Torsion-free rank one sheaves on a semi-stable curve

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Introduction

This work deals with the problem of compactifying the Picard scheme of a semi-stable, reducible curve defined over a field k.

Let $X \to \operatorname{Spec} k$ be a projective curve over a field k, that is, a projective scheme of pure dimension 1 over $\operatorname{Spec} k$. The relative Picard functor is defined by:

$$\operatorname{Pic}_{X/k} : (\operatorname{Sch}/k)^{\mathrm{op}} \to (\mathrm{Sets}), \quad T \mapsto \frac{\operatorname{Pic}(X_T)}{\operatorname{Pic} T}.$$

In the case of a smooth curve X/k with $X(k) \neq \emptyset$, this functor is represented by a k-scheme $\operatorname{Pic}_{X/k}$ called Picard scheme, which is proper over k. This is not the case in general when X is a singular curve. Then, when the relative Picard functor is representable, a compactification $\overline{\operatorname{Pic}_{X/k}}$ of $\operatorname{Pic}_{X/k}$ can be achieved by "adding points" to $\operatorname{Pic}_{X/k}$, i.e. allowing more general sheaves than just invertible ones.

In 2001 Esteves constructed in [2] a compactification of the Picard scheme of a projective, flat family of geometrically-reduced and connected curves $X \to S$, using torsion-free rank one sheaves. These sheaves were only defined for curves over fields as invertible sheaves degenerating at the singular points; the compactification constructed by Esteves is based on coherent sheaves on $X \times_S T$, flat over T, whose fibres over T are torsion-free rank one.

In this work we consider the case of a semi-stable reducible curve X over a field k of genus 1 with two nodal singularities, given by two copies of the projective line $\mathbb{P}^1_k$ meeting at two distinct points. Our aim is to give a definition of torsion-free rank one sheaf on the base change $X \times_k S$ for any k-scheme S. We show that with such definition, these sheaves yield an algebraic stack $\mathbb{T}$ over $(\operatorname{Sch}/k)_{fppf}$. We then consider the open substack $\mathbb{T}' \subset \mathbb{T}$ of simple torsion-free rank one sheaves, following Esteves. The rigidified version of the algebraic stack $\mathbb{T}'$ contains the stack of rigidified invertible sheaves as an open substack, and we show that it is representable by a scheme T' which has an open covering by copies of the original curve X.

Chapter 1 starts with the definition of torsion-free rank one sheaf on X_S for any k-scheme S. An $\mathcal{O}_{X_S}$-module is said to be torsion-free rank one if: a) it is of finite presentation, b) it is $\mathcal{O}_S$-flat, c) it is invertible on the smooth locus of X_S, d) calling $j : X_S^{sm} \to X_S$ the inclusion of the smooth locus, the map $\mathcal{F} \to j_*j^*\mathcal{F}$ is S-universally injective. The two main results in the rest of the chapter are the following:

- Prop. 1.2.5: we show that for a particular finite map $f_S : X_S \to \mathbb{P}^1_S$, an $\mathcal{O}_{X_S}$-

module of finite presentation $\mathcal{F}$ is torsion-free rank one if and only if its push-forward $f_{S*}\mathcal{F}$ is locally free of rank 2, plus a condition assuring that $\mathcal{F}$ has the correct rank on the components of X_S.

- Prop. 1.2.8: we give a proof of the already well known fact that a torsion-free rank one sheaf on X is a pushforward of an invertible sheaf from a partial normalization of X - where by "partial" we mean a normalization at either two, one or no singular points.

In chapter 2 we make torsion-free rank one sheaves into a fibred category $\mathbb{T}$ over $(\mathrm{Sch}/k)_{fppf}$. Using Prop. 1.2.6, we prove that $\mathbb{T}$ is an algebraic stack. So there is a smooth surjective morphism $T \to \mathbb{T}$, where T is some k-scheme.

In chapter 3, we define the stack $\mathbb{T}' \subset \mathbb{T}$ of simple torsion-free rank one sheaves - those whose fibres over the base scheme have one-dimensional endomorphism ring - and a particular open substack of it, which we call $\mathbb{T}^0$. We give an action of $\mathbb{Z} \times \mathbb{Z}$ on $\mathbb{T}'$ and show that the translates of $\mathbb{T}^0$ via this action cover the stack of simple torsion-free rank one sheaves $\mathbb{T}'$. After conveniently rigidifying all our stacks at a smooth point $\epsilon \in X(k)$, we build up some machinery in order to prove the main theorem: inspired by the fact that an elliptic curve E over a field k is isomorphic as a scheme to $\mathrm{Pic}^{(1)}_{E/k}$, and using a similar method for the proof as in Katz and Mazur [8], we show that:

Theorem. *3.4.6 The rigidified stack $\mathbb{T}^0_\epsilon$ is isomorphic to the scheme X.*

It follows that $\mathbb{T}'_\epsilon$ is representable by a scheme T' which is covered via open immersions by copies of the curve X. The forgetful functor $T' \cong \mathbb{T}'_\epsilon \to \mathbb{T}'$ provides the surjective smooth morphism of stacks sought for.

As a final note, I will remark that at the moment of writing I was not aware that Ngo and Laumon used torsion-free rank one sheaves in the context of the Hitchin fibration in their work on the Langlands-Shelstad Fundamental Lemma. They start from a smooth connected curve C and give an equivalence of categories between vector bundles of degree n on C endowed with a twisted endomorphism, and torsion-free rank one sheaves on a "spectral curve" which is an n-covering of C, possibly singular. This resembles the characterization of torsion-free rank one sheaves that we give in Prop. 1.2.5. My plan for the near future is to investigate whether the Hitchin fibration can provide ideas to generalize our results.

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Chapter 1

Torsion-free rank one sheaves

1.1 Definition of torsion-free rank 1 sheaves

Let k be a field and $X \to \text{Spec } k$ a scheme over k built in the following way: given schemes Y, Z both isomorphic to $\mathbb{P}^1_k$, take four distinct k-rational points $P_0, P_1 \in Y$, $Q_0, Q_1 \in Z$. Then let X be the scheme given by Y and Z where P_0 and Q_0 are identified, and P_1 and Q_1 are identified. We will also let $j : W \to X$ be the inclusion of the smooth locus of X, so that W is actually isomorphic to $\text{Spec}(k[x, x^{-1}] \times k[y, y^{-1}])$.

It will often be useful to restrict to affine neighborhoods of the singular points; we have a cover of X by two affines U and V both isomorphic to $\text{Spec } k[x, y]/xy$ and such that $U \cap V = j(W)$.

We give a definition of torsion-free rank one sheaf.

Definition 1.1.1. Let $S \to \text{Spec } k$ be any k-scheme, and $X_S := X \times_{\text{Spec } k} S$ be the base change of X to S. Let $\mathcal{F}$ be an $\mathcal{O}_{X_S}$-module. We say that $\mathcal{F}$ is a torsion-free rank one sheaf if it satisfies the following conditions:

1) $\mathcal{F}$ is quasi-coherent of finite presentation over $\mathcal{O}_{X_S}$;

2) $\mathcal{F}$ is flat over the base scheme S;

3) calling $j' : W_S \to X_S$ the base change of j, $j'^*\mathcal{F}$ is locally free of rank one over $\mathcal{O}_{W_S}$;

4) the canonical morphism of sheaves $\mathcal{F} \to j'_* j'^* \mathcal{F}$ is universally injective, i.e. for every morphism of k-schemes $f : S' \to S$ the base change $f^*\mathcal{F} \to f^* j'_* j'^* \mathcal{F}$ is injective.

The idea behind the definition is to take into consideration more sheaves than just line bundles, by allowing some degeneration at the singular points. We see that by condition 3), a torsion-free rank one sheaf is invertible on the smooth locus W of

X. This explains the *rank one* denomination. Condition 4) implies that there are no sections of $\mathcal{F}$ supported only at the singular points. Any such section would indeed vanish when restricted to the smooth locus via the map j.

We will find later in this chapter an equivalent and more easily stated characterization of torsion-free rank one sheaves. Yet the definition above conveys a better intuition of the kind of objects we want to allow in our study, and we chose to keep it as a first definition of torsion-free rank one sheaf. In the end of the chapter we will prove the most useful result: for $S = \operatorname{Spec} k$, torsion-free rank one sheaves over X are either invertible, or pushforwards of invertible sheaves on a partial normalization X' of X.

1.2 Characterizations of torsion-free rank one sheaves

As a first remark, notice that for a quasi-coherent $\mathcal{O}_{X_S}$-module $\mathcal{F}$ on X_S, with S affine and $R = \mathcal{O}_S(S)$ a k-algebra, the conditions for being torsion-free rank one yield the following conditions on $M = \mathcal{F}(U)$, where $U \cong \text{Spec } R[x,y]/xy$ is an affine open of X_S:

1) M is an $R[x,y]/xy$-module of finite presentation,
2) M is R-flat,
3) $M_x \oplus M_y = M \otimes_{R[x,y]/xy} (R[x,x^{-1}] \times R[y,y^{-1}])$ is locally free of rank 1 over $R[x,x^{-1}] \times R[y,y^{-1}]$,
4) the map $M \to M_x \oplus M_y$ is universally injective, i.e. for any R-algebra R', $M \otimes_R R' \to (M \otimes_R R')_x \oplus (M \otimes_R R')_y$ is injective.

We call an $R[x,y]/xy$-module satisfying the above conditions *torsion-free rank one.*

Remark 1.2.1. Notice that the kernel of the map in condition 4) is simply given by $M[x^\infty, y^\infty]$, the submodule of elements annihilated by some power of x and some power of y. However, the condition $M[x^\infty, y^\infty] = 0$ turns out to be equivalent to the condition $M[x,y] = 0$. One direction of the equivalence is obvious since $M[x,y] \subset M[x^\infty, y^\infty]$; suppose now that $M[x,y] = 0$. Then if $m \in M[x^\infty, y^\infty]$, there are $l, k \in \mathbb{Z}_{\geq 1}$ such that $x^l m = 0$ and $y^k m = 0$. Then, if $l > 1$, $x^{l-1}m$ is annihilated by both x and y (because $yx = 0$) and is therefore zero. Reiterating this argument, it is clear that $xm = ym = 0$ and hence $m \in M[x,y] = 0$. Hence condition 4) can be stated as $(M \otimes_R R')[x,y] = 0$ for all R-algebras R'.

We continue to express condition 4) on universal injectiveness in yet different terms. To do so we may again assume $S = \operatorname{Spec} R$ affine, and restrict to a neighbourhood of the singular section of X_S isomorphic to Spec $R[x,y]/xy$. We have the following lemma.

Lemma 1.2.2. *Let R be a ring and $A = R[x,y]/xy$. Let M be an A-module of finite presentation. Then the following are equivalent* *i) M is R-flat and the morphism of A-modules $M \to M_x \oplus M_y$ is injective and remains so when base changed via any morphism $R \to R'$;*

ii) $M \xrightarrow{x-y} M$ is injective, $M/(x-y)M$ is R-flat and $M_x \oplus M_y$ is R-flat.

Proof.

$i) \Rightarrow ii)$ As pointed out before, the kernel of the map $M_{R'} \to M_{R',x} \oplus M_{R',y}$ is $M_{R'}[x,y]$. Clearly $xM_{R'} \cap yM_{R'}$ is contained in such kernel, and is therefore zero. Suppose now an element $m \in M_{R'}$ is annihilated by $x-y$. Then $xm = ym \in xM_{R'} \cap yM_{R'} = 0$. Therefore $m \in M_{R'}[x,y] = 0$, so $m = 0$ by Remark 1.2.1. This shows that multiplication by $x-y$ on M is universally injective. Then, for every R-algebra R', the exact sequence

$$0 \to M \xrightarrow{x-y} M \to M/(x-y)M \to 0$$

remains exact when the functor $R' \otimes_R \_$ is applied. Since M is R-flat, $\operatorname{Tor}^1_R(M, R') = 0$; then also $\operatorname{Tor}^1_R(M/(x-y)M, R') = 0$ for all $R \to R'$. This implies that for all ideals $I \subset R$ we have $\operatorname{Tor}^1_R(M/(x-y)M, R/I) = 0$. Then $M/(x-y)M$ is R-flat [11, TAG 00M5].

Finally, since M is R-flat and the map $A \to A_x \times A_y$ is flat, the module $M_x \oplus M_y = M \otimes_A (A_x \times A_y)$ is R-flat.

$ii) \Rightarrow i)$ If $m \in \ker(M \to M_x \oplus M_y)$, then for some $l, k \in \mathbb{Z}_{\geq 1}$ $x^l m = y^k m = 0$. Then, if $l > 1$ $(x-y)x^{l-1}m = 0$, so $x^{l-1}m = 0$, and reiterating the argument one finds $xm = ym = 0$. Then $(x-y)m = 0$ and therefore $m = 0$. Hence $\ker(M \to M_x \oplus M_y) = 0$. Now, multiplication by $x-y$ is universally injective, because the cokernel $M/(x-y)M$ is flat. Therefore the argument above holds also when we base change via any $R \to R'$.

It remains to show that M is R-flat. We have an exact sequence

$$0 \to (x-y)M/(x-y)^2M \to M/(x-y)^2M \to M/(x-y)M \to 0.$$

Since $x-y$ is not a zero-divisor in M, $(x-y)M/(x-y)^2M$ is isomorphic to $M/(x-y)M$ and is therefore R-flat. Then also $M/(x-y)^2M$ is flat, and so are all the $M/(x-y)^nM$ for all $n \geq 1$.

Now, M is flat if and only if, for all prime ideals $\mathfrak{p} \subset R$, $M_{\mathfrak{p}+(x,y)}$ is $R_{\mathfrak{p}}$-flat. Indeed, the fact that $M_x \oplus M_y$ is R-flat assures that M is R-flat outside the singular locus. So fix $\mathfrak{p} \subset R$ and let $A' = A_{\mathfrak{p}+(x,y)}$, $M' = M_{\mathfrak{p}+(x,y)}$. By [11, TAG 00HD], M' is $R_{\mathfrak{p}}$-flat if for all finitely generated ideals $I \subset R_{\mathfrak{p}}$, the canonical map $I \otimes_{R_{\mathfrak{p}}} M' \to M'$ is injective . We have a diagram

$$\begin{array}{ccc} I \otimes_{R_{\mathfrak{p}}} M' & \longrightarrow & M' \\ \downarrow{\scriptstyle \varphi_n} & & \downarrow \\ I \otimes_{R_{\mathfrak{p}}} (M'/(x-y)^n M') & \hookrightarrow & M'/(x-y)^n M' \end{array}$$

where injectivity of the lower map follows from $R_{\mathfrak{p}}$-flatness of $M'/(x-y)^n M' \cong (M/(x-y)^n M) \otimes_R R_{\mathfrak{p}}$. If for $a \in I \otimes_{R_{\mathfrak{p}}} M'$ there is an n such that $\varphi_n(a) \neq 0$, then the image of a in $M'/(x-y)^n M'$ is not zero, and therefore a is not in the kernel of $I \otimes_{R_{\mathfrak{p}}} M' \to M'$. Then we just need to show that $\bigcap_{n\in\mathbb{Z}} \ker \varphi_n = 0$. Since

$$\begin{aligned} I \otimes_{R_{\mathfrak{p}}} (M'/(x-y)^n M') &= I \otimes_{R_{\mathfrak{p}}} (M' \otimes_{A'} (A'/(x-y)^n A')) = \\ &= (I \otimes_{R_{\mathfrak{p}}} M') \otimes_{A'} (A'/(x-y)^n A') = \\ &= (I \otimes_{R_{\mathfrak{p}}} M')/(x-y)^n (I \otimes_{R_{\mathfrak{p}}} M') \end{aligned}$$

we find that $\ker \varphi_n = (x-y)^n (I \otimes_{R_{\mathfrak{p}}} M')$. So the thesis becomes

$$\bigcap (x-y)^n (I \otimes_{R_{\mathfrak{p}}} M') = 0.$$

This is a consequence of Artin-Rees lemma when R (and hence A) is a noetherian ring. Indeed, in this case, suppose that the intersection above is $N \neq 0$. Call $P := (I \otimes_{R_{\mathfrak{p}}} M')$. By Artin-Rees lemma, for some $n \geq 0$ we have $((x-y)^{n+1}P) \cap N = (x-y)(((x-y)^n P) \cap N)$, that is, $N = (x-y)N$. The ideal $(x-y)$ is contained in the maximal ideal of the local ring A', hence by Nakayama's lemma we find $N = 0$.

However, in our case R is not necessarily noetherian. Yet, using the finite presentation condition on M we may reduce to the noetherian case as follows: since A is of finite presentation over R and M is an A-module of finite presentation, there exist a directed set $(\Lambda, \leq)$ and a system of rings $\{R_\lambda\}_{\lambda\in\Lambda}$ and $A_\lambda := R_\lambda[x,y]/xy$-modules M_λ, satisfying the following conditions:

- $\mathrm{colim}_\Lambda R_\lambda = R$
- $\mathrm{colim}_\Lambda M_\lambda = M$
- each R_λ is noetherian;
- each M_λ is finite over A_λ;
- for every $\lambda \leq \mu$, $M_\lambda \otimes_{R_\lambda} R_\mu = M_\mu$ and in particular $M_\lambda \otimes_{R_\lambda} R = M$

(for a proof, see [11, TAG 00R1]).

Because $M/(x-y)M$ and $M_x \oplus M_y$ are flat R-modules, there exists λ such that for all $\mu \geq \lambda$ both $M_\mu/(x-y)M_\mu$ and $M_{\mu,x} \oplus M_{\mu,y}$ are flat R_μ-modules [11, TAG 00r6]. Take then the kernel K of $M_\mu \xrightarrow{x-y} M_\mu$. It is a finite R_μ-module (because R_μ is noetherian and M_μ finite). All of its finitely many generators must vanish in some $M_{\mu'}$ with $\mu' \geq \mu$ because $M \xrightarrow{x-y} M$ is injective. For such μ' we can apply what seen above and conclude that M_λ is flat for all $\lambda \geq \mu'$. Therefore M itself is a flat R-module, being the colimit of flat modules.

□

We are now interested in obtaining a more concrete understanding of how torsion-free rank one sheaves look at the singular locus, that is, on the closed subscheme corresponding to the section $S \to X_S$ obtained by base changing the inclusion of the singular points Spec $k \to X$. On an affine patch of X_S isomorphic to Spec $R[x,y]/xy$, the inclusion of the singular locus corresponds then to the surjective map of R-algebras $R[x,y]/xy \to R$ sending x and y to zero.

We first look at the most simple case, that is, when the base is a field. We have the following result.

Lemma 1.2.3. *Let M be a TFR1 (torsion-free rank one) module over $k[x,y]/xy$. Then $M/(x-y)M$ is of dimension 2 over k, and $M/(x,y)M$ is of dimension 1 or 2.*

Proof. We localize at (x,y) the injective map $M \to M_x \oplus M_y$. The localization of M_x at the ideal generated by x is isomorphic to $M \otimes_A k(x)$ (where $A = k[x,y]/xy$). Since M_x is locally-free of rank 1 over A_x, it follows that $M \otimes_A k(x)$ is isomorphic to $k(x)$. Hence, we obtain an injection $M_{(x,y)} \to k(x) \times k(y)$ of $A_{(x,y)}$-modules. Let's look for generators of $M_{(x,y)}$ as an $A_{(x,y)}$-module by identifying $M_{(x,y)}$ with its isomorphic image N in $k(x) \times k(y)$. The image of N in $k(x)$ is a $k[x]_{(x)}$-submodule of $k(x)$. It cannot be the whole $k(x)$, for M is finite over A. Therefore, the image of N in $k(x)$ is of the form $x^n k[x]_{(x)}$ for some $n \in \mathbb{Z}$. Similarly the image of N in $k(y)$ is of the form $y^m k[y]_{(y)}$ for some $m \in \mathbb{Z}$. So there are two elements in $N \subset k(x) \times k(y)$, $\alpha = (x^n, uy^k)$ and $\beta = (x^j, vy^m)$, with $j,k \in \mathbb{Z}$, $j \geq n$, $k \geq m$ and u,v units in $k[y]_{(y)}$.

Suppose that $k > m$. Then multiplying β by $\frac{u}{v}y^{k-m}$ we get $(0, uy^k) \in N$, and therefore also $(x^n, 0) = \alpha - (0, uy^k) \in N$. Since $j \geq n$, it's clear that also $(0, y^m) \in N$. Then $(x^n, 0)$ and $(0, y^m)$ generate N. Clearly, $(x-y)N$ is generated by $(x^{n+1}, 0)$ and $(0, y^{m+1})$ and it follows that $M_{(x,y)}/(x-y)M_{(x,y)} \cong N/(x-y)N$ is of dimension 2 over k.

Let's treat the case where $k = m$ and $j = n$ instead. If $u - v$ is a unit in $k[y]_{(y)}$, then since $\alpha - \beta \in N$ we are again in the case $(x^n, 0), (0, y^m) \in N$. So we can assume $u - v = wy^k$ with w a unit and $k \in \mathbb{Z}_{>0}$. Then $\alpha - \beta = (0, wy^{m+k}) = \frac{w}{v}\beta y^k$. Hence α can be generated by β. It's easy to see then that any other element in N can be generated by β in a similar way, so $N = \beta A_{(x,y)}$ and $M_{(x,y)}$ is free of rank 1. In this case $M_{(x,y)}/(x-y)M_{(x,y)} \cong A_{(x,y)}/(x-y)A_{(x,y)} \cong k[t]/t^2$, hence it is again of dimension 2 over k as we wanted to show.

Finally, $M/(x,y)M$ is a quotient of $M/(x-y)M$ hence it is of lower or equal dimension. It cannot be zero though, since the finite presentation condition implies that it would be zero in a neighborhood, and every open set intersects the smooth locus where M is locally free of rank 1. Hence $M/(x,y)M$ is of dimension 1 or 2. □

We can generalize easily to the case of any base scheme

Corollary 1.2.4. *Let R be a k-algebra, $A = R[x,y]/xy$ and M a TFR1 A-module. Then $M/(x-y)M$ is locally free of rank 2 over R, and $M/(x,y)M$ is locally generated by 1 or 2 elements over R.*

Proof. By lemma 1.2.2, $M/(x-y)M$ is flat over R. Since it is also of finite presentation over R, it is enough to see that it is locally generated by 2 elements. For all $\mathfrak{p} \in$ Spec

R, $M_{\mathfrak{p}}/(x-y)M_{\mathfrak{p}}$ is generated by 2 elements if it is a vector space of dimension 2 when tensored with the field $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$. This happens because of lemma 1.2.3. The same argument works for $M/(x,y)M$. □

We have now all the ingredients to state and prove an equivalent characterization of torsion-free rank one sheaves that will result most useful in the next section, where we prove that TFR1 sheaves on X yield an algebraic stack.

Proposition 1.2.5. *Let R be a ring, $A := R[x,y]/xy$, $B = R[z]$. Let $f : B \to A$ be the morphism of R-algebras given by $z \mapsto x - y$. Let also M be an A-module of finite presentation. Then the following are equivalent:*

i)
- *M is R-flat*
- *$M_x \oplus M_y = M \otimes_A (A_x \times A_y)$ is locally free of rank 1 over $A_x \oplus A_y$*
- *for all k-algebra morphisms $R \to R'$, the map $M \otimes_R R' \to (M \otimes_R R')_x \oplus (M \otimes_R R')_y$ is injective.*

ii)
- *M is locally free of rank 2 as a B-module*
- *the two R-modules $M(1,0) := M \otimes_{A,\varphi_1} R$ and $M(0,1) := M \otimes_{A,\varphi_2} R$ are locally free of rank 1, where $\varphi_1, \varphi_2 : A \to R$ are R-algebra morphisms given by $x \mapsto 1, y \mapsto 0$ and $x \mapsto 0, y \mapsto 1$ respectively.*

Before proving the proposition, we see in the following corollary how it gives an equivalent characterization of torsion-free rank one sheaves.

Corollary 1.2.6. *Let S be a k-scheme, $X_S = X \times_{\operatorname{Spec} k} S$ the base change of X to S, and let $\mathcal{F}$ be a finite presentation $\mathcal{O}_{X_S}$-module. Let also $f : X \to \mathbb{P}^1_k$ be the morphism given on both affine patches of X by $k[z] \to k[x,y]/xy$, $z \mapsto x - y$. Denote by $f' : X_S \to \mathbb{P}^1_S$ the base change of f via $S \to \operatorname{Spec} k$. Let $P, Q : \operatorname{Spec} k \to X$ be two smooth k-points belonging to different components of X. Let $P_S, Q_S : S \to X_S$ be the base changes of P and Q via $S \to \operatorname{Spec} k$. Then $\mathcal{F}$ is torsion-free rank one if and only if the $\mathcal{O}_{\mathbb{P}^1_S}$-module $f_*\mathcal{F}$ is locally free of rank 2, and $P_S^*\mathcal{F}, Q_S^*\mathcal{F}$ are locally free of rank 1.*

The fact that this characterization involves the choice of a map $f : X_S \to \mathbb{P}^1_S$ constitutes one more reason why we chose not to adopt this as our initial definition of torsion-free rank one sheaves.

Proof.

$i) \Rightarrow ii)$ First we check that M is locally-free of rank 2 outside the singular locus, that is on the open subscheme $D(z) \subset \operatorname{Spec} B$. The map

$$B_z \to A_x \times A_y$$

is free of rank 2: indeed $A_x \cong B_z \cong A_y$ as B_z-modules. Since $M_x \times M_y$ is locally free of rank 1 over $A_x \times A_y$, it then follows that it is locally free of rank 2 over B_z.

Local freeness at $z = 0$ can be checked on stalks, being M finitely presented over A and hence over B. Thus, we want to argue that for all primes $\mathfrak{p} \subset R$,

$$M_{\mathfrak{p}+(x,y)} \text{ is free of rank 2 over } R[z]_{\mathfrak{p}+(z)}.$$

This amounts to show that such modules are flat and generated by 2 elements. For the flatness part, we can apply the flatness criterion for fibres [11, TAG 00R7]: it tells us that it is enough to check that $M_{\mathfrak{p}+(x,y)}/\mathfrak{p}M_{\mathfrak{p}+(x,y)}$ is flat over $R[z]_{\mathfrak{p}+(z)}/\mathfrak{p}R[z]_{\mathfrak{p}+(z)} \cong k(\mathfrak{p})[z]_{(z)}$. This latter ring is a principal ideal domain, so in this case flatness is equivalent to torsion-freeness, which is easily checked: by Lemma 1.2.2 multiplication by $z = x - y$ is universally injective, hence the map

$$M_{\mathfrak{p}+(x,y)}/\mathfrak{p}M_{\mathfrak{p}+(x,y)} \xrightarrow{z} M_{\mathfrak{p}+(x,y)}/\mathfrak{p}M_{\mathfrak{p}+(x,y)}$$

is injective and the module is torsion-free, as we wanted to show.

To see that $M_{\mathfrak{p}+(x,y)}$ is generated by 2 elements over $R_{\mathfrak{p}}[z]_{\mathfrak{p}+(z)}$, observe that Lemma 1.2.3 assures that $M_{\mathfrak{p}+(x,y)}/\mathfrak{p} + (x-y)M_{\mathfrak{p}+(x,y)}$ has dimension 2 over $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$. Then by Nakayama's lemma we conclude.

Finally, the fact that the two modules $M(1,0)$ and $M(0,1)$ are locally free of rank 1 over R follows trivially from the assumption that $M_x \oplus M_y$ is locally free of rank 1 over $A_x \times A_y$.

ii) $\Rightarrow$ *i)* The first condition of *i)* follows immediately from flatness of the map $R \to R[z]$.

Next, local freeness of M over $R[z]$ implies that $M \xrightarrow{x-y} M$ is injective. Also, because M is flat over $R[z]$, we deduce that $M/(x-y)M = M/zM$ is flat over $R \cong R[z]/zR[z]$. This, by Lemma 1.2.2, implies the third condition of *i)*.
It remains to show that $M_x \oplus M_y$ is locally free of rank 1 over $A_x \times A_y$. The B_z-module

$${}_{B_z}(M_x \oplus M_y) \cong {}_{B_z}M_x \oplus {}_{B_z}M_y$$

is locally free of rank 2, hence both ${}_{B_z}M_x$ and ${}_{B_z}M_y$ are projective; being also finitely presented, they are locally free B_z-modules. Since the map $B_z \to A_x \times A_y$ induces isomorphisms $B_z \cong A_x \cong A_y$, it is now enough to show that M_x and M_y are of rank 1 to prove that $M_x \oplus M_y$ is locally free of rank 1 over $A_x \times A_y$. We show this for M_x: the rank of a locally free module is locally constant, and M_x is by hypotesis of rank 1 on the closed subscheme given by the section $P : \operatorname{Spec} R \to \operatorname{Spec} A_x$. Since every fiber of the structure morphism $\operatorname{Spec} A_x \to \operatorname{Spec} R$ intersects the closed subscheme P, it is enough to check that such fibers are connected. This is trivially true since the fibers are isomorphic to Spec $K[x, x^{-1}]$ (for some field K), which is the spectrum of a domain.

□

Finally, we would like to have a complete description of torsion-free rank one sheaves on the curve X over k. In order to achieve this, we first refer to [3] and give a practical corollary of [3, Theorem 2.2] that allows us to describe a torsion-free rank one module in terms of its restrictions to the two components Y and Z of X.

Corollary 1.2.7. *Let M be a torsion-free rank one module on $k[x,y]/xy$. Then*

$$M = M/xM \times_{M/(xy)M} M/yM$$

Proof. By [3, Theorem 2.2], there is a canonical surjective morphism of $k[x,y]/xy$-modules $M \to M/xM \times_{M/(xy)M} M/yM$, whose kernel is contained in $xM \cap yM$. The latter is contained in the kernel of $M \to M_x \oplus M_y$, which is zero being M torsion-free rank 1. □

Consider now a partial normalization $n : X' \to X$ of X at one of the two singular points, say P. The scheme X' is given by two copies of $\mathbb{P}^1_k$ meeting at one point. Two distinct non-singular points of X' are mapped to P by n, while on the open subscheme $X \setminus n^{-1}(P)$, n is an isomorphism. Hence the pushforward via n of the structure sheaf of X' is isomorphic to $\mathcal{O}_X$ on $X \setminus \{P\}$, while in a neighbourhood $U \cong \operatorname{Spec} k[x,y]/xy$ of P, $n_*\mathcal{O}_{X'|U} \cong (k[x] \times k[y])^{\sim}$. The structure of $k[x,y]/xy$-module for $k[x] \times k[y]$ is simply

$$f(x,y) \cdot (g(x), h(y)) = (f(x,0)g(x), f(0,y)h(y)).$$

Notice that $n_*\mathcal{O}_{X'}$ is TFR1. Indeed, it is invertible on the smooth locus of X, and $x-y$ is not a zero divisor on $k[x] \times k[y]$. Suprisingly, it turns out that in a neighborhood of a singular point P a torsion-free rank one sheaf is necessarily either invertible or isomorphic to $n_*\mathcal{O}_{X'}$. We prove it in the following proposition.

Proposition 1.2.8. *Let $\mathcal{F}$ be a torsion-free rank 1 sheaf on X. Then $\mathcal{F}$ is either invertible, or it is the pushforward of an invertible sheaf from a normalization of X.*

Proof. Since the normalization map is an isomorphism on the smooth locus of X, where $\mathcal{F}$ is invertible, we may restrict to a neighbourhood isomorphic to $\operatorname{Spec} k[x,y]/xy$ of one of the singular points. There we have that $\mathcal{F}$ is given by a torsion-free rank one A-module M, where $A = k[x,y]/xy$. Outside the origin M is locally-free of rank 1, and up to restricting the neighbourhood we may assume that $M_x \oplus M_y \cong A_x \oplus A_y = k[x,x^{-1}] \oplus k[y,y^{-1}]$.

The injective map $M \to M_x \cong k[x,x^{-1}]$ factors via the the morphism of $k[x]$-modules

$$M/yM \to k[x,x^{-1}]$$

since yM is sent to zero in M_x. This map cannot be surjective, since M/yM is finitely generated over $k[x]$. Hence the image of M/yM in $k[x,x^{-1}]$ is of the form $x^n k[x] \subset k[x,x^{-1}]$ for some $n \in \mathbb{Z}$, and in particular it is isomorphic to $k[x]$. Therefore we have an exact sequence of $k[x]$-modules

$$0 \to (M/yM)[x^\infty] \to M/yM \to k[x] \to 0.$$

Being $k[x]$ free, the sequence splits and we find

$$M/yM \cong (M/yM)[x^\infty] \oplus k[x].$$

Let's denote by N' the $k[x]$-module $(M/yM)[x^\infty]$, by P' the $k[y]$-module $(M/xM)[y^\infty]$ and let $N = k[x] \oplus N'$, $P = k[y] \oplus P'$, so that $N \cong M/yM$ and $P \cong M/xM$. By Corollary 1.2.7, $M \cong N \times_f P$, where $f : N/xN \to P/yP$ is an isomorphism of k-vector space. So the diagram

$$\begin{array}{ccccc} M & & \longrightarrow & & P = k[y] \oplus P' \\ \downarrow & & & & \downarrow \\ N = k[x] \oplus N' & \longrightarrow & N/xN & \xrightarrow{f} & P/yP \end{array}$$

is commutative.

Since the $k[x]$-module N' is finitely generated, there exists a power of x that annihilates it, say x^k. Suppose $k \geq 2$, and take $n \in N'$. The element of M given as an element of $N \times_f P$ by $((0, x^{k-1}n), (0,0))$ is annihilated by both x and y, so it belongs to $M[x] \cap M[y] = 0$. Therefore $x^{k-1}n = 0$. Repeating this argument shows that $xN' = 0$, and similarly $yP' = 0$.

Now, by lemma 1.2.3 the k-vector space $M/(x,y)M \cong N/xN = k \oplus N'/xN' = k \oplus N' \cong k \oplus P'$ is of dimension 1 or 2. In the first case, $N' = P' = 0$, and so $M \cong k[x] \times_k k[y] \cong k[x,y]/xy$. In the second case, $N' \cong k$ and $P' \cong k$. The isomorphism $f : k \oplus N' \to k \oplus P'$ must send the first summand k to P' and N' to k, else M would be isomorphic to $k[x,y]/xy \oplus k$, which is not torsion-free rank one. Therefore we obtain $M \cong N \times_f P \cong k[x] \times k[y]$.

This shows that, in a neighbourhood of the origin, M is isomorphic to $k[x,y]/xy$ or $k[x] \times k[y]$. In the first case, $\mathcal{F}$ is invertible. In the second case, $\mathcal{F}$ is the pushforward of an invertible sheaf from the normalization of X.

□

We have defined earlier a partial normalization $X' \to X$ of the curve X at one singular point. We can also consider a morphism $\mathbb{P}^1_k \sqcup \mathbb{P}^1_k \to X$ sending each copy of $\mathbb{P}^1_k$ to a component of X; we will call this a normalization of X.

Lemma 1.2.9. *Let $\mathcal{F}$ be a torsion-free rank one sheaf on X. Then $\operatorname{End}_{\mathcal{O}_X}(\mathcal{F}) = k$ if $\mathcal{F}$ is invertible or a pushforward of an invertible sheaf from a partial normalization at only one point of X, and $\operatorname{End}_{\mathcal{O}_X}(\mathcal{F}) \cong k \oplus k$ if $\mathcal{F}$ is a pushforward of an invertible sheaf from a normalization of X.*

Proof. Let $\mathcal{F} = n_*\mathcal{L}$ be the pushforward of an invertible sheaf $\mathcal{L}$ via a partial normalization $n : X' \to X$ (where n could be a normalization at either both singular points or only one or none at all - hence the identity in the latter case). We have $\operatorname{End}_{\mathcal{O}_X}(n_*\mathcal{L}) = \Gamma(X, \mathscr{E}nd_{\mathcal{O}_X}(n_*\mathcal{L}))$. We also have a morphism of $\mathcal{O}_X$-modules

$$\varphi : n_*\mathscr{E}nd_{\mathcal{O}_{X'}}(\mathcal{L}) \to \mathscr{E}nd_{\mathcal{O}_X}(n_*\mathcal{L}).$$

We would like to check that it is an isomorphism: then, taking global sections, we would have

$$\operatorname{End}_{\mathcal{O}_{X'}}(\mathcal{L}) \cong \operatorname{End}_{\mathcal{O}_X}(n_*\mathcal{L})$$

and the LHS is isomorphic to $\mathrm{End}_{\mathcal{O}_{X'}}(\mathcal{L} \otimes \mathcal{L}^\vee) = \mathrm{End}_{\mathcal{O}_{X'}}(\mathcal{O}_{X'}) = \mathcal{O}_{X'}(X')$. This is k in the case of n being a partial normalization at one point or the identity, and is isomorphic to $k \oplus k$ in the case of n being a normalization of X, as we wished to show.

Now, letting $U \subset X'$ be the locus of X' where n is an isomorphism, of course $\varphi_{|n(U)}$ is an isomorphism. Then we just need to check that the stalk of φ at the normalized point(s) is an isomorphism.

Let $P \in X(k)$ be a point normalized by n. We have $\mathscr{E}nd_{\mathcal{O}_{X'}}(\mathcal{L}) = \mathscr{E}nd_{\mathcal{O}_{X'}}(\mathcal{O}_{X'}) = \mathcal{O}_{X'}$. As seen in Proposition 1.2.8, for every line bundle $\mathcal{M}$ on X', $(n_*\mathcal{M})_P$ is isomorphic to the $\mathcal{O}_{X,P}$-module $A := k[x]_{(x)} \oplus k[y]_{(y)}$. In particular $(n_*\mathscr{E}nd_{\mathcal{O}_{X'}}(\mathcal{L}))_P \cong A$. On the other hand, $(\mathscr{E}nd_{\mathcal{O}_X}(n_*\mathcal{L}))_P \cong \mathrm{End}_{\mathcal{O}_{X,P}}(A)$. Notice that

$$\mathrm{End}_{\frac{k[x,y]}{xy}}(k[x] \oplus k[y]) =$$
$$\mathrm{End}_{\frac{k[x,y]}{xy}}(k[x]) \oplus \mathrm{End}_{\frac{k[x,y]}{xy}}(k[y]) \oplus \mathrm{Hom}_{\frac{k[x,y]}{xy}}(k[x], k[y]) \oplus \mathrm{Hom}_{\frac{k[x,y]}{xy}}(k[y], k[x])$$

The first two terms give $k[x] \oplus k[y]$ while the second two terms are zero. Hence $\mathrm{End}_{\mathcal{O}_{X,P}}(A) = A$ and φ is an isomorphism. □

Lemma 1.2.10. *Let S be a k-scheme and $\mathcal{F}$ be a torsion-free rank one sheaf on X_S. Let $\mathcal{L}$ be an invertible sheaf on X_S. Then $\mathcal{F} \otimes_{\mathcal{O}_{X_S}} \mathcal{L}$ is torsion-free rank one.*

Proof. The sheaf $\mathcal{F} \otimes \mathcal{L}$ is S-flat and locally free on the smooth locus of X_S. On an affine cover $\{U_i\}$ of X_S where $\mathcal{L}$ is free, it is clear that $(\mathcal{F} \otimes \mathcal{L})_{|U_i} \to j_*j^*((\mathcal{F} \otimes \mathcal{L})_{|U_i})$ is universally injective. □

Chapter 2

The stack of torsion-free rank one sheaves

This chapter is devoted to the setting up of a categorical environment for the study of torsion-free rank one sheaves. We will show that they yield an algebraic stack $\mathbb{T}$. See [11, TAG 0260] for a definition of algebraic stack. So in particular there exists a scheme T and a surjective smooth morphism $T \to \mathbb{T}$. For definitions of some of the terminology appearing, we will give references to [11].

Definition 2.0.11. We denote by $\mathbb{T}$ the category whose objects are pairs $(S, \mathcal{F})$, with

i) S a k-scheme;

ii) $\mathcal{F}$ a torsion-free rank one sheaf on X_S

and whose morphisms $(S, \mathcal{F}) \to (T, \mathcal{G})$ are given by pairs (f, φ) where

- $f : S \to T$ is a morphism of k-schemes;
- $\varphi : \mathcal{G} \to \mathcal{F}$ is an f'-isomorphism of sheaves of modules, where $f' : X_S \to X_T$ is the base change of f.

Then we define a functor

$$p : \mathbb{T} \to (\mathrm{Sch}/k)_{fppf}, \quad (S, \mathcal{F}) \mapsto S, \quad (f, \varphi) \mapsto f$$

Lemma 2.0.12. *Let $f : S \to T$ be a morphism of k-schemes and $(T, \mathcal{F})$ an object of $\mathbb{T}$. Then $(S, (f')^*\mathcal{F})$ is also an object of $\mathbb{T}$.*

Proof. Conditions $1), 2), 3), 4)$ of definition 1.1.1 are stable under base change. □

The following lemma is therefore immediate. Refer to [11, TAG 003T] for the definition of category fibred in groupoids.

Lemma 2.0.13. *The functor* $p : \mathbb{T} \to (\mathrm{Sch}/k)_{fppf}$ *makes* $\mathbb{T}$ *into a category fibred in groupoids.*

Proof. We have to check that the two conditions defining a category fibred in groupoids are satisfied. For i), given $f : S \to T$ and an object $(T, \mathcal{F})$ of $\mathbb{T}$, by lemma 2.0.12 we have the desired morphism $(S, (f')^*\mathcal{F}) \to (T, \mathcal{F})$. For ii), given morphisms

$$(f_1, \varphi_1) : (S_1, \mathcal{F}_1) \to (T, \mathcal{G}),\ (f_2, \varphi_2) : (S_2, \mathcal{F}_2) \to (T, \mathcal{G})$$

and a morphism of k-schemes $g : S_1 \to S_2$, just let $\varphi : \mathcal{F}_2 \to \mathcal{F}_1$ be given by the composition $\varphi_1 \circ \varphi_2^{-1}$.

□

Let now Coh_X be the category fibred in groupoids built in analogous way as $\mathbb{T}$, but whose objects need to satisfy just conditions 1) and 2) of 1.1.1. We will assume without proof that Coh_X is an algebraic stack. For a proof of this, see [9, Theorem 4.6.2.1]

We now use the characterization of torsion-free rank one sheaves given in corollary 1.2.6 to see that $\mathbb{T}$ is an algebraic stack. The proof uses the fact that the fibred product of algebraic stacks in the $(2,1)$-category of categories, is still an algebraic stack. So let's recall what the fibred product of categories is.

Remark 2.0.14. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be categories, $F : \mathcal{A} \to \mathcal{C}$, $G : \mathcal{B} \to \mathcal{C}$ be functors. Then the category $\mathcal{A} \times_{\mathcal{C}} \mathcal{B}$ is defined as follows:

- its objects are triples (A, B, f) with $A \in \mathrm{Ob}(\mathcal{A})$, $B \in \mathrm{Ob}(\mathcal{B})$, and $f : F(A) \to G(B)$ is an isomorphism in $\mathcal{C}$,
- a morphism $(A, B, f) \to (A', B', f')$ is given by a pair (a, b), where $a : A \to A'$ is a morphism in $\mathcal{A}$, $b : B \to B'$ is a morphism in $\mathcal{B}$ and the diagram

$$\begin{array}{ccc} F(A) & \xrightarrow{f} & G(B) \\ \downarrow{\scriptstyle F(a)} & & \downarrow{\scriptstyle G(b)} \\ F(A') & \xrightarrow{f'} & G(B') \end{array}$$

  commutes.

Proposition 2.0.15. *The stack* $\mathbb{T}$ *is algebraic.*

Proof. Let $f : X \to \mathbb{P}^1_k$ be as in Corollary 1.2.6. Then we have a morphism of algebraic stacks $Coh_X \xrightarrow{f_*} Coh_{\mathbb{P}^1}$. Indeed all base changes f' of f are finite and flat, hence f'_* sends finite presentation sheaves flat over the base scheme to finite presentation sheaves flat over the base scheme. We let $LFR2_{\mathbb{P}^1}$ be the algebraic stack of locally-free, rank 2 modules on $\mathbb{P}^1$. This is a full subcategory of $Coh_{\mathbb{P}^1}$. We can take the fibred product

$$\begin{array}{ccc} LFR2_{\mathbb{P}^1} \times_{Coh_{\mathbb{P}^1}} Coh_X & \longrightarrow & Coh_X \\ \downarrow & & \downarrow f_* \\ LFR2_{\mathbb{P}^1} & \longrightarrow & Coh_{\mathbb{P}^1} \end{array}$$

If we let $\mathcal{A}$ be the stack of finite presentation $\mathcal{O}_{X_S}$-modules, flat over S, such that $f_*\mathcal{F}$ is locally free of rank 2, then there is an equivalence of categories $F : \mathcal{A} \to LFR2_{\mathbb{P}^1} \times_{Coh_{\mathbb{P}^1}} Coh_X$ given, for S a k-scheme, by

$$F(S) : \mathcal{F} \mapsto (\mathcal{F}, f_*\mathcal{F}, \mathrm{id}_{f_*\mathcal{F}})$$

and in the obvious way on morphisms. Indeed, consider the functor

$$G(S) : LFR2_{\mathbb{P}^1} \times_{Coh_{\mathbb{P}^1}} Coh_X(S) \to \mathcal{A}(S),\ (\mathcal{F}, \mathcal{G}, \sigma) \mapsto \mathcal{F}.$$

Then the composition $G(S) \circ F(S)$ is the identity on $\mathcal{A}(S)$. The functor from the fibred product category to itself sending

$$(\mathcal{F}, \mathcal{G}, \sigma) \mapsto (\mathcal{F}, \sigma\mathcal{G}, \mathrm{id})$$

constitutes a natural isomorphism between the identity and $F(S) \circ G(S)$, proving that F is an equivalence.

Similarly we can call $\mathcal{B}$ the stack of finite presentation sheaves $\mathcal{F}$ on X_S such that both $P^*\mathcal{F}$ and $Q^*\mathcal{F}$ are invertible on S, where P and Q are the base change of non-singular k-rational points on the two distinct components of X. Then we have a diagram

$$\begin{array}{ccc} LFR1_k \times LFR1_k \times_{Coh_k \times Coh_k} Coh_X & \longrightarrow & Coh_X \\ \downarrow & & \downarrow (P^*, Q^*) \\ LFR1_k \times LFR1_k & \longrightarrow & Coh_k \times Coh_k \end{array}$$

and $\mathcal{B}$ is equivalent to the fibred product category.

Finally, by Corollary 1.2.6, we have $\mathbb{T} \cong \mathcal{A} \times_{Coh_X} \mathcal{B}$. By [11, TAG 04T2] the fibred product of algebraic stacks is an algebraic stack, and by [11, TAG 03YQ] a category equivalent to an algebraic stack is an algebraic stack. Since the categories of quasi-coherent finite presentation sheaves flat over the base scheme and of locally free rank n sheaves are algebraic stacks [9, Theorem 4.6.2.1], we deduce that $\mathbb{T}$ is an algebraic stack. □

This proves that there is smooth, surjective morphism of stacks

$$S \to \mathbb{T}$$

for some k-scheme S. The next chapter will be devoted to describing one such presentation for $\mathbb{T}$.

Chapter 3

A presentation for $\mathbb{T}'$

We have managed to show that $\mathbb{T}$ is an algebraic stack, so there exists a smooth surjective morphism of stacks

$$T \to \mathbb{T}$$

from a scheme T to $\mathbb{T}$, where obviously we identify the scheme T with the stack it represents.

In this chapter we will find an explicit description of an open subscheme of such a scheme T. To do so, we will first restrict our attention to a substack $\mathbb{T}'$ of $\mathbb{T}$, consisting of torsion-free rank one sheaves on X_S satisfying the condition $\mathrm{End}(\mathcal{F}_s) = k(s)$ for all geometric points s of S. These sheaves are introduced in [2] and are called *simple*. The condition is open, and as seen in Lemma 1.2.8, it leaves out those TFR1 sheaves $\mathcal{F}$ that have some fibre $\mathcal{F}_s$ which is a pushforward of a line bundle from a normalization of the fibre X_s. Then $\mathbb{T}'$ is an open substack of $\mathbb{T}$, hence an algebraic stack, and we will find that there is a surjective and smooth morphism $T' \to \mathbb{T}'$ with T' a scheme that can be covered by copies of the original curve, X, via open immersions.

3.1 k-points of $\mathbb{T}$

First we will try to have an intuitive picture of the k-points of $\mathbb{T}'$, that is, the torsion-free rank one simple sheaves on X. We start by looking at invertible sheaves on X.

Remark 3.1.1. By [5, Prop. 3.18], that develops [3, Thm 2.2], given a co-cartesian diagram of schemes

$$\begin{array}{ccc} W & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Z & \longrightarrow & X \end{array}$$

where the horizontal maps are closed immersions and the vertical maps are affine, there is an equivalence of categories

$$\mathcal{C}_X \cong \mathcal{C}_Y \times_{\mathcal{C}_W} \mathcal{C}_Z$$

where $\mathcal{C}$ denotes the category of invertible sheaves on the scheme.

This remark allows us to compute easily the Picard group of X.

Lemma 3.1.2. *The Picard group* $\mathrm{Pic}(X)$ *is isomorphic to* $\mathbb{Z} \times \mathbb{Z} \times k^{\times}$.

Proof. We can compute $\mathrm{Pic}(X) \cong H^1(X, \mathcal{O}_X^{\times})$ via Cech cohomology. Consider the usual cover $\{\mathcal{U}\}$ by two affine U, V of X. We have $\mathrm{Pic}(U) = \mathrm{Pic}(V) = 0$. Indeed, a locally free rank 1 module on $k[x,y]/xy$ is, by remark 3.1.1, the datum of two locally free rank 1 modules on $k[x]$ and $k[y]$ and of an isomorphism between their fibres at $x = 0$ and $y = 0$. But $\mathrm{Pic}(\mathrm{Spec}\, k[x]) = \mathrm{Pic}(\mathrm{Spec}\, k[y]) = 0$, hence $\mathrm{Pic}(\mathrm{Spec}\, k[x,y]/xy) = 0$. Therefore we have $H^1(X, \mathcal{O}_X^{\times}) = H^1(\{\mathcal{U}\}, \mathcal{O}_X^{\times})$, which is the cokernel of the map

$$\mathcal{O}^{\times}(U) \times \mathcal{O}^{\times}(V) \xrightarrow{(f,g) \mapsto f/g} \mathcal{O}^{\times}(U \cap V).$$

We have $\mathcal{O}^{\times}(U) \cong (k[x,y]/xy)^{\times} \cong k^{\times}$, $\mathcal{O}^{\times}(V) \cong (k[u,v]/uv)^{\times} \cong k^{\times}$, while $\mathcal{O}^{\times}(U \cap V) \cong (k[x,x^{-1}] \times k[y,y^{-1}])^{\times} \cong k^{\times} \times \langle x \rangle \times k^{\times} \times \langle y \rangle$. The map sends $(a,b) \in k^{\times} \times k^{\times}$ to a/b, hence the cokernel is isomorphic to $k^{\times} \times \langle x \rangle \times \langle y \rangle \cong k^{\times} \times \mathbb{Z} \times \mathbb{Z}$.

□

Let's now find an explicit group isomorphism $\mathbb{Z} \times \mathbb{Z} \times k^{\times} \to \mathrm{Pic}\, X$. Notice that by Remark 3.1.1 an invertible sheaf on X is the datum of two invertible sheaves $\mathcal{L}_Y$, $\mathcal{L}_Z$ on Y and Z and two isomorphisms $\sigma_1 : \mathcal{L}_Y \otimes k(P_1) \to \mathcal{L}_Z \otimes k(P_1)$ and $\sigma_2 : \mathcal{L}_Y \otimes k(P_2) \to \mathcal{L}_Z \otimes k(P_2)$ where P_1, P_2 are the singular points of X. Hence an invertible sheaf can be expressed in the form $(\mathcal{L}_Y, \mathcal{L}_Z, \sigma_1, \sigma_2)$.

Now, we give a system of coordinates to the projective lines Y and Z, so that the points ∞_Y and ∞_Z correspond to a singular point on X, and the points 0_Y and 0_Z correspond to the other singular point. We also let $P = 1_Y$ and $Q = 1_Z$. If D_Y and D_Z are divisors on Y and Z, the invertible sheaves $\mathcal{O}(D_Y)$ and $\mathcal{O}(D_Z)$ have fibres canonically isomorphic to k at 0 and ∞. Then an isomorphism between such fibres is simply given by an element of $k^{\times}$.

Lemma 3.1.3. *The group homomorphism*

$$\begin{aligned} \mathbb{Z} \times \mathbb{Z} \times k^{\times} &\to \mathrm{Pic}\, X \\ (m, n, \alpha) &\mapsto (\mathcal{O}(mP), \mathcal{O}(nQ), 1, \alpha) \end{aligned}$$

is an isomorphism.

Proof. We first show that the map is injective. If $(\mathcal{O}(mP), \mathcal{O}(nQ), 1, \alpha)$ is trivial, its restriction to each component is trivial, and therefore $m = n = 0$. Now, by remark 3.1.1, an isomorphism $(\mathcal{O}, \mathcal{O}, 1, 1) \to (\mathcal{O}, \mathcal{O}, 1, \alpha)$ is given by isomorphisms $f_Y : \mathcal{O}_Y \to \mathcal{O}_Y$ and $f_Z : \mathcal{O}_Z \to \mathcal{O}_Z$ of the structure sheaves of each component of X, inducing the identity isomorphism at one singular point and α at the other singular point. But f_Y and f_Z are global sections of Y and Z, hence constants, so $\alpha = 1$.

Let's prove that the homomorphism is surjective. Let $\mathcal{L} = (\mathcal{L}_Y, \mathcal{L}_Z, \sigma_1, \sigma_2) \in \operatorname{Pic} X$. Then $\mathcal{L}_Y \cong \mathcal{O}(mP)$ and $\mathcal{L}_Z \cong \mathcal{O}(nQ)$ for some $m, n \in \mathbb{Z}$. These two isomorphism induce new isomorphisms between the fibres at 0 and ∞, which are now elements of $k^\times$. Hence $\mathcal{L} \cong (\mathcal{O}(mP), \mathcal{O}(nQ), \alpha, \beta)$ for some $\alpha, \beta \in k^\times$. Now, the map

$$\mathcal{O}_X = (\mathcal{O}_Y, \mathcal{O}_Z, 1, 1) \to (\mathcal{O}_Y, \mathcal{O}_Z, \alpha, \alpha)$$

is an isomorphism, given by multiplying the sheaf $\mathcal{O}_Z$ by the constant α. Therefore

$$\mathcal{L} \cong (\mathcal{O}(mP), \mathcal{O}(nQ), \alpha, \beta) \otimes (\mathcal{O}_Y, \mathcal{O}_Z, \alpha^{-1}, \alpha^{-1}) \cong (\mathcal{O}(mP), \mathcal{O}(nQ), 1, \beta/\alpha)$$

and the thesis follows. □

Then, since $\mathbb{Z} \times \mathbb{Z} \times k^\times \cong \operatorname{Pic} Y \times \operatorname{Pic} Z \times k^\times$, we can from now on express an invertible sheaf on X as $(\mathcal{L}_Y, \mathcal{L}_Z, \alpha)$, with $\mathcal{L}_Y \in \operatorname{Pic} Y$, $\mathcal{L}_Z \in \operatorname{Pic} Z$ and $\alpha \in k^\times$.

Let now σ be the unique automorphism of X fixing the two singular points and sending P to Q. With our system of coordinates on the components, for a point R on X we can of course define the point $1/R$ as the one with inverted coordinates on the same component. Then we claim that for any $\mathcal{L}_1 \in \operatorname{Pic} Y$, $\mathcal{L}_2 \in \operatorname{Pic} Z$, $\alpha \in k^\times$ and R on the smooth locus of X, there is an isomorphism

$$(\mathcal{L}_1(P - R), \mathcal{L}_2, \alpha) \cong (\mathcal{L}_1, \mathcal{L}_2(Q - \sigma(1/R)), \alpha)$$

where, for a divisor D, $\mathcal{L}(D)$ is short for $\mathcal{L} \otimes \mathcal{O}(D)$. We identify Y with $\operatorname{Proj} k[x, y]$ and Z with $\operatorname{Proj} k[u, v]$ so that the affine ring of a neighborhood of a singular point is $k[\frac{x}{y}, \frac{u}{v}]/\frac{x}{y}\frac{u}{v}$. Suppose that R is the point $x = a, y = 1$ on Y, for some $a \in k^\times$. Then multiplication by the global function given by

$$\frac{x - ay}{x - y} \text{ on } Y \text{ and } \frac{u - v}{u - \frac{1}{a}v} \text{ on } Z$$

yields the isomorphism.

Now, on a heuristic level, as the point R "approaches the singular point x_0" the sheaf $(\mathcal{L}_1(P - R), \mathcal{L}_2, \alpha)$ tends to the torsion-free rank one sheaf

$$\mathcal{F}_0 = (\mathcal{L}_1(P), \mathcal{L}_2, \alpha) \otimes I(x_0)$$

where $I(x_0)$ is the ideal sheaf of the singular point x_0. On the other hand, as R approaches x_0, $\sigma(1/R)$ approaches the other singular point x_1. Yet the sheaf $(\mathcal{L}_1, \mathcal{L}_2(Q - \sigma(1/R)), \alpha)$ tends to

$$\mathcal{F}_1 = (\mathcal{L}_1, \mathcal{L}_2(Q), \alpha) \otimes I(x_1)$$

which is not isomorphic to $\mathcal{F}_0$, although it arises from the same passage to the limit! This strongly suggests that any scheme parametrizing torsion-free rank one sheaves is not separated.

Actually we have not shown yet that the ideal sheaf of a point is torsion-free rank one, but this is rather easy.

Lemma 3.1.4. *Let k be an algebraically closed field, $x \in X(k)$. Let $\mathfrak{m}_x$ be the ideal sheaf of the point x. Then m_x is torsion-free, rank 1.*

Proof. Suppose x belongs to an open affine $U \cong \operatorname{Spec} k[x,y]/xy$. We can assume that on U, $\mathfrak{m}_x$ is given by the ideal $I = (x-a, y)k[x,y]/xy$ for some $a \in k$. Then multiplication by $x - y$ is injective on I (since it is on $k[x,y]/xy$). Moreover, on the smooth locus we have $I_x \oplus I_y \cong (x-a)k[x,x^{-1}] \oplus k[y,y^{-1}]$, which is a free $k[x,x^{-1}] \times k[y,y^{-1}]$-module of rank 1.

□

In the next proposition, we create a bijection between the set of k-points of X and a particular type of torsion-free rank one sheaves on X.

Proposition 3.1.5. *Let k an algebraically closed field, $P, Q \in X(k)$ belonging to the smooth locus of X and lying on different components of X. Let also σ be the unique automorphism of X such that $\sigma(P) = Q$; let finally $\mathcal{O}(P+Q)$ be the invertible sheaf associated to the divisor $P+Q$, and $\mathfrak{m}_{\sigma(x)}$ be the ideal sheaf of the point $\sigma(x)$. We define a map*

$$\varphi : \{k\text{-rational points of } X\} \to \{\text{Torsion-free rank 1 sheaves on } X\}/\cong$$

given by

$$x \mapsto \mathcal{O}(P+Q) \otimes \mathfrak{m}_{\sigma(x)}.$$

Then φ is injective, and its image consists of those TFR1 sheaves $\mathcal{F}$ on X such that

i) $H^0(X, \mathcal{F}) \cong k$,

ii) $H^1(X, \mathcal{F}) = 0$,

iii) the support of the sheaf $\mathcal{F}/\mathcal{O}_X t$ is finite, where t is any non-zero global section of $\mathcal{F}$.

Proof. Let's show that the space of global sections of the sheaf $\mathcal{O}(P+Q) \otimes \mathfrak{m}_{\sigma(x)}$ has dimension 1. Call X_1 the component of X containing P, and X_2 the one containing Q. We can assume without loss of generality that $\sigma(x)$ belongs to X_1. Let t be a global section of the sheaf. Then t, seen as a rational function on X, takes the value zero at $\sigma(x)$. If t is constantly zero on one of the components, it is zero at both singular points. Then it is either zero or has two poles on the other components, but only one pole - P or Q - is allowed. Hence in this case $t = 0$. If t is not constantly zero on any component, t is non-constant on X_1. Hence it has a pole at P. The fact of having exactly one pole at P and one zero at $\sigma(x)$ determines t up to constant on X_1. For any choice of the constant, t takes two distinct values at the singular points. Hence it is non-constant also on X_2, so it has a pole at Q, and this determines it on X_2. Therefore $H^0(X, \mathcal{O}(P+Q) \otimes m_{\sigma(x)}) \cong k$.

To see that $H^1(X, \mathcal{O}(P+Q) \otimes m_{\sigma(x)}) = 0$, consider the exact sequence of sheaves

$$0 \to \mathfrak{m}_{\sigma(x)} \to \mathcal{O}_X \to x_* k \to 0.$$

Taking the tensor product with $\mathcal{O}(P+Q)$ yields

$$0 \to \mathcal{O}(P+Q) \otimes \mathfrak{m}_{\sigma(x)} \to \mathcal{O}(P+Q) \to x_* k \to 0$$

which gives a long exact sequence of cohomology. The map $H^0(X, \mathcal{O}(P+Q)) \to H^0(X, x_* k))$ is surjective, since $1 \in H^0(X, \mathcal{O}(P+Q))$, so we end up with an exact sequence

$$0 \to H^1(X, \mathcal{O}(P+Q) \otimes \mathfrak{m}_{\sigma(x)}) \to H^1(X, \mathcal{O}(P+Q)) \to H^1(X, x_* k) = 0.$$

Hence $\dim_k H^1(X, \mathcal{O}(P+Q) \otimes \mathfrak{m}_{\sigma(x)}) = \dim_k H^1(X, \mathcal{O}(P+Q))$. By Riemann Roch, $\chi(\mathcal{O}(P+Q)) = 2+1-1 = 2$ since the genus of X is 1. Taking dimensions in the short exact sequence

$$0 \to H^0(X, \mathcal{O}(P+Q) \otimes \mathfrak{m}_{\sigma(x)}) \to H^0(X, \mathcal{O}(P+Q)) \to H^0(X, x_* k) = 0$$

we see that $H^0(X, \mathcal{O}(P+Q)$ has dimension 2. It follows that $\dim_k H^1(\mathcal{O}(P+Q)) = \dim_k H^0(\mathcal{O}(P+Q) - \chi(\mathcal{O}(P+Q)) = 0$ as we wanted to show.

Now, choose a non-zero global section t of $\mathcal{O}(P+Q) \otimes \mathfrak{m}_{\sigma(x)}$. Then, t satisfies $t(y) = t(\sigma(y))$ for all $y \in X$. Indeed, this is true when y is either P or one of the two singular points, hence it is true for all $y \in Y$. Therefore t has only x and $\sigma(x)$ (which can coincide, if x is singular) as zeroes. We want to look at the support of the cokernel of the map

$$\mathcal{O}_X \xrightarrow{t} \mathcal{O}_X(P+Q) \otimes_{\mathcal{O}_X} \mathfrak{m}_{\sigma(x)}.$$

If x is non-singular, the support consists only of the point x. If $x = \sigma(x)$ is singular, restrict to an open neighbourhood $U \cong \operatorname{Spec} k[u,v]/uv$ of x. There, by symmetry t is of the form $u+v$. Hence it does not generate the maximal ideal (u,v) at x, and the quotient $(u,v)/(u+v)$ is a k-vector space of dimension 1. So again we get $x_* k$ as cokernel. This shows in particular that $\mathcal{O}_X \xrightarrow{t} \mathcal{O}_X(P+Q) \otimes_{\mathcal{O}_X} \mathfrak{m}_{\sigma(x)}$ has finite support.

What we want to show next is that if $\mathcal{F}$ is a torsion-free rank 1 sheaf on X satisfying the conditions in the statement, then

$$\mathcal{F} \cong \mathcal{O}(P+Q) \otimes_{\mathcal{O}_X} \mathfrak{m}_{\sigma(x)} \text{ for some } x \in X(k).$$

Suppose first that $\mathcal{F}$ is a line bundle, given by $(\mathcal{O}(n), \mathcal{O}(m), \alpha)$ for some $\alpha \in k^\times$, $n, m \in \mathbb{Z}$. If one between n and m is negative, then a global section t of $\mathcal{F}$ is zero on a whole irreducible component of X, and the support of $\mathcal{F}/\mathcal{O}_X t$ is not finite. So we necessarily have $n, m \geq 0$. By Riemann-Roch, we have $\dim_k H^0(X, \mathcal{F}) - \dim_k H^1(X, \mathcal{F}) = m+n+1-g = m+n$. Hence $m+n = 1$, from which we conclude that $\mathcal{F}$ is either $(\mathcal{O}, \mathcal{O}(1), \alpha)$ or $(\mathcal{O}(1), \mathcal{O}, \alpha)$ for some $\alpha \in k^\times$.

Suppose without loss of generality that $\mathcal{F} \cong (\mathcal{O}, \mathcal{O}(1), \alpha)$. Such a line bundle is isomorphic to $(\mathcal{O}, \mathcal{O}(Q), \beta) = (\mathcal{O}(P) \otimes \mathcal{O}(-P), \mathcal{O}(Q), \beta)$ for some $\beta \in k^\times$. Now, we can find a rational function f on the component containing P that has value 1 at a singular point, β at the other, and a pole of order 1 at P. Then f has exactly one zero of order 1 at some point x on the smooth locus of the component. Therefore, multiplication by f on the component containing P induces an isomorphism of $\mathcal{F}$ with

$(\mathcal{O}(P)\otimes\mathcal{O}(-x),\mathcal{O}(Q),1_k)\cong\mathcal{O}(P+Q)\otimes\mathfrak{m}_x$ for some $x\in X(k)$, as we wanted to show.

Finally we have to treat the case where $\mathcal{F}$ is a torsion-free, rank 1 sheaf which is not a line bundle. Then, by Proposition 1.2.8 $\mathcal{F}$ is the pushforward of a line bundle from a (possibly partial) normalization of X. It cannot come from a normalization $X''\cong\mathbb{P}^1_k\sqcup\mathbb{P}^1_k\xrightarrow{n}X$ though: suppose by contradiction that $\mathcal{F}=n_*\mathcal{L}$ with $\mathcal{L}$ an invertible sheaf on the normalization X''. Then $\mathcal{L}$ is of the form $(\mathcal{O}(d_1),\mathcal{O}(d_2))$ for some $d_1,d_2\in\mathbb{Z}$. Since n is affine, we have $H^i(X'',\mathcal{L})\cong H^i(X,n_*\mathcal{L})$ for all $i\geq 0$. Then we have $1=\chi(\mathcal{L})=\chi(\mathcal{O}(d_1))+\chi(\mathcal{O}(d_2))=1+d_1+1+d_2$ by Riemann-Roch, so that $d_1+d_2=-1$. Then one between d_1 and d_2 is negative, so that on one of the two components of X, $n_*\mathcal{L}$ has no non-zero global sections. This contradicts the condition on finite support.

This shows that $\mathcal{F}$ is the pushforward of an invertible sheaf $\mathcal{L}=(\mathcal{O}(m),\mathcal{O}(n),\alpha)$ on a partial normalization of $X'\xrightarrow{n}X$ of X at one point. Since n is affine, we have $H^0(X',\mathcal{L})=H^0(X,n_*\mathcal{L})=1$ and $H^1(X',\mathcal{L})=H^1(X,n_*\mathcal{L})=0$. We can apply Riemann-Roch on X' which has genus zero, and get

$$\chi(L)=n+m+1,$$

from which we deduce that $n+m=0$. If m or n is less than zero, then any global section of $\mathcal{L}$ is constantly zero on one of the components, so the same is true for global sections of $n_*\mathcal{L}$, and condition iii) on finite support is not satisfied. Therefore $n=m=0$ and we conclude that $\mathcal{F}$ is the pushforward of the structure sheaf from one of the two partial normalizations X' of X.

To see that such a sheaf is of the form $\mathcal{O}(P+Q)\otimes m_x$ with x a singular point, we can give an explicit isomorphism $\mathcal{F}:=n_*\mathcal{O}_{X'}\to\mathcal{G}:=\mathcal{O}(P+Q)\otimes\mathfrak{m}_{\sigma(x)}$: take U and V to be the usual open affine subschemes isomorphic to $\operatorname{Spec}k[x,y]/xy$ and $\operatorname{Spec}k[u,v]/uv$. On the intersection $U\cap V$, the two rings glue via $x\mapsto u^{-1}$ and $y\mapsto v^{-1}$. The pushforward $\mathcal{F}$ is such that

$$\mathcal{F}(U)=k[x]\times k[y],\ \ \mathcal{F}(V)=k[u,v]/uv$$

and the two modules glue on $U\cap V$ via

$$k[x,x^{-1}]\times k[y,y^{-1}]\xrightarrow{\substack{(1,0)\mapsto(1,0)\\(0,1)\mapsto(0,1)}}k[u,u^{-1}]\times k[v,v^{-1}].$$

On the other hand $\mathcal{G}$ is given by

$$\mathcal{G}(U)=\frac{(x,y)}{x+y-1}k[x,y]/xy,\ \ \mathcal{G}(V)=\frac{1}{u+v-1}k[u,v]/uv$$

and the gluing on $U\cap V$ is given by

$$\frac{x}{x-1}k[x,x^{-1}]\times\frac{y}{y-1}k[y,y^{-1}]\xrightarrow{\substack{(x,0)\mapsto(1,0)\\(0,y)\mapsto(0,1)}}\frac{1}{u-1}k[u,u^{-1}]\times\frac{1}{v-1}k[v,v^{-1}].$$

Then the isomorphisms

$$\begin{aligned} &\mathcal{F}(U) \to \mathcal{G}(U), \quad (1,0) \mapsto \frac{x}{x+y-1},\ (0,1) \mapsto \frac{y}{x+y-1} \\ &\mathcal{F}(V) \to \mathcal{G}(V), \quad 1 \mapsto \frac{1}{u+v-1} \end{aligned}$$

are compatible on $U \cap V$ and hence give the desired isomorphism $\mathcal{F} \xrightarrow{\cong} \mathcal{G}$.

We have succeeded in proving that every torsion-free, rank 1 sheaf $\mathcal{F}$ satisfying the conditions in the statement is of the form $\mathcal{O}(P+Q) \otimes_{\mathcal{O}_X} \mathfrak{m}_x$. It just remains to check that the map $x \mapsto \mathcal{O}(P+Q) \otimes \mathfrak{m}_{\sigma(x)}$ is injective. Let then $x, y \in X(k)$ be distinct points. If x is one of the two singular points, $\mathcal{O}(P+Q) \otimes \mathfrak{m}_x$ is not isomorphic to $\mathcal{O}(P+Q) \otimes \mathfrak{m}_y$, just because the latter is locally free at x and the former is not. Else, if x and y are not singular, they map to line bundles $\mathcal{L}$ and $\mathcal{L}'$ which are isomorphic if and only if $\mathcal{L} \otimes \mathcal{L}'^{\vee} \cong \mathcal{O}(x-y)$ is trivial. This cannot be though, since $\Gamma(X, \mathcal{O}(x-y)) = 0$ because no rational function on X can have only one pole. Indeed, such a function would give an isomorphism from the component where y lies to $\mathbb{P}^1$. Therefore it would assume distinct values at the singular points, and therefore have another pole on the other component.

□

3.2 The substacks $\mathbb{T}^0$ and $\mathbb{T}'$

As we have seen, there is a bijection between the set of k-rational points of X and a particular class of torsion-free rank one sheaves on X. This suggests the possibility of extending this correspondence to S-valued points, for any k-scheme S. Inspired by Proposition 3.1.5, we give the following definitions.

Definition 3.2.1. We let $\mathbb{T}'$ be the full subcategory of $\mathbb{T}$ whose objects are those $(S, \mathcal{F})$ with $\mathrm{End}(F_{\overline{s}}) = k(\overline{s})$ for all geometric points $\overline{s}$ of S. The sheaves $\mathcal{F}$ satisfying this condition are called *simple*.

Definition 3.2.2. We let $\mathbb{T}^0$ be the full subcategory of $\mathbb{T}$ whose objects are those $(S, \mathcal{F})$ with $\mathcal{F}$ satisfying the following conditions:

i) $R^1 p_* \mathcal{F} = 0$;

ii) $p_* \mathcal{F}$ is locally free of rank 1;

iii) Letting t be a local basis of $p_* \mathcal{F}$, the cokernel of the morphism

$$\mathcal{O}_{X_S} \xrightarrow{t} \mathcal{F}$$

has support finite over S.

We call an $\mathcal{F}$ satisfying the above conditions *very simple*.

It is straightforward that for TFR1 sheaves on the curve X, being very simple means exactly satisfying the three conditions of Proposition 3.1.5. The reasons for introducing the substack $\mathbb{T}'$, as explained in the introduction to this chapter, is that we want to leave out those sheaves that on fibres are pushforwards of invertible sheaves from normalizations of the fibre.

We need to state a couple of technical lemmas.

Lemma 3.2.3. *Let $S \cong \operatorname{Spec} A$ be an affine k-scheme and $p : X_S \to S$ be base change of $X \to \operatorname{Spec} k$. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite presentation, flat over S. Then*

i) $R^1 p_* \mathcal{F} \otimes_{\mathcal{O}_S} k(s) \cong H^1(X_s, \mathcal{F}_s)$ *for all* $s \in S$.

ii) $H^1(X_s, \mathcal{F}_s) = 0$ *for all* $s \in S \Rightarrow p_* \mathcal{F} \otimes_{\mathcal{O}_S} k(s) \cong H^0(X_s, \mathcal{F}_s)$ *for all* $s \in S$.

iii) $R^n p_* \mathcal{F} = 0$ *for all* $n \geq n_0 \Rightarrow H^n(X_s, \mathcal{F}_s) = 0$ *for all* $s \in S, n \geq n_0$.

iv) B *is a flat* A*-algebra* $\Rightarrow H^n(X \times_S \operatorname{Spec} B, \mathcal{F} \otimes_A B) \cong H^n(X, \mathcal{F}) \otimes_A B$.

v) $H^1(X_s, \mathcal{F}_s) = 0$ *for all* $s \in S \Rightarrow p_* \mathcal{F}$ *is locally free on* S.

Proof. Let $\{U, V\}$ be the usual cover of X_S by affine opens. We make use of Cech Cohomology.

i) For all $s \in S$ we have an exact sequence

$$\mathcal{F}_s(U_s) \times F_s(V_s) \to \mathcal{F}_s((U \cap V)_s) \to H^1(X_s, \mathcal{F}_s) \to 0.$$

Since S is affine, $R^1 p_* \mathcal{F} \otimes_{\mathcal{O}_S} k(s) \cong R^1 p_* \mathcal{F}(S) \otimes_A k(s) \cong H^1(X_S, \mathcal{F}) \otimes_A k(s)$. Applying the right-exact functor $\_ \otimes_A k(s)$ to the exact sequence

$$\mathcal{F}(U) \times F(V) \to \mathcal{F}(U \cap V) \to H^1(X_S, \mathcal{F}) \to 0$$

the thesis follows, since for any affine $W \subset X_S$, $\mathcal{F}(W) \otimes_A k(s) \cong \mathcal{F}_s(W_s)$.

ii) We want to show that $\mathcal{F}(X_S) \otimes_A k(s) \cong \mathcal{F}_s(X_s)$. By point i), we have $R^1 p_* \mathcal{F} \otimes_{\mathcal{O}_S} k(s) = 0$ for all $s \in S$. Since $R^1 p_* \mathcal{F}$ is of finite type, it is zero. Then we have an exact sequence

$$0 \to \mathcal{F}(X_S) \to \mathcal{F}(U) \times \mathcal{F}(V) \to \mathcal{F}(U \cap V) \to 0.$$

Applying $\_ \otimes_A k(s)$, by S-flatness of $\mathcal{F}$, the sequence remains exact; since we also have an exact sequence

$$0 \to \mathcal{F}_s(X_s) \to \mathcal{F}_s(U_s) \times F_s(V_s) \to \mathcal{F}_s((U \cap V)_s) \to 0$$

the thesis follows.

iii) If $n_0 \geq 2$ there is nothing to prove. If $n_0 = 1$, the thesis follows by point i). If $n_0 = 0$ the thesis follows by point i) and ii).

iv) For $n \geq 2$ there is nothing to prove since both sides are zero. For the cases $n = 0$ or 1, we can apply $\_ \otimes_A B$ to the exact sequence

$$0 \to \mathcal{F}(X_S) \to \mathcal{F}(U) \times \mathcal{F}(V) \to \mathcal{F}(U \cap V) \to H^1(X_S, \mathcal{F}) \to 0$$

and the thesis follows by A-flatness of B and by the fact that for any affine $W \subset X_S$, $\mathcal{F}(W) \otimes_A B \cong (\mathcal{F} \otimes_A B)(W \times_S \operatorname{Spec} B)$.

v) Since $X_S \to S$ is finitely presented, we can reduce to the case where A is noetherian. Then we refer to [7, pag 19]

□

Corollary 3.2.4. *Let $p : X_S \to S$ be the base change to a k-scheme S of the structure morphism of X. Then if $R^1 p_* \mathcal{F} = 0$, $p_* \mathcal{F}$ is locally free on S.*

Proof. It follows directly from part i) and v) of Lemma 3.2.3. □

We remark that Lemma 3.2.3 and Corollary 3.2.4 hold more in general for proper maps $X \to Y$ of relative dimension one, but the proof in the general case is not as easy.

Lemma 3.2.5. *Let S be k-scheme and $\mathcal{F}$ a very simple torsion-free rank one sheaf on X_S. Letting t be a local basis of $p_*\mathcal{F}$, the support of the cokernel $\mathcal{G}$ of $\mathcal{O}_{X_S} \xrightarrow{t} \mathcal{F}$ is the image of a section $x : S \to X_S$ of p.*

Proof. First of all, we claim that

$$p_*\mathcal{O}_{X_S} \cong R^1 p_* \mathcal{O}_{X_S} \cong \mathcal{O}_S.$$

The functor $R^i p_*$ commutes with flat base change on the base scheme for any $i \geq 0$, by [11, TAG 02KH]. Then, since $p_*\mathcal{O}_X \cong R^1 p_* \mathcal{O}_X \cong \mathcal{O}_{\operatorname{Spec} k}$, the claim follows.

Now, by assumption $R^1 p_* \mathcal{F} = 0$, and lemma 3.2.3 assures that $p_*\mathcal{F}$ is locally free of rank 1 on S. Let then $\{U_i\}_{i \in I}$ be an affine open cover of S such that $p_*\mathcal{F}_{|U_i} \cong \mathcal{O}_{U_i}$. Let t_i be a basis of $p_*\mathcal{F}_{|U_i}$ for all $i \in I$. Then t_i induces a global section of $\mathcal{F}_{|X_{U_i}}$. This gives a map

$$\mathcal{O}_{X_{U_i}} \xrightarrow{t_i} \mathcal{F}_{|X_{U_i}}.$$

We claim that it is injective. We know that the cokernel, $\mathcal{G}_i$, has finite support over U_i. Then all the fibres $(\mathcal{G}_i)_s$ have finite support over $k(s)$, for all $s \in U_i$. Now, the zero locus of t_s is contained in the support of $(G_i)_s$, and therefore it is finite over $k(s)$. Consider the kernel K of the morphism $\mathcal{O}_{X_s} \xrightarrow{t} \mathcal{F}_s$. Its support is contained in the zero locus of t. But any morphism from a module with finite support to $\mathcal{O}_{X_s}$ is zero, hence $K = 0$. This shows that for all $s \in U_i$, $\mathcal{O}_{X_s} \xrightarrow{t_s} \mathcal{F}_s$ is injective. Then it follows from [11, TAG 05FQ], which uses U_i-flatness of $\mathcal{F}_{X_{U_i}}$ that t_i is (universally) injective.

Let's denote with $\mathcal{G}_i$ the cokernel of such map, so that by assumption $\operatorname{Supp}\mathcal{G}_i \to U_i$ is a finite map. Let also $p_i := p_{|X_{U_i}}$. Applying the left-exact functor p_{i*} to the sequence

$$0 \to \mathcal{O}_{X_{U_i}} \xrightarrow{t_i} \mathcal{F}_{|X_{U_i}} \to \mathcal{G}_i \to 0$$

we get a long exact sequence

$$0 \to \mathcal{O}_{U_i} \to p_{i*}(\mathcal{F}_{|X_{U_i}}) \to p_{i*}\mathcal{G}_i \to R^1 p_{i*}\mathcal{O}_{X_{U_i}} \to R^1 p_{i*}(\mathcal{F}_{|X_{U_i}}).$$

Since the functor p_{i*} commutes with restriction to open subschemes of S we have $p_{i*}(\mathcal{F}_{|X_{U_i}}) \cong (p_*\mathcal{F})_{|U_i} \cong \mathcal{O}_{U_i}$ and $R^1 p_{i*}(\mathcal{F}_{|X_{U_i}}) \cong (R^1 p_*\mathcal{F})_{|U_i} = 0$. Moreover, $R^1 p_{i*}\mathcal{O}_{X_{U_i}} \cong \mathcal{O}_{U_i}$ and it follows that $p_{i*}\mathcal{G}_i \cong \mathcal{O}_{U_i}$.

Now we would like to see that the finite morphisms $\operatorname{Supp}\mathcal{G}_i \to U_i$ are actually isomorphisms. Here $Z_i := \operatorname{Supp}\mathcal{G}_i$ is the schematic support of $\mathcal{G}_i$, i.e. the closed subscheme of X_{U_i} given by the annihilator sheaf of $\mathcal{G}_i$. The assumption that U_i is affine ensures that also Z_i is. So let $U_i = \operatorname{Spec} A$, $Z_i = \operatorname{Spec} B$ and $\mathcal{G}_{i|Z_i} = \widetilde{M}$. The situation is the following: B is an A-algebra and M a faithful B-module which is isomorphic to A when viewed as an A-module. We would like to show that $B \cong A$. We have a morphism of rings

$$A \to B \hookrightarrow \operatorname{End} M$$

where the rightmost morphism is injective by faithfulness of M. Actually the image of B lies inside $\operatorname{End}_B M$ which is in turn contained in $\operatorname{End}_A M \subset \operatorname{End} M$. Since M is isomorphic to A as an A-module, $\operatorname{End}_A M \cong A$ and we have that the composition

$$A \to B \hookrightarrow A \cong \operatorname{End}_A M$$

is the identity of A. Hence the rightmost map is surjective and in particular an isomorphism, inverse to the original map $A \to B$. This proves that the composition of maps of schemes $Z_i \to X_{U_i} \to U_i$ is an isomorphism, and therefore Z_i is the closed subscheme corresponding to a section $x_i : U_i \to X_{U_i}$ of $p_{|U_i} : X_{U_i} \to U_i$.

We claim that the sections $x_i : U_i \to X_{U_i}$ agree on the intersections $U_i \cap U_j$, $i, j \in I$. Indeed, for all $i, j \in I$, $t_{i|U_i\cap U_j} = u_{i,j} t_{j|U_i\cap U_j}$, where the $u_{i,j}$ are elements of $\mathcal{O}(U_i \cap U_j)^\times$ such that $u_{ij}u_{ji} = 1$ and $u_{ij}u_{jk}u_{ki} = 1$ on $U_i \cap U_j \cap U_k$. Then for all $i, j \in I$

$$\mathcal{G}_{i|X_{U_i\cap U_j}} \cong \mathcal{G}_{j|X_{U_i\cap U_j}}.$$

Since restriction to open subschemes commutes with taking support, the claim follows.

This way, the x_i glue to give a section $x : S \to X_S$, whose image is the support of the sheaf $\mathcal{G}$, which is obtained by glueing the sheaves $\mathcal{G}_i$.

□

Further on we will want to show that the stack $\mathbb{T}'$ can be covered by copies of $\mathbb{T}^0$; these copies are translates of $\mathbb{T}^0$ via an action of $\mathbb{Z} \times \mathbb{Z}$ on $\mathbb{T}'$ that we are now going to construct.

Let as usual P and Q be smooth points on the two components of X. We define, for each $(a,b) \in \mathbb{Z}\times\mathbb{Z}$, an invertible sheaf $\mathcal{O}_{X_S}(a,b)$ on X_S as follows: on X, consider the invertible sheaf $(\mathcal{O}(aP),\mathcal{O}(bQ),\mathrm{id},\mathrm{id})$. Then for a k-scheme $f : S \to \operatorname{Spec} k$, define the invertible sheaf

$$\mathcal{O}_{X_S}(a,b) := f^*(\mathcal{O}(aP),\mathcal{O}(bQ),\mathrm{id},\mathrm{id}).$$

We can now give for every k-scheme S an action

$$(\mathbb{Z}\times\mathbb{Z})\times\mathbb{T}'(S)\to\mathbb{T}'(S),\quad (a,b)\cdot\mathcal{F}\mapsto\mathcal{F}(a,b) := \mathcal{F}\otimes_{\mathcal{O}_{X_S}}\mathcal{O}_{X_S}(a,b).$$

Indeed, $\mathcal{F}\otimes_{\mathcal{O}_{X_S}}\mathcal{O}_{X_S}(a,b)$ is still a simple TFR1 sheaf, since for any invertible sheaf $\mathcal{L}$ on X_S and any geometric point s of S, $\mathrm{End}(\mathcal{F}_s\otimes\mathcal{L}_s) = \mathrm{End}(\mathcal{F}_s\otimes\mathcal{L}_s\otimes{\mathcal{L}_s}^{\vee}) = \mathrm{End}(\mathcal{F}_s)$. Moreover the action is, by the way $\mathcal{O}_{X_S}(a,b)$ was constructed, functorial in S.

Lemma 3.2.6. *Let K be an algebraically closed field and $\mathcal{F}$ an object of $\mathbb{T}'(K)$, that is, a simple torsion-free rank one sheaf on X_K. Then there exists a pair $(a,b)\in\mathbb{Z}\times\mathbb{Z}$ such that $\mathcal{F}(a,b)$ is very simple.*

Proof. Proposition 3.1.5 tells us that the objects of $\mathbb{T}^0(K)$ are exactly the sheaves that are of one of the following kinds:

- $(\mathcal{O}(1),\mathcal{O},\alpha)$ for some $\alpha\in K^{\times}$,
- $(\mathcal{O},\mathcal{O}(1),\alpha)$ for some $\alpha\in K^{\times}$,
- $(\mathcal{O}(1),\mathcal{O}(1),1)\otimes m_{x_0}$, where m_{x_0} is the ideal sheaf of a singular point,
- $(\mathcal{O}(1),\mathcal{O}(1),1)\otimes m_{x_1}$, where m_{x_1} is the ideal sheaf of the other singular point.

Over an algebraically closed field, simple torsion-free rank one sheaves $\mathcal{F}$ are either invertible sheaves $(\mathcal{O}(n),\mathcal{O}(m),\alpha)$ or pushforwards of invertible sheaves via one of the two partial normalizations, hence of the form $(\mathcal{O}(n),\mathcal{O}(m),1)\otimes m_{x_i}$. In any case it is straightforward that there is a pair (a,b) that sends $\mathcal{F}$ to a sheaf of one of the four types above.

□

Lemma 3.2.7. *Let S be any k-scheme and $\mathcal{F}$ an object of $\mathbb{T}'(S)$. Then for any $s\in S$ there exists an open neighbourhood $U\subset S$ of s and a pair $(a,b)\in\mathbb{Z}\times\mathbb{Z}$ such that $(\mathcal{F}(a,b))_{|X_U}$ is very simple on X_U.*

Proof. Take $s\in S$ and let $\overline{s}$ be the geometric point lying over s. The fibre $\mathcal{F}_{\overline{s}}$ is simple on $X_{\overline{s}}$, so as seen in Lemma 3.2.6 there exists a pair $(a,b)\in\mathbb{Z}\times\mathbb{Z}$ such that $\mathcal{F}_{\overline{s}}(a,b)$ is very simple. Then $(R^1p_*\mathcal{F}(a,b))_{\overline{s}} \cong H^1(X_{\overline{s}},\mathcal{F}(a,b)_{\overline{s}}) \cong H^1(X_{\overline{s}},\mathcal{F}_{\overline{s}}(a,b)) = 0$. Then there exists a neighbourhood U_0 of s such that $R^1p_*\mathcal{F}(a,b)_{|U_0} = 0$. Similarly, $(p_*\mathcal{F}(a,b))_{\overline{s}}\cong k(\overline{s})$. By Nakayama and Lemma 3.2.4 there is a neighbourhood $U_1\subset U_0$ of s such that $p_*\mathcal{F}(a,b)_{|U_1}$ is locally free of rank 1 - in particular, we can assume

$p_*\mathcal{F}(a,b)_{|U_1} \cong \mathcal{O}_{U_1}$. Choose a basis t of $p_*\mathcal{F}(a,b)_{|U_1}$ and let $\mathcal{G}$ be the cokernel of the map

$$\mathcal{O}_{X_{U_i}} \xrightarrow{t} \mathcal{F}(a,b).$$

We know that $\mathcal{G}_{\overline{s}}$ has finite support. Take now two point $\alpha, \beta \in X(k)$ on distinct components of the smooth locus of X, such that the base changes $\alpha_{k(\overline{s})}, \beta_{k(\overline{s})}$ are not in the support of $\mathcal{G}_{\overline{s}}$. Consider the open subset $U \subset U_1$ where $\alpha^*_{U_1}\mathcal{G} = 0$ and $\beta^*_{U_1}\mathcal{G} = 0$. We want to argue that $\operatorname{Supp}\mathcal{G}_{|U}$ is finite over U. By [4, 8.11.1] a morphism which is proper, locally of finite presentation and with finite fibres is finite. Since $\operatorname{Supp} G_{|U}$ is a closed subscheme of X_U which is proper and finitely presented over U, we just need to check that fibres are finite. For every point $s_0 \in U$, the support of $\mathcal{G}_{\overline{s_0}}$ either contains a whole component of $X_{\overline{s_0}}$ or is finite. Since it does not contain the points $\alpha_{\overline{s_0}}$ and $\beta_{\overline{s_0}}$, it is finite. □

3.3 Rigidification

We would like to find an equivalence of categories between X and the stack of very simple sheaves, but in order to do so we need to modify $\mathbb{T}^0$ a bit. Indeed, for any k-scheme S the fibred category $X(S)$ has only identities as morphisms. This is not the case for $\mathbb{T}^0(S)$, whose objects have plenty of automorphisms. We get rid of these non-trivial automorphisms via the process of rigidification.

Definition 3.3.1. Let $\epsilon \in X(k)$ be a point lying on the smooth locus of X, and let $\epsilon_S : S \to X_S$ be its base change for all S. We let $\mathbb{T}_\epsilon$ be the category of *rigidified torsion free rank 1 sheaves*, defined in the following way:

- its objects are triplets $(S, \mathcal{F}, \alpha)$, where S is a k-scheme, $\mathcal{F}$ is a torsion-free rank one sheaf on X_S, and α is an isomorphism $\alpha : \epsilon^*\mathcal{F} \xrightarrow{\sim} \mathcal{O}_S$,
- a morphism $h : (S, \mathcal{F}, \alpha) \to (S, \mathcal{G}, \beta)$ is given by a morphism $h : \mathcal{F} \to \mathcal{G}$ such that $\beta \circ \epsilon^* h = \alpha$.

We define analogously the categories $\mathbb{T}'_\epsilon$ and $\mathbb{T}^0_\epsilon$.

These new rigidified objects have the following useful property.

Lemma 3.3.2. *An object $(S, \mathcal{F}, \alpha)$ of $\mathbb{T}'_\epsilon$ has no non-trivial automorphisms.*

Proof. Suppose $h : \mathcal{F} \to \mathcal{F}$ is an isomorphism with $\alpha \circ \epsilon^* h = \alpha$. Then $\epsilon^* h = id_{\epsilon^*\mathcal{F}}$. We would like to show that then h is the identity. We know that it maps to the identity via the map

$$\operatorname{Aut}_{\mathcal{O}_{X_S}}(\mathcal{F}) \xrightarrow{\epsilon^*} \operatorname{Aut}_{\mathcal{O}_S}(\epsilon^*\mathcal{F})$$

so let's compare the two sides. Being $\epsilon^*\mathcal{F}$ an invertible sheaf on S, $\operatorname{Aut}_{\mathcal{O}_S}(\epsilon^*\mathcal{F}) = \operatorname{Aut}_{\mathcal{O}_S}(\mathcal{O}_S) = \mathcal{O}_S(S)^\times$.

To show that also $\mathrm{Aut}_{\mathcal{O}_{X_S}}(\mathcal{F}) = \mathcal{O}_S(S)^\times$, it would be enough to show that

$$p_*\mathscr{E}nd(\mathcal{F}) = \mathcal{O}_S.$$

Now, suppose we prove this for very simple sheaves. By Lemma 3.2.7, there is an open cover $\{U_i\}_{i\in I}$ of S and invertible sheaves $\{\mathcal{L}_i\}_{i\in I}$ on X_{U_i} such that, for all $i \in I$, $\mathcal{F}_{|U_i} \otimes \mathcal{L}_i$ is very simple on X_{U_i}. Taking the tensor product by an invertible sheaf does not alter the sheaf of endomorphisms, hence $p_*\mathscr{E}nd(\mathcal{F})_{|U_i} = p_{|X_{U_i},*}\mathscr{E}nd(\mathcal{F}_{|X_{U_i}}) = p_{|X_{U_i},*}\mathscr{E}nd(\mathcal{F}_{|X_{U_i}} \otimes \mathcal{L}_i) = \mathcal{O}_{U_i}$. For all $i \in I$ the isomorphisms $p_*\mathscr{E}nd\mathcal{F}_{|U_i} = \mathcal{O}_{U_i}$ are canonical, so they glue to $p_*\mathscr{E}nd\mathcal{F} = \mathcal{O}_S$.

Hence we can assume that $\mathcal{F}$ is very simple, which implies $p_*\mathcal{F}$ locally free of rank 1. We have a morphism of $\mathcal{O}_S$-modules

$$s : p_*\mathscr{E}nd_{\mathcal{O}_{X_S}}(\mathcal{F}) \to \mathscr{E}nd_{\mathcal{O}_S}(p_*\mathcal{F}) = \mathcal{O}_S.$$

There is also a natural injection $\mathcal{O}_S \hookrightarrow p_*\mathscr{E}nd_{\mathcal{O}_{X_S}}(\mathcal{F})$. Let $\mathcal{H}$ be its cokernel. Then the map s above gives a splitting of the exact sequence

$$0 \to \mathcal{O}_S \to p_*\mathscr{E}nd_{\mathcal{O}_{X_S}}(\mathcal{F}) \to \mathcal{H} \to 0.$$

Hence $p_*\mathscr{E}nd_{\mathcal{O}_{X_S}}(\mathcal{F}) = \mathcal{O}_S \oplus \mathcal{H}$, and $s : \mathcal{O}_S \oplus \mathcal{H} \to \mathcal{O}_S$ is the projection on the first summand. We would like to show that $\mathcal{H} = 0$. Let $U \subset S$ be an open affine where $p_*\mathcal{F}_{|U} \cong \mathcal{O}_U$. Let $a \in \mathcal{H}(U)$. Then $a \in \mathrm{End}_{\mathcal{O}_{p^{-1}(U)}}(\mathcal{F}_{|p^{-1}(U)})$. Since $s(U)(a) = 0$, we have that a induces the zero endomorphism on $\mathcal{F}(p^{-1}(U))$. The latter module is free, isomorphic to $\mathcal{O}(U) \cdot t$ for some $t \in \mathcal{F}(p^{-1}(U)$. Then the composition

$$\mathcal{O}_{p^{-1}(U)} \xrightarrow{t} \mathcal{F}_{|p^{-1}(U)} \xrightarrow{a} \mathcal{F}_{|p^{-1}(U)}$$

is zero, hence a factors via the cokernel $\mathcal{G}$ of $\mathcal{O}_{p^{-1}(U)} \xrightarrow{t} \mathcal{F}_{p^{-1}(U)}$. By the assumption that $\mathcal{F}$ is very simple and by Lemma 3.2.5, $\mathcal{G}$ has support on the image of a section $x : U \to p^{-1}U$. Since $p_*\mathcal{G} \cong \mathcal{O}_U$ and $\mathcal{G} \cong x_*p_*\mathcal{G}$, we find $\mathcal{G} \cong x_*\mathcal{O}_U$. But $\mathrm{Hom}_{\mathcal{O}_{p^{-1}(U)}}(x_*\mathcal{O}_U, \mathcal{F}_{|p^{-1}(U)}) = 0$ since $\mathcal{F}$ is TFR1. This proves that a is zero and so is $\mathcal{H}$. Hence $p_*\mathscr{E}nd_{\mathcal{O}_{X_S}}(\mathcal{F}) = \mathcal{O}_S$ as we wished to show. □

3.4 The main theorem

In this section we prove that the stack $\mathbb{T}^0_\epsilon$ is represented by the scheme X. We first need to prove a few lemmas.

Lemma 3.4.1. *Let S be a k-scheme and $\mathcal{F}$ a very simple torsion free rank one sheaf on X_S. Then if $\mathcal{L}$ is an invertible sheaf on S, $\mathcal{F} \otimes_{\mathcal{O}_{X_S}} p^*\mathcal{L}$ is also very simple.*

Proof. For every $s \in S$, $p^*\mathcal{L}_s$ is trivial. Then by lemma 3.2.3 it follows that for all $s \in S$, $R^1p_*(\mathcal{F} \otimes p^*\mathcal{L}) \otimes k(s) \cong H^1(X_s, \mathcal{F}_s \otimes p^*\mathcal{L}_s) = H^1(X_s, \mathcal{F}_s) = R^1p_*\mathcal{F} = 0$, and $p_*\mathcal{F} \otimes k(s) = p_*(\mathcal{F} \otimes p^*\mathcal{L}) \otimes k(s)$, so conditions i) and ii) are satisfied. For condition iii), finitess of the map can be checked locally on S. Notice that the base change to an open $U \subset S$ of the map $\mathrm{Supp}(\mathcal{F}/t\mathcal{O}_{X_S}) \to S$ is the map $\mathrm{Supp}(\mathcal{F}_{|X_U}/t_{|U}\mathcal{O}_{X_U}) \to U$. Then if $\{U_i\}_{i\in I}$ is an open cover of S such that $\mathcal{L}_{|U_i} \cong \mathcal{O}_{U_i}$ for all $i \in I$, we have $p^*(\mathcal{L})_{|X_{U_i}} \cong p^*(\mathcal{L}_{|U_i}) \cong \mathcal{O}_{X_{U_i}}$ and we conclude. □

Lemma 3.4.2. *Let X_S be the base change of X by a k-scheme S. Let $x : S \to X_S$ be a section of $p : X_S \to S$. Let I_x be the ideal sheaf of the closed subscheme of X_S given by x. Then I_x is TFR1 on X_S.*

Proof. Such an ideal satisfies an exact sequence

$$0 \to I_x \to \mathcal{O}_{X_S} \to x_*\mathcal{O}_S \to 0.$$

Since $\mathcal{O}_{X_S}$ and $x_*\mathcal{O}_S$ are S-flat, I_x is S-flat. To check that I_x is locally-free on the smooth locus of X_S, take an affine $U \cong \operatorname{Spec} R[x, x^{-1}]$ in the smooth locus of X_S, with R the affine ring of some open of S. The restriction of I_x to U is the kernel of a surjection of R-algebras

$$R[x, x^{-1}] \to R_f$$

for some $f \in R$, hence of the form $(x - a)R[x, x^{-1}]$ for some invertible $a \in R_f$, and therefore a free R-module. Finally, restricting to an affine neighborhood of a singular point, the ideal sheaf I_x is given by an ideal $I \subset R[x, y]/xy$ which is the kernel of a surjective morphism of R-algebras $R[x, y]/xy \to R$. It must be checked that the map $I \to I \otimes_R (R[x, x^{-1}] \times R[y, y^{-1}])$ is R-universally injective. Denoting by $I[x, y]$ the elements of I annihilated by both x and y, this is equivalent to showing that for all R-algebras R', we have $(I \otimes_R R')[x, y] = 0$. Now, the complex

$$0 \to I \to R[x, y]/xy \to R \to 0$$

remains exact when tensored with R' over R, because R is obviously R-flat. So we have $I \otimes_R R' \subset R'[x, y]/xy$, and there are no non-zero elements in this ring annihilated by both x and y. □

Let P and Q be the base change to X_S of the two points previously called P and Q in proposition 3.1.5, and σ be the base change to X_S of the unique automorphism of X that sends one point to the other. Any section $x : S \to X_S$ gives a closed subscheme of X_S; let I_x be its ideal sheaf. Let also $\mathcal{O}(P + Q)$ be the invertible sheaf on X_S dual to the ideal sheaf (which is in fact invertible) of the closed subscheme given by P and Q. For any section $x : S \to X_S$, we define

$$\mathcal{F}(x) := \mathcal{O}(P + Q) \otimes I_{\sigma(x)}$$

and

$$\mathcal{G}(x) := \mathcal{F}(x) \otimes p^*(\epsilon^*(\mathcal{F}(x))^\vee).$$

Since $p \circ \epsilon = \mathrm{id}_S$, the pullback $\epsilon^*\mathcal{G}(x)$ is canonically isomorphic to $\mathcal{O}_S$. Therefore we have a canonical rigidification $(\mathcal{G}(x), \mathrm{id})$.

While proving Proposition 3.1.5 we saw that for every point $x \in X(k)$, the sheaf $\mathcal{F}(x)$ fitted into an exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{F}(x) \to x_*k \to 0.$$

We now prove the analogue over a general scheme.

Lemma 3.4.3. *Let T be an affine scheme and $p : X_T \to T$ be the base change of $X \to \operatorname{Spec} k$. Let $x : T \to X_T$ be any section of p. Then, for some invertible sheaf $\mathcal{L}$ on T, the sheaf $\mathcal{F}(x)$ fits into an exact sequence*

$$0 \to \mathcal{O}_{X_T} \to \mathcal{F}(x) \to x_*\mathcal{L} \to 0.$$

Proof. The scheme X_T has a covering by two affine opens U and V isomorphic to $\operatorname{Spec} R[x,y]/xy$ and $\operatorname{Spec} R[u,v]/uv$, glued via $x \mapsto u^{-1}$ and $y \mapsto v^{-1}$ as usual. The preimage of U via the closed immersion $\sigma(x)$ is an affine open in T, isomorphic to $\operatorname{Spec} R'$ for some R-algebra R'. Then the restriction of the ideal sheaf of $I_{\sigma(x)|U}$ is given by the kernel of an R'-algebra surjective morphism

$$R'[x,y]/xy \twoheadrightarrow R',$$

which is of the form

$$(x-\alpha, y-\beta)R'[x,y]/xy$$

for some $\alpha, \beta \in R'$ with $\alpha\beta = 0$. The same can be done on V. There, $I_{\sigma(x)|V}$ is given by

$$(u-\alpha', v-\beta')R''[u,v]/uv$$

with α', β' belonging to an R-algebra R'' and $\alpha'\beta' = 0$. Then, since on the intersection $U \cap V$ we have $x = u^{-1}$ and $y = v^{-1}$, we obtain $\alpha' = \alpha^{-1}$ and $\beta' = \beta^{-1}$ in $R' \times_R R''$.

The sheaf $\mathcal{F}(x)$ has a global section f given by

$$\frac{x+y-\alpha-\beta}{x+y-1} \text{ on } U \text{ and } \frac{u+v-\alpha'-\beta'}{u+v-1} \text{ on } V.$$

Let $\mathcal{G}$ be the quotient of $\mathcal{O}_{X_T} \xrightarrow{f} \mathcal{F}(x)$. Then $\mathcal{G}_{|U}$ is given by the module

$$M = \frac{\frac{(x-\alpha,y-\beta)}{(x-1)(y-1)}R'[x,y]/xy}{\frac{(x+y-\alpha-\beta)}{(x-1)(y-1)}R'[x,y]/xy} \cong \frac{(x-\alpha,y-\beta)R'[x,y]/xy}{(x+y-\alpha-\beta)R'[x,y]/xy}.$$

Now, $y-\beta = \alpha - x$ in M, so the module is generated by just $x-\alpha$. Also, we can substitute $y = \alpha+\beta-x$ and get

$$M = (x-\alpha)\frac{R'[x]}{x^2-\alpha x-\beta x}.$$

Since $\alpha\beta = 0$, $x^2 - \alpha x - \beta x = (x-\alpha)(x-\beta)$, and by the fact that $x-\alpha$ is not a zero-divisor in $R[x]$ we have that

$$M \cong R'[x]/(x-\beta).$$

By comparison, we have

$$\frac{R'[x,y]}{(x-\beta, y-\alpha, xy)} \cong \frac{R'[x]}{(x-\beta, \alpha x)}$$

by substituting $y = \alpha$. But $\alpha x = \alpha(x - \beta)$, so we get indeed

$$M \cong \frac{R'[x, y]}{(x - \beta, y - \alpha, xy)}$$

which is the ring of the skyscraper sheaf at x. Therefore, $\mathcal{F}_{|U} \cong x_{|x^{-1}(U),*}\mathcal{O}_{x^{-1}(U)}$.

By symmetry, the same argument works on V. Hence $\mathcal{F} \cong x_*\mathcal{L}$ for an invertible sheaf $\mathcal{L}$ on T that is free on the affine opens $x^{-1}(U)$ and $x^{-1}(V)$. □

Lemma 3.4.4. *Let $T = \operatorname{Spec} R$, and $x, y : T \to X_T$ be two sections of $p : X_T \to T$. If there exists an isomorphism of $\mathcal{O}_{X_T}$-modules $\sigma : I_x \to I_y$, then $x = y$.*

Proof. Taking the tensor product of the ideal sheaf I_x with $\mathcal{O}(P + Q)$, by Lemma 3.4.3 one gets an exact sequence

$$0 \to \mathcal{O}_{X_T} \xrightarrow{f} \mathcal{O}(P + Q) \otimes I_x \to x_*\mathcal{L} \to 0$$

where f is a global section of $\mathcal{O}(P+Q) \otimes I_x$. The isomorphism σ applied to f gives a global section $\sigma(f)$ of $\mathcal{O}(P+Q) \otimes I_y$. Then we get an isomorphism of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_{X_T} & \xrightarrow{f} & \mathcal{O}(P+Q) \otimes I_x & \longrightarrow & x_*\mathcal{L} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \sigma} & & & & \\ 0 & \longrightarrow & \mathcal{O}_{X_T} & \xrightarrow{\sigma(f)} & \mathcal{O}(P+Q) \otimes I_y & \longrightarrow & y_*\mathcal{M} & \longrightarrow & 0 \end{array}$$

for some invertible $\mathcal{M}$ on T, and in particular an isomorphism $x_*\mathcal{L} \cong y_*\mathcal{M}$. Take an open cover $\{U_i\}$ where both $\mathcal{L}_{|U_i}$ and $\mathcal{M}_{|U_i}$ are trivial. Then $x_*\mathcal{O}_{U_i} \cong y_*\mathcal{O}_{U_i}$ for all $i \in I$. Since $x(U_i)$ is the support of $x_*\mathcal{O}(U_i)$, the morphisms $x_{|U_i}$ and $y_{|U_i}$ have same image; since they are both sections of $p_{|x(U_i)}$ they coincide on points of U_i. Because $x_*\mathcal{O}_{U_i} \cong y_*\mathcal{O}_{U_i}$ it follows that $x_{|U_i}$ and $y_{|U_i}$ yield the same morphism of sheaves and are therefore the same morphism of schemes. Then since x and y coincide on an open cover of T, we have $x = y$. □

In the next lemma we use the dualizing sheaf of the morphism $p : X_S \to S$, as defined in [910, pag. 243], and of Grothendieck duality [1, Thm 4.3.1].

Lemma 3.4.5. *Let S be an affine k-scheme, $p : X_S \to S$ be the base change of the structure morphism of X, and $x : S \to X_S$ be a section of p. Let also $\omega_{X/S}$ be the dualizing sheaf of p. Then*

$$\operatorname{Ext}^1{}_{\mathcal{O}_{X_S}}(x_*\mathcal{O}_S, \omega_{X/S}) = \mathcal{O}_S(S).$$

Proof. Let $D^b(X)$ (resp. $D^+(X)$) be the bounded (resp. bounded below) derived category of quasi-coherent sheaves on X. Grothendieck duality asserts that the diagram

$$\begin{array}{ccc} D^b(X) & \xrightarrow{R\mathscr{H}om(\_,\omega_{X/S}[1])} & D^+(X) \\ \downarrow{\scriptstyle Rp_*} & & \downarrow{\scriptstyle Rp_*} \\ D^b(S) & \xrightarrow{R\mathscr{H}om(\_,\mathcal{O}_S[0])} & D^+(S) \end{array}$$

commutes. Hence applying it to the complex with the sheaf $x_*\mathcal{O}_S$ concentrated in degree zero, we get

$$R\mathscr{H}om(Rp_*(x_*\mathcal{O}_S[0]), \mathcal{O}_S[0]) \cong Rp_*(R\mathscr{H}om(x_*\mathcal{O}_S[0]), \omega_{X/S}[1]).$$

Let's compute the LHS first. The sheaf $x_*\mathcal{O}_S$ is acyclic for the functor p_*. Indeed, letting $I^\bullet$ be an injective resolution of $\mathcal{O}_S$, $x_*I^\bullet$ is an injective resolution of $x_*\mathcal{O}_S$. Then applying the functor p_* gives us back the resolution $I^\bullet$.

Then

$$Rp_*(x_*\mathcal{O}_S[0]) = p_*x_*\mathcal{O}_S[0] = \mathcal{O}_S[0].$$

Therefore we have to compute $R\mathscr{H}om(\mathcal{O}_S[0], \mathcal{O}_S[0])$. Since the complex $\mathcal{O}_S[0]$ has acyclic terms for $\mathscr{H}om(\_, \mathcal{O}_S)$, this is simply

$$\mathscr{H}om^\bullet(\mathcal{O}_S[0], \mathcal{O}_S[0]) = \mathcal{O}_S[0].$$

We now turn to the RHS. We have

$$R\mathscr{H}om(x_*\mathcal{O}_S[0], \omega_{X/S}[1]) = \mathscr{H}om^\bullet(x_*\mathcal{O}_S[0], I^\bullet[1])$$

for an injective complex $I^\bullet$ quasi-isomorphic to $\omega_{X/S}[0]$. This is a complex that in degree n has the sheaf $\mathscr{H}om(x_*\mathcal{O}_X, I^{n+1})$. This is acyclic for the functor p_* (it is the pushforward via x of some sheaf on S, and we can apply the same argument as earlier in the proof). So applying Rp_* we just obtain

$$p_*\mathscr{H}om^\bullet(x_*\mathcal{O}_S[0], I^\bullet[1]).$$

Therefore we have a quasi-isomorphism

$$\mathcal{O}_S[0] \cong p_*\mathscr{H}om^\bullet(x_*\mathcal{O}_S[0], I^\bullet[1])$$

and taking cohomology we obtain

$$p_*\mathscr{E}xt^i_{\mathcal{O}_{X_S}}(x_*\mathcal{O}_S, \omega_{X/S}) = \begin{cases} \mathcal{O}_S, & \text{if } i = 1. \\ 0, & \text{if } i \neq 1. \end{cases}$$

Now, the Grothendieck spectral sequence for composition of the functors $\mathscr{H}om(x_*\mathcal{O}_S, \_)$ and $\Gamma(X_S, \_)$ yields an exact sequence in low degrees

$$\begin{aligned} &0 \to H^1(X_S, \mathscr{H}om_{\mathcal{O}_{X_S}}(x_*\mathcal{O}_S, \omega_{X_S/S})) \to \mathrm{Ext}^1(x_*\mathcal{O}_S, \omega_{X_S/S}) \to \\ &\to \Gamma(X_S, \mathscr{E}xt^1_{\mathcal{O}_{X_S}}(x_*\mathcal{O}_S, \omega_{X_S/S})) \to H^2(X_S, \mathscr{H}om_{\mathcal{O}_{X_S}}(x_*\mathcal{O}_S, \omega_{X_S/S})) \end{aligned}$$

The first and fourth term in the sequence vanish. Indeed, the Hom sheaf is supported on the the image of x, which is isomorphic to the affine scheme S, and on affines coherent sheaves have zero higher cohomology. Hence

$$\begin{aligned} \mathrm{Ext}^1(x_*\mathcal{O}_S, \omega_{X_S/S}) &= \Gamma(X_S, \mathscr{E}xt^1_{\mathcal{O}_{X_S}}(x_*\mathcal{O}_S, \omega_{X_S/S})) = \\ &= \Gamma(S, p_*\mathscr{E}xt^1_{\mathcal{O}_{X_S}}(x_*\mathcal{O}_S, \omega_{X_S/S})) = \mathcal{O}_S(S). \end{aligned}$$

□

We have now all the ingredients to state and prove the main result.

Theorem 3.4.6. *Let S be any scheme over k, $p : X_S \to S$ the base change of the structure morphism $X \to \operatorname{Spec} k$. Then the functor*

$$F_S : X(S) \to \mathbb{T}^0_\epsilon(S)$$

given by

$$F_S(x) = (\mathcal{G}(x), \mathrm{id})$$

is an equivalence of categories.

Proof. Notice first that $X(S) = X_S(S)$, via the bijection sending $x : S \to X$ to $(x, \mathrm{id}) : S \to X \times_{\operatorname{Spec} k} S$.

We check that for any $x \in X(S)$ the sheaf $\mathcal{G}(x)$ is very simple. Ideal sheaves of sections of $p : X_S \to S$ are TfR1 by Lemma 3.4.2, and tensor product of a TFR1 sheaf by an invertible sheaf is TFR1 by Lemma 1.2.10. It follows that $\mathcal{F}(x) = \mathcal{O}(P+Q) \otimes I_{\sigma(x)}$ is TFR1. Now by lemma 3.4.1 it is enough to check that $\mathcal{F}(x)$ satisfies conditions i),ii),iii) of definition 3.2.2 .

Letting $\mathcal{F} = \mathcal{O}(P+Q) \otimes I_{\sigma(x)}$, notice that for all points $s \in S$ the fibre F_s is a torsion-free rank one sheaf on the curve X_s over the field $k(s)$. By part (iv) of lemma 3.2.3, $\dim_{k(s)} H^i(X_s, \mathcal{F}_s) = \dim_{\overline{k(s)}} H^i(X_{\overline{s}}, F_{\overline{s}})$, where $\overline{s}$ is the geometric point above s. This allows us to reconduct to proposition 3.1.5, being $F_{\overline{s}} = \mathcal{O}(P \circ \overline{s} + Q \circ \overline{s}) \otimes I_{\sigma(x) \circ \overline{s}}$, and hence a torsion-free rank 1 sheaf of the kind described in the proposition.
So, for all $s \in S$ and integers $i \geq 2$, $H^i(X_s, \mathcal{F}_s) = 0$; then by lemma 3.2.3 $R^1 p_* \mathcal{F} \otimes k(s) \cong H^1(X_s, \mathcal{F}_s) = 0$. This shows that $R^1 p_* \mathcal{F} = 0$, which in turn implies that $p_* \mathcal{F}$ is locally free by lemma 3.2.4. Now, again by Lemma 3.2.3, $p_* \mathcal{F} \otimes k(s) \cong H^0(X_s, \mathcal{F}_s) \cong k(s)$; hence, by Nakayama's lemma $p_* \mathcal{F}$ is of rank 1. Lastly, too see that $\operatorname{Supp}(\mathcal{F}/\mathcal{O}_{X_S} t) \to S$ is a finite map, we use the fact that finiteness is local on the target. By Lemma 3.4.3, there is an affine cover $\{U_i\}$ of S such that $(\mathcal{F}_{|U_i}/\mathcal{O}_{X_{U_i}} t) \cong x_* \mathcal{O}_{U_i}$. Then $\operatorname{Supp}(x_* \mathcal{O}_{U_i}) \to U_i$ is an isomorphism, hence a finite map.

Next we show that the functor F_S is fully faithful. This is made easier by the fact that the only morphisms in $X(S)$ are the identities. By lemma 3.3.2 elements in $T^0_\epsilon(S)$ have no non-trivial automorphisms. Then we just need to check that for $x \neq y$ objects of $X(S)$, the set $\operatorname{Hom}(F_S(x), F_S(y))$ is empty. The only morphisms in a category fibred in groupoids are isomorphisms. So assume by contradiction that $\mathcal{G}(x) \cong \mathcal{G}(y)$. Then

$$\mathcal{F}(x) \cong \mathcal{F}(y) \otimes p^* \mathcal{L}$$

for some invertible sheaf $\mathcal{L}$ on S, and in particular

$$I_{\sigma(x)} \cong I_{\sigma(y)} \otimes p^* \mathcal{L}.$$

Consider an affine cover $\{U_i\}_{i \in I}$ of S such that $\mathcal{L}_{|U_i} \cong \mathcal{O}_{U_i}$ for all $i \in I$. Then $(p^* \mathcal{L})_{|X_{U_i}} \cong \mathcal{O}_{X_{U_i}}$ and therefore $I_{\sigma(x_{|U_i})} \cong I_{\sigma(y_{|U_i})}$. By lemma 3.4.4, it follows that $x_{|U_i} = y_{|U_i}$ for all $i \in I$ and hence $x = y$, contradicting the hypothesis.

Our objective is now to construct a quasi-inverse J to F_S. Let's see how to associate to every object $(\mathcal{F},\varphi)$ of $\mathbb{T}^0_\epsilon(S)$ a point $J(\mathcal{F},\varphi)$ of $X(S)$.

Lemma 3.2.5 gives a recipe to associate a section $x : S \to X_S$ to a very simple torsion-free rank one sheaf $\mathcal{F}$: we choose an affine cover $\{U_i\}$ of S on which the invertible sheaf $p_*\mathcal{F}$ is trivial. On every U_i we let t_i be a basis of $(p_*\mathcal{F})_{|U_i}$ and it turns out that the cokernel $\mathcal{G}_i$ of $\mathcal{O}_{X_{U_i}} \xrightarrow{t_i} \mathcal{F}_{|X_{U_i}}$ is supported on a section $x_i : U_i \to X_{U_i}$. The sections x_i then glue to give a section $x : S \to X_S$.

We let $J : \mathbb{T}^0_\epsilon(S) \to X(S)$ be the functor given by

$$J : (\mathcal{F},\varphi) \mapsto x$$

as done above. To see that F_S is an equivalence of categories, it is enough to show that for all $(\mathcal{F},\varphi)$ there is a unique isomorphism

$$(\mathcal{F},\varphi) \to F_S(J(\mathcal{F},\varphi)).$$

Actually, because of the rigidification, between two isomorphic objects of $\mathbb{T}^0_\epsilon(S)$ there is exactly one isomorphism, by lemma 3.3.2. So we just need to show that $\mathcal{F}$ is isomorphic to $\mathcal{G}(x)$, where x is the section associated to $\mathcal{F}$. Then $(\mathcal{F},\varphi)$ will necessary be isomorphic to $(\mathcal{G}(x),1)$ (since different rigidifications of the same sheaf are all isomorphic among them).

We consider again the situation over the affines $\{U_i\}$. Since the sheaves $\mathcal{G}_i$ are supported at Z_i, where x_i and $p_{|Z_i}$ are inverse to each other, it follows that $\mathcal{G}_i = x_{i*}p_*\mathcal{G}_i \cong x_{i*}\mathcal{O}_S$. Hence we have an exact sequence

$$0 \to \mathcal{O}_{X_{U_i}} \to \mathcal{F}_{|X_{U_i}} \to x_{i*}\mathcal{O}_{U_i} \to 0.$$

Consider now the sheaf $\mathcal{G}(x) = \mathcal{F}(x) \otimes p^*(\epsilon^*(\mathcal{F}(x))^\vee)$, where $\mathcal{F}(x) = \mathcal{O}(P+Q) \otimes I_{\sigma(x)}$. Up to refining the initial open cover U_i we can assume without loss of generality that the invertible sheaf $p^*(\epsilon^*(\mathcal{F}(x))^\vee)$ is trivial when restricted to the U_i. So we get

$$\mathcal{G}(x)_{|U_i} \cong \mathcal{O}(P_{|U_i} + Q_{|U_i}) \otimes I_{\sigma(x_i)}.$$

Because U_i affine, by lemma 3.4.3 we find that there are exact sequences

$$0 \to \mathcal{O}_{X_{U_i}} \to \mathcal{G}(x)_{|X_{U_i}} \to x_{i*}\mathcal{O}_{U_i} \to 0.$$

We have seen that both $\mathcal{F}_{|X_{U_i}}$ and $\mathcal{G}(x)_{X_{|U_i}}$ are extensions of $x_*\mathcal{O}_{U_i}$ by $\mathcal{O}_{X_{U_i}}$, so they correspond to elements of the extension group $\mathrm{Ext}^1{}_{\mathcal{O}_{X_{U_i}}}(x_*\mathcal{O}_{U_i}, \mathcal{O}_{X_{U_i}})$. We claim that, up to refining the cover $\{U_i\}$, the Ext module is isomorphic to $\mathcal{O}_{U_i}(U_i)$.

Let $\omega_{X_S/S}$ be the dualizing sheaf of $p : X_S \to S$. As seen in lemma 3.4.5, we have a canonical isomorphism $\mathcal{O}_{U_i}(U_i) = \mathrm{Ext}^1{}_{\mathcal{O}_{X_{U_i}}}(x_{i*}\mathcal{O}_{U_i}, \omega_{X_{U_i}/U_i})$. Interpreting the RHS as the extension group of $x_{i*}\mathcal{O}_{U_i}$ by $\omega_{X_{U_i}/U_i}$, we find that it is isomorphic to

$$\mathrm{Ext}^1{}_{\mathcal{O}_{X_{U_i}}}(x_{i*}\mathcal{O}_{U_i} \otimes \omega^\vee_{X_{U_i}/U_i}, \mathcal{O}_{X_{U_i}}).$$

Indeed, tensor product by the invertible (and hence $\mathcal{O}_{X_{U_i}}$-flat) dual sheaf $\omega^{\vee}_{X_{U_i}/U_i}$ sends bijectively extensions

$$0 \to \omega_{X_{U_i}/U_i} \to \mathcal{F} \to x_{i*}\mathcal{O}_{U_i} \to 0$$

to extensions

$$0 \to \mathcal{O}_{X_{U_i}} \to \mathcal{F}' \to x_{i*}\mathcal{O}_{U_i} \otimes \omega^{\vee}_{X_{U_i}/U_i} \to 0.$$

We actually have

$$x_{i*}\mathcal{O}_{U_i} \otimes \omega^{\vee}_{X_{U_i}/U_i} \cong x_{i*}x_i{}^*\mathcal{O}_{X_{U_i}} \otimes x_*x^*\omega^{\vee}_{X_{U_i}/U_i} \cong x_{i*}x_i{}^*\omega^{\vee}_{X_{U_i}/U_i}.$$

Putting together the isomorphisms found so far, we get, for some invertible sheaf $\mathcal{L}$ on U_i,

$$\mathcal{O}_{U_i}(U_i) \cong \mathrm{Ext}^1{}_{\mathcal{O}_{X_{U_i}}}(x_{i*}\mathcal{L}, \mathcal{O}_{X_{U_i}}).$$

Up to refining the cover U_i, we can assume that $\mathcal{L}$ is trivial on each U_i, and hence we find

$$\mathcal{O}_{U_i}(U_i) \cong \mathrm{Ext}^1{}_{\mathcal{O}_{X_{U_i}}}(x_{i*}\mathcal{O}_{U_i}, \mathcal{O}_{X_{U_i}}),$$

which proves our claim.

Now we show that $\mathcal{F}_{|X_{U_i}}$ and $\mathcal{G}(x)_{|X_{U_i}}$ are isomorphic. It is enough to show that they correspond to units $a, b \in R_i = \mathcal{O}_{U_i}(U_i) = \mathrm{Ext}^1{}_{\mathcal{O}_{X_{U_i}}}(x_{i*}\mathcal{O}_{U_i}, \mathcal{O}_{X_{U_i}})$. If this is the case, letting u be the unit in $R_i{}^{\times}$ such that $a = ub$, we get a diagram of exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_{X_{U_i}} & \longrightarrow & \mathcal{F}_{|X_{U_i}} & \longrightarrow & x_{i*}\mathcal{O}_S & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow u & & \\
0 & \longrightarrow & \mathcal{O}_{X_{U_i}} & \xrightarrow{\alpha} & \mathcal{G}(x)_{|X_{U_i}} & \xrightarrow{\beta} & x_{i*}\mathcal{O}_{U_i} & \longrightarrow & 0
\end{array}$$

where the rightmost rectangle is the pullback diagram of u and β, and the map from $\mathcal{O}_{X_{U_i}}$ to $\mathcal{F}_{|X_{U_i}}$ is given by α and the zero map to $x_{i*}\mathcal{O}_S$. Since u is an isomorphism, also the arrow $\mathcal{F}_{|X_{U_i}} \to \mathcal{G}(x)_{|X_{U_i}}$ must be an isomorphism. So to conclude, it is sufficient to show that if an extension $\mathcal{F}$ of $x_{i*}\mathcal{O}_{U_i}$ by $\mathcal{O}_{X_{U_i}}$ is torsion-free rank one, then it corresponds to a unit in $R_i = \mathcal{O}_{U_i}(U_i) = \mathrm{Ext}^1{}_{\mathcal{O}_{X_{U_i}}}(x_*\mathcal{O}_{U_i}, \mathcal{O}_{X_{U_i}})$. So suppose $\mathcal{F}$ corresponds to $s \in R_i$. Let $T = \operatorname{Spec} R_i/sR_i$, $f : T \to U_i$ the map corresponding to the quotient $R_i \to R_i/sR_i$. Pulling back via f an exact sequence

$$0 \to \mathcal{O}_{X_{U_i}} \to \mathcal{F} \to x_{i*}\mathcal{O}_{U_i} \to 0$$

preserves exactness, because $x_{i*}\mathcal{O}_{U_i}$ is obviously U_i-flat. Hence we obtain another exact sequence:

$$0 \to \mathcal{O}_{X_T} \to f^*\mathcal{F} \to x_{i*}\mathcal{O}_T \to 0.$$

The pullback f^* induces then a morphism of R_i-modules

$$\begin{array}{ccc}
R_i & \longrightarrow & R_i/sR_i \\
\| & & \| \\
\mathrm{Ext}^1{}_{\mathcal{O}_{X_{U_i}}}(x_{i*}\mathcal{O}_{U_i}, \mathcal{O}_{X_{U_i}}) & \longrightarrow & \mathrm{Ext}^1{}_{\mathcal{O}_{X_T}}(x_{i*}\mathcal{O}_T, \mathcal{O}_{X_T})
\end{array}$$

which is the quotient by sR_i. Hence the sheaf $\mathcal{F}$ corresponding to s in the left-hand side goes to zero on the right-hand side, i.e. $f^*\mathcal{F} \cong x_{i*}\mathcal{O}_T \oplus \mathcal{O}_{X_T}$. But if $\mathcal{F}$ is TFR1, also $f^*\mathcal{F}$ is, and $x_*\mathcal{O}_T \oplus \mathcal{O}_{X_T}$ is TFR1 if and only if T is the empty scheme, i.e., s is a unit in R_i, as we wished to show.

This shows that for all $i \in I$, $\mathcal{F}_{|X_{U_i}} \cong \mathcal{G}(x)_{|X_{U_i}}$. In particular, for all $i \in I$, there is exactly one such isomorphism inducing an isomorphism on the rigidified sheaves

$$(F_{|X_{U_i}}, \varphi_{|U_i}) \xrightarrow{\sigma_i} (\mathcal{G}(x)_{|X_{U_i}}, 1).$$

Since for all $i, j \in I$ there is exactly one isomorphism

$$(F_{|X_{U_i \cap U_j}}, \varphi_{|U_i \cap U_j}) \to (\mathcal{G}(x)_{|X_{U_i \cap U_j}}, 1)$$

the isomorphisms $\{\sigma_i\}_{i \in I}$ coincide on the intersections $X_{U_i} \cap X_{U_j} \cong X_{U_i \cap U_j}$, and therefore give a global isomorphism

$$\mathcal{F} \xrightarrow{\sigma} \mathcal{G}(x)$$

which yields a unique isomorphism

$$(\mathcal{F}, \varphi) \xrightarrow{\sigma} (\mathcal{G}(x), 1),$$

completing the proof.

□

3.5 A smooth surjective morphism $T \to \mathbb{T}'$

In this section we will use Theorem 3.4.6 to find a smooth and surjective morphism $\mathbb{T}'_\epsilon \to \mathbb{T}'$ and show that $\mathbb{T}'_\epsilon$ is a scheme T covered via open immersions by copies of X.

Proposition 3.5.1. *Let $\mathbb{T}^0$ and $\mathbb{T}'$ be the stacks of very simple and simple torsion-free rank one sheaves. Then*

a) $\mathbb{T}'$ is an open substack of $\mathbb{T}$.

b) $\mathbb{T}^0$ is an open substack of $\mathbb{T}'$.

Proof.

a) It is enough to show that for every scheme S and torsion-free rank one sheaf $\mathcal{F}$ on X_S, the locus of S where the condition

$$\mathrm{End}(F_{\overline{s}}) = k(\overline{s}) \text{ for all geometric points } \overline{s} \text{ of } S$$

holds is open in S.

We first assume that S is locally noetherian, and we show that the complement of the above locus is closed. Notice that for s a geometric point of S, $\mathrm{End}(\mathcal{F}_s)$ has dimension greater than 1 if and only if $\mathcal{F}_s$ is a pushforward from a normalization of X_s (Lemma 1.2.9). Call $x_0, x_1 : \operatorname{Spec} k \to X$ the two singular points. We call with the same name their base changes to any k-scheme. We are looking for those $s \in S$ such that $\dim_{k(s)} x_0^* s^* \mathcal{F} = 2$ and $\dim_{k(s)} x_1^* s^* \mathcal{F} = 2$. There is an isomorphism $x_0^* s^* \mathcal{F} \cong s^* x_0^* \mathcal{F}$, and for all $s \in S$, $s^* x_0^* \mathcal{F}$ is either of dimension 1 or 2. Now, on a locally noetherian base, the dimension of the fiber is an upper semicontinuous function, and this shows that the locus is indeed closed.

In the case of a general base scheme S, we can first reduce to the case where S is affine, and then express S as a limit of noetherian schemes S_i [11, TAG 01Z7]. For each S_i, let U_i be the open locus where condition $\mathrm{End}(F_{\bar{s}}) = k(\bar{s})$ is satisfied. Then the U_i give cartesian diagrams

$$\begin{array}{ccc} \mathbb{T}' & \longrightarrow & \mathbb{T} \\ \uparrow & & \uparrow \\ U_i & \longrightarrow & S_i \end{array}$$

Then also

$$\begin{array}{ccc} \mathbb{T}' & \longrightarrow & \mathbb{T} \\ \uparrow & & \uparrow \\ \lim U_i & \longrightarrow & \lim S_i \end{array}$$

is cartesian, and moreover $\lim U_i \to \lim S_i$ is an open immersion.

b) We first prove that $\mathbb{T}^0$ is a subcategory of $\mathbb{T}'$. Let $\mathcal{F}$ be an object of $\mathbb{T}^0(S)$. For all geometric points s of S, $H^1(X_s, \mathcal{F}_s) = R^1 p_* \mathcal{F} \otimes k(s) = 0$, and $H^0(X_s, \mathcal{F}_s) = p_* \mathcal{F} \otimes k(s) \cong k(s)$. Then $\mathcal{F}_s$ is not a pushforward of a line bundle from a normalization of X_S, hence $\mathrm{End}(\mathcal{F}_s) = k(s)$.

We pass to checking that conditions of Definition 3.2.2 are open on S. The sheaf $R^1 p_* \mathcal{F}$ is of finite presentation and therefore its zero locus is an open subscheme $U_0 \subset S$. On U_0, $p_* \mathcal{F}$ is locally free, and the locus where it is of rank 1 is a connected component V_0 of U_0, hence open in S. Finally, for the last condition, let X_1 and X_2 be the irreducible components of X. Now, for each point $x \in X_i(V_0)$, let $V_{i,x}$ be the zero locus in V_0 of $x^* \mathcal{G}$, where $\mathcal{G} = \mathrm{coker}(\mathcal{O}_{X_S} \xrightarrow{t} \mathcal{F})$. Clearly $V_{i,x}$ is open, and so is $V_i = \bigcup_{x \in X_i(V_0)} V_{i,x}$. The claim is that the locus of V_0 where the support of $\mathcal{G}$ is finite is exactly the open subscheme $V_1 \cap V_2$. This is equivalent to saying that the support of $\mathcal{G}$ is finite over an open subscheme $T \subset V_0$ if and only if there exist two points $x_1 : T \to X_1$ and $x_2 : T \to X_2$ such that $x_1^* \mathcal{G}_{|X_1} = 0$ and $x_2^* \mathcal{G}_{|X_2} = 0$. So let's check this.

Suppose first that there exist two such points. Then for all $s \in T$, the pullback $\mathcal{G}_s$ on X_s has fibre zero at each of the two points $x_{1,s}$ and $x_{2,s}$, which lie on different components of X_s. Then the global section t restricted to X_s cannot be zero on a whole component, otherwise the support of $\mathcal{G}_s = \mathcal{F}_s / \mathcal{O}_{X_s} t$ would contain that component, including one of the two points $x_{1,s}, x_{2,s}$. Hence t has finitely many zeroes and the support of $\mathcal{G}_s$ is finite. Finally, being $X_T \to T$

a proper morphism of finite presentation, the condition $\operatorname{Supp}\mathcal{G} \to T$ finite is implied by $\operatorname{Supp}\mathcal{G}_s$ being finite for all $s \in T$. [4, 8.11.1]

Suppose now that $\operatorname{Supp}\mathcal{G} \to T$ is finite. Then, we saw in the proof of lemma 3.4.6 that the support is a closed subscheme corresponding to a section $x : T \to X_T$. Then there exist $x_1 : T \to X_{1,T}$ and $x_2 : T \to X_{2,T}$ such that $x_1(T) \cap x(T) = \emptyset$ and $x_2(T) \cap x(T) = \emptyset$, so that $x_1^*\mathcal{G} = 0$ and $x_2^*\mathcal{G} = 0$.

This proves that $U := V_1 \cap V_2$ is indeed open.

□

Corollary 3.5.2. *Let $\mathbb{T}^0_\epsilon$ and $\mathbb{T}'_\epsilon$ be the stacks of very simple and simple rigidified torsion-free rank one sheaves. Then*

a) $\mathbb{T}'_\epsilon$ is an open substack of $\mathbb{T}_\epsilon$.

b) $\mathbb{T}^0_\epsilon$ is an open substack of $\mathbb{T}'_\epsilon$.

Proof. The diagram

$$\begin{array}{ccc} \mathbb{T}'_\epsilon & \longrightarrow & \mathbb{T}_\epsilon \\ \downarrow & & \downarrow \\ \mathbb{T}' & \longrightarrow & \mathbb{T} \end{array}$$

where the horizontal arrows are the inclusions and the vertical arrows are the forgetful functors, is cartesian. Indeed, for all k-schemes S, an object of $\mathbb{T}'(S) \times_{\mathbb{T}(S)} \mathbb{T}_\epsilon(S)$ is a triple $(\mathcal{F}, (\mathcal{G}, \varphi), \sigma)$ with $\mathcal{F}$ simple on X_S, $(\mathcal{G}, \varphi)$ a rigidified TFR1 on X_S, and $\sigma : \mathcal{F} \to \mathcal{G}$ an isomorphism. Then the functor

$$\mathbb{T}'_\epsilon(S) \to \mathbb{T}'(S) \times_{\mathbb{T}(S)} \mathbb{T}_\epsilon(S) \quad (\mathcal{F}, \varphi) \mapsto (\mathcal{F}, (\mathcal{F}, \varphi), \mathrm{id})$$

is an equivalence of categories. □

We have seen that $\mathbb{T}^0_\epsilon$ is an open substack of $\mathbb{T}'_\epsilon$, the stack of simple torsion-free rank one sheaves. What we would like to prove next is that $\mathbb{T}'_\epsilon$ can be covered by translates of $\mathbb{T}^0_\epsilon$.

Recall the action of $\mathbb{Z} \times \mathbb{Z}$ on $\mathbb{T}'$ that we constructed in section 3.2. The action can be of course extended to rigidified simple tfr1 sheaves, by setting

$$(a, b) \cdot (\mathcal{F}, \varphi) = (\mathcal{F}(a, b), \varphi).$$

We have then the following.

Lemma 3.5.3. *Let*

$$F : \bigsqcup_{\mathbb{Z} \times \mathbb{Z}} \mathbb{T}^0_\epsilon \to \mathbb{T}'_\epsilon$$

be the morphism of stacks given by

$$F(S) : (a, b, (\mathcal{F}, \varphi)) \mapsto (a, b) \cdot (\mathcal{F}, \varphi).$$

Then F is surjective.

Proof. Of course, being $\mathbb{T}^0_\epsilon$ an open substack of $\mathbb{T}'_\epsilon$, F is representable and we can talk about its being surjective. We have to show that for every S and $(\mathcal{F},\varphi)$ rigidified, simple torsion-free rank 1 sheaf on X_S, in the cartesian diagram

$$\begin{CD} \coprod_{\mathbb{Z}\times\mathbb{Z}} \mathbb{T}^0_\epsilon @>{F}>> \mathbb{T}'_\epsilon \\ @AAA @AA{(\mathcal{F},\varphi)}A \\ (\coprod_{\mathbb{Z}\times\mathbb{Z}} \mathbb{T}^0_\epsilon) \times_{\mathbb{T}'_\epsilon} S @>>> S \end{CD}$$

the lower horizontal arrow is a surjective morphism of schemes. This amounts to prove that for any point s of S, its fibre via the lower horizontal arrow is non-empty. We can of course reduce to geometric points $\overline{s}$ of S. Then we are done if we show that for every algebraically closed field K, any morphism

$$\operatorname{Spec} K \to \mathbb{T}'_\epsilon$$

factors via F. But this follows trivially from Lemma 3.2.6. □

Precomposing the map in 3.5.3 with the inclusion $\mathbb{T}^0_\epsilon \to \coprod_{\mathbb{Z}\times\mathbb{Z}} \mathbb{T}^0_\epsilon$ of $\mathbb{T}^0_\epsilon$ given by the couple (a,b) induces a morphism

$$\mathbb{T}^0_\epsilon \to \mathbb{T}'_\epsilon$$

given by $\mathcal{F} \mapsto (a,b)\cdot\mathcal{F}$. This is still an open immersion. To see this, it is enough to check that for all schemes S, the morphism

$$f : \mathbb{T}'_\epsilon \to \mathbb{T}'_\epsilon,\ \ \mathcal{F} \mapsto (a,b)\cdot\mathcal{F}$$

is an isomorphism. So let $(\mathcal{F},\varphi)$ be an object of $\mathbb{T}'_\epsilon(S)$. Then the diagram

$$\begin{CD} \mathbb{T}'_\epsilon @>{f}>> \mathbb{T}'_\epsilon \\ @A{(-a,-b)\cdot(\mathcal{F},\varphi)}AA @AA{\mathcal{F}}A \\ S @>{\mathrm{id}}>> S \end{CD}$$

commutes and is trivially cartesian.

This, together with lemmas 3.5.1 and 3.5.3, proves that indeed $\mathbb{T}'_\epsilon$ is covered via open immersions by copies of $\mathbb{T}^0_\epsilon$. Then $\mathbb{T}'_\epsilon$ is a scheme, which we denote by T'.

Finally, we show that T' covers smoothly $\mathbb{T}'$.

Lemma 3.5.4. *The map of stacks*

$$T' \cong \mathbb{T}'_\epsilon \to \mathbb{T}'$$

given by forgetting the rigidification datum is smooth and surjective.

Proof. For every k-scheme S and every simple TFR1 $\mathcal{F}$ on X_S, the diagram

$$\begin{CD} \mathbb{T}'_\epsilon @>>> \mathbb{T}' \\ @AAA @AA{\mathcal{F}}A \\ \underline{\mathrm{Isom}}_S(\mathcal{O}_S, \epsilon^*\mathcal{F}) @>>> S \end{CD}$$

is cartesian. Now, for any $s \in S$, take an open neighbourhood $U \subset S$ of s such that $(\epsilon^*\mathcal{F})_{|U} \cong \mathcal{O}_S$. Then we can extend the diagram above to a cartesian diagram

$$\begin{array}{ccc}
\mathbb{T}'_\epsilon & \longrightarrow & \mathbb{T}' \\
\uparrow & & \uparrow{\scriptstyle \mathcal{F}} \\
\underline{\mathrm{Isom}}_S(\mathcal{O}_S, \epsilon^*\mathcal{F}) & \longrightarrow & S \\
\uparrow & & \uparrow \\
\underline{\mathrm{Isom}}_U(\mathcal{O}_U, \mathcal{O}_U) & \longrightarrow & U
\end{array}$$

Now, $\underline{\mathrm{Isom}}_U(\mathcal{O}_U, \mathcal{O}_U)$ is simply $\mathbb{G}_{m,U}$, and the map $\mathbb{G}_{m,U} \to U$ is smooth and surjective. □

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