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UNIVERSITÀ DEGLI STUDI DI PADOVA

FACOLTÀ DI SCIENZE MM. FF. NN.
CORSO DI LAUREA IN MATEMATICA

ELABORATO FINALE

ON THE UBIQUITY OF SIMPLICIAL OBJECTS

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DIPARTIMENTO DI MATEMATICA PURA E APPLICATA

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ANNO ACCADEMICO 2012/2013

Contents

Introduction

In this thesis we introduce simplicial objects and then treat four different areas in which they are used: Homotopy theory, homological algebra, higher category theory and cohomological descent.

The first chapter is a short introduction to the basic definitions of simplicial objects in arbitrary categories.

In the second chapter we specialise to simplicial sets. We will develop some simplicial homotopy theory and then establish an equivalence between simplicial homotopy theory and the classical homotopy theory of topological spaces. Most of our results in this section come from May's *Simplicial Objects in Algebraic Topology*, [10]. We end with a brief discussion on the Kan Loop Group introduced in by Kan in [8].

Chapter three looks at simplicial objects in the setting of homological algebra. We focus on simplicial objects in abelian categories. The main result of this section is the Dold-Kan Theorem (Theorem 3.1.1) which gives a correspondence between simplicial objects and (non-negative) chain complexes. We will see the construction of Eilenberg-MacLane spaces as an application of this theorem.

In chapter four we introduce ∞-categories and look at how certain types of simplicial sets are a good model for ∞-categories. The theory of ∞-categories can be seen as a generalisation of ordinary category theory in many ways, however it is also closely related to homotopy theory. Our primary reference for this section is Lurie's book *Higher Topos Theory*, [9].

Finally, in chapter five, we are interested in cohomological descent. Cohomological descent was studied by Saint-Donat in [13]. Our main reference is Conrad's expository article, [3], which follows the work of Saint-Donat. The theory of cohomological descent gives a method of computing the cohomology of a space in terms of the cohomology of a diagram of spaces (in fact, a simplicial space) lying above it. In particular we will use simplicial methods to "build up" simplicial objects by means of a coskeleton functor. Using this coskeleton functor we construct hypercovers of a topological space S. These hypercovers will be the simplicial spaces we use to compute the cohomology on our original space S.

Acknowledgment: First and foremost I would like to thank Prof B. Chiarellotto for suggesting the topic of this thesis as well as for all his advice, his guidance, his time and his patience. I am also grateful to Michalis Neururer for the support and encouragement he provided throughout the year.

1

Simplicial Objects

We call Δ the category whose objects $[n]$ are sequences of integers $[n] = \{0, 1, \ldots, n\}$, $n \geq 0$. The morphisms of Δ are non-decreasing maps

$$\alpha : [n] \to [m],\ \alpha(i) \leq \alpha(j) \text{ if } i < j.$$

We define face maps δ_i, and degeneracy maps σ_i, in Δ as follows:

$$\delta_i : [n] \to [n+1], \qquad \delta_i(j) = \begin{cases} j & \text{if } j < i \\ j+1 & \text{if } j \geq i \end{cases}$$

$$\sigma_i : [n] \to [n-1], \qquad \sigma_i(j) = \begin{cases} j & \text{if } j \leq i \\ j-1 & \text{if } j > i. \end{cases}$$

Lemma 1.0.1. *Every morphism $\alpha : [n] \to [m]$ in Δ can be written uniquely in the following form:*

$$\alpha = \delta_{i_1} \ldots \delta_{i_s} \sigma_{j_1} \ldots \sigma_{j_t}$$

with $0 \leq i_s < \cdots < i_1 \leq m$ and $0 \leq j_1 < \cdots < j_t < n$. It follows that α has a unique epi-monic factorisation, $\alpha = \delta\sigma$, where $\delta = \delta_{i_1} \ldots \delta_{i_s}$ and $\sigma = \sigma_{j_1} \ldots \sigma_{j_t}$

Proof. Let $i_s < \cdots < i_1$ be the elements of $[m]$ not in $\alpha([n])$. Let $j_1 < \cdots < j_t$ be the elements of $[n]$ such that $\alpha(j_i) = \alpha(j_i + 1)$. Then $\alpha = \delta_{i_1} \ldots \delta_{i_s} \sigma_{j_1} \ldots \sigma_{j_t}$. In particular α factorises as

$$[n] \overset{\sigma}{\twoheadrightarrow} [p] \overset{\delta}{\hookrightarrow} [m]$$

where $p = n - t = m - s$. This factorisation is clearly unique. Given a morphism $\alpha : [n] \to [m]$ with $\alpha([n]) \simeq [l]$ then the monomorphism is necessarily $[l] \hookrightarrow [m]$ with the ith term of $[l]$ mapping to the ith term of the image $\alpha([n]) \subseteq [m]$. If I_i is the subset of $[n]$ mapping to the ith term of the image $\alpha([n])$ then the epimorphism is necessarily $[n] \twoheadrightarrow [l]$ with I_i mapping to $i \in [l]$. $\square$

Definition 1.0.1. A *simplicial object* in a category $\mathcal{C}$ is a contravariant functor $K : \Delta \to \mathcal{C}$.

We also have the following equivalent definition of simplical objects.

Definition 1.0.2. A *simplicial object* in $\mathcal{C}$ is given by a sequence of objects in $\mathcal{C}$, $\{K_n\}_{n\in\mathbb{N}}$, along with face maps

$$d_i : K_n \to K_{n-1},$$

and degeneracy maps

$$s_i : K_n \to K_{n+1},$$

satisfying the following *simplicial identities*:

$$\begin{aligned} d_i d_j &= d_{j-1} d_i, && \text{if } i < j \\ s_i s_j &= s_{j+1} s_i, && \text{if } i \leq j \\ d_i s_j &= s_{j-1} d_i, && \text{if } i < j \\ d_i s_j &= \text{id} = d_{i+1} s_i, && \\ d_i s_j &= s_j d_{i-1}, && \text{if } i > j+1. \end{aligned}$$

This definition provides a more combinatorial description of simplicial objects.

To see the equivalence of the two definitons, consider first a simplicial object K as defined in Definition 1.0.1. We write K_n for $K([n])$ and set $d_i = K(\delta_i)$, $s_i = K(\sigma_i)$. We have a sequence of objects $\{K_n\}$ along with face and degeneracy operators d_i and s_i. These operators satisfy the simplicial identities.
Consider $d_i d_j$, for $i < j$,

$$d_i d_j = K(\delta_i)K(\delta_j) = K(\delta_j \delta_i) = K(\delta_i \delta_{j-1}) = K(\delta_{j-1})K(\delta_i) = d_{j-1} d_i.$$

The rest of the identities follow in a similar manner.
Conversely, given a sequence of objects $\{K_n\}$ in $\mathcal{C}$ and maps d_i, s_i we may define a functor

$$\begin{aligned} K : \Delta &\longrightarrow \mathcal{C} \\ [n] &\longmapsto K_n \\ \alpha = \delta_{i_1} \dots \sigma_{j_t} &\longmapsto s_{j_t} \dots d_{i_1}. \end{aligned}$$

This gives a well-defined contravariant functor.

Let $K_\bullet$ be a simplicial object in a category $\mathcal{C}$. For each n, we call the elements $x \in K_n$ n-*simplices*. An n-simplex $x \in K_n$ is *degenerate* if there exists an $(n-1)$-simplex $y \in K_{n-1}$ with $x = s_i y$ for some i. We say a simplicial object $L_\bullet$ is a *sub-simplicial object* of $K_\bullet$ if L_n is a sub-object of K_n for each n. If K_0 consists of a single element, we say that $K_\bullet$ is *reduced*.

Replacing the contravariant functor of Definition 1.0.1 with a covariant functor, we obtain the definition of a *cosimplicial object*. Analogous to the above, we have the equivalent defintion of a cosimplicial object in a category $\mathcal{C}$ as a sequence of objects, $K^n \in \mathcal{C}$, and maps $d^i : K^n \to K^{n+1}$, $s^i : K^n \to K^{n-1}$, which satisfy the *cosimplicial identities*:

$$
\begin{aligned}
d^j d^i &= d^i d^{j-1}, && \text{if } i < j \\
s^j s^i &= s^i s^{j+1}, && \text{if } i \leq j \\
s^j d^i &= d^i s^{j-1}, && \text{if } i < j \\
s^i d^i &= \mathrm{id} = s^i d^{i+1}, && \\
s^j d^i &= d^{i-1} s^j, && \text{if } i > j+1.
\end{aligned}
$$

Definition 1.0.3. A *simplicial map* $f : K \to L$ between two simplicial objects K, L, in a category $\mathcal{C}$ consists of $\{f_n\}$ where $f_n : K_n \to L_n$ is such that

$$
\begin{cases} f_n d_i = d_i f_{n+1} \\ f_n s_i = s_i f_{n-1}. \end{cases}
$$

It follows immediately from the equivalent definitions of a simplicial object that a simplicial map is a natural transformation between two contravariant functors. In the same way, a *cosimplicial map* between cosimplicial objects is a natural transformation between two covariant functors.

Definition 1.0.4. The cartesian product $K \times L$ of two simplicial objects K and L is defined as

$$
(K \times L)_n = K_n \times L_n
$$

with face and degeneracy operators given by

$$
\begin{aligned}
d_i(x, y) &= (d_i x, d_i y), \\
s_i(x, y) &= (s_i x, s_i y).
\end{aligned}
$$

Example 1 (Constant simplicial object). Let C be an object in a category $\mathcal{C}$. Then the simplest example of a simplicial object in $\mathcal{C}$ is the *constant simplicial object* with value $C \in \mathcal{C}$. In other words we have $K_n = C$ for all $n \geq 0$ and the face and degeneracy maps are simply the identity on C.

2

Simplicial Sets and Topological spaces

In this chapter we focus on simplicial objects in the category of sets, Set. We define Kan complexes and, using these Kan complexes, we develop some simplicial homotopy theory. In Section 2.3 we introduce the geometric realisation functor and we will see, in Theorem 2.3.2, that simplicial sets correspond to topological spaces, up to homotopy. Finally we take a look to the Kan loop group in Section 2.4.

2.1 Simplicial Sets

Definition 2.1.1. A *simplicial set* is a simplicial object in the category Set.

A simplicial set can thus be viewed as a contravariant functor from Δ into Set, or equivalently as a graded set with face and degeneracy maps satisfying the simplicial identities.
We form the category of simplicial sets, denoted Set_Δ. The morphisms between two simplicial sets are the simplicial maps defined in the previous chapter (Definition 1.0.3).

Example 2 (Construction of a Simplicial Set from a Topological space). Let X be a topological space and let Δ^n be the standard n-simplex, $\Delta^n = \{(t_0, \dots, t_n) \mid \sum t_i = 1, t_i \geq 0\}$. A singular n-simplex of X is a continuous map $\gamma : \Delta^n \to X$. Let $\mathsf{Sing}_n(X)$ to be the set of all singular n-simplices of X and define face and degeneracy maps as follows:

$$\begin{cases} (d_i\gamma)(t_0, \dots, t_{n-1}) = \gamma(t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{n-1}) \\ (s_i\gamma)(t_0, \dots, t_{n+1}) = \gamma(t_0, \dots, t_i + t_{i+1}, t_{i+2}, \dots, t_{n+1}) \end{cases}$$

It is straightforward to check that d_i and s_i satisfy the simplicial identities: Let us show the first identity:

$$\begin{aligned} (d_i d_j \gamma)(t_0, \dots, t_{n-2}) &= (d_j\gamma)(t_0, \dots, t_{i-1}, 0, t_i \dots, t_j, \dots, t_{n-2}) \\ &= \gamma(t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{j-2}, 0, t_{j-1}, \dots, t_{n-2}) \\ &= (d_{j-1} d_i \gamma)(t_0, \dots, t_{n-2}) \end{aligned}$$

The rest of the identities follow in a similar manner and $\mathsf{Sing}(X)$ is a simplicial set. In fact, Sing defines a functor from the category of topological spaces to the category of simplicial sets.

$$\begin{aligned} \mathsf{Sing} : \mathsf{Top} &\to \mathsf{Set}_\Delta \\ X &\mapsto \mathsf{Sing}(X) \end{aligned}$$

A continuous map $f : X \to Y$ in Top is sent, via this functor, to the simplicial map $f^* = \{f_n^*\}$,

$$\begin{aligned} f_n^* : \mathsf{Sing}(X)_n &\longrightarrow \mathsf{Sing}(Y)_n \\ \left(\Delta^n \xrightarrow{\gamma} X\right) &\longmapsto \left(\Delta^n \xrightarrow{f\circ\gamma} Y\right). \end{aligned}$$

We now introduce the notion of Kan complexes. We employ the terminology of May in [10]. These Kan complexes will be a key concept in much of what follows.

Definition 2.1.2. Let Λ_k^n be the standard n-simplex Δ^n with the interior and the k^{th} face removed. We call it the k^{th} horn of Δ^n. A simplicial set is a *Kan complex* if, for any $0 \leq k \leq n$, any simplicial morphism $\Lambda_k^n \to K$ can be extended to a morphism $\Delta^n \to K$. We capture this definition in the following diagram:

$$\begin{array}{ccc} \Lambda_k^n & \longrightarrow & K \\ \big\downarrow & \overset{\exists}{\nearrow} & \\ \Delta^n & & \end{array}$$

A more combinatorial definition is the following:

Definition 2.1.3. We say that a collection of n $(n-1)$-simplices

$$x_0, x_1, \ldots, x_{k-1}, -, x_{k+1}, \ldots, x_n$$

satisfy the *compatibility condition* if $d_i x_j = d_{j-1} x_i$, for all $i < j$, $i \neq k$, $j \neq k$. A simplicial set is a *Kan complex* if for every collection of n $(n-1)$-simplices

$$x_0, x_1, \ldots, x_{k-1}, -, x_{k+1}, \ldots, x_n$$

satisfying this compatibility condition, there exists an n-simplex y such that $d_i y = x_i \ \forall\ i \neq k$.

The equivalence between these two formulations of a Kan complex is evident from the following identification: Each vertex of Δ^n corresponds to a 0-simplex of K, each edge to a 1-simplex, and so on. In particular the $n+1$ faces of Δ^n are $(n-1)$-simplices of K. Λ_k^n corresponds to n faces of Δ^n and so $\Lambda_k^n \to K$ corresponds to the n $(n-1)$-simplices of K,

$$x_0, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{n+1}.$$

The faces are "glued" together along their edges. This gluing data corresponds to the requirement that the simplices satisfy the compatibility condition. Once we have made this identification it is easy to see that the existence of an n-simplex, y such that $d_i y = x_i$ for all $i \neq k$ amounts to the existence of a map $\Delta^n \to K$ such that its restriction to Λ_k^n is the given map $\Lambda_k^n \to K$.

Definition 2.1.4. A map of simplicial sets $f : K \to L$ is a *Kan fibration* if for every collection of n $(n-1)$-simplices of K,

$$x_0, \cdots x_{k-1}, -, x_{k+1}, \ldots, x_n$$

which satisfy the compatibility condition of Definition 2.1.3, and for every n-simplex $y \in L_n$ such that $d_i y = f(x_i), i \neq k$, there exists an n-simplex $x \in K_n$ such that $d_i x = x_i$, for all $i \neq k$ and $f(x) = y$.

If we take L to be the simplicial set generated by a single element l_0, then a map of simplicial sets $f : K \to L$ is a Kan fibration if and only if K is a Kan complex. For this reason Kan complexes are sometimes referred to as *fibrant simplicial sets*, for example in [5] and [14].

Example 3. ($\mathsf{Sing}(X)$ is a Kan complex) Let X be a topological space and consider $\mathsf{Sing}(X)$ the simplicial set obtained from X. A simplicial morphism

$$\Lambda^n_k \to \mathsf{Sing}(X)$$

corresponds to a collection of $n-1$ simplices $\gamma_i : d_i\Delta^n \to X$ of $\mathsf{Sing}(X)$, $i \neq k$, coming from the faces $d_i\Delta^n \simeq \Delta^{n-1}$ of Λ^n_k. These simplices are compatible in the sense that they "glue together" to give a map $f : \Lambda^n_k \to X$. Since Λ^n_k is a retract of Δ^n we can, by composing with the retraction, extend f to a map $\Delta^n \to X$. This is an n-simplex $\gamma \in \mathsf{Sing}(X)$ such that $d_i\gamma = \gamma_i$ for $i \neq k$. Thus we can extend our simplicial morphism to $\Delta^n \to \mathsf{Sing}(X)$ as desired and $\mathsf{Sing}(X)$ is a Kan complex.

By the following lemma the class of Kan complexes includes all simplicial groups and simplicial abelian groups.

Lemma 2.1.1. *If G is a simplicial group, then the underlying simplicial set is a Kan complex.*

Proof. Suppose we have

$$x_0, \ldots, x_{k-1}, -, x_{k+1}, \ldots, x_{n+1}$$

elements of G_n such that $d_i x_j = d_{j-1} x_i$ for all $i < j$, $i, j \neq k$. We wish to find a $g \in G_{n+1}$ such that $d_i g = x_i$ for all $i \neq k$. We proceed by induction on r to find $g_r \in G_{n+1}$ such that $d_i g_r = x_i$ for all $i \leq r$, $i \neq k$. We then get our desired g by taking $g = g_n$.
Set $g_{-1} = 1 \in G_{n+1}$. Suppose we have g_{r-1} such that $d_i g_{r-1} = x_i$ for all $i < r-1$, $i \neq k$. Now we look for g_r. If $k = r$ we take $g_r = g_{r-1}$. If $r \neq k$ consider the element $y = x_r^{-1}(d_r g_{r-1}) \in G_n$. Then recalling that $d_i g_{r-1} = x_i$ for $i < r$ we have

$$\begin{aligned} d_i(y) &= d_i(x_r^{-1}) d_i d_r g_{r-1} \\ &= (d_i x_r)^{-1} d_{r-1} d_i g_{r-1} \\ &= (d_i x_r)^{-1} d_{r-1} x_i \\ &= (d_i x_r)^{-1} (d_i x_r) = 1 \end{aligned}$$

for all $i < r$, $i \neq k$ and it follows that $d_i(s_r y) = s_{r-1} d_i y = 1$ for all $i < r$, $i \neq k$. Hence we take $g_r = g_{r-1}(s_r y)^{-1}$. It is clear from the previous calculations along with the definition of y that we have

$$d_i g_r = (d_i g_{r-1}) d_i (s_r y)^{-1} = d_i g_{r-1} = x_i \text{ for all } i < r$$

and

$$d_r g_r = (d_r g_{r-1})(d_r s_r y)^{-1} = (d_r g_{r-1})(d_r g_{r-1})^{-1} x_r = x_r.$$

□

2.2 Simplicial Homotopy Theory

Definition 2.2.1. Let K be a simplicial set. Two n-simplices, $x, x' \in K_n$, are homotopic if

$$d_j x = d_j x', \quad \forall \ 0 \le j \le n$$

and there exists $y \in K_{n+1}$ such that

$$d_n y = x,\ d_{n+1} y = x',\ \text{and}\ d_i y = s_{n-1} d_i x = s_{n-1} d_i x',\ \text{for}\ 0 \le i < n.$$

We write $x \sim x'$ and we call the $(n+1)$-simplex y a homotopy from x to x'.

Lemma 2.2.1. *If K is a simplicial set satisfying the Kan condition then the homotopy relation $\sim$ is an equivalence relation on K_n for each $n \ge 0$.*

Proof. Consider $x \in K_n$ and let $y = s_n x \in K_{n+1}$. Then, from the simplicial identities we have $d_n y = d_{n+1} y = x$, and for $0 \le i < n$, $d_i y = d_i s_n x = s_{n-1} d_i x$. So $x \sim x$ and the relation is reflexive.
Suppose $x, x', x'' \in K_n$ are such that $x' \sim x$ and $x'' \sim x$. Let y' be a homotopy from x' to x and y'' a homotopy from x'' to x. We have

$$d_i x' = d_i x = d_i x'' \text{ for } 0 \le i < n.$$

$$d_i y' = \begin{cases} s_{n-1} d_i x' & \text{if } 0 \le i < n \\ x' & \text{if } i = n \\ x & \text{if } i = n+1 \end{cases}$$

$$d_i y'' = \begin{cases} s_{n-1} d_i x' & \text{if } 0 \le i < n \\ x'' & \text{if } i = n \\ x & \text{if } i = n+1 \end{cases}$$

For $0 \le j < n$ we set $z_j = s_{n-1} s_{n-1} d_j x'$. By the simplicial identities, for $0 \le j < n$, $s_{n-1} s_{n-1} d_j = d_j s_n s_n$. Thus for $0 \le i < j < n$ we have

$$\begin{aligned} d_i z_j = d_i s_{n-1} s_{n-1} d_j x' &= d_i d_j s_n s_n x' \\ &= d_{j-1} d_i s_n s_n x' = d_{j-1} s_{n-1} s_{n-1} d_j x' = d_{j-1} z_i. \end{aligned}$$

For $0 \le j < n$ we have

$$d_{n+1} z_j = d_{n+1} s_{n-1} s_{n-1} d_j x' = s_{n-1} d_n s_{n-1} d_j x' = s_{n-1} d_j x'$$

and

$$d_n z_j = d_n s_{n-1} s_{n-1} d_j x' = s_{n-1} d_j x'$$

as $d_n s_{n-1} = \text{id}$. Finally we have $d_{n+1} y' = x = d_{n+1} y''$. It follows then that the $n+2$ $(n+1)$-simplices

$$z_0, z_1, \ldots, z_{n-1}, -, y', y''.$$

satisfy the Kan condition. Therefore there exists an $(n+2)$-simplex z such that $d_{n+1} z = y'$, $d_{n+2} z = y''$ and $d_i z = z_i$ for all $0 \le i < n$. Using the simplicial identities one again, it is straightforward to check that $d_n z$ is a homotopy from x'' to x'. Then the special case of $x'' = x$ shows that the relation is symmetric. It follows that the relation is also transitive. □

Let K be a simplicial set, and choose a 0-simplex $k_0 \in K_0$. Then we have a sub-simplicial set of K generated by k_0. For each $n \geq 0$, there is exactly one simplex in degree n

$$s_{n-1}s_{n-2}\dots s_0k_0.$$

We will use k_0 to denote both the sub-simplicial set it generates as well as any of its simplices. If K is a Kan complex, we will call (K, k_0) a *Kan pair*. In the following definitions and results we will see that k_0 plays a role analogous to the base point of a topological space in the classical homotopy theory of spaces.

Definition 2.2.2. Let (K, k_0) be a Kan pair. Then we define

$$\pi_n(K, k_0) = \frac{\{x \in K_n | d_i x = k_0\}}{\sim}$$

where $\sim$ is the equivalence relation described above.

Example 4. Let K be a Kan complex. Then, looking at the case $n = 0$, we have

$$\pi_0(K) = \frac{K_0}{\sim},$$

where for any $y \in K_1$ we have $d_0 y \sim d_1 y$. We call $\pi_0(K)$ the set of path-connected components of K, and K is said to be path-connected if $\pi_0(K)$ contains only a single element.

Proposition 2.2.1. *Let (K, k_0) be a Kan pair. Then $\pi_n(K, k_0)$ is a group for $n \geq 1$.*

Proof. Take $\alpha, \beta \in \pi_n(K, k_0)$. We define the multiplication $\alpha\beta$ in the following way:
Choose a representative x of α and a representative y of β. Then the $n+1$ n-simplices

$$k_0, k_0, \dots, k_0, x, -, y$$

satisfy the Kan condition. Therefore there exists a $(n+1)$-simplex z such that $d_{n+1}z = y$, $d_{n-1}z = y$ and $d_i z = k_0$ for $0 \leq i < n-1$. We define $\alpha\beta$ to be the equivalence class of $d_n z$:

$$\alpha\beta = [d_n z].$$

Let us first verify that the multiplication is well-defined:
Suppose $z' \in K_{n+1}$ also satisfies $d_{n+1}z' = y, d_{n-1}z' = x$ and $d_i z' = k_0$ for $0 \leq i < n-1$. Then the $n+2$ $(n+1)$-simplices

$$k_0, \dots, k_0, s_n d_{n-1} z, -, z, z'$$

satisfy the Kan condition and there exists $w \in K_{n+2}$ such that $d_i w = k_0$ for $0 \leq i < n-1$, $d_{n-1}w = s_n d_{n-1} z$, $d_{n+1}w = z$ and $d_{n+2}w = z'$. It is straightforward to verify that $d_n w$ is a homotopy from z to z' and so $[d_n z'] = [d_n z]$.
Now suppose we choose another representative y' of β. Then

$$\alpha\beta = [d_n z']$$

with $d_i z' = k_0$ for $0 \leq i < n-1$ and

$$d_{n-1}z' = x,\ d_{n+1}z' = y'.$$

There is a homotopy, w from y' to y and the $n+2$ $(n+1)$-simplices

$$k_0, \ldots, k_0, s_{n-1}x, z', -, w$$

satisfy the Kan condition. Therefore there exists $u \in K_{n+2}$ such that

$$\begin{aligned} d_i u &= k_0, \\ d_{n-1}u &= s_{n-1}x, \\ d_n u &= z'', \\ d_{n+2}u &= w. \end{aligned}$$

Consider now the $(n+1)$-simplex $v = d_{n+1}u$. We have $d_i v = k_0$ for $0 \leq i < n-1$, and

$$d_{n-1}v = x,\ d_n v = d_n z',\ d_{n+1}v = y.$$

The choice of z' is independent of the choice of representative of β. Similarly it is independent of the choice of representative of α and the multiplication is well-defined.

It is clear that $[k_0] \in \pi_n(K, k_0)$ is the identity element.

We check associativity: Let x, y, z be representatives of $\alpha, \beta, \gamma \in \pi_n(K, k_0)$ respectively. Suppose $[d_n w] = \alpha\beta$ and $[d_n w'] = \beta\gamma$. The $n+1$ n-simplices

$$k_0, \ldots, k_0, d_n w, -, z$$

satisfy the Kan condition and there exists $u \in K_{n+1}$ such that $d_i u = k_0$, for $0 \leq i < n-1$ and

$$d_{n-1}u = d_n w,\ d_{n+1}u = z.$$

Thus, $[d_n u] = [d_n w]\gamma = (\alpha\beta)\gamma$. The $n+2$ $(n+1)$-simplices

$$k_0, \ldots, k_0, w, -, u, w'$$

satisfy the Kan condition and there exists $v \in K_{n+2}$ such that

$$d_{n-1}v = w,\ d_{n+1}v = u,\ , d_{n+2}v = w'.$$

Now $[d_n(d_n v)] = \alpha[d_n w'] = \alpha(\beta\gamma)$. Finally we have,

$$\begin{aligned} \alpha(\beta\gamma) = \alpha[d_n w'] &= [d_n d_n v] \\ &= [d_n d_{n+1} v] = [d_n u] = (\alpha\beta)\gamma \end{aligned}$$

and the multiplication is associative.

It remains to verify left and right divisibility. Choose x, y representatives of α and β in $\pi_n(K, k_0)$ respectively. The simplices

$$k_0, \ldots, k_0, -, y, x$$

satisfy the extension condition and there exists $z \in K_{n+1}$ such that $d_i z = k_0$ for $0 \leq i < n-1$, $d_n z = y$ and $d_{n+1} z = x$. Then

$$[d_{n-1}z]\alpha = [d_n z] = \beta.$$

Left divisibility is shown in a similar manner and $\pi_n(K, k_0)$ is a group for $n \geq 1$. □

We call $\pi_n(K, k_0)$ the n^{th} *simplicial homotopy group of* K (with respect to k_0). If the homotopy groups of a simplicial set K are all trivial then we say that K is *contractible*.

Proposition 2.2.2. *Consider* (K, k_0) *a Kan pair. Then* $\pi_n(K, k_0)$ *is abelian for* $n \geq 2$.

Proof. The proof is similar to that of the previous proposition making repeated use of the Kan condition. Details can be found in [10, Prop. 4.4] □

If K a Kan complex and $L \subset K$ a sub-Kan complex with $l_0 \in L_0$, then we call

$$(K, L, l_0)$$

a *Kan triple*. With this concept we can define the the relative simplicial homotopy groups.

Definition 2.2.3. Let K be a simplicial set, and L a sub-simplicial set of K. Two n-simplices, x, x' of K are *homotopic relative to* L if

$$d_j x = d_j x' \ \ \forall\, 0 \leq j \leq n,$$
$$d_0 x \sim d_0 x' \text{ in } L$$

and there exists $w \in K_{n+1}$ such that

$$d_0 w = y, \ d_n w = x, \ d_{n+1} w = x'$$
$$\text{and } d_i w = s_{n-1} d_i x = s_{n-1} d_i x' \text{ for } 1 \leq i < n,$$

where $y \in K_n$ is a homotopy from $d_0 x$ to $d_0 x'$ in L. We write $x \sim_L x'$ and we call w a relative homotopy from x to x'.

Definition 2.2.4. Let (K, L, l_0) be a Kan triple, then we define

$$\pi_n(K, L, l_0) = \frac{\{x \in K_n | d_0 x \in L_{n-1}, d_i x = l_0 1 \leq i \leq n\}}{\sim_L}.$$

We define multiplication in $\pi_n(K, L, l_0)$ similarly to the non-relative case:
Choose $\alpha, \beta \in \pi_n(K, L, l_0)$, with $n \geq 2$. Let x and y be representatives of α and β respectively. Then $d_0 x, d_0 y \in L_{n-1}$ and, we have that $[d_0 x][d_0 y] = [d_{n-1} z]$ for $z \in L_n, d_i z = l_0, i \leq n-3$ and $d_{n-2} z = d_0 x, d_n z = d_0 y$. The $n+1$ n-simplices

$$z, l_0, \ldots, l_0, x, -, y$$

satisfy the Kan condition and there exists $w \in K_{n+1}$ such that $d_i w = l_0$ for $1 \leq i \leq n-2$ and $d_0 w = z, d_{n-1}z = x, d_{n+1}z = y$. We define

$$\alpha\beta = [d_n w]$$

Analogously to the previous propositions regarding $\pi_n(K, k_0)$ we have that this multiplication is well defined and $\pi_n(K, L, l_0)$ is a group for $n \geq 2$ which is abelian if $n \geq 3$.

Theorem 2.2.1. *Let (K, L, l_0) be a Kan triple. Then we have a long exact sequence*

$$\cdots \to \pi_{n+1}(K, L, l_0) \xrightarrow{d} \pi_n(L, l_0) \xrightarrow{i} \pi_n(K, l_0) \xrightarrow{j} \pi_n(K, L, l_0) \to \cdots$$

Where $d : \pi_n(K, L, l_0) \to \pi_n(L, l_0)$ is defined by $d[x] = [d_0 x]$ and i, j are induced by the inclusions.

Proof. We will show exactness at $\pi_n(L, l_0)$. Exactness in the remaining cases is shown similarly (see [10, Theorem 3.7]).
Im $d \subseteq$ Ker i: Consider $x \in K_{n+1}$ such that $d_0 x \in L_n$ and $d_i x = l_0$ for $1 \leq i \leq n+1$. Then $[x] \in \pi_{n+1}(K, L, l_0)$ and $d[x] = [d_0 x] \in \pi_n(L, l_0)$. The $n+2$ $(n+1)$-simplices

$$-, l_0, \ldots, l_0, x$$

satisfy the Kan condition and there exists $z \in K_{n+2}$ such that $d_i x = l_0$ for $1 \leq i \leq n+1$ and $d_{n+2} z = x$. It is straightforward to check that $d_0 z$ is a homotopy from l_0 to x in K_n and so $i[d_0 x] = [l_0]$.
Ker $i \subseteq$ Im d: Suppose y a representative in Ker i. Then $i[y] = [l_0] \Rightarrow y \sim l_0$ in K_n. Let z be a homotopy from y to l_0. The $n+2$ $(n+1)$-simplices

$$z, l_0, \ldots, l_0, -$$

satisfy the Kan condition and there exists $w \in K_{n+2}$ such that $d_0 w = z$ and $d_i w = l_0$ for $1 \leq i \leq n+1$. Considering $d_{n+2} w$ we see that

$$d_0 d_{n+2} w = y \Rightarrow d[d_{n+2} w] = [y] \Rightarrow [y] \in \text{Im } d$$

and Ker $i \subseteq$ Im d. □

Let us now consider homotopies of simplicial maps, or *simplicial homotopies*. We define a simplicial homotopy in the general setting of simplicial objects in a category $\mathcal{C}$.

Definition 2.2.5. Let K and L be simplicial objects in a category $\mathcal{C}$. Then two simplicial maps $f, g : K \to L$ are simplicially homotopic (f is homotopic to g) if there exist morphisms $h_i : K_n \to L_{n+1}$, for $0 \leq i \leq n$ such that

$$d_0 h_0 = f, \;\; d_{n+1} h_n = g,$$

$$d_i h_j = \begin{cases} h_{j-1} d_i & \text{if } i < j \\ d_i h_{i-1} & \text{if } i = j \neq 0 \\ h_j d_{i-1} & \text{if } i > j+1 \end{cases},$$

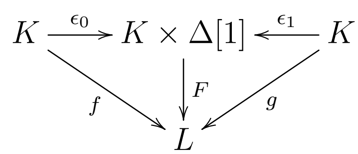
$$s_i h_j = \begin{cases} h_{j+1} s_i & \text{if } i \leq j \\ h_j s_{i-1} & \text{if } i > j \end{cases}.$$

We say $h = \{h_i\}$ is a homotopy from f to g and we write $f \simeq g$.

Specialising to the case where $\mathcal{C}$ is an abelian category or the category of sets, we have a formulation more reminiscent of the classical case given by the following proposition.
Let $\Delta[n]$ be the simplicial set given by the functor $\mathrm{Hom}_\Delta(-,[n])$. For each m, identifying $\lambda \in \mathrm{Hom}([m],[n])$ with its image $\lambda([m])$, an m-simplex is given by a sequence of integers $(a_0,\dots,a_m)$ such that $0 \le a_0 \le \cdots \le a_m \le n\}$. The face and degeneracy maps are given by

$$d_i(a_0,\dots,a_m) = (a_0,\dots a_{i-1},a_{i+1},\dots,a_m)$$
$$s_i(a_0,\dots,a_m) = (a_0,\dots,a_i,a_i,\dots,a_m)$$

Proposition 2.2.3. *Suppose $\mathcal{C}$ is either an abelian category or the category of sets. Let K and L be simplicial objects in $\mathcal{C}$, and $f,g : K \to L$ two simplicial maps. For $i = 0,1$, let $\epsilon_i : K \to K \times \Delta[1]$ be the map induced by $\delta_i : [0] \to [1]$ in Δ. There is a one-to-one correspondence between simplicial homotopies from f to g and simiplicial maps $F : K \times \Delta[1] \to L$ such that the following diagram commutes:*



Proof. We give the proof when $\mathcal{C}$ is the category of sets. For a proof when $\mathcal{C}$ is an abelian category see [14].
Suppose $\{h_i\}$ is a homotopy from f to g. For $x \in K_n$ we define the map $F_n : K_n \times \Delta[1]_n$ by

$$F(x,(0,\dots,0)) = f(x)$$
$$F(x,(1,\dots,1)) = g(x)$$
$$F(x, s_{n-1}\cdots s_{i+1}s_{i-1}\cdots s_0(0,1)) = d_{i+1}h_i(x).$$

This defines a simplicial map and the diagram clearly commutes.
Conversely, given the simplicial map F, we define a morphisms

$$h_i(x) = F(s_i x, s_n \cdots s_{i+1}s_{i-1}\cdots s_0(0,1))$$

for $x \in K_n$, $0 \le i \le n$. Then $h = \{h_i\}$ defines a morphism from f to g. □

One can check that simplicial homotopy is an equivalence relation when $\mathcal{C}$ is an abelian category. When $\mathcal{C}$ is the category of sets, we have that simplicial homotopy is an equivalence relation on the simplicial maps $f : K \to L$ whenever L is a Kan complex. This is less straightforward than the abelian case, see [10, §5] or [5, I.6].

Definition 2.2.6. Let K and L be two simplicial sets, and $f : K \to L$ a simplicial map.
(1) f is a *homotopy equivalence* if there exists a simplicial map $g : L \to K$ such that

$$g \circ f \simeq \mathrm{id}_K$$
$$f \circ g \simeq \mathrm{id}_L.$$

(2) f is a *weak homotopy equvialence* if it induces isomoprhisms

$$\pi_n(K,k_0) \xrightarrow{\sim} \pi_n(L,f(k_0))$$

for all $n \ge 0$ and for all $k_0 \in K_0$.

We say that two simplicial sets K and L are *homotopy equivalent* if there exists a homotopy equivalence $f : K \to L$. Homotopy equivalence implies weak homotopy equivalence. Thus if $K \simeq L$ we have that $\pi_i(K, k_0) \cong \pi_i(L, f(k_0))$.

2.3 Geometric Realisation of a Simplicial Set

Before we make explicit the relation between simplicial sets and topological spaces we observe that many of the results which we have seen regarding simplicial homotopy are analogous to the results of classical homotopy theory. In this section we introduce the geometric realisation functor $| - |$ and we shall see that we have an equivalence (up to homotopy) between Kan complexes and CW-complexes.

Consider the standard n-simplex $\Delta^n = \{(t_0, \ldots, t_n) \mid \sum_{i=0}^n t_i = 1, 0 \leq t_i \leq n\}$. We say that a point $u_n = (t_0, \ldots, t_n) \in \Delta^n$ is interior if $n = 0$ or $0 < t_i < 1$ for all i. Identifying the elements of $[n]$ with the vertices $v_0 = (1, 0, \ldots, 0), \ldots, v_n = (0, \ldots, 0, 1)$ of Δ^n a morphism $\alpha : [n] \to [m]$ in Δ induces a morphism

$$\begin{aligned} \Delta^n &\xrightarrow{\alpha^*} \Delta^m \\ v_i &\longmapsto v_{\alpha(i)} \end{aligned}$$

Let K be a simplicial set. We construct the *geometric realisation* $|K|$ *of* K, as follows:
Topologise the product $K_n \times \Delta^n$ as the disjoint union of copies of Δ^n indexed by the elements of K_n. Then we define

$$|K| = \frac{\sqcup_{n \geq 0} K_n \times \Delta^n}{\sim}$$

where $(\alpha^*(x), u) \sim (x, \alpha^*(u))$. The class of (x_n, u_n) is denoted $|x_n, u_n|$.
Clearly we can ignore all elements of the form $(s_i(x), u)$, $0 \leq i \leq n$. In particular, we consider only the (x, u) where x is non-degenerate. We say that an element $(x, u) \in \bigsqcup K_n \times \Delta^n$ is non-degenerate if x is non-degenerate and u is an interior point of Δ^n.

Lemma 2.3.1. *Every* $(x, u) \in \bigsqcup K_n \times \Delta^n$ *is equivalent to a unique non-degenerate element of* $\bigsqcup K_n \times \Delta^n$.

Proof. Every $x_n \in K_n$ can be written in the form

$$s_{j_p} \cdots s_{j_1} x_{n-p},$$

where $0 \leq j_1 < \cdots < j_p < n$ and $x_{n-p} \in K_{n-p}$ is non-degenerate. Similarly, every $u_n \in \Delta^n$ can be written uniquely in the form

$$\delta_{i_q} \ldots \delta_{i_1} u_{n-q},$$

where $0 \leq i_1 < \cdots < i_q \leq n$ and u_{n-q} is an interior point of Δ^{n-q}. We define morphisms λ and ρ as follows:

$$\begin{aligned} \lambda : \sqcup_{n \geq 0} K_n \times \Delta^n &\to \sqcup_{n \geq 0} K_n \times \Delta^n \\ \lambda(x_n, u_n) &= (k_{n-p}, \sigma_{j_1} \ldots \sigma_{j_p} u_n), \\ \rho : \sqcup_{n \geq 0} K_n \times \Delta^n &\to \sqcup_{n \geq 0} K_n \times \Delta^n, \\ \rho(x_n, u_n) &= (\delta_{i_1} \ldots \delta_{i_q} x_n, u_n), \end{aligned}$$

where $x_n = s_{i_p} \dots s_{i_1} k_{n-p}$ with k_{n-p} non-degenerate and $u_n = \delta_{i_q} \dots \delta_{i_1} u_{n-q}$ with u_{n-q} interior. Then $\lambda \circ \rho$ sends each point to an equivalent non-degenerate point. □

From this lemma we obtain the following result:

Theorem 2.3.1. *The geometric realisation $|K|$ of a simplicial set K is a CW-complex with one n-cell for each non-degenerate n-simplex of K.*

A simplicial morphism $f : K \to L$ induces a morphism of topological spaces,

$$|f| : |K| \longrightarrow |L|$$
$$|x_n, s_n| \longmapsto |f(x_n), s_n|$$

and this gives rise to the geometric realisation functor

$$|-| : \mathsf{Set}_\Delta \to \mathsf{Top}$$
$$K \mapsto |K|$$

The cartesian product $K \times L$ of simplicial sets is the simplicial set with $(K \times L)_n = K_n \times L_n$. Let p_1 be the projection $K \times L \to K$ and p_2 the projection $K \times L \to L$. We define a map

$$|p_1| \times |p_2| : |K \times L| \to |K| \times |L|.$$

In fact this map is a bijection and moreover, if the the product of the geometric realisations $|K| \times |L|$ is a CW-complex, then it is a homeomorphism, [10, Theorem 14.3].

Recall that two simplicial maps $f, g : K \to L$ are homotopic if there exists a simplicial map $F : K \times \Delta[1] \to L$ with $F(x, (0)) = f(x)$ and $F(x, (1)) = g(x)$ for $x \in K_n$ and $(0) = (0, \dots, 0), (1) = (1, \dots, 1)$ in $\Delta[1]_n$. The geometric realisation $|\Delta[1]|$ corresponds to the interval $I = [0, 1]$. Under this correspondence we have $|F| : |K \times \Delta[1]| \to |L|$. Combining this with the homeomorphism described above we see that our definition of simplicial homotopy corresponds to the geometric approach.
This brings us to the main result of this section. We show that the singular simplex functor Sing and the geometric realisation functor $|-|$ are adjoint. Furthermore they provide an equivalence between the category of Kan complexes and the category of CW complexes (taking morphisms up to homotopy). This allows us to work with simplicial homotopy theory in place of classical homotopy theory. While we have already seen in the previous section that they are closely aligned, the following result makes this explicit.

Theorem 2.3.2. *The singular simplex functor $\mathsf{Sing} : \mathsf{Top} \to \mathsf{Set}_\Delta$ and the geometric realisation functor $|-| : \mathsf{Set}_\Delta \to \mathsf{Top}$ are adjoint. Furthermore, for K a simplicial set and X a topological space, we have a one-to-one correspondence between the homotopy classes of simplicial maps $K \to \mathsf{Sing}(X)$ and the homotopy classes of continuous maps $|K| \to X$. In particular $\pi_i(X, x_0) = \pi_i(\mathsf{Sing}(X), \mathsf{Sing}(X_0))$ for all $i \geq 0$.*

Proof. For ease of notation we will denote the functor Sing simply by S, and the functor $|-|$ by T.
Consider a topological space X, and a simplicial set K. We define

$$\phi : \mathrm{Hom}_{\mathsf{Set}_\Delta}(K, S(X)) \longrightarrow \mathrm{Hom}_{\mathsf{Top}}(T(K), X)$$

by $\phi(f)(|k_n, u_n|) = f(k_n)(u_n)$, and

$$\psi : \mathrm{Hom}_{\mathsf{Top}}(T(K), X) \longrightarrow \mathrm{Hom}_{\mathsf{Set}_\Delta}(K, S(X))$$

by $\psi(g)(k_n)(u_n) = g(|k_n, u_n|)$.
We have a one-to-one correspondence between ϕ and the natural transformation

$$\Phi : TS \to 1_{\mathsf{Top}}$$

given by

$$\Phi(X) = \phi(1_{S(X)}) \tag{2.1}$$
$$\phi(f) = \Phi(X) \circ T(f) \tag{2.2}$$

for $f \in \mathrm{Hom}_{\mathsf{Set}_\Delta}(K, S(X))$. Similarly we have a one-to-one correspondence between ψ and the natural transformation

$$\Psi : 1_{\mathsf{Set}_\Delta} \to ST$$

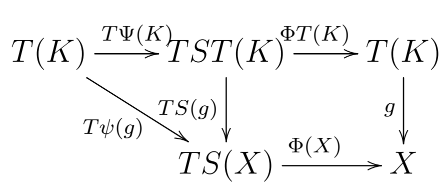
given by

$$\Phi(K) = \psi(1_T)$$
$$\psi(g) = S(g) \circ \Psi(K)$$

for $g \in \mathrm{Hom}_{\mathsf{Top}}(T(K), X)$.
Then T and S define adjoint functors if the compositions

$$T \xrightarrow{T\Psi} TST \xrightarrow{\Phi T} T \tag{2.3}$$
$$S \xrightarrow{\Psi S} STS \xrightarrow{S\Phi} S \tag{2.4}$$

are the identity transformations 1_T and 1_S respectively. We now show that the composition in (2.3) is the identity transformation 1_T if and only if $\phi \circ \psi = 1$. If $g \in \mathrm{Hom}_{\mathsf{Top}}(T(K), X)$ then the following diagram is commutative:



and $\Phi(X) \circ T\psi(g) = \phi \circ \psi(g)$ by (2.2). It is thus clear that $\Phi T(K) \circ T\Psi(K) = 1_{T(K)}$ implies $\phi \circ \psi(g) = g$. Conversely, if $\phi \circ \psi(g) = g$ then taking $g = 1_{T(K)}$ implies $\Phi T(K) \circ T\Psi(K) = 1_{T(K)}$.
Similarly we have that the composition in (2.4) is the identity transformation 1_{Set_Δ} if and only

if $\psi \circ \phi = 1$. It is clear from the definitions that $\phi \circ \psi = 1$ and $\psi \circ \phi = 1$ and so we have that T and S, that is the geometric realisation functor and the singular simplex functor, are adjoint. Finally we show that we have a one-to-one correspondence of homotopy classes of maps. Let $f, g : K \to L$ be two simplicial maps. Then if $f \simeq g$ we have a simplicial map $F : K \times \Delta[1] \to L$. By [ref] we have that $T(f), T(g) : T(K) \to T(L)$ are homotopic. Since $\phi(F) = \Phi(X) \circ T(f)$, it follows that $\phi(f) \simeq \phi(g)$. Similarly, given two homotopic maps of spaces f', g' we have that $\psi(f') \simeq \psi(g')$. □

From the unit and co-unit natural transformations (given by Ψ and Φ in the proof of our adjunction) we have

$$K \to \mathsf{Sing}|K|$$
$$|\mathsf{Sing}X| \to X$$

and these maps are weak equivalences, [10, Theorem 16.6], [5, 11.1]. In other words, they induce isomorphisms

$$\pi_n(K, k_0) \xrightarrow{\sim} \pi_n(\mathsf{Sing}|K|, \mathsf{Sing}|k_0|)$$
$$\pi_n(|\mathsf{Sing}X|, |\mathsf{Sing}(x_0)|) \xrightarrow{\sim} \pi_n(X, x_0)$$

for all simplicial sets K, $k_0 \in K_0$ and for all topological spaces X, $x_0 \in X$. We can obtain a category $\mathrm{Ho}(\mathsf{Set}_\Delta)$ from Set_Δ by formally inverting the weak equivalences. Similarly, we may also form the category $\mathrm{Ho}(\mathsf{Top})$ by formally inverting the weak homotopy equivalences. Now $\mathrm{Ho}(\mathsf{Top})$ is equivalent to the category of CW complexes with homotopy classes of maps. In the same way, $\mathrm{Ho}(\mathsf{Set}_\Delta)$ is equivalent to the category of Kan complexes with simplicial homotopy classes of maps. In other words, we may take as objects, Kan complexes, and for any two Kan complexes, K, L the set of morphisms from K to L is defined to be the homotopy class of maps $K \to L$. It follows then from the preceeding discussion that the functors Sing and $| - |$ induce an equivalence of categories between $\mathrm{Ho}(\mathsf{Set}_\Delta)$ and $\mathrm{Ho}(\mathsf{Top})$, [5, I.11].
So working up to homotopy we can replace topological spaces with Kan complexes. This equivalence allows us to define the homotopy groups for *any* simplicial set K.

Definition 2.3.1. Let K be a simplicial set and $k_0 \in K_0$. Then we define $\pi_n(K, k_0) = \pi_n(\mathsf{Sing}|K|, \mathsf{Sing}|k_0|)$.

2.4 Kan Loop Groups

In this section we will introduce path and loop spaces of a simplicial set, and then look at the construction of GK the Kan loop group of a simplicial set K. This construction was introduced in [8] in order to provide an alternative definition of the homotopy groups of K.

Let us first consider the general case of a simplicial object in a category $\mathcal{C}$. We can construct a functor

$$P : \Delta \to \Delta$$

sending $[n]$ to $[n+1]$. Each $j \leq n$ is sent to $j+1$ and we formally add the initial element 0 to the image. Given any simplicial object K, recalling that K is a contravariant functor $K : \Delta \to \mathcal{C}$ and composing with P we obtain a simplicial object which we denote PK.

Definition 2.4.1. Let K be a simplicial object in $\mathcal{A}$. Then the *path space* of K is the simplicial object PK.

Thus PK is a simplicial object with $(PK)_n = K_{n+1}$, and the i^{th} face and degeneracy operators of PK, d_i^P and s_i^P are the d_{i+1} and s_{i+1} operators of K. We have a simplicial map

$$p : PK \to K$$

coming from the maps $d_0 : K_{n+1} \to K_n$.

Lemma 2.4.1. *Let K be a simplicial object. Then PK, the path space of K, is homotopy equivalent to the constant simplicial object at K_0.*

Proof. We define a simplicial map $s : K_0 \to PK$ by taking $s_n = s_0^{n+1} : K_0 \to K_{n+1} = (PK)_n$. Similarly we define $d : PK \to K_0$ by $d_n = d_1^{n+1} : K_{n+1} \to K_0$. By the simplicial identities we find that the composition ds is the identity on K_0. Now let us show that we have a simplicial homotopy $\text{id}_{PK} \simeq sd$. Define simplicial maps $h_i : (PK)_n \to (PK)_{n+1}$ for $0 \leq i \leq n$ by setting

$$h_j = s_0^{j+1} d_1^j.$$

Now, by the simplicial identities

$$d_0^P h_0 = d_0^P s_0 = d_1 s_0 = \text{id},$$

and

$$d_{n+1}^P h_n = d_{n+2} s_0^{n+1} d_1^n = s_0^{n+1} d_1^{n+1} = sd.$$

The rest of the identities required to show that these h_i define a homotopy from id_{PK} to sd follow in a similar way. □

Now let us restrict to simplicial sets. If K is reduced (i.e. K_0 consists of a single element), then $\pi_0(K_0) = \{\text{pt}\}$ and K is path-connected (see Example 4). By the above result, PK is thus homotopy equivalent to the simplicial object generated by a single element, and we see that the homotopy groups of PK are trivial. Hence PK is contractible. This corresponds to the geometric notion of contractibility. Taking the geometric realisation we have $|K_0| = \{\text{pt}\}$ and so $|PK|$ is homotopic to a point.

If K is a reduced Kan complex, then $p : PK \to K$ is a Kan fibration, as defined in Definition 2.1.4. Since for each n we have $(PK)_n = K_{n+1}$, this follows from the fact that K is Kan. We define LK to be the fibre $p^{-1}(k_0)$ where k_0 is the sub-simplicial set generated by the single element $k_0 \in K_0$. We call LK the *loop space* of K. Since p is a Kan fibration LK and PK are Kan: Given $n+1$ compatible n-simplices of LK,

$$x_0, \ldots, x_{k-1}, -, x_{k+1}, \ldots, x_{n+1},$$

$p(x_i) = k_0$ for each i (where we take k_0 to denote any simplex of the sub-simplicial set $k_0 \subset K$) then, since p is a Kan fibration, we can find $x \in (PK)_{n+1}$ such that $d_i x = x_i$ and $p(x) = k_0$. Thus we have $x \in (LK)_{n+1}$ and the Kan condition is satisfied. Similarly, using the properties of the Kan fibration, and that K is Kan one can check that PK is also Kan, [10, §7]. We will see that under certain hypotheses, LK is an example of a Kan loop group.

For a topological space X we can define the loop space ΩX. One of the fundamental properties of this loop space is the "shift" in homotopy groups. That is, for a topological space X, we have isomorphisms

$$\pi_n(\Omega X) \cong \pi_{n+1}(X)$$

for all $n \geq 0$, see [12, Chapter 11 (§3)].
The Kan loop group can be considered as a simplicial analogue of the loop space in the sense that it shares this property. We will restrict to the case of reduced simplicial sets. For the general case, see [8]. Before we can define this loop group we must first introduce principle twisted cartesian products.

Definition 2.4.2. Let K be a simplicial set, G a simplicial group. A *principal twisted cartesian product*, is a simplicial set $G \times_\tau K$, where τ is a *twisting function*: $\tau_n : K_n \to G_{n-1}$, $n > 0$, such that for every $k \in K_n$,

$$\begin{aligned} d_0\tau(k) &= (\tau(d_0k))^{-1}\tau(d_1k) \\ d_i\tau(k) &= \tau(d_{i+1}k), i > 0 \\ s_i\tau(k) &= \tau(s_{i+1}k), i \geq 0 \\ \tau(s_0k) &= e_n, \end{aligned}$$

for e_n is the identity in G_n.
The simplicial set $G \times_\tau K$ is given by$(G \times_\tau K)_n = G_n \times K_n$ with face and degeneracy operators defined by

$$\begin{aligned} d_0(g, k) &= (\tau(k) \cdot d_0g, d_0k), \\ d_i(g, k) &= (d_ig, d_ik), i > 0 \\ s_i(g, k) &= (s_ig, s_ik), i \geq 0. \end{aligned}$$

The identities on τ ensure that we do in fact obtain a simplicial set. A principal twisted cartesian product is a special case of the more general twisted cartesian product, [10, IV].

Definition 2.4.3. A simplicial group G is a *Kan loop group* of the simplicial set K if there exists a principle twisted cartesian product $G \times_\tau K$ which is contractible.

If K is a reduced Kan simplicial set, then LK is a Kan loop group of K if it is a simplicial group [10, §23]. The twisted cartesian product is the simplicial set PK which we saw to be contractible. In particular, if K is a reduced simplicial group, then LK is a loop group.
We can construct a loop group of a general reduced simplicial set as follows.

Definition 2.4.4. Let K be a reduced simplicial set. The Kan loop group of K is the simplicial group GK where $(GK)_n$ is defined to be the free group generated by the elements of K_{n+1} modulo the relation $s_0 k_n = e_n$ for $k_n \in K_n$. Let $\alpha(k)$ denote the class of k in $(GK)_n$. We define face and degeneracy operators on the generators of GK as follows:

$$\alpha(d_0 k) d_0 \alpha(k) = \alpha(d_1 k)$$
$$d_i \alpha(k) = \alpha(d_{i+1} k) \text{ if } i > 0$$
$$s_i \alpha(k) = \alpha(s_{i+1} k) \text{ if } i \geq 0.$$

It is clear that GK is a simplicial group. The function α can be verified to be a twisting function. To see that the simplicial set $GK \times_\alpha K$ is contractible see [10, §26], or [8, §13].
In the case of simplicial sets, this GK construction is seen to provide the simplicial analogue of the loop space in the sense that we have the desired "shift" in homotopy groups. Since we are dealing only with reduced simplicial sets, we no longer refer to the simplicial basepoint and we denote the homotopy groups of K by $\pi_i(K)$. For a simplicial group G, we denote by e the sub-simplicial set consisting of the identity elements $e_n \in G_n$ for each n. We denote $\pi_i(G, e)$ simply by $\pi(G)$.

Proposition 2.4.1. *Let K be a reduced simplicial set. Then for all $n \geq 1$, we have isomorphisms*

$$\pi_n(K) \cong \pi_{n-1}(GK).$$

Proof. We sketch the outline of the proof. For further details see [8] or [10, §26].
We have a fibre sequence

$$GK \xrightarrow{q} GK \times_\tau K \xrightarrow{p} K$$

where p is the projection onto K and for $x \in (GK)_n$ $q(x) = (x, s_{n-1} \dots s_0 k_0)$ where k_0 is the unique 0-simplex $k_0 \in K_0$. This gives rise to a long exact sequence of homotopy groups,

$$\dots \to \pi_n(GK) \to \pi_n(GK \times_\tau K) \to \pi_n(K) \to \pi_{n-1}(GK) \to \dots$$

$$\dots \to \pi_0(K) \to 1$$

Since $\pi_n(GK \times_\tau K)$ is contractible, it follows that we have isomorphisms

$$\pi_n(K) \cong \pi_{n-1}(GK) \tag{2.5}$$

for all $n > 0$. □

In [8], Kan introduces the loop group as a means of providing an alternative definition of homotopy groups. In particular, as we will see in the following chapter, for simplicial groups we can define the homotopy groups in terms of an associated chain complex. Then, since the loop group GK of a simplicial set K is a simplicial group, we can make use of the isomorphisms (2.5) to define the homotopy groups $\pi_i(K)$ in terms of the chain complex which will be associated to GK.

3

Simplicial Objects and Homological Algebra

In this chapter we work with abelian categories. Throughout, let $\mathcal{A}$ be an abelian category and let $A = A_\bullet$ be a simplicial object in $\mathcal{A}$. The category of simplicial objects in $\mathcal{A}$ is denoted Simp($\mathcal{A}$). The main result of this chapter is the Dold-Kan correspondence (Theorem 3.1.1) which gives an equivalence between the category Simp($\mathcal{A}$) and the category of (bounded below) chain complexes in $\mathcal{A}$.

Definition 3.0.5. The *associated chain complex* of A, $C(A)$ is the chain complex

$$\cdots \to C_n \to C_{n-1} \to \cdots \to C_0 \to 0 \to 0 \to \cdots$$

with $C_n = A_n$ and differential $\partial_n = \sum_i^n (-1)^i d_i : C_n \to C_{n-1}$, where d_i are the face maps of A.

The simplicial identities satisfied by the d_i give us $\partial_n \circ \partial_{n+1} = 0$ and the chain complex is well defined. We can similarly describe the *associated cochain complex* of a cosimplicial object $B \in$ Cosimp($\mathcal{A}$). It is zero in negative degrees and for $n \geq 0$, the degree n term is B_n:

$$0 \to B_0 \to B_1 \to \cdots \to B_n \to n_{n+1} \to \cdots$$

The differential $\partial^n = \sum_{i=0}^{n+1} (-1)^i d^i : B_n \to B_{n+1}$ where the d^i are the coface maps of B.

Example 5. (Singular chain complex) Consider the cosimplicial space $\{\Delta^n\}$ defined by the covariant functor

$$\Delta \to \mathsf{Top}$$
$$[n] \to \Delta^n$$

where the elements of $[n]$ are associated with the vertices $v_1 = (1, 0, \ldots, 0), \ldots, v_n = (0, \ldots, 0, 1)$ of Δ^n in the obvious way. Applying $\mathrm{Hom}_{\mathsf{Top}}(-, X)$ to this cosimplicial space we obtain a simplicial set $S(X)$ with $S(X)_n = \{\Delta^n \to X\}$. That is, the singular simplicial set which we defined combinatorially in the previous chapter.
Taking R a commutative ring we define a functor Set $\to$ R-mod sending a set X to the free R-module on X. If X is a simplicial set, then we can define a simplicial R-module $R[X]_\bullet$

with $R[X]_n = R[X_n]$. Returning to the singular simplicial set $S(X)$ we obtain the simplicial R-module $R[S(X)]_\bullet$. The associated chain complex

$$\cdots \to R[S(X)_n] \to R[S(X)_{n-1}] \to \ldots$$

is precisely the chain complex used to compute the singular homology groups of X. More generally, for any simplicial set X, the associated chain complex of the simplicial R-module $R[X]_\bullet$ is the chain complex used to calculate the simplicial homology of $|X|$.

Definition 3.0.6. The *simplicial homology* (with coefficients in a commutative ring R) of a simplicial set X is the homology of the associated chain complex of the simplicial module $R[X]_\bullet$,

$$H_\bullet(X; R) := H_\bullet(|X|; R)$$

It follows immediately from the previous example that the simplicial homology of the singular simplicial set $S(X)$, for any topological space X, is equal to the singular homology of X.

Definition 3.0.7. The *normalised chain complex* of A is the chain complex with

$$N_n(A) = \bigcap_{i=0}^{n-1} \mathrm{Ker}(d_i : A_n \to A_{n-1})$$

in degree n and differential

$$\partial = (-1)^n d_n$$

where the d_i are the face operators of the simplicial object A.

If $x \in N_n(A)$ then $x \in A_n$ and $d_n x \in A_{n-1}$. By the simplicial identites, $d_i d_n x = d_{n-1} d_i x = 0$ for $0 \leq i < n$ so $d_n x \in N_{n-1}(A) = \bigcap_{i=0}^{n-1} \mathrm{Ker}\ d_i : A_{n-1} \to A_{n-2}$. Finally, $\partial \circ \partial x = (-1)^{n-1}(-1)^n d_{n-1} d_n x = 0$ and the chain complex is well-defined.
Moreover, the normalised chain complex forms a functor N from the category of simplicial objects in $\mathcal{A}$ to the category of non-negative chain complexes in A. That is,

$$N : \mathrm{Simp}(\mathcal{A}) \to \mathrm{Ch}_{\geq 0}(\mathcal{A}).$$

Given a simplicial morphism $f : A \to B$, $A, B \in \mathrm{Simp}(\mathcal{A})$, we have a collection of maps $f_n : A_n \to B_n$. Then $N(f) : N(A) \to N(B)$ is the chain map where

$$N(f)_n = f_n|_{N_n(A)} : N_n(A) \to N_n(B).$$

Lemma 3.0.2. *Let A be a simplicial object in $\mathcal{A}$. Let $C(A)$ be the associated chain complex, $N(A)$ the normalised chain complex, and $D(A)$ the degenerate subcomplex of $C(A)$, where $D_n(A) = \sum s_i(C_{n-1}(A))$. Then $C(A) = N(A) \oplus D(A)$.*

Proof. See [14, Lemma 8.3.7]. □

In the previous chapter we defined the homotopy groups $\pi_n(K)$ for a (Kan) simplicial set K, (Definitions 2.2.1, 2.3.1) . Given a simplicial object A in any abelian category $\mathcal{A}$ we define

$$\pi_n(A) = H_n(N(A)).$$

Proposition 3.0.2. *Let A be a simplicial object of an abelian category $\mathcal{A}$. Then, for all $n \geq 0$, the homotopy groups $\pi_n(A)$ are isomorphic to the homology groups $H_n(C)$ of the associated chain complex $C = C(A)$.*

Proof. We need to show that the homology of the normalised chain complex $N(A)$ is isomorphic to the homology of the chain complex $C(A)$. Then we have

$$\pi_n(A) = H_n(N(A)) \cong H_n(C(A)).$$

From the previous lemma, it suffices to prove that the degenerate chain complex $D(A)$ is acyclic. This is shown in [14, Thm 8.3.8]. □

Example 6 (Classifying space). Let G be a group. We construct a simplicial set BG by taking $BG_0 = \{1\}$ and more generally $BG_n = G^n$. We define face maps,

$$d_i(g_1, \dots, g_n) = \begin{cases} (g_2, \dots, g_n), & i = 0 \\ (g_1, \dots, g_i g_{i+1}, \dots, g_n), & 0 < i < n \\ (g_1, \dots, g_{n-1}) & i = n \end{cases}$$

and degeneracy maps,

$$s_i(g_1, \dots, g_n) = (g_1, \dots, g_i, 1, g_{i+1}, \dots, g_n).$$

It is straightforward to verify that these maps satisfy the simplicial identites and so BG is a simplicial set. We call the geometric realisation $|BG|$ the *classifying space* of G. In fact, BG is a Kan complex, and so we have $\pi_n(|BG|) = \pi_n(BG)$. Now

$$N_1(BG) = \mathrm{Ker}(d_0 : G \to \{1\}) = G,$$
$$N_n(BG) = \bigcap_{i=0}^{n-1} \mathrm{Ker}(d_i : BG_n \to BG_{n-1}) = \{1\},\ n > 1.$$

So the normalised chain complex of BG is the following

$$\cdots \to 1 \to \cdots \to 1 \to G \to 1,$$

and we have

$$\pi_n(BG) = H_n(N(BG)) = \begin{cases} 1 & \text{if } n \neq 1 \\ G & \text{if } n = 1 \end{cases}.$$

Definition 3.0.8. Let G be a group. An *Eilenberg-MacLane space* of type $K(G, n)$ is a Kan complex K such that

$$\pi_i(K) = \begin{cases} G & \text{if } i = n \\ 0 & \text{otherwise.} \end{cases}$$

From the previous example we see that for any group G, BG is an Eilenberg-MacLane space of type $K(G, 1)$. In fact, we will see in Example 7 how the Dold-Kan correspondence will enable us to construct Eilenberg-MacLane spaces of type $K(G, n)$ for all n.

3.1 The Dold-Kan Correspondence

Theorem 3.1.1 (Dold-Kan)**.** *For any abelian category $\mathcal{A}$, the normalised chain functor*

$$N : \mathit{Simp}(\mathcal{A}) \longrightarrow \mathit{Ch}_{\geq 0}(\mathcal{A})$$

is an equivalence of categories. Moreover under this correspondence simplicial homotopy corresponds to homology and simplicially homotopic maps correspond to chain homotopic maps.

Proof. We first construct the inverse functor, $K : \mathrm{Ch}_{\geq 0}(\mathcal{A}) \to \mathrm{Simp}(\mathcal{A})$. Given a chain complex $C \in \mathrm{Ch}_{\geq 0}(\mathcal{A})$ define

$$K_n(C) = \bigoplus_{\eta:[n]\twoheadrightarrow[p],p\leq n} C_\eta$$

where for $\eta : [n] \twoheadrightarrow [p]$, $C_\eta = C_p$, the degree p entry of C. GIven a morphism $\alpha : [n] \to [m]$ in Δ we define a morphism $K(C)(\alpha) : K_m(C) \to K_n(C)$ by its restriction to each of its summands C_η. Consider the unique epi-monic factorisation of $\eta \circ \alpha$ for each $\eta : [m] \twoheadrightarrow [p]$, $p \leq m$.

$$\begin{array}{ccc} [n] & \xrightarrow{\alpha} & [m] \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\eta} \\ [q] & \xhookrightarrow{\delta} & [p] \end{array}$$

If $p = q$, $\eta\alpha = \sigma$ and we define $K(C)(\alpha, \eta)$ to be the identification of $C_\eta = C_p$ with $C_\sigma = C_p$. If $p = q + 1$ and $\delta = \delta_p$, then we set $K(C)(\alpha\eta) = \partial : C_p \to C_{p-1} = C_\sigma \subseteq K_n(C)$, where ∂ is the differential of our chain complex C. Otherwise we set $K(C)(\alpha, \eta) = 0$. The restriction of $K(C)(\alpha)$ to each of the summands C_η is defined to be $K(C)(\alpha, \eta)$.
We have to verify that $K(C)$ is a simplicial object of $\mathcal{A}$. For any two composable morphisms $\alpha : [l] \to [m]$, $\beta : [m] \to [n]$ in Δ consider the map $K(C)(\beta \circ \alpha) : K_n(C) \to K_l(C)$. For any $\eta : [n] \twoheadrightarrow [p], p \leq n$ we look at the restriction $K(C)(\beta \circ \alpha, \eta)$.

$$\begin{array}{ccccc} [l] & \xrightarrow{\alpha} & [m] & \xrightarrow{\beta} & [n] \\ \downarrow & & \downarrow{\scriptstyle\eta'} & & \downarrow{\scriptstyle\eta} \\ [q] & \longrightarrow & [q'] & \longrightarrow & [p] \end{array}$$

If $q \leq p - 2$ it is clear that $K(C)(\beta \circ \alpha, \eta) = 0 = K(C)(\alpha, \eta') \circ K(C)(\beta, \eta)$ as at least one of these must be 0. If $q = p - 1$ and $\delta = \delta_p$ then $K(C)(\beta \circ \alpha) = \partial$. Furthermore, $q' = p - 1$ or p and $K(C)(\alpha, \eta') \circ K(C)(\beta, \eta) = id \circ \partial$ or $\partial \circ id$ respectively. Hence in either case, $K(C)(\beta \circ \alpha) = K(C)(\alpha, \eta') \circ K(C)(\beta, \eta)$. Finally if $q = p$ then clearly $q = q' = p$, $\eta = \eta'$ and $K(C)(\beta \circ \alpha) = id = K(C)(\alpha, \eta') \circ K(C)(\beta, \eta)$. We conclude that $K(C)(\beta \circ \alpha) = K(C)(\alpha) \circ K(C)(\beta)$. So we have defined a functor

$$\begin{aligned} K : \mathrm{Ch}_{\geq 0}(\mathcal{A}) &\to \mathrm{Simp}(\mathcal{A}) \\ C &\mapsto K(C) \end{aligned}$$

where $K(C)$ is the simplicial object defined above, with $K_n(C) = \bigoplus_{\eta:[n]\twoheadrightarrow[p],p\leq n} C_\eta$.

It remains to check that K is inverse to N. Let us look first at $NK(C)$. For each surjection $\eta : [n] \twoheadrightarrow [p], p \leq n$ we can write $\eta = \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_t}$ by Lemma 1. It follows that, for $n \neq p$, we can write $C_\eta = (s_{i_t}\cdots s_{i_1})C_{id_p}$ where the s_{i_j} are the degeneracy operators of the simplicial object $K(C)$ corresponding to the σ_{i_j}. So C_η lies in the degenerate complex $DK(C)$. Now for $n = p$, we have

$$d_i|_{C_{id_n}} = K(C)(\delta_i, id_n) = \begin{cases} d & \text{if } i = n \\ 0 & \text{otherwise} \end{cases}$$

and $N_n(K(C)) = C_{id_n} = C_n$. So we have a natural isomorphism

$$id \to NK.$$

Now let us define the natural transformation $KN \to id$. For a simpicial object $A \in \text{Simp}(A)$, we define

$$\psi_n : K_nN(A) \to A_n$$

on it's restrictions to the summands of $K_nN(A)$. For each $\eta : [n] \to [p]$ the corresponding summand $N_\eta(A) = N_p(A)$ is a subobject of A_p. We take $\psi_n|_{N_\eta(A)}$ to be the composition

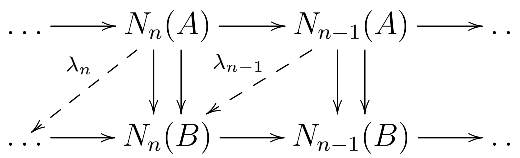
$$N_\eta(A) = N_p(A) \hookrightarrow A_p \stackrel{A(\eta)}{\to} A_n.$$

Given any $\alpha : [m] \to [n]$ in Δ let $\delta\circ\sigma$ be the epi-monic factorisation of $\eta\circ\alpha$. Then the following diagram commutes

$$\begin{array}{ccccccc} K_nN(A) & \hookleftarrow & N_p(A) & \hookrightarrow & A_p & \xrightarrow{A(\eta)} & A_n \\ \downarrow{\scriptstyle K(\alpha)} & & \downarrow{\scriptstyle K(\delta)} & & \downarrow{\scriptstyle A(\delta)} & & \downarrow{\scriptstyle A(\alpha)} \\ K_mN(A) & \hookleftarrow & N_q(A) & \hookrightarrow & A_q & \longrightarrow & A_m \end{array}$$

and ψ is a simplicial map, natural in A. Clearly $\psi_0 : K_0(N(A)) = N_0(A) = A_0 \stackrel{\sim}{\to} A_0$ an isomorphism. Proceeding by induction we show that ψ is an isomorphism. Recall that $A_n = N_n(A) \oplus D_n(A)$. Now $N_n(A)$ is in the image of ψ_n as ψ_n-$N_n(A)$ is simply the inclusion $N_n(A) \hookrightarrow A_n$. Now take $y \in D_n(A)$. Then $y = s_jx$ for some $x \in A_{n-1}$, for some $0 \leq j \leq n-1$. By the inductive hypothesis $\psi_{n-1} : K_{n-1}N(A) \stackrel{\sim}{\to} A_{n-1}$ so we have $x \in \text{Im}\psi_{n-1} \Rightarrow y = s_jx \in \text{Im}\psi_m$. So $D_n(A)$ is also contained in the image of ψ and ψ is surjective. Consider $(x_\eta) \in K_nN(A)$ and suppose $(x_\eta) \mapsto 0$. Now $x_\eta \in N_\eta(A)$, for $\eta : [n] \twoheadrightarrow [p]$. If $p = n$, then $x_\eta = x_{id_n} = 0$ as $\psi|_{N_n(A)}$ is simply the inclusion $N_n(A) \hookrightarrow A_n$. If $p < n$ then there exists $\epsilon : [p] \to [n]$ such that $\eta\circ\epsilon = id_p$ and $K(\epsilon, \eta)(x_\eta) = x_{id_p}$ in $K_p(N(A))$. Now $\psi_p(x_{id_p}) = 0$ and by the inductive hypothesis ψ_p an isomorphism. Thus $x_{id_p} = 0$ in $K_p(N(A))$. It follows then that $x_\eta = 0$ in $K_n(N(A))$ and ψ_n is injective. We have a natural isomorphism $KN \to id$ and the functor K is the desired inverse of N proving the equivalence of categories.

Let A and B be two simplicial objects in $\mathcal{A}$, and $f, g : A \to B$ two simplicial maps. Suppose $f \simeq g$. Replacing g with $g - f$ we may assume $f = 0$, and we are reduced to the case $0 \simeq g$. Let h be a homotopy from 0 to g. We must show that the corresponding chain map $g_* : N(A) \to N(B)$ is chain homotopic to the zero map. Define $\lambda_n = \sum_{j=0}^{n}(-1)^jh_j$. It is straightforward to show that $\lambda_n\left(N_n(A) \subseteq N_{n+1}(B)\right)$. So we have



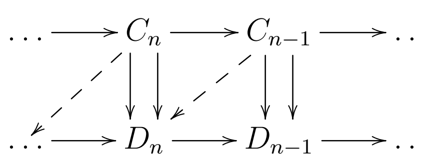
Now

$$\begin{aligned}
\partial_{n+1}\lambda_n &= (-1)^{n+1} d_{n+1} \sum (-1)^j h_j \\
&= (-1)^{n+1} \sum_{j=0}^{n} (-1)^j d_{n+1} h_j \\
&= (-1)^{n+1} \sum_{j=0}^{n-1} (-1)^j h_j d_n + (-1)^{n+1}(-1)^n d_{n+1} h_n \\
&= (-1)^{n+1} \sum_{j=0}^{n-1} (-1)^j h_j d_n - g.
\end{aligned}$$

and

$$\begin{aligned}
\lambda_{n-1}\partial_n &= \left(\sum (-1)^j h_j\right)(-1)^n d_n \\
&= (-1)^n \sum_{j=0}^{n-1} (-1)^j h_j d_n \\
&= -\partial_{n+1}\lambda_n - g.
\end{aligned}$$

So $\{(-1)^n \lambda_n\}$ is a chain homotopy from 0 to g_*.
Let C and D be non-negative chain complexes in $\mathcal{A}$, and $f', g' : C \to$ chain maps. Suppose that $\lambda = \{\lambda_n\}$ is a chain homotopy from f' to g',



We wish to show that $K(f'), K(g') : K(C) \to K(D)$ are simplicially homotopic. We define

$$h_i : K_n(C) = \bigoplus_{\eta:[n]\twoheadrightarrow[p]} C_\eta \to K_{n+1}(D) = \bigoplus_{\eta':[n+1]\twoheadrightarrow[p]} D_{\eta'}$$

by its restriction to each of the summands of $K_n(C)$. On $C_{id_n} = C_n$ we set

$$h_i|_{C_n} = \begin{cases} s_i f & \text{if } i < n-1 \\ s_{n-1} f - s_n \lambda_{n-1} d & \text{if } i = n-1 \\ s_n(f - \lambda_{n-1} d) - \lambda_n & \text{if } i = n, \end{cases}$$

with d the differential of the chain complex $C_\bullet$. On the remaining summands $C_\eta, \eta : [n] \twoheadrightarrow [p]$ we define $h_i|_{C_\eta}$ by induction on $n-p$. Let j be the largest element of $[n]$ such that $\eta(j) = \eta(j+1)$, then $\eta = \eta'\eta_j$ and $C_{\eta'} \cong C_\eta$ via s_j. By induction we have defined the h_i on $C_{\eta'}$. let $\tilde{h}_i = h_i \circ s_j$. Then we define the h_i on C_η as follows:

$$h_i|_{C_\eta} = \begin{cases} s_j\tilde{h}_{i-1} & \text{if } j < i \\ s_{j+1}\tilde{h}_i & \text{if } j \geq i. \end{cases}$$

Some straightforward calculations show that $h = \{h_i\}$ is a simplicial homotopy from $K(f)$ to $K(g)$. □

Since $\mathrm{Cosimp}(\mathcal{A}) = \mathrm{Simp}(\mathcal{A}^{op})^{op}$, by the Dold-Kan theorem we have

$$\mathrm{Cosimp}(\mathcal{A}) \simeq \mathrm{Ch}_{\geq 0}(\mathcal{A})(\mathcal{A}^{op})^{op} = \mathrm{Coch}_{\geq 0}(\mathcal{A})$$

where $\mathrm{Coch}_{\geq 0}(\mathcal{A})$ is the category of non-negative cochain complexes in $\mathcal{A}$. Thus we obtain an equivalence

$$Q : \mathrm{Cosimp}(\mathcal{A}) \to \mathrm{Coch}_{\geq 0}(\mathcal{A}).$$

Let $B_\bullet$ be a cosimplicial object of $\mathcal{A}$. We set

$$Q(B)^n = \mathrm{Coker}\left(\oplus_{i=0}^{n-1} d^i : B_{n-1} \to B_n\right)$$
$$Q(B)^0 = B_0.$$

Taking the differential to be

$$\partial^n = (-1)^{n+1} d^{n+1} : Q(B)^n \to Q(B)^{n+1}$$

we obtain a cohcain complex $Q(B)$

$$0 \to Q(B)^0 \to Q(B)^1 \to \cdots$$

We call this the *normalised cochain complex* associated to $B_\bullet$. The dual statement of the Dold-Kan Theorem is thus the following:

Theorem 3.1.2. *For any abelian category $\mathcal{A}$, the functor*

$$Q : \mathit{Cosimp}(\mathcal{A}) \to \mathit{Coch}_{\geq 0}(\mathcal{A})$$

which sends a cosimplicial object to its associated normalised cochain complex, is an equivalence of categories.

The following example shows how we can use the Dold-Kan correspondence to construct Eilenberg-MacLane spaces.

Example 7 (Eilenberg-MacLane spaces). Let G be an abelian group and let $G(n)$ denote the chain complex that is G concentrated in degree n. The homology groups $H_i(G(n))$ of this chain complex vanish for all $i \neq n$ and $H^n(G(n)) = G$. From the Dold-Kan correspondence we

obtain a simplicial abelian group $K(G(n))$. Since homology corresponds to homotopy under this correspondence, $K(G(n))$ is an Eilenberg-MacLane space of type $K(G,n)$.
Moreover, if a simplicial abelian group A is an Eilenberg-MacLane space of type $K(G,n)$ then it has the same simplicial homotopy type as the simplicial abelian group $K(G(n))$. To see this consider the subcomplex $\tau_{\geq n}(N(A))$ of the normalised chain complex $N(A)$ obtained by truncating at n. We have:

$$\tau_{\geq n}(N(A))_i = \begin{cases} 0 & \text{if } i < n \\ \text{Ker } d_n & \text{if } i = n \\ N_i(A) & \text{if } i > n. \end{cases}$$

We define the chain map $f : \tau_{\leq n}(N(A)) \to G(n)$, by $f_i = 0$ and $f_n : \text{Ker}(d_n) \to G$ is given by the following composition

$$\text{Ker}(d_n) \subseteq N_n(A) \to H_n(N(A)) \cong \pi_n(A) = G.$$

By the Dold-Kan correspondence we have an associated simplicial abelian group $\tilde{K} = K(\tau_{\geq n}(N(A)))$, and a simplicial map

$$K(f) : \tilde{K} \to K(G(n))$$

which induces an isomorphism on homotopy groups. Similarly, we have a simplicial map

$$\tilde{K} \to A$$

obtained by applying the functor K to the inclusion of chain complexes $\tau_{\leq n}(N(A)) \to NA$. This too induces isomorphisms on homotopy groups, and so we conclude that A has the same homotopy type as $K(G(n))$.

3.2 Simplicial Path and Loop spaces in an Abelian Category

Let A be a simplicial object in an abelian category $\mathcal{A}$. We recall that in Definition 2.4.1 of Chapter 2 we defined the path space of a simplicial object, PA. It was the simplicial object with $(PA)_n = A_{n+1}$ and the face and degeneracy operators d_i^P, s_i^P were the "shifted" face and degeneracy operators of A, d_{i+1}, s_{i+1}. Moreover, the face maps d_0 gave us a simplicial map

$$p : PA \to A.$$

In the case of (reduced) simplicial sets we went on to define the loop space and loop group of a simplicial set. Let us now look at the loop space of our simplicial object A.

Definition 3.2.1. Let A be a simplicial object in $\mathcal{A}$. We define the loop space of A, LA to be the following simplicial object,

$$LA = \text{Ker}\,(p : PA \to A)\,.$$

In terms of our original simplicial object A, we have that in degree n,

$$(LA)_n = \mathrm{Ker}\left(d_0 : A_{n+1} \to A_n\right).$$

Remark that this definition coincides with the definition of Chapter 2.4 for the loop space LK of a reduced simplicial set K, whenever K is a reduced simplicial *group*.
The following result shows that this construction is the simplicial analogue of the topological loop space $\Omega(X)$.

Lemma 3.2.1. *Let A be a simplicial object in $\mathcal{A}$. Then, for all $n > 0$ we have the following isomorphisms,*

$$\pi_n(LA) \cong \pi_{n+1}(A).$$

Proof. We show that the components of the corresponding normalised chain complexes are such that $N_n(LA) \cong N_{n+1}(A)$. Then, we have that

$$\pi_i(LA) = H_i(N(LA)) \cong H_{i+1}(N(A)) = \pi_{i+1}(A).$$

From the construction of LA we see that $(LA)_n$ is a subobject of $(PA)_n$ and its face and degeneracy operators come from the face and degeneracy operators of the path space. In particular, the d_i^L and s_i^L are the d_{i+1} and s_{i+1} of our simplicial object A. The normalised chain complex has entries

$$\begin{aligned} N_n(LA) &= \bigcap_{i=0}^{n-1} \mathrm{Ker}\left(d_i^L : (LA)_n \to (LA)_{n-1}\right) \\ &= \bigcap_{i=0}^{n-1} \mathrm{Ker}\left(d_{i+1} : (\mathrm{Ker}(d_0 : A_{n+1} \to A_n)) \to (\mathrm{Ker}(d_0 : A_n \to A_{n-1}))\right) \\ &= \bigcap_{i=0}^{n} \mathrm{Ker}\left(d_i : A_{n+1} \to A_n\right) \\ &= N_{n+1}(A). \end{aligned}$$

□

For a simplicial set K we constructed the loop group GK (Definition 2.4.4) to obtain the equivalent result. However if K is a reduced simplicial group the loop space of Chapter 2.4 (which coincides with the construction above) is a loop group and our results here are consistent with those of the previous Chapter. In light of how we defined the homotopy groups of A, and the isomorphisms (2.5) in Chapter 2.4, we can define the homotopy groups of a reduced simplicial set K as follows:

$$\pi_n(K) = H_{n-1}\left(N(GK)\right).$$

This is alternative definition of Kan in [8].

4

∞-Categories

In this chapter we follow the material in the opening chapter of Lurie's book *Higher Topos Theory*, [9]. The theory of higher categories involves the notion of an n-category. Roughly speaking, an n-category, $n \geq 0$, consists of a collection of objects, morphisms between objects, 2-morphisms between morphisms, and also k-morphisms for every $k \leq n$.

The following is an example of a 2-category.

Example 8. Consider the category of (small) categories. This can be viewed as a 2-category:
• we have a collection of objects, $\mathcal{C}, \mathcal{D}, \ldots$
• for every pair of objects $\mathcal{C}, \mathcal{D}$ we have morphisms between these objects, given by functors $F : \mathcal{C} \to \mathcal{D}$
• for every pair of morphisms F, G we have morphisms between these morphisms, or 2-morphisms, given by natural transformations of functors.

Let us now look at n-categories, for any $n \geq 0$.

Example 9. (Fundamental n-groupoid) Let X be a topological space. Then for $0 \leq n \leq \infty$ we may define an n-category $\Pi_{\leq n} X$ as follows:
• objects are the points of X
• morphisms between $x, y \in X$ are continuous paths $[0, 1] \to X$ from x to y
• 2-morphisms are homotopies of paths
• 3-morphisms are homotopies of homotopies
• $\ldots$ and so on up to n-morphisms between $(n-1)$-morphisms.
If $n < \infty$ we identify two n-morphisms with each other if and only if they are homotopic. Taking $n = 0$ we find the set of all path components of X. When $n = 1$ we obtain the fundamental groupoid of X. We call $\Pi_{\leq n} X$ the *fundamental n-groupoid of X* as every k-morphism is invertible (up to homotopy).

The fundamental n-groupoid is an example of an $(n, 0)$-category: for all $0 < k \leq n$ the k-morphisms are invertible. More generally, an (n, r)-category is an n-category in which the k-morphisms are invertible for all $r < k \leq n$. Taking $n = \infty$, the fundamental ∞-groupoid of a topological space X, $\Pi_{\leq \infty}(X)$ gives us our first example of an ∞-category. In particular it is an $(\infty, 0)$-category which we call an *∞-groupoid*. To every topological space we can associate an ∞-groupoid. Conversely, we would like that all ∞-groupoids have the form $\Pi_{\leq \infty}(X)$ for a topological space X. This is known as the *homotopy hypothesis*, [2].

The homotopy hypothesis: There is an equivalence between ∞-groupoids and topological spaces (up to weak homotopy equivalence).

We will see later (in Corollary 4.2.1), that ∞-groupoids are given by Kan complexes which, as we saw in Chapter 2, correspond to topological spaces (Section 2.3).

Before we continue let us briefly introduce the concept of *enriched categories*. The notion of an enriched category is a generalistion of a category where, instead of having merely a set of morphisms between each pair of objects, we have an additional structure, for example a topological space, a group or a simplicial set. Given a suitable category $\mathcal{D}$, a category enriched over $\mathcal{D}$, is a category $\mathcal{C}$ which consists of a collection of objects, and for each pair of objects $x, y \in \mathcal{C}$ we have a mapping-object $\mathrm{Map}_{\mathcal{C}}(x, y) \in \mathcal{D}$. For formal details on enriched categories and on the "suitability" conditions of $\mathcal{D}$ see [9, A.1.3; A.1.4].

It is natural to consider an $(\infty, 1)$-category as an enriched category. We have objects, 1-morphisms, and all higher morphisms (k-morphisms for $k > 1$) are invertible. Thus, if we consider the morphisms $\mathrm{Hom}(X, Y)$ between any pair of objects in an $(\infty, 1)$-category, they form an ∞-groupoid: the 1-morphsims between $\mathrm{Hom}(X, Y)$ are given by the 2-morphisms of our $(\infty, 1)$-category and are hence invertible. Continuing in this way, we see that all the k-morphisms between $\mathrm{Hom}(X, Y)$ are given by the $(k+1)$-morphisms of our $(\infty, 1)$-category and are invertible for all $k \geq 1$.

The homotopy hypothesis then leads us to the theory of topological categories: we have a collection of objects, and for each pair of objects X, Y we have a topological space $\mathrm{Map}(X, Y)$. Using the correspondence of Chapter 2 (Theorem 2.3.2) between topological spaces and simplicial sets it is not surprising that we can also look at simplicial categories where the enrichement is over the category of simplicial sets. We will see later (Proposition 4.1.1) that it is precisely those simplicial categories whose mapping objects are Kan complexes that correspond to $(\infty, 1)$-categories.

More generally, if we consider an (∞, r)-category, for any pair of objects X, Y in this category the collection of morphisms (1-morphisms) between a pair of objects should form an $(\infty, r-1)$-category.

For the rest of this chapter we focus solely on $(\infty, 1)$-categories. They are also known as quasi-categories, [7] or weak Kan complexes. Following Lurie's terminology in [9] we will call them ∞-categories.

Definition 4.0.2. A *topological category* is a category which is enriched over Top, the category of topological spaces.

A topological category $\mathcal{C}$ consists of a collection of objects and, for each pair of objects $X, Y \in \mathcal{C}$, we have a topological space $\mathrm{Map}_{\mathcal{C}}(X, Y)$. These spaces are equipped with a composition law given by the continuous maps

$$\mathrm{Map}_{\mathcal{C}}(X,Y) \times \mathrm{Map}_{\mathcal{C}}(Y,Z) \to \mathrm{Map}_{\mathcal{C}}(X,Z).$$

The theory of topological categories is one possible approach to dealing with ∞-categories and we have the following definition.

Definition 4.0.3. An ∞-category is a topological category.

This approach is quite difficult to work with. It turns out that one should not require the composition law in ∞-categories to be strictly associative, [9, §1.1.1, §1.1.2]. We will introduce an alternative definition of ∞-categories via the theory of simplicial sets and then show the equivalence of this approach with the topological categories defined above.

We have just seen how ∞-categories are closely related to topological spaces, by means of ∞-groupoids. We can also associate to any ordinary category $\mathcal{C}$, an ∞-category (having no non-trivial higher morphisms). Recall from Chapter 2 that a simplicial set satisfying the Kan condition provided a model for a topological space. The following results will show how simplicial sets are also closely related to categories, thus motivating their use in the theory of ∞-categories.

Definition 4.0.4. The *nerve of a category* $\mathcal{C}$ is the simplicial set $\mathcal{N}(\mathcal{C})$ where $\mathcal{N}(\mathcal{C})_n$ is the set of all functors $[n] \to \mathcal{C}$, with $[n]$ viewed as a category in the obvious way.

It is clear that $\mathcal{N}(\mathcal{C})_0$ can be identified with $\mathrm{ob}(\mathcal{C})$, and $\mathcal{N}(\mathcal{C})_1$ with the set of morphisms in $\mathcal{C}$. More generally, we see that $\mathcal{N}(\mathcal{C})_n$ can be seen as the set of all chains of composable morphisms of length n.

Proposition 4.0.1. *Let K be a simplicial set. Then the following are equivalent:*
(1) *K is isomorphic to the nerve of some (small) category $\mathcal{C}$*
(2) *For each $0 < i < n$, every simplicial map $f_i : \Lambda^n_i \to K$ admits a unique extension $f : \Delta^n \to K$.*

Note the similarity between condition (2) and the Kan condition. Here we only require the extension condition for Λ^n_i when $0 < i < n$. We call these the *inner horns*. However, our condition requires the extension to be unique, which is not required for the Kan condition. Clearly, as with the Kan condition, we can reformulate (2) in a more combinatorial way. We have:
(2) for every collection of n $(n-1)$-simplices, $x_0, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n, k \neq 0, k \neq n$, with $d_i x_j = d_{j-1} x_i \forall i < j, i, j \neq k$ there exists *a unique* n-simplex y such that $d_i y = x_i$ for all $i \neq k$.

Proof. $(\Rightarrow)$: Suppose $K \cong \mathcal{N}(\mathcal{C})$ for some category $\mathcal{C}$. Consider a simplicial map $f_i : \Lambda^n_i \to K$, $0 < i < n$. Let $(\Delta^n)_0$ denote the set of vertices of Δ^n and $(\Lambda^n_i)_0$ denote the set of vertices of Λ^n_i. Then remarking that these sets of vertices coincide, and using the correspondence between $\mathcal{N}(\mathcal{C})_0$ and $\mathrm{ob}(\mathcal{C})$ we have, in degree 0,

$$(f_i)_0 : (\Lambda^n_i)_0 \to K_0 \cong \mathrm{ob}(\mathcal{C}).$$

Let us denote by $X_k \in \mathrm{ob}(\mathcal{C})$ the image of the vertex $\{k\} \in (\Lambda^n_i)_0$ for each $0 \leq k \leq n$. The restriction of f_i to the 1-simplex $\{k, k+1\} \subset \Lambda^n_i$ gives us a morphism $g_k : X_k \to X_{k+1}$ in $\mathcal{C}$. The g_k, $0 \leq k \leq n$ form a chain of composable morphisms in $\mathcal{C}$

$$X_0 \overset{g_0}{\to} X_1 \overset{g_1}{\to} \dots \overset{g_{n-1}}{\to} X_n.$$

Hence, they determine a n-simplex of K_n, that is a map $f : \Delta^n \to K$. Let us show that this f is such that the following diagram is commutative:

$$\begin{array}{ccc} \Lambda^n_i & \xrightarrow{f_i} & K \\ \downarrow & \nearrow_{f} & \\ \Delta^n & & \end{array}$$

In particular, we will show that f and f_i coincide on the restrictions to all the faces of Λ^n_i; i.e. for all $j \neq i$, $0 \leq j \leq n$,

$$f|_{\{0,\dots,j-1,j+1,\dots,n\}} = f_i|_{\{0,\dots,j-1,j+1,\dots,n\}}. \tag{4.1}$$

It suffices to show that $f|_{\{k,k'\}} = f_i|_{\{k,k'\}}$ for each adjacent pair $\{k, k'\}$ of $\{0, \dots, j-1, j+1, \dots, n\}$. If $k' = k+1$ then, by construction of f the restrictions coincide. It follows that for $j = 0$ and $j = n$ the restrictions coincide. In particular

$$f|_{\{1,2,\dots,n\}} = f_i|_{\{1,2,\dots,n\}} \text{ and } f|_{\{0,\dots,n-2,n-1\}} = f_i|_{\{0,\dots,n-2,n-1\}}. \tag{4.2}$$

If $0 < j < n$, and $k = j-1$, $k' = j+1$ we are dealing with the case $n > 2$ (otherwise $j = 1 = i$, a contradiction). Then either $k > 0$ or $k' < n$. Suppose $k > 0$. It follows that $\{k, k'\} \subset \{1, 2, \dots, n\}$, and by (4.2) $f|_{\{k,k'\}} = f_i|_{\{k,k'\}}$. The case $k' < n$ is similar, and f renders the above diagram commutative. Now, suppose we have $f' : \Delta^n \to K$ such that $f'|_{\Lambda^n_i} = f_i$. Then f', as an element of $\mathcal{N}(\mathcal{C})_n$, corresponds to the chain of composable morphisms $X_0 \to \dots \to X_n$ and so $f' = f$. Thus the map f is unique K satisfies (2).

($\Leftarrow$) : Suppose K satisfies (2). We will show that K is isomorphic to the nerve of the category $\mathcal{C}$ which we define as follows:

- The objects of $\mathcal{C}$ are the 0-simplices of K, that is $\mathrm{ob}(\mathcal{C}) := K_0$
- For any pair of objects $x_0, x_1 \in \mathrm{ob}(\mathcal{C})$ we define $Hom_{\mathcal{C}}(x_0, x_1)$ to be the set of all 1-simplices, $y \in K$ such that $d_0 y = x_0$ and $d_1 y = x_1$.

Composition will follow from condition (2). Suppose we have two morphisms in $\mathcal{C}$, $x_0 \to x_1$, $x_1 \to x_2$. These correspond to 1-simplices of K, $y, z \in K_1$ with $d_0 y = x_0, d_1 y = x_1$ and $d_0 z = x_1, d_1 z = x_2$. Looking at this collection of 1-simplices $\{y, -, z\}$, since $d_0 z = x_1 = d_1 y$, by (2) there exists a unique 2-simplex w such that $d_0 w = y$ and $d_2 w = z$. Now $d_1 w$ provides a morphism from x_0 to x_2, and is the composition $z \circ y$. Let us check that this composition is associative. For $x_i \in \mathrm{ob}(\mathcal{C})$, consider the morphisms

$$f : x_0 \to x_1,\ g : x_1 \to x_2,\ h : x_2 \to x_3.$$

Let $d_1 t$ be the composition $g \circ f$ and let $d_1 u$ be the composition $h \circ g$. Let $d_1 v$ be the composition $h \circ (d_1 t) = h \circ (g \circ f)$. Now $d_0 v = d_1 t$, $d_0 u = d_2 t$ and $d_2 u = d_2 v$ so looking at the three 2-simplices

$$t, -, v, u$$

we find a unique 3-simplex τ such that $d_0\tau = t, d_2\tau = v$ and $d_3\tau = u$. Moreover $d_1(d_1\tau)$ is the composition $(d_1 u) \circ f = (h \circ g) \circ f$. Now

$$(h \circ g) \circ f = d_1 d_1 \tau = d_1 d_2 \tau = d_1 v = h \circ (g \circ f)$$

and so the composition is associative. Finally, for any $x \in \mathrm{ob}(\mathcal{C})$ the identity morphism is given by the 1-simplex $s_0(x)$. If follows that $\mathcal{C}$ is a well-defined category and we have a simplicial morphism

$$K \to \mathcal{N}(\mathcal{C}).$$

In degrees 0 and 1 this is an isomorphism by construction. Using the fact that K and $\mathcal{N}(\mathcal{C})$ both satisfy (2) and proceeding by induction it is straightforward to check that K is isomorphic to $\mathcal{N}(\mathcal{C})$. □

These results bring us to the condition required of a simplicial set to be a good model for an ∞-category. We need something "inbetween" the Kan condition and condition (2) of the proposition above.

Definition 4.0.5. A simplicial set K satisfies the *weak Kan condition* if for each $0 < i < n$, any simplicial map $f_i : \Lambda^n_i \to K$ extends to a simplicial map $\Delta^n \to K$.

Here we have dropped the unicity requirement in the proposition, and we have dropped the extension condition for *outer horms* Λ^n_n, Λ^n_0 in our definition of the Kan condition. We can define the weak Kan condition combinatorially as follows

Definition 4.0.6. A simplicial set K satisfies the weak Kan condition if for every collection of n $(n-1)$-simplices

$$x_0, \ldots, x_{k-1}, -, x_{k+1}, \ldots, x_n$$

with $k \neq 0, k \neq n$ satisfying the compatibility condition there exists an n-simplex y such that $d_i y = x_i$ for all $i \neq k$.

Definition 4.0.7. An ∞-category is a simplicial set K which satisfies the weak Kan condition.

From this point on we reserve the notation ∞-category for a simplicial set satisfying the weak Kan condition.

Example 10. A Kan complex is an ∞-category. For any topological space X, we saw that the singular complex $\mathsf{Sing}(X)$ is a Kan complex (Example 3), thus we can associate an ∞-category to our topolgical space via the Sing functor. Moreover, by the discussion following Theorem 2.3.2, $\mathsf{Sing}(\mathrm{X})$ determines X up to weak homotopy equivalence. Consequently the theory of ∞-categories can be seen as a generalisation of classical homotopy theory (via the simplicial homotopy theory of Chapter 2).

Example 11. The nerve of any category $\mathcal{C}$ is an infinity category. Moreover it determines $\mathcal{C}$ up to isomorphism. In this way an ∞-category is a kind of generalisation of an ordinary category. We will see later (in Section 4.2) that we can talk about objects and morphisms in ∞-categories. In fact, many of the basic ideas of category theory (limits, adjunctions, ...) have generalisations in the theory of ∞-categories. See, for example, [9, Chapters 4 and 5]

4.1 Topological Categories, Simplicial Categories and ∞-Categories

Following the approach of Lurie, we will show the equivalence of the two definitions discussed in the previous section by means of simplicial categories.

Definition 4.1.1. A *simplicial category* is a category enriched over Set_Δ, the category of simplicial sets.

A simplicial category is a simplicial object in the category Cat with a constant simplicial set of objects. Converslеy, a simplicial object of Cat forms a simplicial category if the underlying simplicial set is constant. We also remark that any ordinary category can be considered as a simplicial category by taking each of the simplicial sets $\mathrm{Hom}(x, y)$ to be constant.

Let us now recall the adjunction from Chapter 2 between the category of simplicial sets and the category of topological spaces,

$$\mathsf{Set}_\Delta \underset{\mathsf{Sing}}{\overset{||}{\rightleftarrows}} \mathsf{Top}\ .$$

We can in a similar way define adjoint functors between the category of simplicial categories Cat_Δ and the category of topological categories $\mathsf{Cat}_{\mathsf{Top}}$.

Given a simplicial category $\mathcal{C}$ we define a topological category $|\mathcal{C}|$ as follows:
- Objects of $|\mathcal{C}|$ are the objects of $\mathcal{C}$,
- $\mathrm{Map}_{|\mathcal{C}|}(x, y) = |\mathrm{Map}_{\mathcal{C}}(x, y)|$, for $x, y \in \mathcal{C}$,
- Composition in $|\mathcal{C}|$ comes from the composition in $\mathcal{C}$ by applying the geometric realisation functor $| - |$.

Similarly, by applying the singular complex functor Sing to each of the morphism spaces of a topological category $\mathcal{D}$, we obtain a simplicial category $\mathsf{Sing}(\mathcal{D})$. These constructions determine the adjunction between Cat_Δ and $\mathsf{Cat}_{\mathsf{Top}}$. The adjunction $(\mathsf{Sing}, | - |)$ of Chapter 2 provided an equivalence between the homotopy category of simplicial sets and the homotopy category of topological spaces. We wish to formulate a similar statement for Cat_Δ and $\mathsf{Cat}_{\mathsf{Top}}$.
Before we proceed we must define the relevant homotopy categories and introduce a notion of equivalence in our higher categorical setting.

Consider the topological category CW whose objects are CW-complexes, and with morphism spaces given by the set of continuous maps with the compactly generated compact-open topology, $\mathrm{Map}(x, y) = \mathrm{Hom}_{\mathsf{Top}}(x, y)$.

Definition 4.1.2. We define the *homotopy category of spaces*, $\mathcal{H}$, to be the following category:
- Objects of $\mathcal{H}$ are the objects of CW.
- $\mathrm{Hom}_{\mathcal{H}}(x, y) = \pi_0 \mathrm{Map}_{\mathsf{CW}}(x, y)$, for $x, y \in \mathsf{CW}$.
- Composition comes from the composition in CW and applying π_0.

By the CW approximation theorem, for any $X \in \mathsf{Top}$ there exists a weak homotopy equivalence $Y \to X$ where Y is a CW-complex. This Y is unique *up to homotopy equivalence* and so

$$X \mapsto [X] = Y$$

determines a functor

$$\Psi : \mathsf{Top} \to \mathcal{H}.$$

Clearly Ψ sends weak homotopy equivalences to isomorphisms in $\mathcal{H}$. We can describe $\mathcal{H}$ as the category obtained from Top by formally inverting all the weak homotopy equivalences, $\mathrm{Ho}(\mathsf{Top})$. Recalling (from Chapter 2.3) that the maps

$$S \to \mathsf{Sing}|S|,$$
$$|\mathsf{Sing}X| \to X$$

are weak homotopy equivalences it follows that $\mathcal{H}$ is equivalent to $\mathrm{Ho}(\mathsf{Set}_\Delta)$, the category obtained from Set_Δ by formally inverting weak homotopy equivalences. We shall use $\mathcal{H}$ to denote each of these equivalent categories.

Definition 4.1.3. Let $\mathcal{C}$ be a topological category. Then the homotopy category $\mathrm{h}\mathcal{C}$ is a $\mathcal{H}$-enriched category obtained by applying the functor $\mathsf{Top} \to \mathcal{H}$ to the mapping spaces $\mathrm{Map}_{\mathcal{C}}(X, Y)$ of $\mathcal{C}$.

Definition 4.1.4. Let $\mathcal{C}$ be a simplicial category. Then the homotopy category $\mathrm{h}\mathcal{C}$ is a $\mathcal{H}$-enriched category. It is obtained by applying the functor $\mathsf{Set}_\Delta \to \mathcal{H}$ to the morphism spaces $\mathrm{Map}_{\mathcal{C}}(x, y)$ of $\mathcal{C}$.

For a simplicial category $\mathcal{C}$ and a topological category $\mathcal{D}$ we have isomorphisms

$$\mathrm{h}\mathcal{C} \cong \mathrm{h}|\mathcal{C}|$$
$$\mathrm{h}\mathcal{D} \cong \mathrm{h}\mathsf{Sing}(\mathcal{D}).$$

Definition 4.1.5. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between topological (resp. simplicial) categories. Then F is an equivalence of topological (resp. simplicial) categories if the induced functor $\mathrm{h}\mathcal{C} \to \mathrm{h}\mathcal{D}$ is an equivalence of $\mathcal{H}$-enriched categories.

It follows, that for $\mathcal{C}, \mathcal{C}'$ simplicial categories, a functor $\mathcal{C} \to \mathcal{C}'$ is an equivalence if and only if the geometric realisation $|\mathcal{C}| \to |\mathcal{C}'|$ is an equivalence of topological categories. In particular, the maps,

$$\mathcal{C} \to \mathsf{Sing}|\mathcal{C}|$$
$$|\mathsf{Sing}\mathcal{D}| \to \mathcal{D}$$

induce an equivalence of homotopy categories. This gives us the desired equivalence between simplicial categories and topological categories. In other words, working up to equivalence, we can replace a simplicial category $\mathcal{C}$ with its geometric realisation $|\mathcal{C}|$, or a topological category $\mathcal{D}$ with the simplicial category $\mathsf{Sing}\mathcal{D}$.

Let us now compare our construction of ∞-categories (as simplicial sets satisfying the weak Kan condition) with simplicial categories. For this purpose we introduce the simplicial nerve structure. Recall that the nerve of an ordinary category was characterised by

$$\mathrm{Hom}_{\mathsf{Set}_\Delta}(\Delta^n, \mathcal{N}(\mathcal{C})) = \mathrm{Hom}_{\mathsf{Cat}}([n], \mathcal{C}).$$

Let $\mathcal{C}$ be a simplicial category. We will replace $[n]$, in the characterisation of the ordinary nerve, with a suitable *simplicial* category in order to capture the simplicial structure of $\mathcal{C}$.

Definition 4.1.6. Let J be a finite nonempty linearly ordered set, viewed as a category. For $i, j \in J$ we define P_{ij} to be the partially ordered set consisting of all subsets $I \subseteq J$ such that $i, j \in I$ and for all $k \in I, i \leq k \leq j$. We consider P_{ij} as a category in the obvious way. We define the category $\mathfrak{S}[\Delta^J]$ as follows:

- The objects of $\mathfrak{S}[\Delta^J]$ are the elements of J.
- For $i, j \in J$ we set $\mathrm{Map}_{\mathfrak{S}[\Delta^J]}(i, j) = \begin{cases} \emptyset & \text{if } j < i \\ N(P_{ij}) & \text{if } i \leq j \end{cases}$.
- For $i \leq j \leq k$ the composition

$$\mathrm{Map}_{\mathfrak{S}[\Delta^J]}(i, j) \times \mathrm{Map}_{\mathfrak{S}[\Delta^J]}(j, k) \to \mathrm{Map}_{\mathfrak{S}[\Delta^J]}(i, k)$$

is induced by the map

$$\begin{aligned} P_{ij} \times P_{jk} &\to P_{ik} \\ (I_1, I_2) &\mapsto I_1 \cup I_2. \end{aligned}$$

We define $\mathfrak{S}[n]$ to be the simplicial category $\mathfrak{S}[\Delta^J]$ when $J = [n]$. We have a functor

$$\mathfrak{S} : \Delta \to \mathsf{Cat}_\Delta$$

The simplicial category $\mathfrak{S}[n]$ is what we will use to replace $[n]$. Remark that they share the same collection of objects.

Definition 4.1.7. Let $\mathcal{C}$ be a simplicial category. The *simplicial nerve* of $\mathcal{C}$ is the simplicial set $\mathcal{N}(C)$ characterised by

$$\mathrm{Hom}_{\mathsf{Set}_\Delta}(\Delta^n, \mathcal{N}(\mathcal{C})) = \mathrm{Hom}_{\mathsf{Cat}_\Delta}(\mathfrak{S}[n], \mathcal{C}).$$

It is the simplicial set where $\mathcal{N}(C)_n$ is the set of all simplicial functors $\mathfrak{S}[n] \to \mathcal{C}$.

Definition 4.1.8. Let $\mathcal{C}$ be a topological category. The *topological nerve* of $\mathcal{C}$ is the simplicial nerve of $\mathsf{Sing}(\mathcal{C})$.

Note that the simplicial nerve of a simplicial category $\mathcal{C}$ does not coincide with the ordinary nerve of $\mathcal{C}$ (viewed as an ordinary category by "forgetting" all the simplices of positive dimension). Similarly the topological nerve of a topological category does not coincide with the ordinary nerve of the underlying category. From this point on, if $\mathcal{C}$ is a simplicial (resp. topological) category, then $\mathcal{N}(\mathcal{C})$ will denote the simplicial (resp. topological) nerve of $\mathcal{C}$, unless stated otherwise.

Proposition 4.1.1. *Let $\mathcal{C}$ be a simplicial category such that the simplicial set $Map_{\mathcal{C}}(X,Y)$ is a Kan complex for every pair of objects $X, Y \in \mathcal{C}$. Then the simplicial nerve $\mathcal{N}(\mathcal{C})$ is an ∞-category.*

Proof. For details, see [9, §1.1.5]. By the characterisation of the simplicial nerve functor, a morphism $F : \Delta^n \to \mathcal{N}(\mathcal{C})$ corresponds to a simplicial functor $\mathfrak{S}[n] \to \mathcal{C}$. The proof then follows from observing that the objects of $\mathfrak{S}[\Lambda^n_i]$ are the objects of $\mathfrak{S}[\Delta^n] = \mathfrak{S}[n]$ and the simplicial set $\mathrm{Map}_{\mathfrak{S}[\Lambda^n_i]}(x,y)$ coincides with $\mathrm{Map}_{\mathfrak{S}[n]}(x,y)$ for all $(x,y) \neq (0,n)$. Thus one only has to check that the simplicial map $\mathrm{Map}_{\mathfrak{S}[\Lambda^n_i]}(x,y) \to \mathrm{Map}_{\mathcal{C}}(F(0),F(n))$ extends to a map $\mathrm{Map}_{\mathfrak{S}[\Delta^n]}(x,y) \to \mathrm{Map}_{\mathcal{C}}(F(0),F(n))$ □

Recalling that the singular complex of any topological space is a Kan complex we get the following corollary:

Corollary 4.1.1. *Let $\mathcal{C}$ be a topological category. The topological nerve $\mathcal{N}(\mathcal{C})$ is an ∞-category.*

Since Cat_Δ admits colimits, we can obtain an identification between functors $\Delta \to \mathsf{Cat}_\Delta$ and functors $\mathsf{Set}_\Delta \to \mathsf{Cat}_\Delta$ which preserve colimits. We can thus extend the functor $\mathfrak{S} : \Delta \to \mathsf{Cat}_\Delta$ to a functor $\mathfrak{S} : \mathsf{Set}_\Delta \to \mathsf{Cat}_\Delta$. For a simplicial set S, we can think of $\mathfrak{S}(S)$ as a "free" simplicial category generated by S. The n-simplices of S correspond to simplicial functors $\mathfrak{S}[n] \to \mathfrak{S}(S)$. In particular the objects of the simplicial category $\mathfrak{S}(S)$ correspond to the 0-simplices of S. For a detailed discussion on the the simplicial category $\mathfrak{S}(S)$ see [11]. This "extended" $\mathfrak{S}$ is left adjoint to the simplicial nerve functor. Similarly, composing with the geometric realisation (of simplicial categories) we have that $|\mathfrak{S}|$ is left adjoint to the topological nerve functor. Let us denote by T the functor $|\mathfrak{S}|$.

Definition 4.1.9. Let $\mathcal{C}$ be an ∞-category (or more generally a simplicial set). The homotopy category $\mathrm{h}\mathcal{C}$ is the homotopy category $\mathrm{h}\mathfrak{S}[\mathcal{C}]$ of the simplicial category $\mathfrak{S}[\mathcal{C}]$.

We can view the homotopy category to be a category enriched over $\mathcal{H}$ as we did for the homotopy category of simplicial categories.

Definition 4.1.10. A map of simplicial sets $\mathcal{C} \to \mathcal{C}'$ is a *categorical equivalence* if the induced map of homotopy categories $\mathrm{h}\mathcal{C} \to \mathrm{h}\mathcal{C}'$ is an equivalence of $\mathcal{H}$-enriched categories.

Clearly, a map of ∞-categories $\mathcal{C} \to \mathcal{C}'$ is a categorical equivalence if and only if $\mathfrak{S}[\mathcal{C}] \to \mathfrak{S}[\mathcal{C}']$ is an equivalence of simplicial categories, and thus if and only if $T(\mathcal{C}) \to T(\mathcal{C}')$ is an equivalence of topological categories.

Theorem 4.1.1. *Let $\mathcal{C}$ be a topological category. Let X, Y be objects of $\mathcal{C}$. Then the map*

$$|Map_{\mathfrak{S}[\mathcal{N}(\mathcal{C})]}(X,Y)| \to Map_{\mathcal{C}}(X,Y)$$

is a weak homotopy equivalence of topological spaces.

Proof. See [9, §2.2.4]. □

Theorem 4.1.2. *Let $h\mathsf{Cat}_{\mathsf{Top}}$ denote the category of topological categories with morphisms up to equivalence. Let $h\mathsf{Set}_{\Delta}$ denote the category of simplicial sets with morphisms up to categorical equivalence. The adjoint functors $(T, \mathcal{N})$ provide an equivalence between the these two categories. In particular, we have an equivalence between $h\mathsf{Cat}_{\mathsf{Top}}$ and the $h\mathsf{Cat}_{\infty}$, the category of ∞-categories up to categorical equivalence.*

Proof. We note that a map of simplicial sets $\mathcal{C} \to \mathcal{C}'$ is a categorical equivalence if and only if $T(\mathcal{C}) \to T(\mathcal{C}')$ is an equivalence. Similarly $\mathcal{N}$ takes equivalences to categorical equivalences. Let $\mathcal{D}$ be a topological category. By the preceding theorem, the counit map

$$T\mathcal{N}(\mathcal{D}) \to \mathcal{D}$$

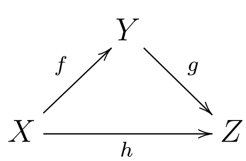
is an equivalence of topological categories. For any simplicial set S, the unit map of simplicial sets

$$S \to \mathcal{N}T(S)$$

is a categorical equivalence if and only if $T(S) \to T\mathcal{N}T(S)$ is an equivalence of topological categories and, by Theorem 4.1.1, this is the case. Hence, we have an equivalence between the theory of simplicial sets up to categorical equivalence and that of topological categories up to equivalence.
Finally, for every simplicial set S, the map $\mathcal{N}T(S) \to S$ is a categorical equivalence since, again by Theorem 4.1.1, $T\mathcal{N}T(S) \to T(S)$ is an equivalence of topological categories. By Corollary 4.1.1 the nerve of a topological category is an ∞-category, and every simplicial set is categorically equivalent to an ∞-category. Thus we have an equivalence between $h\mathsf{Cat}_{\mathsf{Top}}$ and $h\mathsf{Cat}_{\infty}$. □

4.2 Homotopy Category of an ∞-Category

Recalling that the nerve of any ordinary category is an ∞-category it is possible to see the theory of ∞-categories as a generalisation of standard category theory. We can consider objects and morphisms in an ∞-category. In particular the objects of an ∞-category are the 0-simplices. The morphisms are given by the 1-simplices: a 1-simplex f corresponds to a morphism from $d_0 f = X$ to $d_1 f = Y$. A 2-simplex can be viewed as a diagram:



which captures the commutativity $g \circ f \cong h$. It is the 2-simplex arising from the set of compatible simplices $\{g, -, f\}$.
In fact, many of the important concepts in category theory have their counterparts in the theory of ∞-categories. Let us look now more closely at the homotopy category of an ∞-category.

Proposition 4.2.1. *The nerve functor $\mathcal{N} : \mathsf{Cat} \to \mathsf{Set}_{\Delta}$ is right adjoint to the functor $h : \mathsf{Set}_{\Delta} \to \mathsf{Cat}$ which associates to each simplicial set K its homotopy category hK. Moreover $h\mathcal{N}(\mathcal{C}) \cong \mathcal{C}$ for any $\mathcal{C} \in \mathsf{Cat}$.*

Proof. See [9, Prop. 1.2.3.1]. □

We recall the definition of homotopic simplices from Chapter 2:

Definition 4.2.1. Let K be a simplicial set. Two n-simplices, $x, x' \in K_n$, are homotopic if

$$d_j x = d_j x' \quad \forall\ 0 \leq j \leq n$$

and there exists $y \in K_{n+1}$ such that

$$d_n y = x,\ d_{n+1} y = x', \text{ and } d_i y = s_{n-1} d_i x = s_{n-1} d_i x, \text{ for } 0 \leq i < n.$$

We write $x \sim x'$ and we call the $n+1$-simplex y a homotopy from x to x'.

We saw that this defined an equivalence relation on Kan complexes and we defined the homotopy groups $\pi_i(K, k_0)$ for a Kan complex K, with respect to a "basepoint" $k_0 \in K_0$. We also saw that, in degree 0, we had $\pi_0(K) = \frac{K_0}{\sim}$.
Let us consider an ∞-category $\mathcal{C}$ and construct a category $\pi(\mathcal{C})$. The objects of $\pi(\mathcal{C})$ are the elements of K_0. Given ϕ, a 1-simplex of $\mathcal{C}$ we write $\phi : C \to C'$ where $C = d_1\phi$ and $C' = d_0\phi$. For each $C \in \mathcal{C}$ we take $s_0 C : C \to C$ to be id_C.
Given two arrows $\phi, \phi' : C \to C'$ then, by the preceding definition (with $n = 1$), ϕ and ϕ' are homotopic if there exists a 2-simplex σ such that $d_1\sigma = \phi, d_2\sigma = \phi$; and $d_0\sigma = id_{C'}$.

Lemma 4.2.1. *Let $\mathcal{C}$ be an ∞-category. Let C and C' be objects of $\pi(\mathcal{C})$. Then the homotopy relation $\sim$ is an equivalence relation on the arrows joining C to C'.*

Proof. A close look at the proof of Lemma (2.2.1) reveals that only the hypothesis of the *weak* Kan condition is needed. In other words, If K is an ∞-category, the homotopy relation $\sim$ is an equivalence relation on K_n, for $n \geq 0$. In particular, taking $n = 1$ we get the desired result. □

So we will take $\pi(\mathcal{C})$ to be $\pi_0(\mathcal{C})$, defined now for a simplicial set satisfying the *weak* Kan condition. For $C, C' \in \mathcal{C}$ we define $\mathrm{Hom}_{\pi(\mathcal{C})}(C, C')$ to be the set of homotopy classes of arrows $\phi : C \to C'$, and we let $[\phi]$ denote the morphism $C \to C'$ in $\pi(\mathcal{C})$. For each $C \in \mathcal{C}$, the identity morphism id_C is given by $[s_0 C]$. We define a composition law on $\pi(\mathcal{C})$ similar to the way we did for the homotopy groups: Suppose C, C', C'' are objects in $\mathcal{C}$, and suppose we have $\phi : C \to C'$ and $\phi' : C' \to C''$ then the existence of a 2-simplex σ follows from the fact that $\mathcal{C}$ is an ∞-category and hence satisfies the weak Kan condition. We define $[\phi'] \circ [\phi] = [d_1\sigma]$. It follows from looking at the proof of Proposition (2.2.1), which only makes use of the weak Kan condition, that this composition law is well-defined.

Proposition 4.2.2. *If $\mathcal{C}$ is an ∞-category, then $\pi(\mathcal{C})$ is a category.*

The proof follows similar procedures to that of Proposition 2.2.1 and so we omit some details.

Proof. We must verify that id_C is an identity with respect to the composition law, and that the composition is associative. It is straightforward to check that $[s_0 C] \circ [\phi] = [d_1 s_1 \phi] = [\phi]$ for any $\phi : C' \to C$. Hence $\mathrm{id}_C = [s_0 C]$ is a left identity. A similar argument shows that id_C is also a right identity.
Suppose we have $[\phi] : C \to C'$, $[\phi'] : C' \to C''$ and $[\phi''] : C'' \to C'''$. Let $[\psi] = [\phi'] \circ [\phi]$ and

$[\psi'] = [\phi''] \circ [\phi']$. Following again the proof of Proposition (2.2.1), and noting once more that it only makes use of the weak Kan condition, we find a 3-simplex σ such that

$$[d_1 d_2 \sigma] = [\phi''] \circ [\psi]$$
$$[d_1 d_1 \sigma] = [\psi'] \circ [\phi].$$

So

$$[\phi''] \circ ([\phi'] \circ [\phi]) = [\phi''] \circ [\psi] = [\psi'] \circ [\phi] = ([\phi''] \circ [\phi']) \circ [\phi]$$

and the composition law is associative. □

Finally one can show that $\pi(\mathcal{C})$ is naturally isomorphic to the homotopy category $h\mathcal{C}$.

Proposition 4.2.3. *Let $\mathcal{C}$ be an ∞-category. Then there exists an isomorphism of categories $F : h\mathcal{C} \to \pi(C)$.*

Proof. See [9, Prop. 1.2.3.9]. □

Given two morphisms f, g in an ∞-category (i.e. two 1-simplices), then by the previous results they are homotopic if they determine the same morphism in the homotopy category $h\mathcal{C}$. We say that a morphism f is an equivalence if it determines an isomorphism in the homotopy category.

In the introduction to this chapter we saw the notion of ∞-groupoids where all morphisms are invertible. In the language of ∞-categories (as simplicial sets) an ∞-category $\mathcal{C}$ is an ∞-groupoid if all the morphisms in $\mathcal{C}$ are equivalences. We arrive at the following precise definition.

Definition 4.2.2. An ∞-category $\mathcal{C}$ is called an ∞-groupoid if the homotopy category $h\mathcal{C}$ is a groupoid.

The weak Kan condition of ∞-categories (Definition 4.0.7) required the horn extension property for inner horns only. The following result, due to Joyal, [7], gives us a characterisation of equivalences in terms of the outer horns.

Proposition 4.2.4. *Let $\mathcal{C}$ be an ∞-category. A morphism f of $\mathcal{C}$ is an equivalence if and only if every map $f_0 : \Lambda^n_0 \to \mathcal{C}$, with $f_0|_{\{0,1\}} = f$, admits an extension $\Delta^n \to \mathcal{C}$ for all $n \geq 2$.*

Equivalently we have the analogous statement for the Λ^n_n. From this it follows that all ∞-groupoids are given by Kan complexes, and consequently, by the results of Section 2.3, we have the assertion that all ∞-groupoids are given by topological spaces. This is in keeping with our original discussion of ∞-groupoids in the introduction to this chapter.

Corollary 4.2.1. *An ∞-category $\mathcal{C}$ is an ∞-groupoid if and only if $\mathcal{C}$ is a Kan complex.*

5

Cohomological Descent

In this Chapter, we follow Conrad's expository article on the subject,[3], which is based largely on Saint-Donat's *Techniques de descente cohomologique*, [13]. We begin by introducing the coskeleton functor (Theorem 5.1.1), which provides a means of "building up" a simplicial object. Then we will use these simplicial methods to construct suitable hypercovers of a topological space. Finally, we look at the mechanism of cohomological descent and how this allows us to the compute the cohomology of a space X by means of certain hypercovers, (Theorem 5.2.2).

5.1 Coskeleta and Hypercovers

Recall Δ the category whose objects $[n]$ are sequences of integers for $n \geq 0$ and whose morphisms are non-decreasing maps. We define Δ^+ to be the category whose objects are the $[n] \in \Delta$ along with $[-1] = \emptyset$. For $[n], [m] \in \Delta^+$, $\mathrm{Hom}_{\Delta^+}([n], [m])$ is the set of non-decreasing maps $[n] \to [m]$. We have that Δ is the full subcategory of Δ^+ with objects $[n], n \geq 0$. Note that $[-1]$ is an initial object in Δ^+, while Δ does not admit an initial object.

Definition 5.1.1. An *augmented simplicial object* in a category $\mathcal{C}$ is a contravariant functor $\Delta^+ \to \mathcal{C}$ where Δ^+ is the category defined above.

The category of augmented simplicial objects in $\mathcal{C}$ is denoted $\mathrm{Simp}^+(\mathcal{C})$.

We can thing of an augmented simplicial object as an "extension" of an ordinary simplicial object. To give an augmented simplicial object amounts to giving a simplicial object $X_\bullet$ and a map $X_0 \to X_{-1}$ such that all the composite maps

$$X_n \to X_{-1}$$

coincide. We denote an augmented simplicial object by $X_\bullet \to S$, or sometimes $X_\bullet/S$, with S the degree -1 part, and $X_\bullet$ the "ordinary" simplicial part. However, for simplicity of notation, when there is little chance of ambiguity we will denote the entire augmented simplicial object simply by $X_\bullet$.

Definition 5.1.2. An n-truncated simplicial object in $\mathcal{C}$ is a contravariant functor $\Delta_{\leq n} \to \mathcal{C}$, where $\Delta_{\leq n}$ is the full subcategory of Δ whose objects are given by $[k]$ for $k \leq n$.

The n-truncated simplicial objects in $\mathcal{C}$ form a category $\mathrm{Simp}_n(\mathcal{C})$. We can also consider $\mathrm{Simp}_n^+(\mathcal{C})$, the category of augmented n-truncated objects in $\mathcal{C}$. In the augmented case we restrict to $\Delta_{\leq n}^+$, the category whose objects are the $[k] \in \Delta_{\leq n}$ along with $[-1] = \emptyset$.

Given $X_\bullet : \Delta \to \mathcal{C}$ in $\mathrm{Simp}(\mathcal{C})$ we can "restrict" it to $\Delta_{\leq n}$ to obtain a n-truncated simplicial object. In this way we obtain a functor

$$\mathrm{sk}_n : \mathrm{Simp}(\mathcal{C}) \to \mathrm{Simp}_n(\mathcal{C}).$$

We can think of $\mathrm{sk}_n(X_\bullet)$ as the first n degrees of $X_\bullet$ along with the necessary face and degeneracy maps. We call sk_n the *skeleton functor* or *truncation functor* and we call $\mathrm{sk}_n(X_\bullet)$ the *n-skeleton of* $X_\bullet$. Similarly, we define the skeleton functor for the augmented case and we denote it

$$\mathrm{sk}_n^+ : \mathrm{Simp}^+(\mathcal{C}) \to \mathrm{Simp}_n^+(\mathcal{C}).$$

An interesting question at this point is, given an n-truncated simplicial object, can we "build up" an ordinary simplicial object?

Theorem 5.1.1. *Let $\mathcal{C}$ be a category admitting finite products and finite fibre products. Then the truncation functor $sk_n : Simp(\mathcal{C}) \to Simp_n(\mathcal{C})$ admits a right adjoint $cosk_n$ for all $n \geq 0$. Similarly sk_n^+ admits a right adjoint $cosk_n^+$ for all $n \geq -1$.*

Before proving this theorem for all n let us consider the $n = 0$ case. We wish to construct a right adjoint, cosk_0, to the 0-truncation functor

$$\mathrm{sk}_0 : \mathrm{Simp}(\mathcal{C}) \to \mathcal{C}.$$

That is, given an object $Y_0 \in \mathcal{C}$ we are looking for a pair $(X_\bullet, \phi_0)$, where $\mathrm{sk}_0(X_\bullet) = X_0$ and $\phi_0 : X_0 \to Y_0$ such that for any $X'_\bullet \in \mathrm{Simp}(\mathcal{C})$ with $\phi'_0 : X'_0 \to Y_0$ there exists a simplicial map $f : X'_\bullet \to X_\bullet$ such that $\phi_0 \circ f_0 = \phi'_0$.
Given $\phi'_0 : X'_0 \to Y_0$ we have

$$X'_1 \underset{d_0}{\overset{d_1}{\rightrightarrows}} X'_0 \longrightarrow Y_0$$

giving us a map $f'_1 : X'_1 \to Y_0 \times Y_0$. We also have compatibility with the face and degeneracy maps between X'_0 and X'_1 in the sense that the following diagrams commute:

$$\begin{array}{ccc} X'_1 & \longrightarrow & Y_0 \times Y_0 \\ {\scriptstyle d_1}\downarrow\;\downarrow{\scriptstyle d_0} & & {\scriptstyle p_1}\downarrow\;\downarrow{\scriptstyle p_0} \\ X'_0 & \longrightarrow & Y_0 \end{array}$$

where p_1 and p_2 are the obvious projections and the d_i are the degeneracy maps.

$$\begin{array}{ccc} X'_1 & \longrightarrow & Y_0 \times Y_0 \\ {\scriptstyle s_0}\uparrow & & \uparrow \\ X'_0 & \longrightarrow & Y_0 \end{array}$$

where s_0 is the face map, and the vertical map $Y_0 \to Y_0 \times Y_0$ the diagonal $y \mapsto (y, y)$.
Similarly, we have a map $f_2' : X_2' \to Y_0 \times Y_0 \times Y_0$ coming from compostion with the degeneracy maps,

$$X_2' \overset{d_2}{\underset{d_0}{\rightrightarrows}} X_1' \overset{d_1}{\underset{d_0}{\rightrightarrows}} X_0' \longrightarrow Y_0 \ .$$

Again we have compatibility with the relevant face and degeneracy maps.
Continuing we find that we can take $\operatorname{cosk}_0(Y_0) = X_\bullet$ with $X_n = Y_0^{n+1}$ and $\phi_0 = id_{Y_0}$.

The augmented case is similar, replacing the products with fibre products over Y_{-1}.

Once we move to higher n this construction becomes increasingly difficult. Even trying to construct $\operatorname{cosk}_1(Y_0)$ explicitly is combinatorially tricky with many maps and commutative diagrams to keep track of. We construct cosk_n in the general case by using inverse limits.

Proof. Take a m-truncated simplicial object $Y = Y_{\leq m} \in \operatorname{Simp}_m(\mathcal{C})$. We will begin by constructing the simplicial object $\operatorname{cosk}_m(Y)$ degree by degree. Fix $n \geq 0$ and let $\Delta([n])$ denote the contravariant functor $\operatorname{Hom}_\Delta(-, [n])$. So $\Delta([n])$ is a simplicial set with $\Delta([n])_k = \operatorname{Hom}_\Delta([k], [n])$. We can view $\Delta([n])$ as a category with non-decreasing maps $\phi : [k] \to [n]$ as objects. A morphism between ϕ and $\phi' : [k'] \to [n]$ is given by a morphism $\alpha : [k] \to [k']$ in Δ such that $\phi' \circ \alpha = \phi$. Now $\operatorname{sk}_m\Delta([n])$ is a finite full subcategory of $\Delta([n])$ with objects the $\phi : [k] \to [n]$ with $k \leq m$. For each $\phi : [k] \to [n] \in \operatorname{sk}_m\Delta([n])$ we define $Y_\phi := Y_k$. A morphism $\alpha : \phi \to \phi'$ gives a map $Y(\alpha) : Y_\phi' \to Y_\phi$. The Y_ϕ thus define a contravariant functor $Y_* : \operatorname{sk}_m\Delta([n]) \to \mathcal{C}$. We take the finite inverse limit

$$\varprojlim_{\operatorname{sk}_m\Delta([n])} Y_\phi =: \tilde{Y_n}^{(m)}.$$

This will serve as the degree n part of $\operatorname{cosk}_m(Y)$.
Let us check that this does indeed define a a functor $\operatorname{Simp}_m(\mathcal{C}) \to \operatorname{Simp}(\mathcal{C})$. Firstly, we show that the construction above gives a simplicial object in $\mathcal{C}$. Take a map $\alpha : [n'] \to [n]$ in Δ. For each $\phi' : [k'] \to [n']$ in $\operatorname{sk}_m\Delta([n'])$, there exists a projection

$$\epsilon_{\phi'} : \tilde{Y_n}^{(m)} \to Y_{k'}$$

where $Y_{k'} = Y_{\alpha\circ\phi'}$. Taking the inverse limit over these ϕ' we find

$$\tilde{Y}^{(m)}(\alpha) : \tilde{Y_n}^{(m)} \to \tilde{Y_{n'}}^{(m)},$$

and we have a contravariant functor $\Delta \to \mathcal{C}$. We will denote this simplicial object $\tilde{Y}^{(m)}$.
Finally we check that this $\tilde{Y}^{(m)}$ serves as $\operatorname{cosk}_m(Y)$. The adjunction map $\operatorname{sk}_m\operatorname{cosk}_m \to 1$ gives the map

$$\operatorname{sk}_m\tilde{Y}^{(m)} \to Y.$$

But for $k' \leq m$, $\operatorname{sk}_m\Delta([k'])$ has an initial object $id_{k'} : [k'] \to [k']$ and so

$$\tilde{Y_{k'}}^{(m)} := \varprojlim_{\operatorname{sk}_m\Delta([n])} Y_\phi = Y_{\mathrm{id}_{k'}} = Y_{k'}.$$

The adjunction map is a natural isomorphism of $\mathrm{Simp}_m(\mathcal{C})$.
Now a simplicial map $X_\bullet \to \tilde{Y}^{(m)}$ corresponds to a collection of maps, $X_n \to \tilde{Y_n}^{(m)}$ for all n. Since $\tilde{Y_n}^{(m)}$ is an inverse limit this amounts to giving a map

$$X_n \to Y_\phi = Y_k$$

for all $\phi : [k] \to [n], k \le m$ such that the following diagram commutes:

$$\begin{array}{ccc} X_n & \xrightarrow{\epsilon_\phi} & Y_k \\ {\scriptstyle X(\phi)}\downarrow & & \| \\ X_k & \xrightarrow[\epsilon_{id_k}]{} & Y_k \end{array}$$

Thus defining $\epsilon_\phi = X(\phi)\epsilon_{id_k}$ gives rise to the bijection

$$\mathrm{Hom}(X_\bullet, \tilde{Y}^{(m)}) \longrightarrow \mathrm{Hom}\left(\mathrm{sk}_m(X_\bullet), Y\right),$$

and $\tilde{Y}^{(m)} = \mathrm{cosk}_m(Y)$.
It remains to consider the augmented case. Since $\mathcal{C}$ admits fibre products, we can consider simplicial objects augmented by a fixed S and restrict to the category $\mathcal{C}_{/S}$; the category with objects given by the morphisms $f \in \mathrm{Hom}_{\mathcal{C}}(X, S)$ for all $X \in \mathrm{ob}(\mathcal{C})$. Morphisms from $f : X \to S$ to $f' : X' \to S$ are given by $g \in \mathrm{Hom}_{\mathcal{C}}(X, X')$ such that $f' \circ g = f$. Replacing $\mathcal{C}$ with the category $\mathcal{C}_{/S}$ in the preceding argument, we see that we have only to treat the case $m = -1$. For $m = -1$ the right adjoint to $sk_{-1} : \mathrm{Simp}^+(\mathcal{C}) \to \mathcal{C}$ is given by the functor which, to each $X \in \mathcal{C}$ associates the constant augmented simplicial object $X_\bullet$ where $X_n = X$ for all $n \ge -1$ and the augmentation is simply the identity. □

Corollary 5.1.1. *The adjunction map*

$$sk_n cosk_n Y \to Y$$

is an isomorphism, for all $y \in Simp_n(\mathcal{C})$.

Corollary 5.1.2. *For $0 \le n \le m$ the natural map*

$$\rho_{m,n} : cosk_n \to cosk_m sk_m cosk_n$$

is an isomorphism.

Proof. Fix an n-truncated simplicial object $Y \in \mathrm{Simp}_n(\mathcal{C})$. For $X \in \mathrm{Simp}(\mathcal{C})$ we consider a map

$$f : X \to \mathrm{cosk}_m \mathrm{sk}_m \mathrm{cosk}_n Y.$$

By the Yoneda lemma, if we show that

$$\mathrm{Hom}(X, \mathrm{cosk}_n Y) \to \mathrm{Hom}(X, \mathrm{cosk}_m \mathrm{sk}_m \mathrm{cosk}_n Y)$$
$$g \mapsto \rho_{m,n} \circ g$$

is a bijection then we have that $\rho_{m,n}$ is an isomorphism. In particular, we need to show that there exists a unique $g : X \to \mathrm{cosk}_n Y$ such that $\rho_{m,n} \circ g = f$. Since we are checking equality between two maps mapping to m-coskeleta it suffices to check equality on the m-truncations. In other words, given

$$f' : \mathrm{sk}_m X \to \mathrm{sk}_m \mathrm{cosk}_m \mathrm{sk}_m \mathrm{cosk}_n Y \cong \mathrm{sk}_m \mathrm{cosk}_n Y$$

we need to show that there exists a unique $g : X \to \mathrm{cosk}_n Y$ such that $\mathrm{sk}_m(g) = f'$. Now g is a map to an n-coskeleton so it suffices to give its restriction $g' = g|_{\mathrm{sk}_n(X)} : \mathrm{sk}_n(X) \to \mathrm{sk}_n \mathrm{cosk}_n(Y) \cong Y$. Our only choice for this restriction, if we require $\mathrm{sk}_m(g) = f'$, is to take $g' = f'|_{\mathrm{sk}_n(X)}$. The desired result is then a consequence of the following lemma.

□

Lemma 5.1.1. *For $0 \leq n \leq m$ let h, h' be two maps*

$$h, h' : sk_m X \to sk_m cosk_n Y.$$

If h and h' coincide on n-skeleta, then they are equal.

Proof. If $m = n$ then there is nothing to show. Suppose we have equality on the $m-1$-skeleta. Then we only need to show that the maps coincide in degree m. In other words, we need to show that the two maps

$$h_m, h'_m : X_m \to (\mathrm{cosk}_n Y)_m$$

are equal. Now, by the construction of $\mathrm{cosk}_n Y$ we have

$$(\mathrm{cosk}_n Y)_m = \varprojlim_{\mathrm{sk}_n \Delta([m])} Y_\phi$$

where the limit is taken over $\phi : [k] \to [m]$ with $k \leq n$. By the universal property of the limit, it suffices to show that, for each Y_ϕ, the compositions of h_m and h'_m with the projections to Y_ϕ coincide. For $\phi : [k] \to [m]$, $k \leq n$ we have that $Y_\phi = Y_k \simeq (\mathrm{cosk}_n Y)_k$ and the projection map from $(\mathrm{cosk}_n Y)_m$ to Y_k is identified with

$$(\mathrm{cosk}_n Y)(\phi) : (\mathrm{cosk}_n Y)_m \to (cosk_n Y)_k.$$

Recall that a simplicial map between simplicial objects, $f_\bullet : K_\bullet \to L_\bullet$ is such that $L(\alpha) \circ f_m = f_k \circ K(\alpha)$, for all $\alpha : [k] \to [n]$ in Δ with $k \leq n$. We thus deduce that h_m and h'_m are equal if $h_k \circ X(\phi)$ and $h'_k \circ X(\phi)$ coincide. Since $k \leq n$ and $\mathrm{sk}_n(h) = \mathrm{sk}_n(h')$ it follows that $h = h'$. If $m > n$ for a fixed n, then proceeding by induction on m gives the result. □

We are now able to introduce the notion of hypercovers. Throughout we assume that the category $\mathcal{C}$ admits finite products and finite inverse products (so we can construct coskeleta). $\mathbf{P}$ will denote a class of morphisms in $\mathcal{C}$ which is stable under base change, preserved under composition and contains all isomorphisms. For example, we may consider the case $\mathcal{C} = \mathsf{Top}$ and take $\mathbf{P}$ to be the class of proper maps.

Definition 5.1.3. A simplicial object $X_\bullet \in \mathrm{Simp}(\mathcal{C})$ is a $\mathbf{P}$-hypercovering if, for all $n \geq 0$ the adjunction map

$$X_\bullet \to \mathrm{cosk}_n \mathrm{sk}_n X_\bullet$$

induces a map

$$X_{n+1} \to (\mathrm{cosk}_n \mathrm{sk}_n X_\bullet)_{n+1}$$

which is in $\mathbf{P}$ and surjective.
In the augmented case, for $X_\bullet \in \mathrm{Simp}^+(\mathcal{C})$ we extend the above definition to include the case $n = -1$, and we say that $X_\bullet$ is a $\mathbf{P}$-hypercovering of X_{-1}.

When we take $\mathcal{C}$ to be a suitable category of spaces and $\mathbf{P}$ to be the class of proper maps we say that we have a *proper hypercovering*. Similarly we can consider *étale hypercoverings* or *fppf hypercoverings* where we take $\mathbf{P}$ to be the class of étale morphisms, or morphisms which are faithfully flat and (locally) of finite presentation.

Lemma 5.1.2. *Let X and S be topological spaces. Then the augmented simplicial object $cosk_0(X \to S)$ is a $\mathbf{P}$-hypercovering of S if and only if the augmentation map $X \to S$ is in $\mathbf{P}$.*

Proof. If $\mathrm{cosk}_0(X \to S)$ is a $\mathbf{P}$-hypercovering of S, then taking $n = -1$ and recalling that $\mathrm{cosk}_{-1}(S)$ is the constant simplicial space S, it follows from the definition that the augmentation map $X \to S$ is in $\mathbf{P}$.
Now suppose the augmentation map is in $\mathbf{P}$. Let $Y_\bullet = \mathrm{cosk}_0(X \to S)$. The natural adjunction map

$$Y_\bullet \to \mathrm{cosk}_n \mathrm{sk}_n(Y_\bullet)$$

is an isomorphism by Corollary (5.1.2). Thus for all $n \geq -1$ the induced maps

$$Y_{n+1} \to (\mathrm{cosk}_n \mathrm{sk}_n(Y_\bullet))_{n+1}$$

are isomorphisms and in $\mathbf{P}$. Hence $Y_\bullet = \mathrm{cosk}_0(X \to S)$ is a $\mathbf{P}$-hypercovering of S. ∎

Example 12. Let $\mathcal{C} = \mathsf{Top}$ and $\mathbf{P}$ be the class of proper surjective maps. Take $X \in \mathcal{C}$, $\{U_i\}_{i \in I}$ an open cover of X and let $U = \bigsqcup_{i \in I} U_i$. Then by the previous lemma $Y_\bullet = \mathrm{cosk}_0(U \to X)$ is a $\mathbf{P}$-hypercovering of X.

5.2 Cohomological Descent

Throughout this section "spaces" are topological spaces. In particular a simplicial space is a simplicial object in the category Top.
We begin by defining a sheaf of sets on a simplicial space $X_\bullet$.

Definition 5.2.1. Let $X_\bullet$ be a simplicial space. A *sheaf of sets on* $X_\bullet$, consists of a sheaf of sets $\mathcal{F}^n$ over each X_n, and for any $\phi : [n] \to [m]$ we have a morphism of sheaves

$$[\phi]_{\mathcal{F}} : X(\phi)^*(\mathcal{F}^n) \to \mathcal{F}^m$$

with $X(\phi)^*$ the pullback via $X(\phi) : X_m \to X_n$. We denote by $\mathcal{F}^\bullet$ this sheaf of sets on $X_\bullet$.

These sheaves of sets on $X_\bullet$ form a category which we denote $\mathrm{Sh}_{X_\bullet}$. The $\mathcal{F}^\bullet$, are *not* cosimplicial objects as the $\mathcal{F}^n$ do not all "live" in the same space. They are defined over the different X_n. However in the following example we see how, if we take all the $X_n = S$ they become cosimplicial.

Example 13 ($\mathrm{Sh}_{S_\bullet} \cong \mathrm{Cosimp}(\mathrm{Sh}_S)$). Consider the constant simplicial object $S_\bullet$. Let $\mathcal{F}^\bullet$ be a sheaf of sets on $S_\bullet$. Then $\mathcal{F}$ corresponds to a cosimplicial object in the category of sheaves over S. Each $\mathcal{F}^n$ defines a sheaf over S and for every morphism $\phi : [n] \to [m]$ in Δ we have a morphism of sheaves $\mathcal{F}^n \to \mathcal{F}^m$. Composition is simply $[\phi] \circ [\psi] = [\phi \circ \psi]$ for all composable ϕ, ψ in Δ.

An important question to ask at this point is whether the category of abelian sheaves on $X_\bullet$, which we will denote $\mathrm{AbSh}_{X_\bullet}$, has enough injectives. To resolve this we use the fact that the category of abelian sheaves on a site has enough injectives, and consider the following *equivalent* construction of $\mathrm{Sh}_{X_\bullet}$:

Definition 5.2.2. Fix a simplicial space $X_\bullet$. Consider the site with étale maps $U \to X_n$ for $n \geq 0$ as objects. A morphism from $U \to X_n$ to $V \to X_m$ is given by a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{f} & V \\ \downarrow & & \downarrow \\ X_n & \xrightarrow[X(\phi)]{} & X_m \end{array}$$

where $\phi : [m] \to [n]$ a map in Δ and f a map in $\mathcal{C}$. A covering of $U \to X_n$ is given by a covering of U in $\mathcal{C}$. Then we define $\mathrm{Sh}_{X_\bullet}$ to be the category of sheaves of sets on this site and $\mathrm{AbSh}_{X_\bullet}$ the subcategory of abelian sheaves.

The following lemma, which is proved in [3, §6], gives an idea of what an injective object in $\mathrm{AbSh}_{X_\bullet}$ looks like and will be useful later.

Lemma 5.2.1. *Let $\mathcal{I}^\bullet$ be an injective object in $AbSh_{X_\bullet}$. Then for each $n \geq 0$ $\mathcal{I}^n$ is an injective object in $AbSh_{X_n}$.*

Given $X_\bullet$, $Y_\bullet \in \mathrm{Simp}(\mathsf{Top})$ and a simplicial map $f_\bullet : X_\bullet \to Y_\bullet$ we can define functors

$$u_{\bullet *} : \mathrm{Sh}_{X_\bullet} \to \mathrm{Sh}_{Y_\bullet}$$
$$u_\bullet^* : \mathrm{Sh}_{Y_\bullet} \to \mathrm{Sh}_{X_\bullet}$$

term-by-term via the regular pushforward and pullback, u_{n*}, u_n^*.
Now consider an augmented simplicial space

$$X_\bullet \xrightarrow{a} S.$$

Then there exists a unique simplicial map

$$a_\bullet : X_\bullet \to S_\bullet$$

with $a_n = a \circ d_0 \circ \cdots \circ d_{n-1} : X_n \to S$. Using the correspondence between $\mathrm{Sh}_{S_\bullet}$ and $\mathrm{Cosimp}(\mathrm{Sh}_S)$ we define adjoint functors

$$\begin{aligned} a^* : \mathrm{Sh}_S &\to \mathrm{Sh}_{X_\bullet} \\ \mathcal{F} &\mapsto (a^*\mathcal{F})^\bullet \end{aligned}$$

where $(a^*\mathcal{F})^n = a_n^*\mathcal{F}$, and

$$\begin{aligned} a_* : \mathrm{Sh}_{X_\bullet} &\to \mathrm{Sh}_S \\ \mathcal{F}^\bullet &\mapsto a_*\mathcal{F}^\bullet \end{aligned}$$

where $a_*\mathcal{F}^\bullet = \mathrm{Ker}\ \left(a_{0_*}\mathcal{F}^0 \stackrel{d^1-d^0}{\to} a_{1_*}\mathcal{F}^1\right)$.

Recalling that a^* is exact and a_* left exact, on the level of derived categories we get the total derived functors

$$a^* : D^+(S) \to D^+(X_\bullet), \quad \mathbf{R}a_* : D^+(X_\bullet) \to D^+(S).$$

Definition 5.2.3. Let $a : X_\bullet \to S$ be an augmented simplicial space. Then a is said to be a *morphism of cohomological descent* if the adjoint functors (a_*, a^*) are such that the natural transformation

$$\mathrm{id}_{D^+(S)} \longrightarrow \mathbf{R}a_* \circ a^*$$

is an isomorphism.

In particular, at the level of abelian sheaves, the map $a : X_\bullet \to S$ is said to be a *morphism of cohomological descent* if, for any $\mathcal{F} \in \mathrm{Ab}(\mathrm{Sh}(S))$,

$$\mathcal{F} \cong a_*a^*\mathcal{F} \ \text{ and } \ \mathrm{R}^i a_*(a^*\mathcal{F}) = 0, \ \forall\, i > 0.$$

Example 14. Let X be a topological space. Consider the constant simplicial space $X_\bullet$ where $X_n = X$ for all indices n. With a the identity map on X we have a constant augmented simplicial space $X_\bullet \stackrel{a}{\to} X$. We recall that in this case $\mathrm{Sh}_{X_\bullet} \cong \mathrm{Cosimp}(\mathrm{Sh}_X)$.

Using the correspondence $\mathrm{Sh}_{X_\bullet} \cong \mathrm{Cosimp}(\mathrm{Sh}_X)$ we have

$$\begin{aligned} a_* : \mathrm{Cosimp}(\mathrm{Sh}_X) &\to \mathrm{Sh}_X \\ \mathcal{F}^\bullet &\mapsto \mathrm{Ker}\left(a_{0_*}\mathcal{F}^0 \stackrel{\delta}{\to} a_{1_*}\mathcal{F}^1\right) \end{aligned}$$

with $\delta = d^1 - d^0$, d^i the face maps on $\mathcal{F}^\bullet$.

By the cosimplicial analogue of the Dold-Kan corresondence (Theorem 3.1.2) the functor that sends a cosimplicial sheaf $\mathcal{G}$ to its normalised cochain complex $Q(\mathcal{G})$ is an equivalence of categories. Moreover, dualising the results of Chapter 3 (Proposition 3.0.2), we also have $H^i(Q(\mathcal{G})) = H^i(S(\mathcal{G}))$ with $S(\mathcal{G})$ the associated cochain complex of $\mathcal{G}$. Under this corresopondence we see that a_* corresponds to the H^0 functor:

$$\begin{aligned} a_*\mathcal{F}^\bullet &= \mathrm{Ker}\left(a_{0_*}\mathcal{F}^0 \stackrel{\delta}{\to} a_{1_*}\mathcal{F}^1\right) \\ &= H^0\left(a_{0_*}\mathcal{F}^0 \to a_{1_*}\mathcal{F}^1 \to \cdots \to a_{n_*}\mathcal{F}^n \to \dots\right) \\ &= H^0\left(C(\mathcal{F}^\bullet)\right). \end{aligned}$$

The functor $\mathcal{F} \mapsto \mathcal{F}[0]$, sending $\mathcal{F}$ to the cochain complex with $\mathcal{F}$ in degree 0 and 0 elsewhere, is adjoint to H^0. So we find

$$\mathcal{F} \mapsto a_* a^* \mathcal{F} \cong H^0(\mathcal{F}[0]) = \mathcal{F}$$

and

$$R^i a_*(a^*\mathcal{F}) \cong H^i(\mathcal{F}[0]) = 0, \text{ for } i > 0.$$

It follows that $a : X_\bullet \to X$ is of cohomological descent.

Definition 5.2.4. Let $a : X_\bullet \to S$ be an augmented simplicial space. Then a is *universally of cohomological descent* if for every base change $S' \to S$ the augmentation

$$a_{/S'} : X_\bullet \times_S S' \to S'$$

is of cohomological descent.

We say that a map of spaces $a_0 : X_0 \to S$ is (universally) of cohomological descent if the augmented simplicial space

$$\mathrm{cosk}_0(a_0) : \mathrm{cosk}_0(X_0/S) \to S$$

is (universally) of cohomological descent.

Theorem 5.2.1. *Let $X_\bullet$ be a simplicial space, and $a : X_\bullet \to S$ an augmentation. Call $a_p : X_p \to S$ the induced map. Then for all $K \in D^+(S_\bullet)$ there exists a canonical spectral sequence*

$$E_1^{p,q} = R^q a_{p*}(K|_{X_p}) \Rightarrow R^{p+q} a_*(K).$$

This spectral sequence is functorial in $a : X_\bullet \to S$.

Proof. For $K \in D^+(X_\bullet)$, let $K \to I^\bullet$ be a quasi isomorphism to a bounded below complex of injective abelian sheaves on $X_\bullet$ and consider the (1st quadrant) double complex with (p,q) entry $a_{p*} I^q|_{X_p}$. The horizontal differentials

$$\partial_H^{p,q} : a_{p*} I^q|_{X_p} \to a_{p+1*} I^q|_{X_{p+1}}$$

are induced by the alternating sum of the pullbacks of the relevant face maps on $X_\bullet$, that is the map

$$\sum_{i=0}^{p+1} (-1)^i d_i^* : I^q|_{X_p} \to I^q|_{X_{p+1}}.$$

The vertical differentials $\partial_V^{p,q} : a_{p*} I^q|_{X_p} \to a_{p*} I^{q+1}|_{X_p}$ are induced by the differentials of the complex $I^\bullet$. The commutativity conditions are satisfied. Now we consider the two spectral sequences arising from this double complex by looking at the filtration by rows, and the filtration by columns. Filtering by columns gives rise to a spectral sequence

$$^I E_1^{p,q} = H^q a_{p*}(I^\bullet|_{X_p}) \Rightarrow H^{p+q}\left(\mathrm{Tot}\left(a_{\bullet *} I^\bullet|_{X_\bullet}\right)\right). \tag{5.1}$$

As $K \to I^\bullet$ is a quasi-isomorphism, we have that $K|_{X_p} \to I^\bullet|_{X_p}$ is a quasi-isomorphism for each p. We recall (by Lemma 5.2.1) that each of the I^q are injective in $\mathrm{Ab}(X_\bullet)$, hence each of the $I^q|_{X_p}$ are injective in $\mathrm{Ab}(X_p)$. It follows that $K|_{X_p} \to I^\bullet|_{X_p}$ is a quasi-isomorphism to a bounded below complex of injectives and we conclude that

$$H^q a_{p*}(I^\bullet|_{X_p}) = R^q a_{p*}(K|_{X_p}). \tag{5.2}$$

Let us now look at the filtration by columns. This gives rise to the spectral sequence

$$\begin{aligned} {}^{II}E_1^{p,q} &= H^q\left(a_{0*}I^p|_{X_0} \to a_{1*}I^p_{X_1} \to \dots \to a_{k*}I^p_{X_k} \to \dots\right) \\ &= H^q\left(a_{\bullet*}I^p|_{X_\bullet}\right) \\ &\Rightarrow H^{p+q}\left(\mathrm{Tot}\left(a_{\bullet*}I^\bullet|_{X_\bullet}\right)\right). \end{aligned}$$

Let us show that the complex

$$a_{0*}I^p|_{X_0} \to a_{1*}I^p_{X_1} \to \dots \to a_{k*}I^p_{X_k} \to \dots$$

is acyclic outside of degree 0. Consider the functor

$$\begin{aligned} \mathrm{Ab}(X_\bullet) &\to \mathrm{Ch}_{\geq 0}((Ab)(S)) \\ I^p &\to \{a_{i*}I^p|_{X_i}\}_{i\geq 0}. \end{aligned}$$

This admits an exact left adjoint given by

$$\left(\mathcal{F}^0 \to \mathcal{F}^1 \to \dots \mathcal{F}^i\right) \to \left(a_i^*\mathcal{F}^i\right).$$

Hence we have that that the complexes are acyclic outside of degree 0. Now since $H^0\left(a_{\bullet*}I^p|_{X_\bullet}\right) = a_*I^p$ we have that ${}^{II}E_1^{\bullet,0} = a_*I^\bullet$. The spectral sequence collapses and we have that

$$H^{p+q}\left(\mathrm{Tot}\left(a_{\bullet*}I^\bullet|_{X_\bullet}\right)\right) = H^{p+q}\left(a_*I^\bullet\right) = R^{p+q}a_*K. \tag{5.3}$$

Combining (5.1), (5.2), and (5.3) we get the spectral sequence

$$E_1^{p,q} = R^q a_{p*}\left(K|_{X_p}\right) \Rightarrow R^{p+q}a_*K$$

as desired. □

Theorem 5.2.2. *Let $X_\bullet$ be a simplicial space. For any $K \in D^+(X_\bullet)$ there is a natural cohomological spectral sequence*

$$E_1^{p,q} = \mathbf{H}^q\left(X_p, K|_{X_p}\right) \Rightarrow \mathbf{H}^{p+q}\left(X_\bullet, K\right) \tag{5.4}$$

with horizontal differentials of $E_1^{p,q}$ induced by the differentials of the associated cochain complex of $X_\bullet$.
If $a : X_\bullet \to S$ an augmented simplicial space with a of cohomological descent then for any $K' \in D^+(S)$ there is a spectral sequence

$$E_1^{p,q} = \mathbf{H}^q\left(X_p, a_p^*K'\right) \Rightarrow \mathbf{H}^{p+q}\left(S, K'\right) \tag{5.5}$$

Proof. Working with $\Gamma(X_\bullet, -)$ and $\Gamma(X_p, -)$ in place of a_* and a_{p*} we can follow the procedure of the proof to the above theorem to obtain the natural spectral sequence

$$E_1^{p,q} = \mathbf{H}^q\left(X_p, K|_{X_p}\right) \Rightarrow \mathbf{H}^{p+q}\left(X_\bullet, K\right)$$

for any $K \in D^+(X_\bullet)$.
Suppose we have an augmentation $a : X_\bullet \to S$ of cohomological descent. We note that $\Gamma(X_\bullet, \mathcal{F}) = \Gamma(S, a_*\mathcal{F})$ for any abelian sheaf $\mathcal{F}$ on $X_\bullet$ and, since a is of cohomological descent, the natural adjunction map $\mathrm{id} \to \mathbf{R}a_*a^*$ is an isomorphism. Thus, for any $K' \in D^+(S)$

$$\mathbf{R}\Gamma(S, K') \cong \mathbf{R}\Gamma(S, \mathbf{R}a_*(a^*K')) \cong \mathbf{R}(\Gamma(s, -) \circ a_*)(a^*K') = \mathbf{R}\Gamma(X_\bullet, a^*K').$$

It follows then that $\mathbf{H}^i(S, K') \cong \mathbf{H}^i(X_\bullet, a^*K')$. Taking $K = a^*K'$, we have $K|_{X_p} = a_p^*K'$ and hence, by the spectral sequence (5.4), we have the desired spectral sequence

$$E_1^{p,q} = \mathbf{H}^q\left(X_p, a_p^*K'\right) \Rightarrow \mathbf{H}^{p+q}\left(S, K'\right).$$

□

5.3 Criteria for Cohomological Descent

It is natural to ask at this point which maps are of cohomological descent. In this final section we will look at some criteria and properties of cohomological descent. Then we will look at proper maps and establish the fact that an augmented proper hypercovering of a topological space S is of cohomological descent.

Theorem 5.3.1. *Let $f : X \to S$ be a map of topological spaces which admits a section locally on S. Then f is universally of cohomological descent.*

Proof. Let us show that f is of cohomological descent. Then, as the hypotheses are perserved under base change it follows that f is universally of cohomological descent.
Taking $a = \mathrm{cosk}_0(f) : \mathrm{cosk}_0(X \to S) = X_\bullet \to S$ we wish to show that at the level of abelian sheaves, we have

$$\mathcal{F} \to a_*a^*\mathcal{F}, \text{ and } R^ia_*(a^*\mathcal{F}) = 0, \ \forall i > 0$$

for $\mathcal{F}$ an abelian sheaf on S.
Working locally on S, we may assume that f admits a section $\epsilon : S \to X$.
By Theorem 5.2.1 we obtain the following spectral sequence,

$$E_1^{p,q} = R^qa_{p*}(\mathcal{F}'|_{X_p}) \Rightarrow R^{p+q}a_*(\mathcal{F}')$$

For $\mathcal{F}' \in D^+(X_\bullet)$. The horizontal differentials $\partial^{p,q}$ are induced by the simplicial structure of $X_\bullet$. Set $a_{-1} = \mathrm{id}_S$. Then we can extend the rows $E_1^{\bullet,q}$ to an augmented complex and $E_1^{-1,q} - R^qa_{-1*}(\mathcal{F}'|_S) = 0$ for all $q > 0$. Define maps $h_p : X_p \to X_{p+1}$ as follows,

$$h_p = \epsilon \times \mathrm{id}_{X_p} : X_p = X \times_S^{(p+1)} = S \times_S X \times_S^{(p+1)} \to X \times_S^{(p+2)} = X_{p+1},$$

and $h_{-1} = \epsilon : S \to X$. Then we have the following equations

$$\begin{cases} d_0 \circ h_p = \mathrm{id}_{X_p} \; \forall \, p \geq 0 \\ d_{j+1} \circ h_p = h_{p-1} \circ d_j \; \forall \, p \geq 1, 0 \leq j \leq p-1 \end{cases} \tag{5.6}$$

where the d_i are the face maps of $X_\bullet$. Now, taking $\mathcal{F}' = a^*\mathcal{F}$, and making repeated use of the equations (5.6), we have isomorphisms

$$h_p^*(\mathcal{F}'|_{X_{p+1}}) = h_p^*(a_{p+1}^*\mathcal{F}) \cong a_p^*\mathcal{F} = \mathcal{F}'|_{X_p}$$

giving rise to maps

$$E_1^{p,q} \to E_1^{p-1,q}$$

which define a chain homotopy on the augmented complex $E_1^{\bullet,q}$ between the identity map and 0. In particular we have found that our chain complex is acylic. Thus, for $q > 0$, $E_1^{\bullet,q}$ is exact in all degrees since $E_1^{-1,q} = 0$ for all $q > 0$. Now lets consider the case $q = 0$.

$$E_1^{\bullet,0} = \mathcal{F} \to a_{0*}a_0^*\mathcal{F} \to a_{1*}a_1^*\mathcal{F} \to \ldots$$

is exact in positive degrees. So our sequence collapses and we have that

$$\begin{cases} R^i a_* a^* \mathcal{F} = 0 \; \forall \, i > 0 \\ a_* a^* \mathcal{F} \cong \mathrm{Ker}\,(a_{0*}a_0^*\mathcal{F} \to a_{1*}a_1^*\mathcal{F}) \cong \mathcal{F}. \end{cases}$$

□

Example 15. Let X be a topological space, and $\mathcal{U} = \{U_i\}_{i \in I}$ and open covering of X. We set $U = \coprod_{i \in I} U_i$. Let f be a covering map $U \to X$. Then, taking $\mathbf{P}$ to be the class of covering maps we construct Y, a $\mathbf{P}$-hypercovering of X,

$$Y = \mathrm{cosk}_0(U \to X).$$

Observing that covering maps admit a section locally, we have that f is universally of cohomological descent. The spectral sequence of Theorem 5.2.2, gives the following

$$E_1^{p,q} = \mathbf{H}^q(Y_p, f_p^*\mathcal{F}') \Rightarrow \mathbf{H}^{p+q}(X, \mathcal{F}).$$

For $\mathcal{F}' \in D^+(X)$. Taking $\mathcal{F}$ a sheaf on X and $\mathcal{F}'$ to be the complex of sheaves with $\mathcal{F}$ concentrated in degree 0 (and zero elsewhere), then hypercohomology coincides with cohomology and we get the following spectral sequence,

$$E_1^{p,q} = H^q(Y_p, f_p^*\mathcal{F}) \Rightarrow H^{p+q}(X, \mathcal{F}).$$

Thus as a special case we obtain the Čech-to-Derived Functor Spectral sequence:

$$E_2^{p,q} = \check{H}^p\left(\mathcal{U}, \mathcal{H}^q(X, \mathcal{F})\right) \Rightarrow H^{p+q}(X, \mathcal{F}),$$

where $\mathcal{H}^q(X, \mathcal{F})$ is the presheaf $U \mapsto H^q(U, \mathcal{F})$, and $\mathcal{U} = \{U_i\}$.

The following result shows that, in our definition of **P**-hypercoverings, we can take **P** to be the class of morphisms universally of cohomological descent.

Theorem 5.3.2. (1) *Consider a cartesian diagram of topological spaces*

$$\begin{array}{ccc} X' & \xrightarrow{p'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow[p]{} & S \end{array}$$

Suppose p is universally of cohomological descent. Then f is universally of cohomological descent if and only if f' is.
(2) *If $f : X \to Y$ and $g : Y \to Z$ are maps of topological spaces with $g \circ f$ universally of cohomological descent then g is of cohomological descent.*
(3) *If $f : X \to Y$ and $g : Y \to Z$ are maps of topological spaces each universally of cohomological descent, then the composition $g \circ f$ is universally of cohomological descent.*
(4) *If $f : X' \to X$ and $g : Y' \to Y$ are maps of topological spaces over S, each universally of cohomological descent, then $f \times g : X' \times_S Y' \to X \times_S Y$ is universally of cohomological descent.*

Proof. (1): If f is universally of cohomological descent, it follows from the definition that f' is of cohomological descent. To show the other direction is more complicated and involves the theory of bisimplicial objects, we refer the reader to [13, §3.3] or [3, Theorem 7.5].
(2): Consider the cartesian diagram

$$\begin{array}{ccc} X \times_Z Y & \xrightarrow{h} & Y \\ {\scriptstyle s}\uparrow\downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[g\circ f]{} & Z \end{array}$$

with $s = \mathrm{id}_X \times f$. The composition $g \circ h$ is assumed to be universally of cohomological descent. Then, by (1) it follows that g is universally of cohomological descent if g' is. Now $g' \circ s = \mathrm{id}_X$ and so by Theorem 5.3.1 we have our result.
(3): Since g is universally of cohomological descent in our cartesian diagram above it follows from (1), this time with base change map $g : Y \to Z$, that the composition $g \circ f$ is of universally of cohomological descent if and only if h is. Now $h \circ s = f$ is universally of cohomological descent thus it follows from (2) that h is also universally of cohomological descent.
(4): We factorise our map as follows:

$$X' \times_S Y' \overset{\mathrm{id}_{X'} \times g}{\longrightarrow} X' \times_S Y \overset{f \times \mathrm{id}_Y}{\longrightarrow} X \times_S Y.$$

Then the result follows from (3).

□

Now we wish to show that a proper hypercovering of a topological space S is universally of cohomological descent. We begin by showing that a proper map of spaces

$$f : X \to S$$

is universally of cohomological descent. For this we will make use of the proper base change theorem.

Theorem 5.3.3 (Proper Base Change Theorem)**.** *Consider the following cartesian diagram of topological spaces,*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y \end{array}$$

and suppose f is proper. Then for an abelian sheaf $\mathcal{F}$ on X we have the following canonical isomorphism,

$$g^* R^i f_* \mathcal{F} \xrightarrow{\sim} R^i f'_* g'{*}\mathcal{F}.$$

Proof. See Theorem 4.1.1 in [13, §4.1]. □

Theorem 5.3.4. *Let $f : X \to S$ be a proper surjective map of topological spaces. Then f is a map of cohomological descent. Moreover, it remains so after base change to any other topological space S'.*

Proof. Since proper surjective maps are stable under base change it suffices to show that, for $\mathcal{F}$ an abelian sheaf on S and $a : X_\bullet = \operatorname{cosk}_0(X/S) \to S$ we have that $\mathcal{F} \to a_* a^* \mathcal{F}$ is an isomorphism and $R^i a_*(a^*\mathcal{F}) = 0$ for all $i > 0$.
Since f is proper and $a_p : X_p \to S$ is simply the $(p-1)^{\text{th}}$ fibre power of f it follows that each of the a_p are proper. By Theorem 5.2.1 we have the following spectral sequence,

$$R^q {a_p}_* ({a_p}^* \mathcal{F}) \Rightarrow R^{p+q} a_*(a^*\mathcal{F}).$$

If follows from looking at its construction that the spectral sequence is compatible with base change on S. Applying the proper base change theorem to the a_p's, we then have that the direct image functors $R^i a_*$ are compatible with base change. It follows that we can reduce to the case in which S is a point. In this case $f : A \to S$ admits a section and so by Theorem 5.3.1 we are done. □

If $X_\bullet$ is a proper hypercovering of a topological space S. Then, by definition, for all $n \geq -1$ the maps

$$X_{n+1} \to (\operatorname{cosk}_n \operatorname{sk}_n X_\bullet)_{n+1}$$

are proper surjective maps hence, by the previous result, they are of cohomological descent. The desired result will thus follow from the following theorem.

Theorem 5.3.5. *Suppose $a : X_\bullet \to S$ an augmented simplicial space such that, for all $n \geq -1$, each map of spaces*

$$X_{n+1} \to (cosk_n sk_n X_\bullet)_{n+1}$$

is universally of cohomological descent. Then $a : X_\bullet \to S$ is universally of cohomological descent.

The proof follows from an inductive argument combining the following two results.

Lemma 5.3.1. *Suppose $a : X_\bullet \to S$ is an augmented simplicial space such that, for large enough n, the augmented simplicial spaces*

$$cosk_n sk_n(X_\bullet) \to S$$

are (universally) of cohomological descent. Then a is (universally) of cohomological descent.

Proof. It is enough to show the result for cohomological descent. Then the universal case will just follow by base change.

By Theorem 5.2.1 we have a spectral sequence associated to $a : X_\bullet \to S$,

$$E_1^{p,q} = R^q a_{p*}(a_p^* \mathcal{F}) \Rightarrow R^{p+q} a_*(a^* \mathcal{F}).$$

Choose $N \geq 1$ large enough such that the map

$$\mathrm{cosk}_N \mathrm{sk}_N(X_\bullet) \to S \tag{5.7}$$

is of cohomological descent. Let $Y_\bullet = \mathrm{cosk}_N \mathrm{sk}_N X_\bullet$. Again by Theorem 5.2.1 we have the spectral sequence

$$E'^{p,q}_1 = R^q a'_{p*}(a'^*_p \mathcal{F}) \Rightarrow R^{p+q} a'_*(a'^* \mathcal{F}).$$

where a' is the augmentation $Y_\bullet \to S$. The map of augmented simplicial spaces (over S)

$$X_\bullet \to Y_\bullet$$

is an isomorphism in all degrees $k \leq N$, and it induces maps on the components of the spectral sequences

$$E_1^{p,q} \to E'^{p,q}_1$$

which are isomorphisms for all $p \leq N$. The $E_\infty^{p,q}$ terms of the spectral sequences depend only on the (p', q')-components for $p' + q' \leq (p + q) + 1$. It follows that we have isomorphisms

$$R^i a_*(a^* \mathcal{F}) \cong R^i a'_*(a'^* \mathcal{F})$$

for $i \leq N - 1$. Since a' is of cohomological descent we have that $\mathcal{F} \cong a_* a^* \mathcal{F}$ and $R^i a_*(a^* \mathcal{F}) = 0$ for $i \leq N - 1$. We may take N as large as we like and so we are done. □

Lemma 5.3.2. *Suppose $a : X_\bullet \to S$ an augmented simplicial space such that, for all $-1 \leq k < n$, each map of spaces*

$$X_{k+1} \to (cosk_k sk_k X_\bullet)_{k+1}$$

is universally of cohomological descent, then $cosk_n sk_n(X_\bullet) \to S$ is universally of cohomological descent.

Proof. In the case $n = -1$ we have nothing to show and the conclusion is simply that $S_\bullet \to S$ is of cohomological descent, where $S_\bullet$ is the constant simplicial set at S. The proof follows by induction on n. See [13], or [3]. □

Summarising, the above results prove the following theorem.

Theorem 5.3.6. *A proper hypercovering of a topological space is universally of cohomological descent.*

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