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Master Thesis

On the sup-norm of holomorphic cusp forms

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Chapter 0

Introduction

0.1 Statement of the problem, known results and applications

The goal of this thesis is to study different techniques to find an upper bound for the sup-norm of holomorphic cuspidal eigenforms of weight k over $\Gamma_0(q)$, where q is a prime number that tends to infinity. If f_0 is such a cuspidal eigenform, it is shown that

$$\|y^{k/2} f_0\|_\infty \ll q^{-\frac{1}{22}+\epsilon} \|f_0\|_2$$

This problem has been recently studied in a more general framework by V. Blomer and R. Holowinsky [BH10], and by N. Templier [Tem10]. Namely they studied upper bounds for the sup-norm of Hecke-Maass eigenforms on arithmetic surfaces. Let f be an Hecke-Maass cuspidal newform of squarefree-level N and bounded Laplace eigenvalue. Blomer and Holowinsky provided a non-trivial bound when f is non-exceptional:

$$\|f\|_\infty \ll q^{-\frac{1}{37}} \|f\|_2$$

They proved also that this bound holds true for $f = y^{k/2} F$, where F is a holomorphic cusp-form of weight k.
In his article Templier improved this bound and extended this result also for a large class of f. He used a different approach in that he relies on the geometric side of the trace formula, obtaining

$$\|f\|_\infty \ll q^{-\frac{1}{23}} \|f\|_2$$

In my work I have recovered the same bound as Templier in the case in which $f = y^{k/2} F$, with F a holomorphic cusp-form of weight k and prime level q.

Since f is cuspidal, then it is bounded on any neighborhood of any cusp, so it is uniformly bounded on the whole hyperbolic plane $\mathbb{H}$. So it is a natural question to try to quantify this boundness in an effective way. Such bounds play an important role for instance in connection with subconvexity of L-functions [HM06, Proposition 4], and in connection with the mass equidistribution conjecture [Rud05, Appendix].

The problem of bounding cusp forms is also related to bounding periods of automorphic forms [MV06], and hence it is implicitly related to the subconvexity problem for automorphic L-functions.

As one can see in [JK04] the best possible bound for $N \to \infty$ is

$$\sup_{z\in\mathbb{H}} \sum_{F\in\mathcal{B}(N,1)} y^2|F(z)|^2 = \mathcal{O}(1)$$

Nothing beyond this average bound is known. For an individual Hecke cusp-form it recovers only the trivial bound, that comes from the Fourier expansion. On the other hand, since $Vol(F_0(N))$ is about N, one might conjecture

$$\|f\|_\infty \ll N^{-\frac{1}{2}}\|f\|_2$$

There is no real evidence for the validity of such a bound, except that it is trivially true for old forms of level 1. It is a very optimistic and a very strong conjecture, since it would imply the most optimistic bound for the L^p norms, and the Lindelof Hypotesis for automorfhic L-functions in the level aspect for $L(1/2, f)$, since $f(i/\sqrt{N}) \approx L(1/2, f)N^{-1/2}\|f\|_2$. This shows that in order to derive some subconvex bound for $L(1/2, f)$ in the level aspect by this method, one would need already a relatively strong pointwise bound $\|f\|_\infty \ll N^{-\frac{1}{4}-\delta}\|f\|_2$.

Note that as I have said above the conjecture holds true when f is an old form that comes from a level 1 form, so if we consider the prime level q, all the oldforms comes from level 1, and so it is reasonable to consider f a newform.

0.2 Structure of the thesis

This thesis is organized as follows:

The first chapter describes the theory of Modular Forms. In particular it focus on the algebraic structure of the set of cusp forms for congruence subgroups $\mathcal{S}_k(\Gamma_0(q))$. It is shown that this set is in fact a vector space which has an orthogonal basis made of simultaneous eigenforms for the Hecke operators $T(n)$, for all n coprime with q. This space can be decomposed as a direct sum of two subspaces (newforms and oldforms) which are stable under the Hecke operators $T(n)$, for n coprime with q, namely $\mathcal{S}_k(\Gamma_0(q)) = \mathcal{S}_k^{old}(\Gamma_0(q)) \oplus \mathcal{S}_k^{new}(\Gamma_0(q))$. Moreover we will see that there exists $\{f_1, \cdots, f_n\}$ orthogonal basis of $\mathcal{S}_k^{old}(\Gamma_0(q))$ made of eigenforms for all $T(m)$ such that $(m, q) = 1$, and $\{f_{n+1}, \cdots, f_J\}$ orthogonal basis of $\mathcal{S}_k^{new}(\Gamma_0(q))$ made of arithmetically normalized eigenforms for all $T(m)$. It is denoted by $\mathcal{S}_k^{P}(\Gamma_0(q))$ and it is unique up to permutation. More about that can be found in [Lan95], [Kob93] and [DS05].

In the second chapter we shall introduce the main theorem

Theorem 0.2.1 (The Main Theorem)**.** *Let q be a prime number and let $k \geq 4$ be a fixed integer. Let $f_0 \in \mathcal{S}_k^P(\Gamma_0(q))$ be an arithmetically normalized newform of weight k. Denote $g(z) = y^{k/2}|f_0(z)|$, where $z = x + iy \in \mathbb{H}$, then*

$$\|g\|_\infty \ll q^{\frac{1}{2}}q^{-\frac{1}{22}+\epsilon}$$

and some preliminaries that will be used throughout the whole thesis, namely the theory of the Atkin-Lehner operators [AL70], that together with the study of the Siegel sets will allow us to restrict the domain of our cusp form f_0 to $\mathfrak{S}_q$. Thus our problem becames easier since the area of investigation is sensitively smaller, and we obtain some useful informations for z.

The third chapter describes the two trivial methods to find an upper bound for the sup-norm of our cusp form. Namely, the first method is based on the properties of the coefficients of the Fourier expansion of f_0, while the second one uses the pre-trace formula described in appendix 1.

With chapters four and five we introduce the amplification method. This technique will be no sufficient in chapter four to improve the previous bounds, but it is so powerful that added to a Diophantine argument it will allow us to improve the trivial bound, at most for those z which lie in a certain reagion of our domain $\mathfrak{S}_q$: it will be splitted into two reagions, one "away from the cusps" and one "closed to the cusps". The amplification method fails for those z which are closed to the cusps, but in this reagion the trivial bound found with the Fourier coefficients will be good enough to improve the upper bound for $f_0(z)$ in the whole $\mathbb{H}$.

Chapter 1

Modular Forms and Hecke Algebra

In this chapter I want to recall some definitions and basic facts about *modular forms* and the *Hecke algebra* associated to them. In particular I am interested in the *cusp forms* over $\Gamma_0(q)$ which will be studied in detail.

1.1 Modular Forms

Denote by $SL_2(\mathbb{Z})$ the group of all matrices of determinant 1 with coefficients in $\mathbb{Z}$. It is called the *Modular Group*, and it is generated by the two matrices

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

If we consider the **hyperbolic plane** $\mathbb{H} = z \in \mathbb{C} : \Im z > 0$, there is an action of the modular group on this space given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(z) = \frac{az+b}{cz+d}$$

Definition 1.1.1. *Let k be an integer. A **modular form of weight k** is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ satisfing the relation*

$$f(z) = \frac{1}{(cz+d)^k} f(\gamma z) \textit{ for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$$

Moreover it has to be holomorphic at ∞.

To define this last notion, recall that the traslation matrix $T \in SL_2(\mathbb{Z})$, hence $f(Tz) = f(z+1) = f(z)$, so f is $\mathbb{Z}$-periodic. Since f is also holomorphic it induces an holomorphic function f_∞ on the punctured disc defined by $f_\infty(q) = f_\infty(e(z)) = f(z)$, where $e(z) = e^{2\pi i z}$. Also $e : \mathbb{H} \to D^* = \{q \in \mathbb{C} : 0 < |q| < 1\}$ is holomorphic and $\mathbb{Z}$-periodic, so we can write f_∞ as

$$f_\infty(q) = \sum_{n \in \mathbb{Z}} a_n q^n$$

The relation $|q| = e^{-2\pi \Im z}$ shows that $q \to 0$ as $\Im z \to \infty$. So thinking of ∞ as lying far in the imaginary direction, we have the following

Definition 1.1.2. *f is said to be **holomorphic at** ∞ if f_∞ extends holomorphically to the punctured point $q = 0$.*

Remark 1.1.3. *With this property the Laurent series of f_∞ sums over $n \in \mathbb{N}$. This means that f has a Fourier expansion*

$$f(z) = \sum_{n=0}^{\infty} a_n(f)e(nz)$$

Remark 1.1.4. *For simplify the notation we call $j(\gamma, z) = cz + d$ where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Moreover we define $f[\gamma]_k(z) = \frac{1}{j(\gamma,z)^k} f(\gamma z)$. With this notation $f : \mathbb{H} \to \mathbb{C}$ is a modular form of weight k if it is an holomorphic function on $\mathbb{H}$, holomorphic at ∞ and $f[\gamma]_k(z) = f(z)$ $\forall \gamma \in SL_2(\mathbb{Z})$. Moreover for all $\gamma, \gamma' \in SL_2(\mathbb{Z})$ and $z \in \mathbb{H}$ we have the following:*

- $j(\gamma\gamma', z) = j(\gamma, \gamma' z) j(\gamma', z)$;
- $(\gamma\gamma')(z) = \gamma(\gamma')(z)$;
- $[\gamma\gamma']_k = [\gamma]_k [\gamma']_k$ *as operators;*
- $\Im(\gamma(z)) = \frac{\Im z}{|j(\gamma,z)|^2}$.

Remark 1.1.5. *If k is an odd integer then considering $\gamma = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ we have $f(z) = (-1)^k f(\gamma z) = -f(z)$, thus the only modular form of weight k odd is the zero function. For this reason we will consider k even.*

The set of all modular forms of weight k is denoted by $\mathcal{M}_k(SL_2(\mathbb{Z}))$. In fact it is a $\mathbb{C}$-vector space. Moreover the product of a modular form of weight k with a modular form of weight l is a modular form of weight $k + l$. Thus the sum

$$\mathcal{M}(SL_2(\mathbb{Z})) = \bigoplus_{k \in \mathbb{Z}} \mathcal{M}_k(SL_2(\mathbb{Z}))$$

is a graded ring.

Definition 1.1.6. *A **cusp form of weight** k f is a modular form of weight k whose Fourier expansion has leading coefficient $a_0(f) = 0$,i.e.*

$$f(z) = \sum_{n=1}^{\infty} a_n(f)e(nz)$$

The set of all cusp forms is denoted by $\mathcal{S}_k(SL_2(\mathbb{Z}))$ and it is a $\mathbb{C}$-vector subspace of $\mathcal{M}_k(SL_2(\mathbb{Z}))$. Moreover the graded ring

$$\mathcal{S}(SL_2(\mathbb{Z})) = \bigoplus_{k \in \mathbb{Z}} \mathcal{S}_k(SL_2(\mathbb{Z}))$$

is an ideal of $\mathcal{M}(SL_2(\mathbb{Z}))$,

Remark 1.1.7. *A cusp form is a modular form such that $\lim_{\Im z \to \infty} f(z) = 0$. The limit point ∞ of $\mathbb{H}$ is called the cusp of $SL_2(\mathbb{Z})$.*

1.2 Modular Forms for Congruence Subgroups

Definition 1.2.1. *Let N be a positive integer. The* ***principal congruent subgroup of level*** *N is*

$$\Gamma(N) = \left\{ \gamma \in SL_2(\mathbb{Z}) : \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod N \right\}$$

In particular $\Gamma(1) = SL_2(\mathbb{Z})$. Being the kernel of the natural homomorphism $SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/N\mathbb{Z})$ $\Gamma(N)$ is normal in $SL_2(\mathbb{Z})$. In fact this map is a surjection, hence we have

$$[SL_2(\mathbb{Z}) : \Gamma(N)] = |SL_2(\mathbb{Z}/N\mathbb{Z})| = N^3 \prod_{p|N} \left(1 - \frac{1}{p^2}\right)$$

Definition 1.2.2. *A subgroup Γ of $SL_2(\mathbb{Z})$ is a* ***congruence subgroup*** *if $\exists N$ positive integer such that $\Gamma(N) \subset \Gamma$, in which case Γ is a congruence subgroup of level N. In particular any congruence subgroup has finite index in $SL_2(\mathbb{Z})$.*

Besides the principal congruence subgroups, the most important congruence subgroups are

$$\Gamma_0(N) = \left\{ \gamma \in SL_2(\mathbb{Z}) : \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod N \right\}$$

and

$$\Gamma_1(N) = \left\{ \gamma \in SL_2(\mathbb{Z}) : \gamma \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod N \right\}$$

Note that $\Gamma(N) \subset \Gamma_1(N) \subset \Gamma_0(N) \subset \Gamma(1)$.

Remark 1.2.3. *The map*

$$\Gamma_1(N) \to \mathbb{Z}/N\mathbb{Z}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto b \pmod N$$

is a surjection with kernel $\Gamma(N)$, hence $[\Gamma_1(N) : \Gamma(N)] = N$.

Remark 1.2.4. *Similarly the map*

$$\Gamma_0(N) \to (\mathbb{Z}/N\mathbb{Z})^*, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto d \pmod N$$

is a surjection with kernel $\Gamma_1(N)$, hence $[\Gamma_0(N) : \Gamma_1(N)] = \varphi(N)$.

Each congruence subgroup Γ contains a traslation matrix of th form

$$\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$$

for some minimal $h \in \mathbb{Z}_{>0}$. If an holomorphic function $f : \mathbb{H} \to \mathbb{C}$ satisfies the condition $f[\gamma]_k(z) = f(z)\ \forall \gamma \in \Gamma$ therefore is $h\mathbb{Z}$-periodic and thus $f(z) = f_\infty(e(z/h))$ where $f_\infty : D^* \to \mathbb{C}$ (D^* is the punctured disc) is holomorphic in D^*, so it has a Fourier expansion.

Definition 1.2.5. *We say that f is **holomorphic at** ∞ if f_∞ extends holomorphically to 0.*

Thus f has a Fourier expansion

$$f(z) = \sum_{n=0}^{\infty} a_n(f) e(z/h)^n$$

The idea for a congruence subgroup Γ is to consider $\hat{\mathbb{H}} = \mathbb{H} \cup \{\infty\} \cup \mathbb{Q}$, and then to identify adjoined points under Γ-equivalence. Each Γ-equivalence class of points in $\{\infty\} \cup \mathbb{Q}$ is called a **cusp of** Γ. Since each rational number s takes the form $s = \alpha(\infty)$ for some $\alpha \in SL_2(\mathbb{Z})$, then the number of cusps is at most the number of cosets $\Gamma\alpha$ in $SL_2(\mathbb{Z})$, a finite number since $[SL_2(\mathbb{Z}) : \Gamma]$ is finite.

Definition 1.2.6. *Let Γ be a congruence subgroup of $SL_2(\mathbb{Z})$, and let k be an integer. A function $f : \mathbb{H} \to \mathbb{C}$ is a **modular form of weight k with respect to** Γ if*

1. *f is holomorphic;*
2. *$f\,[\gamma]_k\,(z) = f(z)\ \forall \gamma \in \Gamma$;*
3. *$f\,[\gamma]_k$ is holomorphic at ∞ for all $\gamma \in SL_2(\mathbb{Z})$.*

The modular forms of weight k with respect to Γ are denoted by $\mathcal{M}_k(\Gamma)$.

Condition (3) means that f should be holomorphic at all cusps.

Definition 1.2.7. *A **cusp form of weight** k **with rispect to** Γ is a modular form of weight k with respect to Γ such that the Fourier expansion of $f\,[\alpha]$ has the first coefficient $a_0 = 0$ for all $\alpha \in SL_2(\mathbb{Z})$.*
The cusp forms of weight k with respect to Γ are denoted $\mathcal{S}_k(\Gamma)$.

1.2.8 The congruence subgroup $\Gamma_0(q)$

I want to give some more results about the congruence subgroup $\Gamma_0(q)$, where q denotes a prime number, being the group we are more interested in. These results will be useful later.

Lemma 1.2.9. *The index of $\Gamma_0(q)$ in $SL_2(\mathbb{Z})$ is $[SL_2(\mathbb{Z}) : \Gamma_0(q)] = q + 1$*

Proof. In the previous section we have seen that

$$[SL_2(\mathbb{Z}) : \Gamma(N)] = N^3 \prod_{p|N} \left(1 - \frac{1}{p^2}\right)$$

$$[\Gamma_1(N) : \Gamma(N)] = N$$

and

$$[\Gamma_0(N) : \Gamma_1(N)] = \varphi(N)$$

Since in this case q is prime, we obtain

$$[SL_2(\mathbb{Z}) : \Gamma(q)] = q(q^2 - 1)$$

$$[\Gamma_1(q) : \Gamma(q)] = q$$

and

$$[\Gamma_0(q) : \Gamma_1(q)] = q - 1$$

The conclusion follows immediately. $\square$

In the next chapter, when we will introduce Hecke operators, we will have to work with the group

$$G_n(q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) : ad - bc = n,\ q|c,\ (a,q) = 1 \right\}$$

In particular $\Gamma_0(q)$ acts on $G_n(q)$ by left multiplication, hence we are interested in $\Gamma_0(q) \setminus G_n(q)$.

Proposition 1.2.9.1. *The set* $\Delta_n(q) = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in M_2(\mathbb{Z}) : ad = n,\ 0 \leq b < d,\ (a,q) = 1 \right\}$ *is a system of representatives of* $\Gamma_0(q) \setminus G_n(q)$*.*

Proof. We prove first that for all $A \in G_n(q)$ $\exists B \in \Gamma_0(q)$ such that $BA \in \Delta_n(q)$: Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $ad - bc = n$, $q|c$ and $(a,q) = 1$. Define $\gamma = \frac{c}{(a,c)}$ and $\delta = \frac{-a}{(a,c)}$, thus $q|\gamma$. Let α and β be such that $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0(q)$, call it B. So we have $\alpha\delta - \beta\gamma = \frac{-\alpha a - \beta c}{(a,c)} = 1$, hence $\alpha a + \beta c = -(a,c)$. We have

$$BA = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -(a,c) & \alpha b + \beta d \\ 0 & -n/(a,c) \end{pmatrix}$$

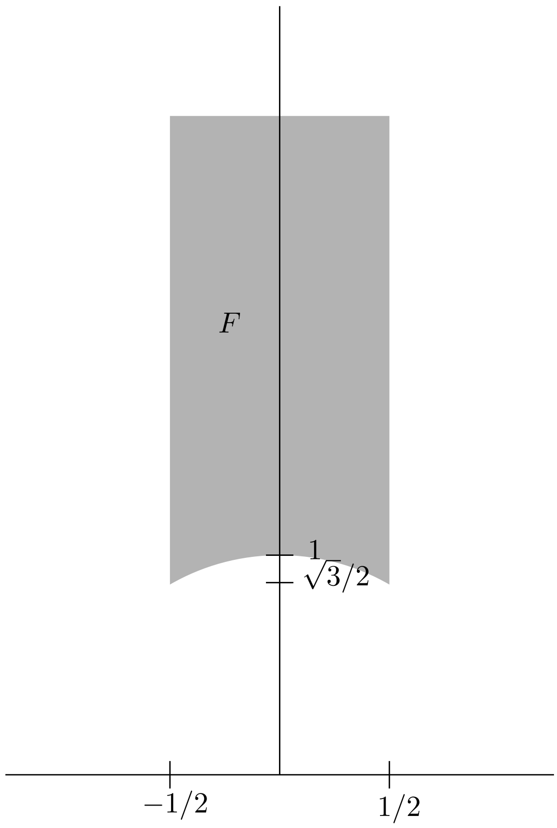
Since $(a,q) = 1$ then $(-(a,c), q) = 1$. Moreover the determinant is exactly n, and multiplying BA on the left by the matrix $T^m = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$ we can obtain $0 \leq \alpha b + \beta d < -n/(a,c)$. Note that $T^m B \in \Gamma_0(q)$.
To prove that $\Delta_n(q)$ is a system of representatives we have to check that two elements of $\Delta_n(q)$ can not represent the same left coset: suppose

$$AB = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} = C$$

with $A \in \Gamma_0(q)$ and B, $C \in \Delta_n(q)$, then $\gamma a = 0$, hence $\gamma = 0$. So $\alpha\delta = 1 \Rightarrow \alpha = \delta = \pm 1$. If $\alpha = \delta = 1$ then $d = z$, $a = x$ and $0 \leq b + \beta d = y < z = d$ which implies $\beta = 0$ and $A = id$. If $\alpha = \delta = -1$ similarly we obtain $A = -id$. This proves the proposition. $\square$

To conclude this section I want to describe a fundamental domain for $\Gamma_0(q)$ in $\mathbb{H}$. In general $SL_2(\mathbb{Z})$ acts on $\mathbb{H}$ as usual, so it is divided into equivalence classes; two points are said to be in the same equivalence class if there exists $\gamma \in SL_2(\mathbb{Z})$ that sends one to the other. In particular if Γ is a subgroup of $SL_2(\mathbb{Z})$, then we say that two points are Γ-equivalent if an element of Γ send one to the other. A closed (and usually is also required simply connected) region in $\mathbb{H}$, call it F, is said to be a **fundamental domain for** Γ if every $z \in \mathbb{H}$ is Γ-equivalent to a point of F, but no

two distinct points in the interior of F are Γ-equivalent. The most famous fundamental domain of $SL_2(\mathbb{Z})$ is
$$F = \left\{ z \in \mathbb{H} : -\frac{1}{2} \leq \Re z \leq \frac{1}{2} \text{ and } |z| \geq 1 \right\}$$



Remark 1.2.10. *If we consider $\mathbb{H} \cup \infty \cup \mathbb{Q} = \hat{\mathbb{H}}$, one can easily see that ∞ could be send to any rational number by an element of $SL_2(\mathbb{Z})$, so a fundamental domain for $\hat{\mathbb{H}}$ is $\hat{F} = F \cup \infty$.*

Suppose now to have a subgroup Γ of $SL_2(\mathbb{Z})$ of finite index; we want to construct a fundamental domain F' of Γ starting from a fundamental domain F of $SL_2(\mathbb{Z})$: since it is of finite index (with respect to the left multiplication), say $[SL_2(\mathbb{Z}) : \Gamma] = n$, there exists $\alpha_1, \cdots, \alpha_n$ such that $SL_2(\mathbb{Z}) = \coprod_{i=1}^n \alpha_i \Gamma$.

Lemma 1.2.11. *$F' = \coprod_{i=1}^n \alpha_i^{-1} F$.*

Proof. First of all we verify that any $z \in \mathbb{H}$ is Γ-eqiuvalent to an element of F': since F is a fundamental domain of $SL_2(\mathbb{Z})$, there exists $\gamma \in SL_2(\mathbb{Z})$ such that $\gamma z \in F$. Then for some i we have $\gamma = \alpha_i \gamma'$ with $\gamma' \in \Gamma$, hence $\gamma' z \in \alpha_i^{-1} F \in F'$.

Now, if two points z_1 and z_2 in the interior of F' are Γ-equivalent, then there exists $\gamma \in \Gamma$ such that $\gamma_{z_1} = z_2$. Moreover there exists i and j such that $z_1 = \alpha_i^{-1} w_1$ and $z_2 = \alpha_j^{-1} w_2$ for some w_1, $w_2 \in F$. So we have $w_2 = \alpha_j \gamma \alpha_i^{-1} w_1$, that is a contradiction because $\alpha_j \gamma \alpha_i^{-1} \in SL_2(\mathbb{Z})$ and w_1, $w_2 \in F$. $\square$

In the case of $\Gamma_0(q)$ we have that its fundamental domain is the disjoint union of $q+1$ transformation of F. We want to describe the structure of these transformation:

Proposition 1.2.11.1. *Let n be a positive integer. A set of inequivalent cusps for $\Gamma_0(n)$ is given by the following fractions:*

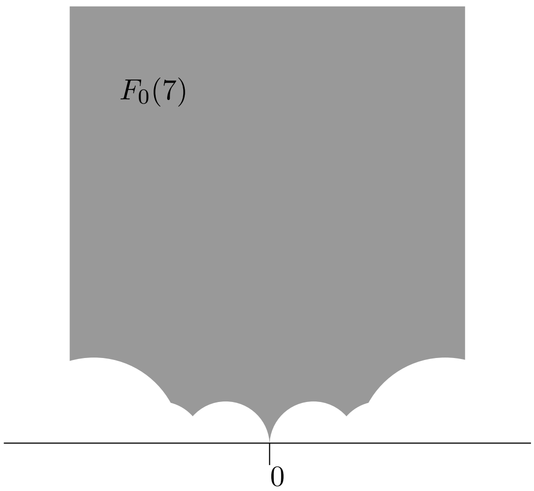
$$\frac{u}{v} \text{ with } q|v \ , \ (u,v) = 1 \ , \ u \pmod{(v, q/v)}$$

Hence the number of inequivalent cusps is

$$\sum_{vw=q} \varphi((v,w))$$

Remark 1.2.12. *A proof for the above proposition can be found in [Iwa97, p. 36].*

So in the case of q prime we have two cusps that could be represented by ∞ and 0. Note that ∞ and 0 are not $\Gamma_0(q)$-equivalent. So the fundamental domain $\overline{F_0(q)}$ of $\Gamma_0(q)$ in $\hat{\mathbb{H}}$ is the following



1.3 Hecke operators and Petersson inner product

1.3.1 Hecke operators

In a general setting, the *Hecke operators* are averaging operators over a suitable finite collection of double cosets with respect to a group, therefore a great deal of the Hecke theory belongs to linear algebra.
In this section I want to present the theory of *Hecke operators* in the context of $SL_2(\mathbb{Z})$ and of the congruence subgroup $\Gamma_0(q)$.

Assume k to be a fixed integer. For any $\alpha \in GL_2^+(\mathbb{R})$ the operator $[\alpha]_k$ is defined on $f : \mathbb{H} \to \mathbb{C}$ by

$$f[\alpha]_k(z) = (det\alpha)^{k/2} \frac{1}{j(\alpha, z)^k} f(\alpha z)$$

For a positive integer n define

$$G_n = \{\alpha \in GL_2(\mathbb{Z}) : det\alpha = n\}$$

The modular group $SL_2(\mathbb{Z})$ acts on G_n from both sides, so that $G_n = SL_2(\mathbb{Z})G_n = G_n SL_2(\mathbb{Z})$.

Lemma 1.3.2. *The collection*

$$\Delta_n = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : ad = n,\ 0 \leq b < d \right\}$$

is a complete set of right coset representatives of G_n modulo $SL_2(\mathbb{Z})$

The proof is very similar to that we have done in the previuos section to find a system of representatives of $G_n(q)$ modulo $\Gamma_0(q)$.

Definition 1.3.3. *Let $f \in \mathcal{M}_k(SL_2(\mathbb{Z}))$, n be a positive integer, then the operator $T(n)$ defined by*

$$T(n)f = \frac{1}{\sqrt{n}} \sum_{\gamma \in \Delta_n} f[\gamma]_k$$

*is called **Hecke operator for** $SL_2(\mathbb{Z})$.*

Remark 1.3.4. *Usually the exponent of n in this definition is different from $-1/2$, and it depends on k. The reason of this normalization will be explain later.*

Remark 1.3.5.

- *Hecke operators maps modular forms to modular forms;*
- *Hecke operators maps cusps forms to cusps forms;*
- *$T(n)T(m) = T(m)T(n)$.*

Consider now the congruence subgroup $\Gamma_0(q)$;

Recall 1.3.6. $\Delta_n(q) = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : ad = n,\ 0 \leq b < d,\ (a, q) = 1 \right\}$ *is system of representatives of $G_n(q)$ modulo $\Gamma_0(q)$.*

Definition 1.3.7. *Let $f \in \mathcal{M}_k(\Gamma_0(q))$, n be a positive integer, then the operator $T(n)$ defined by*

$$T(n)f = \frac{1}{\sqrt{n}} \sum_{\gamma \in \Delta_n(q)} f[\gamma]_k$$

is called ***Hecke operator for*** *$\Gamma_0(q)$.*

Remark 1.3.8. *Hecke operators send modular forms (resp. cusp forms) with respect to $\Gamma_0(q)$ to modular forms (resp. cusp forms) with respect to $\Gamma_0(q)$.*

To analize more properties about these operators we need to define an inner product in the space of cusps forms, namely the *Petersson inner product*.

1.3.9 Petersson inner product

To study the space of cusp forms $\mathcal{S}_k(\Gamma_0(q))$ further, we make it into an inner product space. The inner product will be defined as an integral. Let f and $g \in \mathcal{S}_k(\Gamma_0(q))$; we define the **Petersson inner product**

$$\langle f, g\rangle = \int_{F_0(q)} f(z)\overline{g(z)} y^k \frac{dxdy}{y^2}$$

where $z = x + iy$ and $F_0(q)$ is a fundamental domain of $\Gamma_0(q)$ in $\mathbb{H}$.

Clearly this product is linear in f and conjugate linear in g, Hermitian, symmetric and positive definite.

Remark 1.3.10. *$\frac{dxdy}{y^2}$ is called* ***hyperbolic measure*** *and it is denoted by $d\mu(z)$. It is invariant under $GL_2^+(\mathbb{R})$ meaning for all $\alpha \in GL_2^+(\mathbb{R})$ $d\mu(\alpha z) = d\mu(z)$ for all $z \in \mathbb{H}$. In particular it is $SL_2(\mathbb{Z})$-invariant.*

Remark 1.3.11. *Since $\mathbb{Q} \cup \infty$ is countable it has measure zero, hence it is enough for integrating over $\hat{\mathbb{H}}$.*

Remark 1.3.12. *For any continuos bouded function $\varphi : \mathbb{H} \to \mathbb{C}$ and any $\alpha \in SL_2(\mathbb{Z})$, the integral $\int_F \varphi(\alpha z) d\mu(z)$ converges. In particular it converges in $\mathcal{S}_k(\Gamma_0(q))$ but not necessary in $\mathcal{M}_k(\Gamma_0(q))$, so it cannot be extended to it.*

I recall that if T is an operator over a inner product space V, then its adjoint T^* is the unique operator such that $\langle Tv, w\rangle = \langle v, T^*w\rangle$ for all $v, w \in V$. The following proposition describe the adjoints of $T(n)$ over $\Gamma_0(q)$.

Proposition 1.3.12.1. *If $(n, q) = 1$ and $f, g \in \mathcal{S}_k(\Gamma_0(q))$ then*

$$\langle T(n)f, g\rangle = \langle f, T(n)g\rangle$$

i.e. the Hecke operators $T(n)$ with $(n, q) = 1$ are self adjoints.

An obvious consequence of this result is that $T(n)$'s are normal, and from the spectral theorem of linear algebra, given a commutative family of normal operators of a finite dimensional inner product space, the space has an orthogonal basis of simultaneous eigenvectors for the operators. In our case the eigenvectors are called eigenforms, thus we have the following

Theorem 1.3.13. *$\mathcal{S}_k(\Gamma_0(q))$ has an orthogonal basis of simultaneous eigenforms for the Hecke operators $T(n)$ such that $(n,q)=1$.*

Let $f \in \mathcal{M}_k(\Gamma_0(q))$ be an eigenform for $T(n)$ with n and q coprime, then there exists $\lambda(n) \in \mathbb{C}$ such that $T(n)f = \lambda(n)f$. Since the operator $T(n)$ is self adjoint we obtain $\lambda(n) = \overline{\lambda(n)}$, so the eigenvalues are all real.
For our task I want to normalize the Fourier coefficients of any cusp forms, to have a good relation between these coefficients and the eigenvalues. As we have already seen any cusp form has a Fourier expansion

$$f(z) = \sum_{n\geq 1} a_n(f)e(nz)$$

We define the **normalized Fourier coefficients** as

$$\psi_f(n) = \frac{a_n(f)}{n^{\frac{k-1}{2}}}, \text{ for all } n \geq 1$$

In particular we obtain

$$f(z) = \sum_{n\geq 1} \psi_f(n) n^{\frac{k-1}{2}} e(nz)$$

From the definition we compute the action of $T(m)$ on the Fourier expansion at infinity of a modular form:

$$\begin{aligned}
T(m)f(z) &= m^{-1/2} \sum_{\gamma \in \Delta_0(q)} f[\gamma] \\
&= m^{-1/2} \sum_{\substack{ad=m \\ (a,q)=1}} m^{k/2} d^{-k} \sum_{0\leq b<d} f\left(\frac{az+b}{d}\right) \\
&= \frac{1}{m^{\frac{k+1}{2}}} \sum_{\substack{ad=m \\ (a,q)=1}} a^k \sum_{0\leq b<d} \sum_{n\geq 0} \psi_f(n) n^{\frac{k-1}{2}} e\left(n\frac{az+b}{d}\right) \\
&= \sum_{n\geq 0} \psi_f(n) \frac{n^{\frac{k-1}{2}}}{m^{\frac{k+1}{2}}} \sum_{\substack{ad=m \\ (a,q)=1}} a^k e\left(\frac{anz}{d}\right) \sum_{0\leq b<d} e\left(\frac{nb}{d}\right) \\
&= \sum_{n\geq 0} \psi_f(n) \frac{n^{\frac{k-1}{2}}}{m^{\frac{k+1}{2}}} \sum_{\substack{ad=m \\ (a,q)=1 \\ dl=n}} a^k e\left(\frac{anz}{d}\right) d
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n\geq 0} \psi_f(n) \frac{n^{\frac{k-1}{2}}}{m^{\frac{k-1}{2}}} \sum_{\substack{ad=m \\ (a,q)=1 \\ dl=n}} a^{k-1} e\left(\frac{anz}{d}\right) \\
&= \sum_{n\geq 0} \sum_{\substack{ad=m \\ (a,q)=1 \\ al=n}} \psi_f\left(\frac{nd}{a}\right)\left(\frac{dn}{am}\right)^{\frac{k-1}{2}} a^{k-1} e(nz) \\
&= \sum_{n\geq 0} \sum_{\substack{ad=m \\ (a,q)=1 \\ al=n}} \psi_f\left(\frac{nm}{a^2}\right)\left(\frac{n}{a^2}\right)^{\frac{k-1}{2}} a^{k-1} e(nz) \\
&= \sum_{n\geq 0} \left(\sum_{\substack{d|(n,m) \\ (d,q)=1}} \psi_f\left(\frac{nm}{d^2}\right) n^{\frac{k-1}{2}} \right) e(nz)
\end{aligned}
$$

From this formula we obtain an important property of the eigenforms, namely

Proposition 1.3.13.1. *Let m be a positive integer, f an eigenform for $T(m)$ with eigenvalue $\lambda_f(m)$, then*

$$\lambda_f(m)\psi_f(1) = \psi_f(m)$$

Proof. $T(m)f = \lambda_f(m)f$ hence the coefficient of $e(z)$ is $\lambda_f(m)\psi_f(1)$; on the other hand by the previous formula we obtain $\lambda_f(m)\psi_f(1) = \sum_{d|(1,m),(d,q)=1} \psi_f\left(\frac{m}{d^2}\right) = \psi_f(m)$ □

Remark 1.3.14. *If f is an eigenform of $T(m)$ for all m then we can conclude that $\psi_f(1) \neq 0$ otherwise f is the zero function.*

Remark 1.3.15. *The relation between Hecke operators and Fourier coefficients of a modular forms is a proof of the commutativity property of the Hecke operators. We can obtain*

$$T(m)T(n) = \sum_{d|(n,m),(d,q)=1} T\left(\frac{nm}{d^2}\right)$$

In particular if $(n,m) = 1$ then $T(nm) = T(n)T(m)$. Therefore each $T(n)$ is the product of Hecke operators of the form $T(p^j)$ for some prime numbers p.

Remark 1.3.16. *We have*

$$T(p^{j+1}) = T(p)T(p^j) - T(p^{j-1})$$

Remark 1.3.17. *Always from the above formula it follows that*

$$\psi_f(m)\psi_f(n) = \psi_f(1) \sum_{d|(n,m),(d,q)=1} \psi_f\left(\frac{mn}{d^2}\right)$$

1.4 The Structure of $\mathcal{S}_k(\Gamma_0(q))$: Oldforms and Newforms

As we have seen before there exists a basis of $\mathcal{S}_k(\Gamma_0(q))$ consisting of eigenforms for Hecke operators $T(m)$ with m coprime with q. It would be more interesting to have a basis of eigenforms for all the Hecke operators without exceptions. In this case we are sure that the first coefficient of the Fourier expansion is non zero. Unfortunately in general it is not possible to find such a basis, but the problem in partially solved thanks to Atkin-Lehner theory, that we are going to explain.
The basic idea is that we can move from $\mathcal{S}_k(\Gamma_0(M))$ to $\mathcal{S}_k(\Gamma_0(N))$ where $M|N$ with two natural operators:

1. $\iota : \mathcal{S}_k(\Gamma_0(M)) \hookrightarrow \mathcal{S}_k(\Gamma_0(N))$ the natural inclusion;
2. For $d|\frac{N}{M}$, $[d] : \mathcal{S}_k(\Gamma_0(M)) \to \mathcal{S}_k(\Gamma_0(N))$ given by $f[d](z) = f(dz)$.

Remark 1.4.1. *Hecke operators $T(m)$ with m coprime with q commute with $[d]$, hence if f is an eigenform also $f[d]$ is an eigenform. However $f[d](z) = \sum_{n\geq 1} a_n(f)e(dnz)$, hence the first Fourier coefficient of $f[d]$ vanishes if $d > 1$.*

The theory of *newforms* remedies to this defect by considering that cusp forms such as $f[d] \in \mathcal{S}_k(\Gamma_0(N))$ are not really of level N but come from lower level. We shall study the case $N = q$ prime, but this results are true for all integers N.

Definition 1.4.2. *Let $\mathcal{S}_k^{old}(\Gamma_0(q))$ be the subspace of $\mathcal{S}_k(\Gamma_0(q))$ spanned by all cusp forms of the type $\iota(f)$ and $f[q]$ where $f \in \mathcal{S}_k(\Gamma_0(1))$.*
Let $\mathcal{S}_k^{new}(\Gamma_0(q))$ its orthogonal with respect to the inner product. Thus we have the orthogonal decomposition

$$\mathcal{S}_k(\Gamma_0(q)) = \mathcal{S}_k^{old}(\Gamma_0(q)) \oplus \mathcal{S}_k^{new}(\Gamma_0(q))$$

Proposition 1.4.2.1. *The subspaces $\mathcal{S}_k^{old}(\Gamma_0(q))$ and $\mathcal{S}_k^{new}(\Gamma_0(q))$ are stable under the Hecke operators $T(m)$ with $(m, q) = 1$.*

As a consequence, we have that each of this two subspaces has an orthonormal bases consisting of eigenforms of Hecke operators $T(m)$ with $(m, q) = 1$.

Definition 1.4.3. *$f \in \mathcal{S}_k^{new}(\Gamma_0(q))$ that is an eigenform of $T(m)$ for all positive integer m is called* ***newform****.*
$\mathcal{S}_k^{new}(\Gamma_0(q))$ is called the ***space of newforms of level*** *q and $\mathcal{S}_k^{old}(\Gamma_0(q))$ is called the* ***space of oldforms of level*** *q.*

Theorem 1.4.4 (Main Lemma)**.** *Let $f \in \mathcal{S}_k(\Gamma_0(q))$ sucht that $f(z) = \sum_{n\geq 1} a_n(f)e(nz)$ with $a_n(f) = 0$ for all n coprime with q, then $f \in \mathcal{S}_k^{old}(\Gamma_0(q))$*

Theorem 1.4.5. *$\mathcal{S}_k^{new}(\Gamma_0(q))$ has an orthogonal basis of normalized newforms.*

Proof. We already know that $\mathcal{S}_k^{new}(\Gamma_0(q))$ has an orthogonal basis consisting of eigenforms for $T(m)$ for all m coprime with q, in particular we know that for all such an m $\psi_f(m) = \lambda_f(m)\psi_f(1)$, where $\psi_f(i)$ are the normalized coefficient. If $\psi_f(1) = 0$ then by the main lemma $f \in \mathcal{S}_k^{old}(\Gamma_0(q))$ that is in contradiction with the hypotesis, so $\psi_f(1) \neq 0$ and we may assume it to be 1. Now for all

$m \in \mathbb{Z}_{>0}$ define $g_m = T(m)f - \psi_f(m)f \in \mathcal{S}_k^{new}(\Gamma_0(q))$. It is clearly an eigenform for all $T(n)$ such that $(n, q) = 1$ and $\psi_{g_m}(1) = \psi_{T(m)f}(1) - \psi_{\psi_f(m)f}(1) = \psi_f(m) - \psi_f(m) = 0$, hence by the main lemma each $g_m \in \mathcal{S}_k^{old}(\Gamma_0(q)) \cap \mathcal{S}_k^{new}(\Gamma_0(q)) = 0$, thus

$$T(m)f = \psi_f(m)f \text{ for all } m \in \mathbb{Z}_{>0}$$

□

Remark 1.4.6. *Note that if f is a normalized newform, i.e. $\psi_f(1) = 1$ then the eigenvalues $\lambda_f(m)$ for $T(m)$ are exactly the normalized Fourier coefficients $\psi_f(m)$.*

Theorem 1.4.7 (Multiplicity One Property)**.** *Let $f \in \mathcal{S}_k^{new}(\Gamma_0(q))$ be a non zero eigenform of $T(m)$ for all m coprime with q. If g satisfies the same conditions as f and has the same $T(m)$-eigenvalues, then $g = cf$ for some constant c.*

This theorem implies that for each eigenvalue there exists exactly one normalized newform associated to it. The set of normalized newform in the space $\mathcal{S}_k^{new}(\Gamma_0(q))$ is an orthogonal basis of the space and it is unique.

Lemma 1.4.8. *If $f \in \mathcal{S}_k(\Gamma_0(q))$ is an eigenform for all $T(m)$, then it is either an oldform or a newform.*

Proof. If $\psi_f(1) = 0$ then $f = 0$ because it is an eigenform. Assume $\psi_f(1) \neq 0$ and assume f is normalized, hence $\psi_f(1) = 1$. Then $T(n)f = \psi_f(m)f$ for all m. f can be write as $f = g + h$ with $g \in \mathcal{S}_k^{old}(\Gamma_0(q))$ and $h \in \mathcal{S}_k^{new}(\Gamma_0(q))$. Thus we have $\psi_f(m)f = \psi_f(m)g + \psi_f(m)h$, so g and h are eigenforms with the same eigenvalues of f. If $h = 0$ then $f = g$ is an oldform, otherwise $h \neq 0$ implies $\psi_h(1)$ because it is a newform, and $T(m)h = \frac{\psi_h(m)}{\psi_h(1)}h$. Hence $\psi_f(m) = \frac{\psi_h(m)}{\psi_h(1)}$ thus by the multiplicity one property $f = \frac{1}{\psi_h(1)}h$ is a newform. □

Resuming, we have seen that

$$\mathcal{S}_k(\Gamma_0(q)) = \mathcal{S}_k^{old}(\Gamma_0(q)) \oplus \mathcal{S}_k^{new}(\Gamma_0(q))$$

Moreover there exists $\{f_1, \cdots, f_n\}$ orthogonal basis of $\mathcal{S}_k^{old}(\Gamma_0(q))$ made of eigenforms for all $T(m)$ such that $(m, q) = 1$, and $\{f_{n+1}, \cdots, f_J\}$ orthogonal basis of $\mathcal{S}_k^{new}(\Gamma_0(q))$ made of arithmetically normalized eigenforms for all $T(m)$. It is denoted by $\mathcal{S}_k^P(\Gamma_0(q))$ and it is unique up to permutation.

Chapter 2

The Main Theorem

In this chapter I want to introduce the problem, stating the main theorem, and I want to reduce it using algebraic properties of the cusp forms, that allow us to reduce their domain to a specific fundamentale domain.

2.1 The main theorem

Let q be a prime number, $k \geq 4$ a positive integer, and $f_0 \in \mathcal{S}_k^P(\Gamma_0(q))$ an arithmetically normalized newform, which is an eigenform for all the Hecke operators.

Let $\|f_0\|_2$ be the L^2-norm of f_0, namely

$$\|f_0\|_2^2 = \langle f_0, f_0 \rangle = \int_{F_0(q)} |f_0|^2 \, y^k d\mu(z)$$

where $z = x + iy$. We define $g(z) = |f(z)|y^{k/2}$. It will be the main object of my studies, indeed

$$\|f_0\|_2^2 = \int_{F_0(q)} g(z)^2 d\mu(z) \leq \|g\|_\infty^2 Vol_\mu(F_0(q))$$

where $Vol_\mu(F_0(q))$ is the volume of the fondamental domain $F_0(q)$ with respect to the hyperbolic measure $d\mu(z)$, so by definition $Vol_\mu(F_0(q)) = \int_{F_0(q)} d\mu(z)$.

Remark 2.1.1. *Note that giving a bound for $g(z)$ is equivalent of giving a bound for $f_0(z)$, since y depends directly from z.*

Lemma 2.1.2. *With the above notations we have*

$$Vol_\mu(F_0(q)) \asymp q$$

Proof. By lemma 1.2.11 we have that

$$F_0(q) = \coprod_{i=1}^{n} \alpha_i^{-1} F$$

where F is a fundamental domain for $SL_2(\mathbb{Z})$, $n = [SL_2(\mathbb{Z}) : \Gamma_0(q)]$, and $\{\alpha_i\}_i$ left representatives of $\Gamma_0(q) \setminus SL_2(\mathbb{Z})$. Moreover by lemma 1.2.9 $n = q + 1$. Therefore we have

$$\int_{F_0(q)} d\mu(z) = (q+1) \int_F d\mu(z)$$

Now,

$$\int_F d\mu(z) \leq \int_{x=-1}^{1} \int_{y=\frac{\sqrt{3}}{2}}^{\infty} \frac{1}{y^2} dy dx = \frac{4\sqrt{3}}{3}$$

So $Vol_\mu(F_0(q)) = \int_{F_0(q)} d\mu(z) \asymp q$. □

Remark 2.1.3. *As a consequence of this result we obtain that*

$$\|f_0\|_2^2 \leq \|g\|_\infty^2 Vol_\mu(F_0(q)) \asymp \|g\|_\infty^2 q$$

Therefore the most optimistic upper bound for $\|g\|_\infty$ that we could obtain is

$$\|g\|_\infty \ll_\epsilon q^{\epsilon - 1/2} \|f\|_2$$

Our purpose is to find an upper bound for $\|g\|_\infty$ using different techniques, namely

- Special properties of the coefficients of the Fourier expansion of f_0;
- Relations between f_0 and the automorphic kernel $h_m(z, w)$;
- The amplification method;
- A geometric approach.

Theorem 2.1.4 (The Main Theorem)**.** *Let q be a prime number and let $k \geq 4$ be a fixed integer. Let $f_0 \in \mathcal{S}_k^P(\Gamma_0(q))$ be an arithmetically normalized newform of weight k. Denote $g(z) = y^{k/2} f_0(z)$, where $z = x + iy \in \mathbb{H}$, then*

$$\|g\|_\infty \ll q^{\frac{1}{2}} q^{-\frac{1}{22}+\epsilon}$$

The first step consists to reduce the domain of f_0 to a subregion of $\mathbb{H}$, called Siegel set, coming from the invariance of $g(z)$ for a special subgroup $A_0(q)$ of $SL_2(\mathbb{Z})$.

2.2 Atkin-Lehner operators and $A_0(q)$

I recall that in our context q is a fixed prime number, and $\Gamma_0(q)$ is the congruence subgroup of $SL_2(\mathbb{Z})$ of all the matrices with the third entry divisible by q.

Definition 2.2.1. *For $M|q$, pick a matrix $w_M \in M_2(\mathbb{Z})$ such that*

$$\begin{cases} det(w_M) = M \\ w_M \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{q} \\ w_M \equiv \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} \pmod{M} \end{cases}$$

*We call it an **Atkin-Lehner matrix**.*

Definition 2.2.2. *Let w_M be an Atkin-Lehner matrix, for $M|q$, then scaling by $1/\sqrt{M}$ we obtain a matrix $W_M = \frac{1}{\sqrt{M}} w_M \in SL_2(\mathbb{R})$ that is called an* ***Atkin-Lehner operator****.*

In our context since q is prime, we have that the Atkin-Lehenr matrices are w_1 that is the identity, and $w_q = \begin{pmatrix} 0 & -1 \\ q & 0 \end{pmatrix}$. Therefore the Atkin-Lehner operators are

$$W_1 = id \text{ and } W_q = \begin{pmatrix} 0 & -\sqrt{q}/q \\ \sqrt{q} & 0 \end{pmatrix}$$

Definition 2.2.3. *$A_0(q)$ is the subgroup of $SL_2(\mathbb{R})$ generated by $\Gamma_0(q)$ and all the Atkin-Lehner operators.*

Proposition 2.2.3.1. *$\Gamma_0(q)$ is a normal subgroup of $A_0(q)$ and $A_0(q)/\Gamma_0(q)$ is an abelian group of order 2.*

Proof. It is enough to prove that W_q normalizes $\Gamma_0(q)$. First note that $W_q^2 = -id$, hence we have to prove that for $V \in \Gamma_0(q)$, $W_q V W_q \in \Gamma_0(q)$: if $V = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then one has

$$W_q V W_q = V = \begin{pmatrix} -d & c/q \\ qb & -a \end{pmatrix}$$

Note that $V \in \Gamma_0(q)$, so $q|c$, hence the above matrix belongs to $\Gamma_0(q)$. □

Remark 2.2.4. *$A_0(q)$ has a central role in our discussion; indeed as we have seen the congruence subgroup $\Gamma_0(q)$ is normal in $A_0(q)$, so it is contained in the normalizer of $\Gamma_0(q)$ in $SL_2(\mathbb{R})$, call it $N_0(q)$. Therefore for any $\rho \in N_0(q)$ and $f \in \mathcal{M}_k(\Gamma_0(q))$, $f[\rho]_k \in \mathcal{M}_k(\rho^{-1}\Gamma_0(q)\rho) = \mathcal{M}_k(\Gamma_0(q))$. In particular if f is invariant under the action of a subgroup of $N_0(q)$, then one could restrict the fundamental domain on which f take values. Suppose the subgroup for which f is invariant is $A_0(q)$, then this restriction of the fundamental domain will be good enough, since $[N_0(q) : A_0(q)]$ is finite.*

To complete this section we shall see some important properties of the Atkin-Lehner operators which will be useful later. Atkin-Lehner operators act as usual on $\mathcal{S}_k^{new}(\Gamma_0(q))$. in particular

$$f_0[W_q]_k(z) = \frac{1}{(q^{1/2}z)^k} f(W_q.z)$$

Moreover, for each $f \in \mathcal{S}_k^{new}(\Gamma_0(q))$, $f[W_q]_k$ is still in $\mathcal{S}_k^{new}(\Gamma_0(q))$. In fact we can know more about this action, specially in the case of newform, indeed

Lemma 2.2.5. *Let $f \in \mathcal{S}_k(\Gamma_0(q))$, and let p be a prime different from q, then*

$$(T(p)f)[W_q]_k = T(p)(f[W_q]_k)$$

Proposition 2.2.5.1. *Newforms are eigenvectors for the Atkin-Lehner operators, with eigenvalues ± 1.*

Proof. To prove this proposition we need the following fact:

1. If two newforms have the same eigenvalues for all $T(p)$ with p prime different from q then they are equal;

2. If $f \in \mathcal{S}_k^{new}(\Gamma_0(q))$ is an eigenform of $T(p)$ for all p prime different from q, then f is a constant multiple of a newform f_i;

3. If $f \in \mathcal{S}_k(\Gamma_0(q))$ is an eigenform for all $T(m)$ for all m such that $(m, q) = 1$ then it is either an oldform or a newform. (Lemma 1.4.8)

Now, form the previous lemma $T(p)(f[W_q]_k) = (T(p)f)[W_q]_k = \lambda_f(p)f[W_q]_k$; hence by fact (3) $f[W_q]_k$ is either an oldform or a newform. If $f[W_q]_k \in \mathcal{S}_k^{old}(\Gamma_0(q))$ then $f = (f[W_q]_k)[W_q]_k$ is both a newform and an oldform, hence it is 0. So $f[W_q]_k \in \mathcal{S}_k^{new}(\Gamma_0(q))$ by fact (1) and (2) we have $f[W_q]_k = c_q f$ for some non zero constant c_q. Therefore $f = (f[W_q]_k)[W_q]_k = c_q^2 f$, hence $c_q = \pm 1$. □

In particular we obtain the following

Corollary 2.2.6. *Let f as above, $f[W_q]_k = c_q f$, $c_q = \pm 1$, then*

$$c_q = 1 \Leftrightarrow f \in \mathcal{S}_k(A_0(q))$$

Lemma 2.2.7. *The operator $[W_q]_k$ is hermitian with respect to the Petersson scalar product, i.e.*

$$\langle f[W_q]_k, g\rangle = \langle f, g[W_q]_k\rangle$$

where f and $g \in \mathcal{S}_k(\Gamma_0(q))$

Proof. It is immediate sice $[W_q]_k[W_q]_k = [id]_k$ and for an operator L $\langle f[L], g[L]\rangle = \langle f, g\rangle$. □

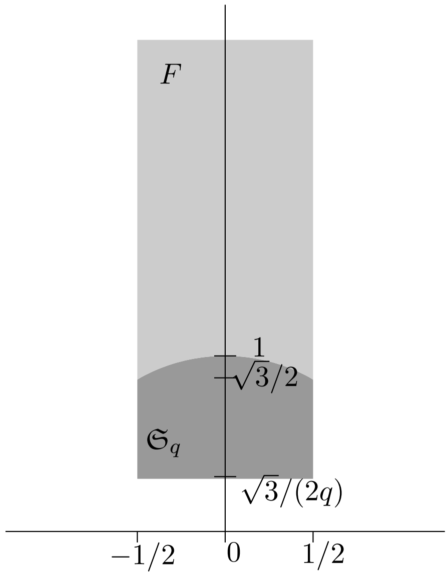
2.3 The Siegel sets

Now we have all the ingredients to define a new fundamental domain for f_0,namely

Definition 2.3.1. *Let N be a positive integer. A **Siegel set** $\mathfrak{S}_N$ is a rectangle of the form*

$$\mathfrak{S}_N := \left\{ x + iy \in \mathbb{C} : x \in [0, 1)\ ,\ y \in [\frac{\sqrt{3}}{2N}, \infty) \right\}$$

Note that for $N = 1$, $\mathfrak{S}_1$ is more or less the fundamental domain F of SL_2; in particular $F \subseteq \mathfrak{S}_1$, as we can see in the following picture:



Definition 2.3.2. *Let N be a positive integer. A* ***generalized Siegel set*** *is $\overline{\mathfrak{S}_N} = \mathfrak{S}_N \cup \{\infty\}$.*

Proposition 2.3.2.1. *Let q be our fixed prime number, then*

1. *For any couple of coprime integers a and c, there exist $b, d \in \mathbb{Z}$, $M \in 1, q$ and $\gamma \in \Gamma_0(q)$ such that*
$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \gamma w_M \begin{pmatrix} 1/M & 0 \\ 0 & 1 \end{pmatrix}$$
2. $\overline{\mathbb{H}} = \bigcup_{\delta \in A_0(q)} \delta.\overline{\mathfrak{S}_q}$

Proof. 1) If q divides c just take $M = 1$. Since a and c are coprime then certenly there exist integers b and d such that $ad - bc = 1$. Take $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(q)$.

If q does not divides c then take $M = q$. We have $w_q \begin{pmatrix} 1/q & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. I claim that there exist $b, d \in \mathbb{Z}$ such that
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} b & -a \\ d & -c \end{pmatrix} \in \Gamma_0(q)$$
i.e. there exist b, d such that $ad - bc = 1$ and $q|d$. Indeed, certainly there exist $\xi, \eta \in \mathbb{Z}$ such that $a\eta - c\xi = 1$.
Since $(q, c) = 1$ then there exist $u, v \in \mathbb{Z}$ such that $uq + vc = 1$. So
$$1 = (a\eta - c\xi)(uq + vc) = a(qu\eta) + c(-\xi vc + va\eta - \xi uq)$$

Just call $d = qu\eta$ and $b = -\xi vc + va\eta - \xi uq$.
2) As I have said above, $\overline{\mathfrak{S}_1}$ contains the fundamental domain $\overline{F}$ for $SL_2(\mathbb{Z})$ in $\overline{\mathbb{H}}$. So

$$\overline{\mathbb{H}} = \bigcup_{\rho \in SL_2(\mathbb{Z})} \rho.\overline{\mathfrak{S}_1}$$

Now,form the first part of the proposition we have that any element ρ of $SL_2(\mathbb{Z})$ can be written as $\rho = \delta \begin{pmatrix} 1/M & 0 \\ 0 & 1 \end{pmatrix}$, where $\delta \in A_0(q)$. Note also that if $z \in \overline{\mathfrak{S}_1}$, then $\begin{pmatrix} 1/M & 0 \\ 0 & 1 \end{pmatrix} z = \frac{z}{M} \in \overline{\mathfrak{S}_M} \subset \overline{\mathfrak{S}_q}$.
Hence

$$\overline{\mathbb{H}} = \bigcup_{\delta \in A_0(q)} \delta \begin{pmatrix} 1/M & 0 \\ 0 & 1 \end{pmatrix} .\overline{\mathfrak{S}_1} \subset \bigcup_{\delta \in A_0(q)} \delta\overline{\mathfrak{S}_q}$$

□

Thanks to this proposition one can concludes that if a function f defined on $\mathbb{H}$ is invariant under some action of $A_0(q)$, then to study the values taken from f in the whole $\mathbb{H}$ it is enough to study the values that it takes on $\mathfrak{S}_q$. This is exaclty the case of our function $g(z) = y^{k/2}|f_0(z)|$, namely

Lemma 2.3.3. *$g(z)$ is $A_0(q)$-invariant.*

Proof. It is enough to prove that $g(z)$ is invariant under $\gamma \in \Gamma_0(q)$ and under W_q.
Let first $\gamma \in \Gamma_0(q)$. So

$$g(\gamma.z) = (\Im(\gamma.z))^{k/2}|f_0(\gamma.z)|$$

It is a general fact that $\Im(\gamma.z) = \frac{\Im(z)}{|j(\gamma,z)|^2}$ and by definition of modular form we have $f_0(\gamma.z) = f_0(z)j(\gamma,z)^k$. Therefore we obtain

$$g(\gamma.z) = \left(\frac{\Im(z)}{|j(\gamma,z)|^2}\right)^{k/2} |f_0(z)j(\gamma,z)^k| = y^{k/2}|f_0(z)| = g(z)$$

Consider now the Atkin-Lehner operator W_q:

$$\begin{aligned} g(W_q.z) = (\Im(W_q.z))^{k/2}|f_0(W_q.z)| &= \left(\Im\left(-\frac{1}{qz}\right)\right)^{k/2} |f_0(-\frac{1}{qz})| \\ &= \left(\frac{\Im(z)}{q|z|^2}\right)^{k/2} |f_0(-\frac{1}{qz})| = \frac{y^{k/2}}{|j(W_q,z)|^k}|f_0(-\frac{1}{qz})| \\ &= y^{k/2}|f_0[W_q]_k(z)| = y^{k/2}|f_0(z)| = g(z) \end{aligned}$$

where the second-last equality comes from tha fact that newforms are eigenvectors for Atkin-Lehner operators, of eigenvalues ± 1, as I stated in the proposition 2.2.5.1. □

This important Lemma and the proposition above, allow us to restrict the domain of $g(z)$ to $\mathfrak{S}_q$. Indeed if $z \in \mathbb{H}$ then there exist $w \in \mathfrak{S}_q$ and $\delta \in A_0(q)$ such that $z = \delta.w$. Hence $g(z) = g(\delta w) = g(w)$. In particular $\|g\|_\infty = \left\|g \restriction_{\mathfrak{S}_q}\right\|_\infty$.

Chapter 3

Bound via Fourier Expansion and Pre-Trace Formula

In the previous chapter we have seen how to reduce the domain of our function $g(z)$ to $\mathfrak{S}_q$. In this chapter we are ready to apply two techniques to establish a bound for $g(z)$. The first one uses the Fourier expansion of $f_0(z)$ and the behaviour on average of its Hecke eigenvalues; the second one is a little more sofisticated and uses a particular relation between th functions $g(z)$ and an automorphic kernel; this relation is called pre-trace formula. I recall

$$g(z) = |y^{k/2} f_0(z)| \text{ and } \mathfrak{S}_q = \left\{ x + iy \in \mathbb{H} : x \in [0,1), y \in [\frac{\sqrt{3}}{2q}, \infty) \right\}$$

where $z = x + iy$.

3.1 Bound via Fourier expansion

Since f_0 is a cusp form, it decays rapidly at infinity. This allows us to split the Fourier expansion of f_0 into a finite sum and a negligible tail. To do this we need the following result which is derived from the analytic properties of the Rankin-Selberg L-function,

Theorem 3.1.1 (Rankin and Selberg)**.** *Let $f \in \mathcal{S}_k^P(\Gamma_0(q))$ be a primitive cusp form, and denote by $\lambda_f(n)$ its Hecke eigenvalues, then*

$$\sum_{1 \le n \le X} |\lambda_f(n)|^2 \ll_\epsilon X(qX)^\epsilon$$

Proposition 3.1.1.1. *Let $0 < \eta \le 1$ be a real number, then uniformly on $z \in \mathfrak{S}_{\eta q}$*

$$g(z) = y^{k/2}|f_0(z)| \ll_\epsilon q^\epsilon (\eta q)^{1/2}$$

for all $\epsilon > 0$.
In particular for $z \in \mathfrak{S}_q$, $g(z) \ll_\epsilon q^{1/2+\epsilon}$.

Proof. I recall that $f_0(z) = \sum_{n\geq 1} n^{\frac{k-1}{2}} \psi_{f_0}(n)e(nz)$, where $\psi_{f_0}(n)$ are the normalized Fourier coefficients. Moreover, since f_0 is primitive then $\psi_{f_0}(n) = \lambda_{f_0}(n)$, if $\lambda_{f_0}(n)$ are the Hecke eigenvalues. So we have

$$g(z) = y^{k/2} \left| \sum_{n\geq 1} n^{\frac{k-1}{2}} \psi_{f_0}(n)e(nz) \right| = y^{1/2} \left| \sum_{n\geq 1} (ny)^{\frac{k-1}{2}} \psi_{f_0}(n)e(nz) \right| \leq y^{1/2} \sum_{n\geq 1} \frac{(ny)^{\frac{k-1}{2}}}{e^{2\pi ny}} |\psi_{f_0}(n)|$$

Focus on the argument in the last formula: the tail of the sum, when $ny > q^{\epsilon}$, is negligible because of the rapidly decay of the argument. Indeed we have the following

Lemma 3.1.2.

$$\psi_{f_0}(N) \ll N^{1/2}$$

Proof. As we know

$$f_0(z) = \sum_{n\geq 1} n^{\frac{k-1}{2}} \psi_{f_0}(n)e(nz) = \sum_{n\geq 1} n^{\frac{k-1}{2}} \psi_{f_0}(n)e(nx)e^{-2\pi ny}$$

Moreover, since f_0 is cuspidal, then uniformly on z we have $|y^{k/2} f_0(z)| \ll 1$. So consider for a fixed N

$$\int_{x=0}^{1} f(x+iy)e(-Nx)dx = \sum_{n\geq 1} n^{\frac{k-1}{2}} \psi_{f_0}(n)e^{-2\pi ny} \int_{x=0}^{1} e(x(n-N))dx$$

The integral is clearly different from 0 only when $n = N$, where it assumes the value 1. Hence we obtain

$$= N^{\frac{k-1}{2}} \psi_{f_0}(N)e^{-2\pi Ny}$$

Therefore it follows that

$$\begin{aligned} \psi_{f_0}(N) &\ll N^{\frac{1-k}{2}} e^{2\pi Ny} \int_{x=0}^{1} f(z)e^{-Nx}dx \\ &\ll N^{\frac{1-k}{2}} e^{2\pi Ny} \frac{1}{y^{k/2}} \end{aligned}$$

We can take $y = \frac{1}{N}$ obtaining

$$\psi_{f_0}(N) \ll N^{\frac{1-k}{2}} N^{k/2} = N^{1/2}$$

$\square$

So we have

$$g(z) \ll y^{1/2} \sum_{1\leq n\leq q^{\epsilon}/y} \frac{(ny)^{\frac{k-1}{2}}}{e^{2\pi ny}} |\psi_{f_0}(n)|$$

We apply the Cauchy-Schwarz inequality to this sum, obtaining

$$g(z)^2 \ll y \left(\sum_{1\leq n\leq q^{\epsilon}/y} |\psi_{f_0(n)}|^2 \right) \left(\sum_{1\leq n\leq q^{\epsilon}/y} \frac{(ny)^{k-1}}{e^{4\pi ny}} \right)$$

fronm Rankin-Selberg's theorem

$$\sum_{1\le m\le n} |\psi_{f_0}(m)|^2 = \sum_{1\le m\le n} |\lambda_f(m)|^2 \ll_\epsilon n(qn)^\epsilon$$

hence we can bound the sum in the first parenthesis as

$$\sum_{1\le n\le q^\epsilon/y} |\psi_{f_0(n)}|^2 \ll \frac{q^\epsilon}{y}\left(\frac{q^{\epsilon+1}}{y}\right)^\epsilon$$

For the sum in the second parenthesis we have trivially

$$\sum_{1\le n\le q^\epsilon/y} \frac{(ny)^{k-1}}{e^{4\pi ny}} \le \frac{q^\epsilon}{y}(q^\epsilon)^{k-1}$$

This implies

$$g(z)^2 \ll y\frac{q^\epsilon}{y}\left(\frac{q^{\epsilon+1}}{y}\right)^\epsilon \frac{q^\epsilon}{y}(q^\epsilon)^{k-1} \ll \frac{q^\epsilon}{y}$$

Therefore

$$g(z) \ll_\epsilon \frac{q^\epsilon}{y^{1/2}}$$

Now, if $z = x+iy \in \mathfrak{S}_{\eta q}$ then $y \gg \frac{1}{\eta q}$, so we conclude

$$g(z) \ll_\epsilon q^\epsilon(\eta q)^{1/2}$$

□

Remark 3.1.3. *This result tells us that if one fixes a real number $0<\eta<1$, then he find a subreagion $\mathfrak{S}_{\eta q}$ of our domain $\mathfrak{S}_q$ in which the bound for the supnorm of $g(z)$ is in some sense 'non-trivial', namely strictly less then $q^{1/2}$. The problem is that in the remaining part of our domain with this method we can only find the 'trivial result'. So our aim is to find a method to improve the result of the above proposition in the reagion $\mathfrak{S}_q - \mathfrak{S}_{\eta q}$, and then to find a suitable value η which makes the bound as good as possible.*

3.2 Bound via Pre-Trace formula

The technique that we use in this section is essentially to compare our function $f_0(z)$ with the automorphic kernel

$$h(z,w) = \sum_{\rho\in\Gamma_0(q)} \frac{1}{j(\rho,z)^k(w+\rho.z)^k}$$

and to find a bound working on this last function.
From appendix 1 we obtain the pre-trace formula

$$h(z,w) = C_k \sum_{j=1}^{J} \frac{1}{\langle f_j, f_j\rangle} \overline{f_j(-\bar{w})} f_j(z)$$

where $\{f_1, \cdots, f_J\}$ is the orthogonal basis described at the end of chapter 1. C_k is a constant depending only on k, which is the fixed degree. In particular, choosing $w = -\bar{z} \in \mathbb{H}$ one obtains

$$h(z,-\bar{z}) = C_k \sum_{j=1}^{J} \frac{|f_j(z)|^2}{\langle f_j, f_j\rangle}$$

Now, since f_0 is by definition an element of this orthogonal basis, then

$$\frac{|f_0(z)|^2}{\langle f_0, f_0\rangle} = C_k h(z,-\bar{z}) \ll h(z,-\bar{z})$$

i.e.

$$|f_0(z)|^2 \ll \langle f_0, f_0\rangle\, h(z,-\bar{z})$$

This alows us to study the function $h(z,-\bar{z})$, to give an upper bound for $|f_0(z)|$. For what is concerning the factor $\langle f_0, f_0\rangle$ we have the following

Lemma 3.2.1. *Let f be a primitive cusp form with respect to $\Gamma_0(q)$, then $\langle f, f\rangle \ll_\epsilon q^{1+\epsilon}$*

Remark 3.2.2. *A proof for this result can be found in [Iwa90].*

Definition 3.2.3. *For any matrix $\rho \in M_2(\mathbb{Z})$ we define the following function on $\mathbb{H}$*

$$u_\rho(z) = \frac{j(\rho,z)(\bar{z} - \rho.z)}{y}$$

where $z = x + iy$.

Remark 3.2.4. *This new function u_ρ is strongly related to the hyperbolic distance between an element $z \in \mathbb{H}$ and its image under ρ,*

$$u(z,\rho.z) = \frac{|z-\rho.z|^2}{4y\Im(\rho.z)}$$

Coming back to our problem, we define the function $K(x) = \frac{1}{x^k}$ for real x. Hence we obtain

$$y^k h(z, -\bar{z}) = \sum_{\rho \in \Gamma_0(q)} \frac{1}{j(\rho, z)^k (-\bar{z} + \rho.z)^k} = \sum_{\rho \in \Gamma_0(q)} K(u_\rho(z))$$

therefore

$$\begin{aligned} g(z)^2 = y^k |f_0(z)|^2 &\ll q^{1+\epsilon} y^k |h(z, -\bar{z})| \\ &\ll q^{1+\epsilon} \left| \sum_{\rho \in \Gamma_0(q)} K(u_\rho(z)) \right| \\ &\ll q^{1+\epsilon} \sum_{\rho \in \Gamma_0(q)} K(|u_\rho(z)|) \end{aligned}$$

At this point our aim is to study this last object

$$\sum_{\rho \in \Gamma_0(q)} K(|u_\rho(z)|)$$

The idea is to study carefully this sum using the Stiltjes integral, and noting that there is a strictly positive lower bound for $|u_\rho(z)|$, for any matrix $\rho \in \Gamma_0(q)$ and any z in our domain. This fact makes our sum finite. First of all we have to define the following function:

Definition 3.2.5. *Let $z \in \mathbb{H}$ and δ a positive real number, then*

$$M(z, \delta) = \sharp\{\rho \in \Gamma_0(q) : |u_\rho(z)| < \delta\}$$

This function is the heart of our bound, indeed a careful bound of it permits us to give a bound for the function $\sum_{\rho \in \Gamma_0(q)} K(|u_\rho(z)|)$ (and so for $g(z)$), simply using Stieltjes integral.

3.2.6 A Bound for $M(z, \delta)$

I recall that $M(z, \delta)$ is the cardinality of $\{\rho \in \Gamma_0(q) : |u_\rho(z)| < \delta\}$; to bound it we have to split this set into two subsets, distinguishing the case in which the matrices have the third entry equal 0, and the case in which the third entry is not 0, namely we define

Definition 3.2.7.

$$M_0(z, \delta) = \sharp\{\rho = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \Gamma_0(q) : |u_\rho(z)| < \delta\}$$

Definition 3.2.8.

$$M_\star(z, \delta) = \sharp\{\rho = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(q) : c \neq 0 \ , \ |u_\rho(z)| < \delta\}$$

So $M(z, \delta) = M_0(z, \delta) + M_\star(z, \delta)$, and we bound separately these two terms.

Lemma 3.2.9. *Let $z = x + iy \in \mathbb{H}$ and δ a positive real number, then*

$$M_0(z, \delta) \ll \delta y$$

Proof. Let $\gamma \in \{\rho = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \Gamma_0(q) : |u_\rho(z)| < \delta\}$, then $a = d = \pm 1$. Without lost of generality we may assume $a = d = 1$, so $\gamma = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$. In particular $M_0(z, \delta)$ is the double of the possible choiches for b. We have

$$|u_\gamma(z)| = \frac{|\gamma.z - \bar{z}||j(\gamma, z)|}{y} = \frac{|b + 2iy|}{y} < \delta$$

and considering the real part of this last inequality we obtain

$$|b| < \delta y$$

Therefore $M_0(z, \delta) \ll \delta y$. □

Lemma 3.2.10. *Let $z = x + iy \in \mathbb{H}$ and δ a positive real number, then*

$$M_\star(z, \delta) \ll \frac{\delta^3}{yq}$$

Proof. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(q)$ be such that $|u_\gamma(z)| < \delta$ and $c \neq 0$, then a direct computation gives

$$|u_\gamma(z)| = |b - c|z|^2 + x(a - d) + iy(a + d)|\frac{1}{y} < \delta$$

Considering the imaginary part one obtains

$$|a + d| < \delta$$

Considering the real part one has

$$|b - c|z|^2 + x(a - d)|\frac{1}{y} < \delta$$

Now substitute $b = \frac{ad-1}{c}$ and note that $|cz + d|^2 = c^2|z|^2 + d^2 + 2cx$, so we obtain

$$\begin{aligned} \delta|c|y &> |c^2|z|^2 - cx(a - d) - ad + 1| \\ &= |(c^2|z|^2 + d^2 + 2cdx) - d^2 - cx(a + d) - ad + 1| \\ &= ||cz + d|^2 + 1 - (a + d)(cx + d)| \end{aligned}$$

We have seen above that $|a + d| < \delta$, so

$$||cz + d|^2 + 1| < \delta|c|y + |a + d||cx + d| \ll \delta|cz + d|$$

which implies

$$|cz + d| \ll \delta$$

As a consequence we obtain

$$|cx + d| \ll \delta \text{ and } |c|y \ll \delta$$

In particular $|c| \ll \frac{\delta}{y}$.
On the other hand $c \equiv 0 \pmod q$, so we can say that the number of possible choiches for c is

$$\sharp\{c\} \ll \frac{\delta}{yq}$$

Let now I be an interval of lenght $\ll \delta$, containing $a + d$ and $cx + d$. Then

$$a - d - 2cx = a + d - 2(cx + d) \in I$$

So, once c is fixed we have to find the triple (a, b, d) such that

$$\begin{cases} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(q) \\ a + d \in I \\ a - d - 2cx \in I \end{cases}$$

Let $A := a + d$ and $D := a - d - [2cx]$. Since b depends on the other coefficients of the matrix, then the number of triples (a, b, d) is equal to the possible choices for the pairs (a, d), which is equivalent to count the number of pairs (A, D) such that

$$\begin{cases} A, D \in I \\ 2A[2cx] + 4bc = D^2 - 4 - [2cx]^2 - A^2 \end{cases}$$

In particular weaking again our conditions the number of possible pairs (A, D) is

$$\sharp\{(A, D)\} \ll |I|^2 \ll \delta^2$$

Recalling that the number of possible choiches for c is $\ll \frac{\delta}{yq}$ we can conclude that

$$M_\star(z, \delta) \ll \frac{\delta^3}{yq}$$

□

Remark 3.2.11. *Note that the bound for $M_0(z, \delta)$ does not depend on the level q. This is reasonable since the only parameter connacted with q is c, which is 0.*

Remark 3.2.12. *It is important to note that the buond for $M_0(z, \delta)$ is directly proportional to the imaginary part of z, while the bound for $M_\star(z, \delta)$ is inversely proportional to it. So, if we want to give a simultaneous bound for both, independent on $y = \Im z$, then we have to restrict y to an interval upper and lower bounded.*

Proposition 3.2.12.1. *Let $z = x + iy \in \mathbb{H}$ and δ a positive real number, then*

$$M(z,\delta) \ll \delta y + \frac{\delta^3}{yq}$$

Proof. It is an immediate consequence of the above two lemmas, indeed

$$M(z,\delta) = M_0(z,\delta) + M_\star(z,\delta) \ll \delta y + \frac{\delta^3}{yq}$$

□

Corollary 3.2.13. *Let $z \in \mathfrak{S}_q - \bigcup_{\alpha \in A_0(q)} \alpha.\mathfrak{S}_{\eta q}$, $\eta > q^{-1}$, δ a positive real number, then*

$$M(z,\delta) \ll \delta^3$$

3.2.14 A Lower Bound for $u_\rho(z)$

In the appendix 2 we have seen that it is possible to classify the motions $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL2(\mathbb{R})$ depending on the value of the trace, namely

$$\begin{cases} |a+d| = 2 \Leftrightarrow \gamma \text{ is parabolic (i.e. it fixes exactly one point in } \bar{\mathbb{R}}) \\ |a+d| > 2 \Leftrightarrow \gamma \text{ is hyperbolic (i.e. it fixes exactly two points in } \bar{\mathbb{R}}) \\ |a+d| < 2 \Leftrightarrow \gamma \text{ is elliptic (i.e. it fixes exactly one point in } \bar{\mathbb{H}}, \text{ and the complex conjugate)} \end{cases}$$

Taking now $\gamma \in \Gamma_0(q)$ and $z \in \mathfrak{S}_q$, I want to give a lower bound for $u_\gamma(z)$ (i.e. an upper bound for $K(u_\gamma(z))$) with respect to the classification of γ, using the following inequality:

$$\begin{aligned} |u_\gamma(z)| &= \frac{|\gamma.z - \bar{z}||j(\gamma,z)|}{y} \\ &= \frac{|az + b - \bar{z}(cz+d)|}{y} \\ &= \frac{|b - c|z|^2 + az - d\bar{z}|}{y} \\ &= \frac{|b - c|z|^2 + x(a-d) + iy(a+d)|}{y} \end{aligned}$$

Now, considering the imaginary part of $u_\gamma(z)$, one has

$$|u_\gamma(z)| \geq \Im(u_\gamma(z)) = |a+d|$$

This implies that $|u_\gamma(z)|$ in strongly related to the classification of motions. In fact we have:

1. If γ is parabolic or hyperbolic then $|u_\gamma(z)| \geq |a+d| \geq 2$, then

$$\frac{1}{|u_\gamma(z)|} \leq \frac{1}{2}$$

2. If γ is elliptict then either $|a+d|=1$ or $|a+d|=0$:

   - If $|a+d|=1$ then $|u_\gamma(z)| \geq |a+d| = 1$ and so

$$\frac{1}{|u_\gamma(z)|} \leq 1$$

   - If $|a+d|=0$ then $a=-d$ and $b=-\frac{d^2+1}{c}$; hence bubstituting this values into our formula for $u_\gamma(z)$ we obtain

$$\begin{aligned} |u_\gamma(z)| &= \frac{|d^2+1+c^2|z|^2+2cdx|}{|c|y} \\ &= \frac{|1+(cx+d)^2+c^2y^2|}{|c|y} \\ &\geq |c|y \end{aligned}$$

Now, if $c=0$ then γ is parabolic. So we may assume $c \neq 0$. Moreover since $\gamma \in \Gamma_0(q)$ then $|c| \geq q$ and since $z \in \mathfrak{S}_q$ then $y \geq \frac{\sqrt{3}}{2q}$; therefore

$$|u_\gamma(z)| \geq |c|y \geq \frac{\sqrt{3}}{2}$$

and in particular

$$\frac{1}{|u_\gamma(z)|} \leq \frac{2\sqrt{3}}{3}$$

3.2.15 The Total Bound

In the discussion above we have seen that $K(|u_\gamma(z)|) = \frac{1}{|u_\gamma(z)|^k}$ is bounded for each $\gamma \in \Gamma_0(q)$, so our hope is that also $\sum_{\rho \in \Gamma_0(q)} K(|u_\rho(z)|)$ can be bounded.

$K(x) = \frac{1}{x^k}$ is continous and bounded in $[\frac{\sqrt{3}}{2}, \infty)$, and $M(z,\delta)$ is obviously a monotonically increasing function (with respect to δ). So we are in the hypotesis to apply the Stieltjes integral:

$$\sum_{\rho \in \Gamma_0(q)} K(|u_\rho(z)|) = \int_{\frac{\sqrt{3}}{2}}^{\infty} K(\delta) dM(z,\delta)$$

Remark 3.2.16. *Since $|u_\gamma| \geq \frac{\sqrt{3}}{2}$ then $M(z,\delta)=0$ for all $\delta < \frac{\sqrt{3}}{2}$, and so the integration starts from $\frac{\sqrt{3}}{2}$ instead of* 0.

Applying the integration by parts we obtain

$$\int_{\frac{\sqrt{3}}{2}}^{\infty} K(\delta) dM(z,\delta) = \left[\frac{M(z,\delta)}{\delta^k}\right]_{\delta=\frac{\sqrt{3}}{2}}^{\infty} + k\int_{\frac{\sqrt{3}}{2}}^{\infty} \frac{M(z,\delta)}{\delta^{k+1}} d\delta$$

Now, we have seen in the subsection 3.2.6 that $M(z,\delta) \ll \delta^3$, hence

$$\begin{aligned}
\left[\frac{M(z,\delta)}{\delta^k}\right]_{\delta=\frac{\sqrt{3}}{2}}^{\infty} + k\int_{\frac{\sqrt{3}}{2}}^{\infty} \frac{M(z,\delta)}{\delta^{k+1}} d\delta &\ll \left[\frac{1}{\delta^{k-3}}\right]_{\delta=\frac{\sqrt{3}}{2}}^{\infty} + k\int_{\frac{\sqrt{3}}{2}}^{\infty} \frac{1}{\delta^{k-2}} d\delta \\
&= \left[\frac{1}{\delta^{k-3}}\right]_{\delta=\frac{\sqrt{3}}{2}}^{\infty} - \frac{k}{k-3}\left[\frac{1}{\delta^{k-3}}\right]_{\frac{\sqrt{3}}{2}}^{\infty} \\
&= -\frac{3}{k-3}\left[\frac{1}{\delta^{k-3}}\right]_{\frac{\sqrt{3}}{2}}^{\infty} \\
&= \frac{3}{k-3}\left(\frac{2\sqrt{3}}{3}\right)^k \ll 1
\end{aligned}$$

Proposition 3.2.16.1. *Let $\frac{1}{q} < \eta \leq 1$ be a real number, then uniformly on $z \in \mathfrak{S}_q - \bigcup_{\alpha\in A_0(q)} \alpha.\mathfrak{S}_{\eta q}$*

$$g(z) \ll_\epsilon q^{1/2+\epsilon}$$

for all $\epsilon > 0$.

Proof. So far we have seen that

$$g(z)^2 \ll q^{1+\epsilon} \sum_{\rho\in\Gamma_0(q)} K(|u_\rho(z)|)$$

and in the last proposition we have proved that in fact

$$\sum_{\rho\in\Gamma_0(q)} K(|u_\rho(z)|) \ll 1$$

Therefore

$$g(z) \ll q^{1/2+\epsilon}$$

□

Remark 3.2.17. *This method is completely different from the previous one, since it uses algebraic properties (holomorphic kernel) instead of analytic properties (Fourier expansion), but it gives the same bound in the reagion $\mathfrak{S}_q - \mathfrak{S}_{\eta q}$.*

Chapter 4

The Amplification Method

The aim of this chapter is to try to improve the bound previously found, especially in the reagion $\mathfrak{S}_q - \mathfrak{S}_{\eta q}$, using another method, the so called *amplification method.*

In general this method takes an established estimate involving an arbitrary object, as a function, and obtains a stronger (amplified) estimate by trasforming the object in a well chosen manner into a new object, applying the estimate to that new object and seeing what the estimate says about the original object. In our specific case the idea is the following: suppose to have a family of functions $\{u_j(z)\}_{j\in I}$, and suppose that our goal is to find an estimate for a function u_0 belonging to this family. Suppose to know an esimate involving this family of functions of the form

$$\sum_{j\in I} \lambda_j(m)\lambda_j(n)|u_j(z)|^2 \ll F_1(n,m)$$

for some formula $F(n,m)$ depending on integers n,m, and some complex numbers $\lambda_j(m)$ and $\lambda_j(n)$. Let a_n's be complex numbers, and N be a positive integer, then one can obtain

$$\sum_{j\in I} |\sum_{n\leq N} a_n\lambda_j(n)|^2 |u_j(z)|^2 \ll F_2(a_1,\cdots,a_N)$$

The trick consists in finding a good estimate for the right hand-side, and then since the terms in the left hand-side are non negative then one can define the linear form $L_j = \sum_{n\leq N} a_n\lambda_j(n)$ which gives

$$|L_0|^2|u_0|^2 \ll F_2(a_1,\cdots,a_N)$$

The linear form L_0 is used to amplify the contribution of the selected function u_0, so the goal is to choose the a_n's that makes L_0 big and $F_2(a_1,\cdots,a_N)$ small.

In our case the family of functions will be the orthogonal basis of eigenforms for $\mathcal{S}_k(\Gamma_0(q))$ described in chapter 1, and we will give the bound for the right hand-side of the inequality using another time the pre-trace formula, and then applying two times some Hecke operators.

For our purposes I recall three important fact: let f be an element of our basis and ℓ a positive integer, then

1. $T(\ell)f = \ell^{-1/2}\sum_{\rho\in G_\ell(q)/\Gamma_0(q)} f[\rho]$;

2. $T(\ell_1)\circ T(\ell_2)=\sum_{d|(\ell_1,\ell_2)} T(\frac{\ell_1\ell_2}{d^2})$;

3. $h(z,w):=\sum_{\gamma\in\Gamma_0(q)}\frac{1}{j(\gamma,z)^k(w+\gamma.z)^k}=C_k\sum_{j=1}^J\frac{1}{<f_j,f_j>}f_j(z)\overline{f_j(-\bar{w})}$.

Considering this last equation and applying the Hecke operator $T(\ell)$ to the left hand-side, with ℓ coprime with q, we obtain

$$\begin{aligned}
T(\ell)(h(.,w))(z) &= \ell^{-1/2}\sum_{\rho\in G_\ell(q)/\Gamma_0(q)}\sum_{\gamma\in\Gamma_0(q)}\frac{1}{j(\gamma,\rho.z)^k(w+\gamma\rho.z)^k}\frac{\ell^{k/2}}{j(\rho,z)^k}\\
&= \ell^{\frac{k-1}{2}}\sum_{\rho\in G_\ell(q)}\frac{1}{j(\rho,z)^k(w+\rho.z)^k}
\end{aligned}$$

and to the right hand-side we obtain

$$T(\ell)(h(.,w))(z)=C_k\sum_{j=1}^J\frac{1}{<f_j,f_j>}\lambda_j(l)f_j(z)\overline{f_j(-\bar{w})}$$

Now, let ℓ_1 and ℓ_2 be two positive integers coprime with q. The idea is to apply both the Hecke operators $T(\ell_1)$ and $T(\ell_2)$ to the equation. In fact we apply $T(\ell_1)\circ T(\ell_2)$ to the right hand-side, and $\sum_{d|(\ell_1,\ell_2)}T(\frac{\ell_1\ell_2}{d^2})$ to the left hand-side, we obtain

$$\sum_{d|(\ell_1,\ell_2)}\left(\frac{\ell_1\ell_2}{d^2}\right)^{\frac{k-1}{2}}\sum_{\rho\in G_{\frac{\ell_1\ell_2}{d^2}}(q)}\frac{1}{j(\rho,z)^k(-\bar{z}+\rho.z)^k}=C_k\sum_{j=1}^J\frac{1}{<f_j,f_j>}\lambda_j(\ell_1)\lambda_j(\ell_2)|f_j(z)|^2$$

Consider a positive integer L, and complex variables $x_1,\cdots,x_L$ to which we will assign a value later, and look at the following sum:

$$\begin{aligned}
&C_k\sum_j|\sum_{1\le\ell\le L}x_\ell\lambda_{f_j}(\ell)|^2\left|\frac{y^{k/2}f_j(z)}{\sqrt{<f_j,f_j>}}\right|^2\\
&=C_k\sum_j\sum_{1\le\ell_1,\ell_2\le L}x_{\ell_1}\bar{x_{\ell_2}}\lambda_{f_j}(\ell_1)\lambda_{f_j}(\ell_2)\left|\frac{y^{k/2}f_j(z)}{\sqrt{<f_j,f_j>}}\right|^2\\
&=C_k\sum_{1\le\ell_1,\ell_2\le L}x_{\ell_1}\bar{x_{\ell_2}}\sum_j\lambda_{f_j}(\ell_1)\lambda_{f_j}(\ell_2)\left|\frac{y^{k/2}f_j(z)}{\sqrt{<f_j,f_j>}}\right|^2\\
&=\sum_{1\le\ell_1,\ell_2\le L}x_{\ell_1}\bar{x_{\ell_2}}\sum_{d|(\ell_1,\ell_2)}\left(\frac{\ell_1\ell_2}{d^2}\right)^{\frac{k-1}{2}}\sum_{\rho\in G_{\frac{\ell_1\ell_2}{d^2}}(q)}\frac{y^k}{j(\rho,z)^k(-\bar{z}+\rho.z)^k}\\
&=\sum_{1\le\ell_1,\ell_2\le L}x_{\ell_1}\bar{x_{\ell_2}}\sum_{d|(\ell_1,\ell_2)}\left(\frac{\ell_1\ell_2}{d^2}\right)^{\frac{k-1}{2}}\sum_{\rho\in G_{\frac{\ell_1\ell_2}{d^2}}(q)}K(|u_\rho(z)|)
\end{aligned}$$

This equality will allow us to give an upper bound for the right hand-side of the equation given by the amplification, and choosing in a proper way the variables x_i's we can also find a lower bound for the left hand-side, as it is required.

Looking to the last formula above, we see that first of all our goal to give an upper bound for $\sum_{\rho \in G_\ell} K(|u_\rho(z)|)$. This situation is similar to that of the previous chapter, but now we have to sum over all matrices in $G_\ell(q)$ instead of all matrices in $\Gamma_0(q)$.

Remark 4.0.18. *I recall that*

$$G_\ell(q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) : ad - bc = \ell,\ q|c,\ (a, q) = 1 \right\}$$

Note that for $\ell = 1$ we obtain $G_\ell(q) = \Gamma_0(q)$.

Before starting with this bound, we need some more preliminaries, namely

Definition 4.0.19. *Let $z \in \mathbb{H}$, ℓ a positive integer, and δ a positive real number, then*

$$M(z, \ell, \delta) := \sharp\{\rho \in G_\ell(q) : |u_\rho(z)| < \delta\}$$

It is just the generalization of the previous counting function $M(z, \delta)$.

We have now to find a good estimate of it; this is a very important step of this chapter.

4.1 A Bound for $M(z, \ell, \delta)$

First of all we need a very useful lemma, that we shall use several time in this section:

Lemma 4.1.1. *Let r and s two positive integers,$(r, s) = c_1^2 c_2$ and c_2 squarefree, then*

$$card\{\xi \pmod{s} : \xi^2 \equiv r \pmod{s}\} \ll s^\epsilon c_1$$

Proof. Let ξ be a solution of $x^2 \equiv r \pmod{s}$, then $c_1^2 c_2$ divides ξ^2 since it divides both r and s. c_2 is squarefree hence $c_1^2 c_2^2$ divides ξ^2 thus $c_1 c_2$ divides ξ. So $\xi = k c_1 c_2$ for $k < \frac{s}{c_1 c_2}$.
Moreover if $\left(c_2, \frac{s}{c_1^2 c_2}\right) \neq 1$ then also $\left(c_2, \frac{r}{c_1^2 c_2}\right) \neq 1$ since c_2 divides $\frac{\xi^2}{c_1^2 c_2}$ and it is not possible because $\left(\frac{r}{c_1^2 c_2}, \frac{s}{c_1^2 c_2}\right) = 1$. So we have that c_2 is invertible $\pmod{\frac{s}{c_1^2 c_2}}$.
This means that finding a solution $\xi \pmod{s}$ of $x^2 \equiv r \pmod{s}$ is equivalent of finding a solution k of $x^2 \equiv \frac{r}{c_1^2 c_2} c_2^{-1} \pmod{\frac{s}{c_1^2 c_2}}$.

Now $\left(\frac{r}{c_1^2 c_2} c_2^{-1}, \frac{s}{c_1^2 c_2}\right) = 1$, so we can apply the Chinese Remainder theorem to reduce the problem of solving the sistem of congruences

$$x^2 \equiv \frac{r}{c_1^2 c_2} c_2^{-1} \pmod{p^h}$$

for all $p^h || \frac{s}{c_1^2 c_2}$ powers of prime. There are at most two solutions for each equation, so the number of total solutions is $\ll 2^{\omega(\frac{s}{c_1^2 c_2})} \leq 2^{\omega(s)} \leq \tau(s) \ll s^\epsilon$ where ω is the function that counts the number of prime divisors of s, hence counts the number of congruences of the above system.

Suppose now that k is a fixed solution, then $kc_1c_2 = \xi_0$ is a fixed solution of $x^2 \equiv r \pmod s$. We want to count the number of $\xi \pmod b$ which are congruent to ξ_0 $(\text{mod } \frac{s}{c_1^2 c_2})$ and that are solutions of $x^2 \equiv r \pmod s$. In this way we can find all the solutions of this congruence.
We have $\xi = \xi_0 + h\frac{s}{c_1^2 c_2}$ for some $h \in \mathbb{Z}$, and we impose

$$\left(\xi_0 + h\frac{s}{c_1^2 c_2}\right)^2 \equiv \xi_0^2 \pmod s$$

that means

$$s | s\left(\frac{h^2 b}{c_1^4 c_2^2} + \frac{2\xi_0 h}{c_1^2 c_2}\right)$$

i.e.

$$\left(\frac{h^2}{c_1^2 c_2}\frac{s}{c_1^2 c_2} + \frac{2\xi_0 h}{c_1^2 c_2}\right) \in \mathbb{Z}$$

c_2 must divides h^2 since it does not divides $\frac{s}{c_1^2 c_2}$. In particular c_2 must divides h.
Moreover if c_1 divides $\left(\frac{s}{c_1^2 c_2}, \frac{\xi_0}{c_1 c_2}\right)$ then it divides also $\frac{r}{c_1^2 c_2}$ and it is not possible because $\left(\frac{r}{c_1^2 c_2}, \frac{s}{c_1^2 c_2}\right) = 1$. So $c_1^2 c2$ must divides h^2, thus $c_1 c_2$ must divides h.

We conclude that

$$\xi = \xi_0 + (c_1 c_2 h_1)\frac{s}{c_1^2 c_2} = \xi_0 + h_1\frac{s}{c_1}$$

hence there are at most c_1 possibilities for ξ once ξ_0 is fixed.
Therefore the total number of possibilities for a general solution $\pmod s$ is $\ll s^\epsilon c_1$. □

We shall study the bound for $M(z, \ell, \delta)$ giving at first a bound with no restriction on ℓ, and then we will study separately the case in which ℓ is a perfect square.

4.1.2 The general case

We split the problem in two separate bounds: set

$$M(z, \ell, \delta) = M_0(z, \ell, \delta) + M_\star(z, \ell, \delta)$$

where $M_0(z, \ell, \delta)$ is the cardinality of the matrices with the third entry $c = 0$ and $M_\star(z, \ell, \delta)$ is the cardinality of the matrices with the third entry non zero.

First we shall find an estimate for $M_\star(z, \ell, \delta)$, which require more details:

$$|u_\rho(z)| = |az + b - \bar{z}(cz + d)|\frac{1}{y} = |\ell + |cz + d|^2 - (cz + d)(a + d)|\frac{1}{cy} \leq \delta$$

Considering the imaginary part we obtain

$$|a + d| \leq \delta$$

and considering the real part we obtain

$$|\ell + |cz + d|^2 - (cx + d)(a + d)| \leq \delta|cy|$$

then

$$|\ell + |cz+d|^2| \le \delta(|cy| + |cx+d|) \ll \delta |cz+d|$$

therefore

$$|cz+d| \le \delta$$

In particular one has $|cx+d| \ll \delta$, $cy \ll \delta$ and $|a-d-2cx| \ll \delta$. Let I be an interval of length $\ll \delta$. Define $A = a-d$ and $D = a+d$. So the problem is equivalent of counting the number of triple (A, D, c) such that

$$\begin{cases} D \in I \\ A \in 2cx + I \\ A^2 \equiv D^2 - 4\ell \pmod{c} \\ q \le c \ll \frac{\delta}{y} \\ c \equiv 0 \pmod{q} \end{cases}$$

Suppose c and D to be fixed, we want to count the number of possibilities for A: Applying lemma 4.1.1 to our equation $A^2 \equiv D^2 - 4\ell \pmod{c}$ and writing $(D^2 - 4\ell, c) = c_0 = c_1^2 c_2$ we have that the number of solutions for $A \pmod{c}$ are $\ll c^\epsilon c_1$.
Let A_0 one fixed solution $\pmod{c}$, the number of $A \in I$ such that $A \equiv A_0 \pmod{c}$ is $\ll \frac{|I|}{c} + 1$. Hence the total number of possibilities for A when c and D are fixed is

$$\sharp\{A\} \ll c^\epsilon c_1 (\frac{|I|}{c} + 1) \ll c^\epsilon c_1 (\frac{\delta}{c} + 1)$$

Once A, D and c are found, then of course b is determined.
I recall that $q|c$ then $c = qm \ll \frac{\delta}{y}$, so

$$m \ll \frac{\delta}{yq} =: W \ll \delta$$

Remark 4.1.3. *Above we have defined $(D^2 - 4\ell, c) = (D^2 - 4\ell, mq) = c_0 = c_1^2 c_2$; note that q can not divides $D^2 - 4\ell$: If it is not true then $D^2 = 4\ell + hq$ for some integer h. In my case I have that $D \ll \delta \ll q^\epsilon \sqrt{\ell}$. So we have*

$$\ell + q \ll D^2 \ll q^{2\epsilon} \ell$$

In our case $\ell \le L$ and L will be a small power of q, in particular $\ell \le q^{1/2}$. So we obtain $q^{1/2} + q \ll q^{2\epsilon + 1/2}$, that gives a contradiction. Therefore we have that

$$(D^2 - 4\ell, m) = c_0$$

We are ready to count the number of possible choices for the quadruples (A, D, c, b), indeed

$$M_\star(z,\ell,\delta) \ll \sum_{1 \le m \le W} (mq)^\epsilon \sum_{c_0 | m} c_1 (1 + \frac{\delta}{mq}) \sum_{D \ll \delta,\ D^2 - 4\ell \equiv 0 \pmod{c_0}} 1$$

Using lemma 4.1.1 another time, in the last sum we obtain

$$\sum_{D \ll \delta,\ D^2 - 4\ell \equiv 0 \pmod{c_0}} 1 \ll c_0^\epsilon b_1 \left(1 + \frac{\delta}{c_0}\right)$$

where $b_0 = b_1^2 b_2 = (4\ell, c_0)$. So the total sum becomes

$$\ll \sum_{1\le m\le W} (mq)^\epsilon \sum_{c_0|m} c_1 \left(1+\frac{\delta}{mq}\right) c_0^\epsilon b_1 \left(1+\frac{\delta}{c_0}\right)$$

Since $c_0 \le m$ I obtain

$$\ll \sum_{1\le m\le W} (m^2q)^\epsilon \left(1+\frac{\delta}{mq}\right) \sum_{c_0|m} c_1 b_1 \left(1+\frac{\delta}{c_0}\right)$$

If we set

$$S_1(m) = \sum_{c_0|m} c_1 b_1$$

and

$$S_2(m) = \sum_{c_0|m} \frac{c_1 b_1}{c_0}$$

we can write the sum above as

$$M_\star(z,\ell,\delta) \ll \sum_{1\le m\le W} (m^2q)^\epsilon \left(1+\frac{\delta}{mq}\right)(S_1(m)+\delta S_2(m))$$
$$\ll (W^2q)^\epsilon \left(\sum_{1\le m\le W} S_1(m) + \delta \sum_{1\le m\le W} S_2(m) + \frac{\delta}{q}\sum_{1\le m\le W}\frac{S_1(m)}{m} + \frac{\delta^2}{q}\sum_{1\le m\le W}\frac{S_2(m)}{m}\right)$$

So the problem is reduced to bound $\sum_{1\le m\le W} S_j(m)$ and $\sum_{1\le m\le W} \frac{S_j(m)}{m}$ for $j=1,2$.

Lemma 4.1.4.

$$S_1(m) \le L^{[j/2]}\sqrt{m}\tau(m)$$

Proof.

$$S_1(m) = \sum_{c_0|m} c_1 b_1$$

The key point is to give a good approximation for b_1. What we know is that ℓ will be of the form $r^a s^b$, where r and s are two primes in $[L, 2L]$ (maybe not distinct) different from q. a and b are non negative integers ≤ 2. I prefer working with (ℓ, m) instead of $(4\ell, m)$ for semplicity of notation; in terms of estimate it doesn't change the result.
So $(\ell, m) = r^\xi s^\eta$ with ξ and η smaller then a and b respectively. To give an estimate of b_1 I need to define the parameter $j = \xi + \eta$. In this way

$$(4\ell, m) \asymp L^j$$

and

$$b_1 \asymp L^{[j/2]}$$

where [ ] indicates the floor function.
Moreover I define

$$P(j) := \{m \leq W : (m,\ell) = r^{\xi}s^{\eta}, j = \xi + \eta\}$$

The idea is to split the sum over m in five sums with respect to j (j runs from 0 to 4). Note that j depends only on m, since ℓ is fixed.
$c_1 \leq \sqrt{m}$ hence

$$S_1(m) = \sum_{c_0|m} c_1 b_1 \asymp L^{[j/2]} \sum_{c_0|m} c_1 \leq L^{[j/2]} \sqrt{m}\tau(m)$$

$\square$

Lemma 4.1.5.

$$\sum_{1\leq m\leq W} S_1(m) \ll W^{3/2+\epsilon}$$

Proof. From the bound above we obtain

$$\sum_{1\leq m\leq W} S_1 \leq \sum_{1\leq m\leq W} L^{[j/2]}\sqrt{m}\tau(m)$$

I recall the notation: $(\ell,m) = r^{\xi}s^{\eta}$, $j = \xi+\eta$, so $(4\ell,m) \asymp L^j$. Moreover I have defined $P(j) := \{m \leq W : (m,\ell) = r^{\xi}s^{\eta}, j = \xi+\eta\}$. The key point is that $L^j \asymp (4\ell,m)|m$, so

$$\sharp P(j) \leq \frac{W}{L^j}$$

Note that for $m \in P(j)$ I can write $m \asymp L^j h$ where $m = (4\ell,m)h$. I obtain

$$\begin{aligned}
\sum_{1\leq m\leq W} S_1(m) &\leq \sum_{1\leq m\leq W} L^{[j/2]}\sqrt{m}\tau(m) \\
&= \sum_{0\leq j\leq 4}\sum_{m\in P(j)} L^{[j/2]}\sqrt{m}\tau(m) \\
&\leq W^{\epsilon}\sum_{0\leq j\leq 4}\sum_{m\in P(j)} L^{[j/2]}\sqrt{m} \\
&\leq W^{\epsilon}\sum_{0\leq j\leq 4} L^{[j/2]}\sum_{m\in P(j)}\sqrt{m} \\
&\asymp W^{\epsilon}\sum_{0\leq j\leq 4} L^{[j/2]}\sum_{1\leq h\leq W/L^j}\sqrt{L^j h} \\
&= W^{\epsilon}\sum_{0\leq j\leq 4} L^{[j/2]}L^{j/2}\sum_{1\leq h\leq W/L^j}\sqrt{h} \\
&= W^{\epsilon}\sum_{0\leq j\leq 4} L^{[j/2]}L^{j/2}\left(\frac{W}{L^j}\right)^{3/2} \\
&= W^{3/2+\epsilon}\sum_{0\leq j\leq 4}\frac{L^{[j/2]}}{L^j} \\
&\ll W^{3/2+\epsilon}
\end{aligned}$$

□

Lemma 4.1.6.
$$\sum_{1\leq m\leq W}\frac{S_1(m)}{m}\ll W^{1/2+\epsilon}$$

Proof. It follows immediatly from the previous lemma, using the Abel summation formula:

$$\sum_{1\leq m\leq W}\frac{S_1(m)}{m}=(\sum_{1\leq m\leq W}S_1(m))\frac{1}{W}+\int_1^W(\sum_{1\leq m\leq t}S_1(m))\frac{1}{t^2}dt$$

By the previous lemma we have

$$\begin{aligned}\sum_{1\leq m\leq W}\frac{S_1(m)}{m}&\ll\frac{W^{3/2+\epsilon}}{W}+W^{\epsilon}\int_1^W\frac{t^{3/2}}{t^2}dt\\&=W^{1/2+\epsilon}+W^{\epsilon}\left[2t^{1/2}\right]_1^W\\&\ll W^{1/2+\epsilon}\end{aligned}$$

□

Lemma 4.1.7.
$$\sum_{1\leq m\leq W}S_2(m)\ll W^{1+\epsilon}$$

Proof. By definition c_1 and b_1 are smaller then $\sqrt{c_0}$ hence

$$\frac{c_1b_1}{c_0}\leq 1$$

Therefore
$$S_2=\sum_{c_0|m}\frac{c_1b_1}{c_0}\leq\tau(m)$$

So we can conclude
$$\sum_{1\leq m\leq W}S_2\leq\sum_{1\leq m\leq W}\tau(m)\asymp W\ln(W)\ll W^{1+\epsilon}$$

□

Remark 4.1.8. *Even if for this bound we have used a trivial argument, a better result is not expected, indeed*
$$\sum_{1\leq m\leq W}S_2\geq\sum_{1\leq m\leq W}\sigma_{-1}(m)\asymp W$$

Lemma 4.1.9.
$$\sum_{1\leq m\leq W}\frac{S_2(m)}{m}\ll W^{\epsilon}$$

Proof. We use another time the Abel summation formula:

$$\sum_{1\le m\le W}\frac{S_2(m)}{m}=(\sum_{1\le m\le W}S_2(m))\frac{1}{W}+\int_1^W(\sum_{1\le m\le t}S_2(m))\frac{1}{t^2}dt$$

By the previous lemma we have

$$\begin{aligned}\sum_{1\le m\le W}\frac{S_2(m)}{m}&\ll\frac{W^{1+\epsilon}}{W}+W^\epsilon\int_1^W\frac{t}{t^2}dt\\&=W^\epsilon+W^\epsilon\left[\log(t)\right]_1^W\\&\ll W^\epsilon\end{aligned}$$

□

Proposition 4.1.9.1. *Let $z\in\mathfrak{S}_q$, ℓ a positive integer coprime with q, and δ a positive real number, then*

$$M(z,\ell,\delta)\ll(q\delta^3)^\epsilon\delta^2+\tau(\ell)\delta(1+y)$$

Proof. So far: we have set

$$M(z,\ell,\delta)=M_0(z,\ell,\delta)+M_\star(z,\ell,\delta)$$

and we have proved that

$$M_\star(z,\ell,\delta)\ll\sum_{1\le m\le W}(mq)^\epsilon\sum_{c_0|m}c_1(1+\frac{\delta}{mq})\sum_{D\ll\delta,\ D^2-4\ell\equiv0\pmod{c_0}}1$$

In the general case (no restriction on ℓ) we have seen that

$$M_\star(z,\ell,\delta)\ll(W^2q)^\epsilon\left(\sum_{1\le m\le W}S_1(m)+\delta\sum_{1\le m\le W}S_2(m)+\frac{\delta}{q}\sum_{1\le m\le W}\frac{S_1(m)}{m}+\frac{\delta^2}{q}\sum_{1\le m\le W}\frac{S_2(m)}{m}\right)$$

Reminding that $W\ll\delta$ and from lemmas 4.1.5 , 4.1.6 ,4.1.7 and 4.1.9 it follows that

$$\begin{aligned}M_\star(z,\ell,\delta)&\ll(W^2q)^\epsilon\left(W^{3/2+\epsilon}+\delta W^{1+\epsilon}+\frac{\delta}{q}W^{1/2+\epsilon}+\frac{\delta^2}{q}W^\epsilon\right)\\&\ll(\delta^2q)^\epsilon\left(\delta^{3/2+\epsilon}+\delta^{2+\epsilon}+\frac{\delta^{3/2+\epsilon}}{q}+\frac{\delta^{2+\epsilon}}{q}\right)\\&\ll(\delta^3q)^\epsilon\delta^2\end{aligned}$$

For what in concerning $M_0(z,\ell,\delta)$ we note that if a matrix $\gamma=\begin{pmatrix}a&b\\0&d\end{pmatrix}$ is in $G_\ell(q)$, then d is determined from a, since $ad=\ell$. Therefore the number of choices for (a,d) is $\ll\tau(\ell)$. It remains to count the number of possibilities for b:

$$|u_\gamma(z)|=\frac{|az+b-\bar{z}d|}{y}<\delta$$

If we consider the imaginary part we obtain

$$|a+d| < \delta$$

Moreover a and d have the same sign, since $ad = \ell$, then

$$|a-d| \leq |a+d| < \delta$$

On the other hand, considering the real part we have

$$|(a-d)x + b| < y\delta$$

which implies

$$|b| < \delta(|x| + y) \ll \delta(1+y)$$

So

$$M_0(z, \ell, \delta) \ll \tau(\ell)\delta(1+y)$$

The conclusion follows immediatly summing the two results just found. $\square$

4.1.10 A special case

In the previous section we have established a bound for $M(z, \ell, \delta)$ that depends on δ. To do that we have not put any restriction on the size of δ, obtaining

$$M(z, \ell, \delta) \ll (q\delta^3)^\epsilon \delta^2 + \tau(\ell)\delta(1+\epsilon)$$

In this section I want to do something a little bit different: suppose we know something more about δ, namely that it can not be too big, and assume that ℓ is perfect square. Then we can find a better bound for $M(z, \ell, \delta)$. The reason of this technique is the following: thanks to the first rude but general bound we are able to add an assumption on δ, namely that it could be small enough to allows us to use the second better bound for $M(z, \ell, \delta)$ in the next step.

Proposition 4.1.10.1. *For $\delta < q^\epsilon\sqrt{\ell}$ and ℓ a perfect square*

$$M(z, \ell, \delta) \ll \tau(\ell)\delta(1+\epsilon)$$

Proof. As in the general case we can reduce the counting lemma to count the number of possible choices for the parameters D, A, b, c, such that

$$\begin{cases} D \in I \\ A \in 2cx + I \\ A^2 = D^2 - 4\ell - 4bc \\ q|c \\ q \leq c \leq \frac{\delta}{y} \end{cases}$$

where I is an interval centred in 0 of length $\ll \delta$, and ℓ is a perfect square less then $L < q^{1/2}$. The added assumption that distinguishes the general case from that is

$$\delta \ll q^\epsilon\sqrt{\ell} \leq q^{1/2}$$

From the above conditions,in particular from the third one, we obtain the weaker condition

$$(D-A)(D+A) \equiv 4\ell \pmod q$$

Since 4ℓ is a square then we have two possibilities: either $D+A$ and $D-A$ are both squares or they are both non-squares.

1) If they are both squares then

$$\begin{cases} D-A \equiv \alpha^2 \pmod q \\ D+A \equiv \beta^2 \pmod q \end{cases} \tag{4.1.1}$$

for some $0 \le \alpha, \beta \le q^{1/2}$. Moreover if α or β are equal to 0 or $q^{1/2}$ then $4\ell \equiv 0 \pmod q$, but this is in contraddiction with the size of ℓ. So $0 < \alpha, \beta < q^{1/2}$.
Therefore

$$4\ell \equiv (D-A)(D+A) \equiv \alpha^2\beta^2 \pmod q \tag{4.1.2}$$

from which it follows that

$$(\alpha\beta + 2\sqrt{\ell})(\alpha\beta - 2\sqrt{\ell}) \equiv 0 \pmod q$$

This means that either $\alpha\beta + 2\sqrt{\ell} \equiv 0 \pmod q$ or $\alpha\beta - 2\sqrt{\ell} \equiv 0 \pmod q$.
Since $\alpha\beta \le (q^{1/2}-1)^2$ and $\ell < q^{1/2}$ then $\alpha\beta \pm 2\sqrt{\ell} < q$ therefore $\alpha\beta = \pm 2\sqrt{\ell}$. I recall that $\sqrt{\ell}$ is a product of almost two primes, then the possibilities for α and β are at most 16.
Let now α and β be fixed, then the above sistem (4.1.1) gives a unique solution for D an $A \pmod q$. For each such a solution $\pmod q$ we want now to count the number of integral solutions. But since both D and A belong to an interval of length $\delta < q$ then there are at most one integral solution.

2) If they are both non squares $\pmod q$ then there exists a positive integer

$$x_0 \le q^{\frac{1}{4\sqrt{e}}+\epsilon} \le q^{\frac{1}{4}} \tag{4.1.3}$$

such that x_0 is a quadratic nonresidue $\pmod q$. So from (4.1.2) it follows that

$$x_0(D-A)x_0(D+A) \equiv 4\ell x_0^2 \pmod q$$

Now, $x_0(D \pm A)$ are quadratic residues $\pmod q$, so as above there exists $0 \le \alpha, \beta \le q^{1/2}$ such that

$$\begin{cases} x_0(D-A) \equiv \alpha^2 \pmod q \\ x_0(D+A) \equiv \beta^2 \pmod q \end{cases} \tag{4.1.4}$$

As above, if α or β are equal to 0 or $q^{1/2}$ then $4\ell x_0^2 \equiv 0 \pmod q$. Another time, because of the size of $x_0^2 < q^{1/2}$ and $\ell < q^{1/2}$, this gives a contraddiction. So $0 < \alpha, \beta < q^{1/2}$.
We obtain that

$$(\alpha\beta + 2\sqrt{\ell}x_0)(\alpha\beta - 2\sqrt{\ell}x_0) \equiv 0 \pmod q$$

But $\alpha\beta \pm 2\sqrt{\ell}x_0 \le (q^{1/2}-1)^2 + 2q^{1/4}q^{1/4} < q$, so $\alpha\beta = \pm 2\sqrt{\ell}x_0$. Now, since α and β are divisors of $2\sqrt{\ell}x_0$ the number of possible choices for them is $\ll \tau(x_0) \ll q^{\epsilon}$. Fix α and β, then as in the previous case there exists a unique solution (mod q) for D and A in (4.1.4), and so a unique solution for D and A in $\mathbb{Z}$.

So far, we have proved that the contribution of D and A in the counting lemma is at most q^ϵ. Now, suppose c to be fixed, then there are $\ll q^\epsilon$ solutions for D and A and b is given from this solutions. This allows us to say that the counting lemma can be reduce to count the number of possible choices for c, that is $\ll \delta$. So we can conclude that

$$M_\star(z, \ell, \delta) \ll q^\epsilon \delta$$

for $\delta \ll q^\epsilon \sqrt{\ell}$.

As in the general case

$$M_0(z, \ell, \delta) \ll \tau(\ell)\delta(1 + y)$$

and so

$$M(z, \ell, \delta) \ll \tau(\ell)\delta(1 + y)$$

□

Remark 4.1.11. *The problem of finding the best bound for the first quadratic nonresidue in the interval* $[1, q-1]$ *for a large* q *is very famous, and one can find references in [LW08]. Anyway, I want to justify without details the above formula (4.1.3): probabilistic heuristics suggests that this number, call it* n_q*, should have size* $\mathcal{O}(\log q)$*, and indeed Vinogradov conjectured that* $n_q = \mathcal{O}_\epsilon(q^\epsilon)$ *for any* $\epsilon > 0$*. Using Polya-Vinogradov inequality one can get bound* $n_q \ll \sqrt{q}\log q$ *and can improve it to* $\sqrt{q}$ *using smoothed sums. Combining this with a sieve theory argument one can boost this to* $n_q \ll q^{\frac{1}{2\sqrt{e}}} \log^2 q$*. Finally, inserting Burgess's amplification trick one can boost this to* $n_q \ll q^{\frac{1}{4\sqrt{e}}+\epsilon}$*.*

4.2 A bound for $\sum_{\rho \in G_\ell(q)} K(|u_\rho(z)|)$

So far we have found a general bound for $M(z, \ell, \delta)$, namely

$$M(z, \ell, \delta) \ll (q\delta^3)^\epsilon \delta^2 + \tau(\ell)\delta(1 + y)$$

and if δ is small enough and ℓ a perfect square we have that

$$M(z, \ell, \delta) \ll \tau(\ell)\delta(1 + y)$$

In fact for our purposes we can simplify this bound. Indeed, as I remarked several times, we are interested in an upper bound for $g(z)$, where $z \in \mathfrak{S}_q - \mathfrak{S}_{\eta q}$, for same $0 < \eta < 1$. This is the reagion in which we have not found a non-trivial bound for our function. So we may assume $y \ll \frac{1}{\eta q}$. Moreover we will see later that ℓ will be smaller then a small positive power of q, in particular $\tau(\ell) \ll q^\epsilon$. With these two reductions the second term $\tau(\ell)\delta(1+y)$ is negligible with respect to the first one in both the general and the special case. So we may assume for the general case

$$M(z, \ell, \delta) \ll (q\delta^3)^\epsilon \delta^2 \ll q^\epsilon \delta^2$$

and if $\delta \leq q^\epsilon \sqrt{\ell}$ and ℓ a perfect square then

$$M(z, \ell, \delta) \ll q^\epsilon \delta$$

Now we have all the ingredients to bound $\sum_{\rho \in G_\ell(q)} K(|u_\rho(z)|)$.

Proposition 4.2.0.1.

$$\sum_{\rho\in G_\ell(q)} K(|u_\rho(z)|) \ll \frac{q^\epsilon}{\ell^{\frac{k}{2}-1}}$$

Moreover, if ℓ is a perfect square we have

$$\sum_{\rho\in G_\ell(q)} K(|u_\rho(z)|) \ll \frac{q^\epsilon}{\ell^{\frac{k-1}{2}}}$$

Proof. The idea is to split the sum into two sums in which one of the two will be negligible. For this reason define

$$\sigma_1 = \sum_{\rho\in G_\ell(q), |u_\rho(z)|\le U} K(|u_\rho(z)|)$$

and

$$\sigma_2 = \sum_{\rho\in G_\ell(q), |u_\rho(z)|> U} K(|u_\rho(z)|)$$

where U will be choosen such that S_2 will be negligible: using the Stieltjes integral we have

$$\begin{aligned}\sigma_2 = \int_U^\infty \frac{1}{\delta^k} dM(z,\ell,\delta) &= \left[\frac{M(z,\ell,\delta)}{\delta^k}\right]_U^\infty + k\int_U^\infty \frac{M(z,\ell,\delta)}{\delta^{k+1}} d\delta \\ &\ll \left[\frac{q^\epsilon\delta^\epsilon}{\delta^{k-2}}\right]_U^\infty + k\int_U^\infty \frac{q^\epsilon\delta^\epsilon}{\delta^{k-1}} d\delta\end{aligned}$$

Note that in the above computation we have used the general bound for $M(z,\ell,\delta)$. From the appendix 2 we have that for all $\rho\in G_\ell(q)$,

$$|u_\rho(z)| \gg \sqrt{\ell}$$

So the parameter U has to be $\gg \sqrt{\ell}$. To make σ_2 negligible we need to make q^ϵ disappear from the numerator. Since $k\ge 4$ by our initial hypotesis, then it is enough to take $U = q^\epsilon\sqrt{\ell}$. So we may restrict to study

$$\sigma_1 = \sum_{\rho\in G_\ell(q), |u_\rho(z)|\le q^\epsilon\sqrt{\ell}} K(|u_\rho(z)|)$$

Since $|u_\rho(z)| \gg \sqrt{\ell}$ for all $\rho\in G_\ell(q)$ then

$$K(|u_\rho(z)|) \ll \ell^{-k/2}$$

for all $\rho\in G_\ell(q)$. So we can conclude that

$$\sum_{\rho\in G_\ell(q), |u_\rho(z)|\le q^\epsilon\sqrt{\ell}} K(|u_\rho(z)|) \ll \ell^{-k/2} M(z,\ell,\sqrt{\ell}q^\epsilon)$$

Since $\delta = q^\epsilon\sqrt{\ell} < q$, if ℓ is a perfect square then we can use the better bound for $M(z,\ell,\delta)$ and we obtain

$$\begin{aligned}\sum_{\rho\in G_\ell(q),|u_\rho(z)|\le q^\epsilon\sqrt{\ell}} K(|u_\rho(z)|) &\ll \ell^{-k/2}M(z,\ell,\sqrt{\ell}q^\epsilon)\\ &\ll \ell^{-k/2}q^\epsilon\sqrt{\ell}\\ &\ll \frac{q^\epsilon}{\ell^{(k-1)/2}}\end{aligned}$$

If ℓ is not a perfect square we have to use the general bound for $M(z,\ell,\delta)$ obtaining

$$\begin{aligned}\sum_{\rho\in G_\ell(q),|u_\rho(z)|\le q^\epsilon\sqrt{\ell}} K(|u_\rho(z)|) &\ll \ell^{-k/2}M(z,\ell,\sqrt{\ell}q^\epsilon)\\ &\ll \ell^{-k/2}q^\epsilon\left(\sqrt{\ell}q^\epsilon\right)^2\\ &\ll \frac{q^\epsilon}{\ell^{\frac{k}{2}-1}}\end{aligned}$$

$\square$

4.3 The Amplifier

So far we have found the following equality

$$C_k\sum_j|\sum_{1\le l\le L^2} x_\ell\lambda_{f_j}(\ell)|^2\left|\frac{y^{k/2}f_j(z)}{\sqrt{< f_j,f_j >}}\right|^2 = \sum_{1\le\ell_1,\ell_2\le L^2} x_{\ell_1}\bar{x_{\ell_2}}\sum_{d|(\ell_1,\ell_2)}\left(\frac{\ell_1\ell_2}{d^2}\right)^{\frac{k-1}{2}}\sum_{\rho\in G_{\frac{\ell_1\ell_2}{d^2}}(q)} K(|u_\rho(z)|)$$

and we have established an upper bound for $\sum_{\rho\in G_{\frac{\ell_1\ell_2}{d^2}}(q)} K(|u_\rho(z)|)$, which is $\frac{q^\epsilon}{\ell^{k/2-1}}$.

Let's define the amplifier: Let f be a cuspform of level q, and L a positive real number. Let

$$\Lambda := \{p \text{ prime} : (p,q) = 1, p\in[L,2L]\}$$

be a large set of primes. Define

$$x_\ell = \begin{cases} sgn(\lambda_f(\ell)), & \text{if } \ell\in\Lambda\cup\Lambda^2\\ 0, & \text{otherwise}\end{cases}$$

where $\lambda_f(\ell)$ is the eigenvalue of f with respect to $T(\ell)$.
The main property of this amplifier is that

Lemma 4.3.1.

$$|\sum_{1\le\ell\le L^2} x_\ell\lambda_{f_0}(\ell)| \gg L^{1-\epsilon}$$

Proof. Let $\ell \in \Lambda$, then by the remark 1.3.17 we have that

$$\lambda_{f_0}(\ell)^2 - \lambda_{f_0}(\ell^2) = 1$$

In particular

$$max\{|\lambda_{f_0}(\ell)|, |\lambda_{f_0}(\ell^2)|\} \geq \frac{1}{2}$$

Therefore

$$|\sum_{1 \leq \ell \leq L^2} x_\ell \lambda_{f_0}(\ell)| \gg \sum_{\ell \in \Lambda} 1 \gg \pi(L) \gg L^{1-\epsilon}$$

□

Before proving the bound for $g(z)$ we need to introduce another variable, that will be useful during the proof: for all ℓ positive integers we set

$$y_\ell := \sum_{d|(\ell_1,\ell_2),\ \ell = \frac{\ell_1 \ell_2}{d^2}} x_{\ell_1} \bar{x_{\ell_2}}$$

Lemma 4.3.2. *$y_\ell = 0$ for $\ell \gg L^4$ and $|y_\ell| \leq 2$ for all $\ell \neq 1$. Moreover $y_1 \ll \frac{L}{\log L}$*

Proof. Since $\ell = \frac{\ell_1 \ell_2}{d^2}$, where $d|(\ell_1, \ell_2)$, then it follows from the definition of x_ℓ that $y_\ell = 0$ for $\ell \gg L^4$.
Now, $|y_\ell| \leq \sharp\{\ell_1, \ell 2 \in \Lambda \cup \Lambda^2 : \ell = \frac{\ell_1 \ell_2}{d^2}\}$.
Note that ℓ_1 and ℓ_2 are either primes or square of primes. So $\ell \neq 1$ could only be either of the form p^i, with $1 \leq i \leq 4$, or $p^i q^j$, with $1 \leq i, j \leq 2$. We now study each case:

- $\ell = p$ then the only possibilities are $\ell_1 = p^2$, $\ell_2 = p$, $d = p$, and the symmetric case;
- $\ell = p^2$ then the only possibilities are $\ell_1 = \ell_2 = p^2$,$d = p$, and $\ell_1 = \ell_2 = p$, $d = 1$;
- $\ell = p^3$ then the only possibilities are $\ell_1 = p^2$, $\ell_2 = p$, $d = 1$, and the symmetric case;
- $\ell = p^4$ then the only possibility is $\ell_1 = \ell_2 = p^2$,$d = 1$;
- if ℓ is divisible by two different primes, then $d = 1$ and ℓ_1 and ℓ_2 are uniquely determined up to symmetry.

For what is concerning y_1, it occurs whenever $\ell_1 = \ell_2 = d = p^i$, for each prime $p \in \Lambda$ and $i = 1, 2$. So there are $\ll \sharp\Lambda \asymp \frac{L}{\log L}$ possibilities for ℓ_1 and ℓ_2 so that $\ell = 1$. □

Lemma 4.3.3. *The number of ℓ such that $y_\ell \neq 0$ is $\ll \frac{L^2}{logL}$.*

Proof. $\ell = \frac{\ell_1 \ell_2}{d^2}$, so a direct computation gives that y_ℓ is not zero only if ℓ is of the kind

$$\{p, p^2, p^3, p^4, pq, p^2 q, p^2 q^2\}$$

where p and q are primes in $[L, 2L]$. So for each prime in this interval we have 4 possibilities (p, p^2, p^3, p^4) and for each pair of primes we have other 4 possibilities $(pq, p^2 q, p^2 q^2)$. Let $\pi(L)$ be the function that counts the number of primes less then L, then the number of ℓ such that $y_\ell \neq 0$ is at most $4(\pi(2L) - \pi(L) + (\pi(2L) - \pi(L))^2) \ll \pi(2L)^2 \ll \frac{L^2}{logL}$. □

Proposition 4.3.3.1. *Let $0 < \eta \leq 1$ be a real number, then uniformly on $z \in \mathfrak{S}_q - \bigcup_{\alpha \in A_0(q)} \alpha.\mathfrak{S}_{\eta q}$*

$$g(z) \ll_\epsilon q^{1/2+\epsilon}$$

Proof. As a consequence of the equation at the begining of this section, one has

$$\begin{aligned}
|\sum_{1 \leq \ell \leq L^2} x_\ell \lambda_{f_0}(\ell)|^2 \left| y^{k/2} f_0(z) \right|^2 &\ll \langle f_0, f_0 \rangle \sum_{1 \leq \ell_1, \ell_2 \leq L^2} x_{\ell_1} \bar{x_{\ell_2}} \sum_{d|(\ell_1,\ell_2)} \left(\frac{\ell_1 \ell_2}{d^2} \right)^{\frac{k-1}{2}} \sum_{\rho \in G_{\frac{\ell_1 \ell_2}{d^2}}(q)} K(|u_\rho(z)|) \\
&\ll q^{1+\epsilon} \sum_{1 \leq \ell_1, \ell_2 \leq L^2} \sum_{d|(\ell_1,\ell_2)} x_{\ell_1} \bar{x_{\ell_2}} \left(\frac{\ell_1 \ell_2}{d^2} \right)^{\frac{k-1}{2}} \sum_{\rho \in G_{\frac{\ell_1 \ell_2}{d^2}}(q)} K(|u_\rho(z)|) \\
&\ll q^{1+\epsilon} \sum_{1 \leq \ell \ll L^4} y_\ell \ell^{\frac{k-1}{2}} \sum_{\rho \in G_\ell(q)} K(|u_\rho(z)|)
\end{aligned}$$

At this point, we split the sum dividing the case in which ℓ is a perfect square from the other:

$$\ll q^{1+\epsilon} \left[\sum_{1 \leq \ell \ll L^3} y_\ell \ell^{\frac{k-1}{2}} \sum_{\rho \in G_\ell(q)} K(|u_\rho(z)|) + \sum_{\substack{1 \leq \ell \ll L^4 \\ \ell \text{ square}}} y_\ell \ell^{\frac{k-1}{2}} \sum_{\rho \in G_\ell(q)} K(|u_\rho(z)|) \right]$$

From previous section we obtain

$$\begin{aligned}
&\ll q^{1+\epsilon} \left[\sum_{1 \leq \ell \ll L^3} y_\ell \ell^{\frac{k-1}{2}} q\epsilon \frac{1}{\ell^{\frac{k}{2}-1}} + \sum_{\substack{1 \leq \ell \ll L^4 \\ \ell \text{ square}}} y_\ell \ell^{\frac{k-1}{2}} q\epsilon \frac{1}{\ell^{\frac{k-1}{2}}} \right] \\
&\ll q^{1+\epsilon} \left[\sum_{1 \leq \ell \ll L^3} y_\ell \ell^{1/2} + \sum_{\substack{1 \leq \ell \ll L^4 \\ \ell \text{ square}}} y_\ell \right]
\end{aligned}$$

Since $|y_\ell|$ is bounded for $\ell \neq 1$, and $y_1 \ll \frac{L}{\log L}$ from lemmas 4.3.2 and 4.3.3, it follows that

$$\begin{aligned}
&\ll q^{1+\epsilon} \left(y_1 + L^{3/2} \sharp\{y_\ell \neq 0\} \right) \\
&\ll q^{1+\epsilon} \left(\frac{L}{\log L} + L^{3/2} \frac{L^2}{\log L} \right) \\
&\ll q^{1+\epsilon} \frac{L^{7/2}}{\log L}
\end{aligned}$$

By lemma 4.3.1 we conclude that

$$L^{2-\epsilon} g(z)^2 \ll \left| \sum_{1 \leq \ell \leq L^2} x_\ell \lambda_{f_0}(\ell) \right|^2 g(z)^2 \ll q^{1+\epsilon} \frac{L^2}{\log L}$$

Hence

$$g(z) \ll q^{1/2+\epsilon} \frac{L^{3/2}}{\log L}$$

Since L has to be bigger then 1 the proposition is proved. $\square$

Chapter 5

The Diophantine Argument

In this chapter we shall use essentially the same ideas of the previous one, adding a Diophantine argument to improve the bound for $M(z, l, q^\epsilon\sqrt{L})$. More precisely, for all z in a certain reagion of our fundamental domain we shall use *Dirichlet approximation* on $\Re z$ to introduce two new parameters (H and Q) that will allows us to improve the upper bound for $\sum_{l=1}^{L} M(z, l, q^\epsilon\sqrt{L})$.

First of all we recall the *Dirichlet theorem* and we use it to find an important property for all $z \in \mathfrak{S}_q - \bigcup_{\delta\in A_0(q)} \delta\mathfrak{S}_{\eta q}$:

Let H and Q be fixed positive integers, $Q \leq H$.

Theorem 5.0.4 (Dirichlet's approximation theorem)**.** *For all $x \in \mathbb{R}$ there exists s and $t \in \mathbb{Z}$ coprime, $1 \leq t \leq H$ such that*

$$\left|x - \frac{s}{t}\right| \leq \frac{1}{tH}$$

Proof. The proof uses the pigeonhole principle: consider for all $t = 1, \cdots, H$ the integers $t\alpha - [t\alpha]$. All this integers are smaller then 1. Divide the interval $[0, 1]$ into H intervals of length $1/H$, and call them $G_1, \cdots, G_H$. So if at least one of the $t\alpha - [t\alpha]$'s belongs to G_1, then call $s = [t\alpha]$ and we have done. Otherwise by the pigeonhole principle there exist $t_1 \neq t_2$ such that $t_1\alpha - [t_1\alpha]$ and $t_2\alpha - [t_2\alpha]$ belong to the seme subinterval G_i. In particular $|t_1\alpha - [t_1\alpha] - t_2\alpha - [t_2\alpha]| < \frac{1}{H}$, and the conclusion follows immediatly. □

We want to establish if a real number x in well approximated or not, with respect to the size of t, indeed

Definition 5.0.5. *Let $x \in \mathbb{R}$, and let $\frac{s}{t}$ be an approximation as in the Dirichlet theorem. We say that x is* ***well approximated*** *if $1 \leq t \leq Q$. We say that x is* ***not well approximated*** *if $t \geq Q$.*

Lemma 5.0.6. *Assume $H^2 \geq \frac{2q}{\eta}$, $z = x + iy \in \mathfrak{S}_q - \bigcup\limits_{\delta\in A_0(q)} \delta.\mathfrak{S}_{\eta q}$. Then any approximation $\frac{s}{t}$ of x in the sense of Dirichlet theorem satisfies*

$$\sqrt{2}t > \eta^{3/2} q^{1/2}$$

Proof. Let $z \in \mathfrak{S}_q - \mathfrak{S}_{\eta q}$.

Assume that there exists $\frac{s}{t}$ such that $\sqrt{2}t \leq \eta^{3/2}q^{1/2}$. Then by proposition 2.3.2.1 there exists $b, d \in \mathbb{Z}$ such that

$$\begin{pmatrix} s & b \\ t & d \end{pmatrix} = \gamma W_m \begin{pmatrix} 1/m & 0 \\ 0 & 1 \end{pmatrix}$$

for some $\gamma \in \Gamma_0(q)$ and $m \in \{1, q\}$.
I claim that

$$z \in \gamma W_m \begin{pmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{pmatrix} \mathfrak{S}_{\eta q}$$

If the claim is true then we have that $z \in \bigcup_{\delta \in A_0(q)} \delta \mathfrak{S}_{\eta q}$ because these three matrices belongs to $A_0(q)$; this clearly conclude the proof of the lemma. It only remains to prove our claim:
It is clearly equivalent of proving that

$$z \in \begin{pmatrix} s & b \\ t & d \end{pmatrix} \begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{pmatrix} \mathfrak{S}_{\eta q}$$

Since $m \in \{1, q\}$ it suffices to prove that

$$z \in \begin{pmatrix} s & b \\ t & d \end{pmatrix} \begin{pmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{pmatrix} \mathfrak{S}_{\eta}$$

because the matrix $\begin{pmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{pmatrix}$ is just a horizontal translation, and the matrix $\begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix}$ is the multiplication by m. So if an element $w \in \mathfrak{S}_{\eta q}$, then $mw \in \mathfrak{S}_{\eta}$ because $m \leq q$. For these reasons it suffices to prove that $\Im \left(\begin{pmatrix} s & b \\ t & d \end{pmatrix}^{-1} z \right) \geq \frac{\sqrt{3}}{2\eta}$.
It follows immediatly with a direct computation, namely:

$$\Im \left(\begin{pmatrix} s & b \\ t & d \end{pmatrix}^{-1} z \right) = \frac{\Im z}{|-tz + s|^2} \tag{5.0.1}$$

$$\geq \frac{\frac{\sqrt{3}}{2q}}{|s - tx|^2 + \frac{t^2}{q^2\eta^2}} \tag{5.0.2}$$

$$\geq \frac{\frac{\sqrt{3}}{2q}}{\frac{1}{H^2} + \frac{t^2}{q^2\eta^2}} \tag{5.0.3}$$

$$\geq \frac{\sqrt{3}}{4} min \left\{ \frac{H^2}{q}, \frac{\eta^2 q}{t^2} \right\} \tag{5.0.4}$$

$$\geq \frac{\sqrt{3}}{2\eta} \tag{5.0.5}$$

The first inequality comes from $z \in \mathfrak{S}_q - \mathfrak{S}_{\eta q}$; the second by Dirichlet theorem; the third follows from the general fact $\frac{1}{\frac{1}{i} + \frac{1}{j}} \geq \frac{min\{i,j\}}{2}$; the last one follows directly from the hypotesis and from the assumption $\sqrt{2}t \leq \eta^{3/2}q^{1/2}$. □

From now on we assume

$$2Q^2 \leq \eta^3 q \tag{5.0.6}$$
$$\frac{2q}{\eta} \leq H^2 \tag{5.0.7}$$

Remark 5.0.7. *The first inequality allows us to apply the Lemma above, and the second one with the Lemma itself allow us to say that* $\Re z$ *is not well approximable in the sense of Dirichlet, for any* $z = x + iy \in \mathfrak{S}_q - \bigcup_{\delta \in A_0(q)} \delta.\mathfrak{S}_{\eta q}$.

Before going on I want to recall the situation and our goal:
as in the previous chapter we shall use the amplification method; we have found the following equality that allows us to find an upper bound for $g(z) = |y^{k/2} f_0(z)|$ studying the right hand-side:

$$C_k \sum_j |\sum_{1\leq \ell \leq L} x_\ell \lambda_{f_j}(\ell)|^2 \left| \frac{y^{k/2} f_j(z)}{\sqrt{< f_j, f_j >}} \right|^2 = \sum_{1 \leq \ell_1, \ell_2 \leq L} x_{\ell_1} \bar{x_{\ell_2}} \sum_{d|(\ell_1,\ell_2)} \left(\frac{\ell_1 \ell_2}{d^2}\right)^{\frac{k-1}{2}} \sum_{\rho \in G_{\frac{\ell_1 \ell_2}{d^2}}(q)} K(|u_\rho(z)|)$$

The central point of our computation in to give an upper bound for

$$M(z, \ell, q^\epsilon \sqrt{\ell}) = \sharp\{\gamma \in G_\ell(q) : |u_\gamma(z)| \leq q^\epsilon \sqrt{\ell}\}$$

Indeed I remind that by section 4.1 we have obtain that

$$\sum_{\rho \in G_\ell(q)} K(|u_\rho(z)|) \ll \sum_{\rho \in G_\ell(q), |u_\rho(z)| \leq q^\epsilon \sqrt{\ell}} K(|u_\rho(z)|) \ll \frac{1}{\ell^{k/2}} M(z, \ell, q^\epsilon \sqrt{\ell})$$

because $|u_\rho(z)| \gg \sqrt{\ell}$ from the appendix, and so

$$K(|u_\rho(z)|) \ll \frac{1}{\ell^{k/2}}$$

While in the previous chapter I have studied directly $M(z, \ell, q^\epsilon \sqrt{L})$, now the approach is a little bit different: we want to find an upper bound for

$$\sum_{1 \leq \ell \leq L} M(z, \ell, q^\epsilon \sqrt{\ell})$$

Now we can go on: first of all we need to lemmas used to count the number of matrices involved in our sum with the third entry equal 0.

Lemma 5.0.8. *Let* $1 \leq L \leq \eta^2 q^{2-\epsilon}$. *For all* $z \in \mathbb{H} - \bigcup_{\delta \in A_0(q)} \delta \mathfrak{S}_{\eta q}$ *the only parabolic matrix* $\gamma \in G_\ell(q)$ *such that* $|u_\gamma(z)| \leq \sqrt{L} q^\epsilon$ *is* $\pm \begin{pmatrix} \sqrt{\ell} & 0 \\ 0 & \sqrt{\ell} \end{pmatrix}$.

Proof. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a parabolic matrix, i.e. $|a+d| = 2\sqrt{\ell}$. Then γ fixes a cusp, call it $\mathfrak{a}$. Moreover by lemma 2.3.2.1 there exists $\sigma \in A_0(q)$ such that $\sigma(\infty) = \mathfrak{a}$.
Consider $\sigma^{-1}\gamma\sigma = \gamma'$, then $\gamma'(\infty) = \infty$ and it is a conjugate of γ, hence it is parabolic; so we have that $\gamma' = \begin{pmatrix} \sqrt{\ell} & r \\ 0 & \sqrt{\ell} \end{pmatrix}$ for some $r \in \mathbb{Z}$. In particular l is a square. So to conclude it is enough to prove that $r = 0$.
Define $w = \sigma^{-1}(z)$. Then as I have already proved

$$|u_{\gamma'}(w)| = |u_\gamma(z)| \leq \sqrt{L}q^\epsilon$$

In particular $w \notin \bigcup_{\delta \in A_0(q)} \delta\mathfrak{S}_{\eta q}$ because z is not in it, and $\sigma \in A_0(q)$. As a consequence if $w = x' + y'$ then $y' \leq \frac{\sqrt{3}}{2\eta q}$.
So we have

$$|u_{\gamma'}(w)| = \frac{|\sqrt{\ell}w + r - \sqrt{\ell}\bar{w}|}{y'} = \frac{|\sqrt{\ell}2iy' + r|}{y'} \geq \frac{|r|}{y'} \geq |r|\eta q$$

In particular $|r|\eta q < \sqrt{L}q^\epsilon$ hence $|r| < \sqrt{L}\eta^{-1}q^{\epsilon-1} \leq 1$ by the hypotesis. So $r = 0$ and the lemma is proved. $\square$

Lemma 5.0.9. *Assume* $\sqrt{L}q^\epsilon < min\left\{Q, \frac{\eta q}{H}\right\}$. *For all* $z = x + iy \in \mathfrak{S}_q - \bigcup_{\delta \in A_0(q)} \delta.\mathfrak{S}_{\eta q}$, *the only matrix* $\gamma = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in G_\ell(q)$ *such that* $|u_\gamma(z)| \leq \sqrt{L}q^\epsilon$ *is* $\begin{pmatrix} \sqrt{\ell} & 0 \\ 0 & \sqrt{\ell} \end{pmatrix}$.

Proof.

$$|u_\gamma(z)| = \frac{|(a-d)x + b + iy(a+d)|}{y} \leq \sqrt{L}q^\epsilon$$

In particular considering the imaginary part we have that $|a+d| \leq \sqrt{L}q^\epsilon$. Now, $det\gamma = ad = \ell > 0$ hence a and d have the same sign and so

$$|a-d| < |a+d| \leq \sqrt{L}q^\epsilon < Q$$

Claim: $|a-d| = 0$
Since $1 \leq |a-d| < Q$, then it follows that $|(a-d)x + b| \geq \frac{1}{H}$ because as we have seen, with our assumptions on L any approximation (in the sense of Dirichlet) of $\Re z = x$ is a not well approximation, indeed: suppose $|(a-d)x + b| < \frac{1}{H}$ then $|x - \frac{b}{d-a}| < \frac{1}{|d-a|H}$. By our initial assumpitions $|a-d| \leq q^\epsilon\sqrt{L} < Q$ and by lemma 5.0.6 we have

$$2|d-a| > \eta^3 q \geq 2Q$$

then $|d-a| > Q$, which gives a contraddiction.
On the other hand, considering the above inequality for $|u_\gamma(z)|$, and considering the real part we obtain

$$|(a-d)x + b| \leq \sqrt{L}q^\epsilon y \leq \sqrt{L}q^\epsilon \frac{1}{\eta q}$$

Therefore we have

$$\frac{1}{H} \leq |(a-d)x+b| \leq \frac{\sqrt{L}q^{\epsilon}}{\eta q}$$

and so $\sqrt{L}q^{\epsilon} \geq \frac{\eta q}{H}$, in contraddiction with our hypotesis. Hence $a-d=0$, then γ is parabolic and the conclusion follows immediatly from the previous lemma, that can be applied since $\sqrt{L}q^{\epsilon} < \eta q$. $\square$

We have now all the ingredients to prove the following

Proposition 5.0.9.1. *Let* $z \in \mathbb{H} - \bigcup_{\delta \in A_0(q)} \delta \mathfrak{S}_{\eta q}$, *then*
i)

$$\sum_{\ell=1}^{L} M(z,\ell,q^{\epsilon}\sqrt{\ell}) \ll q^{5\epsilon} L \left(\frac{L^{1/2}}{Q}+1\right)\left(\frac{HL^{1/2}}{q}+\frac{L^{1/2}}{H}+1\right)$$

ii)

$$\sum_{\ell=1,\ell \text{ is a square}}^{L} M(z,\ell,q^{\epsilon}\sqrt{\ell}) \ll q^{4\epsilon} L^{1/2} \left(\frac{L^{1/2}}{Q}+1\right)\left(\frac{HL^{1/2}}{q}+\frac{L^{1/2}}{H}+1\right)$$

Proof. i) The counting problem is invariant by conjugation by $A_0(q)$, so we may assume $z \in \mathfrak{S}_q$. Moreover $z = x+iy$ and by lemma 5.0.6 we may assume $\left|x - \frac{s}{t}\right| \leq \frac{1}{tH}$ with $Q < t \leq H$.

If $c=0$, by lemma 5.0.9 $\gamma = \begin{pmatrix} \sqrt{\ell} & 0 \\ 0 & \sqrt{\ell} \end{pmatrix}$, and it is possible only if ℓ is a square,then the contribution of such matrices is at most $L^{1/2}$.

If $c \geq 1$ we have for some $1 \leq \ell \leq L$

$$|u_{\gamma}(z)| = \frac{|\ell + |cz+d|^2 - (cz+d)(a+d)|}{cy} \leq \sqrt{\ell}q^{\epsilon}$$

Considering the imaginary part we obtain

$$|a+d| \leq \sqrt{\ell}q^{\epsilon}$$

and considering the real part we have

$$|\ell + |cz+d|^2 - (cx+d)(a+d)| \leq \sqrt{\ell}q^{\epsilon}cy$$

and so

$$|\ell + |cz+d|^2| \leq \sqrt{\ell}q^{\epsilon}(|cx+d|+cy) \ll \sqrt{\ell}q^{\epsilon}|cz+d|$$

which gives

$$|cz+d| \ll \sqrt{\ell}q^{\epsilon}$$

As a consequence we have

$$|cx+d| \ll \sqrt{\ell}q^{\epsilon}$$
$$c \ll \frac{\sqrt{\ell}q^{\epsilon}}{y}$$

and

$$|a - d - 2cx| = |a + d - 2(cx + d)| \ll \sqrt{\ell} q^{\epsilon}$$

Let I be an interval of length $\ll \sqrt{L} q^{\epsilon}$ centered in 0 and sucht that $a - d \in 2cx + I$.

For each such integer c we are reduced to counting the number of quadruples of integers (a, b, d, ℓ) such that

$$\begin{cases} a - d \in 2cx + I \\ a + d \in I \\ 1 \leq \ell \leq L \\ ad = \ell + bc \end{cases}$$

Note that there are at most $\frac{L^{1/2} q^{\epsilon}}{qy}$ possible choices for c, since it is divisible by q.

Set $u := [2cx]$; $A = a - d - u$ and $D := a + d$. Thus we need to count the number of quadruples (A, D, b, ℓ) such that

$$\begin{cases} A \in I \\ D \in I \\ 1 \leq \ell \leq L \\ 2Au + 4bc = D^2 - 4\ell - u^2 - A^2 \end{cases}$$

First we look for the equation satisfied by (A, b). All the possible integers $D^2 - 4\ell - u^2 - A^2$ belong to an interval K of length $\ll Lq^{2\epsilon}$, because of the size of each summand. So we want to count the number of pairs (A, b) such that

$$\begin{cases} A \in I \\ b \in \mathbb{Z} \\ 2Au + 4bc \in K \end{cases}$$

Let $J = \left[-\frac{1}{tH}, \frac{1}{tH}\right]$, so $x \in \frac{s}{t} + J$.
Consider

$$\begin{cases} Au + 2bc \in K \\ x \in \frac{s}{t} + J \end{cases}$$

Multiplying the second equation by $2Ac$ we obtain

$$\begin{cases} Au + 2bc \in K \\ Au \in \frac{2Acs}{t} + c.I.J \end{cases}$$

which gives

$$\frac{Acs}{t} + bc \in c.I.J + K$$

Multiplying this condition by $\frac{t}{c}$ we obtain the weak condition that (A, b) has to satisfy

$$As + bt \in \frac{t}{c} K + t.I.J$$

This last interval has length

$$\begin{aligned}&\ll \frac{t}{c}Lq^{2\epsilon} + tL^{1/2}q^{\epsilon}\frac{1}{tH} = q^{\epsilon}\left(\frac{t}{c}Lq^{\epsilon} + \frac{L^{1/2}}{H}\right)\\ &\ll q^{2\epsilon}\left(\frac{t}{c}L + \frac{L^{1/2}}{H}\right)\end{aligned}$$

Let ξ be an element of this interval. Since s and t are coprime, s is invertible mod t and so there is a unique solution of $As + bt \equiv \xi \pmod{t}$. Therefore the number of solutions of A is at most $\ll \frac{|I|}{t} + 1 \ll \frac{L^{1/2}q^{\epsilon}}{t} + 1$.

We can conclude that the total number of pairs (A, b) satisfying our conditions is at most

$$\sharp\{(A,b)\} \ll q^{2\epsilon}\left(\frac{t}{c}L + \frac{L^{1/2}}{H} + 1\right) q^{\epsilon}\left(\frac{L^{1/2}}{t} + 1\right)$$

Once A and b have been given, we choose $D \in I$ arbitrarily, and thus ℓ is given. I recall that $Q < t < H$.

To conclude the proof consider

$$\begin{aligned}\sum_{1\le \ell\le L} M(z,\ell,q^{\epsilon}\sqrt{L}) &\ll \sum_{c}\sum_{D} q^{2\epsilon}\left(\frac{t}{c}L + \frac{L^{1/2}}{H} + 1\right) q^{\epsilon}\left(\frac{L^{1/2}}{t} + 1\right)\\ &= q^{3\epsilon}\left(\frac{L^{1/2}}{t} + 1\right)\left[\left(\frac{L^{1/2}}{H} + 1\right)\sum_{c,D} 1 + tL\sum_{c}\frac{1}{c}\sum_{D} 1\right]\end{aligned}$$

Since $D \in I$ then $\sharp\{D\} \le L^{1/2}q^{\epsilon}$. Moreover $c \ll \frac{L^{1/2}q^{\epsilon}}{y}$ and it is divisible by q. So writing $c = mq$ and noting that

$$\sum_{m=1}^{L^{1/2}q^{\epsilon}} \frac{1}{m} \ll q^{\epsilon}$$

we obtain

$$\begin{aligned}&\ll q^{3\epsilon}\left(\frac{L^{1/2}}{t} + 1\right)\left[\left(\frac{L^{1/2}}{H} + 1\right)Lq^{2\epsilon} + tL\sum_{m=1}^{L^{1/2}q^{\epsilon}}\frac{1}{mq}L^{1/2}q^{\epsilon}\right]\\ &\ll q^{5\epsilon}\left(\frac{L^{1/2}}{t} + 1\right)\left[\left(\frac{L^{1/2}}{H} + 1\right)L + \frac{tL}{q}L^{1/2}\right]\\ &\ll Lq^{5\epsilon}\left(\frac{L^{1/2}}{t} + 1\right)\left(\frac{L^{1/2}}{H} + 1 + \frac{t}{q}L^{1/2}\right)\\ &\ll Lq^{5\epsilon}\left(\frac{L^{1/2}}{Q} + 1\right)\left(\frac{L^{1/2}}{H} + \frac{H}{q}L^{1/2} + 1\right)\end{aligned}$$

ii) For this inequality the steps until the bound for the number of choices for the pair (A,b) is the same. Then instead of choosing $D \in I$ arbitrarily, we observe that

$$2Au + 4bc + u^2 + A^2 = D^2 - 4\ell = (D - 2\sqrt{\ell})(D + 2\sqrt{\ell})$$

If the left hand-side is $\neq 0$ then the number of possibilities for the pair (D,ℓ) is $\ll q^\epsilon$ because $D^2 - 4\ell \ll q^\epsilon L \ll q$ and the number of possible pairs (D,ℓ) depends linearly from the number of possible divisors of $D^2 - 4\ell$ that is $\ll (D^2 - 4\ell)^\epsilon \ll q^\epsilon$.
If the left hand-side is 0 then $|a+d| = 2\sqrt{\ell}$, then γ is parabolic and from lemma 5.0.8 $\gamma = \begin{pmatrix} \sqrt{\ell} & 0 \\ 0 & \sqrt{\ell} \end{pmatrix}$ so we have that the number of pairs (D,ℓ) is at most $\ll L^{1/2}$. Moreover $A = a - d - u = -u$ so if A is fixed then also c is fixed.
Therefore can conclude that

$$\sum_{\ell=1, \ell \text{ is a square}}^{L} M(z,\ell,q^\epsilon\sqrt{L}) \ll q^{4\epsilon} L^{1/2} \left(\frac{L^{1/2}}{Q} + 1\right)\left(\frac{HL^{1/2}}{q} + \frac{L^{1/2}}{H} + 1\right)$$

□

Proposition 5.0.9.2. *Let $z \in \mathfrak{S}_q - \bigcup_{\delta \in A_0(q)} \delta\mathfrak{S}_{\eta q}$. Assume*

$$2Q^2 \leq \eta^3 q \leq \frac{2q}{\eta} \leq H^2$$

and

$$L^{1/2} q^\epsilon < min\left\{Q, \frac{\eta q}{H}\right\}$$

for some ϵ arbitrary small. Then, uniformly

$$\sum_{1 \leq \ell \leq L} \ell^{\frac{k-1}{2}} \sum_{\rho \in G_\ell(q)} K(|u_\rho(z)|) \ll L^{1/2} q^{5\epsilon} \left(\frac{L^{1/2}}{Q} + 1\right)\left(\frac{HL^{1/2}}{q} + \frac{L^{1/2}}{H} + 1\right)$$

and the same summation with ℓ restricted to be a perfect square

$$\sum_{1 \leq \ell \leq L,\ \ell \text{ square}} \ell^{\frac{k-1}{2}} \sum_{\rho \in G_\ell(q)} K(|u_\rho(z)|) \ll q^{4\epsilon} \left(\frac{L^{1/2}}{Q} + 1\right)\left(\frac{HL^{1/2}}{q} + \frac{L^{1/2}}{H} + 1\right)$$

Proof. This result follows from the previous proposition, applying Abel summation formula: at the bigginig of this chapter I have recall that

$$\sum_{\rho \in G_\ell(q)} K(|u_\rho(z)|) \ll \sum_{\rho \in G_\ell(q), |u_\rho(z)| \leq q^\epsilon\sqrt{\ell}} K(|u_\rho(z)|) \ll \frac{1}{\ell^{k/2}} M(z,\ell,q^\epsilon\sqrt{\ell})$$

hence

$$\sum_{1 \leq \ell \leq L} \ell^{\frac{k-1}{2}} \sum_{\rho \in G_\ell(q)} K(|u_\rho(z)|) \ll \sum_{1 \leq \ell \leq L} \frac{1}{\ell^{1/2}} M(z,\ell,q^\epsilon\sqrt{\ell})$$

Applying Abel summation formula we obtain

$$\sum_{1\leq\ell\leq L}\frac{1}{\ell^{1/2}}M(z,\ell,q^{\epsilon}\sqrt{\ell})\ll\left(\sum_{1\leq\ell\leq L}M(z,\ell,q^{\epsilon}\sqrt{\ell})\right)\frac{1}{L^{1/2}}+\frac{1}{2}\int_1^L\left(\sum_{1\leq\ell\leq s}M(z,\ell,q^{\epsilon}\sqrt{\ell})\right)\frac{1}{s^{3/2}}ds$$
$$\ll L^{1/2}q^{5\epsilon}\left(\frac{L^{1/2}}{Q}+1\right)\left(\frac{L^{1/2}}{H}+\frac{H}{q}L^{1/2}+1\right)+\frac{1}{2}q^{5\epsilon}\int_1^L\frac{1}{s^{1/2}}\left(\frac{s^{1/2}}{Q}+1\right)\left(\frac{s^{1/2}}{H}+\frac{H}{q}s^{1/2}+1\right)ds$$

To conclude the proof it is enough to show that the second term is bounded by the first one, indeed:

$$q^{5\epsilon}\int_1^L\frac{1}{s^{1/2}}\left(\frac{s^{1/2}}{Q}+1\right)\left(\frac{s^{1/2}}{H}+\frac{H}{q}s^{1/2}+1\right)ds=$$
$$q^{5\epsilon}\frac{1}{Q}\int_1^L s^{1/2}\left(\frac{1}{H}+\frac{H}{q}\right)+1ds+q^{5\epsilon}\int_1^L\left(\frac{1}{H}+\frac{H}{q}\right)+\frac{1}{s^{1/2}}ds$$
$$\ll q^{5\epsilon}\left[\frac{1}{Q}L^{3/2}\left(\frac{1}{H}+\frac{H}{q}\right)+\frac{L}{Q}+\left(\frac{1}{H}+\frac{H}{q}\right)L+L^{1/2}\right]$$
$$=L^{1/2}q^{5\epsilon}\left(\frac{L^{1/2}}{Q}+1\right)\left(\frac{L^{1/2}}{H}+\frac{H}{q}L^{1/2}+1\right)$$

So the first part of the theorem is proved.

For the second part the coputation is exactly the same, so the theorem is proved. □

To conclude this chapter finding a bound for $g(z)$ we need other two steps: the first one consists on finding a formula that describe an upper bound for $g(z)$ depending on the variables Q, H and L for all $z\in\mathfrak{S}_q-\bigcup_{\delta\in A_0(q)}\delta\mathfrak{S}_{\eta q}$. Note that this variables depend all from η. Then, considering the bound via Fourier coefficients studied in the second chapter, for $z\in\bigcup_{\delta\in A_0(q)}\delta\mathfrak{S}_{\eta q}$ we shall give to each parameter a value that allows us to improve the trivial bound for the sup-norm of $g(z)$.

For the first goal we shall use the amplification method, with the same amplifier used in the previous chapter. Let's recall the definition of the amplifier: Let f be a cuspform of level q, and L a positive real number. Let

$$\Lambda:=\{p\text{ prime}:\ (p,q)=1,p\in[L,2L]\}$$

be a large set of primes. Define

$$x_\ell=\begin{cases}sgn(\lambda_f(\ell)), & \text{if }\ell\in\Lambda\cup\Lambda^2\\ 0, & \text{otherwise}\end{cases}$$

where $\lambda_f(\ell)$ is the eigenvalue of f with respect to $T(\ell)$.

Proposition 5.0.9.3. *Let $z\in\mathfrak{S}_q-\bigcup_{\delta\in A_0(q)}\delta\mathfrak{S}_{\eta q}$. Assume*

$$2Q^2\leq\eta^3q\leq\frac{2q}{\eta}\leq H^2$$

and

$$L^2 q^\epsilon < min\left\{Q, \frac{\eta q}{H}\right\}$$

for some ϵ arbitrary small. Then

$$g(z)^2 \ll L^{-1/2} q^{1+\epsilon} \left(\frac{L^{3/2}}{Q} + 1\right)\left(\frac{HL^{3/2}}{q} + \frac{L^{3/2}}{H} + 1\right)$$

Proof. As we have seen in the proof of the proposition 4.3.3.1 we have

$$|\sum_{1\le\ell\le L^2} x_\ell \lambda_{f_0}(\ell)|^2 \left|y^{k/2} f_0(z)\right|^2 \ll q^{1+\epsilon} \sum_{1\le\ell\ll L^4} y_\ell \ell^{\frac{k-1}{2}} \sum_{\rho\in G_\ell(q)} K(|u_\rho(z)|)$$

Since $|y_\ell|$ is bounded and it vanishes for $\ell \ge L^3$ if ℓ is not a square, it becomes

$$\ll q^{1+\epsilon} \sum_{1\le\ell\ll L^3} \ell^{\frac{k-1}{2}} \sum_{\rho\in G_\ell(q)} K(|u_\rho(z)|) + q^{1+\epsilon} \sum_{1\le\ell\ll L^4,\ \ell \text{ square}} \ell^{\frac{k-1}{2}} \sum_{\rho\in G_\ell(q)} K(|u_\rho(z)|)$$

Note that now the sum is over all $\ell \ll L^3$, while in the previous proposition it is over $\ell \ll L$. So both in the hypotesis and the formulas we have to substitute L with L^4 in the case of ℓ square, and L^3 in the case of ℓ not square. Using the previous proposition we obtain

$$\begin{aligned}
&\ll q^{1+\epsilon}\left(L^{3/2}\left(\frac{L^{3/2}}{Q}+1\right)\left(\frac{HL^{1/2}}{q}+\frac{L^{3/2}}{H}+1\right)+\left(\frac{L^2}{Q}+1\right)\left(\frac{HL^2}{q}+\frac{L^2}{H}+1\right)\right)\\
&\ll q^{1+\epsilon}\left(L^{3/2}\left(\frac{L^{3/2}}{Q}+1\right)\left(\frac{HL^{1/2}}{q}+\frac{L^{3/2}}{H}+1\right)+L\left(\frac{L^{3/2}}{Q}+1\right)\left(\frac{HL^2}{q}+\frac{L^{3/2}}{H}+1\right)\right)\\
&\ll q^{1+\epsilon}L^{3/2}\left(\frac{L^{3/2}}{Q}+1\right)\left(\frac{HL^{1/2}}{q}+\frac{L^{3/2}}{H}+1\right)
\end{aligned}$$

We have also seen that

$$|\sum_{1\le\ell\le L^2} x_\ell \lambda_{f_0}(\ell)|^2 \gg L^{2-\epsilon}$$

Therefore

$$g(z)^2 \ll q^{1+\epsilon} L^{-1/2} \left(\frac{L^{3/2}}{Q} + 1\right)\left(\frac{HL^{1/2}}{q} + \frac{L^{3/2}}{H} + 1\right)$$

□

At this point we are ready to prove the main theorem: I recall that from proposition 3.1.1.1 we have

$$g(z)^2 \ll q^{1+\epsilon}\eta$$

for $z \in \bigcup_{\delta\in A_0(q)} \delta\mathfrak{S}_{\eta q}$, $0 < \eta < 1$ real number. So our aim is to find η, Q, H, L satisfying all the conditions used in our computation, and that make smaller as possible

$$max\{\eta, L^{-1/2}\left(\frac{L^{3/2}}{Q} + 1\right)\left(\frac{HL^{1/2}}{q} + \frac{L^{3/2}}{H} + 1\right)\}$$

Theorem 5.0.10 (The Main Theorem)**.** *Let q be a prime number, $k \geq 4$ an integer, both fixed. Let $f_0 \in \mathcal{S}_k^P(\Gamma_0(q))$ be an arithmetically normalized newform of weight k. Denote $g(z) = y^{k/2} f_0(z)$, where $z = x + iy \in \mathbb{H}$, then*

$$\|g\|_\infty \ll q^{\frac{1}{2}} q^{-\frac{1}{22}+\epsilon}$$

Proof. As I said above, our aim is to find η, Q, H, L satisfying all the conditions used in our computation, and that make smaller as possible

$$max\{\eta, L^{-1/2}\left(\frac{L^{3/2}}{Q}+1\right)\left(\frac{HL^{1/2}}{q}+\frac{L^{3/2}}{H}+1\right)\}$$

First of all I want to study the situation for $z \in \mathfrak{S}_q - \bigcup_{\delta \in A_0(q)} \delta \mathfrak{S}_{\eta q}$: the bound for $g(z)^2$ in this reagion is

$$L^{-1/2}\left(\frac{L^{3/2}}{Q}+1\right)\left(\frac{HL^{1/2}}{q}+\frac{L^{3/2}}{H}+1\right)$$

where the parameters have to satisfy:

$$\begin{cases} 2Q^2 \leq \eta^3 q \\ \frac{2q}{\eta} \leq H^2 \end{cases}$$

and

$$L^2 q^\epsilon < min\left\{Q, \frac{\eta q}{H}\right\}$$

Moreover $0 < \eta < 1$.

To simplify our computation I write all the parameters as a rational power of q. It is always possible since they are all bigger then 0. Moreover to choose a rational exponent is not reductive since $\mathbb{Q}$ is dense in $\mathbb{R}$. Moreover I define a common denominator for the rational exponent, say α, to make our new system of conditions just a system of inequalities in $\mathbb{Z}$. So we put
$H = q^{a/\alpha}$, $Q = q^{b/\alpha}$, $L = q^{c/\alpha}$, $\eta = q^{d/\alpha}$.
With this choice the conditions becomes

$$\begin{cases} 2b < 3d + \alpha \\ \alpha - d < 2a \\ 2c < b \\ 2c < d + \alpha - a \\ d < 0 \end{cases}$$

that is equivalent to

$$\begin{cases} 2b \leq 3d + \alpha - 1 \\ \alpha - d + 1 \leq 2a \\ 2c + 1 \leq b \\ a \leq d + \alpha - 1 - 2c \\ d < 0 \end{cases}$$

My choice is to make L as big as possible, so that $L^{-1/2}$ becomes small. For this reason from the above conditions (the first and the third) I choose the limit case

$$c = \frac{\alpha + 3d - 3}{4}$$

This implies

$$b = \frac{\alpha + 3d - 1}{2}$$

The second and the fourth conditions gives possible the choice

$$a = \frac{\alpha - d + 1}{2}$$

At this point i want to study

$$L^{-1/2}\left(\frac{L^{3/2}}{Q} + 1\right)\left(\frac{HL^{1/2}}{q} + \frac{L^{3/2}}{H} + 1\right)$$

More precisely I want to find the term with higher weigth between

$$\{L^{-1/2}, \frac{L}{H}, \frac{L}{Q}, \frac{LH}{q}, \frac{L^{5/2}}{HQ}, \frac{L^{5/2}H}{Qq}\}$$

Note that the last term is negligible because it is bounded by the third one. So writing this term in terms of exponentials we obtain

$$\{\frac{-c}{2\alpha}, \frac{c-b}{\alpha}, \frac{c+a-\alpha}{\alpha}, \frac{c-a}{\alpha}, \frac{5c}{2\alpha} - \frac{a+b}{\alpha}\}$$

and between these terms I want to find the bigger; it is clearly equivalent of finding the smaller between

$$\{\frac{c}{2}, b-c, \alpha-c-a, a-c, a+b-\frac{5}{2}c\}$$

Substituting a, b, c with the values above depending only on d and α it turns out that the smaller term is

$$\frac{c}{2\alpha} = \frac{\alpha + 3d - 3}{8\alpha}$$

and so

$$g(z)^2 \ll q^{1+\epsilon} q^{\frac{-\alpha-3d+3}{8\alpha}}$$

for $z \in \mathfrak{S}_q - \bigcup_{\delta \in A_0(q)} \delta\mathfrak{S}_{\eta q}$.

For $z \in \bigcup_{\delta \in A_0(q)} \delta\mathfrak{S}_{\eta q}$ we have

$$g(z)^2 \ll q^{1+\epsilon}\eta = q^{1+\epsilon} q^{\frac{d}{\alpha}}$$

So we want to find d and α so that

$$max\{\frac{d}{\alpha}, \frac{-\alpha-3d+3}{8\alpha}\}$$

is smallest as possible.

$$\frac{d}{\alpha} \geq \frac{-\alpha - 3d + 3}{8\alpha} \Leftrightarrow \alpha \geq -11d + 3$$

So once d is fixed, if we choose $\alpha \geq -11d + 3$ then we have to make smaller as possible

$$\frac{d}{\alpha} = \frac{d}{-11d + 3}$$

This is an increasing function of $d < 0$, so as d is smaller the functions is smaller. For this reason we can say that

$$g(z)^2 \ll q^{1+\epsilon} q^{d/\alpha} \ll q^{1+\epsilon} \lim_{d \to -\infty} q^{\frac{d}{-11d+3}} = q^{1+\epsilon} q^{-\frac{1}{11}}$$

If we choose $\alpha \leq -11d + 3$ then we have to make smaller as possible

$$\frac{-\alpha - 3d + 3}{8\alpha}$$

Once d is fixed this happens for $\alpha = -11d + 3$ and so the result becomes the same. Therefore we can conclude that

$$g(z) \ll q^{1/2+\epsilon} q^{-\frac{1}{22}}$$

$\square$

Appendix A

Pre-Trace Formula on $\Gamma_0(q)$

The relation between the basis of eigenform of $\mathcal{S}_k(\Gamma_0(q))$ and an automorphic kernel plays a central role in this thesis. In the following appendix we shall describe it, inspired by a similar relation that one can find in [Lan95, Appendix Zagier].

Let $F_0(q)$ be a fundamental domain for $\Gamma_0(q)$ in $\mathbb{H}$. We fix an even weight $k \geq 4$, and let $T_k(m)$ be the Hecke operator on $\mathcal{S}_k(\Gamma_0(q))$. Let $h(z,w)$ be a function of two variables z and w in $\mathbb{H}$,and assume that h is a cusp form of weight k as a function of each variable.If $f \in \mathcal{S}_k(\Gamma_0(q))$ we define $f * h$ as a function of w by

$$(f * h)(w) = \int_{F_0(q)} f(z)\overline{h(z, \overline{-w})}(\Im z)^k d\mu(z) \tag{A.0.1}$$

where $d\mu(z) = \frac{dx\,dy}{y^2}$ denote the hyperbolic measure. This operation is merely the Petersson inner product of f and h, viewed as function of the first variable z.

We know from the first chapter that there exists an orthogonal basis $\mathcal{B} = \{f_1, \cdots, f_J\}$ of $\mathcal{S}_k(\Gamma_0(q))$ made of eigenforms for the Hecke operators $\{T(m) : (q,m) = 1\}$, i.e

$$f_i(z) = \sum_{n=1}^{+\infty} a_n(f_i)e(nz) \Longrightarrow T(m)f_i = \lambda_i(m)f_i$$

where $\lambda_i(m)$ are the eigenvalues of f_i with respect to $T_k(m)$. Define

$$h(z,w) = \sum_{\gamma \in \Gamma_0(q)} \frac{1}{(j(\gamma,z))^k} \frac{1}{(w + \gamma z)^k} \tag{A.0.2}$$

Proposition A.0.10.1. *$h(z,w)$ is a holomorphic cusp form in each variable separately.*

Proof. I divide the proof in four steps:

- h is holomorphic;
- $h[\rho]_k = h$ for all $\rho \in \Gamma_0(q)$;
- $h[\alpha]$ is holomorphic at infinity for all $\alpha \in SL_2(\mathbb{Z})$;

- h vanishes at each cusp, i.e. the Fourier expansion of $h[\alpha]_k$ has the first coefficient equal 0 for all $\alpha \in SL_2(\mathbb{Z})$.

I shall prove this for z and w separately.

$$h(z,w) = \sum_{\gamma\in\Gamma_0(q)} \frac{1}{j(\gamma,z)^k}\frac{1}{(w+\gamma.z)^k}$$

i) As a function of z: both $\Im(j(\gamma,z))$ and $\Im((w+\gamma.z))$ are strictly bigger then 0. So $\frac{1}{j(\gamma,z)^k}\frac{1}{(w+\gamma.z)^k}$ which is a rational polynomial is holomorphic on $\mathbb{H}$. For $k \geq 4$ $\sum_{\gamma\in\Gamma_0(q)} \frac{1}{j(\gamma,z)^k}\frac{1}{(w+\gamma.z)^k}$ converges, and so h is holomorphic.

As a function of w the argument is exactly the same.

ii)As a function of z: let $\rho \in \Gamma_0(q)$, then for all $\gamma \in \Gamma_0(q)$

$$\frac{1}{j(\gamma,\rho.z)j(\rho,z)} = \frac{}{j(\gamma\rho,z)}$$

Hence

$$\begin{aligned} h[\rho](z,w) &= \sum_{\gamma\in\Gamma_0(q)} \frac{1}{j(\rho,z)^k}\frac{1}{j(\gamma,\rho.z)^k}\frac{1}{(w+\gamma\rho.z)^k} \\ &= \sum_{\gamma\in\Gamma_0(q)} \frac{1}{j(\gamma\rho,z)^k}\frac{1}{(w+\gamma\rho.z)^k} \end{aligned}$$

As a function of w: Let $\rho = \begin{pmatrix} u & v \\ r & s \end{pmatrix} \in \Gamma_0(q)$, and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ running through $\Gamma_0(q)$, then we have

$$\begin{aligned} h[\rho](z,w) &= \sum_{\gamma\in\Gamma_0(q)} \frac{1}{j(\gamma,z)^k}\frac{1}{j(\rho,w)^k}\frac{1}{(\rho.w+\gamma.z)^k} \\ &= \sum_{\gamma\in\Gamma_0(q)} \frac{1}{(cz+d)^k}\frac{1}{(rz+s)^k}\frac{1}{\left(\frac{uw+v}{rw+s}+\frac{az+b}{cz+d}\right)^k} \\ &= \sum_{\gamma\in\Gamma_0(q)} \frac{1}{[(cz+d)(uw+v)+(az+b)(rw+s)]^k} \\ &= \sum_{\gamma\in\Gamma_0(q)} \frac{1}{[j(\rho'\gamma,z)(w+\rho'\gamma.z)]^k} \\ &= h(z,w) \end{aligned}$$

where $\rho' = \begin{pmatrix} s & v \\ r & u \end{pmatrix}$.

iii) As function of z: for all $\alpha \in SL_2(\mathbb{Z})$ the trasformed function $h[m]_k(z,w)$ is holomorphic and weight-k invariant under $\alpha^{-1}\Gamma_0(q)\alpha$, and therefore it has a Laurent expansion

$$h[\alpha]_k(z,w) = \sum_{n\in\mathbb{Z}} a_n e(\frac{nz}{q})$$

So it is enough to show that
$$\lim_{\left|e(\frac{z}{q})\right|\to 0} h[\alpha]_k(z,w)=0$$
for all $\alpha\in SL_2(\mathbb{Z})$.

This proves also (iv): $\left|e(\frac{z}{q})\right|\to 0$ is equivalent of $y\to+\infty$, where $z=x+iy$ as usual. Now,
$$h[\alpha]_k(z,w)=\sum_{\gamma\in\Gamma_0(q)}\frac{1}{j(\alpha,z)^k}\frac{1}{j(\gamma,\alpha.z)^k}\frac{1}{(w+\gamma\alpha.z)^k}=\sum_{\gamma\in\Gamma_0(q)}\frac{1}{j(\gamma\alpha,z)^k}\frac{1}{(w+\gamma\alpha.z)^k}$$
Concentrate for a moment on $j(\gamma\alpha,z)$: it tends to ∞ as y tends to $+\infty$, so if the third entry of $\gamma\alpha$ is not 0, then
$$\lim_{\left|e(\frac{z}{q})\right|\to 0}\sum_{\gamma\in\Gamma_0(q)}\frac{1}{j(\gamma\alpha,z)^k}\frac{1}{(w+\gamma\alpha.z)^k}=0$$
since $w+\gamma\alpha.z$ does not tend to 0, because $\gamma\alpha.z\in\mathbb{H}$. If the third entry of $\gamma\alpha$ is 0 then $\gamma\alpha\in\Gamma_0(q)$ and so $\alpha\in\Gamma_0(q)$. So $h[\alpha]_k=h$ which tends to 0 as y tends to $+\infty$.

As a function of w: as in the case of z it is enough to prove that
$$\lim_{\eta\to+\infty} h[\alpha]_k(z,w)=0$$
for all $\alpha\in SL_2(\mathbb{Z})$, where $w=\xi+i\eta$:
if $\alpha=\begin{pmatrix} s & v\\ r & u\end{pmatrix}$, consider $\alpha'=\begin{pmatrix} u & v\\ r & s\end{pmatrix}\in SL_2(\mathbb{Z})$.
As in (ii) one has
$$h[\alpha]_k(z,w)=\sum_{\gamma\in\Gamma_0(q)}\frac{1}{j(\alpha'\gamma,z)^k}\frac{1}{(w+\alpha'\gamma.z)^k}$$
and it tends to 0 as η tends to $+\infty$. This proves also (iv). $\square$

Theorem A.0.11. *Let* $C_k=\frac{(-1)^{k/2}\pi}{2^{(k-3)}(k-1)}$, *then*

(i) $\forall f\in\mathcal{S}_k(\Gamma_0(q))$ *we have*
$$(f*h)(w)=C_kf(w)$$

(ii) *We have the identity*
$$C_k^{-1}h(z,w)=\sum_{i=1}^{J}\frac{f_i(z)\overline{f_i(-\bar{w})}}{\langle f_i,f_i\rangle}$$

Proof. Note first that if $\gamma=\begin{pmatrix} a & b\\ 0 & d\end{pmatrix}\in\Gamma_0(q)$ then
$$(c\bar{z}+d)^{-k}f(z)y^k=f(\gamma z)\Im(\gamma z)^k$$

where $z = x + iy$: $\Im(\gamma z) = \frac{\Im(z)}{|cz+d|^2}$ and since $f \in S_k(\Gamma_0(q)$ then

$$f(z) = f\,[\gamma]_k\,(z) = \frac{1}{(cz+d)^k} f(\gamma z)$$

so

$$\begin{aligned} \frac{1}{(c\bar{z}+d)^k} f(z) y^k &= \frac{1}{|cz+d|^k} f(\gamma z) \Im(z)^k \\ &= f(\gamma z) \Im(\gamma z)^k \end{aligned}$$

Therefore from (A.0.2) we have

$$\begin{aligned} f(z)\overline{h(z,w)}y^k &= \sum_{\gamma \in \Gamma_o(q)} \frac{1}{j(\gamma,\bar{z})^k} \frac{1}{(\bar{w}+\gamma\bar{z})^k} f(z) y^k \\ &= \sum_{\gamma \in \Gamma_o(q)} f(\gamma z) \Im(\gamma z)^k \frac{1}{(\bar{w}+\gamma\bar{z})^k} \end{aligned}$$

Hence

$$\begin{aligned} (f * h)(w) &= \int_{F_0(q)} \sum_{\gamma \in \Gamma_o(q)} f(\gamma z) \Im(\gamma z)^k \frac{1}{(-w+\gamma\bar{z})^k} \frac{dxdy}{y^2} \\ &= \sum_{\gamma \in \Gamma_o(q)} \int_{\gamma F_0(q)} f(z) \Im(z)^k \frac{1}{(-w+\bar{z})^k} \frac{dxdy}{y^2} \\ &= 2 \int_{y=0}^{\infty} \int_{x=-\infty}^{\infty} f(x+iy) y^{k-2} \frac{1}{(-w+x-iy)^k} dxdy \end{aligned}$$

The first equality comes from the $\Gamma_0(q)$-invariance of $dxdy/y^2$, and the second one comes from the fact that the upper half plane $\mathbb{H}$ is equal to the union of transforms of the fundamental domain under $\Gamma_0(q)$, disjoint exept for boundary points of measure zero, and exept for the fact that $\pm\gamma \in \Gamma_0(q)$ give the same transform, whence the factor of 2. Cauchy formula and the fact that f is a cusp form, hence holomorphic and sufficiently small at infinity, imply that

$$\int_{-\infty}^{\infty} f(x+iy) \frac{1}{(-w+x-iy)^k} dx = \frac{2\pi i}{(k-1)!} f^{(k-1)}(2iy+w)$$

where $f^{(k-1)}$ denotes the $(k-1)$-derivative of f. Therefore

$$\begin{aligned} (f * h)(w) &= \frac{4\pi i}{(k-1)!} \int_0^{\infty} y^{k-2} f^{(k-1)}(2iy+w) dy \\ &= \frac{4\pi i}{(k-1)!} \int_0^{\infty} \frac{1}{(2i)^{k-2}} \left(\frac{d}{dt}\right)^{k-2} f'(2ity+w)\,|_{t=1}\, dy \\ &= \frac{4\pi i}{(k-1)!} \frac{1}{(2i)^{k-2}} \left(\frac{d}{dt}\right)^{k-2} \int_0^{\infty} f'(2ity+w) dy\,|_{t=1} \\ &= \frac{4\pi i}{(k-1)!} \frac{1}{(2i)^{k-2}} \left(\frac{d}{dt}\right)^{k-2} \left(\frac{-f(w)}{2it}\right) |_{t=1} \\ &= C_k f(w) \end{aligned}$$

This proves the first part. Part *(ii)* follows essentially from linear algebra. The function h, being a cusp form with respect to z and w, can be written as

$$h_m(z,w) = \sum_{j=1}^{J} c_j f_j(z)$$

where c_j depends on w. So, applying the Petersson scalar product one obtains

$$\langle h(\_,w), f_j\rangle = c_j \langle f_j, f_j\rangle$$

On the other hand

$$\begin{aligned}
\langle h(_,w), f_j\rangle &= \int_{z\in F_0(q)} h(z,w)\overline{f_j(z)}y^k d\mu(z) \\
&= \overline{\int_{z\in F_0(q)} \overline{h(z,w)} f_j(z) y^k d\mu(z)} \\
&= \overline{(f_j * h)(-\bar{w})} \\
&= C_k \overline{f_j(-\bar{w})}
\end{aligned}$$

So we obtain

$$c_j = \frac{\langle h(\_,w), f_j\rangle}{\langle f_j, f_j\rangle} = \frac{C_k \overline{f_j(-\bar{w})}}{\langle f_j, f_j\rangle}$$

and we can conclude that

$$h(z,w) = C_k \sum_{j=1}^{J} \frac{f_j(z)\overline{f_j(-\bar{w})}}{\langle f_j, f_j\rangle}$$

$\square$

Appendix B

Classification of motions in $M_2(\mathbb{R}, \ell)$ and a Lower Bound for $|u_\rho(z)|$

In this appendix we want to classify the elements of $M_2(\mathbb{R}, \ell)$, the 2x2 matrices with entries in $\mathbb{R}$ and determinant ℓ, and then to deduce from that a lower bound for $|u_\rho(z)|$, where $\rho \in G_\ell(q)$. This is a generalization of the classification of motion for $SL_2(\mathbb{R})$ that one can find in [Iwa97].

B.1 Classification of motions

Let $\rho = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R}, \ell)$, and consider the conjugacy class

$$\bar{\rho} := \{g^{-1}\rho g : g \in SL_2(\mathbb{R})\}$$

Remark B.1.1. *Two elements of the same conjugacy class have the same trace*

Remark B.1.2. *The number of fixed points in $\mathbb{H}$ under the action of two elements of the same conjugacy class is invariant, indeed: suppose that $\rho.z = z$, then $g^{-1}\rho g.(g^{-1}.z) = g^{-1}.z$.*

So this are two equivalent criterium for our classification.
Let $\rho = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and let z be a fixed point for ρ. If $c = 0$ then $\frac{az+b}{d} = z$. Therefore ρ fixes just one point in $\bar{\mathbb{R}}$.
If $c \neq 0$ then $\frac{az+b}{cz+d} = z$, which gives the equation

$$cz^2 + z(d - a) - b = 0$$

Looking at the discriminant $(a + d)^2 - 4\ell$ we can classify:

- $|a + d| = 2\sqrt{\ell}$ then ρ fixes one point in $\bar{\mathbb{R}}$, and we say that ρ is *parabolic*;
- $|a + d| > 2\sqrt{\ell}$ then ρ fixes two distinct points in $\bar{\mathbb{R}}$, and we say that ρ is *hyperbolic*;
- $|a + d| < 2\sqrt{\ell}$ then ρ fixes one point in $\bar{\mathbb{H}}$, and we say that ρ is *elliptic*;

B.2 A Lower Bound

In this section I want to prove the following

Proposition B.2.0.1. *Let $\rho \in G_\ell(q)$, $z \in \bar{\mathbb{H}}$, then $|u_\rho(z)| \gg \sqrt{\ell}$.*

Proof. Let $\rho = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_\ell(q)$ and consider

$$|u_\rho(z)| = \frac{|b - c|z|^2 + az - d\bar{z}|}{y}$$

Considering the imaginary part we have

$$|\Im u_\rho(z)| = |a + d|$$

If rho is parabolic or hyperbolic, then

$$|u_\rho(z)| \geq |\Im u_\rho(z)| = |a + d| > \sqrt{\ell}$$

It remains to study the case in which ρ is elliptic: first of all note that if ρ is elliptic, then $c \neq 0$, otherwise it would be parabolic. In particular $c \geq q$, since $\rho \in G_0(q)$.

Remark B.2.1. *$\rho \in M_2(\mathbb{R},\ell)$, and $g \in SL_2(\mathbb{Z})$, then $|u_{g^{-1}\rho g}(z)| = |u_\rho(g.z)|$*

Proof. Set $w := g.z$;

$$\begin{aligned}
|u_{g^{-1}\rho g}(z)| &= \frac{|g^{-1}\rho g.z - \bar{z}||j(g^{-1}\rho g, z)|}{\Im z} \\
&= \frac{|g^{-1}\rho.w - g^{-1}.\bar{w}||j(g^{-1}\rho, g.z)j(g,z)|}{\Im z} \\
&= \frac{|\rho.w - \bar{w}||j(\rho,w)j(g,z)|}{\Im z|j(g^{-1},\bar{w})|} \\
&= \frac{|\rho.w - \bar{w}||j(\rho,w)|}{\Im w} = |u_\rho(w)| = |u_\rho(g(z))|
\end{aligned}$$

□

This allows us to study $|u_\rho(z)|$ taking for ρ any elliptic motion. So take

$$\rho = \begin{pmatrix} \sqrt{\ell}\cos\vartheta & \sqrt{\ell}\sin\vartheta \\ -\sqrt{\ell}\sin\vartheta & \sqrt{\ell}\cos\vartheta \end{pmatrix}$$

for some $\vartheta \neq 0$ Hence we find

$$\begin{aligned}
|u_\rho(z)| &= \frac{|\sqrt{\ell}(\cos\vartheta.z + \sin\vartheta) - \bar{z}\sqrt{\ell}(-\sin\vartheta.z + \cos\vartheta)|}{y} \\
&= \frac{\sqrt{\ell}|\sin\vartheta(1 + |z|^2) + \cos\vartheta(z - \bar{z})|}{y}
\end{aligned}$$

So considering the imaginary part

$$|\Im u_\rho(z)| = \frac{\sqrt{\ell}|2y\cos\vartheta|}{y} = 2|\cos\vartheta|\sqrt{\ell} \gg \sqrt{\ell}$$

So we conclude

$$|u_\rho(z)| \geq |\Im u_\rho(z)| \gg \sqrt{\ell}$$

$\square$

Bibliography

[AL70] A. O. L. Atkin and J. Lehner, *Hecke operators on* $\Gamma_0(m)$, Math. Ann. **185** (1970), 134–160. MR 0268123 (42 #3022)

[BH10] Valentin Blomer and Roman Holowinsky, *Bounding sup-norms of cusp forms of large level*, Invent. Math. **179** (2010), no. 3, 645–681. MR 2587342 (2011d:11095)

[DS05] Fred Diamond and Jerry Shurman, *A first course in modular forms*, Graduate Texts in Mathematics, vol. 228, Springer-Verlag, New York, 2005. MR 2112196 (2006f:11045)

[HM06] Gergely Harcos and Philippe Michel, *The subconvexity problem for Rankin-Selberg L-functions and equidistribution of Heegner points. II*, Invent. Math. **163** (2006), no. 3, 581–655. MR 2207235 (2007j:11063)

[Iwa90] Henryk Iwaniec, *Small eigenvalues of Laplacian for* $\Gamma_0(N)$, Acta Arith. **56** (1990), no. 1, 65–82. MR 1067982 (92h:11045)

[Iwa97] ———, *Topics in classical automorphic forms*, Graduate Studies in Mathematics, vol. 17, American Mathematical Society, Providence, RI, 1997. MR 1474964 (98e:11051)

[JK04] J. Jorgenson and J. Kramer, *Bounding the sup-norm of automorphic forms*, Geom. Funct. Anal. **14** (2004), no. 6, 1267–1277. MR 2135167 (2005m:11071)

[Kob93] Neal Koblitz, *Introduction to elliptic curves and modular forms*, second ed., Graduate Texts in Mathematics, vol. 97, Springer-Verlag, New York, 1993. MR 1216136 (94a:11078)

[Lan95] Serge Lang, *Introduction to modular forms*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 222, Springer-Verlag, Berlin, 1995, With appendixes by D. Zagier and Walter Feit, Corrected reprint of the 1976 original. MR 1363488 (96g:11037)

[LW08] Yuk-Kam Lau and Jie Wu, *On the least quadratic non-residue*, Int. J. Number Theory **4** (2008), no. 3, 423–435. MR 2424331 (2009e:11191)

[MV06] Philippe Michel and Akshay Venkatesh, *Equidistribution, L-functions and ergodic theory: on some problems of Yu. Linnik*, International Congress of Mathematicians. Vol. II, Eur. Math. Soc., Zürich, 2006, pp. 421–457. MR 2275604 (2008g:11085)

[Rud05] Zeév Rudnick, *On the asymptotic distribution of zeros of modular forms*, Int. Math. Res. Not. (2005), no. 34, 2059–2074. MR 2181743 (2006k:11099)

[Tem10] Nicolas Templier, *On the sup-norm of Maass cusp forms of large level*, Selecta Math. (N.S.) **16** (2010), no. 3, 501–531. MR 2734340

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