Erasmus Mundus ALGANT
Master thesis

Zero-cycles on surfaces

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Contents

1 Notation and conventions

$X^{(d)}$	d-th symmetric power of a variety X
$Z_k(X)$	The group of k-dimensional algebraic cycles of variety X
$Z^k(X)$	The group of algebraic cycles of codimension k of variety X
$\square_{\mathrm{rat}}$	The subgroup of cycles which are rationally equivalent to 0
$\square_{\mathrm{hom}}$	The subgroup of cycles which are homologically equivalent to 0 (with respect to singular cohomology)
$\square_{\mathrm{alb}}$	The subgroup of cycles which are killed by Albanese map
$CH_k(X)$	$Z_k(X)/Z_k(X)_{\mathrm{rat}}$
$CH^k(X)$	$Z^k(X)/Z^k(X)_{\mathrm{rat}}$
$\square^\vee$	Dual vector space
$k(x)$	Residue field of a point x

All schemes are locally noetherian. We work exclusively with separated schemes over the field of complex numbers $\mathbf{C}$, except section 3.2.1 where arbitrary schemes are considered. Everywhere except sections 3.2.1 and 3.2.2 schemes are either of finite type over $\mathbf{C}$ or are specra of localizations of rings of finite type over $\mathbf{C}$.

A quasi-projective scheme is a scheme which admits an open embedding into a projective scheme over $\mathbf{C}$. A variety is a reduced scheme of finite type over $\mathbf{C}$. Varieties are not assumed to be irreducible or smooth.

By default $\times$ means cartesian product over $\mathbf{C}$. Similarly, $\mathbf{P}^n$ and $\mathbf{A}^n$ denote projective (respectively affine) n-dimensional space *over* $\mathbf{C}$. If $f_1 \colon X \to Y_1$ and $f_2 \colon X \to Y_2$ are morphisms then (f_1, f_2) denotes the induced morphism $X \to Y_1 \times Y_2$. If $f_1 \colon X_1 \to Y_1$ and $f_2 \colon X_2 \to Y_2$ are morphisms then $f_1 \times f_2$ denotes the induced morphism $X_1 \times X_2 \to Y_1 \times Y_2$.

If $X \to Z$ is a morphism and $f \colon Y \to Z$ is another morphism then $f^{-1}X$ denotes $X \times_Z Y$. If $f \colon X \to Y$ is a morphism and $y \in Y$ a point, then X_y is the fiber of f over y.

All other conventions and notation are standard, and are kept consistent with books [5], [6], [7], [9] and [15].

2 The conjecture of Bloch

Let X be a smooth irreducible projective surface. The group $CH_1(X)$ of divisors on X up to rational equivalence fits into an exact sequence

$$0 \to CH_1(X)_{\text{hom}} \to CH_1(X) \xrightarrow{\text{cl}} H^2(X, \mathbf{Z}) \cap H^{1,1}(X) \to 0,$$

where the map cl associates to a divisor its cohomology class. The subgroup $CH_1(X)_{\text{hom}}$ can be described using the Abel-Jacobi isomorphism

$$CH_1(X)_{\text{hom}} \longrightarrow J^1(X),$$

where $J^1(X) = H^{0,1}(X)/H^1(X, \mathbf{Z})$ is the Jacobian of X. So one can view $CH_1(X)$ as an extension of a discrete group by an abelian variety. The behaviour of $CH_1(X)$ is completely determined by Hodge theory of X.

Let us move to $CH_0(X)$. Similarly to the case of $CH_1(X)$ one has an exact sequence

$$0 \to CH_0(X)_{\text{hom}} \to CH_0(X) \xrightarrow{\text{cl}} H^4(X, \mathbf{Z}) \to 0$$

and an Albanese morphism

$$CH_0(X)_{\text{hom}} \xrightarrow{\text{alb}} \operatorname{Alb} X,$$

where $\operatorname{Alb} X = J^3(X) = H^{1,2}(X)/H^3(X, \mathbf{Z})$. Albanese morphism is surjective.

It was conjectured that the Albanese morphism is an isomorphism, until in 1968 Mumford [11] has demonstrated that if $h^{2,0}(X) > 0$ then the group

$$CH_0(X)_{\text{alb}} = \ker(CH_0(X)_{\text{hom}} \xrightarrow{\text{alb}} \operatorname{Alb} X)$$

is not only nonzero but enormously large (precise meaning of "enormously large" is explained below). So, Hodge structures can not provide a complete description of $CH_0(X)$. The question arises: which language is appropriate to work with objects such as $CH_0(X)$?

The groups $CH_0(X)$ are closely related to correspondences. A correspondence between smooth irreducible projective varieties X and Y is a cycle in $X \times Y$ of dimension $\dim X$. In particular, graphs of morphisms or rational maps $X \to Y$ are correspondences. One can view correspondences as multivalued morphisms.

A correspondence $\Gamma \colon X \to Y$ between varieties of dimensions n and m respectively induces a map of sets $\Gamma_* \colon X \to CH_0(X)$, a group homomorphism $\Gamma_* \colon CH_0(X) \to CH_0(Y)$, and also cohomological pullbacks $\Gamma^* \colon H^k(Y, \mathbf{Z}) \to H^k(X, \mathbf{Z})$, and pushforwards $\Gamma_* \colon H^k(X, \mathbf{Z}) \to H^{k+2(m-n)}(Y, \mathbf{Z})$, which are compatible with Hodge decomposition. Pushforwards and pullbacks are controlled by cohomology class $[\Gamma] \in H^{m,m}(X \times Y)$.

The group $CH_0(Y)$ behaves with respect to the map $\Gamma_*\colon X \to CH_0(Y)$ as if it is something similar to an algebraic variety (or, better, a birational equivalence class of varieties). The fibers of $\Gamma_*\colon X \to CH_0(Y)$ are countable unions of closed subvarieties.

The subsets of $CH_0(Y)$ which have the form $\Gamma_* X$ for some variety X and correspondence $\Gamma\colon X \to Y$ play a special role. Such subsets are called finite-dimensional in Roitman sense (see original work of Roitman [13], and also chapter 10 of [15]). If X is a surface of positive genus $h^{2,0}(X)$, then $CH_0(X)_{\mathrm{alb}}$ is enormously large in the sense that there is no such variety W and correspondence $\Gamma\colon W \to X$ that $CH_0(X)_{\mathrm{alb}} \subset \Gamma_* W$.

Mumford based his argument about the size of $CH_0(X)_{\mathrm{alb}}$ on the following theorem (see [11]):[1]

Theorem A. *Let X, Y be smooth irreducible projective varieties, and $\Gamma\colon X \to Y$ a correspondence. If $\Gamma_*\colon X \to CH_0(Y)$ is zero, then some integer multiple of Γ is rationally equivalent in $CH_*(X \times Y)$ to a cycle which is supported on $X' \times Y$, where $X' \subset X$ is a proper subvariety.*

This theorem immediately implies the following result, also due to Mumford [11]:[2]

Theorem B. *Let X, Y be smooth irreducible projective varieties, and $\Gamma\colon X \to Y$ a correspondence. If $\Gamma_*\colon X \to CH_0(Y)$ is zero, then for every $p \geqslant 0$ the pullback morphism $\Gamma^*\colon H^{p,0}(Y) \to H^{p,0}(X)$ is zero.*

So, Γ_* controls the cohomology class $[\Gamma]$. Conversely, Bloch conjectured in [2] that $[\Gamma]$ determines the behaviour of Γ_*.

To formulate this conjecture we first need to introduce a decreasing filtration on CH_0, which is also due to Bloch. Let X be a smooth irreducible projective variety of arbitrary dimension. Consider a 3-step filtration:

$$\begin{aligned} F^0 CH_0(X) &= CH_0(X), \\ F^1 CH_0(X) &= CH_0(X)_{\mathrm{hom}}, \\ F^2 CH_0(X) &= CH_0(X)_{\mathrm{alb}}. \end{aligned}$$

The adjoint graded object with respect to this filtration is

$$\mathrm{gr}_F\, CH_0(X) = \mathbf{Z} \oplus \mathrm{Alb}\, X \oplus CH_0(X)_{\mathrm{alb}}.$$

[1] Mumford's article does not contain a result which is phrased exactly as our theorem A. Mumford works in terms of morphisms into symmetric powers of varieties, and actually never mentions correspondences. So he does not address explicitely the question to which our theorem A answers. But theorems A and B follow directly from what Mumford did in [11], as we will show in section 5.1 of this text.

[2] Here the same remark applies as in the case of theorem A.

This filtration is functorial with respect to correspondences.

Conjecture (Bloch). *Let X be a variety, Y a surface, and $\Gamma\colon X \to Y$ a correspondence. The pushforward map $\operatorname{gr}_F^p \Gamma_*\colon \operatorname{gr}_F^p CH_0(X) \to \operatorname{gr}_F^p CH_0(Y)$ is zero if and only if $\Gamma^*\colon H^{p,0}(Y) \to H^{p,0}(X)$ is zero.*

The conjecture of Bloch is deep and hard. Still, there is a lot of evidence which supports it. For example, if $X = Y$ is a surface, and $\Gamma = \Delta$, then Bloch's conjecture implies the following hypothetical result (see [2], lecture 1, proposition 1.11):

Let X be a smooth irreducible projective surface. If $h^{2,0}(X) = 0$, then the group $CH_0(X)_{\mathrm{alb}}$ is zero, i.e. the Albanese morphism is an isomorphism.

This result can be viewed as a converse of Mumford's infinite-dimensionality theorem. It was verified by Bloch, Kas and Lieberman [3] for all surfaces which are not of general type. The full conjecture of Bloch is still far from being solved.

Let X be a K3 surface, and $i\colon X \to X$ a symplectic involution, i.e. such an involution that the pullback morphism $i^*\colon H^{2,0}(X) \to H^{2,0}(X)$ is the identity. Bloch's conjecture predicts that i_* must act as the identity on $CH_0(X)$. This prediction was recently verified by Voisin in [14]. Our goal is to work out her argument, proving on the way all auxillary claims, and providing the context which is necessary to understand her work.

The ideas behind Voisin's solution are quite simple and geometric in nature. Let us explain them briefly.

A symplectic involution i on a K3 surface X is much easier to study than an arbitrary correspondence between arbitrary surfaces. First of all, i is a morphism, and so $\operatorname{gr}_F^0 i_* = \mathrm{id}$. Second, X has no global holomorphic 1-forms. As a consequence $CH_0(X)_{\mathrm{alb}} = CH_0(X)_{\mathrm{hom}}$, and $\operatorname{Alb} X = 0$. Finally, there is an explicit correspondence Δ which acts on $CH_0(X)_{\mathrm{hom}}$ exactly as Bloch's conjecture predicts for i_* to act. Thus, it is enough to show that the correspondence $\Gamma = \Delta - \Gamma_i$ acts trivially on $CH_0(X)_{\mathrm{hom}}$ (here Γ_i is the graph of i).

To do so it is a good idea to fit a given zero-cycle $z \in CH_0(X)_{\mathrm{hom}}$ into a smooth irreducible curve $C \subset X$, which is mapped by i to itself. We then get an induced action of i_* on $CH_0(C)_{\mathrm{hom}}$, which is much easier to study because $CH_0(C)_{\mathrm{hom}}$ is a complex torus, the Jacobian of C.

Voisin does it as follows. Consider the quotient Σ of X by i, which is a normal irreducible projective surface. A general ample curve on Σ will be smooth, and its preimage in X will also be smooth, and connected. We thus obtain a family of curves in X which are i-invariant. Let g be the dimension of this family. A general effective zero-cycle of degree g fits exactly into one of these curves.

Acting by Γ on this cycle we obtain another cycle, which is sent to zero by the quotient morphism $X \to \Sigma$.

Therefore we obtain a rational map $X^g \to \mathcal{P}(\widetilde{\mathcal{C}}/\mathcal{C})$, where $\mathcal{C}$ is our family of cuvres in Σ, $\widetilde{\mathcal{C}}$ is the family of their preimages in X, and $\mathcal{P}(\widetilde{\mathcal{C}}/\mathcal{C})$ is the corresponding Prym fibration, i.e. the fibration in kernels of morphisms $J(\widetilde{C}) \to J(C)$, where C is a curve from $\mathcal{C}$, and $\widetilde{C}$ is its preimage in X. Direct computation shows that $\dim \mathcal{P}(\widetilde{\mathcal{C}}/\mathcal{C}) = 2g - 1$ (this property is specific for K3 surfaces), while $\dim X^g = 2g$. Thus, a general fiber of $X^g \to \mathcal{P}(\widetilde{\mathcal{C}}/\mathcal{C})$ is positive-dimensional, which implies that a general fiber of $\Gamma_*\colon X^g \to CH_0(X)_{\mathrm{hom}}$ is positive-dimensional.

Since X is a K3 surface, it follows that the group $\Gamma_*(CH_0(X)_{\mathrm{hom}})$ is finite-dimensional. To prove this implication one uses the fact that X has a special effective zero-cycle of degree 1, the zero-cycle of Beauville-Voisin (see section 6.2, and also proposition 53). Then the factorization theorem of Voisin (see theorem 3) is used to show that Γ_* restricted to $CH_0(X)_{\mathrm{hom}}$ factors through $\operatorname{Alb} X = 0$, and so must be zero.

3 Algebraic correspondences

Correspondences make precise the notion of a family of (classes of) algebraic cycles, and so introduce additional structure on Chow groups $CH_0(X)$. Their careful study is therefore vital for this work.

3.1 Correspondences and families of zero-cycles

Definition 1. Let W be a smooth irreducible quasi-projective variety, and X a smooth irreducible projective variety. A correspondence $\Gamma\colon W \to X$ is a cycle Γ of codimension $\dim X$ in $W \times X$.

This defintion is well-known. We include it only to fix conventions. Note that we do not require correspondences to be finite or even surjective over their domains.

In the literature a domain W of a correspondence is usually assumed to be projective. For our purposes there is a significant technical benefit in allowing W to be just quasi-projective, while all properties relevant to our study continue to hold in this case too.

A correspondence Γ induces a map $\Gamma_*\colon W \to CH_0(X)$ by sending a point $w \in W$ to the cycle $(i_w \times \mathrm{id})^*\Gamma \in CH_0(X)$, where $i_w\colon w \to W$ is the evident inclusion.

Let $p\colon W\times X\to W$ and $q\colon W\times X\to X$ be the projections. Clearly, $\Gamma_* w = p^*w\cdot\Gamma$ interpreted as a cycle in $CH_0(\{w\}\times X) = CH_0(X)$. We can define a map $\Gamma_*\colon Z_0(W)\to CH_0(X)$:

$$\Gamma_*\Big(\sum_i n_i w_i\Big) = n_i(p^*w_i\cdot\Gamma).$$

Assume in addition that W is projective. In this situation the projection q is proper, and so we get a homomorphism $q_*\colon CH_0(W\times X)\to CH_0(X)$. By construction, $\Gamma_*\alpha = q_*(p^*\alpha\cdot\Gamma)$. Moreover, $q_*Z_0(W\times X)_{\mathrm{rat}} = 0$, and so Γ_* factors through $Z_0(W)_{\mathrm{rat}}$ and induces a homomorphism $\Gamma_*\colon CH_0(W)\to CH_0(X)$.

The notation for the maps $\Gamma_*\colon W\to CH_0(X)$ and $\Gamma_*\colon CH_0(W)\to CH_0(X)$ clearly conflicts, so let us introduce a convention. *If $\Gamma\colon W\to X$ is a correspondence, then by default the symbol Γ_* will denote the map $\Gamma_*\colon W\to CH_0(X)$.*

Let $\Delta\colon X\to X$ be the diagonal correspondence, and let d_+ and d_- be non-negative integers. Consider a correspondence $\Sigma_{d_+,d_-}\colon X^{d_+}\times X^{d_-}\to X$ defined as

$$\Sigma_{d_+,d_-} = \sum_{i=1}^{d_+}\Delta\circ p_i - \sum_{i=1}^{d_-}\Delta\circ p_{i+d_+}$$

where $p_i\colon X^{d_+}\times X^{d_-}\times X\to X\times X$ are projections

$$p_i(x_1,\dots,x_{d_++d_-},x) = (x_i,x).$$

Σ_{d_+,d_-} is called the natural correspondence. Let $\sigma_{d_+,d_-} = (\Sigma_{d_+,d_-})_*$. Clearly

$$\sigma_{d_+,d_-}(x_1^+,x_2^+,\dots,x_{d_+}^+,x_1^-,x_2^-,\dots,x_{d_-}^-) = \\ x_1^+ + x_2^+ + \dots + x_{d_+}^+ - x_1^- - x_2^- - \dots - x_{d_-}^-.$$

3.1.1 Fibers over points

Definition 2. Let W be a smooth irreducible quasi-projective variety, X a smooth irreducible projective variety, and $\Gamma\colon W\to X$ a correspondence. Let Z_i be an irreducible component of Γ. We say that Z_i is vertical if the natural projection $Z_i\to W$ is not surjective. We say that Γ is vertical if all its irreducible components are vertical.

Definition 3. Let W be a smooth irreducible quasi-projective variety, X a smooth irreducible projective variety, $\Gamma\colon W\to X$ a correspondence, and $w\in W$ a point. We say that Γ is finite over w if each its irreducible component is. Similarly, we say that Γ is flat over w if each its irreducible component is.

Proposition 1. *Let W be a smooth irreducible quasi-projective variety, X a smooth irreducible projective variety, and $\Gamma\colon W \to X$ a correspondence.*

(a) The set of points over which Γ is flat is Zariski-open and nonempty.

(b) Let U be the set of points over which Γ is finite. The set U is Zariski-open and nonempty. For every irreducible component Z_i of Γ the pullback of $Z_i \to W$ to U is a finite morphism.

Proof. The statement (a) follows from EGA IV 11.3.1 and generic flatness.

Let $Z_i \to W$ be an irreducible component of Γ. Since $Z_i \to W$ is proper, Chevalley's theorem tells that fiber dimension function for this morphism is upper semi-continuous. In particular, the set U of points over which fiber dimension is at most 0 is open. If this set is empty, then $Z_i \to W$ is of relative dimension at least 1, and so $\dim Z_i > \dim W$, a contradiction. Zariski's main theorem together with the fact that $Z_i \to W$ is proper imply that the pullback of $Z_i \to W$ to U is a finite morphism. □

Proposition 2. *Let C be a smooth irreducible quasi-projective curve, X a smooth irreducible projective variety, and $\Gamma\colon C \to X$ a correspondence. If Γ has no vertical components, then it is finite and flat over C.*

Proof. Let Z_i be an irreducible component of Γ. By assumptions, the projection $Z_i \to C$ is surjective. Since Z_i is integral, it follows that Z_i is flat over C, and so each fiber of $Z_i \to C$ is zero-dimensional, i.e. it is quasi-finite. As it is also proper, Zariski's main theorem implies that it is finite. □

Definition 4. Let W be a smooth irreducible quasi-projective variety, X a smooth irreducible projective variety, $\Gamma\colon W \to X$ a correspondence, and $w \in W$ a closed point over which Γ is finite. Write Γ as a sum of irreducible components $\Gamma = \sum_{i=1}^{k} n_i Z_i$. Define the fundamental cycle of the fiber of Γ over w as

$$\Gamma_w = \sum_i n_i (Z_i)_w$$

where $(Z_i)_w$ are fundamental cycles of fibers of respective irreducible components (for the definition of a fundamental cycle of a scheme see [6], chapter 1, paragraph 1.5).

Proposition 3. *Let W be a smooth irreducible quasi-projective variety, X a smooth irreducible projective variety, and $\Gamma\colon W \to X$ a correspondence. Consider a closed point $w \in W$. If Γ is finite over w then $\Gamma_* w$ is defined in $Z_0(X)$ (i.e. not up to rational equivalence). If moreover Γ is flat over w, then $\Gamma_* w = \Gamma_w$ in $Z_0(X)$.*

Proof. Let $f\colon X \to W \times X$ be the inclusion induced by $w \in W$. In the notation of Fulton [6] (chapter 8), $\Gamma_* w = [X] \cdot_f \Gamma$. Clearly it is enough to deal with the case when Γ consists of a unique irreducible component with multiplicity 1.

By our assumptions, Γ_w is a zero-dimensional scheme, so that in particular the intersection $[X] \cdot_f \Gamma$ is proper. Let x be a generic point of Γ_w, and A the local ring of x in Γ. It is known (cf. [6], chapter 7, proposition 7.1), that

$$1 \leqslant i(x, [X] \cdot_f \Gamma) \leqslant \operatorname{length}_A(A/I),$$

where $i(x, [X] \cdot_f \Gamma)$ is the respective intersection multiplicity, and I is the ideal of Γ_w in A. But clearly $I(A/I) = 0$, and so $\operatorname{length}_A(A/I) = \operatorname{length}_{A/I}(A/I)$ i.e. the coefficient of x in the fundamental cycle of Γ_w.

Assume now that Γ is flat over w. Let R be the local ring of W at w, let $\operatorname{Spec} R \to W$ be the natural morphism, and let $\Gamma_R = \Gamma \times_W \operatorname{Spec} R$. By construction $\Gamma_R = \operatorname{Spec} B$ for some ring B, and B is finite and flat over R. Hence B is free over R as a module. Since B is also equidimensional, it follows that B is Cohen-Macaulay ([5], chapter 18, paragraph 4, corollary 18.17). Therefore A is Cohen-Macaulay too, and so proposition 7.1 from [6] implies, that

$$i(x, [X] \cdot_f \Gamma) = \operatorname{length}_A(A/I),$$

which is exactly what we need. □

Since each correspondence Γ is finite and flat over some nonempty Zariski-open subset, this proposition implies, that for some open $U \subset W$ the map Γ_* is actually defined as $\Gamma_*\colon U \to Z_0(X)$, $w \mapsto \Gamma_w$.

3.1.2 Flat points

Definition 5. Let W be a smooth irreducible quasi-projective variety, X a smooth irreducible projective variety, and $\Gamma\colon W \to X$ a correspondence. The degree $[\Gamma : W]$ of Γ is defined as follows. If Γ consists of a unique irreducible component of multiplicity 1, then $[\Gamma : W]$ is the degree of the projection $\Gamma \to W$. In particular, $[\Gamma : W] = 0$ if Γ is vertical. The degree $[\Gamma : W]$ is extended to arbitrary correspondences by additivity.

Let d be a nonnegative integer. The variety $X^{(d)}$ can be viewed as a space of effective 0-cycles of degree d because there is a map $\sigma_{d,0}\colon X^{(d)} \to Z_0(X)$ which defines a bijection between closed points of $X^{(d)}$ and effective 0-cycles of degree d in X. If d_+, d_- are nonnegative integers, then we can also consider the variety $X^{(d_+)} \times X^{(d_-)}$ as a space of differences of effective cycles, but this time the map $\sigma_{d_+,d_-} : X^{(d_+)} \times X^{(d_-)} \to Z_0(X)$ is not injective.

If $\Gamma\colon W \to X$ is a correspondence and w is a closed point over which Γ is finite, then, as we know, $\Gamma_* w$ is well-defined as an element of $Z_0(X)$. If moreover Γ

is effective then the cycles $\Gamma_* w$ may be unambiguously identified with closed points of $X^{(d)}$, and therefore we can introduce a definition:

Definition 6. Let W be a smooth irreducible quasi-projective variety, X a smooth irreducible projective variety, and $\Gamma\colon W \to X$ an effective correspondence of degree d, which is finite and flat over W. Let $f\colon W \to X^{(d)}$ be a map which sends a closed point w to the cycle $\Gamma_* w$ considered as a closed point of $X^{(d)}$. We call f the map induced by Γ.

By construction the map $\Gamma_*\colon W \to CH_0(X)$ splits as

$$W \xrightarrow{\ f\ } X^{(d)} \xrightarrow{\ \sigma_{d,0}\ } CH_0(X).$$

Next proposition shows that the induced map is actually a morphism of algebraic varieties. So, correspondences can be seen as multivalued algebraic maps, or equivalently, as families of 0-cycles.

Proposition 4. *Let W be a smooth irreducible quasi-projective variety, X a smooth irreducible projective variety, and $\Gamma\colon W \to X$ an effective correspondence of degree d. If Γ is finite and flat over W, then the induced map $f\colon W \to X^{(d)}$ is a morphism of algebraic varieties.*

Proof. Write $\Gamma = \sum_{i=1}^k n_i Z_i$ as a sum of its irreducible components and consider the closed subscheme $Z = \operatorname{supp}\Gamma = \bigcup_i Z_i \subset W \times X$. Let $g\colon Z \to W$ be the projection to the first factor.

We want to show that $g\colon Z \to W$ together with the data provided by Γ defines a family of algebraic cycles in the sense of Kollár (see [7], chapter I, section I.3, definition 3.10).

Each irreducible component Z_i of Z is finite over W by assumption. Moreover, for each Z_i we have the corresponding coefficient n_i. What remains to be done is to provide for each (not necessarily closed) point $w \in W$ a cycle-theoretic fiber $g^{[-1]}(w)$, which is an element of $Z_0(g^{-1}(w)) = Z_0(g^{-1}(w)_{\mathrm{red}})$.

Consider the restrictions $g_i\colon Z_i \to W$ of g to irreducible components of Z. Let $p_i\colon g_i^{-1}(w)_{\mathrm{red}} \to g^{-1}(w)_{\mathrm{red}}$ be the corresponding closed embedding, and let $(Z_i)_w$ be the fundamental cycle of $g_i^{-1}(w)$ in $Z_0(g^{-1}(w))$. We define

$$g^{[-1]}(w) = \sum_i n_i p_{i*}((Z_i)_w).$$

The conditions (3.10.1) – (3.10.3) of Kollár's definition are clearly satisfied, and the only remaining condition is (3.10.4).

Let $w \in W$ be a point (not necessarily a closed one), let T be a spectrum of a DVR with t the closed point, and $h\colon T \to W$ a morphism, which sends t to w.

Let $h_t\colon t \to w$ be the morphism of closed points induced by h. Since each Z_i is flat over W, its pullback by h is also flat over T, and hence

$$\lim_{h\to w}(Z_i/W) = (h^*Z_i)_t = h_t^*(Z_i)_w.$$

But then

$$\sum_i n_i \lim_{h\to w}(Z_i/W) = h_t^* g^{[-1]}(w),$$

and so the condition (3.10.4) is satisfied.

By construction, Z is a closed subscheme of $W \times X$, and so $g\colon Z \to W$ together with the coefficients n_i and cycle-theoretic fibers $g^{[-1]}(w)$ is a well-defined family of proper algebraic 0-dimensional cycles of X in the sense of Kollár's definition 3.11 (see [7], chapter I, section I.3). Moreover, the family $g\colon Z \to W$ is nonnegative since Γ is effective. Since $[\Gamma : W] = d$, all the cycles in the family have degree d.

Thus, by theorem I.3.21 of Kollár's book [7], g defines a morphism $W \to \mathrm{Chow}_{0,d}(X)$, where $\mathrm{Chow}_{0,d}(X)$ is the Chow variety of effective 0-cyles of degree d in X. It is known, that $\mathrm{Chow}_{0,d}(X) = X^{(d)}$ (see [7], excercise 3.22). By construction, the morphism $W \to X^{(d)}$ sends each point w to the corresponding cycle-theoretic fiber $g^{[-1]}(w)$. If w is a closed point then $g^{-1}(w) = \Gamma_w$, and so, by proposition 3, the morphism $W \to X^{(d)}$ agrees with the map $f\colon W \to X^{(d)}$ induced by Γ. $\square$

If $\Gamma\colon W \to X$ is a correspondence which is not necessarily finite and flat over W, then we can restrict it to the open subset $U \subset W$ over which it is finite and flat, and hence obtain a morphism $f\colon U \to X^{(d)}$. From now on we will consider this map f as a rational map from W to $X^{(d)}$. At this stage we do not know yet if f agrees with Γ_* at every point at which f is defined.

Proposition 5. *Let W be a smooth irreducible quasi-projective variety, X a smooth irreducible projective variety, d a nonnegative integer, and $f\colon W \to X^{(d)}$ a morphism. There exists an effective correspondence $\Gamma\colon W \to X$ of degree d such that Γ is finite over W, and the map induced by Γ coincides with f at every closed point over which Γ is flat.*

Proof. By definition of the functor $\mathrm{Chow}_{0,d}(X)$ the map $f\colon W \to X^{(d)}$ induces a family $g\colon Z \to W$ of nonnegative 0-cycles of degree d in X in the sense of Kollár. The supporting scheme Z is by construction a closed subscheme of $W \times X$, and each irreducible component Z_i of Z is equipped with an integer multiplicity n_i. Hence we have an algebraic cycle $\Gamma \subset W \times X$, $\Gamma = \sum_i n_i Z_i$. Since all the fibers of $g\colon Z \to W$ are finite, we conclude that Γ is finite over W.

Moreover, for every point $w \in W$ the cycle $g^{[-1]}(w)$ coincides with $f(w)$ considered as a point of $X^{(d)}_{k(w)}$. If $d > 0$, then we conclude that every fiber of

$g\colon Z \to W$ is nonempty, and hence all irreducible components of Γ have dimension $\dim W$. If $d = 0$, then all the fibers of g are empty, i.e. $Z = \varnothing$ and $\Gamma = 0$. In any case, Γ is a correspondence, which is finite over W.

Let $w \in W$ be such a closed point that Γ is flat over it. Let T be a spectrum of a DVR, and $h\colon T \to W$ a morphism, which sends the closed point $t \in T$ to w. Let $h_t\colon t \to w$ be the induced morphism of points. By definition of $g^{[-1]}(w)$,

$$h_t^* g^{[-1]}(w) = \sum_i n_i \lim_{h \to w} (Z_i/W).$$

Each Z_i is flat over W at w, and so

$$\lim_{h \to w} (Z_i/W) = h_t^*(Z_i)_w.$$

Therefore

$$h_t^* g^{[-1]}(w) = h_t^* \Gamma_w,$$

and we conclude that $g^{[-1]}(w) = \Gamma_w = \Gamma_* w$. □

3.1.3 Pullbacks

It is not easy to see that the rational map induced by a correspondence Γ agrees with Γ_* at all closed points over which Γ is finite. Main obstruction is the definition of a cycle-theoretic fiber $g^{[-1]}(w)$. Basically, the only way to control it is to take a "limit" over some curve which passes through w. To compare this limit with $\Gamma_* w$ we need to pull the correspondence back to the curve.

Let X be a smooth irreducible projective variety. Let T and W be smooth irreducible quasi-projective varieties, $\Gamma\colon W \to X$ a correspondence, and $f\colon T \to W$ a morphism. In this situation we can not define the composition $\Gamma \circ f$ as described in the chapter 16 of [6], since the projection $T \times W \times X \to T \times X$ is not proper in general. Nevertheless, we have a Gysin pullback morphism $(f \times \mathrm{id})^*\colon CH_*(W \times X) \to CH_*(T \times X)$.

Definition 7. We define the composition $\Gamma \circ f$ as $(f \times \mathrm{id})^* \Gamma$.

If T and W are projective, then our $\Gamma \circ f$ evidently coincides with the one defined in [6]. If $f\colon w \to W$ is an embedding of a closed point, then $\Gamma \circ f = \Gamma_* w$.

Proposition 6. *Let T, W be smooth irreducible quasi-projective varieties, X a smooth irreducible projective variety, $\Gamma\colon W \to X$ a correspondence, and $f\colon T \to W$ a morphism. If Γ is finite over W, then $\Gamma \circ f$ is defined as a cycle (not a class of rational equivalence), and is finite over T.*

Proof. Clearly it is enough to consider the case when Γ consists of a single irreducible component with multiplicity one.

The intersection class $\Gamma \circ f = (f \times \mathrm{id})^*\Gamma$ is computed as follows (see [6], chapter 8). First, we take the graph morphism $\gamma = \gamma_{(f\times\mathrm{id})}\colon T \times X \to T \times W \times X$. Since $W \times X$ is smooth, γ is a regular embedding. We consider the cross product $[T] \times \Gamma$, where $[T]$ is the fundamental cycle of T. Since $[T]$ is just T with multiplicity one, and Γ is irreducible, the cross product $[T] \times \Gamma$ is the fundamental cycle of closed subscheme $T \times \Gamma \subset T \times W \times X$. Next, we form a cartesian square

$$\begin{array}{ccc} \gamma^{-1}T \times \Gamma & \longrightarrow & T \times \Gamma \\ \big\downarrow & & \big\downarrow \\ T \times X & \xrightarrow{\gamma} & T \times X \times W. \end{array}$$

Since γ is regular, it induces a vector bundle N on $T \times X$, the normal bundle of an embedding. The closed subscheme $\gamma^{-1}T \times X$ of $T \times X$ has a normal cone C, which is in a natural way a purely-dimensional closed subscheme of N. There is a homomorphism $s^*\colon CH_*(N) \to CH_*(T \times X)$ which, informally speaking, intersects classes with the zero section of N (see [6], chapter 3, definition 3.3). By definition, $(f \times \mathrm{id})^*(\Gamma) = s^*(C)$.

We have an evident cartesian diagram

$$\begin{array}{ccccc} \gamma^{-1}T \times \Gamma & \longrightarrow & T \times \Gamma & \longrightarrow & \Gamma \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ T \times X & \xrightarrow{\gamma} & T \times W \times X & \longrightarrow & W \times X \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ T & \xrightarrow{\gamma_f} & T \times W & \longrightarrow & W, \end{array}$$

where γ_f is the graph of f. The composition $T \xrightarrow{\gamma_f} T \times W \to W$ is just $T \xrightarrow{f} W$, and so $\gamma^{-1}T \times \Gamma = f^{-1}\Gamma$. By assumption, Γ is finite and surjective over W, and so $f^*\Gamma$ is finite and surjective over T. Therefore, each irreducible component Z_i of $f^{-1}\Gamma$ has dimension at most $\dim T$.

On the other hand, lemma 7.1 from chapter 7 of [6] applied to the regular embedding γ and closed subscheme $T \times \Gamma$ shows, that each irreducible component Z_i of $f^{-1}\Gamma$ has dimension at least $\dim(T \times \Gamma) - \dim W = \dim T$. Thus, the intersection $f^{-1}\Gamma$ is proper.

In this situation the canonical decomposition of the intersection class $s^*(C)$ has the form $\sum_i m_i[Z_i]$, where m_i are respective intersection multiplicities (see discussion just below the lemma 7.1 from [6], and also definition 6.1.2 from the chapter 6 of the same book). Hence $\Gamma \circ f$ is defined as an element of $Z_*(T \times X)$, and is finite over T. $\square$

Proposition 7. *Let W_1, W_2, W_3 be smooth irreducible quasi-projective varieties. Let $f\colon W_1 \to W_2$ and $g\colon W_2 \to W_3$ be regular embeddings, and $Z \in Z_k(W_3)$ a cycle. Assume that both pullbacks $g^*(Z)$ and $f^*(g^*(Z))$ are proper, i.e. defined as elements of $Z_*(W_2)$ and $Z_*(W_1)$ respectively. In this situation $(g \circ f)^*Z$ is also proper, and $(g \circ f)^*Z = f^*(g^*Z)$ in $Z_*(W_1)$. The same is true under assumption that pullbacks $(g \circ f)^*Z$ and g^*Z are proper.*

Proof. It is enough to assume that Z is irreducible of multiplicity 1. Consider a cartesian diagram

$$\begin{array}{ccccc} f^{-1}g^{-1}Z & \longrightarrow & g^{-1}Z & \longrightarrow & Z \\ \downarrow & & \downarrow & & \downarrow \\ W_1 & \xrightarrow{f} & W_2 & \xrightarrow{g} & W_3. \end{array}$$

Here we view Z as a closed subscheme of W_3. Since $(g \circ f)^{-1}Z = f^{-1}g^{-1}Z$, we conclude that $(g \circ f)^*$ is defined as an element of $Z_*(W_1)$.

Let Z_r^g be the irreducible components of $g^{-1}Z$ and Z_s^f be the irreducible components of $f^{-1}g^{-1}Z$. There is an associativity formula for intersection multiplicities (see [6], chapter 7, example 7.1.8):

$$i(Z_s^f, W_1 \cdot Z; W_3) = \sum_r i(Z_s^f, W_1 \cdot Z_r^g; W_2) \cdot i(Z_r^g, W_2 \cdot Z; W_3).$$

On the other hand, by definition of intersection multiplicities

$$\begin{aligned} g^*(Z) &= \sum_r i(Z_r^g, W_2 \cdot Z; W_3) \cdot Z_r^g, \\ f^*(Z_r^g) &= \sum_s i(Z_s^f, W_1 \cdot Z_r^g; W_2) \cdot Z_s^f, \\ (g \circ f)^*(Z) &= \sum_s i(Z_s^f, W_1 \cdot Z; W_3) \cdot Z_s^f. \end{aligned}$$

A substitution of formulas shows that $(g \circ f)^*Z = f^*(g^*(Z))$. $\square$

Proposition 8. *Let T, W be smooth irreducible quasi-projective varieties, $t \in T$ a closed point, $z \in Z_*(W)$ a cycle, and $j\colon W \to T \times W$ the regular embedding induced by t. The pullback $j^*([T] \times z)$ is defined as an element of $Z_*(W)$, and is equal to z.*

Proof. It is enough to assume that z consists of a unique irreducible component Z with multiplicity 1. Consider a cartesian square

$$\begin{array}{ccc} j^{-1}T \times Z & \longrightarrow & T \times Z \\ \downarrow & & \downarrow \\ W & \xrightarrow{i} & T \times W. \end{array}$$

Clearly, $j^{-1}T \times Z = Z$, so that the intersection is proper, and $j^*([T] \times [Z]) = m[Z]$ in $Z_*(W)$, where m is the respective intersection multiplicity. Let us show that $m = 1$.

Let $\eta \in Z$ be the generic point, and let A be the local ring of $T \times Z$ at $j(\eta)$, and I the ideal of $j(Z)$ in A. As in the proof of proposition 3, we notice that

$$1 \leqslant i(\eta, [W] \cdot_j [T \times Z]; T \times W) \leqslant \operatorname{length}_A(A/I).$$

Since $I(A/I) = 0$, we conclude that $\operatorname{length}_A(A/I) = \operatorname{length}_{A/I}(A/I)$. But A/I is the local ring of Z at η, which is a field, and so $\operatorname{length}_{A/I}(A/I) = 1$. $\square$

Proposition 9. *Let T, W be smooth irreducible quasi-projective varieties, X a smooth irreducible projective variety, $f\colon T \to W$ a morphism, and $\Gamma\colon W \to X$ a correspondence which is finite over W. If $t \in T$ is a closed point, then $(\Gamma \circ f)_* t = \Gamma_* f(t)$ in $Z_0(X)$.*

Proof. It is enough to assume that Γ consists of a unique irreducible component with multiplicity 1.

As in the proof of proposition 6, consider the graph morphism $\gamma = \gamma_{(f \times \mathrm{id})}\colon T \times X \to T \times W \times X$. Let $i\colon t \to T$ be the embedding of the closed point, and let $j = (i \times \mathrm{id})\colon X \to T \times X$ be the induced embedding of X. Consider a cartesian diagram

$$\begin{array}{ccccc} j^{-1}\gamma^{-1}T \times \Gamma & \longrightarrow & \gamma^{-1}T \times \Gamma & \longrightarrow & T \times \Gamma \\ \downarrow & & \downarrow & & \downarrow \\ X & \xrightarrow{j} & T \times X & \xrightarrow{\gamma} & T \times W \times X. \end{array}$$

From proposition 6 we know, that $\gamma^* T \times \Gamma$ is proper. To see that $j^*(\gamma^* T \times \Gamma)$ is proper, consider a cartesian diagram

$$\begin{array}{ccccc} j^{-1}\gamma^{-1}T \times \Gamma & \longrightarrow & T \times \Gamma & \longrightarrow & \Gamma \\ \downarrow & & \downarrow & & \downarrow \\ X & \xrightarrow{\gamma \circ j} & T \times W \times X & \longrightarrow & W \times X. \end{array}$$

Since the composition $X \xrightarrow{\gamma \circ j} T \times W \times X \to W \times X$ is just $(f \circ i) \times \mathrm{id}$, and since Γ is finite over $f(t)$, we conclude that $j^*(\gamma^* T \times \Gamma)$ is also proper. Hence, by proposition 7, $j^*(\gamma^* T \times \Gamma) = (\gamma \circ j)^* T \times \Gamma$.

For the sake of brevity, let us introduce notation $u = (f \circ i) \times \mathrm{id}\colon X \to W \times X$, and $v = i \times \mathrm{id}\colon W \times X \to T \times W \times X$. Consider a cartesian diagram

$$\begin{array}{ccccc} u^{-1}v^{-1}T \times \Gamma & \longrightarrow & v^{-1}T \times \Gamma & \longrightarrow & T \times \Gamma \\ \downarrow & & \downarrow & & \downarrow \\ X & \xrightarrow{u} & W \times X & \xrightarrow{v} & T \times W \times X. \end{array}$$

Both u and v are regular embeddings, and $v \circ u = \gamma \circ j$. Moreover, proposition 8 states that $v^*(T \times \Gamma) = \Gamma$ in $Z_*(W \times X)$.

Applying proposition 7 once again, we obtan an equality

$$u^*(v^*T \times \Gamma) = (v \circ u)^*T \times \Gamma = (\gamma \circ j)^*T \times \Gamma = j^*(\gamma^*T \times \Gamma)$$

But by definition $(\Gamma \circ f)_*t = j^*(\Gamma \circ f) = j^*(\gamma^*T \times \Gamma)$, while $u^*(v^*T \times \Gamma) = u^*(\Gamma) = \Gamma_*f(t)$. □

Proposition 10. *Let T, W be smooth irreducible quasi-projective varieties, X a smooth irreducible projective variety, $\Gamma \colon W \to X$ a correspondence, $f \colon T \to W$ a morphism, and $t \in T$ a closed point. $(\Gamma \circ f)_*t = \Gamma_*f(t)$ in $CH_0(X)$*

Proof. Let $i \colon t \to T$ be the inclusion of point. In the notation of Fulton's chapter 8, $(f \times \mathrm{id})^*\Gamma = [T \times X] \cdot_{(f\times\mathrm{id})} \Gamma$, and $(i \times \mathrm{id})^*z = [X] \cdot_{(i\times\mathrm{id})} z$. Proposition 8.1.1 from chapter 8 of [6] shows that there is an equality in $CH_0(X)$:

$$\begin{aligned}(i \times \mathrm{id})^*((f \times \mathrm{id})^*\Gamma) = [X] \cdot_{(i\times\mathrm{id})} ([T \times X] \cdot_{(f\times\mathrm{id})} \Gamma) = \\ ([X] \cdot_{(i\times\mathrm{id})} [T \times X]) \cdot_{(f\circ i\times\mathrm{id})} \Gamma = (i \times \mathrm{id})^*[T \times X] \cdot_{(f\circ i\times\mathrm{id})} \Gamma.\end{aligned}$$

Proposition 8 shows that $(i \times \mathrm{id})^*[T \times X] = [X]$, and so

$$(i \times \mathrm{id})^*((f \times \mathrm{id})^*\Gamma) = ((f \circ i) \times \mathrm{id})^*\Gamma. \quad \square$$

3.1.4 Finite points

Proposition 11. *Let W be a quasi-projective variety, X a projective variety, and $\Gamma \colon W \to X$ an effective correspondence of degree d. Let $U \subset W$ be the Zariski-open subset over which Γ is flat, and let $f \colon U \to X^{(d)}$ be the induced morphism. Assume that Γ is finite over W. If f is defined at a closed point $w \in W$, then $f(w) = \Gamma_*w$.*

Proof. Let $C_0 \subset W$ be an irreducible curve which contains w and meets U, and let $i \colon C \to W$ be the normalization of C_0. By proposition 6, the correspondence $\Gamma \circ i$ is defined as a cycle and is finite over C. By proposition 9, if $c \in C$ is a closed point, then $(\Gamma \circ i)_*c = \Gamma_*i(c)$ in $Z_0(X)$.

By proposition 2, $\Gamma \circ i$ is flat over C. Hence the morphism g induced by $\Gamma \circ i$ is defined everywhere, and agrees with Γ_* at all closed points of C. Let $u \in i^{-1}(U)$ be a closed point. Since $(\Gamma \circ i)_*u = \Gamma_*i(u)$ in $Z_0(X)$, and since $\Gamma \circ i$ is effective, it follows that $g(u) = f(i(u))$. We conclude that $g|_{i^{-1}U} = f \circ i|_{i^{-1}U}$ because our schemes are of finite type over $\mathbf{C}$. Moreover, the rational map $f \circ i$ is defined everywhere because C is smooth, and so $g = f \circ i$.

By construction, there exists such a closed point $c \in C$ that $i(c) = w$. Hence, $\Gamma_*w = (\Gamma \circ i)_*c = g(c) = f(w)$. □

Proposition 12. *Let W be a quasi-projective variety, X a projective variety, and $\Gamma\colon W \to X$ an effective correspondence of degree d. Let $U \subset W$ be the Zariski-open subset over which Γ is flat, and let $f\colon U \to X^{(d)}$ be the induced morphism. If Γ is finite over W, then f can be extended to W.*

Proof. We need to show that there exists a family $g\colon Z \to W$ of nonnegative effective 0-cycles of X, which agrees with the one defined by f over U.

Let $Z = \operatorname{supp}\Gamma$ and $g\colon Z \to W$ the projection. The cycle Γ provides us with multiplicities for irreducible components of Z_i of Z, and we have already constructed $g^{[-1]}(w)$ for every point $w \in W$ over which Γ is flat.

By assumptions, W is smooth (in particular normal), and $g\colon Z \to W$ satisfies conditions (3.10.1) – (3.10.3) of Kollár's definition 3.10 (see [7], chapter I, section 3). We now apply theorem 3.17 from chapter I of [7], which shows that g is a well-defined family of algebraic cycles of X, i.e. that we can define $g^{[-1]}(w)$ for all points $w \in W$. Over flat points of W this family coincides with the one defined by f, since in this situation $g^{[-1]}(w)$ is unambiguously determined by scheme-theoretic fibers of $g|_{Z_i}$ over w, as we have seen in the proof of proposition 4. $\square$

Proposition 13. *Let W be a smooth irreducible quasi-projective variety, X a smooth irreducible projective variety, and $\Gamma\colon W \to X$ a correspondence which has no vertical components. If there exists such a nonempty Zariski-open subset $U \subset W$ that Γ is finite over U and $\Gamma_* u = 0$ in $Z_0(X)$ for every closed point $u \in U$, then $\Gamma = 0$ in $Z_*(W \times X)$.*

Proof. Restrict Γ to $U \times X$ and write it as a sum of distinct irreducible components

$$\Gamma|_U = \sum_{i=1}^{k} n_i Z_i.$$

Assume that $k > 0$, and consider a cycle $\Gamma' = \sum_{i=1}^{k-1} n_i Z_i$. For every $u \in U$ the fiber $Z_k \cap \{u\} \times X$ is contained in $\operatorname{supp}\Gamma' \cap \{u\} \times X$ because $\Gamma'_* u = -n_k (Z_k)_* u$. Thus $Z_k \subset \operatorname{supp}\Gamma'$, and so it must coincide with one of the Z_i's, which is impossible if $k > 0$. Hence, $\Gamma|_U = 0$ in $Z_*(U \times X)$, which in turn means that $\Gamma \subset (W \backslash U) \times X$, i.e. that no component of Γ is surjective. Since there are no vertical components by assumption, we conclude that $\Gamma = 0$ in $Z_*(W \times X)$. $\square$

We next study closures (extensions) of correspondences.

Proposition 14. *Let W be a smooth irreducible quasi-projective variety, X a smooth irreducible projective variety, $U \subset W$ a Zariski-open subset, and $n = \dim W$. There exists an injective homomorphism $Z_n(U \times X) \to Z_n(W \times X)$, $\Gamma \mapsto \widetilde{\Gamma}$, called the closure homomorphism. It has the property that $\widetilde{\Gamma}|_U = \Gamma$ for every $\Gamma \in Z_n(U \times X)$. Moreover if Γ has no vertical components, then $\widetilde{\Gamma}$ has no vertical components too.*

Proof. Clearly, it is enough to define the operation $\Gamma \mapsto \widetilde{\Gamma}$ for irreducible Γ of multiplicity 1. In this case we let $\widetilde{\Gamma}$ be the closure of Γ in $W \times X$. If Γ is not vertical, then $\widetilde{\Gamma}$ surjects onto W, since the image of $\widetilde{\Gamma}$ under the projection $\widetilde{\Gamma} \to W$ contains a dense open subset U, which is in turn the image of $\Gamma \subset \widetilde{\Gamma}$. $\square$

Proposition 15. *Let W be a smooth irreducible quasi-projective variety, X a smooth irreducible projective variety, and $\Gamma\colon W \to X$ an effective correspondence of degree d, which has no vertical components. Let U be the Zariski-open subset over which Γ is finite, and $f\colon U \to X^{(d)}$ the induced morphism. Consider a correspondence $\Gamma_f\colon U \to X$ induced by f and take its extension $\widetilde{\Gamma}_f\colon W \to X$. The correspondences Γ and $\widetilde{\Gamma}_f$ coincide.*

Proof. By proposition 11, $(\widetilde{\Gamma}_f)_* u = \Gamma_* u$ for every closed point $u \in U$. Moreover Γ_f is finite over U by construction, and so $\widetilde{\Gamma}_f$ has no vertical components. Proposition 13 then shows that $\widetilde{\Gamma}_f = \Gamma$. $\square$

Proposition 16. *Let W be a smooth irreducible quasi-projective variety, X a smooth irreducible projective variety, $\Gamma\colon W \to X$ an effective correspondence of degree d, which has no vertical components, and $f\colon W \to X^{(d)}$ the induced* rational *map. Let $U \subset W$ be the Zariski-open subset over which Γ is finite, and $V \subset W$ the Zariski-open subset over which f is defined. The sets U and V coincide.*

Proof. From proposition 12 we know, that $U \subset V$. Let Z be the correspondence induced by f over V. Proposition 15 tells us that $\Gamma|_V = Z$. Since Z is finite over V by construction, $V = U$. $\square$

3.1.5 Arbitrary points

What remains is to analyze the behaviour of Γ_* at points over which Γ_* is not finite. As it turns out in this case the assumption that the base W is projective makes things a lot easier. Fortunately, the general quasi-projective case can be reduced to the projective one.

Proposition 17. *Let W be a smooth irreducible projective variety, $U \subset W$ a nonempty Zariski-open subset, and $z \in CH_0(W)$ a cycle. There exsists a cycle $z' \in CH_0(X)$, which is rationally equivalent to z and supported on U.*

Proof. There exists a curve $C \subset W$ such that $\operatorname{supp} z \subset C$ and C meets U. Let $f\colon \widetilde{C} \to X$ be the normalization of C. We can lift the cycle z to a divisor $\widetilde{z} \in CH_0(\widetilde{C})$. The subset $f^{-1}U$ is nonempty and its complement $C \backslash f^{-1}U$ is just a finite number of points. Hence there is a divisor $\widetilde{z}'$ which is equivalent to $\widetilde{z}$ and is supported in $f^{-1}U$. Correspondingly, $f_*\widetilde{z}'$ is equivalent to z and is supported in U. $\square$

Proposition 18. *Let W, X be smooth irreducible projective varieties, $\Gamma\colon W \to X$ a correspondence. If there exists a nonempty Zariski-open subset $U \subset W$ such that if $\Gamma_*|_U = 0$, then $\Gamma_* = 0$.*

Proof. Let $w \in W$ be a closed point. Since W is projective, $\Gamma_* w = q_*(p^* w \cdot \Gamma)$, where $p\colon W \times X \to W$ and $q\colon W \times X \to X$ are the projections, and the map Γ_* in this form is defined not just on W, but on $CH_0(W)$. By proposition 17, there exists a cycle z which is equivalent to w and supported in U, so that $\Gamma_* z = 0$. $\square$

As a corollary, if a correspondence Γ is vertical, then $\Gamma_* = 0$.

Proposition 19. *Let W be a smooth irreducible quasi-projective variety, X a smooth irreducible projective variety, $\Gamma\colon W \to X$ an effective correspondence of degree d, and $f\colon W \to X^{(d)}$ the rational map induced by Γ. Let $p\colon W \times X^{(d)} \to W$ and $q\colon W \times X^{(d)} \to X^{(d)}$ be the projections, and $G \subset W \times X^{(d)}$ the closure of the graph of f. If $z \in G$ is a closed point, then $q(z)$ is equal in $CH_0(X)$ to $\Gamma_* p(z)$. In particular, if $z_1, z_2 \in G$ are such points that $p(z_1) = p(z_2)$, then $q(z_1) = q(z_2)$ in $CH_0(X)$.*

Proof. Let us embed W as an open subvariety into an irreducible projective variety Y, and let $\pi\colon \widetilde{Y} \to Y$ be the resolution of singularities of Y. Because W is smooth, π restricted to $\pi^{-1}W$ is an isomorphism, and so we can assume that W is an open subvariety of a smooth irreducible projective variety Y.

If $\Gamma\colon W \to X$ is a correspondence, then we can take its extension $\widetilde{\Gamma}\colon Y \to X$. Let $\widetilde{f}\colon Y \to X^{(d)}$ be the rational map induced by $\widetilde{\Gamma}$. By construction of extension $\widetilde{f}$ agrees with $f\colon W \to X$ induced by Γ, and so the graph $\widetilde{G} \subset Y \times X^{(d)}$ of $\widetilde{f}$ restricted to $W \times X^{(d)}$ is just the graph of f. We can therefore assume that W is projective.

Let $z \in G$ be a closed point. Let $U \subset W$ be the nonempty Zariski-open subset over which Γ is finite. Take a curve $C \subset G$ which passes through z and meets $p^{-1}U$. Let $i\colon \widetilde{C} \to G$ be the normalization of C. Let $g = p \circ i$ and $h = q \circ i$.

Because W is projective, we have a correspondence $\Gamma \circ g \in CH_*(C \times X)$, as defined in chapter 16 of [6]. Let $Z\colon C \to X$ be the correspondence induced by the map h.

Let $c \in g^{-1}U$ be a point. By definition, $Z_* c = h(c)$ in $Z_0(X)$, and $(\Gamma \circ g)_* c = \Gamma_* g(c)$ in $CH_0(X)$. Since $g(c) \in U$, the function f is defined at $g(c)$, and so $\Gamma_* g(c) = f(g(c))$ in $Z_0(X)$. On the other hand, $h(c) = q(i(c)) = f(g(c))$ by definition of the graph G of f.

So, Z_* and $(\Gamma \circ g)_*$ agree over an open subset $g^{-1}U$, and proposition 18 implies, that they agree everywhere. By construction of $\widetilde{C}$, there exists $c \in \widetilde{C}$, such that $i(c) = z$. Hence, $\Gamma_* p(z) = Z_* c = q(z)$. $\square$

Let us collect all the results obtained before into a single theorem.

Theorem 1. *Let W be a smooth irreducible quasi-projective variety, X a smooth irreducible projective variety, and d a nonnegative integer. There is a bijection between the sets:*

correspondences $\Gamma\colon W \to X$, which are effective, of degree d, and which have no vertical components	$\Leftrightarrow$	*rational maps $f\colon W \to X^{(d)}$.*

If $\Gamma\colon W \to X$ is such a correspondence, and $f\colon W \to X^{(d)}$ the induced rational map, then f is defined precisely at the points at which Γ is finite. If $w \in W$ is a closed point over which Γ is finite, then $\Gamma_* w$ is defined as an element of $Z_0(X)$, and $\Gamma_* w = f(w)$ in $Z_0(X)$.

Let $G \subset W \times X^{(d)}$ be the closure of the graph of f, and let $p\colon W \times X^{(d)} \to W$, $q\colon W \times X^{(d)} \to X^{(d)}$ be the projections. If $z \in G$ is a closed point, then $\Gamma_* p(z) = q(z)$ in $CH_0(X)$. In particular all cycles in a fiber of $p\colon G \to W$ over a closed point $w \in W$ are rationally equivalent.

Proof. We pass from left to right by taking the induced rational map, and from right to left by taking the induced correspondence and then extening it to W. The fact that it is is a bijection follows from propositions 15 and 5.

The rest follows from propositions 16, 11, 3 and 19. $\square$

3.2 Rational equivalence of zero-cycles

Let X be a smooth irreducible projective variety, and d a nonnegative integer. Closed points of $X^{(d)}$ are in bijection with effective zero-cycles of degree d in X. Consider the set

$$R = \{(\alpha, \beta) \in X^{(d)}(\mathbf{C}) \times X^{(d)}(\mathbf{C}) \mid \alpha = \beta \text{ in } CH_0(X)\}.$$

We are going to show that R can be parametrized by a countable union of subvarieties of the cartesian product $X^{(d)} \times X^{(d)}$. To do it we need to study rational curves in $X^{(d)}$ for all $d \geqslant 0$.

Let us introduce additional conventions for this section.

Every smooth irreducible projective variety X is is assumed to be equipped with an ample line bundle $\mathcal{O}_X(1)$ with respect to which we will compute degrees of cycles.

Let $f\colon \mathbf{P}^1 \to X$ be a morphism. We will say that such a morphism has degree d if $f^*\mathcal{O}_X(1) \cong \mathcal{O}(d)$. By projection formula,

$$d = \deg(f^*\mathcal{O}_X(1)) = \deg(f_*(1) \cdot \mathcal{O}_X(1)) = (\deg f) \cdot \deg(f(\mathbf{P}^1) \cdot \mathcal{O}_X(1)),$$

where $1 \in CH_1(\mathbf{P}^1)$ is the fundamental cycle of $\mathbf{P}^1$. We emphasise that the degree $\deg f$ of f as a morphism onto its image is just a multiple of d.

3.2.1 Specialization of rational curves

For the moment, let us admit arbitrary schemes over arbitrary bases to the study.

Definition 8. Let A be a DVR, $\eta \in \operatorname{Spec} A$ its generic point, and $X \to \operatorname{Spec} A$ a morphism. The closure of X_η in X is called the flat completion of $X \to \operatorname{Spec} A$. The fiber of the flat completion over the closed point of $\operatorname{Spec} A$ is called the flat limit of $X \to \operatorname{Spec} A$.

Proposition 20. *Let A be a DVR, $\eta \in \operatorname{Spec} A$ its generic point, and $: X \to \operatorname{Spec} A$ a morphism. The flat completion of $X \to \operatorname{Spec} A$ is the maximal closed subscheme of X which is flat over $\operatorname{Spec} A$.*

Proof. Let us first show that the flat completion is flat over $\operatorname{Spec} A$.

Let $i\colon X_\eta \to X$ be the inclusion of the fiber over η. By construction, the flat completion is defined by the maximal quasi-coherent subscheaf of the sheaf $\mathcal{I} = \ker(\mathcal{O}_X \to i_*\mathcal{O}_X)$. Notice that in general $\mathcal{I}$ needs not be quasi-coherent.

Let $\operatorname{Spec} B \subset X$ be an affine open subscheme, and let $t \in A$ be an uniformizing parameter. Clearly, $X_\eta|_{\operatorname{Spec} B} = \operatorname{Spec} B_t$, and so $\mathcal{I}|_{\operatorname{Spec} B} = \ker(B \to B_t)$. In particular, $\mathcal{I}$ is a quasi-coherent sheaf.

Let $I = \ker(B \to B_t)$. By construction, B/I embeds into B_t, and so it is torsion free as an A-module. Hence, the flat completion is flat over $\operatorname{Spec} A$.

Let Y be a closed subscheme of X which is flat over $\operatorname{Spec} A$. Let $\operatorname{Spec} B \subset X$ be an affine open subscheme, and let J be the ideal of $Y|_{\operatorname{Spec} B}$. By assumption, B/J is flat over A. Hence, every A-torsion element of B vanishes in B/J, i.e. is contained in J. But I consists exactly of such elements, and so $I \subset J$, which means that Y is a closed subscheme of the flat completion. $\square$

Definition 9. Let X be a scheme over a field k. We say that X is a rational curve, if its normialization is a disjoint union of rational curves $\mathbf{P}^1_k$.

Proposition 21. *Let A be a DVR, and $X \to \operatorname{Spec} A$ a projective morphism. If the generic fiber of X is an irreducible rational curve, then the flat limit of $X \to A$ is a connected rational curve.*

Proof. Let η be the generic point of $\operatorname{Spec} A$, and let $Y \to \operatorname{Spec} A$ be the flat completion of $X \to \operatorname{Spec} A$. By construction, $Y_\eta = X_\eta$ admits a surjection from $\mathbf{P}^1_{k(\eta)}$, so that we have a rational map $\varphi\colon \mathbf{P}^1_A \to Y$ over $\operatorname{Spec} A$.

Since $\dim \mathbf{P}^1_A = 2$, this scheme is a regular fibered surface in terms of definition 8.3.1 from [9]. Moreover, $Y \to \operatorname{Spec} A$ is projective by construction, so that we can apply theorem 9.2.7 from [9] to φ, and conclude that there exists a projective A-scheme $\widetilde{Y}$, and A-morphisms

$$\mathbf{P}^1_A \xleftarrow{\ f\ } \widetilde{Y} \xrightarrow{\ g\ } Y,$$

such that f is a sequence of blowups of closed points, and $\varphi \circ f = g$.

The morphism g is proper because it is a morphism of projective A-schemes. Also, f restricted to $\widetilde{Y}_\eta$ is an isomorphism with X_η. Since $\varphi \circ f = g$, we conclude that the set of points $g(\widetilde{Y})$ is closed and contains Y_η. Therefore, it is equal to Y, and g is surjective.

Let $\mathfrak{m} \in \operatorname{Spec} A$ be the closed point. Since f is a composition of blowups of closed points, the fiber $\widetilde{Y}_\mathfrak{m}$ is a "tree" of projective lines $\mathbf{P}^1_{k(\mathfrak{m})}$. On the other hand, g restricted to $\widetilde{Y}_\mathfrak{m}$ is surjective onto $Y_\mathfrak{m}$, and so $Y_\mathfrak{m} = \lim(X \to \operatorname{Spec} A)$ is a connected rational curve, as an image of such curve. $\square$

3.2.2 Universal families of rational curves

We return back to schemes over $\mathbf{C}$.

Let X be a smooth irreducible projective variety. Consider a Hom-scheme:

$$\operatorname{Hom}(\mathbf{P}^1, X)(Y) = \{f \colon \mathbf{P}^1_Y \to X_Y \mid f \text{ is a morphism over } Y\}.$$

For the sake of brevity let us denote it H. It is known that H is an open subscheme of the Hilbert scheme $\operatorname{Hilb}(\mathbf{P}^1 \times X)$ (see [7], chapter I, theorem 1.10). In general H is not of finite type over $\mathbf{C}$, but every irreducible component of it is quasi-projective. The scheme H is equipped with an evident evaluation morphism $\operatorname{ev} \colon \mathbf{P}^1_H \to X$.

Similarly, we can introduce a Hom-scheme of morphisms of degree d:

$$\operatorname{Hom}_d(\mathbf{P}^1, X)(Y) = \{f \colon \mathbf{P}^1_Y \to X_Y \mid f \text{ is a morphism of degree } d \text{ over } Y\}.$$

This scheme is quasi-projective.

Proposition 22. *Let X be a smooth irreducible projective variety, H the* Hom-*scheme introduced earlier, and $\widetilde{H}$ the normalization of H. There exists a a well-defined family $g \colon \widetilde{R} \to \widetilde{H}$ of nonnegative proper algebraic cycles of X of dimension* 1*, which is universal in the sense that every irreducible rational curve $C \subset X$ occurs as a fiber of g over a suitable point of $\widetilde{H}$.*

Proof. H is representable, and so we get an universal morphism $f\colon \mathbf{P}^1_H \to X_H$. Pulling it back to $\widetilde{H}$ we obtain an universal morphism $f\colon \mathbf{P}^1_{\widetilde{H}} \to X_{\widetilde{H}}$. Let $G \subset \widetilde{H} \times \mathbf{P}^1 \times X$ be its graph. Take its fundamental cycle $[G]$. Theorem 3.18 from chapter I section 3 of [7] shows that $G \to \widetilde{H}$ together with coefficients provided by $[G]$ is a well-defined family of proper nonnegative 1-dimensional algebraic cycles of $\mathbf{P}^1 \times X$.

Let $p\colon \mathbf{P}^1 \times X \to X$ be the projection. Let $\widetilde{R} \to \widetilde{H}$ be the pushforward along p of the family $G \to \widetilde{H}$ (see [7], chapter I, section 6, definition 6.7). All rational curves in X occur as fibers of $\widetilde{R}$ because by construction its support is the image of the universal morphism $f\colon \mathbf{P}^1_{\widetilde{H}} \to X_{\widetilde{H}}$. ☐

There is no guarantee that the image of the universal morphism $f\colon \mathbf{P}^1_H \to X_H$ will be flat over H in general, and so we need to use Chow varieties instead of Hilbert schemes. The key difference between Hilb and Chow as functors from the category of schemes to itself is that Hilb is contravariant (pullbacks of flat families are flat), while Chow is covariant (pushforwards of families of cycles are families of cycles).

So, at this point we have a universal family of rational curves $\overline{R} \to \overline{H}$. Its irreducible components are only quasi-projective, and our next goal is to compactify it.

Proposition 23. *Let X be a smooth irreducible projective variety. There exists a normal scheme $\overline{H}$, and a well-defined family $\overline{g}\colon \overline{R} \to \overline{H}$ of nonnegative proper 1-dimensional algebraic cycles of X with the following properties:*

(a) The morphism $\overline{H} \to \mathrm{Chow}(X)$ induced by $\overline{R}$ is finite.

(b) Every irreducible rational curve $C \subset X$ occurs as some fiber of $\overline{g}$.

(c) Every fiber of $\overline{g}$ is a connected rational curve.

Proof. For the discussion below it is useful to recall that for every family of cycles g and every point w in the base of this family

$$\operatorname{supp} g^{[-1]}(w) = g^{-1}(w)_{\mathrm{red}}$$

(see [7], chapter I, corollary 3.16).

Let $\tau\colon \widetilde{H} \to \mathrm{Chow}(X)$ be the morphism induced by the family $\widetilde{R} \to \widetilde{H}$ from proposition 22. We construct $\overline{H}$ as the normalization of the closure of the image of τ. $\mathrm{Chow}(X)$ is locally noetherian, and so the condition (a) is satisfied. Let $\overline{R} \to \overline{H}$ be the family induced by this normalization morphism $\overline{h}\colon \overline{H} \to$

Chow(X). We have a commutative diagram

$$\begin{array}{ccccc} \widetilde{R} & \longrightarrow & \overline{R} & \longrightarrow & \mathrm{Univ}(X) \\ \Big\downarrow & & \Big\downarrow & & \Big\downarrow \\ \widetilde{H} & \xrightarrow{h} & \overline{H} & \xrightarrow{\overline{h}} & \mathrm{Chow}(X), \end{array}$$

where Univ(X) is the universal family over Chow(X), and $\tau = \overline{h} \circ h$. Notice that h is dominant.

Let $C \subset X$ be an irreducible rational curve. By proposition 22 there exists a closed point $r \in \widetilde{H}$, such that $C = \operatorname{supp} g^{[-1]}(r)$. Since τ factors as $\overline{h} \circ h$, we conclude that $C = \operatorname{supp} \overline{g}^{[-1]}(h(r))$, and so the condition (b) is satisfied.

Let $r \in \overline{H}$ be a point, and let $\xi \in \overline{H}$ be a generic point of some irreducible component which contains r. Since h is dominant, ξ is an image of some point of $\widetilde{H}$, and as earlier we conclude that $\operatorname{supp} \overline{g}^{[-1]}(\xi)$ is an irreducible rational curve.

Let A be a DVR with generic point η, and closed point $\mathfrak{m}$. Let $j\colon \operatorname{Spec} A \to \overline{H}$ a morphism which sends η to ξ, and $\mathfrak{m}$ to r. Let $g_A\colon Z \to \operatorname{Spec} A$ be the Chow pullback of $g\colon \overline{R} \to \overline{H}$ by j (this Chow pullback is just the reduced scheme-theoretic pullback of g by j with appropriate multiplicities of irreducible components; see [7], chapter I, section 3, definition 3.18)

Every irreducible component of Z maps onto $\operatorname{Spec} A$ since $g_A\colon Z \to \operatorname{Spec} A$ is a well-defined family of algebraic cycles. As Z is also reduced, we conclude that $g_A\colon Z \to \operatorname{Spec} A$ is flat. We already know that $\operatorname{supp} g^{[-1]}(\xi) = g^{-1}(\xi)_{\mathrm{red}}$ is an irreducible rational curve, and so ${g_A}^{-1}(\eta)$ is an irreducible rational curve too. Hence, proposition 21 implies, that $g_A^{-1}(\mathfrak{m})_{\mathrm{red}}$ is a connected rational curve, which in turn means that $g^{-1}(r)_{\mathrm{red}}$ is a connected rational curve. $\square$

The scheme $\overline{H}$ is not of finite type over $\mathbf{C}$ in general. Nevertheless, every irreducible component of $\overline{H}$ is projective over $\mathbf{C}$. Indeed, by construction of $\overline{H}$ there is a finite morphism $\overline{H} \to \mathrm{Chow}(X)$, and every irreducible component of Chow(X) is projective over $\mathbf{C}$.

The scheme Chow(X) is a disjoint union of projective varieties $\mathrm{Chow}_{d,d'}(X)$ parametrizing cycles of dimension d and degree d'. Hence $\overline{H}$ splits into a disjoint union of projective varieties $\overline{H}_d$ indexed by degrees of cycles. Restricions of $\overline{R}$ to $\overline{H}_d$ will be denoted $\overline{R}_d$.

To make dependency on X explicit we will also use notation $\overline{H}_d(X)$ and $\overline{R}_d(X)$.

3.2.3 Parametrization of rational equivalence

Next proposition shows that effective cycles are rationally equivalent iff they can be "continuously deformed" into one another, possibly after adding some auxillary effective cycle to both.

Proposition 24. *Let X be a smooth irreducible projective variety, d a nonnegative integer, and α, β effective 0-cycles of degree d in X. The cycles α and β are rationally equivalent if and only if there exists a nonnegative integer m, effective cycle γ of degree m, and a morphism $f\colon \mathbf{P}^1 \to X^{(d+m)}$, such that $f(0) = \alpha + \gamma$ and $f(\infty) = \beta + \gamma$ in $Z_0(X)$.*

Proof. Let $C_i \subset X$ be curves and f_i rational functions on C_i such that

$$\alpha - \beta = \sum_i \operatorname{div} f_i.$$

Let $\gamma' = \sum_i f_i^{-1}(0) - \alpha = \sum_i f_i^{-1}(\infty) - \beta$. Write $\gamma' = \gamma - \gamma''$ as a difference of effective cycles.

Let $\Gamma_i \subset X \times \mathbf{P}^1$ be the graph of f_i. We can view Γ_i as a correspondence $\Gamma_i\colon \mathbf{P}^1 \to X$. Let $p\colon X \times \mathbf{P}^1 \to X$ be the projection. Consider a correspondence $\Gamma\colon \mathbf{P}^1 \to X$ defined as

$$\Gamma = \sum_i \Gamma_i + p^*\gamma''.$$

By construction,

$$\Gamma_* 0 = \sum_i f_i^{-1}(0) + \gamma'' = \alpha + \gamma,$$
$$\Gamma_* \infty = \sum_i f_i^{-1}(\infty) + \gamma'' = \beta + \gamma.$$

From theorem 1 we know that Γ induces a morphism $f\colon \mathbf{P}^1 \to X^{(d+m)}$, such that $f(0) = \alpha + \gamma$ and $f(\infty) = \beta + \gamma$.

Conversely, if $f\colon \mathbf{P}^1 \to X^{(d+m)}$ is a morphism, and γ an effective cycle of degree m, such that $f(0) = \alpha + \gamma$ and $f(\infty) = \beta + \gamma$, then f induces a correspondence $\Gamma\colon \mathbf{P}^1 \to X$. From intersection theory we know, that $\Gamma_* 0 = \Gamma_* \infty$ in $CH_0(X)$, and so $\alpha = \beta$ in $CH_0(X)$ too. □

Let X be a smooth irreducible projective variety. In the following we will need to control degrees of cycles in all symmetric powers $X^{(d)}$ simultaneously. So, let us pick an ample line bundle $\mathcal{O}_X(1)$ on X, and construct from it line bundles on $X^{(d)}$ in the following way. Let $\pi_i\colon X^d \to X$ be the projections. Consider the line bundle $\mathcal{O}_{X^d}(1) = \bigotimes_{i=1}^d \pi_i^* \mathcal{O}_X(1)$ over X^d. It is ample (see proposition 50). Moreover, it is invariant under the natural action of symmetric group on X^d, and so descends to an ample line bundle on $X^{(d)}$ which we will denote $\mathcal{O}_{X^{(d)}}(1)$.

Proposition 25. *Let d_1, d_2 be nonnegative integers, and let $f_1\colon \mathbf{P}^1 \to X^{(d_1)}$, $f_2\colon \mathbf{P}^1 \to X^{(d_2)}$ be morphisms. Let $d = d_1 + d_2$. Consider a morphism $f_1 + f_2\colon \mathbf{P}^1 \to X^{(d)}$ which is a composition of (f_1, f_2) with an evident addition morphism $X^{(d_1)} \times X^{(d_2)} \to X^{(d)}$. In this situation*

$$(f_1 + f_2)^* \mathcal{O}_{X^{(d)}}(1) = f_1^* \mathcal{O}_{X^{(d_1)}}(1) \otimes f_2^* \mathcal{O}_{X^{(d_2)}}(1).$$

In other words the degree of $f_1 + f_2$ is the sum of degrees of f_i.

Proof. (Compare the proof of lemma 2 in [11]).

For $i = 1, 2$ let $q_i\colon X^{d_i} \to X^{(d_i)}$ be the quotient morphism, and S_i the fiber product $\mathbf{P}^1 \times_{X^{(d_i)}} X^{d_i}$ with reduced induced subscheme structure. We have a commutative diagram

$$\begin{array}{ccc} S_i & \longrightarrow & X^{d_i} \\ \big\downarrow & & \big\downarrow{\scriptstyle q_i} \\ \mathbf{P}^1 & \xrightarrow{f_i} & X^{(d_i)}. \end{array} \tag{1}$$

Let S be the fibered product $S_1 \times_{\mathbf{P}^1} S_2$ with reduced induced subscheme structure, and $p\colon S \to \mathbf{P}^1$ the projection. By construction, $p\colon S \to \mathbf{P}^1$ is finite and dominant. Replacing S by a closed subscheme we may assume that each irreducible component of S dominates $\mathbf{P}^1$, and so p is flat.

Let $\widetilde{f_i}$ be the composition of the projection $S \to S_i$ with the morphism $S_i \to X^{d_i}$ from (1). For each $i = 1, 2$ there is a commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{\widetilde{f_i}} & X^{d_i} \\ \big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle q_i} \\ \mathbf{P}^1 & \xrightarrow{f_i} & X^{(d_i)}. \end{array} \tag{2}$$

Consider a commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{(\widetilde{f_1}, \widetilde{f_2})} & X^{d} \\ \big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle q} \\ \mathbf{P}^1 & \xrightarrow{f_1 + f_2} & X^{(d)}, \end{array} \tag{3}$$

where q is the quotient morphism. Let $p_i\colon X^d \to X^{d_i}$ be the projections of $X^d = X^{d_1} \times X^{d_2}$ to respective factors. By construction,

$$\mathcal{O}_{X^d}(1) = p_1^* \mathcal{O}_{X^{d_1}}(1) \otimes p_2^* \mathcal{O}_{X^{d_2}}(1),$$

and so

$$(\widetilde{f_1}, \widetilde{f_2})^* \mathcal{O}_{X^d}(1) = \widetilde{f}_1^* \mathcal{O}_{X^{d_1}}(1) \otimes \widetilde{f_2}^* \mathcal{O}_{X^{d_2}}(1).$$

On the other hand, the diagram (3) commutes, and so

$$p^*(f_1+f_2)^*\mathcal{O}_{X^{(d)}}(1) = (\widetilde{f}_1,\widetilde{f}_2)^* q^*\mathcal{O}_{X^{(d)}}(1) = (\widetilde{f}_1,\widetilde{f}_2)^*\mathcal{O}_{X^d}(1).$$

Diagrams (2) show that

$$\widetilde{f}_i^*\mathcal{O}_{X^{d_i}}(1) = \widetilde{f}_i^* q_i^*\mathcal{O}_{X^{(d_i)}}(1) = p^* f_i^*\mathcal{O}_{X^{(d_i)}}(1).$$

Hence

$$p^*(f_1+f_2)^*\mathcal{O}_{X^{(d)}}(1) = p^*(f_1^*\mathcal{O}_{X^{(d_1)}}(1)\otimes f_2^*\mathcal{O}_{X^{(d_2)}}(1)).$$

The morphism p is finite and flat by construction, and so proposition 52 shows that the kernel of $p^*\colon \operatorname{Pic}\mathbf{P}^1 \to \operatorname{Pic} S$ is torsion. The group $\operatorname{Pic}\mathbf{P}^1 = \mathbf{Z}$ has no torsion, so p^* is injective, and

$$(f_1+f_2)^*\mathcal{O}_{X^{(d)}}(1) = f_1^*\mathcal{O}_{X^{(d_1)}}(1)\otimes f_2^*\mathcal{O}_{X^{(d_2)}}(1). \quad \square$$

Next theorem is the main result of this section. It was taken from the article of Mumford [11] (see lemma 3).

Theorem 2. *Let X be a smooth irreducible projective variety, and d a nonnegative integer. Consider the subset $R \subset (X^{(d)}\times X^{(d)})(\mathbf{C})$:*

$$R = \{(\alpha,\beta) \in (X^{(d)}\times X^{(d)})(\mathbf{C}) \mid \alpha = \beta \text{ in } CH_0(X)\}.$$

There exists a countable collection $\{R_i\}_{i=1}^{\infty}$ of closed irreducible subvarieties $R_i \subset X^{(d)}\times X^{(d)}$, which satisfies two conditions:

(a) $R = \bigcup_i R_i(\mathbf{C})$.

(b) For each R_i there exist a nonnegative integer $m = m(i)$ depending only on i, a quasi-projective variety T, a surjective morphism $e\colon T \to R_i$, and morphisms $g\colon T \to X^{(m)}$, $h\colon T\times\mathbf{P}^1 \to X^{(d+m)}$, such that for every $t \in T(\mathbf{C})$

$$h(t,0) = g(t) + q_1(e(t)),$$
$$h(t,\infty) = g(t) + q_2(e(t)),$$

where $0,\infty \in \mathbf{P}^1(\mathbf{C})$ are the closed points, $q_1, q_2\colon X^{(d)}\times X^{(d)} \to X^{(d)}$ the projections, and the plus sing denotes the addition morphism $X^{(m)}\times X^{(d)} \to X^{(d+m)}$.

Proof. Let $p, k \geqslant 0$ be integers.

In the previous section for every smooth irreducible projective variety Y and index p we constructed projective varieties $\overline{H}_p(Y)$ which are described by proposition 23. In this proof symbols $\overline{H}_{p,k}$ will denote the varieties $\overline{H}_p(X^{(d+k)})$, and symbols $\overline{R}_{p,k}$ will denote the families of cycles $\overline{R}_p(X^{(d+k)})$ over $\overline{H}_p(X^{(d+k)})$. Let $g_{p,k}\colon \overline{R}_{p,k} \to \overline{H}_{p,k}$ be corresponding projections.

By definition, $\overline{R}_{p,k}$ is a closed subscheme of $\overline{H}_{p,k} \times X^{(d+k)}$, and

$$\overline{R}_{p,k}(\mathbf{C}) = \{(r, \alpha) \in (\overline{H}_{p,k} \times X^{(d+k)})(\mathbf{C}) \mid \alpha \in g_{p,k}^{-1}(r)(\mathbf{C})\}.$$

Therefore the subset

$$\{(r, \alpha, \beta, \gamma) \in (\overline{H}_{p,k} \times X^{(d)} \times X^{(d)} \times X^{(k)})(\mathbf{C}) \mid \alpha + \gamma, \beta + \gamma \in g_{p,k}^{-1}(r)(\mathbf{C})\}$$

is a set of $\mathbf{C}$-points of a closed subscheme of $\overline{H}_{p,k} \times X^{(d)} \times X^{(d)} \times X^{(k)}$. Projecting this subscheme down to $X^{(d)} \times X^{(d)}$ we obtain a closed subscheme $Z_{p,k}$ with the following property: $(\alpha, \beta) \in Z_{p,k}(\mathbf{C})$ if and only if there exists such $\gamma \in X^{(k)}(\mathbf{C})$ and $r \in \overline{H}_{p,k}(\mathbf{C})$, that $\alpha + \gamma, \beta + \gamma \in g_{p,k}^{-1}(r)(\mathbf{C})$.

Since each fiber of $\overline{R}_{p,k}$ is a connected rational curve, and since all rational curves in $X^{(d+k)}$ occur in the families $\overline{R}_{p,k}$ for different p, proposition 24 shows, that

$$R = \Delta(\mathbf{C}) \cup \bigcup_{p,k} Z_{p,k}(\mathbf{C}).$$

Hence, to condition (a) is satisfied if we will take R_i to be irreducible components of the $Z_{p,k}$, and Δ.

It remains to verify that the condition (b) also holds for R_i.

Let $(\alpha, \beta) \in R_i(\mathbf{C})$. By construction, there exists $r \in \overline{H}_{p,k}$, and $\gamma \in X^{(k)}$, such that $\alpha + \gamma, \beta + \gamma \in g_{p,k}^{-1}(r)(\mathbf{C})$. Let us write $g_{p,k}^{[-1]}(r)$ as a sum of irreducible components:

$$g_{p,k}^{[-1]}(r) = \sum_{i=1}^{l} k_i C_i,$$

where $l \leqslant p$. Let $f_i \colon \mathbf{P}^1 \to X$ be the normalization of C_i. The degree of f_i is equal to $\deg(C_i \cdot \mathcal{O}_{X^{(d+k)}})$.

Without loss of generality we may assume that $f_1(0) = \alpha + \gamma$ and $f_l(\infty) = \beta + \gamma$. Since $g_{p,k}^{[-1]}(r)$ is connected, we can also assume that $f_i(\infty) = f_{i+1}(0)$ for every $i = 1, \ldots, l-1$, after decreasing l if necessary. Let $\Gamma_i \colon \mathbf{P}^1 \to X$ be correspondences induced by f_i, and let $\Gamma = \sum_{i=1}^{l} \Gamma_i$. By construction,

$$\Gamma_* 0 = \alpha + \gamma + \sum_{i=2}^{l} f_i(0),$$

$$\Gamma_* \infty = \sum_{i=1}^{l-1} f_i(\infty) + \beta + \gamma.$$

Because $f_i(0) = f_{i-1}(\infty)$ for every $i = 2, \ldots, l$, we have an equality

$$\sum_{i=2}^{l} f_i(0) = \sum_{i=1}^{l-1} f_i(\infty).$$

Let $\gamma_1 = \gamma + \sum_{i=2}^{l} f_i(0)$, and let $f\colon \mathbf{P}^1 \to X^{(d+k+(l-1)(d+k))}$ be the morphism induced by Γ. As we have computed,

$$f(0) = \alpha + \gamma_1,$$
$$f(\infty) = \beta + \gamma_1.$$

Moreover, proposition 25 implies that f has degree at most p. By adding constant correspondences to Γ, we can assume that f takes values in $X^{(p(d+k))}$. This operation does not change the degree of f.

Therefore, we have demonstrated that for $m = p(d+k) - d$, and for every $(\alpha, \beta) \in R_i(\mathbf{C})$ there exists $\gamma \in X^{(m)}$, and $f\colon \mathbf{P}^1 \to X^{(d+m)}$ of degree at most p, such that $f(0) = \alpha + \gamma$, and $f(\infty) = \beta + \gamma$.

Let $T_0 = \coprod_{j=1}^{p} \operatorname{Hom}_j(\mathbf{P}^1, X^{(d+m)})$. By construction, T_0 is equipped with the evaluation map $\operatorname{ev}\colon T_0 \times \mathbf{P}^1 \to X^{(d+m)}$. Let $\pi\colon X^{(d)} \times X^{(m)} \to X^{(d+m)}$ be the addition morphism. Let $i_0, i_\infty\colon T_0 \to T_0 \times \mathbf{P}^1$ be the inclusions of T_0 as a closed subscheme which projects onto the closed point 0 or $\infty \in \mathbf{P}^1$ respectively. Let

$$q_t\colon T_0 \times X^{(d)} \times X^{(d)} \times X^{(m)} \to T_0,$$
$$q_a\colon T_0 \times X^{(d)} \times X^{(d)} \times X^{(m)} \to X^{(d)},$$
$$q_b\colon T_0 \times X^{(d)} \times X^{(d)} \times X^{(m)} \to X^{(d)},$$
$$q_c\colon T_0 \times X^{(d)} \times X^{(d)} \times X^{(m)} \to X^{(m)},$$

be the projections to respective factors. Let

$$u_1\colon T_0 \times X^{(d)} \times X^{(d)} \times X^{(m)} \to X^{(d+m)} \times X^{(d+m)},$$
$$u_2\colon T_0 \times X^{(d)} \times X^{(d)} \times X^{(m)} \to X^{(d+m)} \times X^{(d+m)},$$

be morphisms which are defined as

$$u_1 = (\operatorname{ev} \circ i_0 \circ q_t, \pi \circ (q_a, q_c)),$$
$$u_2 = (\operatorname{ev} \circ i_\infty \circ q_t, \pi \circ (q_b, q_c)).$$

For example, if $(t, \alpha, \beta, \gamma)$ is a closed point, then $u_1(t, \alpha, \beta, \gamma) = (\operatorname{ev}(t, 0), \alpha + \gamma)$.

Consider the diagonal subscheme $\Delta \subset X^{(d+m)} \times X^{(d+m)}$ and let

$$T_1 = u_1^{-1}\Delta \cap u_2^{-1}\Delta.$$

If $(t, \alpha, \beta, \gamma)$ is a $\mathbf{C}$-point of $T_0 \times X^{(d)} \times X^{(d)} \times X^{(m)}$, then it belongs to $T_1(\mathbf{C})$ if and only if $\operatorname{ev}(t, 0) = \alpha + \gamma$ and $\operatorname{ev}(t, \infty) = \beta + \gamma$.

By construction, T_1 is equipped with a projection $e_1\colon T_1 \to X^{(d)} \times X^{(d)}$, $e_1 = (q_a, q_b)|_{T_1}$. As we have demonstrated earlier, $R_i(\mathbf{C}) \subset e_1(T_1(\mathbf{C}))$. Moreover, there is a morphism $g_1\colon T_1 \to X^{(m)}$, $g_1 = q_c|_{T_1}$, and a morphism $h_1\colon T_1 \times \mathbf{P}^1 \to X^{(d+m)}$, $h_1 = \operatorname{ev} \circ (q_t|_{T_1} \times \operatorname{id})$.

Consider a cartesian square

$$\begin{array}{ccc} T & \longrightarrow & T_1 \\ \Big\downarrow e & & \Big\downarrow e_1 \\ R_i & \longrightarrow & X^{(d)} \times X^{(d)}. \end{array}$$

The map $e(\mathbf{C})\colon T(\mathbf{C}) \to R_i(\mathbf{C})$ is surjective by construction. Using suitable compositions with the morphism $T \to T_1$, we also get morphisms $g\colon T \to X^{(m)}$ and $h\colon T \times \mathbf{P}^1 \to X^{(d+m)}$ which satisfy relevant equations of condition (b).

Finally, we can replace T by T_{red}, so that it becomes a quasi-projective variety. As it changes no relevant properties, we conclude that the condition (b) is also satisfied. $\square$

3.3 Fibers of induced maps

We are now ready to study maps $\Gamma_*\colon W \to CH_0(X)$ induced by correspondences $\Gamma\colon W \to X$. As we will see their properties resemble closely the relevant properties of morphisms of projective varieties.

Proposition 26. *Let X be a smooth irreducible projective variety, and d a nonnegative integer. Consider the map $\sigma_{d,d}\colon X^{(d)} \times X^{(d)} \to CH_0(X)$. The fiber $\sigma_{d,d}^{-1}(0)$ is a union of at most countably many Zariski-closed subsets.*

Proof. Follows directly from theorem 2, condition (a). $\square$

Proposition 27. *Let W be a smooth irreducible quasi-projective variety, X a smooth irreducible projective variety, and $\Gamma\colon W \to X$ a correspondence. Consider the map $\Gamma_*\colon W \to CH_0(X)$. The fiber $\Gamma_*^{-1}(0)$ is a union of at most countably many Zariski-closed subsets.*

Proof. Without loss of generality we may assume that Γ has no vertical components, since they do not contribute to Γ_*. Write $\Gamma = \Gamma^+ - \Gamma^-$ as a difference of effective correspondences. Let $f^+\colon W \to X^{(d_+)}$ and $f^-\colon W \to X^{(d_-)}$ be the rational maps induced by these correspondences.

If $d_+ \neq d_-$, then evidently $\Gamma_*^{-1}(0) = \varnothing$, so assume that $d_+ = d_- = d$ for some d. Let $G \subset W \times X^{(d)} \times X^{(d)}$ be the closure of graph of the rational map $(f^+ \times f^-)\colon W \to X^{(d)} \times X^{(d)}$. Let $p\colon G \to W$ and $q\colon G \to X^{(d)} \times X^{(d)}$.

Let $w \in W$ be a point. From theorem 1 (more precisely, from proposition 19) we know, that $\Gamma_* w = 0$ if and only if $q(p^{-1}(w)) \subset \sigma_{d,d}^{-1}(0)$. Hence, $\Gamma_* w = 0$ iff $w \in p(q^{-1}(\sigma_{d,d}^{-1}(0)))$. The projection p is proper, and so $p(q^{-1}(\sigma_{d,d}^{-1}(0)))$ is a union of at most countably many Zariski-closed subsets. $\square$

Proposition 28. *Let W be a smooth irreducible quasi-projective variety, X a smooth irreducible projective variety, $\Gamma\colon W \to X$ a correspondence, and $z \in CH_0(X)$ a cycle. Consider the map $\Gamma_*\colon W \to CH_0(X)$. The fiber $\Gamma_*^{-1}(z)$ is a union of at most countably many Zariski-closed subsets.*

Proof. If z does not belong to the image of Γ_*, then there is nothing to prove. So, assume that $z = \Gamma_* w$ for some $w \in W$.

Let $p_1, p_2\colon W^2 \to W$ be projections to respective factors. Consider a correspondence $Z = \Gamma \circ p_1 - \Gamma \circ p_2$. Proposition 10 shows, that it sends $(w_1, w_2) \in W^2$ to $\Gamma_* w_1 - \Gamma_* w_2$. Let $R = \ker(Z_*) \subset W^2$. By proposition 27, R is a union of at most countably many Zariski-closed subsets. The fiber of Γ_* over z is $p_2(R \cap \{w\} \times W)$, and so it is a union of at most countably many Zariski-closed subsets too. $\square$

In particular for every $w \in W$ the dimension of the fiber of Γ_* over $\Gamma_* w$ is well-defined. Next proposition says that in some sense the fiber dimension is an upper semi-continuous function.

Proposition 29. *Let W be a smooth irreducible quasi-projective variety, X a smooth irreducible projective variety, $\Gamma\colon W \to X$ a correspondence, and $k \geqslant -1$ an integer. Consider the map $\Gamma_*\colon W \to CH_0(X)$ and a set*

$$V = \{w \in W \mid \dim W_{\Gamma_* w} \leqslant k\}$$

The set V is an intersection of at most countably many Zariski-open subsets.

Proof. As in the proof of proposition 27, let $p_1, p_2\colon W^2 \to W$ be the projections, let $Z = \Gamma \circ p_1 - \Gamma \circ p_2$, and let $R = \ker(Z_*)$. The fiber of Γ_* over $\Gamma_* w$ is $p_2(\{w\} \times W \cap R)$.

As we know, $R = \bigcup_i R_i$, where each R_i is a Zariski-closed subset. Let $q_i = p_1|_{R_i}$ and let

$$V_i = \{w \in W \mid \dim q_i^{-1}(w) \leqslant k\}.$$

By Chevalley's theorem, V_i is a Zariski-open subset of W. But clearly $V = \bigcap_i V_i$. $\square$

4 Factorization theorem

Let X be a smooth irreducible projective variety. In general it is difficult to compute the group $CH_0(X)_{\mathrm{hom}}$. A rare example of such computation is provided by next theorem, due to Roitman [13]:

Let X be a smooth irreducible projective variety. If there exists such a smooth projective variety W and a correspondence $\Gamma\colon W \to CH_0(X)$ that $\Gamma_* W =$

$CH_0(X)_{\mathrm{hom}}$, *then the Albanese homomorphism* $\mathrm{alb}\colon CH_0(X)_{\mathrm{hom}} \to \mathrm{Alb}\, X$ *is an isomorphism.*

This theorem was generalized by Voisin in [14]:

Let M, X be smooth irreducible projective varieties, and $Z\colon M \to X$ a correspondence. If there exists a smooth projective variety W, and a correspondence $\Gamma\colon W \to X$ such that $Z_*(CH_0(M)_{\mathrm{hom}}) \subset \Gamma_* W$, then the homomorphism $Z_*\colon CH_0(M)_{\mathrm{hom}} \to CH_0(X)$ factors through the Albanese torus $\mathrm{Alb}\, M$.

Our goal is to present a proof of this theorem, following [14].

4.1 Albanese varieties

We sketch briefly some basic facts about Albanese varieties.

Proposition 30. *Let X be a smooth irreducible projective variety of dimension n. Consider a map $H_1(X, \mathbf{Z}) \to H^{1,0}(X)^\vee$ which sends a homology class α to the integration operator $\int_\alpha\colon H^{1,0}(X) \to \mathbf{C}$. This map is well-defined, and fits into a commutative diagram*

$$\begin{array}{ccc} H_1(X, \mathbf{Z}) & \longrightarrow & H^{1,0}(X)^\vee \\ \uparrow & & \uparrow \\ H^{2n-1}(X, \mathbf{Z}) & \longrightarrow & H^{n-1,n}(X), \end{array} \tag{4}$$

where the left vertical arrow is a Poincaré duality isomorphism, the right vertical arrow is a Serre duality isomorphism, and the lower horizontal arrow is the natural projection.

Proof. For well-definedness recall that by $\partial\overline{\partial}$-lemma every global holomorphic form is closed, and so Stokes theorem shows that $\int_\alpha$ depends only on homology class. Next, decompose the map $H_1(X, \mathbf{Z}) \to H^{1,0}(X)^\vee$ as

$$H_1(X, \mathbf{Z}) \to H_1(X, \mathbf{C}) \to H^1(X, \mathbf{C})^\vee \to H^{1,0}(X)^\vee,$$

where the first map is the change of coefficients homomorphism, the second map is induced by pairing of homology and de Rham cohomology classes by integration, and the last map is the projection induced by Hodge decomposition $H^1(X, \mathbf{C}) = H^{1,0}(X) \oplus H^{0,1}(X)$.

Poincaré duality isomorphisms between homology and cohomology groups are natural with respect to change of coefficients, and so we have a commutative diagram:

$$\begin{array}{ccc} H_1(X, \mathbf{Z}) & \longrightarrow & H_1(X, \mathbf{C}) \\ \uparrow & & \uparrow \\ H^{2n-1}(X, \mathbf{Z}) & \longrightarrow & H^{2n-1}(X, \mathbf{C}). \end{array}$$

Also, cup product Poincaré duality isomorphism $H^{2n-1}(X,\mathbf{C}) \to H^1(X,\mathbf{C})^\vee$ splits as

$$H^{2n-1}(X,\mathbf{C}) \longrightarrow H_1(X,\mathbf{C}) \longrightarrow H^1(X,\mathbf{C})^\vee,$$

where the first arrow is the homology-cohomology Poincaré duality isomorphism, and the second one is the isomorphism provided by evaluation of cohomology classes on homology classes, or equivalently the de Rham integration pairing. To show that the diagram (4) commutes it remains to notice that cup product Poincaré duality is compatible with Serre duality on groups $H^{p,q}(X)$. $\square$

Thus $H_1(X,\mathbf{Z})$ modulo torsion can be viewed as a lattice in $H^{1,0}(X)^\vee$. Quotient by this lattice is a complex torus which is called the Albanese variety of X and denoted $\operatorname{Alb} X$. Equivalently, one can describe $\operatorname{Alb} X$ as a quotient of $H^{n-1,n}(X)$ by the group $H^{2n-1}(X,\mathbf{Z})$ considered as a lattice under natural projection map $H^{2n-1}(X,\mathbf{Z}) \to H^{n-1,n}(X)$ induced by Hodge decomposition. It is known that $\operatorname{Alb} X$ is always projective, i.e. it is an abelian variety.

Fixing a closed point $x_0 \in X$ we can construct a morphism $\operatorname{alb}_{x_0}\colon X \to \operatorname{Alb} X$ as follows. Let $x \in X$ be a closed point, and let $\gamma \in C_1(X,\mathbf{Z})$ be a smooth 1-chain such that $\delta(\gamma) = x - x_0$. Consider an integration operator $\int_\gamma\colon H^{1,0}(X) \to \mathbf{C}$, and set $\operatorname{alb}_{x_0}(x) = \int_\gamma$. If γ' is another chain such that $\delta(\gamma) = x - x_0$, then $\gamma - \gamma'$ is a cycle, and so $\int_\gamma$ is equal to $\int_{\gamma'}$ modulo the lattice $H_1(X,\mathbf{Z}) \subset H^{1,0}(X)^\vee$. Thus, the map alb_{x_0} is well-defined. It is known that it is a morphism of algebraic varieties. By adding several copies of alb_{x_0} together we obtain a morphism $\operatorname{alb}_{x_0}\colon X^k \to \operatorname{Alb} X$ for every $k \geqslant 0$. Since alb_{x_0} is invariant with respect to the natural action of symmetric group on X^k, we also get a morphism $\operatorname{alb}_{x_0}\colon X^{(k)} \to \operatorname{Alb} X$. For k large enough $\operatorname{alb}_{x_0}\colon X^k \to \operatorname{Alb} X$ is known to be surjective.

Similarly, we can define a morphism $\operatorname{alb}\colon X^k \times X^k \to \operatorname{Alb} X$ by sending a point $(x_1, x_2) \in X^k \times X^k$ to $\operatorname{alb}_{x_0}(x_1) - \operatorname{alb}_{x_0}(x_2)$. Consider the case $k = 1$. Let $\gamma_1, \gamma_2 \in C_1(X,\mathbf{Z})$ be such 1-chains that $\delta(\gamma_1) = x_1 - x_0$, $\delta(\gamma_2) = x_2 - x_0$. By construction, $\operatorname{alb}(x_1, x_2) = \int_{\gamma_1} - \int_{\gamma_2} = \int_\gamma$, where $\gamma \in C_1(X,\mathbf{Z})$ is such a chain that $\delta(\gamma) = x_1 - x_2$. Therefore, $\operatorname{alb}\colon X^k \times X^k \to \operatorname{Alb} X$ does not depend on the choice of base point. The same argument shows that there is a well-defined group homomorphism $\operatorname{alb}\colon Z_0(X)_{\mathrm{hom}} \to \operatorname{Alb} X$.

Proposition 31. *Let X be a smooth irreducible projective variety. The Albanese homomorphism $\operatorname{alb}\colon Z_0(X)_{\mathrm{hom}} \to \operatorname{Alb} X$ factors through $Z_0(X)_{\mathrm{rat}}$, and descends to a homomorphism $\operatorname{alb}\colon CH_0(X)_{\mathrm{hom}} \to \operatorname{Alb} X$.*

Proof. Let $\alpha - \beta \in Z_0(X)_{\mathrm{hom}}$ be a cycle, which is rationally equivalent to 0. Let $d = \deg\alpha = \deg\beta$. By assumption there exists an effective cycle $\gamma \in Z_0(X)$ of some degree m, and a morphism $f\colon \mathbf{P}^1 \to X^{(d+m)}$, such that $f(0) = \alpha + \gamma$ and $f(\infty) = \beta + \gamma$. Let $x_0 \in X$ be a point, and consider a composition $\operatorname{alb}_{x_0} \circ f\colon \mathbf{P}^1 \to \operatorname{Alb} X$. Every morphism from $\mathbf{P}^1$ to abelian variety is constant, and so $\operatorname{alb}_{x_0}(\alpha+\gamma) = \operatorname{alb}_{x_0}(\beta+\gamma)$, which in turn means that $\operatorname{alb}(\alpha-\beta) = 0$. $\square$

When X is a curve, Abel-Jacobi theorem shows that $\mathrm{alb}\colon CH_0(X)_{\mathrm{hom}} \to \mathrm{Alb}\, X$ induces an isomorphism between the Jacobian of X and its Albanese variety.

Proposition 32. *Let $f\colon X \to Y$ be a morphism of smooth irreducible projective varieties of dimensions n and m respectively. Both diagrams*

$$\begin{array}{ccc} H_1(X,\mathbf{Z}) & \xrightarrow{f_*} & H_1(Y,\mathbf{Z}) \\ \big\downarrow & & \big\downarrow \\ H^{1,0}(X)^\vee & \xrightarrow{(f^*)^\vee} & H^{1,0}(Y)^\vee, \end{array} \qquad \begin{array}{ccc} H^{2n-1}(X,\mathbf{Z}) & \xrightarrow{f_*} & H^{2m-1}(X,\mathbf{Z}) \\ \big\downarrow & & \big\downarrow \\ H^{n-1,n}(X) & \xrightarrow{f_*} & H^{m-1,m}(X) \end{array}$$

commute and are Poncaré duals of each other. Consequently, the morphism f induces a functorial morphism $\mathrm{Alb}(f)\colon \mathrm{Alb}\, X \to \mathrm{Alb}\, Y$.

Proof. Let $\gamma \in H_1(X,\mathbf{Z})$ be a cycle. Clearly, $\int_\gamma \circ f^* = \int_{f_*\gamma}$, and so the left diagram commutes and induces a morphism $\mathrm{Alb}(f)\colon \mathrm{Alb}\, X \to \mathrm{Alb}\, Y$. The right diagram is a Poincaré dual of the left one by definition of cohomology pushforwards. Functoriality of $\mathrm{Alb}(f)$ is evident. $\square$

Proposition 33. *Let $f\colon X \to Y$ be a morphism of smooth irreducible projective varieties. It induces a commutative diagram:*

$$\begin{array}{ccc} CH_0(X)_{\mathrm{hom}} & \xrightarrow{f_*} & CH_0(Y)_{\mathrm{hom}} \\ \big\downarrow{\scriptstyle \mathrm{alb}_X} & & \big\downarrow{\scriptstyle \mathrm{alb}_Y} \\ \mathrm{Alb}(X) & \xrightarrow{\mathrm{Alb}(f)} & \mathrm{Alb}(Y). \end{array}$$

Proof. Let $x_1, x_2 \in X$ be points, and $\gamma \in C_1(X,\mathbf{Z})$ such a chain that $\delta(\gamma) = x_1 - x_2$. By construction $\mathrm{Alb}(f)$ sends a point corresponding to the operator $\int_\gamma$ to $\int_\gamma \circ f^* = \int_{f_*\gamma}$, which in turn is the image of $f_*(x_1) - f_*(x_2) \in Z_0(Y)$ under the Albanese morphism alb_Y. $\square$

Proposition 34. *Let X be a smooth irreducible projective variety of dimension $n \geqslant 2$, $x \in X$ a point, and $\pi\colon \widetilde{X} \to X$ a blowup of X at x. The morphism $\mathrm{Alb}(\pi)\colon \mathrm{Alb}\, \widetilde{X} \to \mathrm{Alb}\, X$ is an isomorphism.*

Proof. Let $U = X\backslash\{x\}$. Consider relative homology groups $H_i(X,U,\mathbf{Z})$. By excision $H_i(X,U,\mathbf{Z}) = H_i(\mathbf{C}^n, \mathbf{C}^n\backslash\{0\}, \mathbf{Z})$. The pair $(\mathbf{C}^n, \mathbf{C}^n\backslash\{0\})$ is a deformation retract of the pair $(\mathbf{C}^n, U')$ where U' is the complement of the standard closed unit ball in $\mathbf{C}^n$. Hence, $H_i(X,U,\mathbf{Z}) \cong H_i(S^{2n},\mathbf{Z})$. Since $n \geqslant 2$, we conclude that $H_1(X,U,\mathbf{Z}) = H_2(X,U,\mathbf{Z}) = 0$. Long exact sequence of the pair (X,U) shows that $H_1(U,\mathbf{Z}) \to H_1(X,\mathbf{Z})$ is an isomorphism.

Consider a commutative diagram

$$\begin{array}{ccc} H_1(\pi^{-1}U, \mathbf{Z}) & \xrightarrow{\pi_*} & H_1(U, \mathbf{Z}) \\ \downarrow & & \downarrow \\ H_1(\widetilde{X}, \mathbf{Z}) & \xrightarrow{\pi_*} & H_1(X, \mathbf{Z}). \end{array}$$

The upper horizontal arrow is an isomorphism, and the right vertical arrows is an isomorphism too as we demonstrated above. Thus, $H_1(\widetilde{X}, \mathbf{Z}) \to H_1(X, \mathbf{Z})$ is surjective.

Consider the pullback morphism $\pi^*\colon H^{1,0}(X) \to H^{1,0}(\widetilde{X})$. It is injective. We want to show that $h^{1,0}(X) = h^{1,0}(\widetilde{X})$. Consider a composition of maps

$$H^0(\widetilde{X}, \Omega^1_{\widetilde{X}}) \to H^0(\pi^{-1}U, \Omega^1_{\pi^{-1}U}) \to H^0(U, \Omega^1_U) \to H^0(X, \Omega^1_X), \tag{5}$$

where the first map is restriction, the second one is induced by isomorphism $\pi\colon \pi^{-1}U \to U$, and the last one is extension, which is well-defined because $X \backslash U$ is of codimension at least 2. The map (5) is clearly injective, and so $h^{1,0}(X) = h^{1,0}(\widetilde{X})$. Thus the pullback π^* is an isomorphism.

Consider a commutative square

$$\begin{array}{ccc} H_1(\widetilde{X}, \mathbf{Z}) & \xrightarrow{\pi_*} & H_1(X, \mathbf{Z}) \\ \downarrow & & \downarrow \\ H^{1,0}(\widetilde{X})^\vee & \xrightarrow{(\pi^*)^\vee} & H^{1,0}(X)^\vee. \end{array}$$

We know that the vertical arrows are injective modulo torsion, the lower horizontal arrow is an isomorphism and the upper horizontal arrow is surjective. Hence it is also injective modulo torsion, and so $\mathrm{Alb}(\pi)$ is an isomorphism. $\square$

Proposition 35. *Let X be a smooth irreducible projective variety, $x \in X$ a point, and $\pi\colon \widetilde{X} \to X$ a blowup of X at x. The morphism $\pi_*\colon CH_0(\widetilde{X}) \to CH_0(X)$ is an isomorphism.*

Proof. The morphism π_* is evidently surjective, so it is enough to prove its injectivity. Let $z \in Z_0(\widetilde{X})$ be such a cycle, that

$$\pi_* z = \sum_{i=1}^{n} \operatorname{div} f_i,$$

where f_i's are rational functions supported on curves $C_i \subset X$. Let $\widetilde{C_i}$'s be strict transforms of these curves and $\widetilde{f_i}$'s corresponding rational fuctions. Consider a cycle

$$z' = z - \sum_{i=1}^{n} \operatorname{div} \widetilde{f_i}.$$

By construction, $\pi_* z' = 0$, and hence $\deg z' = 0$. Moreover, z' is concentrated on the exceptional divisor of the blowup. Since this divisor is a projective space, we conclude that z' and hence z is rationally equivalent to zero. □

In fact both Alb and CH_0 are birational invariants, but we will not need it in such generality.

Proposition 36. *Let C be a smooth irreducible projective curve, and let* $\operatorname{Alb} C$ *be its Albanese variety. There exists a correspondence $A\colon \operatorname{Alb} C \to C$, such that $A_* x = \operatorname{alb}^{-1}(x)$.*

Proof. Let $c_0 \in C$ be a point, and let g be the genus of C. Consider the Albanese morphism $\operatorname{alb}_{c_0}\colon C^g \to \operatorname{Alb} C$. It is known that this morphism is birational and surjective. Notice, that

$$\operatorname{alb}_{c_0}(c_1, \ldots, c_g) = \operatorname{alb}(c_1 + \ldots + c_g - gc_0).$$

Let $f\colon \operatorname{Alb} C \to C^g$ be the inverse rational map. It induces a correspondence $Z\colon \operatorname{Alb} C \to C$. Let $Z_0\colon \operatorname{Alb} C \to C$ be the constant correspondence sending each point of $\operatorname{Alb} C$ to c_0. Consider a correspondence $A = Z - gZ_0$. By construction, $Z_* x$ is such an effective cycle $z \in CH_0(C)$ of degree g, that $x = \operatorname{alb}(z - gc_0)$. Hence, $A_* x = z - gc_0 = \operatorname{alb}^{-1}(x)$. □

Proposition 37. *Let X be a smooth irreducible projective variety, $Y \subset X$ a smooth irreducible hyperplane section. If $\dim X > 2$, then the map $\operatorname{Alb} Y \to \operatorname{Alb} X$ is an isomorphism. If $\dim X = 2$, then it is a surjection and the kernel is connected, i.e. it is an abelian variety.*

Proof. Let $n = \dim X$. By Lefschetz hyperplane section theorem the maps $H_k(Y, \mathbf{Z}) \to H_k(X, \mathbf{Z})$ are isomorphisms when $k < n - 1$ and surjections when $k = n - 1$. Similarly, the maps $H^k(X, \mathbf{C}) \to H^k(Y, \mathbf{C})$ are isomorphisms when $k < n - 1$ and injections when $k = n - 1$. So, if $n > 2$ then the map $\operatorname{Alb} Y \to \operatorname{Alb} X$ is an isomorphism.

In case $n = 1$ consider a commutative diagram

$$\begin{array}{ccc} H_1(Y, \mathbf{Z}) & \longrightarrow & H_1(X, \mathbf{Z}) \\ \downarrow & & \downarrow \\ H^{1,0}(Y)^\vee & \longrightarrow & H^{1,0}(X)^\vee \end{array}$$

The upper horizontal arrow is surjective. Thus, if an element $y \in H^{1,0}(Y)^\vee$ is mapped into $H_1(X, \mathbf{Z})$, then there exists such $\alpha \in H_1(Y, \mathbf{Z})$ that $y - \alpha$ is mapped exactly to zero. □

Notice that a similar claim for CH_0 is completely wrong.

Another well-known property of Albanese varieties which we will not need but have to mention is that every morphism from a smooth irreducible projective variety X to a complex torus factors through $\operatorname{Alb} X$.

4.2 Simple kernels

Let X be a smooth irreducible projective surface, and $C \subset X$ a smooth irreducible ample curve. As we know from proposition 37, the kernel of the induced morphism $\operatorname{Alb} C \to \operatorname{Alb} X$ is an abelian variety. Our goal is to show that for a very general curve in a fixed ample linear system this kernel is actually simple, i.e. it has no nontrivial proper abelian subvarieties.[3]

We recall briefly some properties of Lefschetz pencils needed for further discussion (for a reference see [15], chapters 2 and 3).

Let X be a smooth irreducible projective variety of dimension n.

Definition 10. A Lefschetz pencil X_t, $t \in \mathbf{P}^1$ is a pencil of hypersurfaces in X, such that its base locus is smooth of codimension 2 (in particular, a general member is smooth), and every section has at most one singular point, which is a quadratic singularity.

Let L be an ample invertible sheaf on X. Let $\mathcal{D}_X \subset |L|$ be the subset of points which correspond to singular hyperplane sections. It is known that $\mathcal{D}_X$ is a subvariety of $|L|$, and either of the following holds:

(a) $\dim \mathcal{D}_X \leqslant n-2$,

(b) $\dim \mathcal{D}_X = n-1$, and there exists a dense open subset $\mathcal{D}_X^0 \subset \mathcal{D}_X$, such that all points of $\mathcal{D}_X^0$ correspond to hyperplane sections with a single quadratic singular point.

As a consequence for a projective line $\ell \subset |L|$ to be a Lefschetz pencil it is necessary and sufficient that either (a) ℓ does not meet $\mathcal{D}_X$ or (b) ℓ meets $\mathcal{D}_X$ transversally at points of $\mathcal{D}_X^0$. In particular, for every complete ample linear system $|L|$ a general projective line $\ell \subset |L|$ is a Lefschetz pencil, and every point of $|L|$ which corresponds to a smooth irreducible hyperplane section can be included into a Lefschetz pencil.

Let $X_t, t \in \mathbf{P}^1$ be a Lefschetz pencil. Fix a point $0 \in \mathbf{P}^1$. For every i define a vanishing cohomology group $H^i(X_0, \mathbf{Z})_{\mathrm{van}}$ as the kernel of the pushforward $H^i(X_0, \mathbf{Z}) \to H^{i+2}(X, \mathbf{Z})$ induced by inclusion $X_0 \subset X$. It is known (see [15], chapter 2, corollary 2.25) that if $i \neq n-1$, then $H^i(X_0, \mathbf{Z})_{\mathrm{van}} = 0$. Theory of Lefschetz pencils provides a description of the remaining group $H^{n-1}(X_0, \mathbf{Z})_{\mathrm{van}}$ in terms of vanishing cycles.

Let $U \subset \mathbf{P}^1$ be the set of regular values, i.e. the set of points $t \in \mathbf{P}^1$ such that the hyperplane sections X_t are smooth. Let $\{t_1, \ldots, t_m\}$ be the complement of U. For each $i = 1, \ldots, m$ fix a small disc $\Delta_i \subset \mathbf{P}^1$ centered at t_i, a point $t_i' \in \Delta_i \backslash \{t_i\}$, and a path $\gamma_i' \subset U$ starting at 0 and ending at t_i'.

[3] Author of this text learned this fact from [14].

Let X_{Δ_i} be the restriction of the family $X_t \to \mathbf{P}^1$ to the disc Δ_i, and let $j\colon X_{t'_i} \to X_{\Delta_i}$ be the inclusion. It is known that the cohomology pushforward morphism $H^{n-1}(X_{t'_i}, \mathbf{Z}) \to H^{n+1}(X_{\Delta_i}, \mathbf{Z})$ induced by inclusion $X_{t'_i} \subset X_{\Delta_i}$ is surjective and its kernel is a cyclic subgroup (see [15], chapter 2, corollary 2.17). Let δ'_i be a generator of this kernel, which is well-defined up to sign.

Let $\delta_i \in H^{n-1}(X_0, \mathbf{Z})$ be the transfer of δ'_i along the path γ'_i. Theory of Lefschetz pencils shows that $H^{n-1}(X_0, \mathbf{Z})_{\mathrm{van}}$ is generated by classes δ_i, which are called vanishing cycles. Notice that these vanishing cycles depend on paths γ_i and a choice of signs for δ'_i.

Loops in U based at 0 induce automorphisms of $H^{n-1}(X_0, \mathbf{Q})$, so that we have a monodromy representation $\rho\colon \pi_1(U, 0) \to H^{n-1}(X_0, \mathbf{Q})$. The vanishing cohomology group $H^{n-1}(X_0, \mathbf{Q})_{\mathrm{van}}$ defines a sub-representation of $H^{n-1}(X_0, \mathbf{Q})$. Picard-Lefschetz formula describes monodromy representation $H^{n-1}(X_0, \mathbf{Q})_{\mathrm{van}}$ in terms of vanishing cycles. To write it down we need to fix small loops $l_i \subset \Delta_i$ which are based at t'_i and go once around critical points t_i. Let $\gamma_i = \gamma'_i \cdot l_i \cdot {\gamma'_i}^{-1}$. The loops γ_i generate $\pi_1(U, 0)$.

Select a fundamental class $[X_0] \in H^{2n-2}(X_0, \mathbf{Q})$ and consider cup product pairing $\langle , \rangle$ on $H^{n-1}(X_0, \mathbf{Q})$. Let $\rho\colon \pi_1(U, 0) \to \operatorname{Aut} H^{n-1}(X_0, \mathbf{Q})_{\mathrm{van}}$ be the monodromy representation. Picard-Lefschetz formula states that:

$$\rho(\gamma_i)(\alpha) = \alpha + \varepsilon\langle \alpha, \delta_i\rangle \delta_i,$$

where $\varepsilon = \pm 1$. If the loops l_i are coherently oriented and cycles δ'_i are selected with respect to this orientation, and if n is even, then one has a formula $\varepsilon = -(-1)^{\frac{n(n+1)}{2}}$ (see [15], chapter 3, theorem 3.16, remark 3.18).

For us the key result about the monodromy representation $H^{n-1}(X_0, \mathbf{Q})_{\mathrm{van}}$ is that it is irreducible (see [15], chapter 3, theorem 3.27). Using this fact one can show that a slightly stronger result holds when X is a surface:

Proposition 38. *Let X be a smooth irreducible projective surface, and $X_t, t \in \mathbf{P}^1$ a Lefschetz pencil of very ample curves on X. Let $U \subset \mathbf{P}^1$ be an open subset of regular values (not necessarily the full subset of regular values), and $0 \in U$ a point. Let $G \subset \pi_1(U, 0)$ be a subgroup of finite index. The monodromy representation $\rho\colon G \to \mathrm{Aut}(H^1(X_0, \mathbf{Q})_{\mathrm{van}})$ is irreducible.*

Proof. For each point $t_i \notin U$ we have a vanishing cycle $\delta_i \in H^1(X_0, \mathbf{Q})_{\mathrm{van}}$, and a loop γ_i. If t_i is not a critical value then $\delta_i = 0$, and the loop γ_i acts trivially on $H^1(X_0, \mathbf{Q})_{\mathrm{van}}$ since it can be contracted to a trivial loop in the subset of all regular values. Thus, Picard-Lefschetz formula holds for non-critical values too.

Since X_0 is a curve, the cup product pairing on $H^1(X_0, \mathbf{Q})$ is alternating. Hence, Picard-Lefschetz formula shows that $\rho(\gamma_i)(\delta_i) = \delta_i$, and so for every i, every integer $m \geqslant 0$ and every $\alpha \in H^1(X_0, \mathbf{Q})_{\mathrm{van}}$

$$\rho(\gamma_i^m)(\alpha) = \alpha + m\langle \alpha, \delta_i\rangle \delta_i. \tag{6}$$

Let m be the index of G in $\pi_1(U, 0)$, $V \subset H^1(X_0, \mathbf{Q})_{\mathrm{van}}$ be a subrepresentation of G, and $\alpha \in V$. By formula (6) $\langle \alpha, \delta_i \rangle \delta_i = \frac{1}{m}(\rho(\gamma_i^m)(\alpha) - \alpha) \in V$. Thus, $\rho(\gamma_i)(\alpha) \in V$, which means that V is a subrepresentation of $\pi_1(U, 0)$, and so it is either 0 or coincides with $H^1(X_0, \mathbf{Q})_{\mathrm{van}}$. □

Proposition 39. *Let S be a scheme, $f: A \to S$ an abelian fibration, $\mathcal{O}(1)$ an invertible sheaf on A which is relatively ample with respect to f, and P a Hilbert polynomial. There exists a scheme $g: H \to S$, and a closed subscheme $F \subset H \times_S A$ with following properties:*

(a) H is reduced, and quasi-projective over S.

(b) F is flat over H with Hilbert polynomial P with respect to $\mathcal{O}(1)$.

(c) F is a sub-abelian family of $H \times_S A$ over H.

(d) For every closed point $s \in S$ and every abelian subvariety $A' \subset A_s$ which has Hilbert polynomial P there exists a closed point $h \in g^{-1}(s)$ such that the natural closed immersion $F_h \to A_s$ factors as $F_h \to A' \to A_s$ and the first arrow is an isomorphism.

Proof. Consider a relative Hilbert scheme $\mathrm{Hilb}_P(A/S)$. Let $U \subset \mathrm{Hilb}_P(A/S)$ be the open subscheme over which the fibers of the universal family are smooth and connected, and let $F_U \to U$ be the corresponding family.

Recall that by construction $A \to S$ is equipped with a zero section $s: S \to A$. Consider a cartesian diagram

$$\begin{array}{ccc} U & \xrightarrow{s_U} & U \times_S A \\ \downarrow & & \downarrow \\ S & \xrightarrow{s} & A. \end{array}$$

s_U is a closed immersion by construction. Let us denote U_0 the corresponding closed subscheme of $U \times_S A$.

Let W be the image of $F_U \cap U_0$ in U, and let F_W be the corresponding family. The abelian fibration $A \to S$ comes equipped with a relative subtraction morphism $A \times_S A \to A$ over S. Let $W \times_S \times A \times_S A \to W \times_S A$ be its pullback and consider its restriction to $F_W \times_W F_W \to W \times_S A$. This morphism is proper, and so its image I is a closed subscheme of $W \times_S A$.

Let d be the dimension of subvarieties determined by P, i.e. the degree of P. Since each fiber of F_W meets the zero section s_0, its image under the subtraction morphism contains this fiber, and hence the fibers of $I \to W$ have dimension at least d. The morphism $I \to W$ is proper, and so there exists an open subscheme $H \subset W$ such that fibers of I over U are exactly of dimension d. But then they just coincide with corresponding fibers of F_W, since the latter ones are smooth and connected by construction. Thus, if $h \in H$ is a point then $(F_W)_h$ is an abelian subvariety of A_s, where s is the image of h in S. We equip H with reduced induced subscheme structure.

We let F be the restriction of F_W to H. Properties (a) – (d) are clear from construction. $\square$

Proposition 40. *Let X be a smooth irreducible quasi-projective curve, $f\colon A \to X$ an abelian fibration, and $\mathcal{O}(1)$ an invertible sheaf on A which is relatively ample with respect to f. If for every closed point $x \in X$ the fiber A_x is not a simple abelian variety, then ther exists an étale morphism $g\colon Y \to X$ and a proper nontrivial sub-abelian fibration $A' \subset g^{-1}A$.*

Proof. Let P be a Hilbert polynomial, and consider corresponding scheme $H \to X$ constructed in proposition 39. Since H is quasi-projective, its image in X is constructible, i.e. a closed subset or an open one, as X is a curve. Because all fibers of A over closed points are not simple, the union of the images of schemes H parametrizing nontrivial proper abelian subfamilies contains all points of X. Hence, by Baire category theorem, one of such H's dominates X.

Let H be this scheme, and F corresponding universal family. Replacing H by a nonempty open subscheme we may assume that it is smooth over $\mathbf{C}$. By generic smoothness, there exists a nonempty open $V \subset X$ such that $H_V \to V$ is smooth. A smooth morphism has a section étale-locally, i.e. there exists an étale morphism $g\colon Y \to V$, and a section $s\colon Y \to g^{-1}H_V$. Pulling F_V back to Y by s we obtain a proper abelian subfamily $A' \subset g^{-1}A$ over Y. $\square$

Proposition 41. *Let X be a smooth irreducible projective surface, and L a very ample line bundle on X. Let $C \in |L|$ be a general smooth irreducible curve. The kernel of the Albanese morphism $\operatorname{Alb} C \to \operatorname{Alb} X$ is a simple abelian variety.*

Proof. We first include C into a Lefschetz pencil $C_t, t \in \mathbf{P}^1$. Let $U \subset \mathbf{P}^1$ be the set of regular points of this pencil, let $f\colon \mathcal{C} \to U$ be the corresponding fibration in hyperplane sections with fiber C_t, and let $A \to U$ be the induced fibration in Albanese varieties $\operatorname{Alb}(C_t)$.

The family $A \to U$ comes with an U-morphism $A \to U \times \operatorname{Alb}(X)$, induced by inclusion $\mathcal{C} \to U \times X$. Let $K \to U$ be the kernel of this morphism. From 37 we know that $K \to U$ is an abelian fibration. Assume that every fiber of $K \to U$ is not simple. By proposition 40, there exists an étale morphism $g\colon V \to U$ and a proper nontrivial sub-abelian fibration $K' \subset g^{-1}K$. We are going to show that its existence contradicts proposition 38.

A family of g-dimensional complex tori over a base Y is given by a holomorphic vector bundle H over Y, a local system L, and an injection of sheaves $L \to H$ such that for every closed point $x \in X$ the fiber L_x is a lattice of rank $2g$ in H_x. In case of family $A \to U$ this local system is $R^1 f_* \mathbf{Z}$ with fiber $H^1(C_t, \mathbf{Z})$

over a point $t \in T$, as follows from the definition of Albanese variety as a factor $H^{0,1}(C_t)/H^1(C_t,\mathbf{Z})$. The inclusion $\mathcal{C} \to U \times X$ yields a morphism from $R^1 f_*\mathbf{Z}$ to the constant sheaf $H^3(X,\mathbf{Z})$, which acts as cohomology pushforward $H^1(C_t,\mathbf{Z}) \to H^3(X,\mathbf{Z})$ on fibers. Hence the local system L which corresponds to the abelian fibration K has fibers $H^1(C_t,\mathbf{Z})_{\mathrm{van}} = \ker(H^1(C_t,\mathbf{Z}) \to H^3(X,\mathbf{Z}))$. Sub-abelian fibration $K' \subset g^{-1}K$ yields a sub-local system $L' \subset g^{-1}L$.

Let $0 \in V$ be a point. Recall that local systems M with fiber F over V are in natural bijection with pairs $(\pi_1(V,0) \to \operatorname{Aut} F, M_0 \to F)$ where the first arrow is a representation, and the second one is an isomorphism (see [15], chapter 3, corollary 3.10). In particular, the local system $g^{-1}L$ provides us with a representation $\pi_1(V,0) \to H^1(C_{g(0)},\mathbf{Z})_{\mathrm{van}}$. This representation is reducible, because $g^{-1}L$ admits a proper nontrivial sub-local system L'.

Changing coefficients to $\mathbf{Q}$ we obtain a reducible representation $H^1(C_{g(0)},\mathbf{Q})_{\mathrm{van}}$ of a subgroup $\pi_1(V,0) \subset \pi_1(U,g(0))$, which has finite index in $\pi_1(U,g(0))$, a contradiction to proposition 38. □

4.3 Finite-dimensional groups of zero-cycles

The notion of a finite-dimensional subset of $CH_0(X)$ plays an important role in the works of Mumford, Roitman and Voisin about zero-cycles.

Definition 11. Let X be a smooth irreducible projective variety, and $G \subset CH_0(X)$ a group of 0-cycles of X. We call G finite-dimensional if there exists a smooth projective variety W, and a correspondence $\Gamma\colon W \to X$ such that $G \subset \Gamma_* W$.

Notice that a finite-dimensional group G always lies in $CH_0(X)_{\mathrm{hom}}$. Indeed, $\deg(\Gamma_* w)$ stays constant when w varies in a chosen connected component of W, so that the set $\deg(\Gamma_* W)$ is finite. Since G is a group, it follows that Γ is of degree 0.

Proposition 42. *A group G of zero cycles is finite-dimensional if and only if there exists an integer $d > 0$ such that $G \subset \sigma_{d,d}(X^d \times X^d)$.*

Proof. Just apply theorem 1. □

Proposition 43. *Let X, K be smooth irreducible projective varieties, and W a smooth projective variety, such that $\dim K \geqslant \dim W$. Let $Z\colon K \to X$ and $\Gamma\colon W \to X$ be correspondences. If $Z_* K \subset \Gamma_* W$, then all the fibers of $Z_*\colon K \to CH_0(X)$ have dimension at least $\dim K - \dim W$.*

Proof. Let $p\colon K \times W \to K$ and $q\colon K \times W \to W$ be the projections. As proposition 27 shows, the set $R = \ker(Z \circ p - \Gamma \circ q)$ is a union of at most countably many Zariski-closed subsets $R_i \subset K \times W$.

By assumption, $Z_*K \subset \Gamma_*W$, and so the projection $p\colon R \to K$ is surjective. Baire category theorem shows that for some index i the projection $p|_{R_i}$ is surjective. Hence $\dim R_i \geqslant \dim K \geqslant W$, and so over closed points all fibers of the projection $q\colon R_i \to W$ have dimension at least $\dim R_i - \dim W \geqslant \dim K - \dim W$.

Let $k \in K$ be a closed point. Take any closed point $w \in q(p^{-1} \cap R_i)$. Let $F = p(q^{-1}w \cap R_i)$. By construction Z_* restricted to closed points of F is constant, and $\dim F = \dim(q^{-1}w \cap R_i) \geqslant \dim K - \dim W$. □

Proposition 43 hints that the notion of dimension makes sense for a finite-dimensional subgroup of $CH_0(X)$.

4.4 Factorization theorem

We are finally ready to prove the factorization theorem of Voisin [14].

Proposition 44. *Let A be an abelian variety, G an abstract group (i.e. a set with a group structure), and $f\colon A \to G$ a homomorphism of groups. Let $V \subset A$ be an irreducible closed subvariety which lies in the kernel of f and contains $0 \in A$. There exists an abelian subvariety A' of A such that $V \subset A'$ and $A' \subset \ker f$.*

Proof. Consider the map $V \times V \to A$ which sends a pair (v_1, v_2) to $v_1 - v_2$, and let V' be its image. By construction, $V' \subset \ker f$. Since V is irreducible, V' is irreducible too. Moreover, $V \subset V'$ as $0 \in V$. If $V = V'$ then V is an abelian subvariety. Otherwise $\dim V' > \dim V$, and we finish by induction on dimension. □

Theorem 3. *Let M, X be smooth irreducible projective varieties, $Z\colon M \to X$ a correspondence. If $Z_*(CH_0(M)_{\mathrm{hom}})$ is finite-dimensional, then the homomorphism $Z_*\colon CH_0(M)_{\mathrm{hom}} \to CH_0(X)$ factors through* $\operatorname{Alb} M$.

Proof. Let $z \in CH_0(M)$ be such a cycle, that $\mathrm{alb}_M(z) = 0$. We want to show that $Z_*(z) = 0$ in $CH_0(X)$.

Let $\pi\colon \widetilde{M} \to M$ be the blowup of M at the support of z, and $E \subset \widetilde{M}$ the exceptional divisor. There exists a lift $\widetilde{z}$ of z to $\widetilde{M}$ which is supported on E. Proposition 34 shows that $\mathrm{alb}_{\widetilde{M}}(\widetilde{z}) = 0$ for each lift $\widetilde{z}$.

Assume that $\dim M > 2$. Let $i\colon H \to \widetilde{M}$ be a smooth irreducible hyperplane section, and $E_H = H \cap E$. By construction, for each $z_0 \in \operatorname{supp} z$ the preimage $\pi^{-1}(z_0)$ is of codimension 1 in $\widetilde{M}$. Hypersurface H is ample, and so it must meet $\pi^{-1}(z_0)$, and moreover $H \cap \pi^{-1}(z_0)$ is of codimension 1 in H. Thus, E_H is of codimension 1 in H, and $\operatorname{supp} z \subset \pi(i(E_H))$. In particular, there exists a lift $\widetilde{z}$ of z to H. By propositions 37 and 33, $\mathrm{alb}_H(\widetilde{z}) = 0$ for each lift $\widetilde{z}$ of z to H.

Hence, by induction we may assume that $\dim H = 2$. We take an ample smooth irreducible curve $j\colon C \to H$ as in proposition 41. Again, $\operatorname{supp} z \subset \pi(j^{-1}E')$, and so we may assume that there exists a lift $\widetilde{z}$ of z to C.

Let $A\colon \operatorname{Alb} C \to C$ be the correspondence from proposition 36. The map A_* is the inverse of the Albanese morphism $\operatorname{alb}\colon CH_0(C)_{\mathrm{hom}} \to \operatorname{Alb} C$, and so it is a group homomorphism. By Abel-Jacobi theorem $A_*\colon \operatorname{Alb} C \to CH_0(C)_{\mathrm{hom}}$ is an isomorphism, and so $\widetilde{z} \in A_*(\operatorname{Alb} C)$. Proposition 37 implies that moreover $\widetilde{z} \in A_*(K)$, where $K = \ker(\operatorname{Alb} C \to \operatorname{Alb} H)$. By proposition 41, K is a simple abelian variety.

Let us restrict A to K, and consider a correspondence $\overline{Z}\colon K \to X$, which is defined as

$$\overline{Z} = Z \circ \pi \circ i \circ j \circ A.$$

We want to show that the map $\overline{Z}_*\colon K \to CH_0(X)$ is zero. Notice, that $\overline{Z}_*$ is a group homomorphism because A_* is.

Let $\Gamma\colon W \to X$ be such a correspondence, that $Z_*(CH_0(M)_{\mathrm{hom}}) \subset \Gamma_* W$. By construction, $\overline{Z}_*(K) \subset Z_*(CH_0(M)_{\mathrm{hom}})$, and so $\overline{Z}_*(K) \subset \Gamma_* W$. Taking C to be ample enough we may assume that $\dim K > \dim W$. By proposition 43 all fibers of $\overline{Z}_*\colon K \to CH_0(X)$ are positive-dimensional. In particular, there exists such a positive-dimensional subvariety $V \subset K$ that $0 \in V$ and $\overline{Z}_*|_V = 0$. Proposition 44 implies that $\overline{Z}_*$ vanishes on some positive-dimensional abelian subvariety of K. But K is simple, and so in fact $\overline{Z}_* = 0$. □

5 Results of Mumford

Let W, X be smooth irreducible projective varieties, and $\Gamma\colon W \to X$ a correspondence. If Γ consists solely of vertical components then in terms of the map Γ_* (and so, of the induced rational map) it is indistinguishable from the zero correspondence. It is an interesting question, what we can say about the rational equivalence class of Γ, if we assume that $\Gamma_* = 0$. As it turns out, in this case some integer multiple of Γ is rationally equivalent to a correspondence consisting of vertical components only.

This observation goes back to Mumford [11]. Moreover, in the same article Mumford demonstrated that this observation leads to strong conclusions on the structure of $CH_0(X)$ when X is a surface of positive genus. Our goal is to describe the results of Mumford in this direction.

This section is a variation on the corresponding part of chapter 10 of the book of Voisin [15].

5.1 Bloch-Srinivas decomposition

Proposition 45. *Let X, Y be smooth quasi-projective varieties, and $f\colon X \to Y$ a smooth morphism. If f has a section, then $f^*\colon CH_*(Y) \to CH_*(X)$ is injective.*

Proof. Let $s\colon Y \to X$ be a section. It is necessary a regular embedding, and so induces the usual Gysin pullback $s^*\colon CH_*(X) \to CH_*(Y)$. Proposition 6.5 (b) from chapter 6 of [6] shows, that $s^* \circ f^* = (f \circ s)^* = \mathrm{id}$. □

Proposition 46. *Let T, W, X be smooth irreducible quasi-projective varietes, $f\colon T \to W$ a smooth morphism, and $\Gamma \in CH_*(W \times X)$ a cycle. If $(f \times \mathrm{id})^*\Gamma = 0$, then there exists a nonempty Zariski-open subset $U \subset W$ and a positive integer m, such that $m\Gamma|_{U\times X} = 0$ in $CH_0(U \times X)$.*

Proof. Every smooth morphism has a section étale-locally (EGA IV, 17.16.3). Let $U \subset W$ be an open subset, and $g\colon V \to U$ a surjective étale morphism. Consider a cartesian diagram

$$\begin{array}{ccccc}
T_V \times X & \longrightarrow & T_U \times X & \longrightarrow & T \times X \\
\Big\downarrow{\scriptstyle (f_V\times\mathrm{id})} & & \Big\downarrow{\scriptstyle (f_U\times\mathrm{id})} & & \Big\downarrow{\scriptstyle (f\times\mathrm{id})} \\
V \times X & \xrightarrow{(g\times\mathrm{id})} & U \times X & \longrightarrow & W \times X.
\end{array}$$

All the arrows in this diagram are flat, and so the induced Gysin pullbacks of cycles coincide with flat pullbacks (see [6], chapter 8, proposition 8.1.2 (a)). Also, all the vertical arrows are smooth.

Flat pullbacks commute already on the level of cycles. Thus,

$$(f_V \times \mathrm{id})^*(g \times \mathrm{id})^*\Gamma|_{U\times X} = 0.$$

If $f_V : T_V \to V$ has a section, then proposition 45 implies, that

$$(g \times \mathrm{id})^*\Gamma|_{U\times X} = 0.$$

Applying projection formula we obtain an equality

$$(g \times \mathrm{id})_*(g \times \mathrm{id})^*(\Gamma|_{U\times X}) = m\Gamma|_{U\times X} = 0,$$

where m is the degree of g. □

Next theorem is due to Bloch and Srinivas [4]. Our proof goes back to Mumford [11].

Theorem 4. *Let W be a smooth irreducible quasi-projective variety, X a smooth irreducible projective variety, and $\Gamma\colon W \to X$ a correspondence. If $\Gamma_* = 0$, then there exists a proper subvariety $W' \subset W$, a nonzero integer m and a cycle $\Gamma' \in CH_*(W \times X)$ supported on $W' \times X$, such that*

$$m\Gamma = \Gamma' \text{ in } CH_*(W \times X).$$

Proof. Due to localization exact sequence for Chow groups it is enough to find such a nonempty Zariski-open subset $U \subset W$ that Γ is torsion when restricted to U.

Without loss of generality we can assume that Γ has no vertical components. Write $\Gamma = \Gamma^+ - \Gamma^-$ as a difference of effective cycles. Let $f^+, f^- \colon W \to X^{(d)}$ be induced rational maps (the indices d coincide for f^+ and f^-, since otherwise $\Gamma_* \neq 0$). Let $f = (f^+, f^-) \colon W \to X^{(d)} \times X^{(d)}$.

Replacing W by an open subset we may assume that f is defined everywhere. The condition $\Gamma_* = 0$ now implies that $f(W(\mathbf{C}))$ is a subset of $R \subset (X^{(d)} \times X^{(d)})(\mathbf{C})$, where R is the relation of rational equivalence from theorem 2. Because there exists such a countable collection $\{R_i\}$ of closed irreducible subvarieties $R_i \subset X^{(d)} \times X^{(d)}$ that $R = \bigcup_i R_i(\mathbf{C})$, we can apply Baire category theorem and conclude that f factors through an inclusion $R_i \to X^{(d)} \times X^{(d)}$ for some i.

Part (b) of theorem 2 implies that there exists a quasi-projective variety T, a nonnegative integer k, a surjective morphism $e \colon T \to W$, a morphism $g \colon T \to X^{(k)}$, and a morphism $h \colon T \times \mathbf{P}^1 \to X^{(d+k)}$ such that

$$h \circ i_0 = g + f_+ \circ e, \tag{7}$$
$$h \circ i_\infty = g + f_- \circ e, \tag{8}$$

where $i_0, i_\infty \colon T \to T \times \mathbf{P}^1$ are evident inclusions, and the plus sign denotes the addition morphism $X^{(d)} \times X^{(k)} \to X^{(d+k)}$. Resolving singularities, we may assume that T is smooth. Replacing T by a nonempty Zariski-open subset we may assume that $e \colon T \to W$ is smooth. Of course e needs not be surjective, but it is still dominant.

Let $H \colon T \times \mathbf{P}^1 \to X$ and $G \colon T \to X$ be the correspondences induced by h and g respectively. Proposition 9 shows that for every closed point $t \in T$

$$(H \circ i_0)_* t = (h \circ i_0)(t), (H \circ i_\infty)_* t = (h \circ i_\infty)(t),$$
$$(\Gamma^+ \circ e)_* t = (f^+ \circ e)(t), (\Gamma^- \circ e)_* t = (f^- \circ e)(t)$$

in $Z_0(X)$. Combining it with (7) and (8) we obtain equalities

$$(H \circ i_0)_* t = G_* t + (\Gamma_+ \circ e)_* t,$$
$$(H \circ i_\infty)_* t = G_* t + (\Gamma_- \circ e)_* t.$$

in $Z_0(X)$ for every closed point $t \in T$. Since all the correspondences involved are finite over T, proposition 13 shows that

$$H \circ i_0 = G + \Gamma_+ \circ e,$$
$$H \circ i_\infty = G + \Gamma_- \circ e.$$

in $Z_*(T \times X)$. Hence $\Gamma \circ e = H \circ i_0 - H \circ i_\infty$. But the cycles $H \circ i_0$ and $H \circ i_\infty$ are rationally equivalent, and so proposition 46 finishes the proof. $\square$

In fact, the original method Bloch and Srinivas implies a stronger result:

Theorem 5. *Let W, X be smooth irreducible projective varieties, and $\Gamma\colon W \to X$ a correspondence. If there exists a proper subvariety $i\colon X' \to X$ such that $\Gamma_* CH_0(W) \subset i_* CH_0(X')$, then there exists an integer $m > 0$, a proper closed subset $W' \subset W$, a cycle Γ' supported on $W' \times X$ and a cycle Γ'' supported on $W \times X'$, such that*

$$m\Gamma = \Gamma' + \Gamma'' \text{ in } CH_*(W \times X).$$

For the proof, see [15], chapter 10, theorem 10.19. When $\Gamma = \Delta$, the expression above is called Bloch-Srinivas decomposition of the diagonal. Such a decomposition was generalized by Paranjape [12] and Laterveer [8]:

Theorem 6. *Let X be a smooth irreducible projective variety of dimension n. If the cycle class map $\mathrm{cl}\colon CH_i(X) \otimes \mathbf{Q} \to H^{2n-2i}(X, \mathbf{Q})$ is injective for all $i \leqslant k$, then there exists a positive integer m and a decomposition*

$$m\Delta = Z_0 + \ldots + Z_k + Z',$$

in $CH^n(X \times X)$, where Z_i are supported on $W_i' \times W_i$, $W_i \subset X$ are subvarieties of dimension i, W_i' are subvarieties of dimension $n - i$, and Z' is supported on $W' \times X$, where $W' \subset X$ is a proper subvariety.

(see [15], chapter 10, theorem 10.29).

5.2 Correspondences and differential forms

Let W, X be smooth irreducible projective varieties, and $\Gamma\colon W \to X$ a correspondence. Let $\pi\colon W \times X \to W$ and $q\colon W \times X \to X$ be the projections. Let n be the dimension of X, and let $[\Gamma] \in H^{2n}(W \times X, \mathbf{Z})$ be the cycle class of Γ. For an integer $k \geqslant 1$ consider a map $\Gamma^*\colon H^k(X, \mathbf{Z}) \to H^k(W, \mathbf{Z})$:

$$\Gamma^* \alpha = \pi_*(q^* \alpha \cdot [\Gamma])$$

Replacing $\mathbf{Z}$ by $\mathbf{C}$ we obtain a similar map for complex cohomology.

Since $[\Gamma]$ is a class of an algebraic subvariety, it is in fact of type (n, n). Therefore the map $\Gamma^*\colon H^k(X, \mathbf{C}) \to H^k(W, \mathbf{C})$ preserves Hodge decomposition, and defines a morphism of Hodge structures of bidegree $(0, 0)$. In particular, for each $p \geqslant 0$ we obtain morphisms $\Gamma^*\colon H^0(X, \Omega_X^p) \to H^0(W, \Omega_W^p)$.

If T is another smooth irreducible projective variety, and $Z\colon T \to W$ a correspondence, then the usual computation shows that $(\Gamma \circ Z)^* = Z^* \circ \Gamma^*$. If Γ is a graph of a morphism $f\colon W \to X$, then $\Gamma^* = f^*$.

Following proposition is taken from [15] (see chapter 10, proposition 10.24).

Proposition 47. *Let $\Gamma\colon W \to X$ be a correspondence between smooth irreducible projective varieties. If Γ is vertical, then $\Gamma^*\colon H^0(X,\Omega_X^p) \to H^0(W,\Omega_W^p)$ vanishes for every $p \geqslant 0$.*

Proof. Let $W' \subset W$ be such a closed proper subset that Γ is supported on $W' \times X$. Let $l\colon \widetilde{W}' \to W' \subset W$ be the desingularization of W'. Consider a cycle $\widetilde{\Gamma} = (l \times \mathrm{id})^*\Gamma$. By construction, $(l \times \mathrm{id})_*\widetilde{\Gamma} = \Gamma$.

Let $\pi'\colon \widetilde{W}' \times X \to \widetilde{W}'$ and $q'\colon \widetilde{W}' \times X \to X$ be the projections. We have equalities of morphisms:

$$\pi \circ (l \times \mathrm{id}) = l \circ \pi',$$
$$q \circ (l \times \mathrm{id}) = q'.$$

Let $\alpha \in H^k(X,\mathbf{C})$ be a cohomology class. Using the above equations, projection formula, and the fact that cycle class map is compatible with pullbacks and pushforwards, we conclude that

$$l_*\pi'_*({q'}^*\alpha \cdot [\widetilde{\Gamma}]) = \pi_*(l \times \mathrm{id})_*((l \times \mathrm{id})^*q^*\alpha \cdot [\widetilde{\Gamma}]) = \pi_*(q^*\alpha \cdot [\Gamma]) = \Gamma^*\alpha.$$

Hence Γ^* factors through l_*.

Notice that $\widetilde{\Gamma}$ is not a correspondence in our sense. Indeed, Γ is of codimension $n = \dim X$ in $W \times X$, while $\widetilde{\Gamma}$ is of codimension $n - s$, where s is the codimension of W_0. Moreover, l_* defines a morphism of Hodge structures of positive bidegree (s,s), and so its image misses the groups $H^{p,0}(W)$ for each $p \geqslant 0$. Thus $\Gamma^*\colon H^{p,0}(X) \to H^{p,0}(W)$ is zero. □

In [11] Mumford demonstrated (in slightly different terms) that the following theorem holds:

Theorem 7. *Let W, X be smooth irreducible projective varieties, and $\Gamma\colon W \to X$ a correspondence. If $\Gamma_* = 0$, then for every $p \geqslant 0$ the pullback morphisms $\Gamma^*\colon H^0(X,\Omega_X^p) \to H^0(W,\Omega_W^p)$ are zero.*

According to Mumford, this theorem goes back to work of Severi on 0-cycles (see references in [11]).

Proof of theorem 7. Apply theorem 4 and proposition 47. □

Next proposition is taken almost verbatim from [15], chapter 10 (see proof of lemma 10.25).

Proposition 48. *Let X be a smooth irreducible projective surface, and $d \geqslant 1$ an integer. Consider the variety $Y = X^d \times X^d$. Let $p_1, p_2\colon Y \times Y \to Y$ be the projections, and $\Gamma\colon Y \times Y \to X$ a correspondence defined as $\Gamma = \Sigma_{d,d} \circ p_1 -$*

$\Sigma_{d,d} \circ p_2$. Let R be an irreducible component of the fiber of Γ_* over 0. Assume that p_1 is surjective when restricted to R.

If ω is a nonzero global holomorphic 2-form on X, then its pullback $\Gamma^*\omega$ is symplectic at almost all smooth points of R, and moreover R is lagrangian with respect to this form. In particular, $\dim R = 4d$.

Proof. Let $q_i \colon X^d \times X^d \to X$, $1 \leqslant i \leqslant 2d$ be the projections. Recall, that

$$\Sigma_{d,d} = \sum_{i=1}^{d} \Delta \circ q_i - \sum_{i=d+1}^{2d} \Delta \circ q_i.$$

Hence, if $\omega \in H^{2,0}(X)$, then

$$\Sigma_{d,d}^*\omega = \sum_{i=1}^{d} q_i^*\omega - \sum_{i=d+1}^{2d} q_i^*\omega.$$

Let $\eta = \Sigma_{d,d}^*\omega$ so that $\Gamma^*\omega = p_1^*\eta - p_2^*\eta$.

Consider the desingularization $f \colon \widetilde{R} \to Y^2$ of R. Since $(\Gamma \circ f)_* = 0$ by construction, the theorem 7 implies, that $f^*(\Gamma^*\omega) = 0$.

Assume, that $\omega \neq 0$. Let $r \in \widetilde{R}$ be a general point. The map $p_1 \circ f$ is surjective, and so $p_1(f(r))$ is general too. Hence, we may assume that for each i the form ω is nondegenerate at $q_i(p_1(f(r)))$, which implies that η is nondegenerate at $p_1(f(r))$.

The map $p_1 \circ f$ is generically smooth, and so for a general $r \in R$ it is a submersion at r. Under this assumption the form $f^*p_1^*\eta$ is of rank $4d$ at r, in the sense that the dimension of the image of the map $T_r\widetilde{R} \to (T_r\widetilde{R})^\vee$ induced by $(f^*p_1^*\eta)_r$ is $4d$.

Since $f^*p_1^*\eta = f^*p_2^*\eta$, the form $f^*p_2^*\eta$ is of rank $4d$ at r too. But then η is of rank $4d$ at $p_2(f(r))$, i.e. it is nondegenerate. Therefore $\Gamma^*\omega = p_1^*\eta - p_2^*\eta$ is nondegenerate at $f(r)$.

Assume further that R is smooth at $f(r)$. In this case R is an isotropic subvariety at $f(r)$ with respect to the form $\Gamma^*\omega$, so that

$$\dim_{f(r)} R \leqslant \frac{1}{2} \dim_{f(r)} Y \times Y = 4d.$$

On the other hand, R surjects onto Y, and so $\dim_{f(r)} R = 4d$, i.e. R is lagrangian. $\square$

An immediate corollary of the proposition 48 is the following theorem of Mumford [11]:

Theorem 8. *Let X be a smooth irreducible projective surface, and let $d \geqslant 1$ be an integer. If $h^{2,0}(X) > 0$ then a very general fiber of the map $\sigma_{d,d} \colon X^d \times X^d \to CH_0(X)$ is zero-dimensional.*

Proof. Let $p_1, p_2 \colon (X^d \times X^d)^2 \to X$ be the projections. Consider a correspondence $\Gamma = \Sigma_{d,d} \circ p_1 - \Sigma_{d,d} \circ p_2$, and let $R \subset (X^d \times X^d)^2$ be the subset over which Γ_* vanishes. We know that R is an at most countable union of irreducible subvarieties R_i. Consider a set

$$V = \bigcap_i \{ z \in X^d \times X^d \mid p_1^{-1}(z) \cap R_i \text{ is finite } \}.$$

Clearly, for each $z \in V$ the fiber of $\sigma_{d,d}$ at z is at most countable. By Chevalley's theorem, V is an intersection of at most countably many open subsets. By proposition 48, for each i the projection $p_1|_{R_i}$ is either generically finite or not surjective. Applying Baire category theorem, we conclude that V is dense. $\square$

Let X be a smooth irreducible projective surface. Assume that $CH_0(X)_{\mathrm{hom}}$ is finite-dimensional in the sense of definition 11, i.e. that $\sigma_{d,d}(X^d \times X^d) = CH_0(X)_{\mathrm{hom}}$ for some $d > 0$. Take any $k > d$, and let $z = (z_+, z_-)$ be an arbitrary point of $X^k \times X^k$. By construction, there exist effective cycles z'_+, z'_- of degree d such that $z = z'_+ - z'_-$ in $CH_0(X)_{\mathrm{hom}}$. Since $k > d$, we see that all fibers of $\sigma_{k,k}$ are positive-dimensional, and so, by theorem 8, $h^{2,0}(X) = 0$. Conversely, if $h^{2,0}(X) > 0$ then $CH_0(X)_{\mathrm{hom}}$ is not finite-dimensional in the sense of Roitman.

Similarly, if $h^{2,0}(X) > 0$, then the group

$$CH_0(X)_{\mathrm{alb}} = \ker(\mathrm{alb} \colon CH_0(X)_{\mathrm{hom}} \to \mathrm{Alb}\, X)$$

is not finite-dimensional too. Indeed, it is known that for d large enough the composition $\mathrm{alb} \circ \sigma_{d,d} \colon X^d \times X^d \to \mathrm{Alb}\, X$ is surjective. Hence, for every cycle $z \in CH_0(X)_{\mathrm{hom}}$ there exists a point $(x_1, x_2) \in X^d \times X^d$, such that $z - \sigma_{d,d}(x_1, x_2) \in CH_0(X)_{\mathrm{alb}}$. So, if $CH_0(X)_{\mathrm{alb}}$ is contained in the image of some correspondence $\Gamma \colon W \to X$, then $CH_0(X)_{\mathrm{hom}}$ is contained in the image of a correspondence $Z \colon W \times (X^d \times X^d) \to X$, such that $Z_*(w, x_1, x_2) = \Gamma_* w + \sigma_{d,d}(x_1, x_2)$.

By these reasons theorem 8 is sometimes called the infinite-dimensionality theorem. It shows that in general $CH_0(X)_{\mathrm{hom}}$ can not be represented by a finite-dimensional abelian variety in a reasonable way, which is in stark contrast with the case of $CH^1(X)_{\mathrm{hom}}$.

6 Symplectic involutions of K3 surfaces

Let X be a projective K3 surface, and $i \colon X \to X$ an involution. Such an involution is called symplectic if the induced map $i^* \colon H^{2,0}(X) \to H^{2,0}(X)$ is the identity. Recall that in this case the conjecture of Bloch states, that i_* acts as identity on $CH_0(X)$. In [14] Voisin demonstrated that it is indeed the case. Below we reproduce her argument elaborating some details as it is appropriate for a master thesis.

6.1 Auxillary geometric facts

Proposition 49. *Let X be a smooth irreducible projective surface. Every ample divisor on X is connected.*

Proof. Let D be an ample divisor, and assume by contradiction that $D = D_1 + D_2$ where D_1 and D_2 do not intersect. Then $D_1.D_2 = 0$. Since D is ample, and D_i are effective, $D.D_i = D_i^2 > 0$. But these two inequalities can not hold simultaneously by Hodge index theorem. □

Proposition 50. *For $i = 1, 2$ let X_i be a smooth irreducible projective variety, L_i an ample invertible sheaf on X_i, and $p_i\colon X_1 \times X_2 \to X_i$ the projection. The invertible sheaf $p_1^*L_1 \otimes p_2^*L_2$ is ample.*

Proof. Assume without loss of generality that L_i are very ample, and let $f_i\colon X_i \to \mathbf{P}^{N_i}$ be corresponding embeddings. Let $\sigma\colon \mathbf{P}^{N_1} \times \mathbf{P}^{N_2} \to \mathbf{P}^N$ be a Segre embedding, $N = (N_1 + 1)(N_2 + 1) - 1$. Since $\operatorname{Pic}(\mathbf{P}^{N_1} \times \mathbf{P}^{N_2}) = \operatorname{Pic}\mathbf{P}^{N_1} \oplus \operatorname{Pic}\mathbf{P}^{N_2} = \mathbf{Z}\oplus\mathbf{Z}$, the pullback $\sigma^*\mathcal{O}(1)$ has the form $q_1^*\mathcal{O}(n_1)\otimes q_2^*\mathcal{O}(n_2)$, where q_i are respective projections, and n_i some integers.

Let $x_i \in \mathbf{P}^{N_i}$ be closed points, and let $e_i\colon \mathbf{P}^{N_i} \to \mathbf{P}^{N_1} \times \mathbf{P}^{N_2}$ be corresponding embeddings. By construction $(q_i \circ e_j)^* = \mathrm{id}$ if $i = j$ and 0 otherwise. From definition of σ it is clear that $\sigma \circ e_i$ embeds $\mathbf{P}^{N_i}$ as a linear subspace. Hence $(\sigma\circ e_i)^*\mathcal{O}(1) = \mathcal{O}(1)$. On the other hand, $(\sigma\circ e_i)^*\mathcal{O}(1) = (q_i\circ e_i)^*\mathcal{O}(n_i) = \mathcal{O}(n_i)$. Therefore, $\sigma^*\mathcal{O}(1) = q_1^*\mathcal{O}(1) \otimes q_2^*\mathcal{O}(1)$. As a consequence, $p_1^*L_1 \otimes p_2^*L_2 = (\sigma \circ (f_1 \times f_2))^*\mathcal{O}(1)$, and so this sheaf is ample. □

Proposition 51. *Let X be an irreducible projective variety, L a base point free line bundle on X, and $k \leqslant \dim |L|$ a positive integer. Consider an incidence variety $\Gamma \subset |L| \times X^k$ defined as*

$$\Gamma = \{(D, x_1, \ldots, x_k) \mid x_i \in D\}.$$

Let $p\colon \Gamma \to |L|$ and $q\colon \Gamma \to X^k$ be projections. The following properties hold:

1. *The projection q is surjective.*
2. *If $U \subset |L|$ is a nonempty Zariski-open subset, then $q(p^{-1}(U))$ contains a nonempty Zariski-open subset.*

3. *If $k = \dim |L|$, then there exists a nonempty Zariski-open $U \subset \Gamma$ such that $q|_U$ is an isomorphism onto its image (here the fact that we work over an algebraically closed field of characteristic zero is crucial).*

Proof. Let $I_x \subset \mathcal{O}_X$ be the ideal of x, and let $\mathcal{O}_x = \mathcal{O}_X/I_x$. Consider an exact sequence

$$0 \to H^0(X, L \otimes I_x) \to H^0(X, L) \to H^0(x, L \otimes \mathcal{O}_x).$$

The sheaf $L \otimes \mathcal{O}_x$ is just the fiber of L over x, and the last morphism in this sequence sends a section to its value at x. Since L is base point free, this last morphism is surjective.

Let us denote by H_x the space of all divisors $D \in |L|$ containing x, i.e. the projectivization of the kernel of the restriction morphism $H^0(X, L) \to H^0(X, L \otimes \mathcal{O}_x)$. We have just shown that H_x is a hyperplane. Since an intersection of $k \leqslant \dim |L|$ hyperplanes is always nonempty, we see that q is surjective, i.e. the first property holds.

If $U \subset |L|$ is a Zariski-open dense subset, then $p^{-1}(U)$ is also Zariski-open and dense. The image $q(p^{-1}(U))$ is in general only constructible, i.e. $q(p^{-1}(U)) = \bigcup_{i=1}^m Z_i$, where Z_i are closed in open subsets $U_i \subset X^k$. But q is also surjective and $p^{-1}(U)$ dense, and so some of the Z_i's must be dense too. Then it coincides with the corresponding U_i, and thus the second property holds.

Now, assume that $k = \dim |L|$. Looking at the fibers of p we see, that $\dim \Gamma = kn$, where $n = \dim X$. Since $\dim X^k = kn$ too, and since q is surjective, the theorem of Chevalley tells us that there exists a nonempty Zariski-open subset $U \subset \Gamma$ such that q is quasi-finite over U. Let $u \in U$ be a point. The set $p(q^{-1}(u))$ is finite. At the same time it is an intersection of $k = \dim |L|$ hyperplanes H_x for some $x \in X$, and so must be a linear subspace of $|L|$. Therefore it consists of a single point.

By Zariski main theorem q factors as $U \to U' \to X^k$, where the first arrow is an open immersion and the second one is finite. But we also know, that $U' \to X^k$ is generically bijective, and since we work in characteristic zero, it follows that $U' \to X^k$ is birational. Finally, X^k is normal and so $U' \to X^k$ is in fact an open immersion. □

Proposition 52. *Let $f \colon X \to Y$ be a finite flat morphism of schemes. In this situation* $\ker(f^* \colon \operatorname{Pic} Y \to \operatorname{Pic} X)$ *lies in* $\operatorname{Pic} X[d]$, *where d is the degree of f.*

Proof. By assumptions, $f_*\mathcal{O}_X$ is locally free. If $L \in \operatorname{Pic} Y$ is such that $f^*L = \mathcal{O}_X$, then by projection formula $f_*\mathcal{O}_X = f_*f^*L = L \otimes f_*\mathcal{O}_X$. Taking determinants we conclude, that $L^{\otimes d} = \mathcal{O}_Y$. □

6.2 Special zero-cycle of Beauville-Voisin

Bogomolov, Mumford, Mori and Mukai [10] discovered that an ample linear system on a K3 surface always contains a rational curve. As an immediate corollary one obtains the following theorem of Beauville-Voisin [1]:

Theorem 9. *Let X be a projective K3 surface. It contains a rational curve. If $x, y \in X$ are points supported on rational curves, then $x = y$ in $CH_0(X)$.*

The class of a point supported on a rational curve is denoted $c_0 \in CH_0(X)$ and is called the special 0-cycle of Beauville-Voisin. It has many important properties. For example, it splits the degree homomorphism $CH_0(X) \to \mathbf{Z}$, so that $CH_0(X) = CH_0(X)_{\mathrm{hom}} \oplus \mathbf{Z}c_0$. The intersection pairing on $CH_1(X)$ takes values in $\mathbf{Z}c_0$, and $c_2(X) = 24c_0$ (see [1]).

Proof of theorem 9. Existence is clear. Let $x, y \in X$ be points supported on rational curves C_1, C_2, and let H be an ample divisor which is a rational curve. Since H is ample, all irreducible components of C_1 and C_2 must intersect it, either properly or not. Since H is connected, $x = y$ in $CH_0(X)$. □

Theorem 9 has an important corollary for correspondences on K3 surfaces, due to Voisin [14]:

Proposition 53. *Let X be a projective K3 surface, W a smooth irreducible projective variety, $\Gamma\colon X \to W$ a correspondence. Suppose that for some integer $g \geqslant 1$ all fibers of $\Gamma_*\colon X^g \to CH_0(W)$ are at least one-dimensional. Then every cycle $z \in \Gamma_* CH_0(X)$ can be represented in the form*

$$z = \Gamma_* t_1 - \Gamma_* t_2 + (\deg z)c_0,$$

where t_1, t_2 are effective 0-cycles of degree $g - 1$. In particular, $\Gamma_* CH_0(X)_{\mathrm{hom}}$ is finite-dimensional.

Proof. Let $z = (z_1, \dots, z_g) \in X^g(\mathbf{C})$ be a point and $C \subset X^g$ a curve passing through z, such that $\Gamma_* z = \Gamma_* z'$ for every $z' \in C(\mathbf{C})$.

Let H be an ample rational curve on X. Let $H_g = \sum_{i=1}^{g} \pi_i^* H$, where $\pi_i\colon X^g \to X$ is the projection to i-th factor. The divisor H_g is ample by proposition 50. It therefore intersects C at some point $t = (t_1, \dots, t_g)$. By construction of H_g, some t_i, considered as a 0-cycle of degree 1, is equal in $CH_0(X)$ to the special cycle c_0 of Beauville-Voisin. Without loss of generality we assume, that $t_g = c_0$. Hence, we have shown that in $CH_0(W)$ there is an equality

$$\Gamma_* z = \Gamma_* z_1 + \ldots + \Gamma_* z_g = \Gamma_* t_1 + \ldots + \Gamma_* t_{g-1} + \Gamma_* c_0.$$

Therefore, if $l \geqslant g$ is an integer, then for every $z_1, \dots, z_l \in X$ there exist $t_1, \dots, t_{g-1} \in X(\mathbf{C})$ such that

$$\Gamma_* z_1 + \ldots + \Gamma_* z_l = \Gamma_* t_1 + \ldots + \Gamma_* t_{g-1} + (l - g + 1)\Gamma_* c_0.$$

Now let $z \in CH_0(X)$ be a 0-cycle. Write it in the form $z = z_+ - z_-$ where both z_+, z_- are effective and of degree at least $g-1$. We conclude, that for some $t_1, t_2 \in X^{g-1}(\mathbf{C})$

$$\Gamma_* z = \Gamma_* t_1 - \Gamma_* t_2 + (\deg c)\Gamma_* c_0. \quad \square$$

6.3 Bloch's conjecture for symplectic involutions

Let X be a projective K3 surface, and $i\colon X \to X$ an involution. Such an involution is called symplectic if the induced map $i^*\colon H^{2,0}(X) \to H^{2,0}(X)$ is the identity.

Consider the quotient $\Sigma = X/\{1, i\}$, and let $q\colon X \to \Sigma$ be the quotient map. Σ is a normal projective surface which is not smooth in general. Let $\Sigma' \subset \Sigma$ be the complement of the singular subset. Blowing up the singularities of Σ we obtain a birational morphism $\pi\colon \widetilde{\Sigma} \to \Sigma$, such that $\widetilde{\Sigma}$ is a smooth surface.

Proposition 54. *$\omega_{\Sigma'}$ is trivial, and $h^1(\widetilde{\Sigma}, \mathcal{O}_{\widetilde{\Sigma}}) = 0$.*

Proof. Let $X' = q^{-1}(\Sigma')$. The map q is unramified over Σ', and thus $q^*\Omega^1_{\Sigma'} = \Omega^1_{X'}$. Hence $q^*\omega_{\Sigma'} = \omega_{X'}$ is trivial.

Since Σ' is smooth, the theory of trace implies that $h^0(X', \Omega^p_{X'}) = h^0(\Sigma', \Omega^p_{\Sigma'})$ for $p = 1, 2$. On the other hand $h^0(X', \Omega^p_{X'}) = h^0(X, \Omega^p_X)$ because $X \backslash X'$ is of codimension at least 2. By proposition 52, $\omega_{\Sigma'}$ is 2-torsion. It has a nonzero section, and so must be trivial.

Similarly, $h^0(p^{-1}(\Sigma'), \Omega^1_{\widetilde{\Sigma}}) = h^0(\Sigma', \Omega^1_{\Sigma'}) = 0$, and so $h^0(\widetilde{\Sigma}, \Omega^1_{\widetilde{\Sigma}}) = 0$. Then Hodge theory implies, that $h^1(\widetilde{\Sigma}, \mathcal{O}_{\widetilde{\Sigma}}) = 0$. $\square$

Proposition 55. *Let L be an ample line bundle on Σ. A general divisor $C \in |L|$ is a reduced, irreducible and smooth curve C of genus $g = \dim |L| \geqslant 1$. Moreover, q^*C is connected, $q^*C \to C$ is étale, and $g(q^*C) = 2g - 1$.*

Proof. The theorem of Bertini says, that a general section $C \in |L|$ is irreducible, reduced and its singularities are contained in the singular locus of Σ. This locus consists of a finite number of points, and hence a general section is smooth.

Take a smooth irreducible curve $C \in |L|$ not meeting the singular locus. The curve q^*C is connected since it is an ample divisor. Positivity of $(q^*C)^2$ implies positivity of $g(q^*C)$. Moreover, $q^*C \to C$ is étale, and so $g = g(C) > 0$. Riemann-Hurwitz formula shows, that $g(q^*C) = 2g - 1$.

Consider a short exact sequence

$$0 \to \mathcal{O}_\Sigma \to L \to L|_C \to 0$$

and pull it back to $\widetilde{\Sigma}$:

$$0 \to \mathcal{O}_{\widetilde{\Sigma}} \to \pi^* L \to \pi^* L|_{\pi^* C} \to 0.$$

Since π is not flat this sequence is a priori only right exact. Nevertheless, since the morphism $\mathcal{O}_\Sigma \to L$ is given by a section of L, its pullback is also injective. Next, consider induced cohomology sequence:

$$0 \to H^0(\widetilde{\Sigma}, \mathcal{O}_{\widetilde{\Sigma}}) \to H^0(\widetilde{\Sigma}, \pi^* L) \to H^0(\pi^* C, \pi^* L|_{\pi^* C}) \to 0.$$

It is exact since $h^1(\widetilde{\Sigma}, \mathcal{O}_{\widetilde{\Sigma}}) = 0$. Moreover, $\pi^* L|_{\pi^* C} = L|_C$ since $C \subset \Sigma'$.

We now construct a map $H^0(\widetilde{\Sigma}, \pi^* L) \to H^0(\Sigma, L)$ as follows. First of all, we have a natural injection $H^0(\widetilde{\Sigma}, \pi^* L) \to H^0(\pi^{-1}\Sigma', \pi^* L|_{\Sigma'}) = H^0(\Sigma', L|_{\Sigma'})$. The restriction map $H^0(\Sigma, L) \to H^0(\Sigma', L|_{\Sigma'})$ is an isomorphism as $\Sigma \backslash \Sigma'$ is of codimension at least 2. Composing these maps we obtain what we need. Our map clearly splits the pullback map $H^0(\Sigma, L) \to H^0(\widetilde{\Sigma}, \pi^* L)$, and so this pullback is in fact an isomorphism.

Thus we demonstrated that there exists an exact sequence

$$0 \to H^0(\Sigma, \mathcal{O}_\Sigma) \to H^0(\Sigma, L) \to H^0(C, L|_C) \to 0.$$

By adjunction formula, $\omega_C = (\omega_{\Sigma'} \otimes L)|_C = L|_C$. Hence $h^0(C, L|_C) = g$, and we conclude, that $\dim |L| = g$. □

Proposition 56. *Consider the correspondence $\Gamma \colon X \to X$ defined as $\Gamma = \Delta - \Gamma_i$, where Γ_i is the graph of i. There exists an integer $g \geqslant 1$ and a nonempty Zariski-open subset $U \subset X^g$ such that all fibers of $\Gamma_* \colon U \to CH_0(X)$ are at least one-dimensional.*

Proof. We take an ample line bundle $L \in \operatorname{Pic} \Sigma$ and let $V \subset |L|$ be the open subvariety of curves which are smooth and do not meet the singular locus. Let $\pi \colon \mathcal{C} \to V$ be the universal bundle of curves from $|L|$ over V. By proposition 55 there exists a fibration $\widetilde{\pi} \colon \widetilde{\mathcal{C}} \to \mathcal{C}$, $\widetilde{\mathcal{C}} \subset V \times X$, such that for $C \in V$ the fiber $\widetilde{\pi}^{-1}\pi^{-1}C$ is the curve q^*C. We thus have an induced morphism of Jacobian fibrations $J(\widetilde{\mathcal{C}}) \to J(\mathcal{C})$ over V, and correspondingly a Prym fibration $\mathcal{P}(\widetilde{\mathcal{C}}/\mathcal{C}) = \ker(J(\widetilde{\mathcal{C}}) \to J(\mathcal{C}))$. A priori the fibers of $\mathcal{P}(\widetilde{\mathcal{C}}/\mathcal{C})$ are not necessarily connected, i.e. they are just commutative projective group schemes.

Let $\mathcal{C}^g$ be the g-th cartesian power of $\mathcal{C}$ *over* V. By proposition 51 there exists a nonempty Zariski-open subset $U_0 \subset \Sigma^g$ and a morphism $U_0 \to \mathcal{C}^g$ sending a point $(\sigma_1, \ldots, \sigma_g)$ into the unique curve $C \in V$ which supports σ_i. Put $U = q^{-1}U_0$. There exists an analogous morphism $U \to \widetilde{\mathcal{C}}^g$.

Recall that every fibration in smooth curves $\widetilde{\mathcal{C}}$ is equipped with corresponding Albanese morphism $\operatorname{alb} \colon \widetilde{\mathcal{C}}^g \times_V \widetilde{\mathcal{C}}^g \to J(\widetilde{\mathcal{C}})$ for each $g \geqslant 1$:

$$\operatorname{alb}(\widetilde{C}; x_1, \ldots, x_g; y_1, \ldots, y_g) = \operatorname{alb}_{\widetilde{C}}(x_1 + \ldots + x_g - y_1 - \ldots - y_g) \in J(\widetilde{C}).$$

Thus there exists a morphism $f\colon U \to J(\widetilde{\mathcal{C}})$ defined as follows:

$$f(x_1, \ldots, x_g) = \mathrm{alb}_{\widetilde{C}}(x_1 + \ldots + x_g - i(x_1) - \ldots - i(x_g))$$

(here $\widetilde{C}$ is the unique curve supporting x_i).

Since $q_*(x_1 + \ldots + x_g - i(x_1) - \ldots - i(x_g)) = 0$, the morphism f factors through $\mathcal{P}(\widetilde{\mathcal{C}}/\mathcal{C})$. Assume that there exists a point $x \in U$ such that $\dim_x \mathcal{P}(\widetilde{\mathcal{C}}/\mathcal{C})_{f(x)} = 0$. By upper semi-continuity of fiber dimension we conclude that there exists a nonempty Zariski-open subset $U' \subset U$ of such points. The dimension of U' conincides with that of U, i.e. with $2g$. On the other hand, $\dim \mathcal{P}(\widetilde{\mathcal{C}}/\mathcal{C}) = 2g-1$, which contradicts the existence of U'. Therefore all fibers of f are at least one-dimensional.

It remains to notice that Γ_* is constant along the fibers of f. □

Proposition 57. *Consider the correspondence $\Gamma\colon X \to X$ defined as $\Gamma = \Delta - \Gamma_i$, where Γ_i is the graph of i. There exists an integer $g \geqslant 1$ such that all the fibers of $\Gamma_*\colon X^g \to CH_0(X)$ are at least one-dimensional.*

Proof. By proposition 56 our claim holds for some nonempty Zariski-open subset $U \subset X^g$. Baire category theorem shows that U can not be covered by a countable union of proper closed subsets. Hence, proposition 29 implies that no fiber of $\Gamma_*\colon X^g \to CH_0(X)$ can be zero-dimensional. □

Theorem 10. *i_* acts as identity on $CH_0(X)$, i.e. the conjecture of Bloch holds in this case.*

Proof. Consider $\Gamma = \Delta - \Gamma_i$. By proposition 57 there exists an integer $g \geqslant 1$ such that $\Gamma_*\colon X^g \to CH_0(X)$ has at least one-dimensional fibers, and so $\Gamma_* CH_0(X)_{\mathrm{hom}}$ is finite-dimensional, as proposition 53 demonstrates. Applying theorem 3 we conclude that $\Gamma_*\colon CH_0(X)_{\mathrm{hom}} \to CH_0(X)$ factors through $\mathrm{Alb}\, X = 0$. Moreover, $\Gamma_* c_0 = 0$, where $c_0 \in CH_0(X)$ is the special zero-cycle of Beauville-Voisin, and so $\Gamma_* = 0$. In other words, $i_* = \Delta_* = \mathrm{id}$. □

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References

[1] A. Beauville, C. Voisin. On the Chow ring of a K3 surface. *J. Algebraic Geom.* **13**, 417 – 426. (2004)

[2] S. Bloch. *Lectures on Algebraic Cycles*. Duke Univ. Math. Series IV. Duke University, Mathematics Department, Durham, N.C. (1980)

[3] S. Bloch, A. Kas, D. Lieberman. Zero cycles on surfaces with $p_g = 0$. *Compositio Math.* **33**, 135 – 145. (1976)

[4] S. Bloch and V. Srinivas, Remarks on Correspondences and Algebraic cycles, *Amer. J. Math.* **105**, 1235 – 1253. (1980)

[5] D. Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry*. Graduate Texts in Math. **150**. Springer, Berlin. (1994)

[6] W. Fulton, *Intersection Theory*. Springer, Berlin. (1984)

[7] J. Kollár. *Rational Curves on Algebraic Varieties*. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. Springer, Berlin. (1996)

[8] R. Laterveer. Algebraic varieties with small Chow groups, *J. Math. Kyoto Univ.* **38**, 673 – 694. (1998)

[9] Q. Liu. *Algebraic Geometry and Arithmetic Curves*, Oxford University Press, Oxford. (2006)

[10] S. Mori and S. Mukai. The uniruledness of the moduli space of curves of genus 11. In *Algebraic geometry (Tokyo/Kyoto, 1982)*, volume 1016 of *Lecture Notes in Math.*, 334 – 353. Springer, Berlin, 1983.

[11] D. Mumford. Rational equivalence of zero-cycles on surfaces, *J. Math. Kyoto Univ.* **9**, 195 – 204. (1968)

[12] K. Paranjape. Cohomological and cycle-theoretic connectivity, *Ann. Math.* **139**, 641 – 660. (1994)

[13] A. Roitman. Rational equivalence of zero-cycles, *Math USSR Sbornik* **18**, 571 – 588. (1972)

[14] C. Voisin, Symplectic involutions of K3 surfaces act trivially on CH_0. *Documenta Math.* **17**, 851 – 860. (2012)

[15] C. Voisin. *Complex Algebraic Geometry and Hodge Theory II*. Cambridge studies in advanced mathematics **77**. Cambridge University Press, Cambridge. (2003)