambda_i = \lambda_1$ for all i's and $\rho j(k)(\zeta_{n'})$ is scalar.

So we have a commutative diagram

$$
\begin{array}{ccc}
\mu_{n'}(k) & \xrightarrow{\;j(k)\;} & \mathrm{SL}_n(k) \\
{\scriptstyle \tau(k)}\downarrow & & \downarrow{\scriptstyle \rho(k)} \\
\mathbb{G}_{\mathrm{m}}(k) & \xrightarrow[\;h(k)\;]{} & \mathrm{GL}_m(k).
\end{array}
$$

By Lemma 1.4, since $\mu_{n'}$ is a constant group scheme, we have for any affine group scheme H the equality

$$
\mathrm{Hom}_{\mathbf{GrSch}/\mathbf{k}}(\mu_{n'}, H) = \mathrm{Hom}_{\mathbf{Grp}}(\mu_{n'}(k), H(k))
$$

and by applying this to the whole diagram the conclusion follows.

Step 2. Suppose now char $k = p > 0$, and $n' = p^r$. In this case $\mu_{n'} = \mathrm{Spec}\, S$ as a scheme, with $S \cong k[X]/(X^{p^r} - 1) \cong k[\delta]/\delta^{p^r}$ (the isomorphism is $X \mapsto 1 + \delta$) (see also Example 1.1). The group $\mu_{n'}(S) = \mathrm{Hom}_k(S, S)$ has a canonical element, the identity. Under the identification $\mathrm{Hom}_k(S, S) = \mu_{n'}(S) = \left\{ x \in S \mid x^{p^r} = 1 \right\}$ given by $\psi \mapsto \psi(X)$, the identity corresponds to $1 + \delta$. Notice that in the diagram

$$
\begin{array}{ccc}
\mathrm{Spec}\, S \xrightarrow{\;\mathrm{id}\;} \mu_{n'} & \xrightarrow{\;j\;} & \mathrm{SL}_n \\
& & \downarrow{\scriptstyle \rho} \\
\mathbb{G}_{\mathrm{m}} & \xrightarrow[\;h\;]{} & \mathrm{GL}(V)
\end{array}
$$

the existences of the two dotted arrows are equivalent, hence it is enough to verify the property for the S-point $1+\delta$, *i.e.* we need to prove that $\rho j(S)(1+\delta)$ is a scalar in $\mathrm{GL}_m(S)$. We have that $\rho j(S)(1+\delta) = 1 + A_1 \delta + \cdots + A_{n'-1}\delta^{n'-1}$ for some $A_i \in \mathrm{Mat}_m(k)$, so we need to show that A_i is scalar for all i.

We will use an argument by induction to prove that indeed the A_i's are scalar. Consider the rings $S_t = k[\delta]/\delta^t$ for $t = 0, 1, \ldots, n'$. We have canonical projections $S = S_{n'} \to S_{n'-1} \to \cdots \to S_0 = k$. We can consider $1 + \delta \in \mu_{n'}(S_t)$ for all t, and we have that $\rho(S_t)(1 + \delta) = 1 + \cdots + A_{t-1}\delta^{t-1}$. What we want to prove is then equivalent to saying that $\rho j(S_t)(1 + \delta)$ is scalar in $\mathrm{GL}_m(S_t)$ for all t.

This is clearly true for $t = 1$. Let us suppose it is true for $t \leq t_0$, and let us prove it for $t = t_0 + 1$.

We know that $\rho j(S_{t_0+1})(1 + \delta) = M + A_{t_0}\delta^{t_0}$ for some M a scalar matrix. Suppose that for all k-algebra R, for all $g \in \mathrm{SL}_n(R)$, we have that $A_{t_0}\rho(R)(g) = \rho(R)(g)A_{t_0}$ (notice that $\rho(R)(g) \in \mathrm{GL}_m(R)$, while $A_{t_0} \in \mathrm{Mat}_m(k) \subset \mathrm{Mat}_m(R)$). Let $V_\lambda \neq 0$ be a generalized eigenspace for A_{t_0} with eigenvalue λ.

Then $R \otimes V_\lambda$ is closed under the action of $\mathrm{SL}_n(R)$ for all R because A_{t_0} commutes with all $\mathrm{SL}_n(R)$, *i.e.* V_λ gives a subrepresentation of ρ, so $V_\lambda = V$

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since ρ is irreducible. Then if A_{t_0} is not scalar we have that $\ker(A_{t_0} - \lambda)$ is a proper subrepresentation, which is a contradiction because ρ is irreducible, and we conclude that A_{t_0} is scalar.

We are left with proving that A_{t_0} commutes indeed with all possible $\rho(R)(g)$ for $g \in \mathrm{SL}_n(R)$. We know that $1 + \delta \in \mu_m(S_{t_0+1})$ commutes with all $h \in \mathrm{SL}_n(S_{t_0+1})$, so $\rho j(R \otimes S_{t_0+1})(1 + \delta)$ commutes with $\rho(R \otimes S_{t_0+1})(g)$ inside $\mathrm{GL}_m(R \otimes S_{t_0+1})$ for all $g \in \mathrm{SL}_n(R)$. That means

$$0 = [M + A_{t_0}\delta^{t_0}, \rho(R \otimes S_{t_0+1})(g)] = \delta^{t_0}[A_{t_0}, \rho(R)(g)]$$

inside $\mathrm{Mat}_m(R \otimes S_{t_0+1})$, hence $[A_{t_0}, \rho(R)(g)] = 0$ inside $\mathrm{Mat}_m(R)$ and we are done.

Step 3. Suppose now char $k = p > 0$, and $n' = ep^r$, with $p \nmid e$. In this case we have that $\mu_{n'} = \mu_e \times \mu_{p^r}$ as group schemes, by using $\mathbb{Z}/n'\mathbb{Z} = \mathbb{Z}/e\mathbb{Z} \times \mathbb{Z}/p^r\mathbb{Z}$ and Example 1.3. We have the canonical morphisms $j_e : \mu_e \rightarrowtail \mathrm{SL}_n$ and $j_{p^r} : \mu_{p^r} \rightarrowtail \mathrm{SL}_n$, and by the previous two steps we have commutative diagrams

$$
\begin{array}{ccc}
\mu_e & \xrightarrow{\ j_e\ } & \mathrm{SL}_n \\
{\scriptstyle \tau_e}\downarrow & & \downarrow{\scriptstyle \rho} \\
\mathbb{G}_\mathrm{m} & \xrightarrow[\ h\ ]{} & \mathrm{GL}(V)
\end{array}
$$

and

$$
\begin{array}{ccc}
\mu_{p^r} & \xrightarrow{\ j_{p^r}\ } & \mathrm{SL}_n \\
{\scriptstyle \tau_{p^r}}\downarrow & & \downarrow{\scriptstyle \rho} \\
\mathbb{G}_\mathrm{m} & \xrightarrow[\ h\ ]{} & \mathrm{GL}(V).
\end{array}
$$

From these we obtain a commutative diagram

$$
\begin{array}{ccc}
\mu_{p^r} \times \mu_e & \xrightarrow{\ j\ } & \mathrm{SL}_n \\
{\scriptstyle \tau}\downarrow & & \downarrow{\scriptstyle \rho} \\
\mathbb{G}_\mathrm{m} & \xrightarrow[\ h\ ]{} & \mathrm{GL}(V)
\end{array}
$$

and we are done. $\qquad\square$

Proof. (of Theorem 2.3) We refer to [7], §VII for a treatment of exact sequences of affine group schemes and quotients. What we will use (see [8], §I, Theorem 1.29) is that there is an exact sequence

$$1 \longrightarrow \mu_n \xrightarrow{\ \varphi\ } \mathbb{G}_\mathrm{m} \times \mathrm{SL}_n \longrightarrow \mathrm{GL}_n \longrightarrow 1$$

and that giving a representation $\mathrm{GL}_n \to \mathrm{GL}(V)$ is the same as giving representations $\xi : \mathrm{SL}_n \to \mathrm{GL}(V), \chi : \mathbb{G}_\mathrm{m} \to \mathrm{GL}(V)$ such that $\xi\varphi = \chi\varphi$ and

$\xi(R)(a)\chi(R)(b) = \chi(R)(b)\xi(R)(a)$ in $\mathrm{GL}_n(R)$ for any k-algebra R and any $a, b \in R$. Moreover, such a construction gives an extension of ξ in the sense of the statement of Theorem 2.3.

Denoting by j the natural map $j : \mu_n \rightarrowtail \mathrm{SL}_n$, we can consider the representation $\rho j : \mu_n \to \mathrm{GL}(V)$. By Lemma 2.4 this factors through $\tau : \mu_n \to \mathbb{G}_\mathrm{m}$, and by Corollary 1.2 the map τ extends to $\tilde{\tau} : \mathbb{G}_\mathrm{m} \to \mathbb{G}_\mathrm{m}$. That is, we have a commutative diagram

$$
\begin{array}{ccc}
\mu_n & \xrightarrow{\ j\ } & \mathrm{SL}_n \\
\tau \downarrow & & \downarrow \rho \\
\mathbb{G}_\mathrm{m} \xrightarrow{\ \tilde{\tau}\ } \mathbb{G}_\mathrm{m} & \xrightarrow{\ h\ } & \mathrm{GL}(V).
\end{array}
$$

Define now $\xi = \rho$, and $\chi = h\tilde{\tau}$. Then we have the required properties for ξ and χ (notice that $\varphi = (z, j)$), so this defines a representation $\tilde{\rho}$ of GL_n that extends ρ.

It remains to check that the representation we have defined is irreducible. Suppose that there is an invariant subspace $V' \subset V$, with $\dim V' < \dim V$. Then the action of SL_n on V' must be trivial, because ρ is irreducible. Then it follows that the action of the whole GL_n is trivial, and we are done. $\qquad\square$

Notice that this construction is the only one possible, up to the choice of an extension $\tilde{\tau}$ of τ, which corresponds to choosing an integer with prescribed congruence modulo n.

Unfortunately, we were not able to use this result to prove that all the irreducible representations of the finite group $\mathrm{GL}_n(\mathbb{F}_q)$ over $\overline{\mathbb{F}}_p$ are restrictions of representations of the affine group scheme GL_n. Let us explain where the difficulty lies. Let ρ be an irreducible representation of $\mathrm{GL}_n(\mathbb{F}_q)$ over $\overline{\mathbb{F}}_p$. We can consider the restriction $\rho|_{\mathrm{SL}_n(\mathbb{F}_q)}$, and suppose that this is irreducible. By Theorem 2.2, there is an irreducible algebraic representation $\tilde{\rho}$ of SL_n such that $\tilde{\rho}_q = \rho|_{\mathrm{SL}_n(\mathbb{F}_q)}$, and by Theorem 2.3 this extends to an irreducible algebraic representation of GL_n. The point is that if we now take its restriction to $\mathrm{GL}_n(\mathbb{F}_q)$, there does not seem to be a reason why this should coincide with ρ. We were however not able to find any example of a representation of the finite group $\mathrm{GL}_n(\mathbb{F}_q)$ that does not come from an algebraic representation of the algebraic group.

In the following section we will restrict our attention to the case $n = 2$. Before we get to that, let us treat another useful result, namely:

Lemma 2.5. *Let k be an algebraically closed field. Let $\rho : \mathrm{GL}_n \to \mathrm{GL}(V)$ be an irreducible representation of $\mathrm{GL}_n = \mathrm{GL}_{n,k}$. Let $i : \mathbb{G}_\mathrm{m} \rightarrowtail \mathrm{GL}_n$ and $h : \mathbb{G}_\mathrm{m} \rightarrowtail \mathrm{GL}(V)$ be the canonical maps. Then there is a map $\alpha : \mathbb{G}_\mathrm{m} \to \mathbb{G}_\mathrm{m}$*

such that the diagram

$$\begin{array}{ccc} \mathbb{G}_{\mathrm{m}} & \xrightarrow{\ \ i\ \ } & \mathrm{GL}_n \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\rho} \\ \mathbb{G}_{\mathrm{m}} & \xrightarrow{\ \ h\ \ } & \mathrm{GL}(V) \end{array}$$

commutes.

Proof. Suppose that such a diagram does not exist, *i.e.* that $\mathbb{G}_{\mathrm{m}}$ does not act as scalars on V. We claim that then there exists a $\lambda \in \mathbb{G}_{\mathrm{m}}(k) = k^\times$ such that $\rho(k)(\lambda)$ is not a scalar in $\mathrm{Aut}\, V$. To prove the claim, notice that $\mathbb{G}_{\mathrm{m}}(k)$ is a dense set in the scheme $\mathbb{G}_{\mathrm{m}}$ (see for instance [4], §I.6, Corollaire 6.5.3), so if every element of $\mathbb{G}_{\mathrm{m}}(k)$ acted as a scalar then we would have a commutative diagram

$$\begin{array}{ccc} \mathbb{G}_{\mathrm{m}}(k) & \xrightarrow{\hspace{2cm}} & \mathrm{GL}_n \\ \downarrow & & \downarrow{\scriptstyle\rho} \\ \mathbb{G}_{\mathrm{m}} & \xrightarrow{\ \ h\ \ } & \mathrm{GL}(V) \end{array}$$

and by density the map $\mathbb{G}_{\mathrm{m}}(k) \to \mathbb{G}_{\mathrm{m}}$ would extend to a map $\mathbb{G}_{\mathrm{m}} \to \mathbb{G}_{\mathrm{m}}$, contradiction.

So we have proved the claim, and fix now a $\lambda \in k^\times$ that does not act as a scalar on V. If the endomorphism $\rho(k)(\lambda)$ of V has more than one eigenvalue, then the corresponding eigenspaces are closed under the action of all GL_n, which is absurd because ρ is irreducible. Then $\rho(k)(\lambda)$ has only one eigenvalue μ. If $\rho(k)(\lambda)$ is not diagonalisable, then $\ker(\rho(k)(\lambda) - \mu)$ is an invariant subspace, which is absurd because ρ is irreducible. We conclude that $\rho(k)(\lambda)$ is diagonalisable, so it is a scalar, contradiction. $\qquad\square$

It follows from this result, together with Corollary 1.2, that the map $\mathbb{G}_{\mathrm{m}} \to \mathbb{G}_{\mathrm{m}}$ induced by an irreducible representation of GL_n is of the form $x \mapsto x^N$ for some $N \in \mathbb{Z}$.

3 The groups SL_2 and GL_2

As it was anticipated, we were not able to find an answer to the main question for the groups GL_n in the literature, nor to come up with one based on the knowledge that we have for SL_n. However, in the case $n = 2$, there is a lot that can be said, and the irreducible representations of both SL_2 and GL_2 can be explicitly described. The representation theory of the finite groups $\mathrm{SL}_2(\mathbb{F}_q)$ and $\mathrm{GL}_2(\mathbb{F}_q)$ is known as well, and it turns out that even for $\mathrm{GL}_2(\mathbb{F}_q)$ all the irreducible representations come from irreducible representations of the affine group scheme.

We will only consider the field $k = \overline{\mathbb{F}}_p$, even though most of what follows applies as well to any algebraically closed field of characteristic $p > 0$. For brevity, we will write SL_2 and GL_2 instead of $\mathrm{SL}_{2,\overline{\mathbb{F}}_p}$ and $\mathrm{GL}_{2,\overline{\mathbb{F}}_p}$.

3.1 Representations of the group scheme SL_2

We begin by introducing a class of representations of SL_2. There is a natural representation ρ of SL_2 on the 2-dimensional vector space $V = k^2 = <X, Y>$ over k. It is defined as follows: the choice of the basis we have made for V induces an isomorphism $\varphi : \mathrm{GL}_2 \to \mathrm{GL}(V)$, and if we call $j : \mathrm{SL}_2 \rightarrowtail \mathrm{GL}_2$ the canonical map, we define ρ to be $\rho = j \circ \varphi$. A more explicit description is the following: for any k-algebra R, for every element

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

of $\mathrm{SL}_2(R)$, the action of $\rho(R)(g)$ is given by

$$\rho(R)(g) : X \mapsto aX + cY$$

and

$$\rho(R)(g) : Y \mapsto bX + dY.$$

We will call this representation the standard representation and denote it by Std.

For $j \in \mathbb{N}$, we will call $V_j = \mathrm{Sym}^j$ Std (the j-th symmetric power of Std). Thus V_j is a $(j+1)$-dimensional representation of SL_2, corresponding to the extension of the action given by ρ to the space of homogeneous polynomials of degree j in the variables X, Y. Notice that in particular V_0 is the trivial 1-dimensional representation, and that $V_1 = \mathrm{Std}$.

From the general theory exposed in [6], we can deduce the full classification of irreducible representations of the affine group scheme SL_2. Recall the definition we have given, for a representation V of SL_2, of its i-twisted representation $V^{[i]}$, given by composing the action with the iterate Frobenius endomorphism of SL_2. We have:

Theorem 3.1. *For every* $j \in \mathbb{N}$, *write* $j = \sum_{i=0}^{s} j_i p^i$, *with* $0 \leq j_i < p$ *for every* i, *and define a representation* $L(j)$ *of the affine group scheme* SL_2 *as follows:*

$$L(j) = \bigotimes_{i=0}^{s} V_{j_i}^{[i]}.$$

Then $L(j)$ *is irreducible, and* $L(j) \cong L(h)$ *implies* $j = h$. *Moreover, every irreducible representation of* SL_2 *is isomorphic to* $L(j)$ *for some* $j \in \mathbb{N}$.

Proof. See [5], §2.2 to §2.8 for the general statement and the particular case of SL_2, and [6], §II.2 or [9] for the proof of the general theorem. $\qquad\square$

Notice that in general, a representation of SL_2 is not a direct sum of irreducibles, so describing the irreducibles is not enough to describe all the representations.

Let us move to the analysis of the finite groups $\mathrm{SL}_2(\mathbb{F}_q)$.

3.2   Representations of the finite group $\mathrm{SL}_2(\mathbb{F}_q)$

The classification of irreducible representations of $\mathrm{SL}_2(\mathbb{F}_q)$ over $k = \overline{\mathbb{F}}_p$ is well known, and it was found by Brauer and Nesbitt in [1] (predating Steinberg's more general theory). It is as follows:

Theorem 3.2 (Brauer-Nesbitt, [1], §30)**.** *Let* $q = p^r$ *for* $r \in \mathbb{N}_{\geq 1}$. *For every* r-*uple* $(j_0, \ldots, j_{r-1})$ *with* $j_i \in \mathbb{N}$ *and* $0 \leq j_i < p$ *for every* i, *define the representation* $H(j_0, j_1, \ldots, j_{r-1})$ *of* $\mathrm{SL}_2(\mathbb{F}_q)$ *over* $\overline{\mathbb{F}}_p$ *by*

$$H(j_0, j_1, \ldots, j_{r-1}) = \bigotimes_{i=0}^{r-1} V_{j_i}^{[i]}.$$

Then these representations are irreducible, they are pairwise non-isomorphic, and every irreducible representation of $\mathrm{SL}_2(\mathbb{F}_q)$ *over* $\overline{\mathbb{F}}_p$ *is isomorphic to one of these.*

Notice that by Theorem 2.2, we know that all these representations are irreducible, being clearly restriction of irreducible representations of the group scheme SL_2. However, in this case it is not so hard to prove the whole theorem by hand. The proof we give is essentially the one given in [1], §30, rewritten in a more modern language and in greater detail.

Definition 3.1. Let G be a finite group, and let $g \in G$. If $p \nmid \mathrm{ord}(g)$, then we say that g is a *p-regular element*. If $p \mid \mathrm{ord}(g)$, then we say that g is a *p-singular element*.

Lemma 3.3. *Let* G *be a finite group, and let* $g \in G$. *Then there exist two elements* $a, b \in G$ *such that the following hold:*

- $g = ab = ba$

- $p \nmid \mathrm{ord}(a)$

- $\mathrm{ord}(b) = p^s$ *for some $s \geq 0$.*

Moreover, the elements a and b satisfying these conditions are uniquely determined and they are both powers of g.

Proof. Let $n = \mathrm{ord}(g)$. If $(n, p) = 1$ then $a = g, b = 1$ is clearly the only possible such decomposition. If $n = p^r q$, with $r \geq 1$ and $(p, q) = 1$, then find integers $x, y \in \mathbb{Z}$ such that $1 = xp^r + yq$. Set $a = g^{xp^r}, b = g^{yq}$. It is easy to see that $\mathrm{ord}(a) = q$, $\mathrm{ord}(b) = p^r$, and that these elements are uniquely determined. $\square$

Definition 3.2. Let G be a finite group, and let $g \in G$. Let $g = ab$ as in Lemma 3.3. Then we say that a is the *p-regular factor* of g and that b is the *p-singular factor* of g.

Notice that the conjugate elements of a p-regular element are p-regular. A *p-regular conjugacy class* is the conjugacy class of a p-regular element.

Theorem 3.4. *Let G be a finite group. Then the number of non-isomorphic irreducible representations of G over $\overline{\mathbb{F}}_p$ is equal to the number of p-regular conjugacy classes of G.*

Proof. Omitted, see [11], §7, Theorem 1.9. $\square$

Lemma 3.5. *Let G be a finite group. Let $\rho : G \to \mathrm{Aut}\, V$ be a representation of G over $\overline{\mathbb{F}}_p$. Let $g \in G$, and write $g = ab$ as in Lemma 3.3. Then the elements $\rho(a)$ and $\rho(g)$ of $\mathrm{Aut}\, V$ have the same eigenvalues with the same multiplicities.*

Proof. We can write $\rho(g)$ in Jordan normal form. Then $\rho(a)$ and $\rho(b)$ are powers of $\rho(g)$ by Lemma 3.3, so they are upper triangular matrices. Since $\mathrm{ord}(\rho(b))$ is a power of p, it follows that its diagonal entries must be equal to 1. Then the result follows easily. $\square$

Notice that in particular we do not lose any information about eigenvalues if we only consider the p-regular factor of a given element of the group.

Lemma 3.6. *Let G be a finite group. Let ρ be a representation of G over $\overline{\mathbb{F}}_p$. Let $a, b \in G$ be p-regular elements. Then $\rho(a)$ and $\rho(b)$ have the same characteristic polynomial if and only if they are conjugate.*

Proof. The "if" part is trivial (two conjugate matrices have the same characteristic polynomial). Suppose now that $\rho(a)$ and $\rho(b)$ have the same characteristic polynomial. Since a and b are p-regular, the corresponding matrices $\rho(a)$ and $\rho(b)$ are diagonalisable. But diagonal matrices are determined by their characteristic polynomial, and we are done. $\square$

We will also need a result about the structure of the representations $V_j^{[l]} = (\mathrm{Sym}^j \mathrm{Std})^{[l]}$.

Lemma 3.7. *Let $j, l \in \mathbb{N}$ with $j \geq 1$. There are exact sequences of representations of $\mathrm{SL}_2(\mathbb{F}_q)$ over $\overline{\mathbb{F}}_p$*

$$0 \longrightarrow V_{j-1}^{[l]} \xrightarrow{\varphi} V_j^{[l]} \otimes V_1^{[l]} \xrightarrow{\psi} V_{j+1}^{[l]} \longrightarrow 0 \tag{1}$$

and

$$0 \longrightarrow V_1^{[l+1]} \xrightarrow{\gamma} V_p^{[l]} \xrightarrow{\chi} V_{p-2}^{[l]} \longrightarrow 0. \tag{2}$$

Proof. The maps in (1) are given by:

$$\begin{aligned} \varphi(h) &= hY \otimes X - hX \otimes Y \\ \psi(s \otimes t) &= st. \end{aligned}$$

The map ψ is obviously a surjective map of representations. Moreover, we have that clearly

$$\psi \circ \varphi = 0.$$

Since

$$\dim(V_j^{[l]} \otimes V_1^{[l]}) = 2j + 2 = \dim V_{j-1}^{[l]} + \dim V_{j+1}^{[l]}$$

we conclude that (1) is an exact sequence of vector spaces. Let us check that φ is a map of representations, *i.e.* that given

$$E = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{F}_q)$$

we have

$$\varphi(E.h) = E.\varphi(h).$$

We have

$$\begin{aligned} E.\varphi(h) &= (E.h)(b^{p^l}X + d^{p^l}Y) \otimes (a^{p^l}X + c^{p^l}Y) + \\ &\quad -(E.h)(a^{p^l}X + c^{p^l}Y) \otimes (b^{p^l}X + d^{p^l}Y) = \\ &= (E.h)(\det E)^{p^l}(Y \otimes X - X \otimes Y) = \\ &= \varphi(E.h) \end{aligned}$$

since $\det E = 1$. We have proved that the sequence (1) is indeed an exact sequence of representations of $\mathrm{SL}_2(\mathbb{F}_q)$.

The maps in (2) are given by:

$$\begin{aligned} \gamma(X) &= X^p \\ \gamma(Y) &= Y^p \\ \chi(f) &= \frac{\partial f}{X \partial Y}. \end{aligned}$$

Let us first check that γ is a map of representations. We have, for the same matrix $E \in \mathrm{SL}_2(\mathbb{F}_q)$,

$$
\begin{aligned}
E.\gamma(X) &= E.X^p = (a^{p^l}X + c^{p^l}Y)^p = \\
&= a^{p^{l+1}}X^p + c^{p^{l+1}}Y^p = a^{p^{l+1}}\gamma(X) + c^{p^{l+1}}\gamma(Y) = \\
&= \gamma(E.X)
\end{aligned}
$$

and similarly

$$
E.\gamma(Y) = \gamma(E.Y)
$$

so γ is indeed a map of representations. Moreover, there is clearly an equality of vector spaces

$$
\ker \chi = \gamma(V_1).
$$

What remains to be done is to show that χ is a map of representations. It is enough to show that χ commutes with the matrices

$$
E_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}
$$

$$
E_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}
$$

$$
E_\alpha = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \quad \text{with } \alpha \in \mathbb{F}_q^\times
$$

since they generate $\mathrm{SL}_2(\mathbb{F}_q)$. Indeed, we have for $0 \le i \le p$, that

$$
E_1.\chi(X^{p-i}Y^i) = \sum_{h=0}^{p-i-1} i \binom{p-i-1}{h} X^{p-i-h-1}Y^{i+h-1}
$$

while

$$
\chi(E_1.X^{p-i}Y^i) = \sum_{h=0}^{p-i-1} (i+h) \binom{p-i}{h} X^{p-i-h-1}Y^{i+h-1}
$$

so we are left with comparing the coefficients

$$
i \binom{p-i-1}{h} \quad \text{and} \quad (i+h)\binom{p-i}{h}
$$

inside $\overline{\mathbb{F}}_p$. We will prove the stronger assertion that

$$
(p-i)\binom{p-i-1}{h} = (p-i-h)\binom{p-i}{h}
$$

inside $\mathbb{Z}$. Indeed, the left-hand side is the number of choices of an object from a set of $p-i$, followed by h objects chosen from the $p-i-1$ remaining.

The right-hand side is the number of ways one can choose h objects from the same set of $p - i$, and then choose one among the remaining $p - i - h$. So the two numbers are the same, since the order in which the two choices are made does not matter. So we have proved that

$$E_1.\chi(X^{p-i}Y^i) = \chi(E_1.X^{p-i}Y^i)$$

for all $0 \leq i \leq p$. Let us proceed to compute, still for $0 \leq i \leq p$, the quantities

$$E_2.\chi(X^{p-i}Y^i) \quad = \quad \sum_{k=1}^{i} i \binom{i-1}{k-1} X^{p-k-1}Y^{k-1}$$

and

$$\chi(E_2.X^{p-i}Y^i) \quad = \quad \sum_{k=1}^{i} k \binom{i}{k} X^{p-k-1}Y^{k-1}.$$

We are left again with comparing the coefficients

$$i\binom{i-1}{k-1} \quad \text{and} \quad k\binom{i}{k}$$

inside $\overline{\mathbb{F}}_p$. They are actually equal inside $\mathbb{Z}$. This can be seen with another combinatorical argument, but for the sake of diversity we present the computation

$$k\binom{i}{k} = \frac{i!k}{k!(i-k)!} = \frac{(i-1)!i}{(k-1)!(i-k)!} = i\binom{i-1}{k-1}.$$

We are done proving that

$$E_2.\chi(X^{p-i}Y^i) = \chi(E_2.X^{p-i}Y^i)$$

for all $0 \leq i \leq p$. It remains to prove that χ commutes with the matrices of type E_α. We have, for $0 \leq i \leq p$,

$$\begin{aligned} E_\alpha.\chi(X^{p-i}Y^i) &= E_\alpha.iX^{p-i-1}Y^{i-1} = \\ &= i\alpha^{p^l(p-2i)}X^{p-i-1}Y^i = \chi(\alpha^{p^l(p-2i)}X^{p-i}Y^i) = \\ &= \chi(E_\alpha.X^{p-i}Y^i) \end{aligned}$$

and this concludes the proof of the Lemma. $\qquad\qquad\square$

Now we are ready to prove Theorem 3.2.

Proof. (of Theorem 3.2) We have to prove three facts: that the representations we have given are irreducible, that they are pairwise non-isomorphic, and that there are no other irreducible representations.

Let us start by proving that $H = H(j_0, j_1, \ldots, j_{r-1})$ is irreducible. The vector space H has a basis consisting of the elements

$$g(a_0, \ldots, a_{r-1}) = X_0^{j_0 - a_0} Y_0^{a_0} \otimes \cdots \otimes X_{r-1}^{j_{r-1} - a_{r-1}} Y_{r-1}^{a_{r-1}}$$

where $V_{j_i}^{[i]} = k[X_i, Y_i]_{j_i}$ as a vector space and $0 \le a_i \le j_i$ for every i. Let us take a nonzero element

$$f = \sum \alpha(a_0, \ldots, a_{r-1}) g(a_0, \ldots, a_{r-1}) \in H$$

with every $\alpha(a_0, \ldots, a_{r-1})$ inside $\mathbb{F}_q$, and let us consider the subrepresentation $V(f) \subset H$ generated by f. It is enough to show that $V(f) = H$. We will first reduce to the case

$$f = Y_0^{j_0} \otimes Y_1^{j_1} \otimes \cdots \otimes Y_{r-1}^{j_{r-1}}.$$

For $t \in \mathbb{F}_q$, let us consider the map $\varphi_t : H \to H$ representing the matrix

$$\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{F}_q).$$

For a basis element $g = g(a_0, \ldots, a_{r-1})$ we have then

$$\varphi_t(g) = (X_0 + tY_0)^{j_0 - a_0} Y_0^{a_0} \otimes \cdots \otimes (X_{r-1} + t^{p^{r-1}} Y_{r-1})^{j_{r-1} - a_{r-1}} Y_{r-1}^{a_{r-1}}.$$

In the above expression for $g(a_0, \ldots, a_{r-1})$, consider now t as a formal variable, and write

$$f_t = \varphi_t(f) = \sum \alpha(a_0, \ldots, a_{r-1}) \varphi_t(g(a_0, \ldots, a_{r-1}))$$

as a polynomial in t

$$f_t = \sum_{\nu=0}^{q-1} t^\nu N_\nu$$

where the N_ν's do not depend on t. The indices ν range from 0 to

$$\left(\sum_{i=0}^{r-1} p^i \right) (p - 1) = q - 1.$$

Notice that $f_t \in V(f)$ if we evaluate it at any $t \in \mathbb{F}_q$. We claim that every N_ν is a linear combination of the elements $\{ f_t \mid t \in \mathbb{F}_q \}$. To prove the claim, notice that we have

$$\begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_{\alpha_{q-1}} \end{pmatrix} = M \begin{pmatrix} N_0 \\ N_1 \\ \vdots \\ N_{q-1} \end{pmatrix}$$

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where

$$M = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1^2 & \cdots & 1^{q-1} \\ 1 & \alpha_2 & \alpha_2^2 & \cdots & \alpha_2^{q-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha_{q-1} & \alpha_{q-1}^2 & \cdots & \alpha_{q-1}^{q-1} \end{pmatrix}$$

with $\mathbb{F}_q = \{0, 1, \alpha_2, \ldots, \alpha_{q-1}\}$. The claim is equivalent to the matrix $M \in \mathrm{Mat}_q(\mathbb{F}_q)$ being invertible. Its determinant $\det M$ is the Vandermonde determinant in $\{0, 1, \alpha_2, \ldots, \alpha_{q-1}\}$, which is nonzero since these elements are all distinct. We have then proved the claim that the N_ν's are linear combinations of the $\{f_t \mid t \in \mathbb{F}_q\}$. Let us now order the basis elements $g(a_0, \ldots, a_{r-1})$ by inverse lexicographical order, that is

$$g(a_0, \ldots, a_{r-1}) \le g(a_0', \ldots, a_{r-1}')$$

if and only if the last difference $a_i' - a_i$ that is nonzero is positive. Let $g(b_0, \ldots, b_{r-1})$ be the unique minimal basis element such that in the expression

$$f_t = \sum \alpha(a_0, \ldots, a_{r-1}) \varphi_t(g(a_0, \ldots, a_{r-1}))$$

the coefficient $\alpha(b_0, \ldots, b_{r-1})$ is nonzero. Then the highest exponent ν_{max} of t appearing in the expression

$$f_t = \sum_{\nu=0}^{q-1} t^\nu N_\nu$$

such that $N_{\nu_{max}} \ne 0$ is

$$\nu_{max} = \sum_{i=0}^{r-1} p^i (j_i - b_i)$$

and the corresponding $N_{\nu_{max}}$ is

$$N_{\nu_{max}} = \alpha(b_0, \ldots, b_{r-1}) \cdot Y_0^{j_0} \otimes Y_1^{j_1} \otimes \cdots \otimes Y_{r-1}^{j_{r-1}}.$$

So we conclude that $f' = Y_0^{j_0} \otimes Y_1^{j_1} \otimes \cdots \otimes Y_{r-1}^{j_{r-1}} \in V(f)$, and in particular $V(f') = H$ implies $V(f) = H$. So we are reduced to showing that $V(f') = H$.

Let us then consider another map $\psi_t : H \to H$ representing the matrix

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{F}_q).$$

Define $f_t' = \psi_t(f')$ and compute it:

$$f_t' = (tX_0 + Y_0)^{j_0} \otimes (t^p X_1 + Y_1)^{j_1} \otimes \cdots \otimes (t^{p^{r-1}} X_{r-1} + Y_{r-1})^{j_{r-1}}.$$

We can express f'_t as a polynomial in t, and in this case the coefficients include all the basis elements $g(a_0, \ldots, a_{r-1})$. Now by the same argument we used for the N_ν's, we deduce that all the elements $g(a_0, \ldots, a_{r-1})$ lie inside $V(f')$ and so we are done.

We now proceed to showing that the representations we have listed are pairwise non-isomorphic. Let us order the representations $H(j_0, \ldots, j_{r-1})$ by

$$H(j_0, \ldots, j_{r-1}) \leq H(h_0, \ldots, h_{r-1})$$

if and only if the first difference $h_i - j_i$ that is nonzero is positive. Suppose then that we have

$$H(j_0, \ldots, j_{r-1}) \cong H(j'_0, \ldots, j'_{r-1}) \tag{3}$$

with $(j_0, \ldots, j_{r-1}) \neq (j'_0, \ldots, j'_{r-1})$. We may assume that $H(j_0, \ldots, j_{r-1})$ is (the unique) minimal element in the set of representations that we have given that is isomorphic to another one of them. Without loss of generality, assume that $j_0 = \cdots = j_{i-1} = 0$ and $j_i \neq 0$. Then clearly $(j_i, \ldots, j_{r-1}) \neq (p-1, \ldots, p-1)$ because of minimality. Moreover, if $(j'_i, \ldots, j'_{r-1}) = (p-1, \ldots, p-1)$ then by dimensional reasons $(j_i, \ldots, j_{r-1}) = (p-1, \ldots, p-1)$ which is also a contradiction, so we deduce that $(j'_i, \ldots, j'_{r-1}) \neq (p-1, \ldots, p-1)$.

Tensor both sides of (3) by $V_1^{[i]}$. Suppose now that $j_i \neq p-1$. In this case, by Lemma 3.7, we know that

$$V_1^{[i]} \otimes H(j_0, \ldots, j_{r-1}) = V_1^{[i]} \otimes V_{j_i}^{[i]} \otimes \cdots \otimes V_{j_{r-1}}^{[r-1]}$$

has the representations

$$H(0, \ldots, 0, j_i - 1, j_{i+1}, \ldots, j_{r-1})$$

and

$$H(0, \ldots, 0, j_i + 1, j_{i+1}, \ldots, j_{r-1})$$

as its two composition factors. The right-hand side of (3) tensored with $V_1^{[i]}$ has the same composition factors, hence for dimensional reasons we have

$$H(0, \ldots, 0, j_i - 1, j_{i+1}, \ldots, j_{r-1}) \cong H(0, \ldots, 0, j'_i - 1, j'_{i+1}, \ldots, j'_{r-1})$$

but since $j_i - 1 \neq j'_i - 1$ we have found an example of distinct isomorphic representations with lower indices, which is a contradiction.

Suppose now that $j_i = p - 1$. Then by Lemma 3.7 we have that

$$V_1^{[i]} \otimes H(j_0, \ldots, j_{r-1}) = V_1^{[i]} \otimes V_{p-1}^{[i]} \otimes \cdots \otimes V_{j_{r-1}}^{[r-1]}$$

has the representations

$$H(0, \ldots, 0, p - 2, j_{i+1}, \ldots, j_{r-1})$$

and
$$K = V_p^{[i]} \otimes H(0, \ldots, 0, j_{i+1}, \ldots, j_{r-1})$$

as subquotients. While the first of these is irreducible, the second is not. We can use the second part of Lemma 3.7 to deduce that K has the representations
$$K' = V_1^{[i+1]} \otimes H(0, \ldots, 0, j_{i+1}, \ldots, j_{r-1})$$

and
$$V_{p-2}^{[i]} \otimes H(0, \ldots, 0, j_{i+1}, \ldots, j_{r-1}) = H(0, \ldots, 0, p-2, j_{i+1}, \ldots, j_{r-1})$$

as subquotients. Notice that the latter of the two is irreducible. Suppose that $j_{i+1} \neq p-1$. Then K' has the representations
$$H(0, \ldots, 0, j_{i+1} - 1, \ldots, j_{r-1})$$

and
$$H(0, \ldots, 0, j_{i+1} + 1, \ldots, j_{r-1})$$

as its two composition factors, and we have again found the composition factors of both sides of (3). We find once more a contradiction because there are isomorphic irreducible representations that have lower indices. Suppose instead that $j_{i+1} = p-1$. We can iterate this whole process, and it will eventually stop because, as we have said, $(j_i, \ldots, j_{r-1}) \neq (p-1, \ldots, p-1)$. So when we reach the first i such that $j_i \neq p-1$ we can conclude by contradiction in the same way. This shows that the representations that we have listed are pairwise non-isomorphic.

Let us conclude by showing that the list of irreducible representations that we have given is in fact the full list. It is enough to show that there are exactly q non-isomorphic irreducible representations of $\mathrm{SL}_2(\mathbb{F}_q)$. By Theorem 3.4, it is enough to show that there are exactly q different p-regular conjugacy classes. We will apply Lemma 3.5 and Lemma 3.6 to the standard representation $\mathrm{Std} = V_1$ of $\mathrm{SL}_2(\mathbb{F}_q)$. Notice that, given a polynomial $p_\gamma(T) = T^2 - \gamma T + 1$ with $\gamma \in \mathbb{F}_q$, there is an element $g_\gamma \in \mathrm{SL}_2(\mathbb{F}_q)$ that has it as its characteristic polynomial, for instance the matrix
$$\begin{pmatrix} 0 & -1 \\ 1 & \gamma \end{pmatrix}.$$

By Lemma 3.5 we deduce that there is a p-regular element $g'_\gamma \in \mathrm{SL}_2(\mathbb{F}_q)$ that has $p_\gamma(T)$ as its characteristic polynomial. By Lemma 3.6 we have that the number of different characteristic polynomials of elements of $\mathrm{SL}_2(\mathbb{F}_q)$ is exactly the same as the number of p-regular conjugacy classes. Since the characteristic polynomials in this case are in $\mathbb{F}_q[T]$, are monic of degree 2, have constant term equal to 1, and all such polynomials occur, we conclude that this sought-after number is indeed $|\mathbb{F}_q| = q$. $\qquad\square$

This concludes the discussion of representations of the various incarnations of the group SL_2. We will now move to the analogous discussion for GL_2. It turns out that the situation is not very different, as it was easy to expect given how much the two groups are related.

3.3   Representations of the group scheme GL_2

Most constructions that we have made for SL_2 are still valid, in particular the vector space Std is in a natural way a representation of GL_2, and we can still form its symmetric powers. We will still write V_j for the representation Sym^j Std of GL_2. All the irreducible representations of SL_2 can be seen as representations of GL_2 (since they are tensor products of twists of symmetric powers of Std). Recall from Example 1.5 that GL_2 also has infinitely many 1-dimensional representations D^n given by powers of the determinant. Notice that these are all trivial when restricted to SL_2.

Given that the main difference between invertible matrices and matrices of determinant 1 is the determinant, it is natural to expect that it is also the main difference between the representation theory of GL_2 and of SL_2. This is indeed the case: given an irreducible representation V of SL_2, we can form the representation $V \otimes D^n$ of GL_2 for any value of $n \in \mathbb{Z}$. This turns out to give the complete description of the irreducible representations of GL_2:

Theorem 3.8. *Let $L(j)$ be an irreducible algebraic representation of SL_2, and let $n \in \mathbb{Z}$. Then the representation $L(j,n) = L(j) \otimes D^n$ of GL_2 is irreducible. Moreover, $L(j,n) \cong L(h,m)$ implies $(j,n) = (h,m)$, and every irreducible algebraic representation of GL_2 is isomorphic to $L(h,m)$ for some $h \in \mathbb{N}$, $m \in \mathbb{Z}$. In other words, the irreducible algebraic representations of GL_2 are the following (up to isomorphism):*

$$L(j,n) = D^n \otimes \bigotimes_{i=0}^{s} V_{j_i}^{[i]}$$

for $n \in Z$ and $j = \sum_{i=0}^{s} j_i p^i$, with $0 \le j_i < p$ for every i.

Proof. To prove the first claim, suppose that L' is a proper subrepresentation of $L(j,n)$. Then $L'|_{\mathrm{SL}_2}$ is a subrepresentation of $L(j)$, it is proper by dimension, so it is 0. But for a k-algebra R, if the action of $\mathrm{SL}_2(R)$ on $R \otimes L'$ is trivial, then so is the action of $\mathrm{GL}_2(R)$, and we are done.

Suppose now that $L(j,n) \cong L(h,m)$. Then a matrix

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \in \mathrm{GL}_2(k)$$

with $\alpha \in k^\times$ acts as α^{2n} on $L(j,n)$ and as α^{2m} on $L(h,m)$. Since k is infinite it follows that $n = m$. Then it suffices to apply Theorem 3.1 to $L(j,n)|_{\mathrm{SL}_2}$ and $L(h,n)|_{\mathrm{SL}_2}$ in order to conclude that $(j,n) = (h,m)$.

Suppose now that V is an irreducible algebraic representation of GL_2, and we want to prove that it is of this kind. We claim that $V|_{\mathrm{SL}_2}$ is irreducible. Indeed, let $N \subset V|_{\mathrm{SL}_2}$ be a nonzero irreducible subrepresentation of $V|_{\mathrm{SL}_2}$. Then $N \cong L(h)$ for some h by Theorem 3.1, and by Theorem 2.3 it can be extended to an irreducible representation of GL_2. The construction in the proof of Theorem 2.3 shows that the only possible ways of extending N are by tensoring with a power of the determinant, so the only irreducible representations of GL_2 that are isomorphic to N when restricted to SL_2 are the $L(h) \otimes D^l$ for $l \in \mathbb{Z}$. Now, by Lemma 2.5 we know that $V|_{\mathbb{G}_{\mathrm{m}}} = D^m|_{\mathbb{G}_{\mathrm{m}}}$ for some $m \in \mathbb{Z}$. Then, for that particular value of m we have that $N \otimes D^m \subset V$ is a nonzero subrepresentation of V, so we get $V \cong L(h) \otimes D^m$ since V is irreducible, and we are done. $\qquad\square$

It should be noted that this result follows from the general theory exposed in [6]. However, the explicit description we have given follows much more easily from the knowledge about SL_2 than from actually computing the highest weights of GL_2. On the other hand, the analysis of the representations of $\mathrm{GL}_2(\mathbb{F}_q)$ does not follow from (the general statement of) Theorem 2.2 (recall that SL_2 is simply connected, while GL_2 is not). In this case, however, the relationship with $\mathrm{SL}_2(\mathbb{F}_q)$ produces an explicit description of the irreducible representations of the groups $\mathrm{GL}_2(\mathbb{F}_q)$ over $\overline{\mathbb{F}}_p$.

3.4 Representations of the finite groups $\mathrm{GL}_2(\mathbb{F}_q)$

Let us proceed to the main result of this thesis, namely the description of the irreducible representations of $\mathrm{GL}_2(\mathbb{F}_q)$ over $\overline{\mathbb{F}}_p$. As a corollary, we obtain that Theorem 2.2 also applies to the affine group scheme GL_2.

Theorem 3.9. *Let $q = p^r$ for $r \in \mathbb{N}_{\geq 1}$. For every $0 \leq n < q - 1$, for every r-uple $(j_0, \ldots, j_{r-1})$ with $j_i \in \mathbb{N}$ and $0 \leq j_i < p$ for every i, define the representation $H(j_0, j_1, \ldots, j_{r-1}, n)$ of $\mathrm{GL}_2(\mathbb{F}_q)$ over $\overline{\mathbb{F}}_p$ by*

$$H(j_0, j_1, \ldots, j_{r-1}, n) = D^n \otimes \bigotimes_{i=0}^{r-1} V_{j_i}^{[i]}.$$

Then these representations are irreducible, they are pairwise non-isomorphic, and every irreducible representation of $\mathrm{GL}_2(\mathbb{F}_q)$ over $\overline{\mathbb{F}}_p$ is isomorphic to one of these.

Corollary 3.10. *For every $q = p^r$, with $r \in \mathbb{N}_{\geq 1}$, the restriction of irreducible representations from $\mathrm{GL}_{2, \overline{\mathbb{F}}_p}$ to $\mathrm{GL}_2(\mathbb{F}_q)$ yields all the irreducible representations of $\mathrm{GL}_2(\mathbb{F}_q)$ over $\overline{\mathbb{F}}_p$.*

To prove Theorem 3.9 it is enough to adapt the proof of Theorem 3.2. We need a new version of Lemma 3.7, namely

Lemma 3.11. *Let $j, l \in \mathbb{N}$ with $j \geq 1$. There are exact sequences of representations of $\mathrm{GL}_2(\mathbb{F}_q)$ over $\overline{\mathbb{F}}_p$*

$$0 \longrightarrow D^{p^l} \otimes V_{j-1}^{[l]} \xrightarrow{\varphi} V_j^{[l]} \otimes V_1^{[l]} \xrightarrow{\psi} V_{j+1}^{[l]} \longrightarrow 0$$

and

$$0 \longrightarrow V_1^{[l+1]} \xrightarrow{\gamma} V_p^{[l]} \xrightarrow{\chi} D^{p^l} \otimes V_{p-2}^{[l]} \longrightarrow 0.$$

Proof. The proof is essentially the same of Lemma 3.7. We need to redefine the maps as follows:

$$\begin{aligned}
\varphi(1 \otimes h) &= hY \otimes X - hX \otimes Y \\
\psi(s \otimes t) &= st \\
\gamma(X) &= X^p \\
\gamma(Y) &= Y^p \\
\chi(f) &= 1 \otimes \frac{\partial f}{X \partial Y}.
\end{aligned}$$

We only need to add to the computations that we carried out to prove Lemma 3.7 some details concerning determinants. Specifically, let us verify that the newly defined map φ is a map of representations, showing that it commutes with the action of a generic matrix

$$E = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{F}_q).$$

We have

$$\begin{aligned}
E.\varphi(1 \otimes h) &= (E.h)(\det E)^{p^l}(Y \otimes X - X \otimes Y) = \\
&= \varphi((\det E)^{p^l} \otimes E.h) = \\
&= \varphi(E.1 \otimes h).
\end{aligned}$$

Now we have to show that the map χ in its new version is a map of representations. The computations to show that it commutes with the matrices

$$E_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

and

$$E_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

have already been carried out in the proof of Lemma 3.7. It remains to show that it commutes with matrices of the form

$$E_{\alpha, X} = \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}$$

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and

$$E_{\alpha,Y} = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}$$

for $\alpha \in \mathbb{F}_q^\times$. Notice that these, together with E_1 and E_2, actually generate all of $\mathrm{GL}_2(\mathbb{F}_q)$. We have, for $0 < i < p$,

$$\begin{aligned} E_{\alpha,X}.\chi(X^{p-i}Y^i) &= i(\det E_{\alpha,X})^{p^l} \otimes \alpha^{p^l(p-i-1)}X^{p-i-1}Y^{i-1} = \\ &= \chi(E_{\alpha,X}.X^{p-i}Y^i) \end{aligned}$$

and similarly

$$E_{\alpha,Y}.\chi(X^{p-i}Y^i) = \chi(E_{\alpha,Y}.X^{p-i}Y^i)$$

and the proof is complete. $\qquad\square$

Now the proof of our main result follows quite easily:

Proof. (of Theorem 3.9) Let us first show that the representations we have given are irreducible. We have that $H = H(j_0, \ldots, j_{r-1}, n)$ restricts to $H(j_0, \ldots, j_{r-1})$ as a representation of $\mathrm{SL}_2(\mathbb{F}_q)$. Suppose that $H' \subset H$ is a subrepresentation, such that $\dim H' < \dim H = \dim H|_{\mathrm{SL}_2(\mathbb{F}_q)}$. Then $H'|_{\mathrm{SL}_2(\mathbb{F}_q)}$ is a proper subrepresentation of $H(j_0, \ldots, j_{r-1})$, hence it is 0. Then the action of the whole $\mathrm{GL}_2(\mathbb{F}_q)$ on H is trivial, and we are done.

Let us prove that the representations we have listed are pairwise non-isomorphic. The proof of this fact is essentially the same that we used in Theorem 3.2, and here we use the updated exact sequences of Lemma 3.11. One needs to notice that for $0 \le i \le r-1$, we have that

$$D^{p^i} \otimes V_{p-2}^{[i]}$$

is an irreducible representation of $\mathrm{GL}_2(\mathbb{F}_q)$. Moreover, notice that if

$$H(j_0, \ldots, j_{r-1}, n) \cong H(j'_0, \ldots, j'_{r-1}, n')$$

then $n = n'$ because for $\alpha \in \mathbb{F}_q$, $\alpha^n = \alpha^{n'}$ implies $n = n' \mod (q-1)$. Then we can proceed exactly as in the proof of Theorem 3.2, starting from an isomorphism that is minimal with respect to lexicographic index ordering (excluding the last index), and then deriving a contradiction by showing that some of the composition factors must be isomorphic and have lower indices.

As for showing that the list is a complete list, the argument that we have used for Theorem 3.2 still applies, but we have to consider polynomials of the form $P_{\gamma,\delta}(T) = T^2 - \gamma T + \delta$, with $\gamma \in \mathbb{F}_q$ and $\delta \in \mathbb{F}_q^\times$. Then the number of non-isomorphic irreducible representations of $\mathrm{GL}_2(\mathbb{F}_q)$ turns out to be $|\mathbb{F}_q||\mathbb{F}_q^\times| = q(q-1)$, and we are done. $\qquad\square$

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