On the heart associated to a faithful torsion pair

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To my parents.

Contents

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Introduction

1. Statement of the problem

1.1. Generalities. The present work deals with the following problem. Let R be an associative ring, and Mod-R the category of all right R-modules. Given a torsion pair $(\mathcal{X}, \mathcal{Y})$ in Mod-R, there naturally arises a full subcategory of the derived category of Mod-R, called the *heart* of the torsion pair, and denoted by $\mathcal{H}(\mathcal{X}, \mathcal{Y})$. Its objects are the bounded complex of R-modules $X^\bullet$ such that $H^i(X^\bullet) = 0$ for every $i \neq -1$, 0, and, moreover, $H^{-1}(X^\bullet) \in \mathcal{Y}$ and $H^0(X^\bullet) \in \mathcal{X}$, where H^i denotes the i-th cohomology functor.
In [**BBD82**] it was proved that $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ is abelian. There are two pairs of functors: the two cohomology functors H^{-1}, $H^0 \colon \mathcal{H}(\mathcal{X}, \mathcal{Y}) \to$ Mod-R and the functors T, $T' \colon$ Mod-$R \to \mathcal{H}(\mathcal{X}, \mathcal{Y})$ taking a module M to the complexes $T(M)$, $T'(M)$ defined as follows: $T(M)^i = 0$ for every $i \neq -1$ and $T(M)^{-1}$ is the torsion-free part of M; similarly, $T(M)^i = 0$ for every $i \neq 0$ and $T(M)^0$ is the torsion part of M. The images of these functors define a torsion pair in $\mathcal{H}(\mathcal{X}, \mathcal{Y})$. When the torsion pair $(\mathcal{X}, \mathcal{Y})$ is faithful, i.e., $R_R \in \mathcal{Y}$, then the two pairs of functors define a tilting counter equivalence between $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ and Mod-R.
The main purpose of this work is to give a full and, if possible, self-contained introduction to some properties of the heart, in particular those properties which are related to the properties of the torsion pair $(\mathcal{X}, \mathcal{Y})$, and, furthermore, to exhibit some examples. For instance, a necessary and sufficient condition for the heart to be a Grothendieck category is that the original torsion pair is cotilting, i.e., cogenerated by a cotilting module.

1.2. Known results. In [**CGM07**] it is proved that a necessary condition for the heart $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ to be a Grothendieck category is that $(\mathcal{X}, \mathcal{Y})$ is cotilting. In [**CG09**] this condition is proved to be also sufficient. Moreover, several notable consequences are drawn: for example, the heart $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ is Grothendieck, locally noetherian if and only if $\mathcal{Y}$ is cogenerated by a Σ-pure injective cotilting module (this has been proved, using different methods, also in [**CMT10**]).

1.3. Original contributions. In Theorem 3.12 we prove that the definition of tilting object in the category Mod-R is equivalent to the classical definition of tilting right R-module. Furthermore, in Example 3.13 we exhibit a non-trivial example of tilting module over a ring of matrices. In Example 6.14 we show that there are no abelian groups which are Σ-pure injective, apart from those containing $\mathbb{Q} \oplus \mathbb{Q}/\mathbb{Z}$ as a direct summand. As a consequence, given any faithful, cotilting, non-trivial torsion pair $(\mathcal{X}, \mathcal{Y})$ in Mod-$\mathbb{Z}$, the heart $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ is a Grothendieck category, but it is not locally noetherian.

2. Background for the work

We shall give a short sketch of the development of derived categories, tilting theory and torsion pairs in order to give a better understanding of the tools used in our work.

2.1. Derived categories. Derived categories were introduced in the sixties by Grothendieck and Verdier in the study of derived functors and spectral sequences.
Given an abelian category $\mathcal{C}$, its derived category $\mathrm{D}(\mathcal{C})$ is obtained from the category $\mathrm{C}(\mathcal{C})$ of (cochain) complexes in two stages. First one constructs a quotient $\mathrm{K}(\mathcal{C})$ of $\mathrm{C}(\mathcal{C})$ by identifying chain homotopy equivalent morphisms between complexes. Such a quotient category $\mathrm{K}(\mathcal{C})$ is called the *homotopy category* of $\mathcal{C}$. It is an additive category in which the homotopy equivalences are invertible, but this advantage comes at a cost: $\mathrm{K}(\mathcal{C})$ is not abelian anymore. Consequently, one has to look for an effective substitute for short exact sequences, that should

still have the property that it induces long exact sequences on cohomology. The concept of a triangulated category with its "distinguished triangles" provides a solution. It turns out that the homotopy category $\mathrm{K}(\mathcal{C})$ can be endowed with the structure of a triangulated category. The second step consists in "localizing" $\mathrm{K}(\mathcal{C})$ by inverting quasi-isomorphisms using a calculus of fractions. The goal is the following: we want morphisms in $\mathrm{K}(\mathcal{C})$ which induce isomorphisms on cohomology to be invertible in the category to be constructed. If we want a quasi-isomorphism to become an isomorphism, it has to have an inverse. Unfortunately, there is no candidate for an inverse around. So to define a derived category, we have to use a more elaborate process called localization of categories.

Derived categories have proved to be of fundamental importance in mathematics, in particular they have been shown to be the correct setting for tilting theory: indeed, they allow the main results to be easily formulated and proved, and also offer new insights concerning homological properties shared by the algebras involved. For the general use of derived categories and tilting the reader may consult the article by Keller [**Kel07**].

2.2. Tilting theory. Tilting theory arises from the representation theory of finite dimensional algebras and has proved to be a universal method for constructing equivalences between categories. It is now considered essential in the study of many areas of mathematics, including group theory, commutative and non-commutative algebraic geometry, and algebraic topology.

Tilting theory goes back to the early seventies, when Bernstein, Gelfand and Ponomarev investigated the reflection functors in connection with giving a new proof of Gabriel's theorem (1972) that a path algebra $k\Delta$ of a finite quiver Δ over an algebraically closed field k admits only finitely many isomorphism classes of indecomposable modules precisely when the underlying graph of Δ is a finite disjoint union of Dynkin diagrams (see [**Gab72**]). This work was later generalized by Brenner and Butler in [**BB80**], who introduced the actual notion of a tilting module for finite dimensional and artin algebras A and the resulting tilting theorem between mod-A and the finitely generated modules over the endomorphism ring of a tilting A-module, a vast generalization of the Morita equivalence theorem between categories of modules over a pair of algebras. Later the work of Brenner and Butler was simplified by Happel and Ringel [**HR82**]. They considered additional functors such as $\mathrm{Ext}^1(T,-)$, in order to obtain a much more complete picture. Subsequently, Miyashita [**Miy86**] and Colby and Fuller [**CF90**] showed that if A is an arbitrary ring and V_A is a tilting module, then the tilting theorem holds between Mod-A and Mod-R, where $R = \mathrm{End}\, V_A$. The tilting theorem is basically a pair of equivalences $\mathcal{T} \rightleftarrows \mathcal{Y}$ and $\mathcal{F} \rightleftarrows \mathcal{X}$ between the members of torsion pairs $(\mathcal{T},\mathcal{F})$ of A-modules and $(\mathcal{X},\mathcal{Y})$ of R-modules. In order to generalize the notion of a finitely generated tilting module, Colpi and Fuller [**CF07**] introduced the notion of a tilting object in an arbitrary abelian category $\mathcal{A}$; in that paper a tilting theorem is proved, showing that a tilting object V provides a counter equivalence between the torsion theory $(\mathcal{T},\mathcal{F})$ generated in $\mathcal{A}$ by V and the corresponding tilted torsion pair $(\mathcal{X},\mathcal{Y})$ in Mod-R, with $R = \mathrm{End}_{\mathcal{A}} V$.

Nowadays techniques of tilting theory apply to (derived) geometry of varieties, noncommutative geometry, representation theory (of finite groups, algebraic groups, quantum groups, quivers, ...), cluster algebras, and so on. A precursor for the techniques of tilting theory in geometry is the work of Beilinson [**Beĭ84**] which relates the derived category of coherent sheaves on a projective space to the derived category of a certain finite dimensional noncommutative algebra: the derived category of coherent sheaves on a projective space $\mathbb{P}^n$ is equivalent to the homotopy category of free graded modules with generators in degree $0, 1, \ldots, n$ over a symmetric or exterior algebra in $n+1$ variables. This work was further developed by Bondal [**Bon89**] and has now become a standard tool in the study of derived categories of varieties.

Further aspects of tilting theory and historical references are contained in [**AHHK07**].

2.3. Torsion pairs. The concept of torsion is of fundamental importance in algebra, geometry and topology, because torsion-theoretic methods allow us to study better phenomena having a local structure. The context of torsion pairs provides the proper environment to study the notion of torsion.

Torsion pairs in abelian categories were introduced formally by Dickson [**Dic66**] as a generalization of the well known torsion pair (*Torsion abelian groups*, *Torsion-free abelian groups*) in the category of abelian groups. The reader may also see the book of Stenström [**Ste75**] for a comprehensive treatment. The use of torsion pairs became then indispensable for the study of localization in various context, such as the localization of topological spaces or spectra, the localization theory of rings and categories, the local study of an algebraic variety, and the tilting theory. Let us give a simple example. It is well known that there is a bijection between Gabriel topologies on a ring A and hereditary torsion pairs of A-modules (see [**Ste75**, Chapter VI]). Let A be a commutative ring and let S be a multiplicative subset of A consisting of non-zero divisors. Then $\mathfrak{F} = \{\mathfrak{a} \mid \mathfrak{a}$ is a right ideal of A and $(\mathfrak{a}\colon a) \cap S \neq \emptyset, \forall a \in A\}$ is a Gabriel topology on A. Let M be an A-module and put $M_{\mathfrak{F}} = \varinjlim_{s\in S} \operatorname{Hom}(sA, M/t(M))$, where $t(M) = \{x \in M \mid xs = 0 \text{ for some } s \in S\}$. Then we get $M_{\mathfrak{F}} \cong S^{-1}M$, in particular $A_{\mathfrak{F}} \cong S^{-1}A$. There is an analogous definition of torsion pair for triangulated categories which is closely related to the notion of a t-structure (see [**BR07**]).
More recently, Beligiannis and Reiten in [**BR07**] studied torsion pairs in the general working context of *pretriangulated categories*. A pretriangulated category is essentially an additive category $\mathcal{C}$ equipped with a pair (Σ, Ω) of adjoint endofunctors, and in addition with a class of left triangles Δ and a class ∇ of right triangles which are compatible with each other and with Σ and Ω. Examples are abelian categories (where $\Sigma = 0 = \Omega$) and triangulated categories (where Σ or Ω is an equivalence). Other important sources of examples of pretriangulated categories come from stable categories (namely, $\mathcal{C}/\omega$, where $\mathcal{C}$ is an abelian category and ω is a functorially finite subcategory, see [**BR07**]) and closed model categories in the sense of Quillen [**Qui67**] and their homotopy categories (see the book of Hovey [**Hov99**] for a comprehensive treatment). In recent years stable categories have proved to be essential in homological representation theory, through the work of Happel [**Hap88**], Keller [**Kel90**], Happel-Reiten-Smalø [**HRO96**], Krause [**Kra00**] and others.
Tilting theory is intimately related to torsion pairs in several different ways. For example, when T is a tilting module, there is an associated torsion pair $(\mathcal{T}, \mathcal{F})$, where $\mathcal{T} = \operatorname{Gen} T$. This torsion pair plays an essential role in tilting theory, and it is closely related to a torsion theory for the category of modules over the ring $\operatorname{End} T$. During the last twenty years Morita theory for module categories, which can be regarded as a specialization of tilting theory, has been extended to derived categories, offering new invariants and levels of classifications. The generalization culminated in a Morita Theory for derived categories of rings and DG-algebras, which describes when derived categories of rings or DG-algebras are equivalent as triangulated categories. This provides also connections with algebraic geometry and topology. Beligiannis and Reiten [**BR07**] interpreted Morita theory for derived categories in torsion-theoretic terms, giving simple torsion-theoretic proofs of central results concerning the constructions of derived equivalences.

3. Structure of the work

The work is organized as follows.

In Chapter 1 we recall the definition and the basic properties of the derived category of an abelian category.

Chapter 2 is devoted to the study of torsion pairs and t-structures in the context of abelian and triangulated categories, respectively. For an abelian category $\mathcal{C}$ with a torsion pair $(\mathcal{X}, \mathcal{Y})$ we want to construct an abelian category $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ with a torsion pair $(\mathcal{T}, \mathcal{F})$ such that $\mathcal{X}$ is equivalent to $\mathcal{F}$ and $\mathcal{Y}$ is equivalent to $\mathcal{T}$. This construction is motivated by the connection between tilting and derived categories, hence we try to find an abelian category $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ in $\mathrm{D}^b(\mathcal{C})$ with $(\mathcal{Y}[1], \mathcal{X}[0])$ as a torsion pair. First we show how a torsion pair $(\mathcal{X}, \mathcal{Y})$ in $\mathcal{C}$ gives rise to a t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ on the bounded derived category $\mathrm{D}^b(\mathcal{C})$. More precisely, we take $\mathcal{D}^{\leq 0} = \{X^\bullet \in \mathrm{D}^b(\mathcal{C}) \mid \mathrm{H}^i(X^\bullet) = 0, i > 0, \mathrm{H}^0(X^\bullet) \in \mathcal{X}\}$ and $\mathcal{D}^{\geq 0} = \{X^\bullet \in \mathrm{D}^b(\mathcal{C}) \mid$

$\mathrm{H}^i(X^\bullet) = 0$, $i < -1$, $\mathrm{H}^{-1}(X^\bullet) \in \mathcal{Y}\}$. Then $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is a t-structure on $\mathrm{D}^b(\mathcal{C})$. The heart of this t-structure, the full subcategory $\mathcal{H}(\mathcal{X}, \mathcal{Y}) = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$, is an abelian category, as it was originally proved in [**BBD82**]. Moreover, setting $\mathcal{T} = \mathcal{Y}[1]$ and $\mathcal{F} = \mathcal{X}[0]$, it turns out that $(\mathcal{T}, \mathcal{F})$ is a torsion pair in $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ with equivalences $\mathcal{T} \rightleftarrows \mathcal{Y}$ and $\mathcal{F} \rightleftarrows \mathcal{X}$.

In Chapter 3 the basic ideas of tilting theory are introduced as a tool for the subsequent study of the heart associated to a (faithful) torsion pair in Mod-R. Following [**CF07**], we introduce the notion of tilting object in abelian categories and see how it specializes in the case of Grothendieck categories. La raison d'etre of a tilting object V in an abelian category $\mathcal{A}$ is that it provides a connection between the category $\mathcal{A}$ and the category of modules over the ring $R = \mathrm{End}_{\mathcal{A}}(V)$. This connection is described in the Tilting Theorem. We shall also prove that the notion of tilting object coincide with the classical one (see [**CF04**]) in the case of modules. Here we also exhibit a non-trivial example of tilting module over a ring of matrices. Furthermore, we shall see that the tilting modules over a commutative ring are precisely the progenerators, i.e., the finitely generated projective generators. In this case, the Tilting Theorem reduces to the classical Morita equivalence between two categories of modules.

In Chapter 4 we begin the study of the heart associated to a faithful torsion pair in Mod-R. This chapter yields a motivation for the study of the heart. First, we shall characterize the faithful torsion pairs $(\mathcal{X}, \mathcal{Y})$ in Mod-R as the torsion pairs which are tilted by means of a tilting object V in a suitable cocomplete abelian category $\mathcal{H}$. More precisely, we will show that any tilted torsion pair is faithful and, conversely, given any ring R and any faithful torsion pair $(\mathcal{X}, \mathcal{Y})$ in Mod-R, then $(\mathcal{X}, \mathcal{Y})$ is tilted by a tilting object in a cocomplete abelian category $\mathcal{H}$. In particular, it will be clear that we may choose $\mathcal{H} = \mathcal{H}(\mathcal{X}, \mathcal{Y})$ and $V = R[1]$. Then we will show that $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ is a prototype for any abelian category $\mathcal{H}$ admitting a tilting object V which tilts to $(\mathcal{X}, \mathcal{Y})$ in Mod-R, i.e., any abelian category which tilts to the same torsion pair is equivalent to it. This is obtained as a consequence of the fact that two abelian categories with tilting objects which tilt to the same target $(\mathcal{X}, \mathcal{Y})$ are (Morita) equivalent.

In Chapter 5 we study the AB-properties of the heart $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ that, more generally, are shared by any abelian category with a tilting object which tilts to the same torsion pair $(\mathcal{X}, \mathcal{Y})$. In particular, we shall show that the heart is an AB4 category, i.e., arbitrary coproducts exist and any coproduct of exact sequences is an exact sequence. Moreover, we prove that the cohomological functors H^{-1}, H^0 which realize the tilting counter equivalence commute with coproducts. Furthermore, the heart has also products, i.e., it is an AB3* category. Then we look for some conditions which guarantee that $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ is a Grothendieck category: we will see that a key point is the behaviour of the functor H^{-1} with respect to direct limits. Finally, we show that $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ has an injective cogenerator if and only if $(\mathcal{X}, \mathcal{Y})$ is cogenerated by a cotilting R-module. As a corollary, a necessary condition for the $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ to be Grothendieck is that $(\mathcal{X}, \mathcal{Y})$ is cogenerated by a cotilting R-module.

The main purpose of Chapter 6 is to show that the heart $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ is a Grothendieck category if and only if $(\mathcal{X}, \mathcal{Y})$ is cogenerated by a cotilting R-module. From this, we will be able to draw several important consequences, by applying the techniques of tilting counter equivalences between a Grothendieck category and the category Mod-R. When $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ is Grothendieck, locally noetherian, the functors H^{-1}, $H^0 \colon \mathcal{H}(\mathcal{X}, \mathcal{Y}) \to \text{Mod-}R$ take noetherian objects to finitely generated modules, hence the torsion pair $(\mathcal{X}, \mathcal{Y})$ satisfies the so-called *Ringel's condition* (see [**RR06**]). We show that, when the ring R is noetherian, this condition is actually equivalent to $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ being Grothendieck, locally noetherian; this proves that the Ringel's condition is also equivalent to the fact that the class $\mathcal{Y}$ is cogenerated by a Σ-pure injective cotilting module.

CHAPTER 1

Preliminaries

In this chapter we recall the basic feautures of the derived category of an abelian category needed in the subsequent development. For a full account we refer to [**KS06**]. For standard terminology in category theory, we refer to [**KS06**] and [**Ste75**].

1. Complexes

1.1. Complexes in additive categories. In this section we let $\mathcal{C}$ be an additive category. A **complex** $X^\bullet = (X^k,\, d_X^k)$ over $\mathcal{C}$ is a sequence of objects X^k and morphisms $d_X^k\colon X^k \to X^{k+1}$ $(k \in \mathbb{Z})$

$$\cdots \to X^{k-1} \xrightarrow{d_X^{k-1}} X^k \xrightarrow{d_X^k} X^{k+1} \to \dots$$

such that $d_X^k d_X^{k-1} = 0$ for all $k \in \mathbb{Z}$. A complex is **bounded below** if $X^k = 0$ for all but finitely many $k < 0$. It is **bounded above** if $X^k = 0$ for all but finitely many $k > 0$. It is **bounded** if it is bounded below and bounded above. We denote by $\mathrm{C}(\mathcal{C})$ the additive category of complexes and by $\mathrm{C}^*(\mathcal{C})$ $(* = b,\, +,\, -)$ the full additive subcategory of $\mathrm{C}(\mathcal{C})$ consisting of bounded complexes (resp. bounded below, bounded above). We may consider $\mathcal{C}$ as a full subcategory of $\mathrm{C}^b(\mathcal{C})$ by identifying each object X of $\mathcal{C}$ with the complex $\cdots \to 0 \to X \to 0 \to \dots$ "concentrated in degree 0", where X stands in degree 0.
Let $X \in \mathrm{Ob}(\mathrm{C}(\mathcal{C}))$ and $p \in \mathbb{Z}$. The **shift functor** is defined by

$$(X[p])^n = X^{n+p}, \qquad d^n_{X[p]} = (-1)^p d_X^{n+p}$$

and if $f\colon X \to Y$ is a morphism in $\mathrm{C}(\mathcal{C})$, then

$$(f[p])^n = f^{n+p}.$$

The shift functor $[1]\colon \mathrm{C}(\mathcal{C}) \to \mathrm{C}(\mathcal{C})$, $X \mapsto X[1]$ is an automorphism (that is, an invertible functor) of $\mathrm{C}(\mathcal{C})$.
The **mapping cone** of a morphism $f\colon X \to Y$ in $\mathrm{C}(\mathcal{C})$ is the complex $(\mathrm{Mc}(f),\, d_{\mathrm{Mc}(f)})$, where $(\mathrm{Mc}(f))^k = (X[1])^k \oplus Y^k$ and $d^k_{\mathrm{Mc}(f)} = \begin{pmatrix} d^k_{X[1]} & 0 \\ f^{k+1} & d^k_Y \end{pmatrix}$. There are natural morphisms of complexes $\alpha(f)\colon Y \to \mathrm{Mc}(f)$, $\beta(f)\colon \mathrm{Mc}(f) \to X[1]$ and $\beta(f)\alpha(f) = 0$.
A morphism $f\colon X \to Y$ in $\mathrm{C}(\mathcal{C})$ is said to be **homotopic to zero** if for all $p \in \mathbb{Z}$ there exists a morphism $s^p\colon X^p \to Y^p$ such that $f^p = s^{p+1} d_X^p + d_Y^{p-1} s^p$. Two morphisms $f,\, g\colon X \to Y$ are **homotopic** if the morphism $f - g\colon X \to Y$ is homotopic to zero. A complex X is **homotopic to 0** if the identity morphism 1_X is homotopic to zero.
Let X, Y be two complexes and define

$$\mathrm{Ht}(X,\, Y) = \{f\colon X \to Y \mid f \text{ is homotopic to zero}\}.$$

If $f\colon X \to Y$ and $g\colon Y \to Z$ are morphisms in $\mathrm{C}(\mathcal{C})$ and if f or g is homotopic to zero, then $gf\colon X \to Z$ is homotopic to zero. This allow us to define a new category, the **homotopy category** $\mathrm{K}(\mathcal{C})$, in which the objects are complexes over $\mathcal{C}$, and for all $X,\, Y \in \mathrm{Ob}(\mathrm{K}(\mathcal{C}))$, $\mathrm{Hom}_{\mathrm{K}(\mathcal{C})}(X, Y) = \mathrm{Hom}_{\mathrm{C}(\mathcal{C})}(X, Y)/\,\mathrm{Ht}(X,\, Y)$. In other words, a morphism homotopic to zero in $\mathrm{C}(\mathcal{C})$ becomes the zero morphism in $\mathrm{K}(\mathcal{C})$ and a homotopy equivalence in $\mathrm{C}(\mathcal{C})$ becomes an isomorphism in $\mathrm{K}(\mathcal{C})$. Similarly, we define $\mathrm{K}^*(\mathcal{C})$ for $* = b,\, +,\, -$. These categories are clearly additive, but in general not abelian.

1.2. Complexes in abelian categories. In this section $\mathcal{C}$ denotes an abelian category. In [**KS06**] it is proved that the categories $\mathrm{C}^*(\mathcal{C})$ are abelian categories. Let $X \in \mathrm{Ob}(\mathrm{C}(\mathcal{C}))$. We define the following objects of $\mathcal{C}$:

$$\mathrm{Z}^n(X) = \mathrm{Ker}\, d_X^n, \qquad \mathrm{B}^n(X) = \mathrm{Im}\, d_X^{n-1}.$$

Note that there is a natural morphism $\mathrm{B}^n(X) \to \mathrm{Z}^n(X)$, so we can define the **n-th cohomology object** of X:

$$\mathrm{H}^n(X) = \mathrm{Z}^n(X)/\mathrm{B}^n(X).$$

If $f\colon X \to Y$ is a morphism in $\mathrm{C}(\mathcal{C})$, then it induces morphisms $\mathrm{Z}^n(X) \to \mathrm{Z}^n(Y)$ and $\mathrm{B}^n(X) \to \mathrm{B}^n(Y)$, hence a morphism $\mathrm{H}^n(X) \to \mathrm{H}^n(Y)$. Therefore, we have an additive functor $\mathrm{H}^n\colon\ \mathrm{C}(\mathcal{C}) \to \mathcal{C}$. By [**KS06**, Lemma 12.2.2], this functor can be extended to a functor $\mathrm{H}^n\colon\ \mathrm{K}(\mathcal{C}) \to \mathcal{C}$. A morphism $f\colon X \to Y$ in $\mathrm{C}(\mathcal{C})$ is said to be a **quasi-isomorphism** (for short, a **qis**) if $\mathrm{H}^k(f)\colon \mathrm{H}^k(X) \to \mathrm{H}^k(Y)$ is an isomorphism for all $k \in \mathbb{Z}$. In this case we say that X and Y are **quasi-isomorphic**. Clearly, a complex X is exact if and only if it is quasi-isomorphic to zero.

2. Triangulated categories

In this section we shall recall the definition of triangulated category.
Let $\mathcal{D}$ be an additive category endowed with an automorphism [1]. A **triangle** in $\mathcal{D}$ is a sequence of morphisms

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1].$$

A morphism of triangles is given by the commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha"'] & Y \arrow[r, "g"] \arrow[d, "\beta"'] & Z \arrow[r, "h"] \arrow[d, "\gamma"'] & X[1] \arrow[d, "\alpha[1]"'] \\ X' \arrow[r, "f'"'] & Y' \arrow[r, "g'"'] & Z' \arrow[r, "h'"'] & X'[1]. \end{tikzcd}$$

A **triangulated category** is an additive category $\mathcal{D}$ endowed with an automorphism [1] and a family of triangles, called **distinguished triangles** (for short, **d.t.**), satisfying the following axioms.

(TR0) A triangle isomorphic to a d.t. is a d.t.

(TR1) The triangle $X \xrightarrow{1_X} X \to 0 \to X[1]$ ia a d.t.

(TR2) For all $f\colon X \to Y$ there exists a d.t. $X \xrightarrow{f} Y \to Z \to X[1]$.

(TR3) A triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ is a d.t. if and only if $Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]$ is a d.t.

(TR4) Given two d.t. $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$, $X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \xrightarrow{h'} X'[1]$ and morphisms $\alpha\colon X \to X'$, $\beta\colon Y \to Y'$ with $f'\alpha = \beta f$, there exists a morphism $\gamma\colon Z \to Z'$ giving rise to a morphism of d.t.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha"'] & Y \arrow[r, "g"] \arrow[d, "\beta"'] & Z \arrow[r, "h"] \arrow[d, dotted, "\gamma"'] & X[1] \arrow[d, "\alpha[1]"'] \\ X' \arrow[r, "f'"'] & Y' \arrow[r, "g'"'] & Z' \arrow[r, "h'"'] & X'[1]. \end{tikzcd}$$

(TR5) (*Octahedral axiom*) Given three d.t.

$$X \xrightarrow{f} Y \xrightarrow{h} Z' \to X[1],$$

$$Y \xrightarrow{g} Z \xrightarrow{k} X' \to Y[1],$$

$$X \xrightarrow{gf} Z \xrightarrow{l} Y' \to X[1],$$

there exists a d.t. $Z' \xrightarrow{\varphi} Y' \xrightarrow{\psi} X' \to Z'[1]$ making the diagram below commutative:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "h"] \arrow[d, "g"] & Z' \arrow[r] \arrow[d, dotted, "\varphi"] & X[1] \arrow[d, equal] \\
X \arrow[r, "gf"] \arrow[d, "f"] & Z \arrow[r, "l"] \arrow[d, equal] & Y' \arrow[r] \arrow[d, dotted, "\psi"] & X[1] \arrow[d, "f[1]"] \\
Y \arrow[r, "g"] \arrow[d, "h"] & Z \arrow[r, "k"] \arrow[d, "l"] & X' \arrow[r] \arrow[d, equal] & Y[1] \arrow[d, "h[1]"] \\
Z' \arrow[r, dotted, "\varphi"] & Y' \arrow[r, dotted, "\psi"] & X' \arrow[r, dotted] & Z'[1].
\end{tikzcd}$$

This diagram is often called the *octahedron diagram*, because it can be written using the vertexes of an octahedron (cfr. [**KS06**, Definition 10.1.6])

By [**KS06**, Proposition 10.1.11], if $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ is a d.t., then $gf = 0$.
Let $(\mathcal{D}, [1])$, $(\mathcal{D}', [1]')$ be triangulated categories. A functor $F : \mathcal{D} \to \mathcal{D}'$ is called a **triangulated functor** if:

(i) F is additive;
(ii) $F[1] \cong [1]'F$;
(iii) F sends d.t.'s to d.t.'s.

A subcategory $\mathcal{D}'$ of a triangulated category $(\mathcal{D}, [1])$ is called a **triangulated subcategory** if:

(i) $\mathcal{D}'$ is triangulated;
(ii) the inclusion functor $\mathcal{D}' \to \mathcal{D}$ is a triangulated functor.

Let $(\mathcal{D}, [1])$ be a triangulated category, $\mathcal{C}$ an abelian category and $F : \mathcal{D} \to \mathcal{C}$ an additive functor. We say that F is a **cohomological functor** if for any d.t. $X \to Y \to Z \to X[1]$ in $\mathcal{D}$, the sequence $F(X) \to F(Y) \to F(Z)$ is exact in $\mathcal{C}$.
As a remarkable example, the functors $\mathrm{Hom}_{\mathcal{D}}(W, -)$ and $\mathrm{Hom}_{\mathcal{D}}(-, W)$, for any $W \in \mathcal{D}$, are cohomological functors (see [**KS06**, Proposition 10.1.13]).

3. Derived categories

In this section we construct the derived category of an abelian category $\mathcal{C}$.

3.1. The homotopy category $\mathrm{K}(\mathcal{C})$. Let $\mathcal{C}$ be an additive category. Recall that the homotopy category $\mathrm{K}(\mathcal{C})$ is defined by identifying to zero the morphisms in $\mathrm{C}(\mathcal{C})$ homotopic to zero. If $f : X \to Y$ is a morphism in $\mathrm{C}(\mathcal{C})$, there is a natural triangle

$$Y \xrightarrow{\alpha(f)} \mathrm{Mc}(f) \xrightarrow{\beta(f)} X[1] \xrightarrow{f[1]} Y[1].$$

Such a triangle is called a **mapping cone triangle**. Clearly, a triangle in $\mathrm{C}(\mathcal{C})$ gives rise to a triangle in the homotopy category $\mathrm{K}(\mathcal{C})$. We define a **distinguished triangle** (for short, **d.t.**) in $\mathrm{K}(\mathcal{C})$ to be a triangle isomorphic in $\mathrm{K}(\mathcal{C})$ to a mapping cone triangle. By [**KS06**, Theorem 11.2.6], we have that the category $\mathrm{K}(\mathcal{C})$ endowed with the shift functor [1] and the family of d.t.'s is a triangulated category.

3.2. Derived categories. Let $\mathcal{C}$ be an abelian category. It is shown in [**KS06**] that there exists a triangulated category $\mathrm{D}^*(\mathcal{C})$ ($* = \mathrm{ub}, \mathrm{b}, +, -$), called the **derived category** $\mathcal{C}$, and a functor $Q\colon \mathrm{K}^*(\mathcal{C}) \to \mathrm{D}^*(\mathcal{C})$ such that:

(a) $Q(s)$ is an isomorphism in $\mathrm{D}^*(\mathcal{C})$ whenever s is a quasi-isomorphism;
(b) for any functor $F\colon \mathrm{K}^*(\mathcal{C}) \to \mathcal{A}$ such that $F(s)$ is an isomorphism whenever s is a quasi-isomorphism, there exists a functor $\tilde{F}\colon \mathrm{D}^*(\mathcal{C}) \to \mathcal{A}$ and a natural isomorphism $F \cong \tilde{F} \circ Q$;

(c) if G_1, G_2 are objects in the functor category $\mathrm{Fct}(\mathrm{D}^*(\mathcal{C}), \mathcal{A})$, then the natural map
$$\mathrm{Hom}_{\mathrm{Fct}(\mathrm{D}^*(\mathcal{C}),\mathcal{A})}(G_1, G_2) \to \mathrm{Hom}_{\mathrm{Fct}(\mathrm{K}^*(\mathcal{C}),\mathcal{A})}(G_1 \circ Q, G_2 \circ Q)$$
is a bijection.

Condition (c) means that the functor $- \circ Q\colon \mathrm{Fct}(\mathrm{D}^*(\mathcal{C}), \mathcal{A}) \to \mathrm{Fct}(\mathrm{K}^*(\mathcal{C}), \mathcal{A})$ is fully faithful. This implies that the functor $\tilde{F}$ in (b) is unique up to unique isomorphism.
The objects in $\mathrm{D}^*(\mathcal{C})$ are the objects in $\mathrm{K}^*(\mathcal{C})$.
A morphism $f\colon X \to Y$ in $\mathrm{D}^*(\mathcal{C})$ is an equivalence class of triplets (Y', t, f'), with $t\colon Y \to Y'$ a qis and $f'\colon X \to Y'$:
$$X \xrightarrow{f'} Y' \xleftarrow{t} Y \ ,$$
and the equivalence relation is defined as follows: $(Y', t, f') \sim (Y'', t', f'')$ if and only if there exists (Y''', t'', f''') (t, t', t'' quasi-isomorphisms) and a commutative diagram:
$$\begin{tikzcd} & Y' \arrow[d, dotted] & \\ X \arrow[ur, "f'"] \arrow[r, dotted, "f'''"] \arrow[dr, "f''"'] & Y''' & Y. \arrow[ul, "t"'] \arrow[l, dotted, "t''"'] \arrow[dl, "t'"] \\ & Y'' \arrow[u, dotted] & \end{tikzcd}$$
The composition of two morphisms $(Y', t, f')\colon X \to Y$ and $(Z', s, g')\colon Y \to Z$ is defined by the diagram below with t, s, s' quasi-isomorphisms :
$$\begin{tikzcd} X \arrow[r, "f'"'] & Y' \arrow[dr, dotted, "h"'] & Y \arrow[l, "t"] \arrow[r, "g'"'] & Z' \arrow[dl, dotted, "s'"] & Z. \arrow[l, "s"] \\ & & W & & \end{tikzcd}$$
The functor $Q\colon \mathrm{K}^*(\mathcal{C}) \to \mathrm{D}^*(\mathcal{C})$ is given by $Q(X) = X$, for all $X \in \mathrm{Ob}(\mathcal{C})$, $Q(f) = (Y, 1_Y, f) = X \xrightarrow{f} Y \xleftarrow{1_Y} Y$, for all $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$. Note that for a morphism $f = (Y', t, f')$ in $\mathrm{D}^*(\mathcal{C})$ we have
$$f = Q(t)^{-1}Q(f').$$
Moreover, for two parallel morphisms f, $g\colon X \to Y$ we have the equivalence
$$Q(f) = Q(g) \iff \text{there exists a qis } s\colon Y \to Y' \text{ such that } sf = sg.$$
For later purposes we need the following truncation functors. Let
$$X = \cdots \to X^{n-1} \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1} \to \dots$$
be a complex over $\mathcal{C}$. We define the **truncation functors**
$$\tau^{\leq n}, \widetilde{\tau}^{\leq n}\colon \mathrm{C}(\mathcal{C}) \to \mathrm{C}^-(\mathcal{C}),$$
$$\tau^{\geq n}, \widetilde{\tau}^{\geq n}\colon \mathrm{C}(\mathcal{C}) \to \mathrm{C}^+(\mathcal{C})$$
in the following way:
$$\tau^{\leq n}(X) = \cdots \to X^{n-2} \to X^{n-1} \to \mathrm{Ker}\, d^n \to 0 \to \dots,$$
$$\widetilde{\tau}^{\leq n}(X) = \cdots \to X^{n-1} \to X^n \to \mathrm{Im}\, d^n \to 0 \to \dots,$$
$$\tau^{\geq n}(X) = \cdots \to 0 \to \mathrm{Coker}\, d^n \to X^{n+1} \to X^{n+2} \to \dots,$$
$$\widetilde{\tau}^{\geq n}(X) = \cdots \to 0 \to \mathrm{Im}\, d^{n-1} \to X^n \to X^{n+1} \to \dots.$$
By [**KS06**, Proposition 13.1.5], we have:

PROPOSITION 1.1. (i) *For $n \in \mathbb{Z}$, the functor $H^n\colon \mathrm{D}(\mathcal{C}) \to \mathcal{C}$ is well defined and is a cohomological functor.*

(ii) *A morphism $f\colon X \to Y$ in $\mathrm{D}(\mathcal{C})$ is an isomorphism if and only if $H^n(f)\colon H^n(X) \to H^n(Y)$ is an isomorphism for all n.*

(iii) *For $n \in \mathbb{Z}$, the functors $\tau^{\leq n}, \widetilde{\tau}^{\leq n} \colon \mathrm{D}(\mathcal{C}) \to \mathrm{D}^-(\mathcal{C})$, as well as the functors $\tau^{\geq n}, \widetilde{\tau}^{\geq n} \colon \mathrm{D}(\mathcal{C}) \to \mathrm{D}^+(\mathcal{C})$, are well defined and naturally isomorphic.*

(iv) *For $n \in \mathbb{Z}$, the functor $\tau^{\leq n}$ induces a functor $\mathrm{D}^+(\mathcal{C}) \to \mathrm{D}^b(\mathcal{C})$ and $\tau^{\geq n}$ induces a functor $\mathrm{D}^-(\mathcal{C}) \to \mathrm{D}^b(\mathcal{C})$.*

Let $X \in \mathrm{K}(\mathcal{C})$, with $H^j(X) = 0$ for $j > n$. Then the morphism $\tau^{\leq n}(X) \to X$ in $\mathrm{K}(\mathcal{C})$ is a qis, hence it is an isomorphism in $\mathrm{D}(\mathcal{C})$. It follows that $\mathrm{D}^+(\mathcal{C})$ is equivalent to the full subcategory of $\mathrm{D}(\mathcal{C})$ consisting of objects X satisfying $H^j(X) \cong 0$ for all but finitely many $j < 0$. Similarly, for $* = -$, b, $\mathrm{D}^*(\mathcal{C})$ is equivalent to the full subcategory of $\mathrm{D}(\mathcal{C})$ consisting of objects X satisfying $H^j(X) \cong 0$ for all but finitely many $j > 0$ in case $* = -$, and $H^j(X) \simeq 0$ for all but finitely many $j \in \mathbb{Z}$ in case $* = b$. Furthermore, the category $\mathcal{C}$ is equivalent to the full subcategory of $\mathrm{D}(\mathcal{C})$ consisting of object X satisfying $H^j(X) \cong 0$ for $j \neq 0$ (see [**KS06**, Proposition 13.1.12, (iii)]).

CHAPTER 2

t-structures and torsion pairs

This chapter is devoted to studying the relationship between torsion pairs and t-structures as introduced in [**BBD82**]. We will show how a torsion pair on an abelian category $\mathcal{C}$ gives rise to t-structures of the bounded derived category $\mathrm{D}^b(\mathcal{C})$ of $\mathcal{C}$.

1. Torsion pairs

Throughout this section, $\mathcal{C}$ is an abelian category.

DEFINITION 2.1. A **torsion pair** in $\mathcal{C}$ is a pair of classes of objects $(\mathcal{T}, \mathcal{F})$ of $\mathcal{C}$ such that

(i) $\mathcal{T} = \{T \in \mathcal{C} \mid \mathrm{Hom}_{\mathcal{C}}(T, F) = 0, \text{ for all } F \in \mathcal{F}\}$,

(ii) $\mathcal{F} = \{F \in \mathcal{C} \mid \mathrm{Hom}_{\mathcal{C}}(T, F) = 0, \text{ for all } T \in \mathcal{T}\}$,

If $(\mathcal{T}, \mathcal{F})$ is a torsion pair, then $\mathcal{T}$ is called the **torsion class**, and $\mathcal{F}$ is called the **torsion-free class**. $T \in \mathcal{T}$ is called a **torsion object**, while $F \in \mathcal{F}$ is called a **torsion-free object**.
In [**Dic66**] it is proved that $\mathcal{T}$ and $\mathcal{F}$ are closed under extensions, $\mathcal{T}$ is closed under factor objects and $\mathcal{F}$ is closed under subobjects. Moreover, for each object $X \in \mathcal{C}$ there is a short exact sequence

$$0 \to t(X) \to X \to X/t(X) \to 0,$$

with $t(X) \in \mathcal{T}$ and $X/t(X) \in \mathcal{F}$. We will call $t(X)$ the **torsion part** of X and $X/t(X)$ the **torsion-free part** of X.
Conversely, a class $\mathcal{T}$ of objects of $\mathcal{C}$ is a torsion class if and only if $\mathcal{T}$ is closed under factor objects, (existing) coproducts and extensions; dually, a class $\mathcal{F}$ is a torsion free class if and only if $\mathcal{F}$ is closed under subobjects, (existing) products and extensions (see [**Dic66**, Theorem 2.3]).

EXAMPLE 2.2. In this example we shall use the following subclasses of Mod-$\mathbb{Z}$: (1) $\mathcal{D}$ is the class of *divisible groups*; (2) $\mathcal{R}$ is the class of *reduced groups*; (3) $\mathcal{T}$ is the class of (usual) *torsion groups*; (4) $\mathcal{F}$ is the class of (usual) *torsion-free groups*. Let us prove that $(\mathcal{T}, \mathcal{F})$ and $(\mathcal{D}, \mathcal{R})$ are torsion pairs in Mod-$\mathbb{Z}$.
We start from $(\mathcal{T}, \mathcal{F})$. Observe that given a group homomorphism $f\colon A \to B$, if $na = 0$ in A for some $n \in \mathbb{N}$, then $nf(a) = 0$ in B. Thus if A is a torsion group and B is torsion-free, f must be the zero homomorphism, since 0 is the only cyclic element in B. Now let A be an abelian group such that any homomorphism $A \to B$, where B is torsion-free, is the zero homomorphism. Notice that the set A' of the cyclic elements of A is a subgroup of A and that the quotient A/A' is obviously torsion-free. So if A is not a torsion group, then A' is a proper subgroup of A and there is a non-zero homomorphism $A \to A/A'$, a contradiction. Hence A must be a torsion group. Therefore, $\mathcal{T} = \{A \in \text{Mod-}\mathbb{Z} \mid \mathrm{Hom}_{\mathbb{Z}}(A, B) = 0, \forall B \text{ torsion-free }\}$. Similarly, $\mathcal{F} = \{B \in \text{Mod-}\mathbb{Z} \mid \mathrm{Hom}_{\mathbb{Z}}(A, B) = 0, \forall A \in \mathcal{T}\}$. This proves that $(\mathcal{T}, \mathcal{F})$ is a torsion pair in Mod-$\mathbb{Z}$.
Now let us consider the pair $(\mathcal{D}, \mathcal{R})$. First of all observe that given a divisible abelian group T and a morphism of abelian groups $f\colon T \to F$, the image $f(T)$ is still a divisible group. When F is reduced, $f(T) = 0$, proving that $f = 0$. Let T be an abelian group such that any homomorphism $T \to F$, where F is reduced, is the zero homomorphism. We shall prove that T is divisible. First observe that given a family of divisible subgroups $T_i \subseteq T$, $i \in I$, their sum $\sum_{i \in I} T_i$ is still a divisible subgroup. Indeeed an element $t \in \sum_{i \in I} T_i$ has the form $t_1 + \cdots + t_j$, with $t_k \in T_{i_k}$; given $n \in \mathbb{N}\backslash\{0\}$, choose $s_k \in T_{i_k}$ such that $ns_k = t_k$. It follows that $n(s_1 + \cdots + s_j) = t$. Write T' for the biggest divisible subgroup of T, i.e., the sum of all the divisible subgroups of T. Let us prove that the quotient T/T' is reduced. Denote by

$q\colon T \to T/T'$ the canonical projection. If $X \subseteq T/T'$ is a divisible subgroup, then $q^{-1}(X)$ is divisible since, given $a \in q^{-1}(X)$ and $n \in \mathbb{N} \setminus \{0\}$, we find $[a] \in X$ and thus $[b] \in X$ with $n[b] = [a]$. This means that $b \in q^{-1}(X)$ with $nb - a \in T$. Since T' is divisible, choose $t \in T'$ such that $nb - a = nt$. Then $a = nb - nt = n(b - t)$. But $q(b - t) = q(b) - q(t) = q(b) \in X$, thus $b - t \in q^{-1}(X)$. This proves that $q^{-1}(X)$ is divisible, and thus contained in T'. As a consequence, $X = 0$ and T/T' is reduced. So if T is not a divisible group, then T' is a proper subgroup of T and there is a non-zero homomorphism $T \to T/T'$, a contradiction. Hence T is a divisible group. It follows that $\mathcal{D} = \{T \in \text{Mod-}\mathbb{Z} \mid \text{Hom}_{\mathbb{Z}}(T, F) = 0, \forall F \text{ reduced }\}$. Similarly, $\mathcal{F} = \{F \in \text{Mod-}\mathbb{Z} \mid \text{Hom}_{\mathbb{Z}}(T, F) = 0, \forall T \in \mathcal{D}\}$. We conclude that $(\mathcal{D}, \mathcal{R})$ is a torsion pair in Mod-$\mathbb{Z}$.

2. t-structures

Let $(\mathcal{D}, [1])$ be a triangulated category.

Definition 2.3. A **t-structure** on $\mathcal{D}$ is a pair of strictly full subcategories $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ of $\mathcal{D}$ satisfying the conditions below. Denote $\mathcal{D}^{\leq n} = \mathcal{D}^{\leq 0}[-n]$, $\mathcal{D}^{\geq n} = \mathcal{D}^{\geq 0}[-n]$.

(a) $\mathcal{D}^{\leq 0} \subset \mathcal{D}^{\leq 1}$ and $\mathcal{D}^{\geq 1} \subset \mathcal{D}^{\geq 0}$,
(b) $\text{Hom}_{\mathcal{D}}(X, Y) = 0$ for $X \in \mathcal{D}^{\leq 0}$, $Y \in \mathcal{D}^{\geq 1}$,
(c) for each object $X \in \mathcal{D}$ there is a distinguished triangle

$$A \to X \to B \to A[1],$$

with $A \in \mathcal{D}^{\leq 0}$, $B \in \mathcal{D}^{\geq 1}$.

The **heart** of the t-structure is the full subcategory $\mathcal{H} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ of $\mathcal{D}$.

It is shown in [**BBD82**] that the heart of a t-structure is an abelian category.

Remark 2.4. Let $\mathcal{H}$ be the heart of the t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ of $\mathcal{D}$. Following [**BBD82**], a sequence $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ in $\mathcal{H}$ is short exact if and only if $A^\bullet \to B^\bullet \to C^\bullet \to A^\bullet[1]$ is a triangle in $\mathcal{D}$.

Example 2.5 (The natural t-structure). Let $\mathcal{C}$ be an abelian category and let $\mathcal{D}^b = \mathrm{D}^b(\mathcal{C})$ be its bounded derived category. Denote by $\mathcal{D}^{\leq n}$ (resp. $\mathcal{D}^{\geq n}$) the full subcategory of $\mathcal{D}^b$ formed by complexes $X^\bullet$ with $\mathrm{H}^i(X^\bullet) = 0$ for $i > n$ (resp. $i < n$). Then the pair $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is a t-structure on $\mathcal{D}^b$, called the *natural t-structure on* $\mathcal{D}^b$, with heart $\mathcal{C}$.
First, we must verify conditions (b) and (c). Let us start from condition (b). Let a morphism $f\colon X \to Y$ in $\mathcal{D}^*$ with $X \in \mathcal{D}^{\leq 0}$, $Y \in \mathcal{D}^{\geq 1}$ be represented by a triplet

$$X \xrightarrow{f'} Y' \xleftarrow{t} Y,$$

where t is a quasi-isomorphism. First of all, as $X \in \mathcal{D}^{\leq 0}$, X is quasi-isomorphic to $\tau^{\leq 0}(X)$, hence we may assume that $X^i = 0$ for $i > 0$. Next, as $Y \in \mathcal{D}^{\geq 1}$ and t is a qis, we have $Y' \in \mathcal{D}^{\geq 1}$, so that the natural morphism $r\colon Y' \to \tau^{\geq 0}(Y')$ is a qis and the triplet

$$X \xrightarrow{rf'} \tau^{\geq o}(Y') \xleftarrow{rt} Y$$

also represents the morphism f. Let us prove that $rf' = 0$. Indeed, for $i \neq 0$ we have either $X^i = 0$ or $(\tau^{\geq o}(Y'))^i = 0$, so that $(rf')^i = 0$. For i=0 we have $d^0_{\tau^{\geq o}(Y')}(rf')^0 = (rf')^1 0_{X^0, X^1} = 0$, so that $(rf')^0 = 0$ because $d^0_{\tau^{\geq o}(Y')}$ is a monomorphism. So condition (b) holds. Condition (c) follows from the exact sequence of complexes

$$0 \to A = \tau^{\leq 0}(X) \to X \to X/\tau^{\leq 0}(X) = B \to 0.$$

Finally, by [**KS06**, Proposition 13.1.12, (iii)], the functor $\mathcal{C} \to \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ is an equivalence of categories, so that the heart of the t-structure $\mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ is $\mathcal{C}$.

PROPOSITION 2.6. *Let $(\mathcal{X}, \mathcal{Y})$ be a torsion pair in an abelian category $\mathcal{C}$. Let*

$$\mathcal{D}^{\leq 0} = \{X^\bullet \in \mathrm{D}^b(\mathcal{C}) \mid \mathrm{H}^i(X^\bullet) = 0,\ i > 0,\ \mathrm{H}^0(X^\bullet) \in \mathcal{X}\},$$

$$\mathcal{D}^{\geq 0} = \{X^\bullet \in \mathrm{D}^b(\mathcal{C}) \mid \mathrm{H}^i(X^\bullet) = 0,\ i < -1,\ \mathrm{H}^{-1}(X^\bullet) \in \mathcal{Y}\}.$$

Then $(\mathcal{D}^{\leq 0},\ \mathcal{D}^{\geq 0})$ is a t-structure on $\mathrm{D}^b(\mathcal{C})$.

PROOF. (a) To verify that the first condition of a t-structure is satisfied is straightforward.
(b) Let $X^\bullet \in \mathcal{D}^{\leq 0}$ and $Y^\bullet \in \mathcal{D}^{\geq 1} = \{Y^\bullet \in \mathrm{D}^b(\mathcal{C}) \mid \mathrm{H}^i(X^\bullet) = 0,\ i < 0,\ \mathrm{H}^0(X^\bullet) \in \mathcal{Y}\}$. Assume that there exists $f \in \mathrm{Hom}_{\mathrm{D}^b(\mathcal{C})}(X^\bullet, Y^\bullet)$, with $f \neq 0$. So f can be represented by a triplet

$$X^\bullet \xrightarrow{f^\bullet} Z^\bullet \xleftarrow{s^\bullet} Y^\bullet,$$

where $s^\bullet$ is a qis. Hence $Z^\bullet \in \mathcal{D}^{\geq 1}$ and $f^\bullet \in \mathrm{Hom}_{\mathrm{K}^b(\mathcal{C})}(X^\bullet, Z^\bullet)$ is non-zero. Therefore, we may assume that $f^\bullet$ is given by a morphism of complexes not homotopic to zero.
Using the truncation functors we obtain the following morphism of triangles in $\mathrm{D}^b(\mathcal{C})$:

$$\begin{tikzcd}
\tau^{\leq 0}(X^\bullet) \arrow[r, "\mu^\bullet"] \arrow[d, "\tau^{\leq 0}(f^\bullet)"] & X^\bullet \arrow[r] \arrow[d, "f^\bullet"] & \tau^{\geq 1}(X^\bullet) \arrow[r] \arrow[d, dotted, "h^\bullet"] & \tau^{\leq 0}(X^\bullet)[1] \arrow[d] \\
\tau^{\leq 0}(Z^\bullet) \arrow[r] & Z^\bullet \arrow[r] & \tau^{\geq 1}(Z^\bullet) \arrow[r] & \tau^{\leq 0}(Z^\bullet)[1]
\end{tikzcd}$$

for some $h^\bullet$ which exists by axiom (TR3) of a triangulated category.
By assumption, $\mathrm{H}^i(\tau^{\geq 1}(X^\bullet)) = 0$ for all $i \in \mathbb{Z}$, so $\tau^{\geq 1}(X^\bullet) = 0$ in $\mathrm{D}^b(\mathcal{C})$ and $\mu^\bullet$ is an isomorphism in $\mathrm{D}^b(\mathcal{C})$. In particular, $\tau^{\leq 0}(f^\bullet) \neq 0$ in $\mathrm{K}^b(\mathcal{C})$.
Using the truncation functors we now obtain the following morphism of triangles in $\mathrm{D}^b(\mathcal{C})$:

$$\begin{tikzcd}
\tilde{\tau}^{\leq -1}(X^\bullet) \arrow[r] \arrow[d, "\tilde{\tau}^{\leq -1}(f^\bullet)"] & \tau^{\leq 0}(X^\bullet) \arrow[r] \arrow[d, "\tau^{\leq 0}(f^\bullet)"] & \mathrm{H}^0(X^\bullet) \arrow[r] \arrow[d, dotted, "h"] & \tilde{\tau}^{\leq -1}(X^\bullet)[1] \arrow[d] \\
\tilde{\tau}^{\leq -1}(Z^\bullet) \arrow[r] & \tau^{\leq 0}(Z^\bullet) \arrow[r, "\rho^\bullet"] & \mathrm{H}^0(Z^\bullet) \arrow[r] & \tilde{\tau}^{\leq -1}(Z^\bullet)[1]
\end{tikzcd}$$

for some h which exists by axiom (TR3) of a triangulated category.
Now, $\mathrm{H}^i(\tilde{\tau}^{\leq -1}(Z^\bullet)) = 0$ for all $i \in \mathbb{Z}$, so $\rho^\bullet$ is an isomorphism in $\mathrm{D}^b(\mathcal{C})$. Since $\mathrm{H}^0(X^\bullet) \in \mathcal{X}$ and $\mathrm{H}^0(Z^\bullet) \cong \mathrm{H}^0(Y^\bullet) \in \mathcal{Y}$, we have that $h = 0$. So we conclude that $\tau^{\leq 0}(f^\bullet) = 0$, which gives a contradiction.
(c) Let $X^\bullet = (X^i,\ d^i) \in \mathrm{D}^b(\mathcal{C})$. Since $(\mathcal{X}, \mathcal{Y})$ is a torsion pair in $\mathcal{C}$ we have an exact sequence

$$0 \to T \xrightarrow{\mu} \mathrm{H}^0(X^\bullet) \xrightarrow{\pi} F \to 0,$$

with $T \in \mathcal{X}$, $F \in \mathcal{Y}$.
Consider the following commutative diagram of exact sequences in $\mathcal{C}$ obtained by pullback along μ from the lower sequence:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathrm{Im}\, d^{-1} \arrow[r, "\mu''"] \arrow[d, equal] & E \arrow[r] \arrow[d, "\mu'"] & T \arrow[r] \arrow[d, "\mu"] & 0 \\
0 \arrow[r] & \mathrm{Im}\, d^{-1} \arrow[r] & \mathrm{Ker}\, d^0 \arrow[r] \arrow[d] & \mathrm{H}^0(X^\bullet) \arrow[r] \arrow[d, "\pi"] & 0 \\
 & & \mathrm{H}^0(X^\bullet)/T \arrow[r, equal] \arrow[d] & \mathrm{H}^0(X^\bullet) \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

Let $d^{-1} = i\rho$ be the canonical factorization of d^{-1} through $\operatorname{Im} d^{-1}$ and let $\tilde{d}^{-1} = \mu''\rho$. Let $X'^{\bullet}$ be the following subcomplex of $X^{\bullet}$ defined by $X'^{i} = X^{i}$ for $i \leq -1$, $X'^{0} = E$, $X'^{i} = 0$ for $i > 0$ and $d^{i}_{X'} = d^{i}_{X}$ for $i < -1$, $d^{-1}_{X'} = \tilde{d}^{-1}$, $d^{i}_{X'} = 0$ for $i > 0$:

$$\cdots \to X^{-2} \xrightarrow{d^2} X^{-1} \xrightarrow{\tilde{d}^{-1}} E \to 0 \to \dots .$$

By construction, $X'^{\bullet} \in \mathcal{D}^{\leq 0}$. Let $X''^{\bullet}$ be the quotient complex $X^{\bullet}/X'^{\bullet}$. We obtain a triangle

$$X'^{\bullet} \to X^{\bullet} \to X''^{\bullet} \to X'^{\bullet}[1]$$

in $\mathrm{D}^b(\mathcal{C})$. Next we show that $X''^{\bullet} \in \mathcal{D}^{\geq 1}$. By construction, $\mathrm{H}^i(X''^{\bullet}) = 0$ for $i < 0$. Now, $X''^{0} = X^0/E$, $X''^{1} = X^1$ and we have a commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E & \longrightarrow & X^0 & \longrightarrow & X^0/E & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle d^0} & & \downarrow{\scriptstyle \tilde{d}^0} & & \\ 0 & \longrightarrow & 0 & \longrightarrow & X^1 & = & X^1 & \longrightarrow & 0. \end{array}$$

Thus $\mathrm{H}^0(X''^{\bullet}) = \operatorname{Ker} \tilde{d}^0 \cong \operatorname{Ker} d^0/E \cong \mathrm{H}^0(X^{\bullet})/T \in \mathcal{Y}$, hence the asserion follows.

$\square$

COROLLARY 2.7. *Let $\mathcal{C}$ be an abelian category and $(\mathcal{X}, \mathcal{Y})$ a torsion pair in $\mathcal{C}$. Then the following hold:*

(i) $\mathcal{H}(\mathcal{X}, \mathcal{Y}) = \{X^{\bullet} \in \mathrm{D}^b(\mathcal{C}) \mid \mathrm{H}^i(X) = 0, i \neq 0, -1, \mathrm{H}^0(X^{\bullet}) \in \mathcal{X}, \mathrm{H}^{-1}(X^{\bullet}) \in \mathcal{Y}\}$ *is an abelian category.*

(ii) *The pair $(\mathcal{T}, \mathcal{F})$ of classes of objects $\mathcal{T} = \mathcal{Y}[1]$ and $\mathcal{F} = \mathcal{X}$ of $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ is a torsion pair in $\mathcal{H}(\mathcal{X}, \mathcal{Y})$.*

PROOF. The first assertion follows from Proposition 2.6 and the fact that the heart of a t-structure is an abelian category.
(ii) Clearly, $\mathcal{T} = \mathcal{Y}[1]$ and $\mathcal{F} = \mathcal{X}$ are classes of objects of $\mathcal{H}(\mathcal{X}, \mathcal{Y})$. Let $X \in \mathcal{T}$ and $Y \in \mathcal{F}$, so $X \cong F[1]$ for some $F \in \mathcal{Y}$. Then $\operatorname{Hom}_{\mathcal{B}}(X, Y) = \operatorname{Hom}_{\mathrm{D}^b(\mathcal{C})}(F[1], Y) = \operatorname{Ext}^{-1}_{\mathcal{C}}(F, Y) = 0$, showing that $(\mathcal{T}, \mathcal{F})$ is a torsion pair in $\mathcal{H}(\mathcal{X}, \mathcal{Y})$. $\square$

CHAPTER 3

Tilting theory

In this chapter we introduce the basic ideas of tilting theory as a tool for our subsequent study of the heart associated to a faithful torsion pair.
We shall begin by defining the notion of tilting object in the general setting of abelian categories. The importance of a tilting object V in an abelian category $\mathcal{A}$ is that it provides a connection between the category $\mathcal{A}$ and the category of modules over the ring $R = \operatorname{End}_{\mathcal{A}}(V)$. This connection is described in the Tilting Theorem. Then we shall see how the notion of tilting object specializes in the case of Grothendieck categories. For instance, we will prove that when the category we deal with is that of the modules over a (possibly) non commutative ring, then we obtain exactly the notion of tilting module in the sense of [**CF04**, Chapter 3].
Our main references are [**CF07**], [**Col99**], [**HRO96**] and [**CF04**].

1. Tilting objects in abelian categories

Let V be an object in an abelian category $\mathcal{A}$ that contains arbitrary coproducts of copies of V. We shall denote by $\operatorname{Gen} V$ the full subcategory of $\mathcal{A}$ generated by V, i.e.,

$$\operatorname{Gen} V = \{M \in \mathcal{A} \mid \exists \text{ an exact sequence } V^{(\alpha)} \to M \to 0\}$$

and by $\overline{\operatorname{Gen}} V$ the closure of $\operatorname{Gen} V$ under subobjects: $\overline{\operatorname{Gen}} V$ is the smallest exact abelian subcategory of $\mathcal{A}$ containing $\operatorname{Gen} V$. Moreover we let $\operatorname{Pres} V$ denote the full subcategory of $\operatorname{Gen} V$ which consists of the objects in $\mathcal{A}$ presented by V, i.e.,

$$\operatorname{Pres} V = \{M \in \mathcal{A} \mid \exists \text{ an exact sequence } V^{(\beta)} \to V^{(\alpha)} \to M \to 0\}.$$

We define the functor $\operatorname{Tr}_V : \mathcal{A} \to \mathcal{A}$ by

$$\operatorname{Tr}_V(M) = \sum\{\operatorname{Im} f \mid f \in \operatorname{Hom}_{\mathcal{A}}(V, M)\}.$$

Finally, let $R = \operatorname{End}_{\mathcal{A}}(V)$ and

$$V^{\perp} = \operatorname{Ker} \operatorname{Ext}^1_{\mathcal{A}}(V, -), \quad V_{\perp} = \operatorname{Ker} \operatorname{Hom}_{\mathcal{A}}(V, -).$$

Definition 3.1. An object V in an abelian category $\mathcal{A}$ that contains arbitrary coproducts of copies of V is called a **tilting object** if:

i) V is selfsmall (i.e., $\operatorname{Hom}_{\mathcal{A}}(V, V^{(\alpha)}) \cong R^{(\alpha)}$ for any cardinal α);
ii) $\operatorname{Gen} V = V^{\perp}$;
iii) $\overline{\operatorname{Gen}} V = \mathcal{A}$, i.e., any object of $\mathcal{A}$ embeds in an object of $\operatorname{Gen} V$.

Basic properties of tilting objects are recorded in the next three propositions.

Proposition 3.2. *If* $\operatorname{Gen} V \subseteq V^{\perp}$*, then* Tr_V *is a radical. In particular* $(\operatorname{Gen} V, V_{\perp})$ *is a torsion pair in* $\mathcal{A}$.

Proof. It is clear that Tr_V is an idempotent preradical. Let $M \in \mathcal{A}$ and consider the canonical exact sequence

$$0 \to \operatorname{Tr}_V(M) \to M \to M/\operatorname{Tr}_V(M) \to 0.$$

Set $H_V = \operatorname{Hom}_{\mathcal{A}}(V, -)$. We obtain the exact sequence

$$0 \to H_V(\operatorname{Tr}_V(M)) \xrightarrow{\cong} H_V(M) \to H_V(M/\operatorname{Tr}_V(M)) \to \operatorname{Ext}^1_{\mathcal{A}}(V, \operatorname{Tr}_V(M)) = 0$$

which shows that $H_V(M/\operatorname{Tr}_V(M)) = 0$, i.e., $\operatorname{Tr}_V(M/\operatorname{Tr}_V(M)) = 0$. This proves that Tr_V is a radical. This also shows that for any $M \in \mathcal{A}$, $\operatorname{Tr}_V(M)$ is the unique subobject of M such that $\operatorname{Tr}_V(M) \in \operatorname{Gen} V$ and $M/\operatorname{Tr}_V(M) \in V_{\perp}$, and so $(\operatorname{Gen} V, V_{\perp})$ is a torsion pair in $\mathcal{A}$. □

By Proposition 3.2 it follows that to any tilting object $V \in \mathcal{A}$ is naturally associated a torsion pair $(\mathcal{T}, \mathcal{F})$ in $\mathcal{A}$, namely $\mathcal{T} = V^{\perp}$ and $\mathcal{F} = V_{\perp}$, called the **tilting torsion pair** associated to V.

PROPOSITION 3.3. *If* $\operatorname{Gen} V = V^{\perp}$*, then* $\operatorname{Gen} V = \operatorname{Pres} V$.

PROOF. Let $M \in \operatorname{Gen} V$ and $\alpha = \operatorname{Hom}_{\mathcal{A}}(V, M)$. Then we have the exact sequences

$$0 \to K \to V^{(\alpha)} \xrightarrow{\varphi} M \to 0$$

and

$$H_V(V^{(\alpha)}) \xrightarrow{H_V(\varphi)} H_V(M) \to \operatorname{Ext}^1_{\mathcal{A}}(V, K) \to 0$$

where the morphism $H_V(\varphi)$ is an epimorphism by construction. Therefore $\operatorname{Ext}^1_{\mathcal{A}}(V, K) = 0$, so by assumption $K \in \operatorname{Gen} V$. This proves that $M \in \operatorname{Pres} V$. □

PROPOSITION 3.4. *If* $\overline{\operatorname{Gen}}\, V = \mathcal{A}$*, then the equality* $\operatorname{Gen} V = V^{\perp}$ *is equivalent to the following conditions:*

i) $\operatorname{proj\,dim} V \leq 1$,
ii) $\operatorname{Ext}^1_{\mathcal{A}}(V, V^{(\alpha)}) = 0$ *for any cardinal* α,
iii) *if* $M \in \mathcal{A}$ *and* $\operatorname{Hom}_{\mathcal{A}}(V, M) = 0 = \operatorname{Ext}^1_{\mathcal{A}}(V, M)$*, then* $M = 0$.

PROOF. Let $\overline{\operatorname{Gen}}\, V = \mathcal{A}$ and $\operatorname{Gen} V = V^{\perp}$. Let us prove i), showing that $\operatorname{Ext}^2_{\mathcal{A}}(V, M) = 0$ for any $M \in \mathcal{A}$. Indeed, given a representative of an element $\epsilon \in \operatorname{Ext}^2_{\mathcal{A}}(V, M)$, say

$$(\epsilon) \qquad 0 \to M \to E_1 \xrightarrow{f} E_2 \to V \to 0$$

let $I = \operatorname{Im} f$. Embedding E_1 in a suitable object $X \in \operatorname{Gen} V$, we first have a push-out diagram (dual to [**Ste75**, Proposition 5.1, page 90])

(1)
$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, equal] & E_1 \arrow[r] \arrow[d, hook] & I \arrow[r] \arrow[d, hook] & 0 \\ 0 \arrow[r] & M \arrow[r] & X \arrow[r] & P' \arrow[r] & 0 \end{tikzcd}$$

where X, and so P', are in $\operatorname{Gen} V$. Then we have a second push-out diagram

(2)
$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] \arrow[d, hook] & E_2 \arrow[r] \arrow[d, hook] & V \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & P' \arrow[r] & P'' \arrow[r] & V \arrow[r] & 0 \end{tikzcd}$$

By glueing (1) and (2) together, we derive a commutative diagram with exact rows

(3)
$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, equal] & E_1 \arrow[r] \arrow[d, hook] & E_2 \arrow[r, "f"] \arrow[d, hook] & V \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & M \arrow[r] & X \arrow[r, "g"] & P'' \arrow[r, "\pi"] & V \arrow[r] & 0 \end{tikzcd}$$

where $\operatorname{Im} g = P' \in V^{\perp}$. Then π is epi-split, and so $\epsilon \sim 0$. This proves i). Condition ii) is contained in the hypothesis, and condition iii) follows from Proposition 3.2.
Conversely, let us assume that conditions i), ii) and iii) hold. The first condition assures that $V^{\perp}$ is closed under factors. Therefore, using the second condition we immediately see that $\operatorname{Gen} V \subseteq V^{\perp}$. In order to prove the opposite inclusion, given any $M \in V^{\perp}$, from the exact sequence $0 \to \operatorname{Tr}_V(M) \to M \to M/\operatorname{Tr}_V(M) \to 0$ and using condition i) we obtain the exact sequence

$$\begin{aligned} 0 \to \operatorname{Hom}_{\mathcal{A}}(V, \operatorname{Tr}_V(M)) \xrightarrow{\cong} \operatorname{Hom}_{\mathcal{A}}(V, M) \to \operatorname{Hom}_{\mathcal{A}}(V, M/\operatorname{Tr}_V(M)) \to \\ \operatorname{Ext}^1_{\mathcal{A}}(V, \operatorname{Tr}_V(M)) = 0 = \operatorname{Ext}^1_{\mathcal{A}}(V, M) \to \operatorname{Ext}^1_{\mathcal{A}}(V, M/\operatorname{Tr}_V(M)) \to 0. \end{aligned}$$

Hence $\operatorname{Hom}_{\mathcal{A}}(V, M/\operatorname{Tr}_V(M)) = 0 = \operatorname{Ext}^1_{\mathcal{A}}(V, M/\operatorname{Tr}_V(M))$. Now condition iii) gives $M/\operatorname{Tr}_V(M) = 0$, i.e., $M = \operatorname{Tr}_V(M) \in \operatorname{Gen} V$. This proves that $V^{\perp} \subseteq \operatorname{Gen} V$. □

Therefore, $V \in \mathcal{A}$ is a tilting object if and only if V is selfsmall, $\overline{\operatorname{Gen}} V = \mathcal{A}$ and V satisfies conditions i)-iii) of Proposition 3.4.
Note that if $\mathcal{A}$ is cocomplete with exact coproducts, or $\mathcal{A}$ has enough injectives, then $\overline{\operatorname{Gen}} V = \mathcal{A}$ whenever $\operatorname{Gen} V = V^{\perp}$. Indeed, if $\mathcal{A}$ has enough injectives, then every object of $\mathcal{A}$ embeds in an injective object which, by definition, belongs to $V^{\perp} = \operatorname{Gen} V$. Now let us assume that $\mathcal{A}$ is cocomplete with exact coproducts. Let $M \in \mathcal{A}$ and α be the cardinality of a spanning set for $\operatorname{Ext}^1_{\mathcal{A}}(V, M)$ as a right R-module. Then, arguing as in [**CF04**, Lemma 3.4.4], we can find an exact sequence

$$0 \to M \to X \to V^{(\alpha)} \to 0$$

such that the connecting homomorphism $\operatorname{Hom}_{\mathcal{A}}(V, V^{(\alpha)}) \to \operatorname{Ext}^1_{\mathcal{A}}(V, M)$ is onto. This gives $\operatorname{Ext}^1_{\mathcal{A}}(V, X) = 0$, i.e., $X \in V^{\perp} = \operatorname{Gen} V$, and so it proves that $\overline{\operatorname{Gen}} V = \mathcal{A}$.

Example 3.5. An object $P \in \mathcal{A}$ is called a **progenerator** if P is a selfsmall, projective generator of $\mathcal{A}$. A progenerator P is clearly a tilting object generating a torsion pair that collapses to $(\mathcal{A}, \{0\})$.

1.1. The Tilting Theorem. The importance of a tilting object $V \in \mathcal{A}$ is that it provides a connection between the category $\mathcal{A}$ and the category of modules over $R = \operatorname{End}_{\mathcal{A}}(V)$. This connection is described in the following theorem (see [**CF07**, Theorem 3.2]):

Theorem 3.6 (Tilting Theorem). *Let V be a tilting object in an abelian category $\mathcal{A}$, $R = \operatorname{End}_{\mathcal{A}}(V)$, $H_V = \operatorname{Hom}_{\mathcal{A}}(V, -)$, $H'_V = \operatorname{Ext}^1_{\mathcal{A}}(V, -)$. Then H_V has a left adjoint functor $T_V : \text{Mod-}R \to \mathcal{A}$ such that $T_V(R) = V$. Let $\sigma : 1_{\text{Mod-}R} \to H_V T_V$ and $\rho : T_V H_V \to 1_{\mathcal{A}}$ be respectively the unit and the counit of the adjunction $\langle T_V, H_V\rangle$, and let T'_V the first left derived functor of T_V. Set*

$$\mathcal{T} = \operatorname{Ker} H'_V, \ \mathcal{F} = \operatorname{Ker} H_V, \ \mathcal{X} = \operatorname{Ker} T_V, \ \mathcal{Y} = \operatorname{Ker} T'_V.$$

Then:

a) *$(\mathcal{T}, \mathcal{F})$ is a torsion pair in $\mathcal{A}$ with $\mathcal{T} = \operatorname{Gen} V$, and $(\mathcal{X}, \mathcal{Y})$ is a torsion pair in* Mod-R;
b) *the functors $H_V\restriction_{\mathcal{T}}$, $T_V\restriction_{\mathcal{Y}}$, $H'_V\restriction_{\mathcal{F}}$, $T'_V\restriction_{\mathcal{X}}$ are exact, and they induce a pair of category equivalences $\mathcal{T} \underset{T_V}{\overset{H_V}{\rightleftarrows}} \mathcal{Y}$ and $\mathcal{F} \underset{T'_V}{\overset{H'_V}{\rightleftarrows}} \mathcal{X}$;*
c) *$T_V H'_V = 0 = T'_V H_V$ and $H_V T'_V = 0 = H'_V T_V$;*
d) *there are natural transformations θ and η that, together with the adjoint transformations ρ and σ, yield exact sequences*

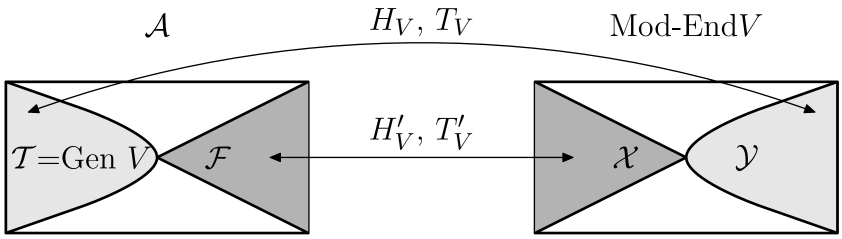
$$0 \to T_V H_V(M) \xrightarrow{\rho_M} M \xrightarrow{\eta_M} T'_V H'_V(M) \to 0$$

and

$$0 \to H'_V T'_V(N) \xrightarrow{\theta_N} N \xrightarrow{\sigma_N} H_V T_V(N) \to 0$$

for each $M \in \mathcal{A}$ and for each $N \in$ Mod-R.

The situation described in the Tilting Theorem can be visualized a follows:



We shall say that the torsion pair $(\mathcal{X}, \mathcal{Y})$ is **tilted by** V or that V **tilts to** $(\mathcal{X}, \mathcal{Y})$.
Moreover, we make the following observations:

1. By condition c) of Theorem 3.6, $H'_V T_V = 0$, so that $\operatorname{Im} T_V \subseteq \operatorname{Ker} H'_V = \mathcal{T}$. As a consequence, for each object $M \in \mathcal{A}$ we have that $T_V H_V(M) \in \mathcal{T}$. Similarly, $T'_V H'_V(M) \in \mathcal{F}$. Since by condition d) the sequence $0 \to T_V H_V(M) \xrightarrow{\rho_M} M \xrightarrow{\eta_M} T'_V H'_V(M) \to 0$ is exact for each $M \in \mathcal{A}$, it follows that the torsion part and the torsion-free part of M are given by
$$t(M) = T_V H_V(M), \qquad M/t(M) = T'_V H'_V(M).$$
2. Dually, we obtain: for each $N \in$ Mod- the torsion part and the torsion-free part of N are given by
$$t(N) = H'_V T'_V(N), \qquad N/t(N) = H_V T_V(N).$$
3. Applying the functor $\operatorname{Hom}_{\mathcal{A}}(V, -)$ to the first exact sequence in d), we easily see that for every $M \in \mathcal{A}$ we have
$$H_V(M) = H_V(t(M)) \quad \text{and} \quad H'_V(M) = H'_V(M/t(M)).$$
4. The existence of the natural transformations θ and η by condition d) ensures that the pair of functors (H'_V, T'_V) is an adjoint pair.

2. Tilting objects in Grothendieck categories

We recall that a **Grothendieck category** is a cocomplete abelian category with a generator and where all direct limits are exact (see [**Ste75**, Chapter V]). Some examples of Grothendieck categories are the category Mod-R, the categories of presheaves and sheaves of abelian groups on a topological space and the full subcategory of the functor category $\operatorname{Fct}(\mathcal{B}^{\mathrm{op}}, \text{Mod-}\mathbb{Z})$ consisting of the additive functors, where $\mathcal{B}$ is a small, preadditive category (see [**Ste75**, Chapter V]).
In the sequel, $\mathcal{G}$ denotes a Grothendieck category. It is well known that, as a consequence of the Gabriel-Popescu Theorem (see [**Ste75**, Theorem 4.1 and Corollary 4.3]), $\mathcal{G}$ has enough injectives. Therefore, an object $V \in \mathcal{G}$ is tilting if and only if V is selfsmall and $\operatorname{Gen} V = V^{\perp}$. Moreover, again by [**Ste75**, Theorem 4.1] and [**Ste75**, Proposition 1.9], $\mathcal{G}$ has an injective cogenerator.

Remark 3.7. For later developments, we need the following definition. A **cotilting module** U_R over a ring R is a right R-module such that $\operatorname{Cogen} U_R = \operatorname{Ker} \operatorname{Ext}^1_R(-, U_R)$. In [**Col99**, Theorem 4.1 a)] it is proved that for a Grothendieck category $\mathcal{G}$ with an injective cogenerator Q and a tilting object V, the tilted torsion pair $(\mathcal{X}, \mathcal{Y})$ in Mod- End V is a **cotilting torsion pair**, i.e., a torsion pair such that there exists a cotilting module U with $\mathcal{Y} = \operatorname{Cogen} U$. The cotilting module cogenerating the class $\mathcal{Y}$ is proved to be the right End V-module $U = \operatorname{Hom}_{\mathcal{G}}(V, Q)$.

As a consequence of the Tilting Theorem, we have:

Proposition 3.8. *Let $V \in \mathcal{G}$ be tilting. Then the functor* $\operatorname{Hom}_{\mathcal{G}}(V, -)$ *preserves direct limits in $\mathcal{G}$.*

Proof. Let $(M_\lambda, f_{\lambda\mu})$ be a direct system in $\mathcal{G}$. As the functor direct limit is exact in $\mathcal{G}$, from the exact sequences
$$0 \to \operatorname{Tr}_V(M_\lambda) \to M_\lambda \to M_\lambda / \operatorname{Tr}_V(M_\lambda) \to 0$$
we get the exact sequence
$$0 \to \varinjlim \operatorname{Tr}_V(M_\lambda) \to \varinjlim M_\lambda \to \varinjlim M_\lambda / \operatorname{Tr}_V(M_\lambda) \to 0.$$
As $\varinjlim \operatorname{Tr}_V(M_\lambda) \in \operatorname{Gen} V = V^{\perp}$, we get the exact sequence
$$0 \to H_V(\varinjlim \operatorname{Tr}_V(M_\lambda)) \to H_V(\varinjlim M_\lambda) \to H_V(\varinjlim M_\lambda / \operatorname{Tr}_V(M_\lambda)) \to 0. \tag{1}$$
From Theorem 3.6 d) we have $M_\lambda / \operatorname{Tr}_V(M_\lambda) \cong T'_V H'_V(M_\lambda)$, therefore we obtain
$$H_V(\varinjlim M_\lambda / \operatorname{Tr}_V(M_\lambda)) \cong H_V(\varinjlim T'_V H'_V(M_\lambda)) \cong H_V T'_V(\varinjlim H'_V(M_\lambda)) = 0,$$

because T'_V preserves direct limits, being a right adjoint, and $H_V T'_V = 0$, as in Theorem 3.6 c). Combining this with (1), we get

$$(2) \qquad H_V(\varinjlim M_\lambda) \cong H_V(\varinjlim \mathrm{Tr}_V(M_\lambda)) \quad \text{canonically.}$$

Now, from Theorem 3.6 d), we have $\mathrm{Tr}_V(M_\lambda) \cong T_V H_V(M_\lambda)$, and so we obtain canonical isomorphisms

$$(3) \qquad H_V(\varinjlim \mathrm{Tr}_V(M_\lambda)) \cong H_V(\varinjlim T_V H_V(M_\lambda)) \cong H_V T_V(\varinjlim H_V(M_\lambda)) \cong \varinjlim H_V(M_\lambda),$$

where the second-last iso follows from [**Col99**, Proposition 1.1 d)] and the last iso follows by Theorem 3.6 b) and from the fact that $\varinjlim H_V(M_\lambda) \in \mathrm{Ker}\, T'_V$. Combining (2) with (3), we get the thesis. □

Recall that an object M of $\mathcal{G}$ is **finitely generated** if whenever $M = \sum M_\lambda$ for a direct family of subobjects $M_\lambda$of M, there exists an index μ such that $M = M_\mu$. Moreover, M is **finitely presented** if it is finitely generated and every epimorphism $L \twoheadrightarrow M$, with L finitely generated, has a finitely generated kernel. Finally, the category $\mathcal{G}$ is called **locally finitely generated** if it has a family of finitely generated generators or, equivalently, if any object M of $\mathcal{G}$ is the sum of its finitely generated subobjects.

COROLLARY 3.9. *Let $\mathcal{G}$ be locally finitely generated. Then any tilting object of $\mathcal{G}$ is finitely presented.*

PROOF. Apply Proposition 3.8 and [**Ste75**, Proposition 3.4, Chapter V]. □

3. Application to tilting modules

In this section we shall characterize the tilting objects in the category Mod-R and we shall see that when the ring R is commutative, then the tilting objects are precisely the progenerators of Mod-R.

3.1. Characterization of tilting modules. A module V_R is called a **tilting module** if it is a tilting object in the category Mod-R.
To prove the main result of this section we need the following lemmas.

LEMMA 3.10. *Let V_R be a tilting module and let $S = \mathrm{End}(V_R)$. Then ${}_S V$ is finitely generated.*

PROOF. Notice that $\mathrm{Gen}\, V_R$ is closed under products. Indeed, $\mathrm{Gen}\, V_R = V_R^{\perp}$ and $\mathrm{Ext}^1_R(V, -)$ commutes with products. In particular, $V^V \in \mathrm{Gen}\, V_R$. Hence there exists an epimorphism $\varphi\colon V^{(I)} \to V^V$. Let $(x_v)_v$ be the element of V^V such that $x_v = v$ for each $v \in V$. Then there exists $(y_i)_i \in V^{(I)}$ with $(x_v)_v = \varphi((y_i)_i) = \sum_{i \in F} \varphi_i(y_i)$, where F is a finite subset of I and $\varphi_i \in \mathrm{Hom}_R(V, V^V)$ for each $i \in F$. For each $i \in I$ and $v \in V$ set $s_i^v = \pi_v \varphi_i$ and notice that $s_i^v \in S$:

$$\begin{tikzcd} V^{(I)} \arrow[r, "\varphi"] & V^V \arrow[d, "\pi_v"] \\ V \arrow[u, hook] \arrow[ur, "\varphi_i"] \arrow[r, "s_i^v"'] & V. \end{tikzcd}$$

Then for each $v \in V$ we have

$$\begin{aligned} v = \pi_v((x_v)_v) &= \pi_v \varphi((y_i)_i) \\ &= \pi_v(\sum_{i \in F} \varphi_i(y_i)) \\ &= \sum_{i \in F} \pi_v \varphi_i(y_i) \\ &= \sum_{i \in F} s_i^v(y_i). \end{aligned}$$

This shows that ${}_S V$ is finitely generated. □

LEMMA 3.11. *Let V_R be finitely generated and assume that* $\operatorname{proj}\dim V \leq 1$. *Then $V_R^{\perp}$ is closed under direct sums and factors.*

PROOF. Since $\operatorname{proj}\dim V \leq 1$, $V_R^{\perp}$ is closed under factors. Moreover, $V \cong X/Y$, where X is a projective module and Y is a finitely generated module. Hence every homomorphism of Y into a direct sum of modules actually maps into a finite direct summand. Therefore, as $V_R^{\perp}$ is closed with respect to finite direct sums, it is closed with respect to arbitrary ones. □

The following theorem yields a characterization of tilting modules:

THEOREM 3.12. *The following conditions are equivalent for a module $V \in$ Mod-R:*

1) *V is a tilting module;*
2) *V satisfies the conditions:*
   (a) *there exists an exact sequence $0 \to R' \to R'' \to V \to 0$, where R' and R'' are direct summands of a direct sum of copies of R_R;*
   (b) $\operatorname{Ext}^1_R(V, V) = 0$*;*
   (c) *there exists an exact sequence $0 \to R_R \to V' \to V'' \to 0$, where V' and V'' are direct summands of a direct sum of copies of V.*

Moreover, condition (a) is equivalent to:

(a') $\operatorname{proj}\dim V \leq 1$ *and V_R is finitely generated.*

PROOF. It is clear that conditions (a) and (a') are equivalent.
1) ⇒ 2). Assume that V_R is a tilting module. By Proposition 3.4 i), $\operatorname{proj}\dim V \leq 1$; moreover, since Mod-$R$ is a Grothendieck locally finitely generated category, Corollary 3.9 ensures that V is finitely presented, so in particular it is finitely generated. Hence (a') holds. By Proposition 3.4 ii), $\operatorname{Ext}^1_R(V, V) = 0$, so condition (b) holds. Let us prove condition (c). Let $S = \operatorname{End}(V_R)$, and notice that ${}_SV$ is finitely generated by Lemma 3.10. Let ${}_SV = \langle v_1, \dots, v_n\rangle$. The correspondence $r \mapsto (v_1 r, \dots, v_n r)$, for every $r \in R$, gives a monomorphism of right R-modules $i\colon R \to V^n$, with $i(1) = (v_1, \dots, v_n)$. So we obtain an exact sequence

$$(*) \qquad 0 \to R \xrightarrow{i} V^n \to V^n/R \to 0.$$

As $V^n/R \in \operatorname{Gen} V_R = \operatorname{Pres} V_R$, there is an exact sequence

$$(**) \qquad 0 \to L \to V_R^{(X)} \to V^n/R \to 0,$$

where $L \in \operatorname{Gen} V_R$ and $\operatorname{Ext}^1_R(V, L) = 0$. Because of (*) and (**), it is enough to prove that $\operatorname{Ext}^1_R(V^n/R, L) = 0$. Applying $\operatorname{Hom}_R(-, L)$ to (*), we get the exact row

$$0 \to \operatorname{Hom}_R(V^n/R, L) \to \operatorname{Hom}_R(V^n, L) \xrightarrow{i^*} L \to \operatorname{Ext}^1_R(V^n/R, L) \to \operatorname{Ext}^1_R(V^n, L) = 0,$$

where $i^* = \operatorname{Hom}_R(i, L)$. Thus it is enough to show that i^* is surjective.
Let $0 \neq x \in L$. As $L \in \operatorname{Gen} V_R$, there are $f\colon V^m \to L$ and $\overline{v} \in V^m$ such that $f(\overline{v}) = x$. As ${}_SV = \langle v_1, \dots, v_n\rangle$, there is $\alpha\colon V^n \to V^m$ such that $\alpha(v_1, \dots, v_n) = \overline{v}$. Thus, from $i(1) = (v_1, \dots, v_n)$, we have $\alpha i(1) = \overline{v}$. Therefore we obtain the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & R \arrow[r, "i"] \arrow[d, "x"] & V^n \arrow[d, "\alpha"] \\ & L & V^m \arrow[l, "f"'] \end{tikzcd}$$

which shows that $f\alpha\colon V^n \to L$ is such that $i^*(f\alpha) = f\alpha i = x$. Hence i^* is surjective.
2) ⇒ 1). Assume that V satisfies conditions (a), (b) and (c). Then by (a'), V_R is finitely generated, hence in particular selfsmall. Let us prove that $\operatorname{Gen} V_R = V_R^{\perp}$. By Lemma 3.11, $V_R^{\perp}$ is closed under direct sums and factors, hence $L \in V_R^{\perp}$. This shows that $\operatorname{Gen} V_R \subseteq V_R^{\perp}$. Now let $0 \neq M \in$ Mod-R be in $V_R^{\perp}$. By Proposition 3.2, $\operatorname{Tr}_V(M) \leq M$ and $\operatorname{Tr}_V(M)$ is the unique subobject of M such that $\operatorname{Tr}_V(M) \in \operatorname{Gen} V_R$ and $M/\operatorname{Tr}_V(M) \in V_{\perp}$. Applying $\operatorname{Hom}_R(V, -)$ to the sequence

$$0 \to \operatorname{Tr}_V(M) \to M \to M/\operatorname{Tr}_V(M) \to 0$$

we get the exact sequence

$$0 \to \operatorname{Hom}_R(V, \operatorname{Tr}_V(M)) \xrightarrow{\cong} \operatorname{Hom}_R(V, M) \to \operatorname{Hom}_R(V, M/\operatorname{Tr}_V(M)) \to$$
$$\operatorname{Ext}^1_R(V, \operatorname{Tr}_V(M)) \to \operatorname{Ext}^1_R(V, M) \to \operatorname{Ext}^1_{\mathcal{A}}(V, M/\operatorname{Tr}_V(M)) \to 0,$$

where $\operatorname{Ext}^1_R(V, \operatorname{Tr}_V(M)) = 0$ because $\operatorname{Tr}_V(M) \in \operatorname{Gen} V_R \subseteq V_R^{\perp}$, and moreover $\operatorname{Ext}^1_R(V, M) = 0$ by hypothesis. So $\operatorname{Hom}_R(V, M/\operatorname{Tr}_V(M)) = 0 = \operatorname{Ext}^1_{\mathcal{A}}(V, M/\operatorname{Tr}_V(M))$. Let us show that $M/\operatorname{Tr}_V(M) = 0$. Applying the functor $\operatorname{Hom}_R(-, M/\operatorname{Tr}_V(M))$ to the sequence in (c), we obtain the exact row

$$0 \to \operatorname{Hom}_R(V'', M/\operatorname{Tr}_V(M)) \to \operatorname{Hom}_R(V', M/\operatorname{Tr}_V(M)) \to \operatorname{Hom}_R(R, M/\operatorname{Tr}_V(M)) \to$$
$$\operatorname{Ext}^1_R(V'', M/\operatorname{Tr}_V(M)) \to \operatorname{Ext}^1_R(V', M/\operatorname{Tr}_V(M)) \to 0,$$

where $\operatorname{Hom}_R(V', M/\operatorname{Tr}_V(M)) = 0 = \operatorname{Ext}^1_R(V'', M/\operatorname{Tr}_V(M))$, because V' and V'' are direct summand of direct sums of copies of V. Hence $M/\operatorname{Tr}_V(M) \cong \operatorname{Hom}_R(R, M/\operatorname{Tr}_V(M)) = 0$. Therefore, $M = \operatorname{Tr}_V(M) \in \operatorname{Gen} V_R$. Thus $V_R^{\perp} \subseteq \operatorname{Gen} V_R$.
This shows that V_R is a tilting module. □

Example 3.13. Let k be an algebraically closed field and let

$$R = \begin{pmatrix} k & k \\ 0 & k \end{pmatrix}$$

be the ring of 2×2 upper triangular matrices with entries in k. Let

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Notice that e_1 and e_2 are idempotents elements of R and that we have the following canonical decomposition of R as right R-module:

$$R_R = e_1R \oplus e_2R,$$

where $e_1R = \begin{pmatrix} k & k \\ 0 & 0 \end{pmatrix}$ and $e_2R = \begin{pmatrix} 0 & 0 \\ 0 & k \end{pmatrix}$. Now e_2R embeds in e_1R via the R-homomorphism $i \colon e_2R \to e_1R$, $\begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} \mapsto \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$. By abuse of notation, we shall denote by e_2R the image of e_2R in e_1R under the map i. We denote by p the canonical projection $e_1R \to e_1R/e_2R$.
Let

$$V_R = e_1R \oplus (e_1R/e_2R).$$

We shall prove that V_R is a tilting module. To do this, we will use the characterization of tilting modules given in Theorem 3.12.
First, there is an exact sequence

$$0 \to e_2R \xrightarrow{i} e_1R \oplus e_1R \xrightarrow{(\mathrm{id},p)} e_1R \oplus (e_1R/e_2R) \to 0,$$

where e_2R and $e_1R \oplus e_1R$ are direct summands of R_R and R_R^2 respectively. Moreover, there is an exact sequence

$$0 \to R = e_1R \oplus e_2R \xrightarrow{(\mathrm{id},i)} e_1R \oplus e_1R \xrightarrow{p} e_1R/e_2R \to 0,$$

where $e_1R \oplus e_1R$ and e_1R/e_2R are direct summands of V^2 and V respectively. Hence it remains to show that $\operatorname{Ext}^1_R(V, V) = 0$. Since $\operatorname{Ext}^1_R(-,-)$ is an additive bifunctor, then we get easily that

$$\begin{aligned} \operatorname{Ext}^1_R(V, V) &= \operatorname{Ext}^1_R(e_1R \oplus (e_1R/e_2R), e_1R \oplus (e_1R/e_2R)) \\ &\cong \operatorname{Ext}^1_R(e_1R, e_1R) \oplus \operatorname{Ext}^1_R(e_1R, e_1R/e_2R) \oplus \\ &\quad \operatorname{Ext}^1_R(e_1R/e_2R, e_1R) \oplus \operatorname{Ext}^1_R(e_1R/e_2R, e_1R/e_2R). \end{aligned}$$

Now e_1R is projective, being a direct summand of R_R, thus

$$\operatorname{Ext}^1_R(e_1R, e_1R) = 0 = \operatorname{Ext}^1_R(e_1R, e_1R/e_2R).$$

Let us show that $\mathrm{Ext}^1_R(e_1R/e_2R, e_1R) = 0 = \mathrm{Ext}^1_R(e_1R/e_2R, e_1R/e_2R)$. Consider the projective resolution of e_1R/e_2R given by

$$(*) \qquad 0 \to e_2R \xrightarrow{i} e_1R \xrightarrow{p} e_1R/e_2R \to 0.$$

Applying the functor $\mathrm{Hom}_R(-, e_1R)$ to (*) we obtain the exact sequence

$$0 \to \mathrm{Hom}_R(e_1R/e_2R, e_1R) \to \mathrm{Hom}_R(e_1R, e_1R) \xrightarrow{i^*} \mathrm{Hom}_R(e_2R, e_1R) \to \\ \mathrm{Ext}^1_R(e_1R/e_2R, e_1R) \to \mathrm{Ext}^1_R(e_1R, e_1R) = 0 = \mathrm{Ext}^1_R(e_2R, e_1R),$$

where $i^* = \mathrm{Hom}_R(i, e_1R)$ and $\mathrm{Ext}^1_R(e_2R, e_1R) = 0$ because e_2R is projective. Therefore

$$\mathrm{Ext}^1_R(e_1R/e_2R, e_1R) \cong \mathrm{Hom}_R(e_2R, e_1R)/\operatorname{Im} i^*.$$

Similarly, applying the functor $\mathrm{Hom}_R(-, e_1R/e_2R)$ to (*) we obtain

$$\mathrm{Ext}^1_R(e_1R/e_2R, e_1R/e_2R) \cong \mathrm{Hom}_R(e_2R, e_1R/e_2R)/\operatorname{Im} i^{*\prime},$$

where $i^{*\prime} = \mathrm{Hom}_R(i, e_1R/e_2R)$.
Now, since e_2 is idempotent, by [**AF92**, Corollary 4.7] the functors $\mathrm{Hom}_R(e_2R, -)$ and $(-)e_2R$ are naturally equivalent. Thus $\mathrm{Hom}_R(e_2R, e_1R) \cong e_1Re_2 \cong \begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix} \cong k$. On the other hand, $\mathrm{Hom}_R(e_1R, e_1R) \cong e_1Re_1 \cong \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix}$, so that given $f \in \mathrm{Hom}_R(e_1R, e_1R)$, we can view f as the left multiplication by a matrix $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$, with $a \in k$. Since i can be viewed as the left multiplication by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, we have that $i^*(f) = fi$ is the left multiplication by the matrix $\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$. Therefore $\operatorname{Im} i^* \cong \begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix} \cong k$, hence

$$\mathrm{Ext}^1_R(e_1R/e_2R, e_1R) = 0.$$

Now let us show that $i^{*\prime}$ is surjective. Let $g \in \mathrm{Hom}_R(e_2R, e_1R/e_2R)$. Since e_2R is projective, there exists $\widetilde{g} \in \mathrm{Hom}_R(e_2R, e_1R)$ such that $p\widetilde{g} = g$. By what we saw above, i^* is surjective, hence there exists $h \in \mathrm{Hom}_R(e_1R, e_1R)$ with $hi = \widetilde{g}$. So the element $ph \in \mathrm{Hom}_R(e_1R, e_1R/e_2R)$ satisfies $i^{*\prime}(ph) = (ph)i = p(hi) = p\widetilde{g} = g$. This proves that $i^{*\prime}$ is surjective. Hence

$$\mathrm{Ext}^1_R(e_1R/e_2R, e_1R/e_2R) = 0.$$

We conclude that V_R is a tilting module.

3.2. The commutative case. We begin with the following result, due to C. Menini (see [**CM93**]):

PROPOSITION 3.14. *Let R be a commutative ring, $M \in \text{Mod-}R$ a finitely presented module such that* $\operatorname{proj\,dim} M \leq 1$ *and* $\mathrm{Ext}^1_R(M, M) = 0$*. Then M is projective.*

PROOF. Since M is finitely presented, by [**Bou61**, Theorem 1, Chapter II, Section 5], it is enough to show that for every maximal ideal $\wp$ of R, $M_\wp$ is projective. Let $\wp$ be a maximal ideal of R. Without loss of generality, we can assume that $M_\wp \neq 0$ so that, being $M_\wp$ a finitely generated $R_\wp$-module, we obtain by Nakayama's lemma that $M_\wp \neq \wp M_\wp$, and hence $M \neq \wp M$. Therefore $M/\wp M$ is a non-zero $R/\wp$-vector space so that we have an epimorphism $M \to R/\wp \to 0$. As $\mathrm{Ext}^1_R(M, M) = 0$ and $\mathrm{Ext}^2_R(M, -) = 0$ we obtain $0 = \mathrm{Ext}^1_R(M, M) = 0 \to \mathrm{Ext}^1_R(M, R/\wp) \to 0$. Therefore $\mathrm{Ext}^1_R(M, R/\wp) = 0$ and by [**CE56**, Proposition 5.1] we obtain

$$\begin{aligned} 0 = \mathrm{Ext}^1_R(M, R/\wp) &\cong \mathrm{Ext}^1_R(M, \mathrm{Hom}_R(R/\wp, E(R/\wp))) \\ &\cong \mathrm{Hom}_R(\mathrm{Tor}^R_1(M, R/\wp), E(R/\wp)), \end{aligned}$$

where $E(R/\wp)$ denotes the injective envelope of $R/\wp$. Since $E_R(R/\wp) \cong E_{R/\wp}(R_\wp/\wp R_\wp)$, we obtain

$$\begin{aligned} 0 &= \mathrm{Hom}_R(\mathrm{Tor}^1_R(M, R/\wp), \mathrm{Hom}_{R_\wp}(R_\wp, E(R_\wp/\wp R_\wp))) \\ &\cong \mathrm{Hom}_{R_\wp}(\mathrm{Tor}^1_R(M, R/\wp)\otimes_R R_\wp, E(R_\wp/\wp R_\wp)). \end{aligned}$$

As $E(R_\wp/\wp R_\wp)$ is a cogenerator of Mod-$R_\wp$, we otain

$$\mathrm{Tor}^1_R(M, R/\wp)\otimes_R R_\wp = 0$$

hence by [**Bou65**, Proposition 9, Chapter X, Section 6 n. 7] we have

$$\mathrm{Tor}^1_{R_\wp}(M_\wp, R_\wp/\wp R_\wp) \cong \mathrm{Tor}^1_R(M, R/\wp)\otimes_R R_\wp = 0.$$

Therefore, in view of [**Bou61**, Corollary 2 to Proposition 5, Chapter II, Section 3], $M_\wp$ is projective. $\square$

In particular, we have:

Corollary 3.15. *A tilting module over a commutative ring is a progenerator.*

CHAPTER 4

The heart associated to a faithful torsion pair

In this chapter we begin the study of the heart associated to a faithful torsion pair. In the first section, we shall characterize the faithful torsion pairs $(\mathcal{X},\mathcal{Y})$ in Mod-R as the torsion pairs which are tilted by means of a tilting object V in a suitable cocomplete abelian category $\mathcal{H}$. In particular, one can choose $\mathcal{H}=\mathcal{H}(\mathcal{X},\mathcal{Y})$ and $V=R[1]$. In the second section, we will show that $\mathcal{H}(\mathcal{X},\mathcal{Y})$ is a prototype for any abelian category $\mathcal{H}$ admitting a tilting object V which tilts to $(\mathcal{X},\mathcal{Y})$ in Mod-R, i.e., any abelian category which tilts to the same torsion pair is equivalent to it.
The reader is referred to [**CGM07**].

1. Representing faithful torsion pairs

Let $\mathcal{A}$ be an abelian category and let $(\mathcal{X},\mathcal{Y})$ be a torsion pair in $\mathcal{A}$. Let $\mathcal{H}(\mathcal{X},\mathcal{Y})$ be the heart of the t-structure in $\mathrm{D}^b(\mathcal{A})$ associated with $(\mathcal{X},\mathcal{Y})$. Let us recall that regarding a map $X^{-1}\xrightarrow{x}X^0$ as a complex $\dots 0\to X^{-1}\xrightarrow{x}X^0\to 0\dots$, the objects of $\mathcal{H}(\mathcal{X},\mathcal{Y})$ are represented, up to isomorphism, by complexes of the form

$$X:\ X^{-1}\xrightarrow{x}X^0 \text{ with } \operatorname{Ker}x\in\mathcal{Y} \text{ and } \operatorname{Coker}x\in\mathcal{X}.$$

If $\mathcal{A}$ has products and coproducts with good behaviour, as in the case of Mod-R, then $\mathcal{H}(\mathcal{X},\mathcal{Y})$ is cocomplete:

LEMMA 4.1. *Let $\mathcal{A}$ be a complete and cocomplete abelian category with exact coproducts, such that for any family of objects the canonical map from their coproduct to their product is monic. Then for any torsion theory $(\mathcal{X},\mathcal{Y})$ in $\mathcal{A}$ the associated heart $\mathcal{H}(\mathcal{X},\mathcal{Y})$ is cocomplete.*

PROOF. Let α be any cardinal. By hypothesis, the diagram

$$\amalg:\prod_{\alpha}\mathcal{A}\rightleftarrows\mathcal{A}:\Delta,$$

where $\amalg$ is the coproduct functor and Δ is the diagonal functor, defines an adjoint pair $\langle\amalg,\Delta\rangle$. This adjunction naturally extends componentwise to the corresponding homotopy categories. Moreover, since both $\amalg$ and Δ are exact, they extend to a pair of functors $\widehat{\amalg}$ and $\widehat{\Delta}$ between the corresponding derived categories. Moreover, thanks to [**Kel07**, § 3], the diagram

$$\widehat{\amalg}:\mathrm{D}^b(\prod_{\alpha}\mathcal{A})\cong\prod_{\alpha}\mathrm{D}^b(\mathcal{A})\rightleftarrows\mathrm{D}^b(\mathcal{A}):\widehat{\Delta}$$

still defines an adjoint pair $\langle\widehat{\amalg},\widehat{\Delta}\rangle$. This shows that $\mathrm{D}^b(\mathcal{A})$ admits arbitrary coproducts, and that they are defined componentwise. Moreover, since the assumptions on $\mathcal{A}$ guarantee that both $\mathcal{X}$ and $\mathcal{Y}$ are closed under arbitrary coproducts, we see that $\mathcal{H}(\mathcal{X},\mathcal{Y})$ is closed under coproducts in $\mathrm{D}^b(\mathcal{A})$. $\square$

For each $M\in$ Mod-R, let us denote by $\operatorname{Ann}_R(M)$ the right annihilator of M, i.e., $\operatorname{Ann}_R(M)=\{r\in R\mid mr=0,\text{ for every } m\in M\}$. Clearly, $\operatorname{Ann}_R(M)$ is a right ideal of R. If $\mathcal{M}$ is a class of objects of Mod-R, we set $\operatorname{Ann}_R(\mathcal{M})=\bigcap\{\operatorname{Ann}_R(M)\mid M\in\mathcal{M}\}$.
By [**CF07**, Theorem 1.4] we have:

PROPOSITION 4.2. *If $(\mathcal{X},\mathcal{Y})$ is a torsion pair in* Mod-R *there is an object $V=(R/\operatorname{Ann}_R(\mathcal{Y}))[1]$ in $\mathcal{H}(\mathcal{X},\mathcal{Y})$ that induces an equivalence*

$$H_V:\mathcal{Y}[1]\rightleftarrows\mathcal{Y}:T_V.$$

Definition 4.3. A torsion pair $(\mathcal{X},\mathcal{Y})$ in Mod-R is **faithful** if $\operatorname{Ann}_R(\mathcal{Y})=0$.

Remark 4.4. Let $(\mathcal{X},\mathcal{Y})$ be a torsion pair in Mod-R. The following conditions are equivalent:

a) $(\mathcal{X},\mathcal{Y})$ is faithful;
b) $R_R\in\mathcal{Y}$;
c) $\mathcal{Y}$ generates Mod-R.

Proof. a) $\Rightarrow$ b). Assume that $(\mathcal{X},\mathcal{Y})$ is faithful, and by contradiction suppose that $R_R\notin\mathcal{Y}$. Then there exist $N\in\mathcal{X}$ and a non-zero R-homomorphism $f\colon N\to R_R$, so there is $\xi\in N$ with $f(\xi)=r\neq 0$. Let $M\in\mathcal{Y}$ and for each $m\in M$ define the map $g_{M,m}\colon N\to M$ by $g_{M,m}=mf(x)$. The verification that $g_{M,m}$ is an R-homomorphism is straightforward. By definition of torsion pair, we have that $g_{M,m}=0$, for every $M\in\mathcal{Y}$ and $m\in M$. In particular, $mr=0$ for every $m\in M$ and $M\in\mathcal{Y}$. Hence $r\in\operatorname{Ann}_R(\mathcal{Y})$, so $r=0$, a contradiction. Therefore, $R_R\in\mathcal{Y}$.
b) $\Rightarrow$ a). Assume that $R_R\in\mathcal{Y}$. Then $0=\operatorname{Ann}_R(R)\supseteq\bigcap_{M\in\mathcal{Y}}\operatorname{Ann}_R(M)=\operatorname{Ann}_R(\mathcal{Y})$, thus $\operatorname{Ann}_R(\mathcal{Y})=0$. This proves that $(\mathcal{X},\,\mathcal{Y})$ is faithful.
b) $\Rightarrow$ c). This is clear, since R_R generates Mod-R.
c) $\Rightarrow$ b). Assume that $\mathcal{Y}$ generates Mod-R. Then there exists an exact sequence $\coprod_\lambda Y_\lambda\to R_R\to 0$, where $Y_\lambda\in\mathcal{Y}$. Since R_R is projective, this sequence splits. It follows that $R_R\in\mathcal{Y}$, because $\mathcal{Y}$ is closed under subobjects. $\square$

Example 4.5. Let $(\mathcal{T},\mathcal{F})$ and $(\mathcal{D},\mathcal{R})$ be as in Example 2.2 and let V_R be the tilting module of Example 3.13. Then $(\mathcal{T},\mathcal{F})$ and $(\mathcal{D},\mathcal{R})$ are both faithful torsion pairs, since $\mathbb{Z}$ is torsion-free and reduced as abelian group, whereas the tilting torsion pair associated to V is not faithful, because $R_R=e_1R\oplus e_2R$ with $e_1R\in\operatorname{Gen}V_R$.

We want to show that when $(\mathcal{X},\mathcal{Y})$ is faithful in Mod-R, the equivalence $H_V:\mathcal{T}\rightleftarrows\mathcal{Y}:T_V$ in Proposition 4.2 is actually induced by a tilting object V with $\operatorname{End}_{\mathcal{H}(\mathcal{X},\mathcal{Y})}(V)=R$.
To do so we need the following

Lemma 4.6. *Let $(\mathcal{X},\mathcal{Y})$ be a torsion pair in $\mathcal{A}$. If $\mathcal{Y}$ generates $\mathcal{A}$, then every object of $\mathcal{H}(\mathcal{X},\mathcal{Y})$ is isomorphic to a complex of the form $Y^{-1}\to Y^0$, with $Y^{-1},Y^0\in\mathcal{Y}$.*

Proof. Let $Z^{-1}\xrightarrow{z}Z^0\in\mathcal{H}(\mathcal{X},\mathcal{Y})$ to obtain exact sequences

$$0\to Y\to Z^{-1}\to I\to 0\quad\text{and}\quad 0\to I\to Z^0\to X\to 0$$

with $I=\operatorname{Im}z$, $Y\in\mathcal{Y}$ and $X\in\mathcal{X}$. Then there are an object $Y^0\in\mathcal{Y}$, an epimorphisms $Y^0\to Z^0$ and a pullback diagram

(1)
$$\begin{tikzcd}
0 \arrow[r] & P \arrow[r] \arrow[d] & Y^0 \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & I \arrow[r] \arrow[d] & Z^0 \arrow[r] \arrow[d] & X \arrow[r] & 0 \\
& 0 & 0 & &
\end{tikzcd}$$

where P is in $\mathcal{Y}$, since $\mathcal{Y}$ is closed under subobjects. Then we obtain a further pullback diagram

(2)
$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r] \arrow[d, equal] & Y^{-1} \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Y \arrow[r] & Z^{-1} \arrow[r] \arrow[d] & I \arrow[r] \arrow[d] & 0 \\
& & 0 & 0 &
\end{tikzcd}$$

where Y^{-1} is in $\mathcal{Y}$, since $\mathcal{Y}$ is closed under extensions. Now (1) and (2) combine to give a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r] \arrow[d, equal] & Y^{-1} \arrow[r] \arrow[d] & Y^0 \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & Y \arrow[r] & Z^{-1} \arrow[r] & Z^0 \arrow[r] & X \arrow[r] & 0 \end{tikzcd}$$

and so the desired quasi-isomorphism. □

This allows us to prove the following

Proposition 4.7. *If $\mathcal{Y}$ generates $\mathcal{A}$, then $\mathcal{T} = \mathcal{Y}[1]$ cogenerates $\mathcal{H}(\mathcal{X}, \mathcal{Y})$.*

Proof. By the last lemma, we know that every object in $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ is isomorphic to a complex of the form $Y^{-1} \xrightarrow{y} Y^0$ with $Y^{-1}, Y^0 \in \mathcal{Y}$. We shall show that

$$\begin{tikzcd} Y^{-1} \arrow[r, "y"] \arrow[d, equal] & Y^0 \arrow[d] \\ Y^{-1} \arrow[r] & 0 \end{tikzcd}$$

is a monomorphism. So suppose that the commutative diagram

$$\begin{tikzcd} Z^{-1} \arrow[r, "z"] \arrow[d, "\varphi^{-1}"] & Z^0 \arrow[d, "\varphi^0"'] \\ Y^{-1} \arrow[r, "y"] \arrow[d, equal] & Y^0 \arrow[d] \\ Y^{-1} \arrow[r] & 0 \end{tikzcd}$$

yields a null-homotopic map, i.e., that there is a map $r^0 : Z^0 \to Y^{-1}$ such that

$$\varphi^{-1} = r^0 z.$$

Let $\gamma = yr^0 - \varphi^0 : Z^0 \to Y^0$ so that

$$\gamma z = yr^0 z - \varphi^0 z = y\varphi^{-1} - y\varphi^{-1} = 0$$

and hence $\operatorname{Im} z \subseteq \operatorname{Ker} \gamma$. But $Z^0/\operatorname{Im} z \in \mathcal{X}$ and $Z^0/\operatorname{Ker} \gamma \in \mathcal{Y}$. Thus $\gamma = 0$ and so

$$\varphi^0 = yr^0.$$

In other words the map

$$\begin{tikzcd} Z^{-1} \arrow[r, "z"] \arrow[d, "\varphi^{-1}"] & Z^0 \arrow[d, "\varphi^0"'] \\ Y^{-1} \arrow[r, "y"] & Y^0 \end{tikzcd}$$

is zero in $\mathcal{H}(\mathcal{X}, \mathcal{Y})$, which proves our assertion. □

Now we can prove:

Theorem 4.8. *A torsion pair $(\mathcal{X}, \mathcal{Y})$ in* Mod-R *is faithful if and only if there is a cocomplete abelian category $\mathcal{H}$ and a tilting object V of $\mathcal{H}$ such that $R = \operatorname{End}_{\mathcal{H}}(V)$.*

Proof. Assume that $(\mathcal{X}, \mathcal{Y})$ is faithful. Then by Remark 4.4, $\mathcal{Y}$ generates Mod-R, so that by Proposition 4.7, $\mathcal{Y}[1]$ cogenerates $\mathcal{H} = \mathcal{H}(\mathcal{X}, \mathcal{Y})$. Now, $\mathcal{H}$ is a cocomplete abelian category by Lemma 4.1. Let $V = (R/\operatorname{Ann}_R(\mathcal{Y}))[1] = R[1]$. By Proposition 4.2, V induces an equivalence $H_V : \mathcal{Y}[1] \rightleftarrows \mathcal{Y} : T_V$, so that by [**CF07**, Theorem 2.4], V is a tilting object in $\overline{\operatorname{Gen}}\, V = \overline{\operatorname{Gen}}\, R[1] = \mathcal{H}$, where the last equality holds because $\mathcal{Y}[1]$ cogenerates $\mathcal{H}$. Finally, by [**CF07**, Theorem 1.4], $\operatorname{End}_{\mathcal{H}}(V) = R$.

Conversely, assume that there is a cocomplete abelian category $\mathcal{H}$ and a tilting object V of $\mathcal{H}$ such that $R = \mathrm{End}_{\mathcal{H}}(V)$. Then $R = H_V(V) \in \mathcal{Y}$, hence $(\mathcal{X}, \mathcal{Y})$ is faithful by Remark 4.4. $\square$

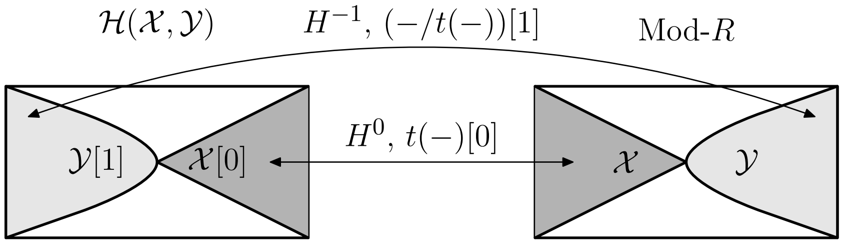
The preceeding theorem asserts that any tilted torsion pair is faithful and that, conversely, given any ring R and any faithful torsion pair $(\mathcal{X}, \mathcal{Y})$ in Mod-R, then $(\mathcal{X}, \mathcal{Y})$ is tilted by means of a tilting object in a suitable cocomplete abelian category $\mathcal{H}(\mathcal{X}, \mathcal{Y})$. In this case we are able to characterize the tilting counterequivalence. More precisely, we have:

Remark 4.9. Let $\mathcal{H}(X, Y)$ be the heart of a faithful torsion pair in Mod-R, and let $V = R[1]$ be the tilting object in $\mathcal{H}(X, Y)$ generating the torsion pair $(\mathcal{Y}[1], \mathcal{X})$. Then the torsion pair $(\mathcal{Y}[1], \mathcal{X})$ tilts to the originary torsion pair $(\mathcal{X}, \mathcal{Y})$ in Mod-R by means of the functors

$$H_V = H^{-1}, \qquad H'_V = H^0,$$

$$T_V(M) = (\iota(M/t(M))[1], \qquad T'_V(M) = \iota(t(M)),$$

where ι denotes the natural embedding of Mod-R into $\mathcal{D}^b(\text{Mod-}R)$.



Proof. Let $M \in \mathcal{H}(\mathcal{X}, \mathcal{Y})$. Then we can view M as a complex $M^{-1} \xrightarrow{m} M^0$, with $\mathrm{Ker}\, m = \mathrm{H}^{-1}(M) \in \mathcal{Y}$ and $\mathrm{Coker}\, m = \mathrm{H}^0(M) \in \mathcal{X}$. By observation 3. of Theorem 3.6, $H_V(M) = H_V(t(M))$ and $H'_V(M) = H'_V(M/t(M))$. Now, the sequence

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & \mathrm{H}^{-1}(M) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & M^{-1} \arrow[r, "m"] \arrow[d] & M^0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \mathrm{H}^0(M) \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

is exact in $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ (see [**BBD82**, 3.1.17]), and $H^{-1}(M) \to 0 \in \mathcal{Y}[1]$, $0 \to H^0(M) \in \mathcal{X}[0]$. Therefore,

$$t(M) = H^{-1}(M) \to 0, \qquad M/t(M) = 0 \to H^0(M).$$

Hence

$$\begin{aligned}
H_V(M) &= H_V(t(M)) \\
&= \mathrm{Hom}_{\mathcal{H}(\mathcal{X},\mathcal{Y})}(R[1], t(M)) \\
&\cong \mathrm{Hom}_R(R, H^{-1}(M)) \\
&\cong H^{-1}(M).
\end{aligned}$$

Similarly,

$$\begin{aligned}
H'_V(M) &= H'_V(M/t(M)) \\
&= \mathrm{Ext}^1_{\mathcal{H}(\mathcal{X},\mathcal{Y})}(R[1], M/t(M)) \\
&\cong \mathrm{Hom}_{\mathrm{D}^b(\mathcal{H}(\mathcal{X},\mathcal{Y}))}(R[1], M/t(M)[1]) \\
&\cong \mathrm{Hom}_R(R, H^0(M)) \\
&\cong H^0(M).
\end{aligned}$$

This shows that $H_V = H^{-1}$ and $H'_V = H^0$. Finally, setting $\widetilde{T_V}(M) = (\iota(M/t(M))[1]$ and $\widetilde{T'_V}(M) = \iota(t(M))$, it is immediate to check that $H_V : \mathcal{Y}[1] \rightleftarrows \mathcal{Y} : \widetilde{T_V}$ and $H'_V : \mathcal{X}[0] \rightleftarrows \mathcal{X} : \widetilde{T'_V}$ are equivalences. The result follows. $\square$

We conclude this section with two results needed in the sequel.

Proposition 4.10. *Let $(\mathcal{X}, \mathcal{Y})$ be a faithful torsion pair in* Mod-R. *Then for any direct system $(X_\lambda, \xi_{\lambda\mu})$ in $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ such that each X_λ is in $\mathcal{T} = \mathcal{Y}[1]$, the object $\varinjlim X_\lambda$ is in $\mathcal{T}$.*

Proof. It follows from the fact that $\mathcal{T}$ is closed under coproducts and factors, and by [**Ste75**, Proposition 8.4, Chapter IV]. $\square$

Proposition 4.11. *Let $\mathcal{Y}$ be a torsion-free class in* Mod-R *cogenerated by a cotilting right R-module. Then $\mathcal{Y}$ is closed under taking direct limits.*

Proof. Let U be a cotilting right R-module cogenerating $\mathcal{Y}$. Let $(X_\lambda, \xi_{\lambda\mu})$ be a direct system of objects in $\mathcal{Y}$ and consider the canonical exact sequence

$$0 \to K \to \coprod X_\lambda \to \varinjlim X_\lambda \to 0.$$

By [**GT06**, Corollary 1.2.7], this sequence is pure, and by a result of Bazzoni [**Baz03**], U is pure injective. So by applying the functor $\mathrm{Hom}_{\mathcal{H}(\mathcal{X},\mathcal{Y})}(-, U)$, we get the long exact sequence

$$\begin{aligned} 0 \to \mathrm{Hom}_{\mathcal{H}(\mathcal{X},\mathcal{Y})}(\varinjlim X_\lambda, U) \to \mathrm{Hom}_{\mathcal{H}(\mathcal{X},\mathcal{Y})}(\coprod X_\lambda, U) \xrightarrow{\alpha} \mathrm{Hom}_{\mathcal{H}(\mathcal{X},\mathcal{Y})}(K, U) \to \\ \mathrm{Ext}^1_{\mathcal{H}(\mathcal{X},\mathcal{Y})}(\varinjlim X_\lambda, U) \to \mathrm{Ext}^1_{\mathcal{H}(\mathcal{X},\mathcal{Y})}(\coprod X_\lambda, U) \to \dots, \end{aligned}$$

with α epic. Since each X_λ is in $\mathcal{Y}$ and $\mathcal{Y}$ is closed under coproducts, we have that $\coprod X_\lambda \in \mathcal{Y}$, hence $\mathrm{Ext}^1_{\mathcal{H}(\mathcal{X},\mathcal{Y})}(\coprod X_\lambda, U) \cong \prod \mathrm{Ext}^1_{\mathcal{H}(\mathcal{X},\mathcal{Y})}(X_\lambda, U) = 0$. We conclude that $\mathrm{Ext}^1_{\mathcal{H}(\mathcal{X},\mathcal{Y})}(\varinjlim X_\lambda, U) = 0$, i.e., $\varinjlim X_\lambda \in \mathcal{Y}$. $\square$

2. Morita equivalence

In this section we show that $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ is a prototype for any abelian category $\mathcal{H}$ admitting a tilting object V which tilts to $(\mathcal{X}, \mathcal{Y})$ in Mod-R, i.e., any abelian category which tilts to the same torsion pair is equivalent to it. This is obtained as a consequence of the fact that two abelian categories with tilting objects which tilt to the same target $(\mathcal{X}, \mathcal{Y})$ are (Morita) equivalent.
If $\mathcal{A}$ is an abelian category with a tilting object $V_\mathcal{A}$, let us denote by $(\mathcal{T}_\mathcal{A}, \mathcal{F}_\mathcal{A})$ the associated torsion theory and by $(\mathcal{X}, \mathcal{Y})$ the "tilted" torsion theory in $R = \operatorname{End} V_\mathcal{A}$.

Lemma 4.12. *If $A \in \mathcal{A}$ and $0 \to A \to A'_1 \to A'_2 \to 0$ and $0 \to A \to A''_1 \to A''_2 \to 0$ are exact sequences with $A'_1, A'_2, A''_1, A''_2 \in \mathcal{T}_\mathcal{A}$, then there are a third exact sequence $0 \to A \to A_1 \to A_2 \to 0$ with $A_1, A_2 \in \mathcal{T}_\mathcal{A}$ and maps making the diagram*

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, equal] & A'_1 \arrow[r] \arrow[d] & A'_2 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A \arrow[r] & A_1 \arrow[r] & A_2 \arrow[r] & 0 \\ 0 \arrow[r] & A \arrow[r] \arrow[u, equal] & A''_1 \arrow[r] \arrow[u] & A''_2 \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

commutative.

Proof. Take for A_1 the push-out of the monomorphisms $A \to A'_1$ and $A \to A''_1$, and complete the sequence with the cokernel. $\square$

Lemma 4.13. *If $f\colon A \to A'$ is a map in $\mathcal{A}$ and $0 \to A \to A_1 \to A_2 \to 0$ is an exact sequences, with $A_1, A_2 \in \mathcal{T}_\mathcal{A}$, then there is a second exact sequence $0 \to A' \to A'_1 \to A'_2 \to 0$, with*

$A'_1, A'_2 \in \mathcal{T}_\mathcal{A}$, and maps $f_1\colon A_1 \to A'_1$ and $f_2\colon A_2 \to A'_2$ making the diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "f"] & A_1 \arrow[r] \arrow[d, "f_1"] & A_2 \arrow[r] \arrow[d, "f_2"] & 0 \\
0 \arrow[r] & A' \arrow[r] & A'_1 \arrow[r] & A'_2 \arrow[r] & 0
\end{tikzcd}$$

commutative.

Proof. A' can be embedded in some $A'_0 \in \mathcal{T}_\mathcal{A}$. Now take for A'_1 the push-out of the monomorphism $A \to A_1$ and of the composed map $A \xrightarrow{f} A' \to A'_0$, and complete the diagram with the cokernel of the new monomorphism $A' \to A'_1$. □

Now we are ready to prove the main result of this section.

Theorem 4.14. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories with tilting objects $V_\mathcal{A} \in \mathcal{A}$ and $V_\mathcal{B} \in \mathcal{B}$. Assume that $\operatorname{End} V_\mathcal{A} \cong \operatorname{End} V_\mathcal{B}$, and that $V_\mathcal{A}$ and $V_\mathcal{B}$ induce a counterequivalence with the same torsion pair $(\mathcal{X}, \mathcal{Y})$. Then $\mathcal{A}$ and $\mathcal{B}$ are equivalent.*

Proof. For the sake of simplicity, instead of $H_{V_\mathcal{A}}$ we write $H_\mathcal{A}$ and similarly for the other three functors involved.
We will explicity define two mutually inverse functors $F\colon \mathcal{A} \to \mathcal{B}$ and $G\colon \mathcal{B} \to \mathcal{A}$, whose actions extend the equivalences $\mathcal{T}_\mathcal{A} \rightleftarrows \mathcal{T}_\mathcal{B}$ and $\mathcal{F}_\mathcal{A} \rightleftarrows \mathcal{F}_\mathcal{B}$ obtained by composing the two torsion theory counter equivalences induced by $V_\mathcal{A}$ and $V_\mathcal{B}$.
Any object $A \in \mathcal{A}$ admits an exact sequence

$$(*) \qquad 0 \to A \to A_1 \xrightarrow{\alpha} A_2 \to 0$$

with $A_1, A_2 \in \mathcal{T}_\mathcal{A}$ (see Definition 3.1 iii)). From (*) we obtain the exact sequence in Mod-R

$$(**) \qquad \begin{tikzcd}
0 \arrow[r] & H_\mathcal{A}A \arrow[r] & H_\mathcal{A}A_1 \arrow[rr, "H_\mathcal{A}(\alpha)"] \arrow[dr, two heads, "\sigma"] & & H_\mathcal{A}A_2 \arrow[r] & H'_\mathcal{A}A \arrow[r] & 0 \\
& & & C \arrow[ur, hook] & & &
\end{tikzcd}$$

and so the commutative diagram in $\mathcal{B}$ with exact rows and columns

$$(***) \qquad \begin{tikzcd}
& & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & T_\mathcal{B}H_\mathcal{A}A \arrow[r, "i_A"] \arrow[d, equal] & F(A) \arrow[r, dotted, "p_A"] \arrow[d, dotted, "j_A"] & T'_\mathcal{B}H'_\mathcal{A}A \arrow[r] \arrow[d, "\delta_A"] & 0 \\
0 \arrow[r] & T_\mathcal{B}H_\mathcal{A}A \arrow[r] & T_\mathcal{B}H_\mathcal{A}A_1 \arrow[r, "T_\mathcal{B}(\sigma)"] \arrow[d, "T_\mathcal{B}H_\mathcal{A}(\alpha)"] & T_\mathcal{B}C \arrow[r] \arrow[d] & 0 \\
& & T_\mathcal{B}H_\mathcal{A}A_2 \arrow[r, equal] \arrow[d] & T_\mathcal{B}H_\mathcal{A}A_2 \arrow[d] & \\
& & 0 & 0 &
\end{tikzcd}$$

where the dashed arrows represent a pullback. Note that the upper exact row shows that $F(A)$ has $T_\mathcal{B}H_\mathcal{A}A$ as its torsion part and $T'_\mathcal{B}H'_\mathcal{A}A$ as its torsionfree part, and the exact colum in the middle shows that $F(A) = \operatorname{Ker} T_\mathcal{B}H_\mathcal{A}(\alpha)$.

Let us now consider any morphism $f\colon A \to A'$ in $\mathcal{A}$ and an arbitrary exact sequence (*) for A. Then Lemma 4.13 gives a commutative diagram with exact rows in $\mathcal{A}$

(+)
$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "f"] & A_1 \arrow[r, "\alpha"] \arrow[d, "f_1"] & A_2 \arrow[r] \arrow[d, "f_2"] & 0 \\ 0 \arrow[r] & A' \arrow[r] & A'_1 \arrow[r, "\alpha'"] & A'_2 \arrow[r] & 0 \end{tikzcd}$$

and so we obtain an analogous diagram in $\mathcal{B}$

(++)
$$\begin{tikzcd} 0 \arrow[r] & F(A) \arrow[r, "j_A"] \arrow[d, "F(f)"] & T_{\mathcal{B}}H_{\mathcal{A}}A_1 \arrow[r, "T_{\mathcal{B}}H_{\mathcal{A}}(\alpha)"] \arrow[d, "T_{\mathcal{B}}H_{\mathcal{A}}(f_1)"] & T_{\mathcal{B}}H_{\mathcal{A}}A_2 \arrow[r] \arrow[d, "T_{\mathcal{B}}H_{\mathcal{A}}(f_2)"] & 0 \\ 0 \arrow[r] & F(A') \arrow[r, "j_{A'}"] & T_{\mathcal{B}}H_{\mathcal{A}}A'_1 \arrow[r, "T_{\mathcal{B}}H_{\mathcal{A}}(\alpha')"] & T_{\mathcal{B}}H_{\mathcal{A}}A'_2 \arrow[r] & 0 \end{tikzcd}$$

where $F(f)$ is the unique map making the diagram commutative.
Now we have to show that the object $F(A)$ does not depend on the choice of (*).
First let us prove that from the naturality of (***) we have the further commutative diagram with exact rows in $\mathcal{B}$

(+++)
$$\begin{tikzcd} 0 \arrow[r] & T_{\mathcal{B}}H_{\mathcal{A}}A \arrow[r, "i_A"] \arrow[d, "T_{\mathcal{B}}H_{\mathcal{A}}(f)"] & F(A) \arrow[r, "p_A"] \arrow[d, "F(f)"] & T'_{\mathcal{B}}H'_{\mathcal{A}}A \arrow[r] \arrow[d, "T'_{\mathcal{B}}H'_{\mathcal{A}}(f)"] & 0 \\ 0 \arrow[r] & T_{\mathcal{B}}H_{\mathcal{A}}A' \arrow[r, "i_{A'}"] & F(A') \arrow[r, "p_{A'}"] & T'_{\mathcal{B}}H'_{\mathcal{A}}A' \arrow[r] & 0 \end{tikzcd}$$

where $F(f)$ is the same map defined in (++). To see that, starting from (+) and considering (**) for A and A', we obtain the two diagrams

$$\begin{tikzcd} 0 \arrow[r] & H_{\mathcal{A}}A \arrow[r] \arrow[d, "H_{\mathcal{A}}(f)"] & H_{\mathcal{A}}A_1 \arrow[r, "\sigma"] \arrow[d, "H_{\mathcal{A}}(f_1)"] & C \arrow[r] \arrow[d, "\varphi"] & 0 \\ 0 \arrow[r] & H_{\mathcal{A}}A' \arrow[r] & H_{\mathcal{A}}A'_1 \arrow[r, "\sigma'"] & C' \arrow[r] & 0 \end{tikzcd}$$

and

$$\begin{tikzcd} 0 \arrow[r] & C \arrow[r] \arrow[d, "\varphi"] & H_{\mathcal{A}}A_2 \arrow[r] \arrow[d, "H_{\mathcal{A}}(f_2)"] & H'_{\mathcal{A}}A \arrow[r] \arrow[d, "H'_{\mathcal{A}}(f)"] & 0 \\ 0 \arrow[r] & C' \arrow[r] & H_{\mathcal{A}}A'_2 \arrow[r] & H'_{\mathcal{A}}A' \arrow[r] & 0 \end{tikzcd}$$

which, respectively, produce the two commutative diagrams

(1)
$$\begin{tikzcd} T_{\mathcal{B}}H_{\mathcal{A}}A_1 \arrow[r, "T_{\mathcal{B}}(\sigma)"] \arrow[d, "T_{\mathcal{B}}H_{\mathcal{A}}(f_1)"'] & T_{\mathcal{B}}C \arrow[d, "T_{\mathcal{B}}(\varphi)"] \\ T_{\mathcal{B}}H_{\mathcal{A}}A'_1 \arrow[r, "T_{\mathcal{B}}(\sigma')"] & T_{\mathcal{B}}C' \end{tikzcd}$$

and

(2)
$$\begin{tikzcd} T'_{\mathcal{B}}H'_{\mathcal{A}}A \arrow[r, "\delta_A"] \arrow[d, "T'_{\mathcal{B}}H'_{\mathcal{A}}(f)"'] & T_{\mathcal{B}}C \arrow[d, "T_{\mathcal{B}}(\varphi)"] \\ T'_{\mathcal{B}}H'_{\mathcal{A}}A' \arrow[r, "\delta_{A'}"] & T_{\mathcal{B}}C' \end{tikzcd}$$

Consider now the diagram

$$\begin{tikzcd}
T_{\mathcal{B}}H_{\mathcal{A}}A \arrow[rr, "i_A"] \arrow[dr, "T_{\mathcal{B}}H_{\mathcal{A}}(f)"'] & & F(A) \arrow[rr, "p_A"] \arrow[dr, "F(f)"] \arrow[dd, "j_A"'] & & T'_{\mathcal{B}}H'_{\mathcal{A}}A \arrow[dr, "T'_{\mathcal{B}}H'_{\mathcal{A}}(f)"] \arrow[dd, dotted, "\delta_A"] & \\
& T_{\mathcal{B}}H_{\mathcal{A}}A' \arrow[rr, "i_{A'}"] & & F(A') \arrow[rr, "p_{A'}"] \arrow[dd, "j_{A'}"] & & T'_{\mathcal{B}}H'_{\mathcal{A}}A' \arrow[dd, "\delta_{A'}"] \\
& & T_{\mathcal{B}}H_{\mathcal{A}}A_1 \arrow[rr, dotted, "T_{\mathcal{B}}(\sigma)"] \arrow[dr, "T_{\mathcal{B}}H_{\mathcal{A}}(f_1)"'] & & T_{\mathcal{B}}C \arrow[dr, dotted, "T_{\mathcal{B}}(\varphi)"] & \\
& & & T_{\mathcal{B}}H_{\mathcal{A}}A'_1 \arrow[rr, "T_{\mathcal{B}}(\sigma')"] & & T_{\mathcal{B}}C'
\end{tikzcd}$$

The front and the back face commute, owing to (***). The left face commutes because of (++), and the right because of (2). Moreover the bottom face commutes thanks to (1). So, by diagram chasing, we see that

$$\delta_{A'}\, T'_{\mathcal{B}}H'_{\mathcal{A}}(f)\, p_A = \delta_{A'}\, p_{A'}\, F(f)$$

and so, since $\delta_{A'}$ is monic, the top right face commutes too. Finally, again by diagram chasing and using (***), we see that

$$j_{A'}\, F(f)\, i_A = T_{\mathcal{B}}H_{\mathcal{A}}(f_1)\, j_A\, i_A = j_{A'}\, i_{A'}\, T_{\mathcal{B}}H_{\mathcal{A}}(f)$$

and so, since $j_{A'}$ is monic, the top left face commutes too. This proves the commutativity of the diagram (+++).
We are at last arrived to see that, for the particular choice $f = \mathrm{id}_A$ in (+), Lemma 4.12 together with the diagram (+++) ensures that $F(A)$ does not depend (up to isomorphism) on the choice of the sequence (*).
Moreover, both (++) and (+++) define the action of F on the maps of $\mathcal{A}$, and it is clear that $F\colon \mathcal{A} \to \mathcal{B}$ is a functor extending both $T_{\mathcal{B}}H_{\mathcal{A}}\restriction_{\mathcal{T}_{\mathcal{A}}}$ and $T'_{\mathcal{B}}H'_{\mathcal{A}}\restriction_{\mathcal{F}_{\mathcal{A}}}$.
In the same way one can define a functor $G\colon \mathcal{B} \to \mathcal{A}$, extending both $T_{\mathcal{A}}H_{\mathcal{B}}\restriction_{\mathcal{T}_{\mathcal{B}}}$ and $T'_{\mathcal{A}}H'_{\mathcal{B}}\restriction_{\mathcal{F}_{\mathcal{B}}}$.
Finally, let us show that F is an equivalence. Take any $A \in \mathcal{A}$ and an exact sequence (*) for A. Then

$$0 \to F(A) \to T_{\mathcal{B}}H_{\mathcal{A}}A_1 \xrightarrow{T_{\mathcal{B}}H_{\mathcal{A}}(\alpha)} T_{\mathcal{B}}H_{\mathcal{A}}A_2 \to 0$$

is an exact sequence of type (*) for $F(A)$ in $\mathcal{B}$. Since the natural transformation $T_{\mathcal{A}}H_{\mathcal{A}} \to \mathrm{id}_{\mathcal{A}}$ induces an isomorphism on $\mathcal{T}_{\mathcal{A}}$ (and similarly for $\mathcal{B}$), we obtain the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d] & A_1 \arrow[r, "\alpha"] \arrow[d, "\cong"] & A_2 \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & GF(A) \arrow[r] & T_{\mathcal{A}}H_{\mathcal{B}}T_{\mathcal{B}}H_{\mathcal{A}}A_1 \arrow[r, "T_{\mathcal{A}}H_{\mathcal{B}}T_{\mathcal{B}}H_{\mathcal{A}}(\alpha)"] & T_{\mathcal{A}}H_{\mathcal{B}}T_{\mathcal{B}}H_{\mathcal{A}}A_2 \arrow[r] & 0
\end{tikzcd}$$

This shows that the left vertical map is a natural isomorphism between A and $GF(A)$. Similarly one can show, for any $B \in \mathcal{B}$, the existence of a natural isomorphism $B \xrightarrow{\cong} FG(B)$. □

Since $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ is an abelian category with a tilting object which tilts to $(\mathcal{X}, \mathcal{Y})$ in Mod-R, we get the following statement.

Corollary 4.15. *Any abelian category $\mathcal{A}$ with a tilting object which tilts to $(\mathcal{X}, \mathcal{Y})$ is equivalent to the heart $\mathcal{H}(\mathcal{X}, \mathcal{Y})$.*

Let us describe the equivalence constructed in Theorem 4.14 in the case in which one of the categories $\mathcal{A}$ and $\mathcal{B}$ is the heart $\mathcal{H}(\mathcal{X}, \mathcal{Y})$. First we need a lemma.

Lemma 4.16. *Let $(\mathcal{X}, \mathcal{Y})$ be a faithful torsion pair in* Mod-R *with associated heart $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ and let $Y', Y'' \in \mathcal{Y}$. Then in $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ the object $M^\bullet$ is isomorphic to the complex $Y' \to Y''$ if and only if the sequence $0 \to M^\bullet \to Y'[1] \to Y''[1] \to 0$ is exact in $\mathcal{H}(\mathcal{X}, \mathcal{Y})$.*

Proof. By Remark 2.4, the sequence

$$0 \to M^\bullet \to Y'[1] \overset{\varphi[1]}{\to} Y''[1] \to 0$$

is exact in $\mathcal{H}(\mathcal{X},\mathcal{Y})$ if and only if

$$M^\bullet \to Y'[1] \overset{\varphi[1]}{\to} Y''[1] \to M^\bullet[1]$$

is a triangle in $\mathrm{D}^b(\text{Mod-}R)$, i.e. if and only if

$$M^\bullet \cong \mathrm{Mc}(\varphi[1])[-1] = Y' \overset{\varphi}{\to} Y''.$$

□

Corollary 4.15 shows the following fact: assume that there is an abelian category $\mathcal{A}$ with a tilting object $V_{\mathcal{A}}$ generating a torsion pair $(\mathcal{T}_{\mathcal{A}},\mathcal{F}_{\mathcal{A}})$ which is counter equivalent to the faithful torsion pair $(\mathcal{X},\mathcal{Y})$ in Mod-R. Then there is an equivalence of categories $\mathcal{A} \cong \mathcal{H}(\mathcal{X},\mathcal{Y})$ making the diagram

$$\begin{tikzcd}
(\mathcal{T}_{\mathcal{A}},\mathcal{F}_{\mathcal{A}}) \subseteq \mathcal{A} \arrow[rr, leftrightarrow, "\cong"] \arrow[dr, leftrightarrow] & & \mathcal{H}(\mathcal{X},\mathcal{Y}) \supseteq (\mathcal{T},\mathcal{F}) \arrow[dl, leftrightarrow] \\
& (\mathcal{X},\mathcal{Y}) \subseteq \text{Mod-}R &
\end{tikzcd}$$

commutative, where the diagonal functors represent the two torsion pair counterequivalences between $(\mathcal{T}_{\mathcal{A}},\mathcal{F}_{\mathcal{A}})$ - respectively $(\mathcal{T},\mathcal{F})$ - and $(\mathcal{X},\mathcal{Y})$. The Morita equivalence between $\mathcal{A}$ and $\mathcal{H}(\mathcal{X},\mathcal{Y})$ is constructed in the proof of Theorem 4.14 in such a way it extends the natural equivalences

$$H_{\mathcal{A}} : \mathcal{T}_{\mathcal{A}} \cong \mathcal{Y} \cong \mathcal{T} : H^{-1} \quad \text{and} \quad H'_{\mathcal{A}} : \mathcal{F}_{\mathcal{A}} \cong \mathcal{X} \cong \mathcal{F} : H^0$$

obtained by composition of the diagonals. The action of this equivalence on the objects is proved to be as follows:

- For every $A \in \mathcal{A}$, since $\mathcal{T}_{\mathcal{A}}$ cogenerates $\mathcal{A}$, there exists an exact sequence $0 \to A \to T' \to T'' \to 0$ with $T', T'' \in \mathcal{T}_{\mathcal{A}}$. From that we get an exact sequence in Mod-R of the form $0 \to H_{\mathcal{A}}(A) \to H_{\mathcal{A}}(T') \to H_{\mathcal{A}}(T'') \to H'_{\mathcal{A}}(A) \to 0$ with $H_{\mathcal{A}}(A) \in \mathcal{Y}$ and $H'_{\mathcal{A}}(A) \in \mathcal{X}$. The complex $A^\bullet = H_{\mathcal{A}}(T') \to H_{\mathcal{A}}(T'')$ in $\mathcal{H}(\mathcal{X},\mathcal{Y})$ is exactly the object in the heart which corresponds to A under this equivalence.
- By Lemma 4.6, every object $M^\bullet \in \mathcal{H}(\mathcal{X},\mathcal{Y})$ is of the form $M^\bullet = Y^{-1} \overset{f}{\to} Y^0$ with $Y^{-1}, Y^0 \in \mathcal{Y}$. Applying Lemma 4.16, we get that the sequence $0 \to M^\bullet \to Y^{-1}[1] \to Y^0[1] \to 0$ is exact in $\mathcal{H}(\mathcal{X},\mathcal{Y})$ and moreover $Y^{-1}[1]$, $Y^0[1] \in \mathcal{T}$. From that we derive the morphism $T_{\mathcal{A}}(Y^{-1}) \overset{T_{\mathcal{A}}(f)}{\to} T_{\mathcal{A}}(Y^0)$ in $\mathcal{T}_{\mathcal{A}}$, whose kernel is exactly the object of $\mathcal{A}$ which corresponds under the equivalence to $M^\bullet$.

CHAPTER 5

Properties of the heart

In this chapter we shall consider the basic properties of the heart associated to a faithful torsion pair. First, we study the AB-properties of the heart $\mathcal{H}(\mathcal{X},\mathcal{Y})$. Then we look for some conditions which guarantee that $\mathcal{H}(\mathcal{X},\mathcal{Y})$ is a Grothendieck category: we will see that a key point is the behaviour of the functor H^{-1} with respect to direct limits. Finally, we show that $\mathcal{H}(\mathcal{X},\mathcal{Y})$ has an injective cogenerator if and only if $(\mathcal{X},\mathcal{Y})$ is cogenerated by a cotilting R-module. As a corollary, a necessary condition for the $\mathcal{H}(\mathcal{X},\mathcal{Y})$ to be Grothendieck is that $(\mathcal{X},\mathcal{Y})$ is cogenerated by a cotilting R-module.
In the sequel, $(\mathcal{X},\mathcal{Y})$ will denote a faithful torsion pair in Mod-R, $\mathcal{H}=\mathcal{H}(\mathcal{X},\mathcal{Y})$ the associated heart, $(\mathcal{T},\mathcal{F})$ the associated counter equivalent torsion pair in $\mathcal{H}$, and $H^{-1},H^0\colon\mathcal{H}\to$ Mod-R, $T,T'\colon$ Mod-$R\to\mathcal{H}$ the involved functors.

NOTE. Since all the results we deal with are of category-theoretic type, in virtue of Corollary 4.15 they remain true if we replace the heart $\mathcal{H}$ by any abelian category $\mathcal{A}$ admitting a tilting object V which tilts to $(\mathcal{X},\mathcal{Y})$.

1. AB-properties of the heart

PROPOSITION 5.1. *$\mathcal{H}$ is an AB3 category, i.e., arbitrary coproducts exist, and the functors H^{-1} and H^0 commute with coproducts.*

PROOF. By Lemma 4.1, arbitrary coproducts in $\mathcal{H}$ exist, and that they are defined componentwise. Therefore both H^{-1} and H^0 preserve them. $\square$

PROPOSITION 5.2. *$\mathcal{H}$ is an AB4 category, i.e., any coproduct of exact sequences is an exact sequence.*

PROOF. Let $0\to X_\lambda\to Y_\lambda\to Z_\lambda\to 0$, with $\lambda\in\Lambda$, be a family of exact sequences in $\mathcal{H}$. Since the coproduct functor is a left adjoint, then it is right exact, so we get exact sequences

$$0\to K\to\coprod X_\lambda\to\coprod Y_\lambda\to\coprod Z_\lambda\to 0 \tag{1}$$

and

$$0\to K\to\coprod X_\lambda\to C\to 0,\qquad 0\to C\to\coprod Y_\lambda\to\coprod Z_\lambda\to 0. \tag{2}$$

From the first sequence in (2) we get the long exact sequence

$$0\to H^{-1}(K)\to H^{-1}(\coprod X_\lambda)\to H^{-1}(C)\to H^0(K)\to H^0(\coprod X_\lambda)\to H^0(C)\to 0 \tag{3}$$

and from the second sequence in (2), using Proposition 5.1, we get the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & H^{-1}(C) \arrow[r] & H^{-1}(\coprod Y_\lambda) \arrow[r] & H^{-1}(\coprod Z_\lambda) \arrow[r] & \cdots \\
0 \arrow[r] & H^{-1}(\coprod X_\lambda)\cong\bigoplus H^{-1}(X_\lambda) \arrow[r] \arrow[u] & \bigoplus H^{-1}(Y_\lambda) \arrow[r] \arrow[u, "\cong"'] & \bigoplus H^{-1}(Z_\lambda) \arrow[r] \arrow[u, "\cong"'] & \cdots
\end{tikzcd}$$

$$\begin{tikzcd}
\cdots \arrow[r] & H^0(C) \arrow[r] & H^0(\coprod Y_\lambda) \arrow[r] & H^0(\coprod Z_\lambda) \arrow[r] & 0 \\
\cdots \arrow[r] & H^0(\coprod X_\lambda)\cong\bigoplus H^0(X_\lambda) \arrow[r] \arrow[u, two heads] & \bigoplus H^0(Y_\lambda) \arrow[r] \arrow[u, "\cong"'] & \bigoplus H^0(Z_\lambda) \arrow[r] \arrow[u, "\cong"'] & 0
\end{tikzcd}$$

which shows that the maps $H^{-1}(\coprod X_\lambda) \to H^{-1}(C)$ and $H^0(\coprod X_\lambda) \to H^0(C)$ are both isomorphisms. Finally, from the exactness of (3) we see that $H^{-1}(K) = 0 = H^0(K)$, and so we can conclude that $K = 0$. Comparing this with (1) we get the thesis. $\square$

The heart $\mathcal{H}$ has also products, as proved in the following proposition.

Proposition 5.3. *$\mathcal{H}$ is an AB3*-category, i.e., arbitrary products in $\mathcal{H}$ exist.*

Proof. First of all, it is known that products exist in $\mathrm{D}^b(\text{Mod-}R)$, and are computed componentwise: if $[C]$ denotes the equivalence class of a bounded complex, the product of a family $([C_\lambda])$ is just $[\prod_\lambda C_\lambda]$, where the product is computed in the category of complexes of R-modules.
We can see $\mathcal{H} = \mathcal{H}(\mathcal{X}, \mathcal{Y})$ as the subcategory of $\mathrm{D}^b = \mathrm{D}^b(\text{Mod-}R)$ consisting of (classes of equivalence of) complexes

$$\cdots \to 0 \to 0 \to M^{-1} \xrightarrow{d^0} M^0 \to 0 \to 0 \to \cdots$$

where $\operatorname{Ker} d^0 \in \mathcal{Y}$ and $\operatorname{Coker} d^0 \in \mathcal{X}$. Denote by $\mathcal{H}'$ the subcategory of D^b defined in the same fashion, but requiring only that $\operatorname{Ker} d^0 \in \mathcal{Y}$. Then $\mathcal{H}'$ is closed under products in D^b, since $\mathcal{Y}$ is closed under products in Mod-R. We denote an object of $\mathcal{H}'$ simply as $[M^{-1} \to M^0]$ (the equivalence class of a complex concentrated in degrees -1 and 0) or by a single letter.
There is a functor $F\colon \mathcal{H}' \to \mathcal{H}$ defined in the following way: given $[M^{-1} \to M^0] \in \mathcal{H}'$, consider the exact sequence

$$0 \to Y \to M^{-1} \to M^0 \to C \to 0$$

and set $Z = \operatorname{Coker}(Y \to M^{-1}) = \operatorname{Ker}(M^0 \to C)$. If X is the torsion part of C with respect to the torsion pair $(\mathcal{X}, \mathcal{Y})$, we can consider the pull-back:

$$\begin{tikzcd}
0 \arrow[r] & Z \arrow[r] \arrow[d, equal] & \hat{M}^0 \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Z \arrow[r] & M^0 \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

and define $F([M^{-1} \to M^0]) = [M^{-1} \to \hat{M}^0]$, where the map $M^{-1} \to \hat{M}^0$ is just the composition of the obvious maps through Z. By the properties of the pull-back, it is clear that this induces a functor $\mathcal{H}' \to \mathcal{H}$, since it is straightforward to verify that this construction takes null-homotopic maps to null-homotopic maps and quasi-isomorphisms to quasi-isomorphisms. Moreover, if $A \in \mathcal{H}$ and $B \in \mathcal{H}'$, any morphism $A \to B$ factors uniquely through $F(B)$. Therefore F is a right adjoint to the inclusion functor and so it creates products. $\square$

Proposition 5.4. *If H^{-1} commutes with direct limits, then $\mathcal{H}$ is AB5, i.e., direct limits are exact.*

Proof. This is a slight adaptation of the proof of Lemma 5.2, taking into account the fact that both H^{-1} and H^0 commute with direct limits, being left adjoint functors. $\square$

Proposition 5.5. *If $\mathcal{H}$ is AB5 then it is a Grothendieck category.*

Proof. Since $V = R[1]$ generates a tilting torsion class, for any object X there are an embedding $X \hookrightarrow Y$ and an epimorphism $\varphi\colon V^{(\Lambda)} \twoheadrightarrow Y$. For any finite subset F of Λ, let φ_F be the restriction of φ to $V^{(F)}$, and consider the commutative diagram

(F)
$$\begin{tikzcd}
X_F \arrow[r, hook] \arrow[d, "\psi_F"] & V^{(F)} \arrow[d, "\varphi_F"] \\
X \arrow[r, hook] & Y
\end{tikzcd}$$

where the upper left corner is the pullback of the lower right corner. Since direct limits are additive and left exact by assumption, they preserve pullbacks; thus, applying the direct limit to (F), we get a pullback diagram

$$\begin{tikzcd} \varinjlim X_F \arrow[r, hook] \arrow[d, "\varinjlim \psi_F"] & V^{(\Lambda)} \arrow[d, two heads, "\varphi"] \\ X \arrow[r, hook] & Y \end{tikzcd}$$

which shows that $\varinjlim \psi_F$ is an epimorphism. This proves that

$$\{Z \mid Z \leq V^n, n \in \mathbb{N}\}$$

is a family of generators for $\mathcal{H}$. Therefore, $\mathcal{H}$ is a Grothendieck category. □

2. When the heart is a Grothendieck category

Our next goal is to give a necessary and sufficient condition for $\mathcal{H}$ being a Grothendieck category. As we have seen above, a key point is to establish when the functor H^{-1} commutes with direct limits.
We begin by showing that there is always a canonical epimorphism

$$\varinjlim H^{-1}X_\lambda \to H^{-1}(\varinjlim X_\lambda) \to 0.$$

We need the following lemma (see [**CGM07**, Lemma 3.5]):

Lemma 5.6. *Let $\mathcal{A}$ be an AB3 category, $\mathcal{B}$ an AB5 category, and $F\colon \mathcal{A} \to \mathcal{B}$ a right exact additive functor commuting with direct limits. For any direct system $(M_\lambda, f_{\lambda\mu})$ in $\mathcal{A}$, let us consider the canonical exact sequence*

$$0 \to K \to \coprod M_\lambda \to \varinjlim M_\lambda \to 0.$$

Then the associated sequence

$$0 \to F(K) \to F(\coprod M_\lambda) \to F(\varinjlim M_\lambda) \to 0$$

is exact.

Now we can prove:

Corollary 5.7. *For any direct system $(X_\lambda, \xi_{\lambda\mu})$ in $\mathcal{H}$ the canonical map*

$$\varphi\colon \varinjlim H^{-1}X_\lambda \to H^{-1}(\varinjlim X_\lambda)$$

is an epimorphism in Mod-R.

Proof. Given the canonical exact sequence $0 \to K \to \coprod X_\lambda \to \varinjlim X_\lambda \to 0$, we get the long exact sequence

$$0 \to H^{-1}K \to H^{-1}(\coprod X_\lambda) \xrightarrow{\alpha} H^{-1}(\varinjlim X_\lambda) \to H^0K \xrightarrow{\beta} H^0(\coprod X_\lambda) \to H^0 \varinjlim(X_\lambda) \to 0.$$

On the other hand, since H^0 is a left adjoint functor, Lemma 5.6 applies to $F = H^0$, proving that β is monic. Therefore α is epic. Finally, the canonical commutative square

$$\begin{tikzcd} \bigoplus H^{-1}X_\lambda \arrow[r] \arrow[d, "\cong"] & \varinjlim H^{-1}X_\lambda \arrow[d, "\varphi"] & \\ H^{-1}(\coprod X_\lambda) \arrow[r, "\alpha"] & H^{-1}(\varinjlim X_\lambda) \arrow[r] & 0 \end{tikzcd}$$

where the first vertical map is an isomorphism by Proposition 5.1, shows the thesis. □

Theorem 5.8. *The following conditions are equivalent:*

a) *$\mathcal{H}$ is a Grothendieck category.*
b) *for any direct system $(X_\lambda, \xi_{\lambda\mu})$ in $\mathcal{H}$ the canonical map*

$$\varphi\colon \varinjlim H^{-1}X_\lambda \to H^{-1}(\varinjlim X_\lambda)$$

is a monomorphism in Mod-R.
c) *the functor H^{-1} commutes with direct limits.*

If $\mathcal{Y}$ is closed under direct limits, then the previous conditions are equivalent to:

d) *the functor TH^{-1} commutes with direct limits.*

Proof. a)⇒c) follows from Proposition 3.8.
(c)⇒(b) is trivial.
(b)⇒(a) By Corollary 5.7 the functor H^{-1} commutes with direct limits. So Proposition 5.4 applies, proving that $\mathcal{H}$ is AB5. Finally Proposition 5.5 shows that $\mathcal{H}$ is Grothendieck.
(c)⇒(d) follows from the fact that T is a left adjoint functor, and so it commutes with direct limits.
Now, let us assume that $\mathcal{Y}$ is closed under direct limits. Assuming (d), let us prove (c). The composition of the canonical isomorphisms $T(\varinjlim H^{-1}X_\lambda) \cong \varinjlim TH^{-1}(X_\lambda) \cong TH^{-1}(\varinjlim X_\lambda)$ gives a canonical isomorphism between the T-images of the two R-modules $\varinjlim H^{-1}X_\lambda$ and $H^{-1}(\varinjlim X_\lambda)$ which belong to $\mathcal{Y}$. Since T induces an equivalence between $\mathcal{Y}$ and $\mathcal{T}$, we conclude that $\varinjlim H^{-1}X_\lambda \cong H^{-1}(\varinjlim X_\lambda)$ canonically. □

We shall state a result that we will need in the sequel (see [**CGM07**, Proposition 3.8 and Corollary 3.9]):

Proposition 5.9. *A faithful torsion pair $(\mathcal{X}, \mathcal{Y})$ in* Mod-R *is cogenerated by a cotilting module if and only if $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ has an injective cogenerator.*

Let us give an outline of the proof of the previous proposition. Let $(\mathcal{X}, \mathcal{Y})$ be a faithful torsion pair in Mod-R and assume that $\mathcal{Y} = \operatorname{Cogen} U$ for a cotilting module U. Set $Q = (\iota(U/t(U)))[1]$ (if $V = R[1]$ is the tilting object in $\mathcal{H}(\mathcal{X}, \mathcal{Y})$, then $Q = T_V(U)$). First one proves that Q is injective in $\mathcal{H}(\mathcal{X}, \mathcal{Y})$. Then, in order to see that Q is a cogenerator, it is only needed to show that, for any $M \in \mathcal{H}(\mathcal{X}, \mathcal{Y})$, $\operatorname{Hom}_{\mathcal{H}(\mathcal{X},\mathcal{Y})}(M, Q) = 0$ implies $M = 0$, since Q is injective. Conversely, assume that $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ has an injective cogenerator Q. Set $U = H^{-1}(Q)$ (or $U = H_V(Q)$). Then one proves that Q cogenerates the class $\mathcal{Y}$ and then that Q is a cotilting module.

Remark 5.10. Let $(\mathcal{X}, \mathcal{Y})$ be a faithful torsion pair in Mod-R. If $(\mathcal{X}, \mathcal{Y})$ is cotilting, then $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ has enough injectives. Indeed, if $(\mathcal{X}, \mathcal{Y})$ is cotilting and $\mathcal{Y} = \operatorname{Cogen} U$ for a cotilting module U, then $Q = T_V(U) = (U/t(U))[1]$ is an injective cogenerator in $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ by Proposition 5.9. Since $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ has products (by Proposition 5.3) and the product of injectives is injective, we conclude that $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ has enough injectives.

From Proposition 5.9 we get immediately:

Corollary 5.11. *Let $(\mathcal{X}, \mathcal{Y})$ be a faithful torsion pair in* Mod-R. *If $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ is a Grothendieck category, then $(\mathcal{X}, \mathcal{Y})$ is cotilting.*

CHAPTER 6

Hearts VS torsion pairs

The main purpose of this chapter is to show that the heart $\mathcal{H}(\mathcal{X},\mathcal{Y})$ is a Grothendieck category if and only if $(\mathcal{X},\mathcal{Y})$ is cogenerated by a cotilting R-module. From this, we will be able to draw several important consequences, by applying the techniques of tilting counter equivalences between a Grothendieck category and the category Mod-R. For instance, we characterize when the heart is Grothendieck, locally noetherian: this is equivalent to the fact that the class $\mathcal{Y}$ is cogenerated by a Σ-pure injective cotilting module.

1. The heart of a cotilting torsion pair is a Grothendieck category

In this section we prove that the heart $\mathcal{H}(\mathcal{X},\mathcal{Y})$ is a Grothendieck category if and only if $(\mathcal{X},\mathcal{Y})$ is cotilting. We begin with the following lemma.

LEMMA 6.1. *Let $(\mathcal{X},\mathcal{Y})$ be a cotilting torsion pair in* Mod-R. *Then there exists a functor $F\colon \mathcal{H}(\mathcal{X},\mathcal{Y}) \to \mathcal{H}(\mathcal{X},\mathcal{Y})$ such that $F(M) \in \mathcal{T}$, and a monic natural transformation $1_{\mathcal{H}(\mathcal{X},\mathcal{Y})} \to F$.*

PROOF. Set $F(M) = W^{\mathrm{Hom}_{\mathcal{H}(\mathcal{X},\mathcal{Y})}(M,W)}$, where W is an injective cogenerator of $\mathcal{H}(\mathcal{X},\mathcal{Y})$ (i.e., $H_V(W)$ is cotilting and cogenerates $\mathcal{Y}$, by [**CGM07**, Proposition 3.8]). This is a covariant functor (the composition of two contravariant ones) and the natural transformation η is defined, for $M \in \mathcal{H}(\mathcal{X},\mathcal{Y})$, by

$$\pi_\xi \eta_M = \xi$$

for every $\xi \in \mathrm{Hom}_{\mathcal{H}(\mathcal{X},\mathcal{Y})}(M,W)$, where $\pi_\xi\colon W^{\mathrm{Hom}_{\mathcal{H}(\mathcal{X},\mathcal{Y})}(M,W)} \to W$ is the canonical projection. □

Now we can prove our main result.

THEOREM 6.2. *Let $(\mathcal{X},\mathcal{Y})$ be a faithful torsion pair in* Mod-R. *The following conditions are equivalent:*

(a) *$(\mathcal{X},\mathcal{Y})$ is cotilting.*
(b) *$\mathcal{H}(\mathcal{X},\mathcal{Y})$ is a Grothendieck category.*

PROOF. (b)⇒(a). This is Corollary 5.11.
(a)⇒(b). Consider a direct system $(X_\lambda, \xi_{\lambda\mu})$ in $\mathcal{H} = \mathcal{H}(\mathcal{X},\mathcal{Y})$; using the previous lemma, form the direct system $(F(X_\lambda), F(\xi_{\lambda\mu}))$ and the exact sequence of direct systems

$$(*) \qquad 0 \to X_\lambda \to F(X_\lambda) \to Y_\lambda \to 0.$$

By applying the functor H_V to each sequence of the system, we get the exact sequence of direct systems in Mod-R

$$0 \to H_V(X_\lambda) \to H_V(F(X_\lambda)) \to H_V(Y_\lambda) \to H'_V(X_\lambda) \to 0.$$

Recalling that $\mathcal{T}$ is closed under direct limits in $\mathcal{H}$ (by Proposition 4.10), if we take the direct limit of the sequences (*) in $\mathcal{H}$, we can write two exact sequences

$$0 \to K \to \varinjlim X_\lambda \to Z \to 0$$
$$0 \to Z \to \varinjlim F(X_\lambda) \to \varinjlim(Y_\lambda) \to 0$$

and, by applying the functor H_V, we can write a commutative diagram with exact rows in Mod-R

$$\begin{array}{ccccccccc}
0 \to & \varinjlim H_V(X_\lambda) & \to & \varinjlim H_V(F(X_\lambda)) & \to & \varinjlim H_V(Y_\lambda) & \to & \varinjlim H'_V(X_\lambda) & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & H_V(Z) & \longrightarrow & H_V(\varinjlim F(X_\lambda)) & \to & H_V(\varinjlim Y_\lambda) & \longrightarrow & H'_V(Z) & \longrightarrow 0
\end{array}$$

and the exact sequence

$$0 \to H_V(K) \to H_V(\varinjlim X_\lambda) \to H_V(Z).$$

The two central vertical arrows are isomorphisms, since H_V induces an equivalence between $\mathcal{T}$ and $\mathcal{Y}$ and $\mathcal{Y}$ is cotilting, hence closed under direct limits by Proposition 4.11. Therefore also the leftmost vertical arrow is an isomorphism, and since that morphism factors through the canonical morphism

$$\varinjlim H_V(X_\lambda) \to H_V(\varinjlim X_\lambda)$$

this one is monic. Therefore, by Proposition 5.4, $\mathcal{H}$ is AB5; moreover it has an injective cogenerator by Proposition 5.9. By the dual of [**Ste75**, Proposition IV.6.6], $\mathcal{H}$ is locally small and so we can apply Proposition 5.5. $\square$

2. Σ-cotilting modules

In this section we shall characterize when the heart is Grothendieck, locally noetherian. To do this, we shall first recall a property of torsion pairs that was first introduced by Ringel [**RR06**]. Let R be a ring and let $(\mathcal{X}, \mathcal{Y})$ be a faithful torsion pair in Mod-R. The *Ringel's condition* is:

(F) For all $Y \in \mathcal{Y}$ and all finitely generated submodules L of Y,
if $Y/L \in \mathcal{X}$, then Y is finitely generated.

For any class $\mathcal{A}$ of right R-modules, denote by $\mathcal{A}^\perp$ the class consisting of those modules M such that $\operatorname{Ext}^1_R(A, M) = 0$, for all $A \in \mathcal{A}$.
We need the following lemma (see [**CG09**, Lemma 4.1]):

LEMMA 6.3. *Let $(\mathcal{X}, \mathcal{Y})$ be a torsion pair over the ring R and denote by $\mathcal{Y}_0$ the class of finitely generated modules in $\mathcal{Y}$. If $(\mathcal{X}, \mathcal{Y})$ satisfies condition* (F) *and $\mathcal{Y}$ is closed under direct limits, then $\mathcal{Y}_0^\perp = \mathcal{Y}^\perp$.*

For a right R-module M, we denote by $\operatorname{Prod}(M)$ the class consisting of all direct summands of products of copies of the module M.

LEMMA 6.4. *If R is right noetherian and $(\mathcal{X}, \mathcal{Y})$ is a cotilting torsion pair in* Mod-R *and $\mathcal{Y}^\perp = \mathcal{Y}_0^\perp$, then the class $\mathcal{Y} \cap \mathcal{Y}^\perp$ is closed under direct sums. In particular, any coproduct of copies of a cotilting module cogenerating $\mathcal{Y}$ is cotilting.*

PROOF. Let (M_λ) be a family of modules in the class and let M be their direct sum. It is clear that $M \in \mathcal{Y}$. If $L \in \mathcal{Y}_0$, then $\operatorname{Ext}^1_R(L, -)$ commutes with direct sums because R is noetherian, therefore

$$\operatorname{Ext}^1_R(L, M) \cong \bigoplus_\lambda \operatorname{Ext}^1_R(L, M_\lambda) = 0,$$

so that $M \in \mathcal{Y}_0^\perp = \mathcal{Y}^\perp$.
We have to show that if U is a cotilting module with $\operatorname{Cogen}(U) = \mathcal{Y}$ and α is any nonempty index set, then $U^{(\alpha)}$ is a cotilting module. It is obvious that $\operatorname{Cogen}(U) = \operatorname{Cogen}(U^{(\alpha)})$ and that $\operatorname{Ext}^1_R(M, U^{(\alpha)}) = 0$ implies $\operatorname{Ext}^1_R(M, U) = 0$. It suffices to show that $M \in \mathcal{Y}$ implies $\operatorname{Ext}^1_R(M, U^{(\alpha)}) = 0$: this is now obvious, because $U \in \mathcal{Y} \cap \mathcal{Y}^\perp$, so that $U^{(\alpha)} \in \mathcal{Y}^\perp$. $\square$

Lemma 6.4 says that given a cotilting torsion pair $(\mathcal{X}, \mathcal{Y})$ in Mod-R, where R is right noetherian, satisfying $\mathcal{Y}_0^\perp = \mathcal{Y}^\perp$, then every cotilting module cogenerating $\mathcal{Y}$ is Σ-cotilting. Now, by a result of Bazzoni [**Baz03**], every cotilting module is pure-injective. Therefore, given a cotilting torsion pair $(\mathcal{X}, \mathcal{Y})$ satisfying $\mathcal{Y}_0^\perp = \mathcal{Y}^\perp$, every cotilting module cogenerating $\mathcal{Y}$ is Σ-pure-injective, provided we assume that R is right noetherian.

In the next result we shall see that it is sufficient that one cotilting module cogenerating $\mathcal{Y}$ is Σ-pure-injective to ensure that all such modules are Σ-pure-injective, hence cotilting, without assuming that the ring is right noetherian.

LEMMA 6.5. *Let $(\mathcal{X}, \mathcal{Y})$ be a cotilting torsion pair over the ring R such that $\mathcal{Y}$ is cogenerated by a Σ-pure-injective cotilting module. Then every cotilting module U cogenerating $\mathcal{Y}$ is Σ-pure-injective and every direct sum of copies of U is a cotilting module.*

PROOF. If U and U' are cotilting modules cogenerating $\mathcal{Y}$ then $U \in \operatorname{Prod}(U')$. If U' is Σ-pure-injective, then any module in $\operatorname{Prod}(U')$ is Σ-pure-injective by Corollary 8.2 in [**JL89**]. In order to prove that $U^{(\alpha)}$ is cotilting for any cardinal number α, we can of course assume that $\alpha > 0$. In this case the following relations hold:

$$\operatorname{Cogen}(U) = \operatorname{Cogen}(U^{(\alpha)}), \qquad \operatorname{Ker}\operatorname{Ext}^1_R(-, U^{(\alpha)}) \subseteq \operatorname{Ker}\operatorname{Ext}^1_R(-, U).$$

Therefore it is sufficient to show that $U^{(\alpha)} \in \operatorname{Prod}(U)$, which is true because U is Σ-pure-injective and so the canonical pure embedding $U^{(\alpha)} \to U^{\alpha}$ splits. □

DEFINITION 6.6. We say that a torsion pair $(\mathcal{X}, \mathcal{Y})$ is Σ-*cotilting* if $\mathcal{Y}$ is cogenerated by a Σ-pure-injective cotilting module.

When the torsion pair $(\mathcal{X}, \mathcal{Y})$ is Σ-cotilting, the objects of $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ have some nice properties.

LEMMA 6.7. *Let $(\mathcal{X}, \mathcal{Y})$ be a Σ-cotilting torsion pair in* Mod-R. *Every coproduct of injective objects in $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ is injective.*

PROOF. Let $(E_\lambda)_{\lambda\in\Lambda}$ be a family of injectives in $\mathcal{H} = \mathcal{H}(\mathcal{X}, \mathcal{Y})$. Then there is a cotilting module U such that $\mathcal{Y} = \operatorname{Cogen}(U)$ and E_λ is a direct summand of $T(U)$, for all λ. Indeed, $H(E_\lambda)$ can be embedded into a suitable direct power of U_0, for some cotilting $U_0 \in \mathcal{Y}$, with cokernel in $\mathcal{Y}$. By taking a sufficiently large power, we can assume that there exist exact sequences $0 \to H(E_\lambda) \to U \to C_\lambda \to 0$, with U cotilting cogenerating $\mathcal{Y}$ and $C_\lambda \in \mathcal{Y}$. Since $E_\lambda \in \mathcal{T}$, we have the claim.
Therefore, $E = \coprod_{\lambda\in\Lambda} E_\lambda$ is a summand of $(T(U))^{(\Lambda)} \cong T(U^{(\Lambda)})$. But $U^{(\Lambda)}$ is cotilting, thanks to our assumptions, Lemma 6.4 and Lemma 6.5, hence $T(U^{(\Lambda)})$ is injective in $\mathcal{H}$. □

PROPOSITION 6.8. *Let $(\mathcal{X}, \mathcal{Y})$ be a Σ-cotilting torsion pair. Then every small object in $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ is noetherian.*

PROOF. (E. Gregorio) Assume $M \in \mathcal{H}(\mathcal{X}, \mathcal{Y})$ is not noetherian and let $0 = X_0 < X_1 < \dots$ be a strictly increasing sequence of subobjects of M and for each $n > 0$ take a non zero morphism $f_n \colon X_n \to W$ which is zero on X_{n-1} $(n > 0)$ and extend it to a morphism $g_n \colon L \to W$, where W is an injective cogenerator of $\mathcal{H}$ and L is the direct limit of the subobjects X_n.
Denote by $i_n \colon X_n \to X_{n+1}$ and $j_n \colon X_n \to L$ the canonical inclusions, and by $k_n \colon W^n \to W^{n+1}$ the morphism such that $\pi_i k_n = \pi_i$ for $i = 1, 2, \dots, n$ and $\pi_{n+1} k_n = 0$.
Now define $h_n \colon X_n \to W^n$ to be the morphism such that

$$\pi_i h_n = g_i j_n \qquad (i = 1, 2, \dots, n).$$

We want to show that, for all $n > 0$, $k_n h_n = h_{n+1} i_n$. Indeed,

$$\pi_i k_n h_n = \begin{cases} \pi_i h_n = g_i j_n & 1 \le i \le n \\ 0 & i = n+1 \end{cases}$$

while $\pi_i h_{n+1} i_n = g_i j_{n+1} i_n = g_i j_n$ $(i = 1, \dots, n, n+1)$. But, for $i = n+1$, we have

$$\pi_{n+1} h_{n+1} i_n = g_{n+1} j_{n+1} i_n = f_{n+1} i_n = 0.$$

Therefore the morphisms $(h_n \colon X_n \to W^n)_{n\in\mathbb{N}}$ form a direct system which induces a morphism $h \colon L \to W^{(\mathbb{N})}$. Since $W^{(\mathbb{N})}$ is injective by Lemma 6.7, this extends to a morphism $\tilde{h} \colon M \to W^{(\mathbb{N})}$ and this shows that M is not small, because all the compositions of $\tilde{h}$ with the projections $W^{(\mathbb{N})} \to W$ are nonzero by construction. □

THEOREM 6.9. *If the torsion pair* $(\mathcal{X}, \mathcal{Y})$ *in* Mod-R *is* Σ*-cotilting, then* $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ *is locally noetherian.*

PROOF. We know from [**CGM07**, Lemma 3.4] that the subobjects of objects of the form V^m, where $V = T(R)$, are a set of generators of $\mathcal{H}(\mathcal{X}, \mathcal{Y})$. Since V is small, it is noetherian and so the subobjects of V^m ($m \in \mathbb{N}$) are noetherian. $\square$

PROPOSITION 6.10. *If the torsion pair* $(\mathcal{X}, \mathcal{Y})$ *in* Mod-R *is* Σ*-cotilting, then every finitely generated module in* $\mathcal{Y}$ *is finitely presented. In particular,* R *is right coherent.*

PROOF. Let $M \in \mathcal{Y}$ be finitely generated and consider an exact sequence $0 \to K \to R^n \to M \to 0$. Then $0 \to T(K) \to T(R^n) \to T(M) \to 0$ is exact; since $T(R^n)$ is noetherian, also $T(K)$ is noetherian and [**Col99**, Lemma 6.1] implies that $K \cong HT(K)$ is finitely generated. Since every finitely generated right ideal of R is in $\mathcal{Y}$, it follows that R is right coherent. $\square$

Next we want to show that any Σ-cotilting torsion pair in Mod-R satisfies Ringel's condition (F). If $(\mathcal{X}, \mathcal{Y})$ is a torsion pair in Mod-R, we define the **Ringel class** $\mathcal{R} = \mathcal{R}(\mathcal{X}, \mathcal{Y})$ as

$$\mathcal{R} = \{Y \in \mathcal{Y} \mid \exists 0 \to L \to Y \to X \to 0 \text{ exact, with } L \text{ finitely generated, } X \in \mathcal{X}\}.$$

We say that such a sequence **witnesses that** $Y \in \mathcal{R}$. Ringel's condition (F) can thus be stated as:

(F) every module in $\mathcal{R}(\mathcal{X}, \mathcal{Y})$ is finitely generated.

LEMMA 6.11. *Let* $(\mathcal{X}, \mathcal{Y})$ *be a faithful torsion pair in* Mod-R *and let* $Y \in \mathcal{Y}$*. Then* $Y \in \mathcal{R}(\mathcal{X}, \mathcal{Y})$ *if and only if* $T(Y)$ *is a quotient of a finite power of* V*, where* $V = T(R) \in \mathcal{H}(\mathcal{X}, \mathcal{Y})$*.*

PROOF. Suppose $Y \in \mathcal{Y}$ and that $0 \to L \to Y \to X \to 0$ witnesses that $Y \in \mathcal{R}$. We apply T and we get the exact sequence $0 \to T'(X) \to T(L) \to T(Y) \to 0$. Since there exists an epimorphism $R^n \to L$, there is also an epimorphism $V^n \cong T(R^n) \to T(L)$ and, by composition, an epic $V^n \to T(Y)$.
Conversely suppose there exists an exact sequence $0 \to K \to V^n \to M \to 0$ in $\mathcal{H}(\mathcal{X}, \mathcal{Y})$; then we get the exact sequences

$$0 \to H(K) \to H(V^n) \to Z \to 0, \qquad 0 \to Z \to H(M) \to H'(K) \to 0.$$

The first one says that Z is finitely generated, since $H(V^n) \cong R^n$; therefore the second one witnesses that $H(M) \in \mathcal{R}$. $\square$

The next corollary characterizes when the heart is Grothendieck, locally noetherian.

COROLLARY 6.12. *Let* $(\mathcal{X}, \mathcal{Y})$ *be a cotilting torsion pair in* Mod-R*. The following conditions are equivalent:*

(a) $(\mathcal{X}, \mathcal{Y})$ *is* Σ*-cotilting.*
(b) $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ *is a locally noetherian Grothendieck category.*
(c) $(\mathcal{X}, \mathcal{Y})$ *satisfies Ringel's condition* (F)*.*

PROOF. It remains to prove only (b)$\Rightarrow$(c). Assuming (b), we need to prove that every module $Y \in \mathcal{R}(\mathcal{X}, \mathcal{Y})$ is finitely generated. By Lemma 6.11, there is an epic $V^n \to T(Y)$. Since V is noetherian, also $T(Y)$ is noetherian. By [**Col99**, Lemma 6.1], we have that $Y \cong HT(Y)$ is finitely generated. $\square$

EXAMPLE 6.13. Let $(\mathcal{T}, \mathcal{F})$ and $(\mathcal{D}, \mathcal{R})$ be as in Example 2.2. Then $(\mathcal{D}, \mathcal{R})$ is not cotilting. Indeed, the Prüfer group $\mathbb{Z}_{p^\infty} = \varinjlim \mathbb{Z}/p^n\mathbb{Z}$ is divisible, but each $\mathbb{Z}/p^n\mathbb{Z}$ is reduced. Therefore, the class $\mathcal{R}$ is not closed under direct limits, hence, by Proposition 4.11, $(\mathcal{D}, \mathcal{R})$ is not cotilting. Now consider the torsion pair $(\mathcal{T}, \mathcal{F})$. By [**Rob96**, § 4.1], $\mathcal{F} = \mathrm{Cogen}(\mathbb{Q} \oplus \hat{\mathbb{Z}})$, and by [**GT00**, Corollary 2.2], $\mathbb{Q} \oplus \hat{\mathbb{Z}}$ is a cotilting abelian group. Thus the heart $\mathcal{H}(\mathcal{T}, \mathcal{F})$ is a Grothendieck category. Let us prove that $\mathcal{H}(\mathcal{T}, \mathcal{F})$ is not locally noetherian by showing that the Ringel's condition (F) fails for $(\mathcal{T}, \mathcal{F})$. Indeed, $\mathbb{Q}$ is an infinitely generated torsion-free abelian group, while $\mathbb{Z}$ is a finitely generated subgroup of $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ is a torsion group. One may ask if $\mathcal{H}(\mathcal{T}, \mathcal{F})$ is locally noetherian. In fact, in the next example we shall see that, apart from the trivial case, this never occurs in the category Mod-$\mathbb{Z}$.

Example 6.14. There are no abelian groups which are Σ-pure injective, apart from those containing $\mathbb{Q}\oplus\mathbb{Q}/\mathbb{Z}$ as a direct summand (notice that $\mathrm{Cogen}(\mathbb{Q}\oplus\mathbb{Q}/\mathbb{Z}) = \text{Mod-}\mathbb{Z}$, since $\mathbb{Q}\oplus\mathbb{Q}/\mathbb{Z}$ is injective). Indeed, by [**JL89**, Example 8.11], the indecomposable Σ-pure injective abelian groups are $\mathbb{Z}/p^n\mathbb{Z}$, $\mathbb{Z}_{p^\infty}$, $\mathbb{Q}$, and any Σ-pure injective abelian group is a direct sum of these. On the other hand, in virtue of [**GT00**, Corollary 2.2], a cotilting abelian group that does not cogenerate all the category Mod-$\mathbb{Z}$ necessarily contains a copy of $\mathbb{Z}_p$, for some prime p, as a direct summand. The results follows.
As a consequence, given any faithful, cotilting, non-trivial torsion pair $(\mathcal{X}, \mathcal{Y})$ in Mod-$\mathbb{Z}$, the heart $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ is a Grothendieck category, but it is not locally noetherian.

APPENDIX A

Grothendieck groups

We shall give an application of tilting theory to Grothendieck groups.
Given any Grothendieck category $\mathcal{G}$, we denote by $K_0(\mathcal{G})$ the Grothendieck group of finitely generated objects of $\mathcal{G}$. $K_0(\mathcal{G})$ is the free abelian group generated by the isomorphism classes $\overline{M}$ of all the finitely generated objects M in $\mathcal{G}$, modulo all the relations of the form $\overline{M}-\overline{L}-\overline{N}$, where $0 \to L \to M \to N \to 0$ is any exact sequence in $\mathcal{G}$.
We recall that an object $M \in \mathcal{G}$ is **noetherian** if the lattice $\mathcal{L}(M)$ of subobjects of M satisfies the ascending chain condition or, equivalently, every nonempty subset of $\mathcal{L}(M)$ has a maximal element. The class of all the noetherian objects of $\mathcal{G}$ is closed under subobjects, factor objects and extensions. The category $\mathcal{G}$ is called **locally noetherian** if it has a family of noetherian generators. In that case, any object of $\mathcal{G}$ is a sum of noetherian subobjects, and every finitely generated object is noetherian. In particular, any locally noetherian category is locally finitely generated.

We need the following lemmas, whose proofs can be found in [**Col99**, § 6].

LEMMA A.1. *Let $V \in \mathcal{G}$ be a tilting object and let $S = \mathrm{End}_{\mathcal{G}}(V)$.*

a) *If $\mathcal{G}$ is locally noetherian, then the functors H_V and H'_V carry finitely generated objects of $\mathcal{G}$ to finitely generated right S-modules.*
b) *If $\mathcal{G}$ is locally finitely generated and S is right noetherian, then the functors T_V and T'_V carry finitely generated right S-modules to finitely generated objects of $\mathcal{G}$.*

LEMMA A.2. *Let $\mathcal{G}$ be a locally finitely generated Grothendieck category, V a tilting object of $\mathcal{G}$ and assume that $S = \mathrm{End}_{\mathcal{G}}(V)$ is right noetherian. Then the position*

$$[N] \mapsto [T'_V(N)] - [T_V(N)]$$

defines an homomorphism $\Psi\ K_0(S) \to K_0(\mathcal{G})$.

Now we can prove:

PROPOSITION A.3. *If $\mathcal{G}$ is a locally noetherian Grothendieck category and V is a tilting object of $\mathcal{G}$ with $S = \mathrm{End}_{\mathcal{G}}(V)$ left noetherian, then*

$$\Phi\,[M] \mapsto [H'_V(M)] - [H_V(M)] \qquad \textit{and} \qquad \Psi\,[N] \mapsto [T'_V(N)] - [T_V(N)]$$

are inverse abelian group isomorphisms between $K_0(\mathcal{G})$ and $K_0(S)$.

PROOF. Ψ is a well defined group homomorphism, because of Lemma A.2. On the other hand, let $0 \to M_1 \to M_2 \to M_3 \to 0$ be an exact sequence in $\mathcal{G}$, with all the M_i finitely generated. Then we get the exact sequence in Mod-S

$$0 \to H_V(M_1) \to H_V(M_2) \to H_V(M_3) \to H'_V(M_1) \to H'_V(M_2) \to H'_V(M_3) \to 0$$

where all the $H_V(M_i)$ and $H'_V(M_i)$ are finitely generated, because of Lemma A.2. This shows that Φ is a well defined group homomorphism. We finish the proof observing that, by Theorem 3.6, it follows that

$$\begin{aligned}\Psi \circ \Phi([M]) &= [T'_V H'_V(M)] - [T'_V H_V(M)] - [T_V H'_V(M)] + [T_V H_V(M)] \\ &= [T'_V H'_V(M)] + [T_V H_V(M)] = [M]\end{aligned}$$

and

$$\begin{aligned}\Phi \circ \Psi([N]) &= [H'_V T'_V(N)] - [H'_V T_V(N)] - [H_V T'_V(N)] + [H_V T_V(N)] \\ &= [H'_V T'_V(N)] + [H_V T_V(N)] = [N].\end{aligned}$$

□

REMARK A.4. As proved by Valenta in [**Val94**, pp. 6052-3], the assuptions $\mathcal{G}$ locally noetherian and $V \in \mathcal{G}$ tilting do not assure in general that $S = \mathrm{End}_{\mathcal{G}}(V)$ is right noetherian, also in the case $\mathcal{G}$ is a full category of modules. Moreover, Trlifaj in [**Trl97**, Example 2.8] has shown that if S is not noetherian, then the corresponding Grothendieck groups $K_0(\mathcal{G})$ and $K_0(S)$ can be really different.

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