Algebra, Geometry and Number Theory

ALGANT Erasmus Mundus Master

2:3

COX RINGS
FOR A
PARTICULAR CLASS
OF
TORIC SCHEMES

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To my sheperd,
that maketh me to lie down
in green pastures
and leadeth me
beside still water.

Preface

The initial aim of this thesis was to read and understand the paper "The Homogeneous Coordinate Ring of a Toric Variety" (1995) by Cox [2]. But it resulted obvious from the beginning that the article was impossible to approach without a clear understanding of what a toric variety is. Hence the author referred to the good book by Cox [4] and to presentation of Fulton [7]. These basics were able to furnish all the instruments in order to understand the main ideas of the article, but had the limit of treating all the theory from the point of view of classical varieties over $\mathbb{C}$. The natural question that arose was: what happens if one wants to define toric varieties not on the base field $\mathbb{C}$, but on other algebraically closed base fields k? And is it possible to extend the theory to not necessarily algebraically closed fields, such as number fields? And, since "appetite comes with eating" the final question was: how to extend the theory on any base ring? Clearly this kind of questions needed to involve the language of schemes. The first thing to do was to translate all the base objects used by Cox into schemes terms. In particular it was necessary to understand what is a variety in this new language, what is an object that is both a group and a variety, what is an action of this objects and so on. After this study it was finally possible to extend the definition of toric variety to the definition of *toric schemes*. Following the usual presentation of the subject, but with a more general point of view, we introduced and studied a particular class of toric schemes, coming from cones and lattices. These are in fact the objects corresponding to the ones considered by Cox in his article. Much time was devoted to understand the properties of toric schemes from fans, inherited from the properties of the base ring R (totally absent in the classical approach) and the properties of the original fan.

Future aims are to understand the theory of divisors on toric schemes with some "good" properties and to apply it to prove Cox theorem in more generality.

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And of course I thank my family: mum, dad, Dani, zio Luca, all my uncles and aunts. They gave me unlimited psychological, emotive and financial support. They have always encouraged me to run after my dreams and they still do. Thank you.

Contents

All rings will be intended to be commutative and with unity[1].

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[1]This is a typical requirement in Algebraic Geometry and in a certain sense, the traditional first sentence of all the Algebraic Geometry books, EGA included.

Introduction

It is common to think that Mathematics is divided (or, better, developed) in many branches. Some of them have a reciprocal influence, some others seem to be so far that it is often impossible for experts in different areas to communicate clearly. For example, there is Combinatorics, whose theorems appear as simple statement about simple objects. On the other side, there is Algebraic Geometry, whose theorems that appear to be so difficult that often have a statement that requires a lot of time to be understood. It seems that this two areas lie in a completely opposite direction in the worldmap of mathematics. But, quite surprisingly, they have stronger connection than one can imagine. A bridge between this two fields is represented by the theory of toric varieties[2].

It seems that the first ideas in the subject appeared in a paper by Demazure entitled *Sous-groupes algébriques de rang maximum du groupe de Cremona* ([6], 1970). he was able to construct a variety (in the sense of Algebraic Geometry) from the datum of a lattice and of some subsets of generators of this lattice. After this paper, people started developing new ideas and a rich literature bloomed. Among them it is worth to mention *Toroidal Embeddings I* by Kempf, Knudsen, Mumford and Saint-Donat ([13], 1973) and *Almost homogeneous algebraic varieties under algebraic torus action* by Miyake and Oda ([20], 1975). In the meanwhile the Russian school produced remarkable papers such as *Newton polyhedra and toroidal varieties* by Khovanskii ([11], 1977) and *Geometry of toric varieties* by Danilov ([5], 1978)[3]. The subject continued its growth and around the nineties some wonderful reference were written. In particular *Convex Bodies an Algebraic Geometry* by Oda ([19], 1988), *Newton polyhedra of principal A-determinants* by Gel'fand, Kapranov and Zelevinsky ([8], 1989), and *Introduction to toric varieties* by Fulton ([7], 1993).
Nowadays toric geometry has developed a wide influence and has applications in Commutative Algebra, Combinatorics and Physics. A survey and a resume on the recent developments in toric geometry can be found in [3].

For what concerns us, we will come back to the definition of toric varieties, trying to present some different approaches that have been studied. The general definition of affine toric variety that will inspire our generalization into scheme words is the following. Here all the varieties studied are varieties over the field of complex numbers,

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[2]A wonderful example of this interplay can be found in a short article of Stanley ([24], 1980), who solved McMullen Conjecture using ideas from toric geometry.

[3]Remark that moreover the translation of the title of the paper of Danilov into English made by Miles Reid represents the origin of the expression "toric varieties".

so a torus over $\mathbb{C}$ is simply the variety $\mathbb{C}^*$ together will the usual componentwise multiplication.

Definition 0.1. An **affine toric variety** is an irreducible affine variety V in some affine space $\mathbb{C}^s$ that contains a torus $(\mathbb{C}^*)^n$ as a Zariski open subset (and so as a dense open set). Moreover one requires that the action of the torus over itself (by componentwise multiplication) extends to an algebraic action over V, i.e. to a map $(\mathbb{C}^*)^n \times V \to V$ that is both an action of group and a morphism of algebraic varieties.

With this definition in mind, we immediately have some examples, like $\mathbb{C}^n$ and $(\mathbb{C}^*)^n$.
A less trivial example is given by the zero set $X = V(x^3 - y^2)$ in $\mathbb{C}^2$. It is classical that this variety is nonnormal (it has a cusp at the origin). Looking to $X \cap (\mathbb{C}^*)^2$ we noticed that it is an open subset of X with the induced Zariski topology and that it is isomorphic to $\mathbb{C}^*$ via the map $\mathbb{C}^* \to X$ such that $t \mapsto (t^2, t^3)$. From this X can be proved to be an affine toric variety; this is the easier example of a nonnormal toric variety.

It is possible to give many other equivalent definitions of affine toric varieties, let's see some of them. Recall that a character of $(\mathbb{C}^*)^n$ is a group homomorphism $(\mathbb{C}^*)^n \to \mathbb{C}^*$. A first example of character is given by a monomial map, as explained in the following example.

Example 0.2. Choose $\underline{m} \in \mathbb{Z}^n$, $\underline{m} = (m_1, m_2, \ldots, m_n)$ and define $\chi^{\underline{m}} : (\mathbb{C}^*)^n \to \mathbb{C}^*$ such that

$$\chi^{\underline{m}}(t_1, t_2, \ldots, t_n) := t_1^{m_1} \cdot t_2^{m_2} \cdot \cdots \cdot t_n^{m_n}.$$

It is clearly a morphism of groups, hence it is by definition a character of $(\mathbb{C}^*)^n$.

One can show that show that in fact all the characters χ over $(\mathbb{C}^*)^n$ are monomial maps, i.e. they all have the form of the morphism in the previous example (see for example [12]).
As a corollary, the characters of $(\mathbb{C}^*)^n$ form (with the componentwise product) a free abelian group of rank n, since it is sufficient to consider the isomorphism of groups $\chi^{\underline{m}} \mapsto \underline{m}$ (it satisfies $\chi^{\underline{m}} \cdot \chi^{\underline{l}} = \chi^{\underline{m}+\underline{l}}$). So, for any $(\mathbb{C}^*)^n$ one can consider its *character lattice* M, whose rank equals n. All this discussion is necessary in order to give this alternative definition of affine toric variety.

Definition 0.3. Choose $\mathcal{A} = \{\chi^{\underline{m_1}}, \chi^{\underline{m_2}}, \ldots, \chi^{\underline{m_s}}\}$ a finite subset of the character lattice M of $(\mathbb{C}^*)^n$. An n-**dimensional affine toric variety** in $\mathbb{C}^s$ is the affine variety in $\mathbb{C}^s$ given by the Zariski closure of the image of the map

$$\phi_{\mathcal{A}} : (\mathbb{C}^*)^n \to \mathbb{C}^s$$

$$\phi_{\mathcal{A}} : t \mapsto (\chi^{\underline{m_1}}(t), \chi^{\underline{m_2}}(t), \ldots, \chi^{\underline{m_s}}(t)).$$

This definition also underline the reason for which at the beginning of the seventies toric varieties were mostly referred to as *toroidal embeddings* (see [13]).

Another way of introducing toric varieties in affine spaces is through toric ideals.

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Using the compact notation $x^{\underline{m}}$ with $\underline{m} = (m_1, m_2, \ldots, m_s)$ to denote $x_1^{m_1} x_2^{m_2} \ldots x_s^{m_s}$, a **toric ideal** is a prime ideal of the form

$$\langle x^\alpha - x^\beta : \alpha, \beta \in \mathbb{N}^s \text{ and } \alpha - \beta \in L \rangle$$

where L is a sublattice in $\mathbb{Z}^s$. Equivalently, a toric ideal is a prime ideal generated by binomials in $\mathbb{C}[x_1, x_2, \ldots, x_s]$. This gives another point of view.

Definition 0.4. An **affine toric variety** in $\mathbb{C}^s$ is the zero set of a toric ideal in $\mathbb{C}[x_1, x_2, \ldots, x_s]$.

At the end, all these constructions define the same objects and some work proves that the three definitions 0.1, 0.3 and 0.4 are equivalent for affine toric varieties (see for example [4, Ch1.1]).
Extending the definition from affine varieties to "abstract" varieties (in a sense that will be cleared out in 1.1.2) one easily obtains the following[4].

Definition 0.5. A **toric variety** is an irreducible algebraic variety V containing a torus $(\mathbb{C}^*)^n$ as a Zariski open subset, with the property that the action of the torus over itself extends to an algebraic action over V.

Among the class of toric varieties over $\mathbb{C}$ there is a special subset: some toric varieties can be in fact obtained via a wonderful construction using cones and fans in a certain vector space, as shown in practically all the text in literature and as we will explain in detail in more general context in Chapter 2. Moreover Oda proved in [19] that all the *normal* toric algebraic varieties over $\mathbb{C}$ are constructed from a fan. Since many authors include in the definition of toric variety the requirement of normality, the study of toric varieties is often reduced to the case of varieties constructed from fans. Remarking that nonnormal toric varieties are not uncommon and even easy to obtain (recall for example the already mentioned cusp in $\mathbb{C}^2$), we will anyway focus on a generalization of toric varieties coming from fans. Our aim is in fact to prove the structure theorem of toric varieties as quotients made in [2]. The proof relies strongly on the theory of divisors on toric varieties. In order to define divisors on a variety and the divisor class group on it, it is necessary to work with normal irreducible varieties. For this reason we will be happy of treating the case of toric varieties from fans. But our efforts will try to work out the construction in more general terms: instead of defining a normal toric variety over $\mathbb{C}$ as commonly done, we will be interested in making the same construction over any base ring R. In order to do this, the language of schemes will be essential.
In the end the structure of the thesis is this:

- in Chapter 1 we will give the naive idea of how to think to varieties, through classical algebraic geometry. All over the exposition, anyway we will prefer to use all the machinery of schemes that will help us both to condense the

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[4]This is not the only point of view anyway. For example Danilov, in [5], starts from a definition that is closer to differential geometry (which is far from the point of view we will develop). In fact he defines a smooth n-dimensional toric variety as an algebraic variety X together with a collection of charts from open subsets of X to $\mathbb{C}^n$ such that the changement of charts are Laurent monomials.

geometric and the algebraic properties in only one object (a topological space together with a sheaf over it) and to treat all the topic in greater generalities. For this reason it will be important to treat the way in which to interpret varieties in the sense of schemes and to extend the concept of the torus to the general setting. It will be important to understand what is a group scheme and what is an action of a group scheme on a scheme. At the end we will extend the definition of toric varieties over $\mathbb{C}$ to toric varieties over any ring R. We will call them R-toric scheme.

- in Chapter 2 we will introduce a particular class of R-toric schemes deduced from cones and fans in a vector space defined starting from a lattice. This part of the study of toric varieties represents a beautiful link between combinatorics and algebraic geometry. We will develop the construction of affine R-toric schemes and then of R-toric schemes; the second ones will be obtained by gluing from affine pieces. We will then pass to the study of some examples and we will discover that many of the ambient spaces used in algebraic geometry, such as affine spaces and projective spaces, are in fact toric varieties. The last part of the chapter will be devoted to the proof of the fact that the fiber products of R-toric schemes constructed from fans is again an R-toric scheme coming from a fan, that we will explicitly determine.

- the aim of Chapter 3 is to study some properties of R-toric schemes constructed from fans. We will discover that some properties are related to the properties of the ring R we will choose and some others come from the properties of the fan. The main focus will be to find conditions such that the R-toric variety constructed from a fan is separated, integral, normal and noetherian, so that it will be possible to define Weil divisors on it.

- in Chapter 4 a brief study of Weil divisors on toric schemes will be treated. After recalling the definition of Weil divisors for a scheme that is separated, integral, normal and noetherian (we will follow Hartshorne's presentation, [10]), we will introduce some particular closed subschemes of a toric scheme coming from a fan. The construction presented will associate to every ray of a fan a prime divisor of the toric scheme deduced from the fan. As a consequence we will try to determine the principal divisor of certain rational function and we will use this results to construct an exact sequence that will allow us to compute some divisor class group for certain toric schemes. Unfortunately here we will have to restrict to the case of toric schemes on an algebraically closed ring.

- in Chapter 5 we will finally approach the article [2] by Cox (1995). We will associate to a toric scheme constructed from a fan a ring, nowadays often called *Cox ring*. The idea will be again to deal with rings instead of the field of complex numbers $\mathbb{C}$. This will be done with the definition of the Cox ring. We will end up stating an interesting result of [2]: any $\mathbb{C}$-toric scheme constructed from a finite fan can be realized as a categorical quotient of some open subset of the spectrum of its Cox ring by the action of a certain group.

Chapter 1

From toric varieties to toric schemes

The aim of this chapter is to present all the tools and the objects that will be irrenunciable in the following, and especially in the definition of toric varieties. We will recall the ideas of varieties in classical algebraic geometry, and we will introduce tori on a field. Then, we will introduce a dictionary to translate these objets in the schemes languages. Everything will be essential in order to give a definition that extends the one of toric variety to toric schemes on any base ring R; this will be done in section 1.5.

1.1 Varieties in classical Algebraic Geometry

We start by recalling some definitions and constructions coming from classical Algebraic Geometry.

1.1.1 Affine varieties

Given a field k and a positive integer n, one can associate to every ideal $I \subseteq k[x_1, x_2, \ldots, x_n]$ the subset

$$V(I) := \{P \in k^n : f(p) = 0 \text{ for all } f \in I\}$$

of k^n. This is what we will call an **affine variety**[1] in k^n. As well-known the family of all the affine varieties in k^n satisfies the same properties of the family of closed subsets of a topology; the topology on k^n whose closed subsets are precisely the set of the form $V(I)$, is called the *Zariski topology* of k^n.
Conversely for any subset X of k^n one can define

$$I(X) := \{f \in k[x_1, x_2, \ldots, x_n] : f(p) = 0 \text{ for all } p \in X\}$$

It is easily proved that $I(X)$ is an ideal in $k[x_1, x_2, \ldots, x_n]$ and that the following properties are true.

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[1]Remark that some authors, like Hartshorne, prefer to call affine variety a closed *irreducible* (with respect to the Zariski topology) subset in k^n of the form in the definition.

Proposition 1.1. *Let k be a field and n a positive integer number. Then:*

(a) if $X_1 \subseteq X_2$ (subsets in k^n), then $I(X_1) \supseteq I(X_2)$.

(b) if $I_1 \subseteq I_2$ (ideals in $k[x_1, x_2, \ldots, x_n]$), then $V(I_1) \supseteq V(I_2)$.

(c) for every X subset of k^n, $V(I(X)) = \overline{X}$ (the Zariski closure of X).

(d) for every I ideal in $k[x_1, x_2, \ldots, x_n]$, $I(V(I)) \supseteq I$. Moreover, if k is algebraically closed, then $I(V(I)) = \sqrt{I}$.

Proof.
See any book of algebraic geometry, for example [10, ChI] or [15]. □

We now focus on the concept of morphism between varieties over an algebraically closed field k. Since varieties are topological spaces, it will be clear that a morphism between varieties will be asked to be a continuous map, but this will not be enough, since a variety also carries a stronger structure. To understand well this let's define before what is a regular function over an affine variety.

Definition 1.2. Let V be an affine variety over a field k, living in the affine space k^n. A function $f : V \to k$ is a **regular function at a point** $P \in V$ if there exists an open neighborhood U of P and there exist two polynomials g, h in $k[x_1, x_2, \ldots, x_n]$ such that h is never zero in U and $f = \frac{g}{h}$ on U. A function $f : V \to k$ is a **regular function in** V if it is regular at any point of V.

With the componentwise addition and multiplication, the set of regular functions over an affine variety V is a ring and it is isomorphic to the ring $\mathcal{O}(V) := k[x_1, x_2, \ldots, x_n]/I(V)$ (see for example [10, Th I.3.2]).

Definition 1.3. Let V_1, V_2 be two affine varieties over the field k. A **morphism of affine varieties** is a continuous map (with respect to the Zariski topology) $\varphi : V_1 \to V_2$ such that for any open subset U of V_2 and for any regular function f on U (i.e. regular at any point of U), one has that $\varphi^* f = f \circ \varphi$ is regular on $\varphi^{-1}(U)$.

In this way we obtain a category whose objects are affine varieties over a fixed algebraically closed field k and whose morphisms are the just defined morphisms between affine varieties. We will call this category the *category of affine variety over k*.

An important object associated to a variety V is the **affine coordinate ring** or **ring of regular functions** of V, defined as

$$\mathcal{O}(V) := k[x_1, x_2, \ldots, x_n]/I(V)$$

It is easily seen that it is a k-algebra and that it is an integral domain if and only if V is an irreducible variety. The most important fact is given by the following.

Theorem 1.4. *$\mathcal{O}$ realizes a contravariant equivalence between the category of affine varieties over k and the category of finitely generated k-algebras without nonzero nilpotents.*

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Proof.

This is the equivalent of [10, Cor I.3.8]. To be explicit, given A a finitely generated k-algebra without nonzero nilpotents, surely A is isomorphic to some $k[x_1, x_2, \ldots, x_n]/I$, for I ideal in $k[x_1, x_2, \ldots, x_n]$ (it is not necessarely prime in our setting). Since the algebra $A \simeq k[x_1, x_2, \ldots, x_n]/I$ does not contain nonzero nilpotents, then we have $I = \sqrt{I}$, so I is a radical ideal. Hence defining $\Phi(A) := V(I)$, one has that

$$(\Phi \circ \mathcal{O})(V) = \Phi(k[x_1, x_2, \ldots, x_n]/I(V)) = V(I(V)) = \overline{V} = V$$

and

$$(\mathcal{O} \circ \Phi)(A) = \mathcal{O}(V(I)) = k[x_1, x_2, \ldots, x_n]/I(V(I)) = k[x_1, x_2, \ldots, x_n]/\sqrt{I} =$$
$$= k[x_1, x_2, \ldots, x_n]/I \simeq A$$

This proves that $\mathcal{O}$ and Φ are one inverse of the others between isomorphism classes of affine varieties and k-algebras. $\square$

This fact allows us to move our attention from varieties to k-algebras, since it immediately implies that two affine varieties are isomorphic if and only if they have isomorphic coordinate rings. Moreover for any finitely generated k-algebra without nonzero nilpotents, we can find an affine variety having it as coordinate ring; this variety is unique up to isomorphisms of affine varieties.

1.1.2 The construction of abstract varieties

In this section we introduce briefly the way of defining *abstract varieties*, i.e. varieties that are not naturally embedded in an affine or projective space. This idea will be particularly important in the discussion about defining a toric variety.

The basic idea is to glue affine varieties in a compatible way in order to obtain something that locally is an affine variety; if we think about it, we notice that this is the way in which many objects in geometry are defined, for example differentiable manifolds, projective varieties and schemes.

Consider a finite collection of affine varieties $\{V_\alpha\}$ over an algebraically closed field k and suppose that for any pair α, β there exist:

- a Zariski open subset of V_α, call it $V_{\beta\alpha} \subseteq V_\alpha$

- a Zariski open subset of V_β, call it $V_{\alpha\beta} \subseteq V_\beta$

- an isomorphism of varieties $g_{\beta\alpha} : V_{\beta\alpha} \to V_{\alpha\beta}$

such that:

- if the two indices are the same, $V_{\alpha\alpha} = V_\alpha$ and $g_{\alpha\alpha} = id_{V_\alpha}$

- for every pair α, β one has $g_{\beta\alpha} = g_{\alpha\beta}^{-1}$

- for every α, β, γ one has that $g_{\beta\alpha}(V_{\beta\alpha} \cap V_{\gamma\alpha}) = V_{\alpha\beta} \cap V_{\gamma\beta}$ and $g_{\gamma\alpha} = g_{\gamma\beta} \circ g_{\beta\alpha}$ on $V_{\beta\alpha} \cap V_{\gamma\alpha} \subseteq V_\alpha$

The requirements we ask for are made in order to ensure that the relation $\sim$ defined on $\sqcup_\alpha V_\alpha$ by setting, for every $x \in V_\alpha$, $y \in V_\beta$

$$x \sim y \iff x \in V_{\beta\alpha}, y \in V_{\alpha\beta} \text{ and } g_{\beta\alpha}(x) = y$$

is an equivalent relation. In fact, we have reflexivity from the first bullet (moreover, if x, y belongs to the same variety, they are in relation if and only if $x = y$), symmetry from the second one and transitivity from the third one[2].

Now we can define the gluing of these varieties.

Definition 1.5. Let $\{V_\alpha\}$ be a finite collection of affine varieties over a field k, such that for any pair of indices α and β there exist two open subsets and an isomorphism satisfying the same properties as above. We call **abstract variety** determined by the data $\big(\{V_\alpha\}, \{V_{\alpha\beta}, \}, g_{\alpha\beta}\big)$ the quotient

$$X = \bigsqcup_\alpha V_\alpha / \sim$$

with $\sim$ as above. We consider this space endowed with the quotient topology.

Remark 1.1. The name "abstract varieties" comes from the fact that these varieties are not naturally embedded in any affine or projective spaces, but they are constructed by gluing a finite number of affine varieties according to a certain equivalent relation.

Remark 1.2. One easily remarks that an abstract variety is locally an affine variety, since for any point P of the topological space X, i.e. an equivalence class, one can consider one of its representative $x \in \sqcup V_\alpha$. It is surely contained in some V_α, that is an open set in $\sqcup V_\alpha$. Then, by definition of quotient topology, $\overline{V_\alpha} = \{[y] : y \in V_\alpha\} = \pi(V_\alpha)$ is an open subset of X. Resuming, for every $P \in X$, there exists an open subset of X containing P and homeomorphic (through the map $\pi_\alpha = \pi_{|V_\alpha}$) to V_α, namely $\overline{V_\alpha}$: the map is injective since, as stated, two elements in V_α are equivalent if and only if they are the same.

We can put a sheaf of rings on this space X simply defining the sheaf of regular functions; for any open subset U of X, take:

$$\mathcal{O}_X(U) = \{f : U \to k \text{ such that } f \circ \pi_\alpha : \pi_\alpha^{-1}(U \cap \overline{V_\alpha}) \to k \text{ is regular for all } \alpha\}$$

This is well defined from the conditions required on $g_{\alpha\beta}$, in particular the fact that it is an isomorphism of varieties.

In the end this clarifies that X is covered by finitely many open subset $\overline{V_\alpha}$ each of which is isomorphic to an affine variety (in fact one puts on it the sheaf of regular functions coming from V_α).

Example 1.6. The simplest nontrivial example of a variety obtained by gluing is the projective space. Let's look at the case $\mathbb{P}^1_k$ for a certain algebraically closed field k. One way to obtain it is to consider two affine lines k and to glue them along k^*, with transition maps given by inversion. To be more precise it means that we consider the two varieties $V_1 = V_2 = \mathbb{A}^1_k$ and we take $V_{12} = V_{21} = \mathbb{A}^1_k - \{0\}$ (they are

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[2]Here one needs both the fact that there is equality between the two sets and that the isomorphisms well behaves under composition.

Zariski open subsets of the two varieties). As isomorphisms, take the identities on each open and the map $g_{21}: V_{21} \to V_{12}$ given by $g_{21}(z) := 1/z$ and $g_{12}: V_{12} \to V_{21}$ given by $g_{12}(z) := 1/z$ (they are bijective maps, they can be seen to be isomorphism of varieties). All the requirement are easily satisfied and one can glue. The result of the gluing is exactly $\mathbb{P}_k^1$.

1.2 The n-dimensional k-torus

Consider an algebraically closed field k. Observe that some open sets in the affine space $\mathbb{A}_k^n$ can also be seen as varieties. In fact, they could *a priori* be isomorphic to some affine variety. For example consider $X = (k^*)^n$. It is an open subset of the affine n-dimensional space since it can be written as $X = k^n - V(x_1 \cdot x_2 \cdot \dots \cdot x_n)$. Moreover take $Y = V(x_1 \cdot x_2 \cdot \dots \cdot x_n \cdot x_{n+1} - 1)$ in k^{n+1} (here we consider a new coordinate x_{n+1}). The projection map $Y \to X$ that forgets the last coordinate gives a bijection with inverse $(x_1, x_2, \dots, x_n) \mapsto (x_1, x_2, \dots, x_n, x_1^{-1} \cdot x_2^{-1} \dots x_n^{-1})$ that is well defined since each coordinate of the element in X is non zero by definition. Explicitly:

$$(x_1, x_2, \dots, x_n) \mapsto (x_1, x_2, \dots, x_n, x_1^{-1} \cdot x_2^{-1} \cdot \dots \cdot x_n^{-1}) \mapsto (x_1, x_2, \dots, x_n)$$

and

$$(x_1, x_2, \dots, x_n, x_{n+1}) \mapsto (x_1, x_2, \dots, x_n) \mapsto (x_1, x_2, \dots, x_n, x_1^{-1} \cdot x_2^{-1} \cdot \dots \cdot x_n^{-1})$$
$$= (x_1, x_2, \dots, x_n, x_{n+1})$$

(the key point is that any point in Y satisfies $x_1 \cdot x_2 \cdot \dots \cdot x_n \cdot x_{n+1} = 1$ by definition). The situation is well visible in $\mathbb{R}^2$, as shown in Figure 1.1.
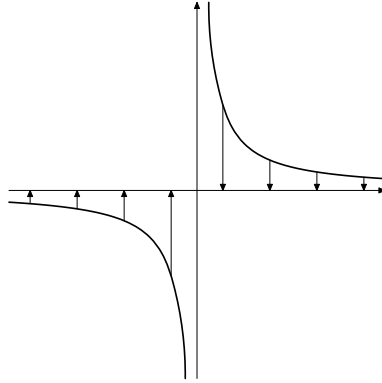
Figure 1.1: The isomorphism between $\mathbb{R}^*$ and the variety $V(xy - 1)$ in $\mathbb{R}^2$.

Anyway the map between Y and X described above is polynomial, hence algebraic and moreover bijective, so it gives X the structure of an affine variety. Its coordinate ring is the same as the one of Y, so:

$$\mathcal{O}((k^*)^n) \simeq k[x_1, x_2, \dots, x_n, x_{n+1}]/V(x_1 \cdot x_2 \cdot \dots \cdot x_{n+1} - 1)$$
$$\simeq k[x_1, x_2, \dots, x_n, x_1^{-1} \cdot x_2^{-1} \dots x_n^{-1}] =$$
$$= k[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}]$$

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where the last equality is easily checked by double inclusion.

But the interesting property of the variety $(k^*)^n$ is that one can endow it with a group structure, given by the obvious componentwise multiplication:

$$(k^*)^n \times (k^*)^n \to (k^*)^n$$

$$(x_1, x_2, \ldots, x_n) \cdot (y_1, y_2, \ldots, y_n) = (x_1 y_1, x_2 y_2, \ldots, x_n y_n)$$

This operation is straightforwardly a morphism of varieties, hence the object we described is in fact an *algebraic group*. Due to its importance we give it a name.

Definition 1.7. Let k be a field. One calls n-**dimensional** k-**torus** any affine variety that is isomorphic to the affine variety $(k^*)^n$ together with the multiplication inherited by the isomorphism.

For example, $\mathbb{C}^*$ is a torus. If k is a finite field, then any k-torus has finitely many elements; a strange case is given by the $\mathbb{F}_2$-tori: they all have only one element!

Definition 1.8. Let T_1, T_2 be two k-tori. A **morphism of tori** is a map $T_1 \to T_2$ that is both a morphism of algebraic varieties and a morphism of groups.

They satisfy some interesting properties as follows.

Proposition 1.9. *Let k be a field and let T_1, T_2 and T be three k-tori. Let also $\varphi : T_1 \to T_2$ be a morphism of tori. Then:*

(a) the image of φ is a torus and it is closed in T_1.

(b) if U is both an irreducible subvariety and a subgroup of T, then U is a torus.

Proof.
Refer for example to [1]. $\qquad\qquad\square$

1.3   Varieties from the schemes point of view

In the language of schemes one can define an affine variety and all the objects defined above in a compact way. The definition of scheme, in fact, allow us to put in an unique object both the topological space and the ring of regular functions over that space. Recall in fact, following for example [10, Ch II] or [9] that for every ring A we can define the topological space $\operatorname{Spec} A$, which is the set of all the prime ideals of A endowed with the Zariski topology whose closed subsets are the set of the form $V(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Spec} A : \mathfrak{p} \supseteq \mathfrak{a}\}$. Moreover this topological space can be given a sheaf of rings defined as follows; for every open subset U of $\operatorname{Spec} A$ define

$$\mathcal{O}(U) := \Big\{ s : U \to \bigsqcup_{\mathfrak{p} \in U} A_{\mathfrak{p}} : s(\mathfrak{p}) \in A_{\mathfrak{p}} \text{ and for all } \mathfrak{p} \in U \text{ there exists } V \subseteq U \text{ open,}$$

$$a, f \in A \text{ such that } \forall \mathfrak{q} \in V \text{ one has } f \notin \mathfrak{q} \text{ and } s(\mathfrak{q}) = a/f \in A_{\mathfrak{p}} \Big\}$$

where $A_{\mathfrak{p}}$ denote the localization of A with respect to the prime ideal $\mathfrak{p}$. Another way to introduce this sheaf is to consider the collection of principal open subset of $\operatorname{Spec} A$ made by $\{D(f)\}_{f \in A}$ where $D(f) = \operatorname{Spec} A - V(\langle f \rangle)$; they are a basis for

the topology on $\operatorname{Spec} A$ and hence it is enough to define a sheaf on them to obtain a unique sheaf on $\operatorname{Spec} A$. The sheaf obtained by setting $\mathcal{O}(D(\langle f \rangle)) = A_f$ (the localization of A in f) gives the same sheaf as before. Hence we obtain a ringed space $(\operatorname{Spec} A, \mathcal{O})$. Moreover recall that the stalk of a scheme $\operatorname{Spec} A$ at a point $\mathfrak{p}$ is defined as $\mathcal{O}_{\mathfrak{p}} = \varinjlim_{\mathfrak{p} \in U} \mathcal{O}(U)$. One has that the stalks of $(\operatorname{Spec} A, \mathcal{O})$ are local rings. We say that $(\operatorname{Spec} A, \mathcal{O})$ is a locally ringed space.

Definition 1.10. An **affine scheme** is a locally ringed space (i.e. a topological space together with a sheaf of ring with local stalks) that is isomorphic as a locally ringed space to $\operatorname{Spec} A$ for some ring A. Moreover a **scheme** is a locally ringed space $(X, \mathcal{O}_X)$ such that for any point $p \in X$ there exists an open neighborhood U of X such that $(U, \mathcal{O}_{X|U})$ is an affine scheme.

This gives us the possibility of defining varieties over arbitrary base rings. Recall that an R-scheme is by definition a scheme X together with a morphism of schemes $X \to \operatorname{Spec} R$.

Definition 1.11. Let R be a ring. An **affine variety** over R is an R-scheme that is isomorphic to the spectrum of a finitely generated R-algebra.

Remark 1.3. In this way an affine variety over an algebraically closed field k is of the form $\operatorname{Spec}(k[x_1, x_2, \ldots, x_n]/J)$ for a certain ideal J, since all the finitely generated k-algebra have the form of the ring in parenthesis. This means that with this definition we are extending a little bit also the idea of affine variety over a field k. In fact in this way we are obtaining all the varieties over k in the previous sense, but we are adding also schemes as $\operatorname{Spec}(k[x]/(x^2))$ for example. Even if this is not considered as an affine variety in classical algebraic geometry, the functor $\mathcal{O}$ of Theorem 1.4 would associate to the finitely generated k-algebra $k[x]/(x^2)$ the zero set $V(x^2)$ that is the same of $V(x)$. According to classic algebraic geometry the two sets are the same, but in the new point of view the two k-varieties are different as schemes! In this way we extend the correspondance of Theorem 1.4 to a correspondance between finitely generated k-algebras and affine k-varieties.

Now we can give the definition of a variety over R.

Definition 1.12. A **variety** over R is a R-scheme that can be covered by a finite number of open subschemes[3] that are affine varieties over R.

Remark 1.4. One sees that the previous definition transfers the concept of abstract variety from classical terms to schemes terms. In fact let's consider a variety over a field k. It is a locally ringed space $(X, \mathcal{O}_X)$ that is a k-scheme covered by a finite number of subschemes $(U_\alpha, \mathcal{O}_{U_\alpha}) = (U_\alpha, \mathcal{O}_{X|U_\alpha})$ each of which is isomorphic to an affine variety over k. This means that locally a variety over R is an affine variety over R, that is exactly the way we constructed an abstract variety.

Conversely, let's take a finite number of affine varieties over k, $(X_\alpha, \mathcal{O}_{X_\alpha})$ and suppose that for any pair of indices α, β there exist two open subschemes $U_{\beta\alpha}$ and $U_{\alpha\beta}$ of X_α and X_β respectively that are isomorphic as locally ringed spaces. Following [10,

---

[3]Recall that an open subscheme of a scheme $(X, \mathcal{O}_X)$ is a scheme of the form $(U, \mathcal{O}_{X|U})$, where U is an open subset of the topological space X.

Ex II.3.5][4] one can glue the schemes along these subschemes and get another scheme: the construction is basically the same as the one made in classical terms, but it results to be more natural, since the gluing conditions are contained in the existence of the isomorphism of locally ringed spaces among the subschemes.

Example 1.13. Using the language of schemes, we can define the projective space on any base ring R. In order to do this, let's recall the alternative construction of the projective line as abstract variety, as seen in Example 1.6. We will define the projective scheme $\mathbb{P}^1_R$ as a gluing. The construction generalizes the one for varieties and goes like this: consider two copies of the affine line $X_1 = X_2 = \mathbb{A}^1_R$, so that $X_1 = \mathrm{Spec}(R[x])$ and $X_2 = \mathrm{Spec}(R[y])$. Consider also two open subschemes X_{21} and X_{12} of X_1 and X_2 respectively, defined via localizations: $X_{21} = \mathrm{Spec}(R[x]_x)$ and $X_{12} = \mathrm{Spec}(R[y]_y)$. Take the isomorphism $X_{21} \to X_{12}$ coming from the isomorphism of R-algebras $y \mapsto x^{-1}$ (recall that this automatically induces an isomorphism between the corresponding affine schemes). In analogy with the case treated for varieties, we will call $\mathbb{P}^1_R$ the scheme obtained by this gluing[5]. It results to be the same as the one obtained with the Proj construction (see [10, Ex II.2.5.1]); moreover this definition can be extended to higher dimensions.

We will prefer to use the scheme notation to introduce the object we will play with; in fact not only it gives the possibility to treat the topic in greater generality, but it also makes some constructions and definitions more "self raising".
For example, the idea of morphism between affine varieties, that doesn't appear to be so natural in definition 1.3, is easier in schemes terms. In fact the classical definition of morphism between two varieties $V_1 = V(J_1)$ and $V_2 = V(J_2)$ is simply the requirement that for any open subset U of V_2 one has a ring homomorphism:

$$\mathcal{O}_{V_2}(U) \to \mathcal{O}_{V_1}(\phi^{-1}(U))$$

$$f \mapsto f \circ \varphi$$

where $\mathcal{O}_{V_i}$ is the sheaf of regular functions on open subsets of V_i, $i = 1, 2$. This can be restated simply asking that there exist a sheaf morphism between the structure sheaves (i.e. sheaf of regular functions) of the two varieties, hence a morphisms of schemes between $\mathrm{Spec}(k[x_1, x_2, \ldots, x_n]/J_1)$ and $\mathrm{Spec}(k[x_1, x_2, \ldots, x_n]/J_2)$. In this way the classical definition of morphism of affine varieties comes from the definition of morphism of schemes. In the same way for abstract varieties.

1.3.1 Normal integral noetherian varieties

In order to define the class of Weil divisors over a variety, a classical requirement is the fact that the variety is normal. generalizing to schemes, one should require (as we will see in Chapter 4) that the scheme is normal, integral and noetherian. In this paragraph we then try to recall this definitions.

---

[4]Here the computations are carried out for two schemes, but the construction is extended to the general case.

[5]Remark that if one chooses the isomorphism of schemes coming from the isomorphism of R-algebras $y \mapsto x$ one obtains the double origin affine line, the simplest example of non separated scheme.

Definition 1.14. A scheme X is called **irreducible** if it is irreducible as a topological space.

Definition 1.15. A scheme X is called an **integral scheme** if it is irreducible and every stalk $\mathcal{O}_{X,p}$ is an integral domain.

Definition 1.16. A scheme X is called locally noetherian if X admits an affine open covering $X = \cup X_i$ such that $\mathcal{O}_X(X_i)$ is a noetherian ring for all i. A **noetherian scheme** is a scheme X that is quasi-compact in the Zariski topology (any open covering of X admit a finite subcovering) and locally noetherian.

Remark 1.5. Equivalently, a scheme is noetherian if it admits a finite open cover by open affine subsets $\operatorname{Spec} A_i$ with all the A_i noetherian rings.

Definition 1.17. A scheme X over a ring R is called **normal** if for any point $P \in X$ the stalk $\mathcal{O}_{X,p}$ is an integrally closed ring.

To have better understanding of the definitions, let's state, without proving, some results and immediate consequences of them for a variety X over a ring R. One can see, referring to Tong for example, that:

- a scheme is normal if and only if for each non empty open subscheme U, the ring $\mathcal{O}_X(U)$ is a normal integral domain. Hence if X is an normal variety over R, then its affine pieces are $\operatorname{Spec} A_i$ with any A_i normal integral domain.

- a scheme is integral if and only if for each non empty open subscheme U, the ring $\mathcal{O}_X(U)$ is integral. Hence if X is an integral variety, then its affine pieces are $\operatorname{Spec} A_i$ with any A_i integral domains.

- if a scheme is noetherian, then so is any open subscheme of it, and moreover $\operatorname{Spec} A$ is noetherian if and only if A is noetherian. So asking that a variety X is noetherian implies that its affine pieces are $\operatorname{Spec} A_i$ with any A_i noetherian ring. Conversely, by definition, if all the affine pieces of the variety have noetherian rings of coordinates, then the variety is locally noetherian.

Now let's present the situation we will have: suppose that the variety X is connected and quasi compact. By definition, it is covered by finitely many affine pieces, $X = \cup X_i$ where each X_i is an affine R-variety with coordinate ring A_i. Suppose moreover that every A_i is a noetherian integral domain. Then we have that X is locally noetherian (simply applying the definition) and hence[6] X is an integral noetherian scheme. This is a typical way of reasoning we will use in Chapter 3.

1.4 The torus from the schemes point of view

In order to extend the definition of toric variety to any base ring R, one have to introduce the n-dimensional torus over a ring R. To do this it is essential to use the language of schemes, as done in the previous section to define varieties over a ring. From the observation that $\mathcal{O}((k^*)^n) = k[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$, the most natural way to

---

[6]For locally noetherian schemes, being integral is equivalent to being a connected scheme covered by spectra of integral domains.

extend the definition to a generic ring is to call n-dimensional R-torus the R-scheme $\mathrm{Spec}(R[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}])$. We would like to continue considering the torus both as a variety and a group; to pursue this aim it is necessary to present the theory of group schemes.

1.4.1   Group schemes

Recall that a scheme over S is a scheme X with a morphism $\pi : X \to S$. From a simple observation on the universal property of the fiber product one could see that $X \times_S S$ and $S \times_S X$ are both isomorphic to X, call j_1 and j_2 the two isomorphisms. Recall also that if X, Y, X', Y' are S-schemes and $f : X \to X'$, $g : Y \to Y'$ are morphism of S-schemes, by the universal property of the fiber product one defines $f \times g : X \times_S Y \to X' \times_S Y'$.

Moreover, if X is an S-scheme, the same universal property gives a morphism $\Delta_X : X \to X \times_S X$, the so called *diagonal embedding.*

Finally, for every S-scheme X call $s_X : X \times_S X \to X \times_S X$ the morphism swapping the factors of the fiber product[7].

We are now ready to define a group scheme.

Definition 1.18. Let S be a scheme and let G be a scheme over S. The scheme G together with three morphisms of S-schemes

$$m : G \times_S G \to G$$

$$i : G \to G$$

$$e : S \to G$$

is called a **group scheme over** S if the following properties are satisfied:

- *(associativity)* $m \circ (m \times id_G) = m \circ (id_G \times m)$

$$
\begin{array}{ccc}
G \times_S G \times_S G & \xrightarrow{m \times id_G} & G \times_S G \\
\downarrow{\scriptstyle id_G \times m} & & \downarrow{\scriptstyle m} \\
G \times_S G & \xrightarrow{\quad m \quad} & G
\end{array}
$$

- *(neutral element)* $m \circ (id_G \times e) = j_1$ and $m \circ (e \times id_G) = j_2$

$$
\begin{array}{ccc}
G \times_S S & \xrightarrow{id_G \times e} & G \times_S G \\
 & {\scriptstyle j_1} \searrow & \downarrow{\scriptstyle m} \\
 & & G
\end{array}
$$

- *(inverse element)* $m \circ (id_G \times i) \circ \Delta_G = e \circ \pi$ and $m \circ (i \times id_G) \circ \Delta_G = e \circ \pi$

$$
\begin{array}{ccc}
G \xrightarrow{\Delta_G} G \times_S G & \xrightarrow{id_G \times i} & G \times_S G \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle m} \\
S & \xrightarrow{\quad e \quad} & G
\end{array}
$$

[7]This is obtained by the universal property of fiber products swapping the morphism from $X \times_S X$ to each one of the two factors X.

Moreover G is called a **commutative group scheme over** S is also the following property is true:

- *(commutativity)* $m \circ s_G = m$

$$G \times_S G \xrightarrow{\ s_G\ } G \times_S G$$

with m going diagonally and m going down to G.

Then, it is not difficult to prove that if G and H are two group schemes over S, then also $G \times_S H$ is a group scheme over S.

As usual, one can introduce the concept of morphism between group schemes.

Definition 1.19. Let S be a scheme and let G, H be two group schemes over S, with multiplication law m_1 and m_2 respectively. A **morphism of group schemes** is a morphism of S-schemes $f : G \to H$ such that $f \circ m_1 = m_2 \circ (f \times f)$.

$$
\begin{array}{ccc}
G \times_S G & \xrightarrow{\ f \times f\ } & H \times_S H \\
\downarrow{\scriptstyle m_1} & & \downarrow{\scriptstyle m_2} \\
G & \xrightarrow{\quad f \quad} & H
\end{array}
$$

1.4.2 The n-dimensional R-torus

Using the terminology developed in the previous paragraph we can now extend the definition of the torus given in Section 1.2 to a larger meaning.

Definition 1.20. Let R be a ring. A **1-dimensional R-torus** is a R-scheme isomorphic (as a group scheme) to $\mathrm{Spec}(R[x^{\pm 1}])$ endowed with the morphisms:

$$m : \mathrm{Spec}(R[x^{\pm 1}]) \times_R \mathrm{Spec}(R[x^{\pm 1}]) \to \mathrm{Spec}(R[x^{\pm 1}])$$

coming from the morphism $x \mapsto x \otimes x$,

$$i : \mathrm{Spec}(R[x^{\pm 1}]) \to \mathrm{Spec}(R[x^{\pm 1}])$$

coming from the morphism $x \mapsto x^{-1}$ and

$$e : \mathrm{Spec}\, R \to \mathrm{Spec}(R[x^{\pm 1}])$$

coming from the morphism $x \mapsto 1$.

It is common to denote the group scheme $\mathrm{Spec}(R[x^{\pm 1}])$ introduced above as $\mathbb{G}_{m,R}$.

Definition 1.21. Let R be a ring. A **n-dimensional split R-torus** is an R-scheme isomorphic (as a group scheme) to the group scheme $\mathbb{G}_{m,R}^n = \mathrm{Spec}(R[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}])$ (the power n is a fiber product power, the group laws are inherited from the ones of $\mathbb{G}_{m,R}$).

Following definition 1.11 one has that the n-dimensional R-torus is an affine variety over R, since it is the spectrum of a finitely generated R-algebra, moreover we endowed it with a group structure. This will be a key point in the definition of a toric scheme. It will be usual to denote the torus $\mathbb{G}_{m,R}^n = \mathrm{Spec}(R[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}])$ also with the notation $\mathbb{T}_{n,R}$, in order to underline is structure as group scheme.

1.4.3 The action of a group scheme

Since a classical toric variety involves the action of a torus that it contains as an open subset, it will be essential to understand how the definition of group action translates in the schemes language. Recall that if X is an S-scheme the fiber product $S \times_S X$ is canonically isomorphic to X, call j_2 the isomorphism.

Definition 1.22. Let S be a ring, let X be a scheme over S and G a group scheme over S with multiplication law m and neutral element e. An **action of the group scheme** G over X is a morphism of schemes

$$a : G \times_s X \to X$$

such that the following holds:

- *(associativity)* $a \circ (m \times id_X) = a \circ (id_G \times a)$

$$
\begin{array}{ccc}
G \times_S G \times_S X & \xrightarrow{\ m \times id_X\ } & G \times_S X \\
{\scriptstyle id_G \times a}\big\downarrow & & \big\downarrow{\scriptstyle a} \\
G \times_S X & \xrightarrow{\qquad a \qquad} & X
\end{array}
$$

- *(action of the neutral element)* $a \circ (e \times id_X) = j_2$

$$
\begin{array}{ccc}
S \times_S X & \xrightarrow{\ e \times id_X\ } & G \times_S X \\
& {\scriptstyle j_2}\searrow & \big\downarrow{\scriptstyle a} \\
& & X
\end{array}
$$

1.5 Toric schemes

We are now ready to give the definition of a toric scheme. This is clearly inspired by Definition 0.5.

Definition 1.23. Let R be a ring. A **toric scheme** over the ring R is a scheme X over R together with an n-dimensional R-torus as an open subscheme such that the multiplication m of the torus extends to an action of the torus on the scheme X.

To be explicit, we mean that an R-scheme is a toric scheme if there exists

$$i : \mathbb{T}_{n,R} \to X$$

such that i is an open immersion and there exists an action a of the group scheme $\mathbb{T}_{n,R}$ over X such that the following diagram commutes (call $S = \mathrm{Spec}(R)$):

$$
\begin{array}{ccc}
\mathbb{T}_{n,R} \times_S \mathbb{T}_{n,R} & \xrightarrow{\ m\ } & \mathbb{T}_{n,R} \\
{\scriptstyle id_{\mathbb{T}} \times i}\big\downarrow & & \big\downarrow{\scriptstyle i} \\
\mathbb{T}_{n,R} \times_S X & \xrightarrow{\ a\ } & X
\end{array}
$$

Remark 1.6. When one considers a field k, one recovers that all the toric varieties over k can be seen as toric schemes.

Example 1.24. Let k be an algebraically closed field and consider the k-schemes $X = \mathrm{Spec}(k[x,y]/(x^2 - y^3))$ and $\mathbb{T} = \mathbb{T}_{1,k}$. It is clear that, set theoretically the two schemes appear as (with abuse of notation):

$$X = \mathrm{Spec}(k[x,y]/(x^2 - y^3))$$
$$= \{\text{classes of prime ideals of } k[x,y] \text{ containing } (x^2 - y^3)\}$$
$$= \{(0), (x - a^3, y - a^2) : a \in k\}$$
$$\mathbb{T} = \mathrm{Spec}(k[z^{\pm 1}]) = \mathrm{Spec}(k[z,w]/(zw - 1))$$
$$= \{\text{classes of prime ideals of } k[z,w] \text{ containing } (zw - 1)\}$$
$$= \{(0), (z - a, w - a^{-1}) : a \in k^*\}$$

Consider now the morphism of schemes $i : \mathbb{T} \to X$ coming from the R-algebras morphism

$$\varphi : k[x,y]/(x^2 - y^3) \to k[z^{\pm 1}]$$
$$\varphi : x \mapsto z^3$$
$$\varphi : y \mapsto z^2$$

It is clear that φ is well defined. Moreover set theoretically it can be seen that:

$$i(\mathbb{T}) = \{(0), i((z - a, w - a^{-1})) : a \in k^*\}$$
$$= \{(0), (x - a^3, y - a^2) : a \in k^*\}$$
$$= X - \{(x, y)\}$$

and so i is an open embedding of schemes. We now define an action of $\mathbb{T}$ on X simply by defining $a : \mathbb{T} \times_S X = \mathrm{Spec}(k[z^{\pm 1}] \otimes_k k[x,y]/(x^2 - y^3)) \to X = \mathrm{Spec}(k[x,y]/(x^2 - y^3))$ coming from the morphism of k-algebras

$$\alpha \mapsto \varphi(\alpha) \otimes \alpha$$

One can check that this is an action in the sense of Definition 1.22, moreover the commutativity of the diagram of Definiton 1.23 comes from the commutativity of the corresponding diagram on the rings of coordinates:

$$
\begin{array}{ccc}
k[z^{\pm 1}] \otimes k[z^{\pm 1}] & \longleftarrow & k[z^{\pm 1}] \\
\uparrow & & \uparrow \\
k[z^{\pm 1}] \otimes k[x,y]/(x^2 - y^3) & \longleftarrow & k[x,y]/(x^2 - y^3)
\end{array}
$$

$$
\begin{array}{ccc}
\varphi(\alpha) \otimes \varphi(\alpha) & \longleftarrow & \varphi(\alpha) \\
\uparrow & & \uparrow \\
\varphi(\alpha) \otimes \alpha & \longleftarrow & \alpha
\end{array}
$$

This proves that the scheme $X = \mathrm{Spec}(k[x,y]/(x^2 - y^3))$ is a k-toric scheme. We also know that this scheme is not normal, so we constructed an example of a *nonnormal k-toric scheme*. This will be an important example in the following.

Chapter 2

A particular class of toric schemes

In this chapter we present a relevant class of toric schemes over a ring R. The ideas we will present are the ones which relate the theory of toric varieties with combinatorics.

2.1 Affine semigroup algebras

A very interesting aspect of the theory of toric varieties is the fact that a wide class of them can be constructed as the spectrum of some special ring, namely of R-semigroup algebras. This is also the starting point of the link between the combinatoric part of toric varieties and the definition from algebraic geometry.

We recall that a lattice is a free abelian group of finite rank. Hence, as a group, it is always isomorphic to $\mathbb{Z}^n$ for some positive integer n.

Definition 2.1. An **affine semigroup** is the datum of a set S and an operation $+$ satisfying:

1. the operation is commutative, associative and it admits a neutral element 0_S.

2. there exists a finite set of elements generating S as a semigroup.

3. there exists an injective morphism of semigroups (i.e. a morphism respecting the operation) from S to a lattice M.

In other words, an affine semigroup is a finitely generated commutative monoid that can be embedded in a lattice. Notice that a subset $\mathcal{A}$ of an affine semigroup S generates, as a semigroup, the set

$$\langle \mathcal{A} \rangle = \left\{ \sum_{a \in \mathcal{A}} n_a a : n_a \in \mathbb{N}, \text{ almost all zeros} \right\}$$

hence, by point two in the definition, all the affine semigroups are of the form $\mathbb{N}\mathcal{A}$ for some finite set $\mathcal{A}$.

Now we want to define an R-algebra structure from the an affine semigroup, where R is a ring. The most natural way consists in building a polynomial R-algebra. In order to do this, let's consider an affine semigroup $(S, *)$ and a field k; we will use

the symbol $*$ for the operation in the semigroup to avoid confusion with the additive operation in k. Then, we define the set:

$$R[S] = \left\{ \sum_{s \in S} c_s T^s : c_s \in R, \text{ almost all zeros} \right\}$$

and we endow it with the operations:

$$\sum_{s \in S} c_s T^s + \sum_{t \in S} d_t T^t = \sum_{s \in S} (c_s + d_s) T^s$$

$$\left(\sum_{s \in S} c_s T^s \right) \left(\sum_{t \in S} d_t T^t \right) = \sum_{u \in S} \sum_{s*t=u} (c_s \cdot d_t) T^u.$$

This simply results to be a R-algebra with 1-element $1_R T^{0_S}$.

Remark 2.1. If the semigroup S is generated by the set $\mathcal{A}$, then $R[S]$ is generated (as an R-algebra) by the set of symbols $\{T^a : a \in \mathcal{A}\}$.

Example 2.2. Let's consider a lattice M. It is true that it is an affine semigroup: in fact, by definition, it is generated, as a group, by a finite set $\{e_1, e_2, \ldots, e_n\}$. As a consequence it is a semigroup generated (as a semigroup) by $\mathcal{A} = \{\pm e_1, \pm e_2, \ldots, \pm e_n\}$; moreover it is a commutative monoid that can be embedded in itself. By the previous remark, for any ring R, $R[M] = R[T^{\pm e_1}, T^{\pm e_2}, \ldots, T^{\pm e_n}]$. The map $R[M] \to R[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$ obtained extending the function on the generators $T^{e_i} \longmapsto x_i$ and $T^{-e_i} \longmapsto x_i^{-1}$ for every $i = 1, 2, \ldots, n$ is easily verified to be an isomorphism of R-algebras. Hence for any lattice M of rank n

$$R[M] \simeq R[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}].$$

This simple example will be the key one to prove that the schemes we will construct are toric schemes and that they satisfies certain properties.

2.2 Convex polyhedral cones

This paragraph, in which we introduce the basic objects of the combinatorial point of view on toric schemes, will be studied following the presentation by Fulton done in [7, Ch 1.2]. We also refer to it for all the proves.

Definition 2.3. Let V be a $\mathbb{R}$-vector space and choose a finite number of vectors $v_1, v_2, \ldots, v_s$ in V. The **convex polyhedral cone** generated by $v_1, v_2, \ldots, v_s$ is defined as the set

$$\sigma = \left\{ \sum_{i=1}^{s} \lambda_i v_i : \lambda_i \geq 0 \right\}.$$

Remark 2.2. A convex polyhedral cone σ defined as above is a convex set (in the sense that for any $x, y \in \sigma$ and for any $0 \leq t \leq 1$, one has $tx + (1-t)y \in \sigma$) and also a cone (in the sense that for any $x \in \sigma$ and for any $t \geq 0$, $tx \in \sigma$).
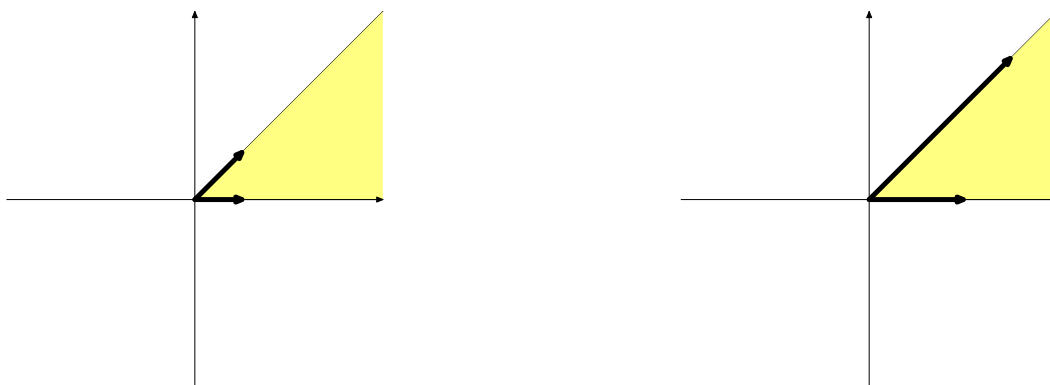
Figure 2.1: A cone in $\mathbb{R}^2$.

A **strongly convex polyhedral cone** is a convex polyhedral cone not containing any linear subspace of the vector space in which it is lying. This is equivalent to the fact that it $\sigma \cap (-\sigma) = \{0\}$.

Observe that different sets of vectors can generate the same cone. For example the cone in figure 1.1 can be generated by $\{(1,0),(1,1)\}$ and also by $\{(2,0),(3,3)\}$. The generators of the cone can be then changed by scalar multiplication. When we will treat cones and lattices, we will add the fact that a basis of the cone can be chosen in some "minimal" way.

Definition 2.4. Let σ be a convex polyhedral cone in V generated by the vectors $v_1, v_2, \ldots, v_s$. The **dimension** of σ is the dimension of the V-subvector space $\sigma + (-1)\sigma = \text{span}(v_1, v_2, \ldots, v_s)$.

Denote with V^* the dual vector space of V, i.e. $V^* := \text{Hom}(V, \mathbb{R})$. There exist a natural pairing $\langle \ , \ \rangle : V^* \times V \to \mathbb{R}$, given by $\langle u, v \rangle := u(v)$. This allows us to define the following fundamental object.

Definition 2.5. Let σ be a convex polyhedral cone in V. The **dual** of σ is the set

$$\sigma^\vee := \{u \in V^* : \langle u, v \rangle \geq 0 \text{ for all } v \in \sigma\}.$$

Example 2.6. Take the first quadrant in the plane. It is the cone generated by the vectors $\{(1,0),(0,1)\}$. We already know that the pairing in $\mathbb{R}^2$ is given by the usual dot product, hence:

$$\sigma^\vee = \{(a,b) \in \mathbb{R}^2 : ax + by \geq 0 \text{ for all } x, y \geq 0\} = \{(a,b) \in \mathbb{R}^2 : a, b \geq 0\} = \sigma$$

since clearly we have one inclusion and conversely it is enough to fix $x = 0$ (or $y = 0$) and take the other variable different from zero.

As seen in the previous example, the dual of a certain cone is again a cone. This is a general fact.

Proposition 2.7. *The dual $\sigma^\vee$ of a convex polyhedral cone σ is a convex polyhedral cone and moreover $(\sigma^\vee)^\vee = \sigma$.*
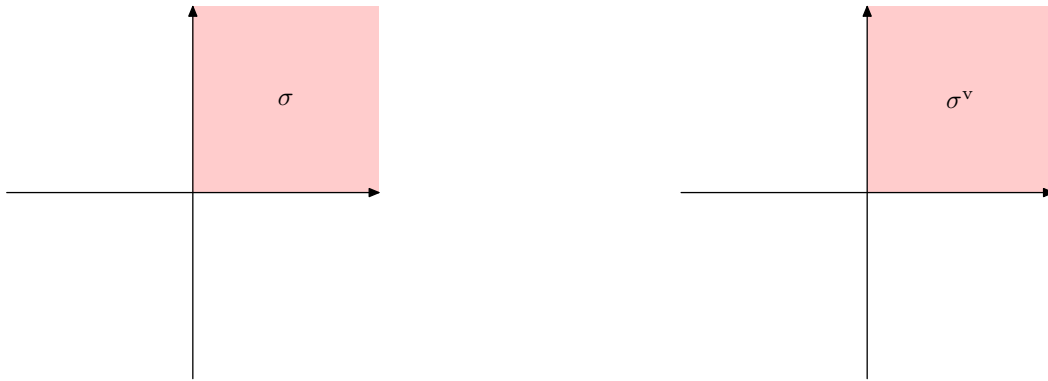
Figure 2.2: The cone of example 2.6 and its dual.

Proof.
Refer to Fulton, [7]. ☐

We now define another object that will be crucial in the theory of toric varieties.

Definition 2.8. Let σ be a convex polyhedral cone in the vector space V. For any $u \in V^*$ denote $u^\perp := \{v \in V : \langle u, v \rangle = 0\}$. A **face** of the cone σ is a subset of σ of the form $\{v \in \sigma : \langle u, v \rangle = 0\} = \sigma \cap u^\perp$ for some $u \in \sigma^\vee$. We will denote the fact that τ is a face of σ writing $\tau \preceq \sigma$.

Remark 2.3. Consider a cone σ and the corresponding dual cone $\sigma^\vee$. Taking $u = 0$ one has that $\sigma = \sigma \cap u^\perp$ is a face of σ, hence any cone is a face of itself. it is customary to call *proper* all the other faces of σ.

Remark 2.4. With this definition one can prove that a cone is strongly convex if and only if $\{0\}$ is a face of the cone.

Figure 2.3 shows that in fact the definition gives rise to the natural idea of face of a cone.

Figure 2.3: The faces of the cone are marked thicker.

In effect, also some other natural and not so difficult results about faces of a cone are true.

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Proposition 2.9. *Let σ be a cone in a vector space V. Then:*

(a) every face of σ is a convex polyhedral cone.

(b) any intersection of faces of σ is again a face of σ.
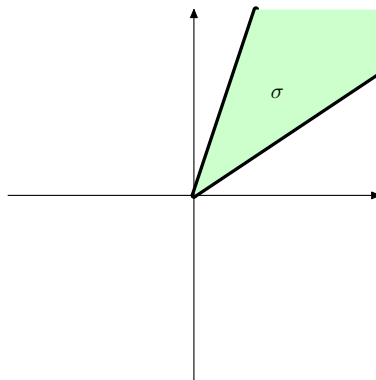
(c) any face of a face of σ (this is meaningful from point (a)) is again a face of σ.

Proof.
Refer again to Fulton, [7]. □

Since the faces of a cone are again cones, one can speak about their dimension. In such a way, one can remark two classes of faces.

Definition 2.10. Let σ be a convex polyhedral cone in the vector space V. A **ray** (or **edge**) of σ is a face of dimension 1. A **facet** of σ is a face of codimension 1.

We will see in the future how important is the fact that we always consider the couple of (dual) vector spaces $(V; V^*)$, as well as the couple of convex polyhedral cones $(\sigma; \sigma^\vee)$. For example this have already allowed us to define the faces of the cone in a simple way.

2.2.1   Cones and lattices

The nice part about cones comes into play when we consider a lattice in the vector space V.

Given a lattice N one can of course gets a $\mathbb{R}$-vector space from it; the simplest way to do it is to consider the tensor product $N_\mathbb{R} := N \otimes_\mathbb{Z} \mathbb{R}$. Moreover, the lattice N comes together with a dual lattice M, that is by definition $M = \text{Hom}(N, \mathbb{Z})$ (as group homomorphisms). Of course we can define the $\mathbb{R}$-vector space $M_\mathbb{R} := M \otimes_\mathbb{Z} \mathbb{R}$; it is true that

$$M_\mathbb{R} = M \otimes_\mathbb{Z} \mathbb{R} = \text{Hom}_{\mathbb{Z}\text{-mod}}(N, \mathbb{Z}) \otimes_\mathbb{Z} \mathbb{R} = \text{Hom}_{\mathbb{R}\text{-mod}}(N, \mathbb{R}) = N_\mathbb{R}^*$$

so the vector space $M_\mathbb{R}$ is the dual of $N_\mathbb{R}$. From now on it will be unforgettable to consider the pair of (dual) vector spaces $(N_\mathbb{R}; M_\mathbb{R})$.

Definition 2.11. Take a lattice N and the corresponding $\mathbb{R}$-vector space $N_\mathbb{R}$. A convex polyhedral cone in $N_\mathbb{R}$ is said to be **rational** (with respect to N) if it can be generated by a set of vectors in N.

Proposition 2.12. *Let N be a lattice and σ a rational convex polyhedral cone in $N_\mathbb{R}$. Then:*

(a) the dual $\sigma^\vee$ is a rational convex polyhedral cone in $M_\mathbb{R}$.

(b) all the faces of σ are rational convex polyhedral cone in $N_\mathbb{R}$.
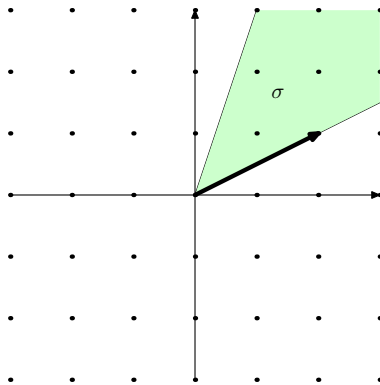
Proof.
Refer to Fulton, [7]. □

Figure 2.4: A cone in $N_{\mathbb{R}}$ with a marked *ray generator*.

The typical example we will always draw is $N = \mathbb{Z}^n$ for some $n \geq 1$. As a consequence $M = \mathrm{Hom}(N, \mathbb{Z}) \simeq \mathbb{Z}^n$, $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{R}^n$ and in the same way $M_{\mathbb{R}} \simeq \mathbb{R}^n$.

If we consider a strongly convex rational polyhedral cone in a vector space deduced from a lattice, it is possible to choose for any ray ρ a unique minimal generator. In fact a ray will be a half line that is still a rational cone, so it is generated by an element of N. Considering the set $\rho \cap N$ one obtains a semigroup of dimension one. The generator can be chosen in a unique way. We will call it the *ray generator* of ρ. It is possible to show that a strongly convex rational polyhedral cone is generated by the ray generators of its rays.

2.3   Affine toric varieties from convex polyhedral cones

Suppose to have the usual pair of dual lattices (N, M) and so of dual vector spaces $(N_{\mathbb{R}}, M_{\mathbb{R}})$.

Consider a convex polyhedral cone σ in the vector space $N_{\mathbb{R}}$. From now on we will always denote with S_{σ} the set

$$S_{\sigma} = \sigma^{\vee} \cap M.$$

Proposition 2.13 (*Gordon's lemma*)**.** *If σ is a rational polyhedral cone, then the set S_{σ} is an affine semigroup with the natural operation over M.*

Proof.
The first remark is that if $m, l \in S_{\sigma}$ then we can sum them (they are in M) and the sum is in $\sigma^{\vee}$ since $\sigma^{\vee}$ is a convex polyhedral cone[1]. Hence, since the operation in M already satisfies all the right properties, S_{σ} has a natural structure of semigroup.
Secondly, there exists a natural injective morphism of semigroups from S_{σ} to a lattice; in fact, it is enough to consider the embedding of S_{σ} into M.
So, the only thing that is left to prove is the fact that S_{σ} is finitely generated as a semigroup. This is the heart of the proposition. From Proposition 2.12 the fact that σ is a rational polyhedral cone in $N_{\mathbb{R}}$ implies that $\sigma^{\vee}$ is a rational convex

---

[1]To be very formal one could say that from convexity $\frac{1}{2}(m + l) \in \sigma^{\vee}$ and from the fact that σ is a cone, the double of the previous element is still in the cone.

polyhedral cone in $M_{\mathbb{R}}$, hence there exists a generating set of $\sigma^{\vee}$ in M, call it $S = \{m_1, m_2, \ldots, m_s\}$. Call

$$K := \left\{ \sum_{i=1}^{s} t_i m_i : t_i \in [0; 1] \right\}$$

and consider the set $K \cap M$; this set is a discrete subset of a compact, hence it is finite. Now the aim is to prove that S_{σ} is generated by the set $S \cup (K \cap M)$ as a semigroup; this set is included in S_{σ} (by definition of S and construction of K). So, let w be an element of S_{σ}: since S_{σ} is a linear combination of elements in S with real nonnegative coefficients, one has that:

$$w = \sum_{i=1}^{s} \lambda_i m_i = \sum_{i=1}^{s} \lfloor \lambda_i \rfloor m_i + \sum_{i=1}^{s} \left(\lambda_i - \lfloor \lambda_i \rfloor \right) m_i$$

so calling:

$$w_1 := \sum_{i=1}^{s} \lfloor \lambda_i \rfloor m_i$$

$$w_2 := \sum_{i=1}^{s} \left(\lambda_i - \lfloor \lambda_i \rfloor \right) m_i$$

one has that $w = w_1 + w_2$; moreover by definition $w_2 \in K$ and w_1 is in the subsemigroup of S_{σ} generated by S (recall that all the λ_i's are nonnegative). The proof is now completed by remarking the fact that $w \in M$, so also $w_2 = w - w_1$ is in M so $w_2 \in K \cap M$. $\qquad\square$

Having proved this fundamental result, we can use one of the definitions given in the previous chapter to define an affine toric variety from a cone.

Definition 2.14. Let R be a ring and let σ be a strongly convex rational polyhedral cone in the vector space $N_{\mathbb{R}}$ (N is a lattice). We call R-**affine toric variety associated to the cone** σ the affine toric variety $U_{\sigma} := \operatorname{Spec}(R[S_{\sigma}])$.

Remark 2.5. Taking the cone σ to be strongly convex is crucial for our aim. We will understand better the sense of this requirement in Proposition 2.18.

Remark 2.6. By definition, the scheme $U_{\sigma} := \operatorname{Spec}(R[S_{\sigma}])$ is an R-scheme and also an affine variety over R; in fact the R-algebra $R[S_{\sigma}]$ is finitely generated since S_{σ} is finitely generated by Gordon's lemma. It is not clear *a priori* that it is a toric variety in the standard sense. We will arrive to the proof of this at the end of the section.

Example 2.15. Consider the cone $\sigma = \{0\}$ in the vector space $N_{\mathbb{R}}$ with dimension, say, n. It is immediate that $\sigma^{\vee} = M_{\mathbb{R}}$ so $S_{\sigma} = M$. This implies that (recall the isomorphism of R-algebras $R[M] \simeq R[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$ of Example 2.2):

$$U_{\{0\}} = \operatorname{Spec}(R[M]) = \operatorname{Spec}(R[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}])$$

that is the n-dimensional R-torus. So *the n-dimensional R-torus is a R-affine toric variety.*

A natural question is to understand in which way the fact that a cone is a face of another one translates to the geometry of the associated toric varieties. A simple observation is the following: since $\tau \subseteq \sigma$, it is immediate that $\sigma^\vee \subseteq \tau^\vee$, hence one obtains $S_\sigma \subseteq S_\tau$, so an injective R-algebras morphism $R[S_\sigma] \to R[S_\tau]$ that, passing to Spec becomes a morphism of schemes $U_\tau \to U_\sigma$. Anyway, proceeding like this, we cannot say anything more precise about the form of this morphism. To get more information will be necessary to pass from a result on faces of convex cones.

Lemma 2.16. *Let σ be a convex rational polyhedral cone in the vector space $N_\mathbb{R}$. All the faces of σ can be written as $\tau = \sigma \cap u^\perp$ for an element $u \in S_\sigma = \sigma^\vee \cap M$. Moreover $S_\tau = S_\sigma + \mathbb{N}(-u)$.*

Proof.
See [7, Prop 2, Ch 1.2]. $\qquad\qquad\square$

Proposition 2.17. *Suppose that σ is a strongly convex rational polyhedral cone in the vector space $N_\mathbb{R}$ and that τ is a face of σ; τ is again a strongly convex rational polyhedral cone. Then the toric variety U_τ is isomorphic to a principal open subset[2] of U_σ.*

Proof.
From the lemma one can write $S_\tau = S_\sigma + \mathbb{N}(-u)$. Our aim is to show that

$$R[S_\tau] = R[S_\sigma]_{T^u}.$$

One inclusion comes from the fact that T^u is multiplicatively invertible in $R[S_\tau]$ since $T^{-u} \in R[S_\tau]$ from the lemma ($0 \in S_\sigma$), so localizing the inclusion $R[S_\sigma] \subseteq R[S_\tau]$:

$$R[S_\sigma]_{T^u} \subseteq R[S_\tau]_{T^u} = R[S_\tau].$$

To prove the converse inclusion, take a R basis of $R[S_\tau]$. It is of the form $T^{s_1}, T^{s_2}, \ldots, T^{s_l}$ for $s_1, s_2, \ldots, s_l$ a basis of the affine semigroup S_τ. But from Lemma 2.16, writing $S_\tau = S_\sigma + \mathbb{N}(-u)$ for a certain $u \in S_\sigma$, one has that every element of the basis can be written in the form $s_i = w_i - p_i \cdot u$, with $w_i \in S_\sigma$ and $p_i \in \mathbb{N}$. This means that a basis of S_τ is $\{T^{w_i - p_i \cdot u}\}_{i=1}^l$, i.e. a finite number of elements of the form

$$\frac{T^{w_i}}{(T^u)^{p_i}}.$$

These are elements in the localization $R[S_\sigma]_{T^u}$ since $T^{w_i} \in R[S_\sigma]$. Hence all the R-basis of $R[S_\tau]$ is contained in this localization, so $R[S_\tau] \subseteq R[S_\sigma]_{T^u}$.
In the end, passing to the spectra one has

$$U_\tau = \mathrm{Spec}(R[S_\tau]) = \mathrm{Spec}(R[S_\sigma]_{T^u}) = D(T^u)$$

so U_τ is a principal open subscheme of U_σ. $\qquad\qquad\square$

---

[2] A principal open subset of an affine scheme $X = \mathrm{Spec}(A)$ is $D(f) = \{\mathfrak{p} \in \mathrm{Spec}(A) : f \notin \mathfrak{p}\}$ for an element $f \in A$. If the structure sheaf of the scheme is $\mathcal{O}_X$, then $\mathcal{O}_X(D(f)) = A_f$.

Remark 2.7. The proposition just proved is strongly depending on the fact that the definition of the toric variety associated to a cone passes through the dual cone of σ and does not involve directly the cone σ itself. This ensures then that there is a face-subvariety correspondance in the sense of the proposition above.

What we proved has an important consequence, that is the fact that the schemes we defined are really affine toric varieties.

Proposition 2.18. *Let σ be a strongly convex rational polyhedral cone in the vector space $N_{\mathbb{R}}$. Then U_{σ} contains an R-torus as an open subscheme, whose multiplication extends to an action on U_{σ}. In other words, U_{σ} is a toric scheme.*

Proof.
Since the cone is strongly convex, then $\{0\}$ is a face of the cone σ, hence by the previous proposition the toric variety $U_{\{0\}}$ is a principal open subset of U_{σ}. But from example 2.15, $U_{\{0\}}$ is the n-dimensional R-torus, where n is the rank of N. Hence referring to the torus as $\mathbb{T}$ instead of $\mathbb{T}_{n,R}$ for simplicity, one has that it is an open subscheme of the variety U_{σ}.
We would now like to prove that the multiplication of the torus extends to an action of the torus on U_{σ}. Recalling Definition 1.23, we then want to find an action a of $\mathbb{T}$ on U_{σ} making the following diagram commutative:

$$
\begin{array}{ccc}
\mathbb{T}_{n,R} \times_S \mathbb{T}_{n,R} & \xrightarrow{\ m\ } & \mathbb{T}_{n,R} \\
\Big\downarrow{\scriptstyle id_{\mathbb{T}} \times i} & & \Big\downarrow{\scriptstyle i} \\
\mathbb{T}_{n,R} \times_S U_{\sigma} & \xrightarrow{\ a\ } & U_{\sigma}
\end{array}
$$

All the schemes involved in the diagram are affine, so the problem is to find a map $a^{\sharp} : R[S_{\sigma}] \to R[M] \otimes R[S_{\sigma}]$ (recall that $\mathbb{T} = U_{\{0\}} = R[M]$) such that it gives rise to an action of $\mathbb{T}$ on U_{σ} and such that the following diagram commute:

$$
\begin{array}{ccc}
R[M] \otimes_R R[M] & \xleftarrow{\ m^{\sharp}\ } & R[M] \\
\Big\uparrow{\scriptstyle id \otimes i^{\sharp}} & & \Big\uparrow{\scriptstyle i^{\sharp}} \\
R[M] \otimes_R R[S_{\sigma}] & \xleftarrow{\ a^{\sharp}\ } & R[S_{\sigma}]
\end{array}
$$

We make two remarks. The first one is the fact that all the morphism are morphisms of R-algebras, so the commutativity of the diagram, and even the definition of the morphism $a^{\sharp}$ will be checked only on an R-algebras basis of $R[S_{\sigma}]$, namely T^u, for certain $u \in M$. As a second observation, one remarks that $i^{\sharp}$ is the morphism that gives the open immersion of the torus $\mathbb{T}$ into the scheme U_{σ}, hence it is by proposition 2.17 the morphism of a certain localization, $T^u \mapsto T^u/1$ that we will prefer to denote, with abuse of notation, by $T^u \mapsto T^u$.
Moreover, let's look of the way in which the morphism $m^{\sharp}$ acts on elements of the form T^u:

$$
m^{\sharp}(T^u) = m^{\sharp}(T^{\lambda_1 e_1^* + \lambda_2 e_2^* + \cdots + \lambda_n e_n^*}) = m^{\sharp}\Big(\prod_{i=1}^{n} T^{\lambda_i e_1^*}\Big) = \prod_{i=1}^{n} m^{\sharp}(T^{e_i^*})^{\lambda_i}.
$$

Recalling the canonical isomorphism $R[M] \simeq R[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$ (obtained by setting $T^{e_i^*}$ into x_i and $T^{-e_i^*}$ into x_i^{-1} for every $i = 1, 2, \ldots, n$), the definition of $m^\sharp$ and the fact that the tensor product of two R-algebras A and B is naturally an R-algebra with product law $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$ we obtain:

$$m^\sharp(T^u) = \prod_{i=1}^n m^\sharp(T^{e_i^*})^{\lambda_i} = \prod_{i=1}^n (T^{e_i^*} \otimes T^{e_i^*})^{\lambda_i} = \prod_{i=1}^n ((T^{e_i^*})^{\lambda_i} \otimes (T^{e_i^*})^{\lambda_i})$$

$$= \left(\prod_{i=1}^n T^{\lambda_i e_i^*} \right) \otimes \left(\prod_{i=1}^n T^{\lambda_i e_i^*} \right) = T^u \otimes T^u.$$

Hence, it is natural to define $a^\sharp : R[S_\sigma] \to R[M] \otimes R[S_\sigma]$ putting on an R-algebra basis of $R[S_\sigma]$

$$a^\sharp(T^u) = T^u \otimes T^u.$$

In fact in this way one has:

$$
\begin{array}{ccc}
T^u \otimes T^u & \xleftarrow{\;\;m^\sharp\;\;} & T^u \\
\uparrow{\scriptstyle id \otimes i^\sharp} & & \uparrow{\scriptstyle i^\sharp} \\
T^u \otimes T^u & \xleftarrow{\;\;a^\sharp\;\;} & T^u
\end{array}
$$

hence an obviously commutative diagram.

It remains to prove that $a^\sharp$ give rise to an action of $\mathbb{T}$ on U_σ. To do this, let's remember Definition 1.22. Since all the schemes are affine, we can check that the corresponding diagrams on the section rings are commutative (and moreover only on the generators of $R[S_\sigma]$ as an R-algebra). In particular, for the associativity:

$$
\begin{array}{ccc}
R[M] \otimes_R R[M] \otimes_R R[S_\sigma] & \longleftarrow & R[M] \otimes_R R[S_\sigma] \\
\uparrow & & \uparrow \\
R[M] \otimes_S R[S_\sigma] & \longleftarrow & R[S_\sigma]
\end{array}
$$

$$
\begin{array}{ccc}
T^u \otimes T^u \otimes T^u & \longleftarrow & T^u \otimes T^u \\
\uparrow & & \uparrow \\
T^u \otimes T^u & \longleftarrow & T^u
\end{array}
$$

and for the action of the neutral element (recall the definition of the map e for the torus, this maps all the elements of the form T^u in 1):

$$
\begin{array}{ccc}
R \otimes_R R[S_\sigma] & \longleftarrow & R[M] \otimes_R R[S_\sigma] \\
& \nwarrow & \uparrow \\
& & R[S_\sigma]
\end{array}
$$

$$
\begin{array}{ccc}
1_R \otimes T^u & \longleftarrow & T^u \otimes T^u \\
& \nwarrow & \uparrow \\
& & T^u
\end{array}
$$

The diagrams are really commutative, so the morphism coming from the map of rings is an action. From the definition, its restriction is the multiplication of the torus. Hence we have the result. □

Remark 2.8. This remarks the fact that the construction we made from cones really gives us a toric scheme, in the sense of definition 1.23. Anyway, to obtain this result the requirement that σ is strongly convex is essential. Moreover the result of the construction is clearly a variety over R, so we obtained a toric variety over R.

2.4 Fans

Definition 2.19. A **fan** in a $\mathbb{R}$-vector space $N_\mathbb{R}$ is a non-empty collection of cones $\Delta = \{\sigma : \text{ cone in } V\}$ such that:

1. each cone is a strongly convex rational polyhedral cone in V.

2. for every cone $\sigma \in \Delta$, each face of σ is in Δ (this is meaningful, since every face of a convex polyhedral cone in V is again a convex polyhedral cone in V).

3. the intersection of two cones σ_1, σ_2 in Δ is a face of both cones, and hence, by 2., is in Δ.

In particular, a fan in V is a not necessarily finite collection of convex polyhedral cones that is closed under considering faces and under intersection. It will be typical (and it is common in literature) to denote with $\Delta(r)$ the set of cones in Δ with dimension r. Moreover we will call **support** of the fan $|\Delta|$ the set of points lying in one of the cones of Δ, $|\Delta| = \cup_{\sigma \in \Delta} \sigma$.

Other terminology we will use for a fan are listed in the following definition.

Definition 2.20. Let N be a lattice and let Δ be a fan in the $\mathbb{R}$-vector space $N_\mathbb{R}$. We say that Δ is:

- **smooth** if every cone in Δ is smooth (i.e. the minimal generators of the cone are part of a $\mathbb{Z}$-basis of N).

- **simplicial** if every cone in Δ is simplicial (i.e. the minimal generators of the cone are $\mathbb{R}$-linearly independent).

- **complete** if the support of the fan is the whole $N_\mathbb{R}$.

2.5 Toric schemes from fans

Let's consider a (not necessarely finite) fan Δ in $N_\mathbb{R}$. For every cone $\sigma \in \Delta$ we can proceed as in the previous paragraphs and construct an affine toric variety over some ring R, more explicitely:
$$U_\sigma = \mathrm{Spec}(R[S_\sigma]).$$

Now we consider the set of this affine varieties U_σ and we proceed constructing the abstract variety as described in Chapter 1. Let's see that it is meaningful: from a fan Δ we automatically obtain a family of R-schemes $\{U_\sigma\}_{\sigma \in \Delta}$. Moreover, take two

of these schemes, say U_{σ_1} and U_{σ_1}. From the definition of fan, to the cones σ_1 and σ_2 we can associate a cone $\sigma_1 \cap \sigma_2$ that is a face of both the cones and it is still in Δ. Then from Proposition 2.17, we have that $U_{\sigma_1 \cap \sigma_2}$ is a principal open subset of both U_{σ_1} and U_{σ_1}. Summarizing, we have a collection of R-schemes such that for any couple there exists an open subscheme of both of them (hence there exist open subschemes of the first and of the second that are isomorphic as locally ringed spaces). As stated in Chapter 1 (in particular in Remark **??** and referring to [10, Ex II.2.12]) we have a collection of schemes U_σ with open subschemes that are pairwise isomorphic via the identity map and moreover the compatibility condition is satisfied. This means that we can coherently define an R-scheme as the glue of all these pieces.

Definition 2.21. Let Δ be a fan in $N_\mathbb{R}$. The **R-toric scheme associated to the fan** Δ is the R-scheme obtained gluing the affine R-toric varieties of the cones of the fan along the principal open subsets coming from the faces of the cones, as described above. We will denote it as $X(R, \Delta)$.

Remark 2.9. This definition is a generalization of the one for affine R-toric variety, since for any cone σ one can consider the fan consisting of σ and of all its faces. The R-toric variety obtained from it is just the affine R-toric variety U_σ, since the definition essentially requires to glue U_σ with open subschemes of itself.

Remark 2.10. If we ask the fan Δ to be finite (i.e. consisting of finitely many cones), the R-toric scheme associated to the fan Δ gives in a variety over R in the sense of Definition **??**; in fact the result of the gluing is an R-scheme that as a topological space is a quotient of the union of the topological spaces of U_σ, $\sigma \in \Delta$. Moreover, as a scheme, $X(R, \Delta)$ is covered by all the X_σ, that are both open subschemes of it and affine R-toric variety. In the end, a R-toric variety $X(R, \Delta)$ is simply the gluing of a finite number of R-affine toric variety, hence it is by definition a variety over R. It will be common in the following to call U_σ's the *affine pieces* of the toric variety $X(R, \Delta)$.

Moreover $X(R, \Delta)$ is a toric scheme according to Definition 1.23. In fact the torus $U_{\{0\}}$ admits an open immersion in $X(R, \Delta)$ (this can be easily proved by considering that the torus is embeddable in any of its affine pieces and hence in the scheme X itself); moreover, the action of the torus can be seen to extend from all the affine pieces to the R-scheme $X(R, \Delta)$ (this is just a question of compatibility of maps, and can be checked).

2.6   Examples

We have already seen in the previous sections that the n-dimensional R-torus is an affine R-toric variety. The aim of this section is to give more interesting examples. In particular we will see that the affine spaces and the projective spaces are toric schemes. This will mean that the concept of toric variety extends the idea of many of the usual ambient spaces used in algebraic geometry.

We will suppose as usual that N is a lattice of rank n, with basis $\{e_1, e_2, \ldots, e_n\}$, that M is its dual lattices with dual basis $\{e_1^*, e_2^*, \ldots, e_n^*\}$ and we will call $N_\mathbb{R}$ and $M_\mathbb{R}$ the corresponding $\mathbb{R}$-vector spaces.

2.6.1 Toric schemes in dimension one

Suppose that N has rank 1, so $N_{\mathbb{R}}$ is isomorphic to $\mathbb{R}$. The only possible strongly convex rational polyhedral cones in this vector spaces are $\{0\}$, the right half line $\mathbb{R}^+ \cdot e_1$ and the left half line $-\mathbb{R}^+ \cdot e_1$ (since $N_{\mathbb{R}}$ itself is not strongly convex). The corresponding affine R-toric varieties are the following:

- $\sigma = \{0\}$: this gives, as a particular case of what already seen, the 1-dimensional R-torus $T_{1,R} = \mathrm{Spec}(R[x^{\pm 1}])$.

- $\sigma = \langle e_1 \rangle$: one easily sees that $\sigma^{\vee} = \langle e_1^* \rangle$, so $S_{\sigma} = M \cap \sigma^{\vee} = \mathbb{N}e_1^*$, but clearly

$$R[S_{\sigma}] = R[\mathbb{N}e_1^*] = R[T^{e_1^*}] \simeq R[x]$$

  hence

$$U_{\sigma} = \mathrm{Spec}(R[S_{\sigma}]) = \mathrm{Spec}(R[x]) = \mathbb{A}_R^1.$$

- $\sigma = \langle -e_1 \rangle$: in the same way as before, one obtains $U_{\sigma} = \mathrm{Spec}(R[x^{-1}])$, that isomorphic to the affine linex.

Hence, the only possible affine R-toric varieties coming from fans in dimension 1 (and so containing the torus $\mathrm{Spec}(R[x^{\pm 1}])$ as an open subset) are the torus itself and the affine space of dimension 1 over R.

Figure 2.5: All the possible fans in dimension 1.

In order to describe all the R-toric varieties coming from fans in dimension 1, we should glue affine varieties of the list above. More precisely, it is clear that all the possible fans in $N_{\mathbb{R}}$ are cones (with their faces) and the collection of cones $\{\text{origin}, \text{right half line}, \text{left half line}\}$. Since the cones give affine R-toric varieties, the only interesting case that is left to study is the one in which the fan consist of two half lines and the origin. In this case, calling $x := T^{e_1^*}$ to simplify the notation, we have to consider the two schemes $X_1 = U_{\sigma_1} = \mathrm{Spec}(R[x])$ and $X_2 = U_{\sigma_2} = \mathrm{Spec}(R[x^{-1}])$ (two affine lines) and glue them along $U_{\{0\}} = \mathrm{Spec}(R[x^{\pm 1}])$. The torus $U_{\{0\}}$ is a principal open subscheme of both the schemes, more precisely it is

$$U_{\{0\}} \simeq X_{21} = \mathrm{Spec}(R[x]_x) = D(x) \subseteq U_{\sigma_1}$$

$$U_{\{0\}} \simeq X_{12} = \mathrm{Spec}(R[x^{-1}]_{x^{-1}}) = D(x^{-1}) \subseteq U_{\sigma_2}$$

and determines an isomorphism $X_{12} \simeq X_{21}$ coming from $R[x]_x \to R[x^{-1}]_{x^{-1}}$ such that $x \mapsto x^{-1}$. Referring to the construction made in 1.13, this is the definition of the projective space $\mathbb{P}_R^1$. In this way we obtained the 1-dimensional projective space over the ring R as a non-affine R-toric scheme.

2.6.2 Affine spaces

Let's consider the cone σ generated by the vectors $e_1, e_2, \ldots, e_n$. Hence it is (for $n = 2$):
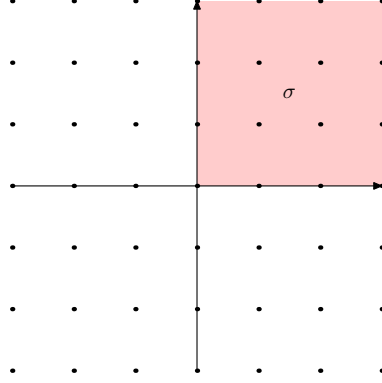

Figure 2.6: The fan in $N_\mathbb{R}$ for the affine space.

The dual cone is given by itself (recall the example 2.6). Hence

$$S_\sigma = M \cap \sigma^\vee = \mathbb{N}e_1^* + \mathbb{N}e_2^* + \cdots + \mathbb{N}e_n^*$$

so

$$R[S_\sigma] = R[\mathbb{N}e_1^* + \mathbb{N}e_2^* + \cdots + \mathbb{N}e_n^*] = R[T^{e_1^*}, T^{e_2^*}, \ldots, T^{e_n^*}] \simeq R[x_1, x_2, \ldots, x_n]$$

so one concludes that

$$U_\sigma = \operatorname{Spec}(R[S_\sigma]) = \operatorname{Spec}(R[x_1, x_2, \ldots, x_n]) = \mathbb{A}_R^n$$

that means that for every n the R-affine space $\mathbb{A}_R^n$ isn affine R-toric variety. This is very believable, since it clearly contains the torus as an open subset. In particular it is evident in the case of fields, with the naive idea of varieties over a field, in which the group law of the torus is simply the componentwise multiplication.

2.6.3 Projective spaces

Let's consider the lattice $\mathbb{Z}^n$ and put $e_0 := -e_1 - e_2 - \cdots - e_n$. Consider all the cones generated by a subset of $\{e_0, e_1, e_2, \ldots, e_n\}$; all together they form a fan. The R-toric variety constructed from this fan is the R-projective space of dimension n. For simplicity, we show this result only in the case $n = 2$.
The cones of the fan are $\sigma_0 = \langle e_1, e_2 \rangle$, $\sigma_1 = \langle e_0, e_2 \rangle$, $\sigma_2 = \langle e_0, e_1 \rangle$ and their faces. The corresponding R-algebras are:

$$R[S_{\sigma_0}] = R[M \cap \sigma_0^\vee] = R[T^{e_1^*}, T^{e_2^*}] \simeq R[x_1, x_2]$$

$$R[S_{\sigma_1}] = R[M \cap \sigma_1^\vee] = R[-\mathbb{N}e_1^* + \mathbb{N}(-e_1^* + e_2^*)] = R[T^{-e_1^*}, T^{-e_1^*+e_2^*}] \simeq R[x_1^{-1}, x_1^{-1}x_2]$$

$$R[S_{\sigma_2}] = R[M \cap \sigma_2^\vee] = R[-\mathbb{N}e_2^* + \mathbb{N}(e_1^* - e_2^*)] = R[T^{-e_2^*}, T^{e_1^*-e_2^*}] \simeq R[x_2^{-1}, x_1x_2^{-1}]$$

so

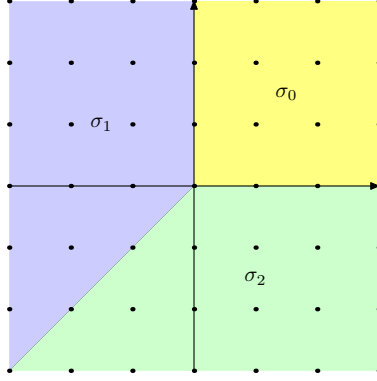$$U_{\sigma_0} = \operatorname{Spec}(R[S_{\sigma_0}]) = \operatorname{Spec}(R[x_1, x_2])$$

Figure 2.7: The fan in $N_\mathbb{R}$ for the projective space.

$$U_{\sigma_1} = \operatorname{Spec}(R[S_{\sigma_1}]) = \operatorname{Spec}(R[x_1^{-1}, x_1^{-1}x_2])$$
$$U_{\sigma_2} = \operatorname{Spec}(R[S_{\sigma_2}]) = \operatorname{Spec}(R[x_2^{-1}, x_1 x_2^{-1}]).$$

Moreover the three one dimensional cones determines R-algebras:

$$R[S_{\tau_0}] = R[M \cap \tau_0^\vee] = R[\mathbb{N}(-e_1^* - e_2^*) + \mathbb{Z}(e_1^* - e_2^*)] \simeq R[x_1^{-1}x_2^{-1}, x_1 x_2^{-1}, x_1^{-1}x_2]$$

$$R[S_{\tau_1}] = R[M \cap \tau_1^\vee] = R[\mathbb{N}e_1^* + \mathbb{Z}e_2^*] \simeq R[x_1, x_2^{\pm 1}]$$
$$R[S_{\tau_2}] = R[M \cap \tau_2^\vee] = R[\mathbb{Z}e_1^* + \mathbb{N}e_2^*] \simeq R[x_1^{\pm 1}, x_2]$$

giving:

$$U_{\tau_0} = \operatorname{Spec}(R[S_{\tau_0}]) = \operatorname{Spec}(R[x_1^{-1}x_2^{-1}, x_1 x_2^{-1}, x_1^{-1}x_2])$$
$$U_{\tau_1} = \operatorname{Spec}(R[S_{\sigma_1}]) = \operatorname{Spec}(R[x_1, x_2^{\pm 1}])$$
$$U_{\tau_2} = \operatorname{Spec}(R[S_{\sigma_2}]) = \operatorname{Spec}(R[x_1^{\pm 1}, x_2]).$$

Moreover, remembering how we defined the embedding of the varieties coming from faces (we study only the case τ_1):

$$U_{\tau_1} = \operatorname{Spec}(R[x_1, x_2^{\pm 1}]) = \operatorname{Spec}(R[x_1, x_2]_{x_2}) = D(x_2) \subseteq U_{\sigma_0}$$

$$U_{\tau_1} = \operatorname{Spec}(R[x_1, x_2^{\pm 1}]) = \operatorname{Spec}(R[x_1 x_2^{-1}, x_2^{-1}]_{x_2^{-1}}) = D(x_2^{-1}) \subseteq U_{\sigma_2}.$$

This gives an isomorphism along which we want to glue U_{σ_0} and U_{σ_2}; remember that their intersection is exactly $U_{\sigma_0} \cap U_{\sigma_2} = U_{\tau_1}$. So we are gluing three affine spaces in which the change of coordinates on the intersection is the inversion: this is exactly the way in which the projective space is defined! The conclusion is that for the fan Δ considered, $X(R, \Delta) = \mathbb{P}_R^2$. In general *any projective space is a toric variety*.

2.6.4 Hirzebruch surface

Let N be $\mathbb{Z}^2$ and $N_\mathbb{R} = \mathbb{R}^2$. Let's consider the fan Δ consisting of the cones $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ in Figure 2.8 and of their faces for a certain $r \in \mathbb{N}$.
We already know two of the dual cones, by previous examples: $\sigma_1^\vee$ is generated by $(1, 0)$ and $(0, 1)$ in $M_\mathbb{R}$, while $\sigma_2^\vee$ is generated by the vectors $(1, 0)$ and $(-1, 0)$ of the same space.
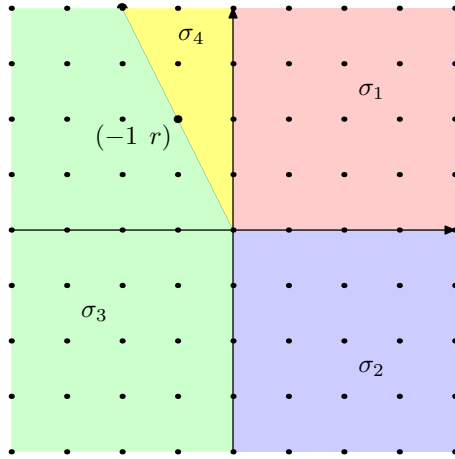
Figure 2.8: The fan for Hirzebruch surface.

Figure 2.9: The cone σ_1 and its dual.

Figure 2.10: The cone σ_2 and its dual.

Let's try to find out the dual cone of σ_4, that is the cone generated by $(-1, r)$ and $(0, 1)$. One sees that:

$$\sigma_4^\vee = \{v \in \mathbb{R}^2 : \langle v; (-\lambda, r\lambda + \mu)\rangle \geq 0 \text{ for all } \lambda, \mu \geq 0\}$$
$$= \{(v_1, v_2) \in \mathbb{R}^2 : -\lambda v_1 + r\lambda v_2 + \mu v_2 \geq 0 \text{ for all } \lambda, \mu \geq 0\}.$$

In particular for $\lambda = 0$ one obtains $v_2 \geq 0$, while for $\lambda \neq 0$ we can rewrite the condition as

$$-v_1 + rv_2 + \frac{\mu}{\lambda}v_2 \geq 0$$

for any positive value of μ and λ. But it is clear that it is sufficient to ask that

$$-v_1 + rv_2 \geq 0,$$

so the cone we obtain is given by the two conditions

$$\begin{cases} v_2 \geq 0 \\ -v_1 + rv_2 \geq 0 \end{cases}$$

that give the cone generated by $(-1, 0)$ and $(r, 1)$. One proceeds in a similar way for σ_3 and so obtains that $\sigma_3^\vee$ is generated by the vectors $(-1, 0)$ and $(-r, -1)$.

Figure 2.11: The cone σ_3 and its dual.

Figure 2.12: The cone σ_4 and its dual.

The four affine R-toric varieties one obtains are all isomorphic to an affine R-plane.

$$U_{\sigma_1} = \operatorname{Spec}(R[x, y])$$

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$$U_{\sigma_2} = \mathrm{Spec}(R[S_{\sigma_2}]) = \mathrm{Spec}(R[T^{e_1^*}, T^{-e_2^*}]) = \mathrm{Spec}(R[x, y^{-1}])$$

$$U_{\sigma_3} = \mathrm{Spec}(R[S_{\sigma_3}]) = \mathrm{Spec}(R[T^{-e_1^*}, T^{-re_1^*-e_2^*}]) = \mathrm{Spec}(R[x^{-1}, x^{-r}y^{-1}])$$

$$U_{\sigma_4} = \mathrm{Spec}(R[S_{\sigma_4}]) = \mathrm{Spec}(R[T^{-e_1^*}, T^{re_1^*+e_2^*}]) \simeq \mathrm{Spec}(R[x^{-1}, x^r y]).$$

Gluing these four copies of the affine space along the open subschemes coming from faces we get what is called the **Hirzebruch surface**.

2.6.5 Fiber product of two toric schemes

The aim of this subsection is to prove that the fiber product of two R-toric schemes constructed from fans is again an R-toric scheme. We will also explicitly construct the fan of the fiber product.

To begin, let's consider two lattices N_1 and N_2 with respectively dual lattices M_1 and M_2 and let's build the corresponding vector spaces $(N_1)_{\mathbb{R}}$, $(N_2)_{\mathbb{R}}$, $(M_1)_{\mathbb{R}}$ and $(M_2)_{\mathbb{R}}$.

Moreover we define the product of two cones in two different vector spaces.

Definition 2.22. Let

$$\sigma = \mathbb{R}^+ v_1 + \mathbb{R}^+ v_2 + \cdots + \mathbb{R}^+ v_n$$

be a cone in a vector space V and

$$\tau = \mathbb{R}^+ w_1 + \mathbb{R}^+ w_2 + \cdots + \mathbb{R}^+ w_m$$

be a cone in the vector space W. The **product** (or the **direct sum**) **of the two cones** is defined as the subset of $V \oplus W$ given by

$$\sigma \oplus \tau = \mathbb{R}^+ v_1 + \mathbb{R}^+ v_2 + \cdots + \mathbb{R}^+ v_n + \mathbb{R}^+ w_1 + \mathbb{R}^+ w_2 + \cdots + \mathbb{R}^+ w_m.$$

Remark 2.11. It is clear from definition that the product of two cones in two different vector spaces is a cone in the direct sum of the two vector spaces.

Lemma 2.23. *Let $\Delta_1 = \{\sigma_i\}_i$ be a fan in $(N_1)_{\mathbb{R}}$ and $\Delta_2 = \{\tau_j\}_j$ be a fan in $(N_2)_{\mathbb{R}}$. Then $\Delta_1 \oplus \Delta_2 = \{\sigma_i \oplus \tau_j\}_{i,j}$ is a fan in $(N_1)_{\mathbb{R}} \oplus (N_2)_{\mathbb{R}}$.*

Proof.
Let σ be a cone in $(N_1)_{\mathbb{R}}$ and τ be a cone in $(N_2)_{\mathbb{R}}$; then a face of $\sigma \oplus \tau$ is obtained "forgetting" some generators of the cone. But this means that the face is the product of two cones, one in $(N_1)_{\mathbb{R}}$ and the other in $(N_2)_{\mathbb{R}}$, obtained from σ and τ respectively "forgetting" some generators. Hence a face of $\sigma \oplus \tau$ is the direct sum of faces of σ and τ. This proves that the faces of the product of two cones are exactly all the possible products of faces of the two cones.

Now we can prove that $\Delta_1 \oplus \Delta_2$ is a fan in $(N_1)_{\mathbb{R}} \oplus (N_2)_{\mathbb{R}}$. In fact:

- it is a collection of rational cones in $(N_1)_{\mathbb{R}} \oplus (N_2)_{\mathbb{R}}$ (use Remark 2.11 and the fact that the generators of the two cones are in N_1 and N_2 respectively).

- since every cone in Δ_1 is strongly convex, $\{0_{N_1}\}$ is a face of it; in the same way $\{0_{N_2}\}$ is a face of every cone in Δ_2. Hence each cone in $\Delta_1 \oplus \Delta_2$ has $\{0_{N_1 \oplus N_2}\}$ as a face and so it is strongly convex.

- for every cone $\sigma \oplus \tau$ in $\Delta_1 \oplus \Delta_2$, each face of it is a product of two faces μ of σ and ν of τ. Since Δ_1 and Δ_2 are fans, $\mu \in \Delta_1$ and $\nu \in \Delta_2$, so also the product $\mu \oplus \nu \in \Delta_1 \oplus \Delta_2$. This proves that faces of cones in $\Delta_1 \oplus \Delta_2$ are again in $\Delta_1 \oplus \Delta_2$.

- take $\sigma_1 \oplus \tau_1$ and $\sigma_2 \oplus \tau_2$ in $\Delta_1 \oplus \Delta_2$ and consider their intersection Clearly

$$(\sigma_1 \oplus \tau_1) \cap (\sigma_2 \oplus \tau_2) = (\sigma_1 \cap \sigma_2) \oplus (\tau_1 \cap \tau_2)$$

  and each of the two intersection is a face of both the cones by definition of fan, so the direct sum is a face of both $\sigma_1 \oplus \tau_1$ and $\sigma_2 \oplus \tau_2$.

Hence $\Delta_1 \oplus \Delta_2$ satisfies all the properties of a fan. $\qquad\square$

Lemma 2.24. *Let R be a ring and consider two semigroups S_1 and S_2. Then $R[S_1 \oplus S_2] = R[S_1] \otimes_R R[S_2]$.*

Proof.
Recall the definition of the tensor product of two R-modules X and Y via universal property. It is defined as an R-algebra $X \otimes_R Y$ together with an R-bilinear map $\varphi : X \times Y \to X \otimes_R Y$ (i.e. a map that is linear in both the arguments and satisfies $\varphi(r \cdot x, y) = \varphi(x, r \cdot y)$ for any $r \in R$) such that for any other R-algebra Z with an R-bilinear map $\psi : X \times Y \to Z$ there exists a unique homomorphism of R-algebras $\theta : X \otimes_R Y \to Z$ such that the following diagram commutes:

$$X \times Y \xrightarrow{\;\varphi\;} X \otimes_R Y$$
$$\psi \searrow \qquad \downarrow \theta$$
$$Z$$

Moreover we know that if it exists, the tensor product is unique up to a unique homomorphism of R-algebras. So, let's check that $R[S_1 \oplus S_2]$ satisfies the universal property. Consider the R-bilinear map $\varphi : R[S_1] \times R[S_2] \to R[S_1 \oplus S_2]$ defined on the R-basis of $R[S_1] \times R[S_2]$ as $\varphi(T^{s_1}, T^{s_2}) = T^{(s_1, s_2)}$. Suppose now there exists another R-algebra Z with a bilinear map $\psi : R[S_1] \times R[S_2] \to Z$. Define $\theta : R[S_1 \times S_2] \to Z$ on an R-basis of $R[S_1 \times S_2]$ putting $\theta(T^{(s_1, s_2)}) = \psi(T^{s_1}, T^{s_2})$. By construction, one has that $\theta \circ \varphi = \psi$ and it is also the only morphism that makes the diagram commutative. Hence one recovers the definition of tensor product. $\qquad\square$

Lemma 2.25. *Let R be a ring, call $S = \operatorname{Spec} R$. Let U_σ and U_τ be affine R-toric schemes, constructed from the convex polyhedral rational cones σ and τ. Then $U_\sigma \times_S U_\tau = U_{\sigma \oplus \tau}$.*

Proof.
This is an easy consequence of Lemma 2.24 and the fact that the dual respect the direct sum. Indeed:

$$U_{\sigma \oplus \tau} = \operatorname{Spec}(R[S_{\sigma \oplus \tau}]) = \operatorname{Spec}(R[(\sigma \oplus \tau)^\vee \cap (M_1 \oplus M_2)])$$
$$= \operatorname{Spec}(R[(\sigma^\vee \cap M_1) \oplus (\tau^\vee \cap M_2)]) = \operatorname{Spec}(R[\sigma^\vee \cap M_1] \otimes_R R[\tau^\vee \cap M_2])$$
$$= \operatorname{Spec}(R[\sigma^\vee \cap M_1]) \times_S \operatorname{Spec}(R[\tau^\vee \cap M_2]) = U_\sigma \times_S U_\tau.$$

$\qquad\square$

Remark 2.12. This lemma proves that the product of two affine R-toric schemes constructed from cones is an affine R-toric scheme constructed from a cone.

We want now to generalize the result for R-toric schemes coming from fans. Of course, Lemma 2.23 will be fundamental.

Lemma 2.26. *Let $\{U_i\}_i$ be a collection of affine schemes over $S = \operatorname{Spec} R$ with some gluing conditions satisfied and $\{V_j\}_j$ be another collection of affine schemes over S with some gluing conditions satisfied. Construct the gluing X from the first collection and Y from the second collection. Also the affine schemes $\{U_i \times_S V_j\}_{i,j}$ satisfy the gluing condition and their gluing is $X \times_S Y$.*

Proof.
This is in fact the way in which Hartshorne proves the existence of the fiber product of two schemes. For a more detailed proof see [10, Ch II, Thm 3.3]. $\qquad\square$

Proposition 2.27. *Let R be a ring, $\Delta_1 = \{\sigma_i\}_i$ be a fan in $(N_1)_\mathbb{R}$ and $\Delta_2 = \{\tau_j\}_j$ be a fan in $(N_2)_\mathbb{R}$. Then, calling $S = \operatorname{Spec}(R)$ one has that $X(R, \Delta_1) \times_S X(R, \Delta_2) = X(R, \Delta_1 \oplus \Delta_2)$ where $\Delta_1 \oplus \Delta_2$ is the fan constructed as in Lemma 2.23.*

Proof.
We know by definition that $X(R, \Delta_1)$ is the scheme constructed from the gluing of the affine schemes $\{U_\sigma\}_{\sigma \in \Delta_1}$ and $X(R, \Delta_2)$ is the scheme constructed from the gluing of the affine schemes $\{U_\tau\}_{\tau \in \Delta_2}$. All the affine schemes involved are schemes over $S = \operatorname{Spec} R$. Using Lemma 2.26, we know that $X(R, \Delta_1) \times_S X(R, \Delta_2)$ is equal to the gluing of all the affine pieces $U_\sigma \times_S U_\tau$, so by Lemma 2.25, we deduce that it is the gluing of the affine pieces $\{U_{\sigma \oplus \tau}\}_{\sigma \in \Delta_1, \tau \in \Delta_2}$. But, from definition of $\Delta_1 \oplus \Delta_2$ we have that $X(R, \Delta_1) \times_S X(R, \Delta_2)$ is obtained by glui ng from the affine pieces $\{U_{\sigma \oplus \tau}\}_{\sigma \oplus \tau \in \Delta_1 \oplus \Delta_2}$, that means by definition that it is $X(R, \Delta_1 \oplus \Delta_2)$. $\qquad\square$

Remark 2.13. Finally we proved that *the product of two R-toric schemes coming from a fan is an R-toric scheme coming from a fan.*

Example 2.28. As an immediate construction one has that all the finite fiber products of affine and projective spaces and tori still are toric schemes. This constructions provide a lot of examples since excluding the product of affine spaces (that gives affine spaces of higher dimension), all the other combinations give new schemes. For example $\mathbb{P}^1_R \times \mathbb{P}^1_R$ (that is not a projective space, but it can be sent to $\mathbb{P}^3_R$ via Segre embedding[3]) is an R-toric scheme coming from a fan. From the previous construction it is also easy to construct its fan: remembering what was studied in subsection 2.6.1, the fan of the R-toric scheme $\mathbb{P}^1_R$ is a collection of two half lines and the origin, so:

$$\mathbb{P}^1_R \times \mathbb{P}^1_R = X(\Delta, R) \times X(\Delta, R) = X(\Delta \times \Delta, R)$$

where $\Delta = \{\text{origin}, \text{right half line}, \text{left half line}\}$ is a fan in $\mathbb{R}^2$. So the fan for the product of two projective lines is simply the fan in $\mathbb{R}^2$ made as in figure 2.13.

---

[3]Recall that the *Segre embedding* $\mathbb{P}^1_R \times \mathbb{P}^1_R \to \mathbb{P}^3_R$ comes from the graded R-algebras morphism $R[z_{00}, z_{01}, z_{10}, z_{11}] \to R[x_0, x_1] \otimes_R R[y_0, y_1]$, $z_{ij} \mapsto x_i \otimes y_j$.
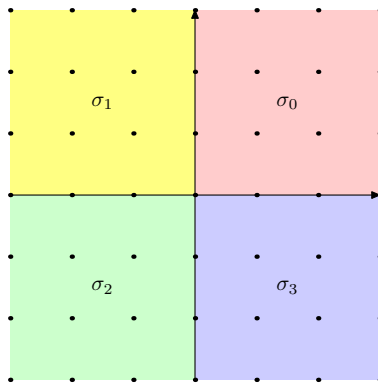
Figure 2.13: The fan in $\mathbb{R}^2$ for $\mathbb{P}^1_R \times \mathbb{P}^1_R$.

Chapter 3

Properties of toric schemes coming from fans

In the next chapters we will deal with divisors on toric schemes. We will see in Chapter 4.1 that in order to define divisors on a scheme it is needed that the scheme is integral, normal and noetherian. For this reason the aim of this chapter is to find sufficient condition on R and on Δ such that the toric scheme $X(R, \Delta)$ is an integral normal noetherian scheme.

3.1 Properties of affine toric varieties

We will begin studying properties of affine R-toric varieties.

3.1.1 Integrality

Proposition 3.1. *If R is an integral domain, any affine R-toric variety associated to a strongly convex polyhedral cone is an integral scheme.*

Proof.
Let σ be a strongly convex polyhedral cone in some vector space. Since the scheme U_σ is an affine scheme, it is sufficient to prove that its coordinate ring $R[S_\sigma]$ is an integral domain. But the fact that S_σ is an affine semigroup implies that it can be embedded in a lattice M, hence surely we have an injective morphisms of R-algebras

$$R[S_\sigma] \to R[M] \simeq R[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}].$$

But the second ring is an integral domain: in fact, R is an integral domain, hence

$$R[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}] = R[x_1, x_2, \ldots, x_n]_{x_1 \cdot x_2 \ldots x_n}$$

is a localization of an integral domain, so an integral domain. So $R[S_\sigma]$ is injectable in an integral domain, it is an integral ring. $\qquad \square$

3.1.2 Normality

Lemma 3.2. *Let R be an integrally closed domain. Then:*

(a) $R[x_1, x_2, \ldots, x_n]$ is an integrally closed domain.

(b) any localization $R_{\mathfrak{p}}$ with respect to a prime ideal $\mathfrak{p}$ is an integrally closed domain.

(c) the intersection of two integrally closed domains with the same field of fractions is again integrally closed.

Proof.

(a) Refer to [23, Lem 10.34.8].

(b) Refer to [23, Lem 10.34.9].

(c) Consider an element x in the common fraction field K of two domains A and B. Suppose it is integral over $A \cap B$. Then it satisfies an integral relation over $A \cap B$. In particular it satisfies an integral relation over A and an integral relation over B, so it is contained in the integral closure of A in K and in the one of B in K. But the rdomains are integrally closed in their field of fractions, so this implies $x \in A$ and $x \in B$.

$\square$

Lemma 3.3. *Let R be an integrally closed domain and let τ be a strongly convex 1-dimensional polyhedral cone in $N_{\mathbb{R}}$. Then $R[S_\tau]$ is an integrally closed domain.*

Proof.
Call n the dimension of the lattice N. Take the unique minimal generator of τ in N, call it u_τ. It can be completed to a $\mathbb{Z}$-basis of the lattice N; let it be $\{u, w_2, \ldots, w_n\}$; compactly we will say $\{w_1, w_2, \ldots, w_n\}$ with $u = w_1$. The dual lattice is M and it is generated by the dual basis $\{w_1^*, w_2^*, \ldots, w_n^*\}$.
By definition:

$$\tau^\vee = \{v \in M_{\mathbb{R}} : \langle v, u \rangle \geq 0 \text{ for all } u \in \tau\}$$
$$= \left\{v \in M_{\mathbb{R}} : \left\langle \sum_i a_i w_i^*, \lambda u_\tau \right\rangle \geq 0 \text{ for all } \lambda \geq 0\right\}$$
$$= \{v \in M_{\mathbb{R}} : \lambda a_i \geq 0 \text{ for all } \lambda \geq 0\}$$
$$= \left\{v \in M_{\mathbb{R}} : v = \sum_i a_i w_i^* \text{ with } a_i \geq 0\right\}$$
$$= \mathbb{R}^+ w_1^* + \mathbb{R} w_2^* + \cdots + \mathbb{R} w_n^*$$

so

$$\tau^\vee \cap N = \mathbb{N} w_1^* + \mathbb{Z} w_2^* + \cdots + \mathbb{Z} e_n^* = \langle w_1^*, \pm w_2^*, \cdots \pm w_n^* \rangle$$

from which

$$R[S_\tau] = R[T^{w_1^*}, T^{\pm w_2^*}, \ldots, {}^{\pm w_n^*}] \simeq R[x_1, x_2^{\pm 1}, \ldots, x_n^{\pm 1}] = R[x_1, x_2, \ldots, x_n]_{x_2 x_3 \ldots x_n}.$$

Now, using Lemma 3.2 (b) one deduces that $R[S_\tau]$ is isomorphic to a localization of the ring $R[x_1, x_2, \ldots, x_n]$, that is integrally closed by Lemma 3.2 (a). Hence it is normal itself. $\square$

Proposition 3.4. *Suppose that R is an integrally closed domain. Then any affine R-toric variety associated to a strongly convex polyhedral cone is a normal scheme.*

Proof.
Recall that an affine scheme $\operatorname{Spec} A$ with A an integral domain is normal if and only if A is an integrally closed domain. So it is sufficient to show that $R[S_\sigma]$ is integrally closed, since the fact that R is a domain already implies that $R[S_\sigma]$ is a domain (see Proposition 3.1).
From the theory of convex polyhedral cone one can prove that (see for example [4, Prop 1.2.8]):

$$\sigma^\vee = \bigcap_{\tau \ 1-\text{dimensional faces of } \sigma} \tau^\vee$$

from which

$$R[S_\sigma] = R[\sigma^\vee \cap M] = R\left[\left(\bigcap \tau^\vee\right) \cap M\right] = R\left[\bigcap(\tau^\vee \cap M)\right] = \bigcap R[\tau^\vee \cap M] = \bigcap R[S_\tau].$$

Now, using Lemma 3.3 and the hypothesis, one deduces that $R[S_\sigma]$ is the intersection of finitely many integrally closed domains (since a cone in a finite dimensional vector space has finitely many faces). But these rings all have the same fraction field, since by the proof of Lemma 3.3 one has, just changing the name of variables:

$$\operatorname{Frac}(R[S_{\tau_i}]) = \operatorname{Frac}(R[x_1^{\pm 1}, \ldots, x_i, \ldots, x_n^{\pm 1}]) = \operatorname{Frac}(R)(x_1, x_2, \ldots, x_n).$$

Now, from Lemma 3.2 (c), the intersection of finitely many integrally closed domains with the same field of fractions is again an integrally closed domain. $\qquad \square$

3.1.3 Noetherianity

Proposition 3.5. *If R is a noetherian ring, any affine R-toric variety associated to a strongly convex polyhedral cone is a noetherian scheme.*

Proof.
By the equivalent definition of noetherian scheme (remark 1.5), one has that an affine scheme is noetherian if and only its section ring is noetherian. So, if σ is a strongly convex polyhedral cone in some vector space, the scheme U_σ is a noetherian if and only if its coordinate ring $R[S_\sigma]$ is a noetherian ring. Now, we recall that $R[S_\sigma]$ is a finitely generated R-algebra by 2.13, so it is isomorphic to a quotient of $R[x_1, x_2, \ldots, x_l]$ for a certain integer l. But from Hilbert basis theorem the noetherianity of R implies the one of $R[x_1, x_2, \ldots, x_l]$ and this one implies the one of the quotient, hence the one of $R[S_\sigma]$. $\qquad \square$

3.2 Properties of toric schemes defined from fans

The first two properties for toric schemes constructed from fans we want to prove will be used to justify some of the other ones.

Proposition 3.6. *If R is an noetherian ring, any R-toric scheme associated to a fan is a locally noetherian scheme.*

Proof.
Just recall that a scheme is said to be locally noetherian if it is covered by spectra of noetherian ring. But it is immediate from the construction that any scheme $X(R,\Delta)$ is covered by schemes isomorphic to U_σ, for σ running in the set of cones in the fan. Since each of this schemes is the spectrum of a noetherian ring by Proposition 3.5 (the ring R is supposed to be noetherian), one gets the result. $\square$

Proposition 3.7. *If the ring R has no idempotents different from 1_R and 0_R, then any R-toric scheme associated to a fan is connected.*

Proof.
A scheme is said to be connected if its underlying topological space is such. Observe now that $X(R,\Delta)$ is constructed as the gluing of affine schemes U_σ, each of which is connected by hypothesis (recall that an affine scheme $\operatorname{Spec} A$ is connected if and only if the ring A possesses no idempotents other than 0 and 1). Moreover, all the cones in Δ intersects in the origin, hence, recalling the correspondance of Proposition 2.17, $U_{\{0\}}$ is a subscheme of all the schemes U_σ coming from the faces. When one glues, this subscheme turns to be an open subset of all the pieces that have to be patched. From the topological point of view, this implies that $X(R,\Delta)$ is the union of connected sets with nonempty intersection. An easy lemma from topology assures that their union is a connected topological space. $\square$

3.2.1 Integrality

Proposition 3.8. *If R is an integral domain, any R-toric scheme associated to a fan is an integral scheme.*

Proof.
Recall that one of the possible definition for an integral scheme X is requiring that the scheme is irreducible and reduced (i.e. every stalk $\mathcal{O}_{X,p}$ has no nonzero nilpotents). Using [23, Lemmas 27.3.2 and 27.3.3] one can see that it is sufficient to prove that the scheme admits an affine open covering $X = \cup U_i$ such that every U_i is irreducible and $\mathcal{O}_X(U_i)$ are reduced rings; moreover it should be verified that each pair of such affine open has nonempty intersection. In other words, it is enough to write the scheme X as the union of integral affine open subsets (see also [23, Lemma 25.12.3]) with intersection pairwise nonempty.
Considered a fan Δ and an integral doman R, let's take the R-toric scheme $X(R,\Delta)$. It can be covered by affine open subsets isomorphic to U_σ, where σ runs over the set of cones in the fan. These open subsets all intersect in the nonempty set $U_{\{0\}}$ since all the cones in the fan have $\{0\}$ as a face. Moreover, since R is supposed to be an integral domain, all the affine pieces result to be integral from Proposition 3.1. $\square$

3.2.2 Normality

Proposition 3.9. *Let R be an integrally closed domain. Then, any R-toric scheme associated to a fan is a normal scheme.*

Proof.
Recall that a scheme X is normal if for any point $P \in X$ the stalk $\mathcal{O}_{X,p}$ is an

integrally closed ring. This means that normality is a local property; but so for any point of a toric scheme coming from a fan we can take the affine piece containing it. It is a normal R-toric schemes by Proposition 3.4, so the point has integrally closed stalk. This proves normality. $\qquad\square$

Remark 3.1. Let k be an algebraically closed field. By what we have just proved, every k-toric scheme from a fan is a normal scheme. But recall that in Example 1.24 we proved the existence of a nonnormal k-toric scheme. Hence, *not all toric schemes can be obtained from fans.*

3.2.3 Quasi-compactness

Recall that a scheme X is said to be **quasi-compact** if it is quasi-compact as a topological space (in the Zariski topology), that is for any open cover of X it is possible to extract from it a *finite* open subcover.

Proposition 3.10. *Let R be a ring and let Δ be a fan in the vector space $N_\mathbb{R}$ consisting of finitely many cones. Then the toric scheme $X(R, \Delta)$ over R is of finite type; in particular it is quasi-compact.*

Proof.
Recall that a scheme of finite type over a base scheme is in particular quasi compact. So it is sufficient to show that the scheme $X(R, \Delta)$ is of finite type.
A scheme Y is of finite type over $\operatorname{Spec} R$ if there exists a finite open covering of X of affine subschemes of the form $\operatorname{Spec} A_i$ with each A_i a finitely generated R-algebra. For R-toric schemes constructed by fans this is immediate from the definition by gluing and from Gordon's lemma. $\qquad\square$

Remark 3.2. If we don't require the fan Δ to be a collection of only finitely many cones, we only have that the toric scheme $X(R, \Delta)$ is *locally of finite type*, meaning that there is an affine open cover of X made by affine $\operatorname{Spec} A_i$ with A_i is finitely generated for any i (in this we don't require the affine open cover to be finite).

3.2.4 Noetherianity

Proposition 3.11. *If R is a noetherian ring, any R-toric scheme associated to a finite fan is a noetherian scheme.*

Proof.
Recall that a noetherian scheme is a scheme that is both locally noetherian and quasi-compact. Using Proposition 3.6 and to Proposition 3.10, one immediately deduces that if R is a noetherian ring and the fan Δ consists of finitely many cones, the toric scheme $X(R, \Delta)$ is noetherian. $\qquad\square$

3.2.5 Separatedness and properness

Let X be an R-scheme. Recall that X is said to be **separated** (over R) if the diagonal morphism coming from the fiber product

$$\Delta_X : X \times_R X \to X$$

is a closed immersion[1].

Moreover, the R-scheme X is said to be **proper** (over R) if it is separated, of finite type and universally closed[2].

The idea of separatedness corresponds to the classical definition of Hausdorff topological space, while the one of completeness corresponds to the classical definition of compact topological space. It is possible to show that, if $R = \mathbb{C}$, considering the R-schemes as complex analytic spaces in the ordinary topology, these categorical definitions give in fact the corresponding topological ones (see the famous paper GAGA by Serre, [22]).

We will now study this scheme properties for R-toric schemes deduced from a fan.

Proposition 3.12. *Any R-toric scheme associated to a fan is a separated scheme.*

Proposition 3.13. *For any base ring R, the R-toric scheme associated to a fan Δ is proper if and only if the fan is complete (i.e. its support covers the entire vector space).*

Example 3.14. A good example of Proposition 3.13 is given by the examples seen in Chapter 2. Looking at the fan in 2.6.3, one sees that it is complete, since it covers all the vector space $\mathbb{R}^2$. Taking $R = \mathbb{C}$ we then expect that the $\mathbb{C}$-toric scheme constructed from it is proper, i.e. compact in the classical topology (it is considered as a complex analytic space): and in fact the $\mathbb{C}$-toric scheme constructed from it is the projective space $\mathbb{P}_{\mathbb{C}}^n$. On the contrary, the fan in 2.6.2 is not complete and in fact the corresponding $\mathbb{C}$-toric scheme is the affine space $\mathbb{A}_{\mathbb{C}}^2$, that is not compact in the usual topology, i.e. it is not proper as a $\mathbb{C}$-scheme.

3.2.6 Smoothness

The most immediate definition of smoothness for schemes requires that the scheme is of finite type over a field k. So, for this paragraph let's consider only k-schemes, with k a field; we will call $\overline{k}$ the algebraic closure of the field k. A k-scheme of finite type X is said to be **smooth** if the base change $X \times_k \operatorname{Spec} \overline{k}$ is a regular scheme, i.e. the stalk of every point is a regular ring[3].

Recall that a fan Δ is said t be *smooth* if every cone in Δ is smooth, i.e. the minimal generators of the cone are part of a $\mathbb{Z}$-basis of the lattice N.

Proposition 3.15. *Let k be a field. The k-toric scheme associated to a finite fan is smooth if and only if the fan is smooth.*

Proof.

First of all the statement is well posed since the fact that the fan is finite implies

---

[1]Recall that a closed immersion of schemes is a morphism of schemes such that the function between topological spaces induces a homeomorphism between the first one and a closed subset of the second one and furthermore the morphism of sheaves is surjective.

[2]Recall that an R-scheme X is universally closed over R if for every other R-scheme Y one has that $X \times_R Y \to Y$ is a closed map on topological spaces.

[3]A noetherian local ring A with unique maximal ideal $\mathfrak{m}$ and residue field k is said to be regular if $\dim(A) = \dim(\mathfrak{m}/\mathfrak{m}^2)$, where the first dimension is the Krull dimension of a ring and the second dimension is as a k-vector space. Observe that the definition is well posed, since each stalk is a local ring by definition of scheme and it is a noetherian ring since the scheme is noetherian (it is a scheme of finite type over a field).

that the k-toric scheme constructed from it is of finite type over k.

We will prove for simplicity only one direction. From the definition, it is clear that smoothness is a local property, hence it will be enough to verify the property on affine k-toric schemes. So, suppose U_σ to be the affine k-toric scheme constructed from the cone σ. Suppose also that the cone σ is smooth. This means that its generators $v_1, v_2, \ldots, v_l$ can be taken as part of a $\mathbb{Z}$-basis of N. So we can complete this set to a basis $w_1 = v_1, \ldots, w_l = v_l, w_{l+1}, \ldots, w_n$ of N. The dual lattice M has dual basis $w_1^*, w_2^*, \ldots, w_n^*$, so

$$S_\sigma = \sigma^\vee \cap M = \left\{ u = \sum_{i=1}^n a_i w_i^* : a_i \in \mathbb{N} \text{ and } u(v) \geq 0 \text{ for all } v \in \sigma \right\}$$

$$= \mathbb{N}w_1^* + \cdots + \mathbb{N}w_l^* + \mathbb{Z}w_{l+1}^* + \cdots + \mathbb{Z}w_n^*$$

hence

$$k[S_\sigma] \simeq k[x_1, \ldots, x_l, x_{l+1}^{\pm 1}, \ldots, x_n^{\pm 1}] = k[x_1, \ldots, x_l] \otimes k[y_1^{\pm 1}, \ldots, y_{n-l}^{\pm 1}].$$

This implies of course

$$U_\sigma = \mathrm{Spec}(k[x_1, \ldots, x_l]) \times_k \mathrm{Spec}(k[y_1^{\pm 1}, \ldots, y_{n-l}^{\pm 1}]) = \mathbb{A}_k^l \times_k \mathbb{T}_{n-l,k}$$

and so it is a smooth k-scheme. $\qquad\square$

Remark 3.3. The statement in the proposition can be extended to any base ring R once one has defined smoothness for schemes over arbitrary base rings.

Remark 3.4. Asking that a fan is smooth in the case $N = \mathbb{Z}^2$ is equivalent to the following requirement: for any two dimensional cone in the fan, the determinant of the matrix having as columns the minimal generators for its two rays must have determinant ± 1.

Remark 3.5. Using the previous remark one could define a way to solve singularities on a toric scheme constructed starting from the lattice $N = \mathbb{Z}^2$. Let's consider a toric scheme constructed from a fan Δ in the vector space $N_\mathbb{R} = \mathbb{R}^2$ and suppose that $X(R, \Delta)$ is not smooth. This means that the scheme has a singular point P or, equivalently, that the fan has at least one cone whose minimal generators are not part of a $\mathbb{Z}$-basis of N. In this case one could refine the fan to a fan Δ'; this means that one can introduce new one dimensional cones in the fan in such a way that the fan results to consist in more cones than before (we are splitting some of the cones). This operation can be done in such a way that every cone in the new fan has minimal generators which are part of a basis of the lattice (this can be done for example following [7, Ch 2.6]). From a geometric point of view, the scheme obtained from Δ' will be, using Proposition 3.15, a smooth R-toric scheme; moreover it can be proved that there exists a proper morphism $X(R, \Delta') \to X(R, \Delta)$ that is an isomorphism outside the locus of singularities. This is what is called a resolution of singularities for the R-toric scheme $X(R, \Delta)$.

Clearly the same resolution of singularities can be extended to higher dimensional cases.
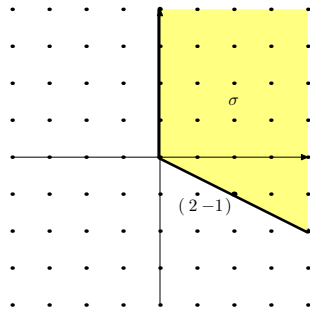
Figure 3.1: A non smooth fan.

Example 3.16. As an example of Remark 3.5 we can consider a field k and the cone σ in $\mathbb{R}^2$ generated by the vectors $(0,1)$ and $(2,-1)$. This cone is not smooth since the two minimal generators are not part of a $\mathbb{Z}$-basis of $\mathbb{Z}^2$ (for example their determinant is -2). So the corresponding affine k-toric scheme U_σ is not smooth by the proposition. Anyway, we can introduce the ray generated by the vector $(1,0)$, as in Figure 3.2. The fan we constructing by adding this ray is smooth, since both the
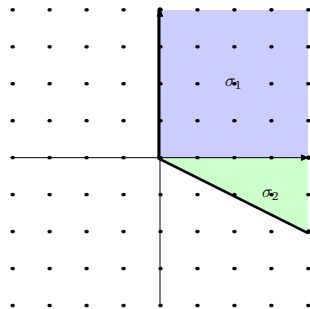

Figure 3.2: A smooth fan refining the one in Figure 3.1.

cones σ_1 and σ_2 are smooth; in fact the determinants of their minimal generators are both 1 up to a signum. Then the k-toric scheme constructed from this fan is a smooth scheme. We say to have "solved the singularities" of U_σ.

3.2.7 Conclusions

The results of this chapter underline the strong connection between the combinatorial data and the toric variety constructed from them. In particular it is evident that the properties of the scheme $X(R, \Delta)$ depends both on algebraic properties of the ring R (such as the fact that it is an integral domain, noetherian or integrally closed) both on the combinatorial properties of the fan (such as the fact that it is a complete fan or a smooth fan). At last, there are properties coming from the construction of the R-scheme itself, such as quasi compactness and separatedness.

What it is worth to remember for future studies is the following result, that resume what has been proved.

Theorem 3.17. *Let R be a noetherian integrally closed domain and let Δ be a fan in the vector space $N_\mathbb{R}$, consisting of finitely many cones. Then the toric scheme $X(R, \Delta)$ is a separated normal integral noetherian scheme.*

Example 3.18. Consider a Dedekind domain R. It is a noetherian integrally closed domain of Krull dimension1. This means in particular that if Δ is a fan consisting of finitely many cones, then the toric scheme $X(R, \Delta)$ is a separated normal integral noetherian scheme. In particular for any number field K, its ring of integers satisfies this property: any $\mathcal{O}_K$-toric scheme coming from a finite fan is a separated normal integral noetherian scheme.

Chapter 4

Weil divisors on toric schemes

The definition of the main object of the thesis is contained in the following chapter. To give it, it is necessary to use the machinery of Weil divisors. In this chapter we will develop the theory needed in the following.

4.1 Divisors on schemes

To begin, recall what is the function field of an integral scheme.

Definition 4.1. Let $(X, \mathcal{O}_X)$ be an integral scheme. The **function field of** X is defined as the ring of fraction of the ring of coordinates $\mathcal{O}(U)$ of one[1] affine Zariski open subset U of X.

We now follow the presentation of Hartshorne [10], for which we refer for all the proofs.

Definition 4.2. A **prime divisor** on a separated normal integral noetherian scheme X is an integral closed subscheme of codimension 1 in X. A **Weil divisor** on a variety is a linear combination of prime divisors, i.e. a finite formal sum of prime divisors with integer coefficients.

Remark 4.1. The naive interpretation for classical abstract varieties is then the following: a *prime divisor* on an irreducible variety X is a closed irreducible subvariety of codimension 1. And a *Weil divisor* on a variety is a linear combination of prime divisors. This will be the way in which we will naively think to Weil divisors.

In this way a divisor on a scheme X is a formal sum $D = \sum n_Y Y$, where $n_Y \in \mathbb{Z}$ for every integral closed subscheme Y of codimension 1 and n_Y almost always zero. A divisor is called **effective** if all the coefficients are nonnegative. The support of a divisor D is the set theoretically union of the subschemes whose coefficient is nonzero. The set of divisors over the scheme X is a group (it is the free abelian group generated by the prime divisors), denoted by $\mathrm{Div}(X)$.
Depending from the fact that the scheme X is also normal, one has that for any

---

[1]For integral schemes, the definition is well posed since all the affine Zariski open subsets of X are seen to give the same fraction field.

prime divisor Y of X, the local ring of the[2] generic point of Y is a discrete valuation ring whose quotient field is the function field of X. The valuation v_Y on the stalk of the generic point can be extended to a valuation on the function field, that we keep calling v_Y with abuse of notation. For any f nonzero element of the function field of X one has an integer $v_Y(f)$; one can prove that this integer is zero for almost all the choices of Y. One calls $(f) = \sum_Y v_Y(f) Y$ the principal divisor associated to the function f. Any divisor which is equal to the divisor of a function is called a **principal divisor**; the set of principal divisors on X is denoted by $\mathrm{Div}_0(X)$ and it is a subgroup of $\mathrm{Div}(X)$.

Definition 4.3. Two divisors D_1, D_2 on X are said to be **linearly equivalent**, $D_1 \sim D_2$, if the divisor $D_1 - D_2 \in \mathrm{Div}_0(X)$. The group of all the divisors over X modulo linear equivalence is called the **divisor class group of** X,

$$\mathrm{Cl}(X) = \mathrm{Div}(X)/\sim = \mathrm{Div}(X)/\mathrm{Div}_0(X).$$

4.2 Star construction: Weil divisors from rays of the fan

We introduce in this section a nice construction of closed subschemes of a toric scheme.

First of all we consider as usual the two dual lattices N and M. Let n be the rank of N. Let Δ be a fan in the vector space $N_\mathbb{R}$. Consider a ray (i.e. a 1-dimensional cone) ρ in the fan Δ. Call N_ρ the sublattice of N generated by $\rho \cap N$, as in figure 4.1. It is clearly a lattice of rank 1, namely $N_\rho = \mathbb{Z} \cdot n_\rho$ if n_ρ is the minimal generator of the ray.
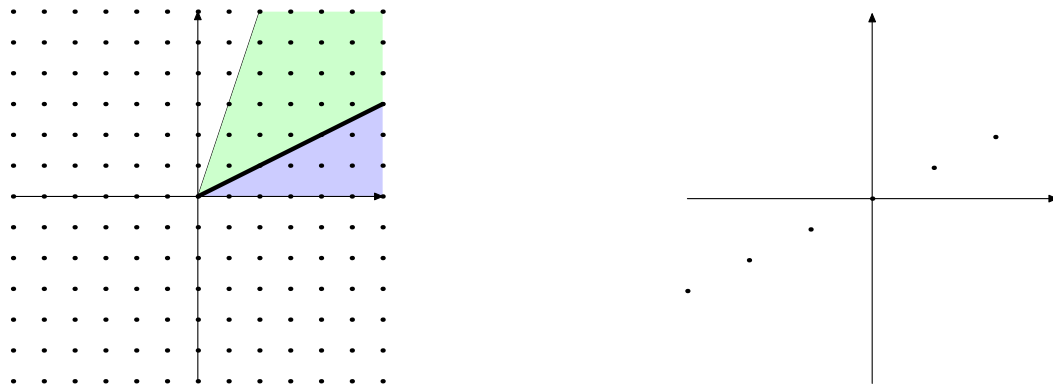

Figure 4.1: The sublattice associated to the highligthed ray.

To this ray we associate the quotient lattice

$$N(\rho) = N/N_\rho.$$

---

[2]Any integral scheme is irreducible. Moreover, there is a one to one correspondence between irreducible components of a scheme S and the generic points of S. Hence here the construction is well posed, since Y is supposed to be irreducible and so has only one generic point.

It is again a lattice and of rank $n-1$; this is because the sublattice is one dimensional and the quotient is still a free group[3]. Since $N_\rho = \mathbb{Z} \cdot n_\rho$, the dual lattice of $N(\rho)$ is

$$
\begin{aligned}
M(\rho) &:= \operatorname{Hom}(N(\rho), \mathbb{Z}) \simeq \{\varphi \in \operatorname{Hom}(N, \mathbb{Z}) : \varphi(N_\rho) = \{0\}\} \\
&= \{m \in M : \langle m, n \rangle = 0 \text{ for all } n \in N_\rho\} = \{m \in M : \langle m, n_\rho \rangle = 0\} \\
&= M \cap \rho^\perp.
\end{aligned}
$$

The isomorphism is given in the following way: construct a basis $\{n_\rho, v_2, \ldots, v_n\}$ of N, then $N(\rho)$ is generated by $\{\overline{v_2}, \overline{v_3}, \ldots, \overline{v_n}\}$ in the quotient, see that any homomorphism of groups from $N(\rho)$ to $\mathbb{Z}$ sends each $\overline{v_i}$ to a certain integer number m_i. The corresponding homomorphism between N and $\mathbb{Z}$ sends n_ρ to 0 and each v_i to m_i. Conversely any morphism $N \to \mathbb{Z}$ that is zero restricted to N_ρ gives a morphism $N(\rho) \to \mathbb{Z}$ by projection. The two constructions are one the inverse of the other. Proceeding as usual we can also associate to this lattices the vector spaces $N(\rho)_\mathbb{R}$ and $M(\rho)_\mathbb{R}$.
We are now ready to define the star construction.

Definition 4.4. Let Δ be a fan in the vector space $N_\mathbb{R}$ and let ρ be a 1-dimensional face of Δ. We call **star of the ray** ρ the collection of $\overline{\sigma} \in N(\rho)$ with σ a cone in Δ containing ρ as a face.

Explicitly it is

$$
\operatorname{Star}(\rho) = \{\overline{\sigma} : \sigma \in \Delta, \rho \preceq \sigma\} \subseteq N(\rho)_\mathbb{R}.
$$

Lemma 4.5. *Let Δ be a fan in the vector space $N_\mathbb{R}$ and let ρ be a 1-dimensional face of Δ. Then $\operatorname{Star}(\rho)$ is a fan in $N(\rho)_\mathbb{R}$.*

Proof.
Consider any set in the collection $\operatorname{Star}(\rho)$. It is the projection of a strongly convex polyhedral rational cone σ in $N_\mathbb{R}$ that contains ρ as a face. But the generators of σ can be chosen to be the ray generators of its rays, so they are $\{n_\rho, v_2, \ldots, v_l\}$. Its projection is generated by $\{\overline{n_\rho}, \overline{v_2}, \ldots, \overline{v_l}\}$ and hence by $\{\overline{v_2}, \overline{v_3}, \ldots, \overline{v_l}\}$. Immediately one sees that it is still a strongly convex rational polyhedral cone, but in the new vector space $N(\rho)_\mathbb{R}$: in fact it is still a cone and it is rational since the generators are $\overline{v_2}, \overline{v_3}, \ldots, \overline{v_l} \in N(\rho)$; finally since ρ is a face of σ, $\{\overline{0}\}$ is a face of $\overline{\sigma}$.
Moreover if $\overline{\sigma}$ is a cone in $\operatorname{Star}(\rho)$, each face of $\overline{\sigma}$ is the projection of a face of σ containing ρ. This claim is enough in order to verify easily that the remaining two properties of a fan are satisfied. $\qquad\square$

By the previous Lemma one obtains that the following definition is coherent.

Definition 4.6. Let R be a ring, Δ be a fan in the vector space $N_\mathbb{R}$ and let ρ be a 1-dimensional face of Δ. Then the **toric divisor associated to** ρ is $D_\rho := X(R, \operatorname{Star}(\rho))$.

---

[3]To see this, notice that n_ρ is a vector with coprime entries ($\gcd(x_1, x_2, \ldots, x_n) = 1$) since it is a *minimal* generator of a ray; as stated in [17] it is a "primitive" vector and can be extended to a basis of the lattice, hence the quotient gives again a free group.

We know verify that it is effectively a prime divisor for the scheme $X(R, \Delta)$. Recall that a prime divisor of a separated normal integral noetherian scheme X is an integral closed subscheme of codimension 1 in X. Let's start checking the fact that it is a closed subscheme of $X(R, \Delta)$.

Proposition 4.7. *Let R be a ring, Δ be a fan in the vector space $N_{\mathbb{R}}$ and let ρ be a 1-dimensional face of Δ. Then D_ρ is a closed subscheme of $X(R, \Delta)$.*

Proof.
Let's understand what happens in the affine case, i.e. when Δ consists of a cone σ. For any 1-dimensional face ρ of σ we have that:

$$\mathrm{Star}(\rho) = \{\overline{\tau} : \rho \preceq \tau \preceq \sigma\} \subseteq N(\rho)_{\mathbb{R}}$$

and so all the affine pieces of $X(R, \mathrm{Star}(\rho))$ are $U_{\overline{\tau}}$, so open subschemes of $U_{\overline{\sigma}}$. Hence:

$$D_\rho = X(R, \mathrm{Star}(\rho)) \simeq U_{\overline{\sigma}} = \mathrm{Spec}(R[M(\rho) \cap \overline{\sigma}^\vee]) \simeq \mathrm{Spec}(R[M \cap \rho^\perp \cap \sigma^\vee])$$

since $\overline{\sigma}^\vee = \{w \in M(\rho) : w(v) \geq 0 \text{ for all } v \in \overline{\sigma}\}$ is isomorphic (under the isomorphism of vector spaces $M(\rho) \simeq M \cap \rho^\perp$) to $\{m \in M \cap \rho^\perp : \langle m, v \rangle \geq 0 \text{ for all } v \in \sigma\} = M \cap \rho^\perp \cap \sigma^\vee$.
Now, consider the morphism of R-algebras $\psi : R[\sigma^\vee \cap M] \to R[M \cap \rho^\perp \cap \sigma^\vee]$ obtained by defining the image of every generator T^u as:

$$\psi(T^u) = \begin{cases} T^u & \text{if } u \in \rho^\perp \\ 0 & \text{if } u \notin \rho^\perp \end{cases}.$$

Then, one has

$$R[M \cap \rho^\perp \cap \sigma^\vee] \simeq R[\sigma^\vee \cap M]/\mathrm{Ker}(\psi) = R[\sigma^\vee \cap M]/\langle T^u : u \notin \rho^\perp \rangle$$

where $\langle T^u : u \notin \rho^\perp \rangle$ denotes the ideal generated by these elements. Hence:

$$D_\rho \simeq \mathrm{Spec}(R[M \cap \rho^\perp \cap \sigma^\vee]) \simeq \mathrm{Spec}(R[\sigma^\vee \cap M]/\langle T^u : u \notin \rho^\perp \rangle) \simeq V(T^u : u \notin \rho^\perp)$$

so $D_\rho = U_{\overline{\sigma}}$ can be seen as a closed subscheme of $\mathrm{Spec}(R[\sigma^\vee \cap M])$.
In the general case (for Δ a general fan), the scheme D_ρ is covered by affine pieces of the type $U_{\overline{\sigma}}$ where σ ranges over the family of cones of Δ containing ρ as a face. Each $U_{\overline{\sigma}}$ is a closed subscheme of U_σ. Compatibility conditions guarantee the fact that $U_{\overline{\sigma}}$ can be glued together to give a closed subscheme of the glue of U_σ. But the glue of the first is $D_\rho = X(R, \mathrm{Star}(\rho))$ while the glue of the second give $X(R, \Delta)$, hence D_ρ is a closed subscheme of $X(R, \Delta)$. $\qquad\square$

Hence D_ρ is a closed subscheme of $X(R, \Delta)$. Moreover if R is taken to be an integral domain, by 3.8 the scheme $X(R, \mathrm{Star}(\rho))$ is an integral scheme.
At last, let's look at the dimension. We suppose R to be noetherian. It follows from the study of R-toric schemes that the scheme $X(R, \mathrm{Star}(\rho))$ contains as a Zariski open subset the torus $U_{\{\overline{0}\}} = \mathrm{Spec}(R[M(\rho)])$: if N has rank n then $N(\rho)$ has been seen to be of rank $n-1$. Hence also the dual $M(\rho)$ has rank $n-1$, so the torus

is $\mathrm{Spec}(R[M(\rho)]) = \mathrm{Spec}(R[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_{n-1}^{\pm 1}])$. This is a scheme of dimension[4] $\dim(R)+n-1$ and it is a dense open subscheme of $X(R, \mathrm{Star}(\rho))$; since the dimension of a topological space is the same as the dimension of any of its open dense subset, one concludes that the dimension of $D_\rho = X(R, \mathrm{Star}(\rho))$ is $\dim(R) + n - 1$. This means that D_ρ has codimension 1 in $X(R, \Delta)$. Indeed this last scheme has dimension $\dim(R) + n$ since it contains the torus $\mathrm{Spec}(R[M]) = \mathrm{Spec}(R[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}])$ as a dense open subset.

To resume, one has the following.

Theorem 4.8. *Let R be a noetherian integrally closed domain and let Δ be a fan in the vector space $N_\mathbb{R}$, consisting of finitely many cones. For every 1-dimensional face ρ of Δ the toric divisor $D_\rho := X(R, \mathrm{Star}(\rho))$ is a prime divisor of $X(R, \Delta)$.*

4.3 Topological structure of a toric scheme coming from a fan

The aim of this paragraph is to understand the topological composition of a toric scheme. This will be essential for the development of the theory of Weil divisors on toric schemes.

Lemma 4.9. *Let ρ be a ray of the cone σ. Then $D_\rho = V(T^{n_\rho^*})$ in U_σ and its generic point is the prime ideal $\langle T^{n_\rho^*} \rangle$.*

Proof.
Since n_ρ is the minimal generator of the ray, we can extend it to a base of N, $\{e_1, e_2, \ldots, e_n\}$, with $e_1 = n_\rho$. Consider the dual basis $\{e_1^*, e_2^*, \ldots, e_n^*\}$ of M. Recall the fact that D_ρ can be seen in U_σ as $D_\rho = V(T^u : u \notin \rho^\perp)$ (see the proof of 4.7). Remarking that $v = \alpha_1 e_1^* + \alpha_2 e_2^* + \cdots + \alpha_n e_n^*$ is in $\rho^\perp$ if and only if $\alpha_1 = 0$, it is easily seen that T^v is in $\langle T^u : u \notin \rho^\perp \rangle$ if and only if it is divided by $T^{e_1^*} = T^{n_\rho^*}$ (recall that necessarily $\alpha_i \geq 0$ since the vector must lie in $\sigma^\vee$). Hence, $\langle T^u : u \notin \rho^\perp \rangle \subseteq \langle T^{n_\rho^*} \rangle$. The converse is evident, so one concludes that

$$\langle T^u : u \notin \rho^\perp \rangle = \langle T^{n_\rho^*} \rangle$$

that means that $D_\rho = V(T^{n_\rho^*})$.
The generic point of D_ρ is then the minimal prime ideal containing $\langle T^{n_\rho^*} \rangle$. But this ideal is already prime. Indeed it is equal to $\langle T^u : u \notin \rho^\perp \rangle$ and this last one is prime: if $T^u T^v$ lies in it, $u + v \notin \rho^\perp$, so $\langle u + v, n_\rho \rangle = \langle u, n_\rho \rangle + \langle v, n_\rho \rangle > 0$; since the two vectors u and v are in $\sigma^\vee$ it must be that at least one of the terms in the sum is strictly positive, say $\langle u, n_\rho \rangle > 0$. In this case $u \notin \rho^\perp$ and so T^u lies in the ideal. $\square$

Now we state without proving a proposition on the structure of the toric scheme $X = X(k, \Delta)$ in the case in which k is a field.

---

[4]This can be proved by induction; prove that $\dim(R[x^{\pm 1}]) = \dim(R) + 1$; in fact $R[x^{\pm 1}]$ is a localization of $R[x]$, which has dimension $\dim(R) + 1$ since R is noetherian (see [25, Prop 2.4.3.12]); now the dimension of this localization cannot be greater than the dimension of the original ring. But it is at least $\dim(R) + 1$ since for example the ideal generated by $x + 1$ is prime.

Proposition 4.10. *Let k be a field and let Δ be a fan in the vector space $N_\mathbb{R}$. Calling $\mathbb{T}$ the torus in $X(k,\Delta)$, set theoretically one has that*

$$\mathbb{T} = X(k,\Delta) - \bigcup_{\rho \in \Delta(1)} D_\rho$$

where each D_ρ is the toric divisors associated to the ray ρ, as explained in Section 4.2.

Proof.
See [16, Thm 6.4(3)], [7, §3.1 page 55] or [4, Thm 3.2.6]. $\qquad\square$

Remark 4.2. This proposition is true also for the ring $\mathbb{Z}$ as proved in [6, §4, Prop 2]. We did not check it for any base ring R, even if this should be a good starting point for extending the theory of next chapter to general base rings.

4.4 Weil divisors on toric schemes

Let N be a lattice, M its dual, $N_\mathbb{R}$ and $M_\mathbb{R}$ the corresponding vector spaces. In the first subsection we will understand some peculiar Weil divisors; we will suppose R to be a noetherian integrally closed domain, Δ a fan consisting of finitely many cones in $N_\mathbb{R}$. In this way the scheme $X := X(R,\Delta)$ is separated, normal, integral and noetherian. We can in this way deal with Weil divisors on X. We will denote with $\Delta(1)$ the collection of rays in the fan and for every ray ρ we will denote with n_ρ its minimal generator, with D_ρ the prime divisors associated to it via star construction, with ν_ρ the valuation of the DVR $\mathcal{O}_{x,\eta}$ (here η is the generic point of the prime divisor D_ρ).
With the same notation we will find a short exact sequence involving the divisor class group of $X(k,\Delta)$ in the case in which k is an algebraically closed field.

4.4.1 Some principal divisors on a toric scheme

One already knows that the cone $\{0\}$ is in the fan (and in fact this implies that a torus is an open subscheme of the toric variety); moreover $U_{\{0\}}$ is an affine subset of $X(R,\Delta)$ and

$$\mathcal{O}_X(U_{\{0\}}) = \mathcal{O}_X(\operatorname{Spec}(R[M])) = R[M].$$

So the function field of X is $K(X) = \operatorname{Frac}(R[M]) \simeq \operatorname{Frac}(R)(x_1, x_2, \dots x_n)$. In any case it is clear that for every $m \in M$ the monomial T^m is an element of $K(X)$, hence one can define the Weil divisor associated to the function T^m.

Lemma 4.11. *For any element $m \in M$, the order of T^m along the toric divisor D_ρ is $\nu_\rho(T^m) = \langle m, n_\rho \rangle$.*

Proof.
The order of a function in the function field can be checked locally, so it is sufficient to verify it on an affine piece of X containing the divisor, let it be $U_\sigma = \operatorname{Spec}(R[M \cap \sigma^\vee])$ with σ containing ρ as a face.
Since n_ρ is the minimal generator of the ray, it is a primitive vector and so we can

extend it to a base of N, $\{e_1, e_2, \ldots, e_n\}$, with $e_1 = n_\rho$. Consider the dual basis $\{e_1^*, e_2^*, \ldots, e_n^*\}$ of M. The local ring of D_ρ in its generic point is then (refer to Lemma 4.9 and to [10, Prop II.2.2]):

$$R[S_\sigma]_{\langle T^{e_1^*}\rangle}$$

with only maximal ideal

$$\mathfrak{M} = \langle T^{e_1^*}\rangle R[S_\sigma]_{\langle T^{e_1^*}\rangle}.$$

The valuation $\nu_\rho(T^m)$ on this stalk is so the maximal integer l for which T^m lies in $\mathfrak{M}^l$. Hence, writing $m = \alpha_1 e_1^* + \alpha_2 e_2^* + \cdots + \alpha_n e_n^*$ (it is a basis of M):

$$
\begin{aligned}
\nu_\rho(T^m) &= \max_l \{T^m \in \mathfrak{M}^l\} = \max_l \left\{T^m \in \langle T^{le_1^*}\rangle R[S_\sigma]_{\langle T^{e_1^*}\rangle}\right\} \\
&= \max_l \left\{T^m/T^{le_1^*} \in R[S_\sigma]_{\langle T^{e_1^*}\rangle}\right\} = \max_l \left\{T^{m-le_1^*} \in R[S_\sigma]_{\langle T^{e_1^*}\rangle}\right\} \\
&= \max_l \left\{T^{(\alpha_1-l)e_1^* + \alpha_2 e_2^* + \cdots + \alpha_n e_n^*} \in R[S_\sigma]_{\langle T^{e_1^*}\rangle}\right\} \\
&= \max_l \{\alpha_1 - l \geq 0\} = \alpha_1 = \langle m, e_1\rangle = \langle m, n_\rho\rangle
\end{aligned}
$$

where we just used the fact that, by definition, in $R[S_\sigma]_{\langle T^{e_1^*}\rangle}$, all the elements of $R[S_\sigma]$ are inverted except the ones in $\langle T^{e_1^*}\rangle$, so $T^{(\alpha_1-l)e_1^*} T^{\alpha_2 e_2^*} \ldots T^{\alpha_n e_n^*}$ is in the localization if and only if the exponent of $T^{e_1^*}$ is nonnegative. $\qquad\square$

Proposition 4.12. *Suppose that R is a ring such that it satisfies the statement of Proposition 4.10. Then, for any element $m \in M$, the principal divisor associated to T^m is*

$$(T^m) = \sum_{\rho \in \Delta(1)} \langle m, v_\rho\rangle D_\rho.$$

Proof.
By definition of principal divisor:

$$\operatorname{div}(T^m) = \sum_{Y \text{ prime divisors}} \nu_Y(T^m) Y.$$

We just have to check that on the torus $\mathbb{T}$ contained in the toric scheme X the function T^m has order zero. By definition of order, we should look at the valuation of the DVR of the stalk at some generic point. Supposing the point P is in the torus, its stalk is $\mathcal{O}_{X,P} = \mathcal{O}_{\mathbb{T},P}$ and so a localization of the section ring of the torus, which is $R[M]$. Call this localization $R[M]_\mathfrak{p}$; it is a local ring with maximal ideal $\mathfrak{p}R[M]_\mathfrak{p}$. We also know that from the fact that this point is chosen as the generic point of a prime divisor, it will be a DVR. Moreover, the valuation of T^m in this ring will be given by the biggest integer l such that $T^m \in \mathfrak{p}^l R[M]_\mathfrak{p}$. The point now is that T^m is already invertible in $R[M]$ (with inverse T^{-m}), so it will be invertible also in $R[M]_\mathfrak{p}$ and so it cannot be contained in any proper ideal of this localization. So necessarily the integer l must be zero, so $\nu_P(T^m) = 0$ for every point P in the torus $\mathbb{T}$. Hence for every prime divisor Y with generic point lying inside the torus, $\nu_Y(T^m) = 0$.

Now, since the toric scheme is $X = \mathbb{T} \sqcup \bigcup D_\rho$ by hypothesis (we suppose Proposition 4.10 to be true), one immediately concludes that, using also Lemma 4.11:

$$\operatorname{div}(T^m) = \sum_{\rho \in \Delta(1)} \nu_\rho(T^m) D_\rho + \sum_{Y \text{ other prime divisors}} \nu_Y(T^m)$$

$$= \sum_{\rho \in \Delta(1)} \nu_\rho(T^m) D_\rho = \sum_{\rho \in \Delta(1)} \langle m, n_\rho \rangle D_\rho.$$

$\square$

Remark 4.3. As seen in Proposition 4.10 and in Remark 4.2, the previous proposition holds surely for fields and for the ring $\mathbb{Z}$.

4.4.2   An exact sequence

In this paragraph we prove that for toric schemes over an algebraically closed field one has a nice exact sequence. Before stating and proving the result, we need to review a result. Recall that if X is a noetherian topological space, every closed subset Z of X can be written in a unique way as union of *finitely many* irreducible closed subsets $Z = Z_1 \cup Z_2 \cup \cdots \cup Zn$ with $Z_i \nsubseteq Z_j$ for any $i \neq j$ (see for example [10, Prop I.1.5]).

Lemma 4.13. *Let X be a noetherian integral normal separated scheme. Let $Z \neq X$ be a closed subset of the topological space underlying X and let $U = X - Z$. Call $D_1, D_2, \ldots, D_s$ the irreducible components of Z that are prime divisors (since a noetherian scheme is a noetherian topological space they are in finite number). Then there exists an exact sequence of groups*

$$\bigoplus_{i=1}^{s} \mathbb{Z} D_i \longrightarrow \operatorname{Cl}(X) \longrightarrow \operatorname{Cl}(U) \longrightarrow 0.$$

Proof.
Good reference for this proof are [10, Prop II.6.5] and [4, Thm 4.0.20]. We will describe the general idea without checking all the details.
Let's start proving exactness in $\operatorname{Cl}(U)$. Consider a divisor D in X; it is of the form $D = \sum n_Y Y$ with Y ranging on prime divisors of X. The map that send D to $\sum n_Y (Y \cap U)$ (ignoring the sum whenever $Y \cap U = \emptyset$) is well defined between $\operatorname{Div}(X)$ and $\operatorname{Div}(U)$. In fact if Y is an integral closed subscheme of X with codimension 1, then $Y \cap U$ is the empty set or an integral closed subscheme of U with codimension 1 (notice that it is a closed subset of U with the induced topology, integrality comes from the fact that reduceness is a local property and the set remains irreducible, dimension statements come from the density of U in X). This map is moreover surjective, since for every divisor $\sum n_{Y'} Y'$ in U one can consider $\sum n_{Y'} \overline{Y'}$, where $\overline{Y'}$ denotes the Zariski closure of Y' in X. This is a divisor on X since it can be verified that every $\overline{Y'}$ is an integral closed subscheme of X with codimension 1.
Once we constructed the surjective morphism $\operatorname{Div}(X) \to \operatorname{Div}(U)$, we see that any principal divisor is sent to another principal divisor, since for every element f in the function field of X, $(f) \mapsto (f_{|U})$. So the previous morphism induces a surjective

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group morphisms on the quotients $\mathrm{Cl}(X) \to \mathrm{Cl}(U)$.

What is left is to study the kernel of this morphism. Surely the kernel contains the image of the projection, $\pi(\bigoplus \mathbb{Z}D_i)$. Indeed

$$\sum n_i D_i \mapsto \sum n_i [D_i] \mapsto \sum n_i [D_i \cap U] = 0$$

since by definition $D_i \cap U = \emptyset$ for every $i = 1, 2, \ldots, s$. Conversely, if $\sum n_Y [Y]$ in $\mathrm{Cl}(X)$ is sent to zero, then it means that $\sum n_Y [Y \cap U] = 0$, so $n_Y = 0$ for every prime divisor Y intersecting U, i.e. not contained in $Z = X - U$. It easily implies that the only prime divisors that can appear with a nonzero coefficient are the ones lying inside Z, that are $D_1, D_2, \ldots, D_s$ by definition. So $\sum n_Y [Y]$ must be the projection of some divisor of the form $\sum n_i D_i$. Hence we have proved exactness also in $\mathrm{Cl}(X)$. $\qquad\square$

We need also a little fact about the divisor class group of an affine scheme.

Lemma 4.14. *Let A be a noetherian integral domain and let $X = \mathrm{Spec}(A)$. If A is a unique factorization domain then $\mathrm{Cl}(X) = \{0\}$.*

Proof.
See a stronger form in [10, Prop II.6.2]. $\qquad\square$

From the lemma one obtains the following.

Theorem 4.15. *Let k be an algebraically closed field and let Δ be a fan in $N_\mathbb{R}$ consisting of finitely many cones. Call $X = X(k, \Delta)$ the k-toric scheme associated to the fan Δ, M the dual of the lattice N and v_ρ the minimal generator of the ray ρ. Then there exists an exact sequence:*

$$M \longrightarrow \bigoplus_{\rho \in \Delta(1)} \mathbb{Z}D_\rho \longrightarrow \mathrm{Cl}(X) \longrightarrow 0$$

where the first arrow is give by the map $\alpha : m \mapsto \sum_{\rho \in \Delta(1)} \langle m, v_\rho \rangle D_\rho$ and the second one is the canonical projection π to the divisor class group.

Moreover if the minimal generators of the rays in the fan span $N_\mathbb{R}$, one has an exact sequence

$$0 \longrightarrow M \longrightarrow \bigoplus_{\rho \in \Delta(1)} \mathbb{Z}D_\rho \longrightarrow \mathrm{Cl}(X) \longrightarrow 0.$$

Proof.
First of all, since k is an algebraically closed field and Δ consists in finitely many cones the scheme X is a noetherian integral normal separated scheme. So we can speak without problems about Weil divisors and we can apply the previous lemma. Let's take U to be the torus $U_{\{0\}}$ contained as an open subset of X. Applying Lemma 4.13 and Proposition 4.10 (that can be applied in the case of fields) we get immediately the exact sequence

$$\bigoplus_{\rho \in \Delta(1)} \mathbb{Z}D_\rho \longrightarrow \mathrm{Cl}(X) \longrightarrow \mathrm{Cl}(U) \longrightarrow 0.$$

But now $U = \mathrm{Spec}(k[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}])$. Since k is a field, then $k[x_1, x_2, \ldots, x_n]$ is a UFD and, since localizations of UFD are still UFD, $k[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}] =$

$k[x_1, x_2, \ldots, x_n]_{x_1 x_2 \ldots x_n}$ is a UFD. By Lemma 4.14 one has that $\mathrm{Cl}(U) = \{0\}$. As a consequence, the previous sequence gives:

$$\bigoplus_{\rho \in \Delta(1)} \mathbb{Z} D_\rho \longrightarrow \mathrm{Cl}(X) \longrightarrow 0$$

that proves the exactness of the left part of the sequence in the statement.
Then, the map $\alpha : M \to \bigoplus \mathbb{Z} D_\rho$ can be easily seen to be the map $m \mapsto \mathrm{div}(T^m)$ (it is enough to apply Proposition 4.12). In this way it is obvious that $\pi \circ \alpha = 0$ so $\mathrm{Im}(\alpha) \subseteq \mathrm{Ker}(\pi)$. So it is left to prove that the other inclusion holds. Then, let's consider an element in the kernel of π: it is obviously of the form (f) for some element of the function field; moreover, it has the property that the element f has order zero everywhere outside the torus, from Proposition 4.10. It can be seen [5] that necessarily f is of the form cT^m where $c \in k^*$ and $m \in M$. Since

$$\alpha(m) = \mathrm{div}(T^m) = \mathrm{div}(cT^m) = \mathrm{div}(f),$$

this implies that (f) lies in the image of α. So, also the inclusion $\mathrm{Ker}(\pi) \subseteq \mathrm{Im}(\alpha)$ holds and then the sequence is also exact in $\bigoplus \mathbb{Z} D_\rho$.
At last, it remains to prove what happens if the minimal generators of the rays in the fan span $N_\mathbb{R}$. In this case, suppose that $\alpha(m) = 0$. Then $\langle m, v_\rho \rangle = 0$ for every ρ and so, since they span the vector space, $\langle m, v \rangle = 0$ for every $v \in N_\mathbb{R}$ and in particular for each vector of the canonical basis e_i. Hence each component of m is zero, so $m = 0$. This proves the injectivity of α and so the exactness of the sequence

$$0 \longrightarrow M \longrightarrow \bigoplus_{\rho \in \Delta(1)} \mathbb{Z} D_\rho \longrightarrow \mathrm{Cl}(X) \longrightarrow 0.$$

$\square$

4.5 Computing the divisor class group of certain toric schemes

In this section we will use the exact sequence worked out in 4.4.2 to compute the class group of some toric schemes. We will assume k to be an algebraically closed field. Moreover we will always suppose the lattice M to be $\mathbb{Z}^n$, with action on N given by the dot product.
So, let's consider a fan Δ in $N_\mathbb{R}$ and let X be the toric scheme $X(k, \Delta)$. The general strategy for computing the divisor class group is the following. Consider all the rays of the fan, $\rho_1, \rho_2, \ldots, \rho_l$ and call their minimal generators $\eta_1 = \eta_{\rho_1}, \eta_2 = \eta_{\rho_2}, \ldots, \eta_l =$

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[5]The intuitive idea is more or less this: f is a nonzero function in the function field of X, that is isomorphic to $\mathrm{Frac}(k[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}])$ (think to the torus embedded in the scheme). So $f = g/h$ where g and h are two coprime nonzero Laurent polynomials with coefficients in k (coprimality is meaningful since $k[M]$ is isomorphic to the localization of a UFD). It must be true that g/h has no zero nor poles on the torus $(k^*)^n$ and this forces the zeros and poles of g and h to be at 0 (otherwise f would have a pole or a zero outside 0 from coprimality). Since k is algebraically closed, the polynomial g has a zero or pole only in 0 if and only if $g = c_1 T^{m_1}$. The polynomial is nonzero, so $c_1 \neq 0$. The same must be true for h and so $f = g/h$ will have the form cT^m.

η_{ρ_l}; for each $i = 1, 2, \ldots, l$ let moreover $D_{\rho_i} = D_i$ be the toric divisor associated to the ray ρ_i. We have a short exact sequence:

$$\mathbb{Z}^n \longrightarrow \mathbb{Z}D_1 \oplus \mathbb{Z}D_2 \oplus \cdots \oplus \mathbb{Z}D_l \longrightarrow \mathrm{Cl}(X) \longrightarrow 0.$$

Here the first map is $\alpha : \mathbb{Z}^n \to \bigoplus \mathbb{Z}D_i$, sending m to $\sum \langle m, \eta_i \rangle D_i$, while the second map is the canonical projection π. The fact that the projection π is surjective means that $\mathrm{Cl}(X)$ is generated (as an additive group) by the classes $[D_1], [D_1], \ldots, [D_l]$ (every element in fact must come from a linear integer combination of D_i, so it is a linear combination, with the same coefficients of the $[D_i]$'s). Moreover the fact that the sequence is exact requires that the image of α is equal to the kernel of π. Now:

$$\begin{aligned}
\mathrm{Im}(\alpha) &= \{\alpha(m_1, \ldots, m_n) : \underline{m} \in \mathbb{Z}^n\} = \left\{\sum \langle \underline{m}, \eta_i \rangle D_i : \underline{m} \in \mathbb{Z}^n\right\} \\
&= \left\{\sum_{i,j} m_j \eta_{i,j} D_i : \underline{m} \in \mathbb{Z}^n\right\} = \left\{\sum_j m_j \sum_i \eta_{i,j} D_i : m_j \in \mathbb{Z}\right\} \\
&= \mathbb{Z}\Big(\sum_i \eta_{i,1} D_i\Big) + \mathbb{Z}\Big(\sum_i \eta_{i,2} D_i\Big) + \cdots + \mathbb{Z}\Big(\sum_i \eta_{i,n} D_i\Big).
\end{aligned}$$

So this must be also the kernel of π. This means that an element of the sum $\bigoplus \mathbb{Z}D_i$ is sent to zero in the divisor class group if and only if it lies in the previous subgroup. But the subgroup is generated by $\sum_i \eta_{i,2} D_i$, so it is necessary and sufficient to ask that this generators are sent to zero. In other words, one asks that $\sum_i \eta_{i,j}[D_i] = 0$ for every $j = 1, 2, \ldots, n$.

We then obtain a *presentation* of the group $\mathrm{Cl}(X)$ in the sense of the following definition.

Definition 4.16. A group G is said to have the **presentation** $\langle S | R \rangle$ if it is isomorphic to the quotient of a free group with set of generators S by the normal subgroup generated by the relations R.

Every group admits a presentation. The case we study naturally gives rise to such a presentation. In fact, proceeding as above one easily concludes that:

$$\mathrm{Cl}(X) \text{ has presentation } \Big\langle [D_i] \Big| \sum_i \eta_{i,j}[D_i] = 0 \Big\rangle. \tag{4.1}$$

We will now study some examples, in order to deduce the divisor class group of particular toric schemes.

4.5.1 The affine space

Recall that the fan from which one obtains the affine plane is the one in Figure 4.2, in which for every $i = 1, 2, \ldots, n$ we call ρ_i the ray generated by the canonical vector of the basis e_i.

Hence the group $\mathrm{Cl}(\mathbb{A}_k^n)$ is generated by $\{[D_i]\}_i$, while the relations are given by the fact that

$$\alpha(m_1, m_2, \ldots, m_n) = \sum \langle \underline{m}, e_i \rangle D_i = \sum m_i D_i$$
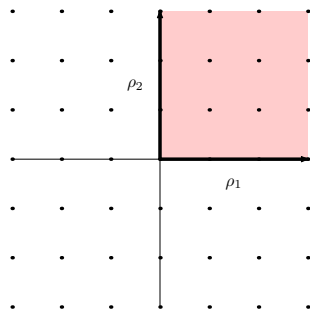
Figure 4.2: The fan in $N_{\mathbb{R}}$ for the affine space.

so the kernel of π is the group generated by all the D_i. In this way it is obvious that since the presentation is

$$\langle [D_1], [D_2], \ldots, [D_n] \,|\, [D_1], [D_2], \ldots, [D_n] \rangle$$

then the group is trivial, so $\mathrm{Cl}(\mathbb{A}_k^n) = \{0\}$. This agrees with Lemma 4.14, since $\mathbb{A}_k^n = \mathrm{Spec}(k[x_1, x_2, \ldots, x_n])$, that is the spectrum of a unique factorization domain.

4.5.2 The projective space

Recall that the fan for the projective plane is the one in Figure 4.3, in which the rays are ρ_i generated by the canonical vector e_i for every $i = 1, 2, \ldots, n$ and ρ_0 generated by $-e_1 - e_2 - \cdots - e_n$.
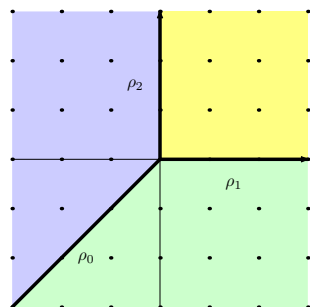

Figure 4.3: The fan in $N_{\mathbb{R}}$ for the projective space.

Using the general proceeding as above, from Equation 4.1 we obtain the presentation:

$$\langle [D_0], [D_1], \ldots, [D_n] \,|\, [D_i] - [D_0] \text{ for all } i = 1, 2, \ldots, n \rangle$$

so the group generators must all coincide, hence $\mathrm{Cl}(\mathbb{P}_k^n) \simeq \mathbb{Z}$.

4.5.3 $\mathbb{P}^1 \times \mathbb{P}^1$

Recall that the fan for the scheme $\mathbb{P}_k^1 \times \mathbb{P}_k^1$ is the one in Figure 4.4, in which the rays are $\rho_1, \rho_2, \rho_3, \rho_4$ generated by the canonical vectors $e_1, e_2, -e_1, -e_2$ respectively. From Equation 4.1 one has the presentation:

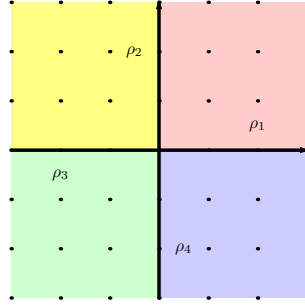$$\langle [D_1], [D_2], [D_3], [D_4] \,|\, [D_1] - [D_3], [D_2] - [D_4] \rangle$$

Figure 4.4: The fan in $\mathbb{R}^2$ for $\mathbb{P}^1 \times \mathbb{P}^1$.

so the group generators are pairwise the same, from which $\mathrm{Cl}(\mathbb{P}^1_k \times \mathbb{P}^1_k) \simeq \mathbb{Z}^2$.

4.5.4 Hirzebruch surface

Recall that the Hirzebruch surface is defined as the toric scheme constructed from the fan in Figure 4.5.
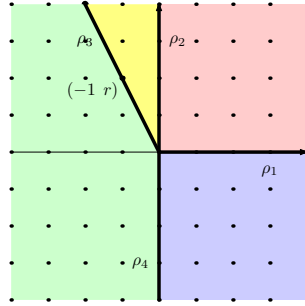

Figure 4.5: The fan in $\mathbb{R}^2$ for the Hirzebruch surface.

This fan has four rays: $\rho_1, \rho_2, \rho_3, \rho_4$ generated by the vectors $(1,0), (0,1), (-1,r), (0,-1)$ respectively.

The map α acts sending $\underline{m} = (a,b)$ to:

$$\alpha(a,b) = (a,b) \cdot (1,0)D_1 + (a,b) \cdot (0,1)D_2 + (a,b) \cdot (-1,r)D_3 + (a,b) \cdot (0,-1)D_4$$
$$= a(D_1 - D_3) + b(D_2 + rD_3 - D_4).$$

Hence Equation 4.1 gives us the presentation:

$$\langle [D_1], [D_2], [D_3], [D_4] \big| [D_1] - [D_3], [D_2] + r[D_3] - [D_4] \rangle$$

from which $[D_1] = [D_3]$ and so the group is presented as:

$$\langle x, y, z \big| x + ry - z \rangle.$$

This is the divisor class group of the Hirzebruch surface, expressed via presentation theory.

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4.5.5   A affine example with torsion

As last, illustrative example, we consider the cone σ in Figure 4.6, generated by the two vectors $v_1 = (d, -1)$ and $v_2 = (0, 1)$ for a certain positive integer $d \in \mathbb{N}$. The two rays are ρ_1 generated by v_1 and ρ_2 generated by v_2. The corresponding toric scheme is the affine toric scheme U_σ.
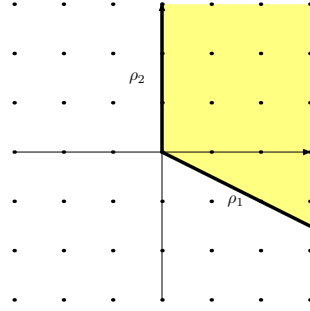

Figure 4.6: A cone in $\mathbb{R}^2$.

The map α sends $\underline{m} = (a, b)$ to:

$$\alpha(a, b) = (a, b) \cdot (d, -1) D_1 + (a, b) \cdot (0, 1) D_2 = (da - b) D_1 + b D_2 = adD_1 + b(D_2 - D_1).$$

Hence Equation 4.1 gives us the presentation:

$$\langle [D_1], [D_2] \big| d[D_1], [D_2] - [D_1] \rangle.$$

This means that in particular $[D_1] = [D_2]$ and so we have only one generator. Moreover $d[D_1] = 0$ so the (only) generator $[D_1]$ has order d. Hence:

$$\mathrm{Cl}(U_\sigma) \simeq \mathbb{Z}/d.\mathbb{Z}$$

so we have an affine toric scheme with a non-free divisor class group.

Chapter 5

The Cox ring for a toric scheme associated to a fan

We want to reinterpret one of the definition and results of [2] in the language of schemes. As usual we will deal with a lattice N and its dual M, hence with the two dual $\mathbb{R}$-vector spaces $N_\mathbb{R} := N \otimes_\mathbb{Z} \mathbb{R}$ and $M_\mathbb{R} := M \otimes_\mathbb{Z} \mathbb{R}$. Since we are going to use the theory of Weil divisors, we will restrict to the case in which R is a noetherian integrally closed domain, and Δ is a finite fan in the vector space $N_\mathbb{R}$.

5.1 Motivation

We saw in Chapter 2 that a projective space is a toric variety. We also know that the projective space can be obtained as a quotient of some affine space minus a closed subset by the action of a group. This is indeed one of the traditional construction of $\mathbb{P}^n_k$ for an algebraically closed field k:

$$\mathbb{P}^n_k = \frac{k^{n+1} - \{0\}}{\sim}$$

where $\sim$ is the equivalence relation given by $x \sim y$ if and only if $x = \lambda y$ for some $\lambda \neq 0$ in k. In other words, if the action of k^* on k^{n+1} is defined by $\lambda.x = \lambda x$, then:

$$\mathbb{P}^n_k = \frac{\mathbb{A}^{n+1}_k - \{0\}}{k^*}.$$

The idea is that this construction is possible for any toric variety. In order to do this it will be important to understand what is a quotient in the category of schemes. For this purpose we study categorical quotients in section 5.2.

5.2 Categorical quotients

There are in fact many different possible definition for the quotient of a scheme by a group. As far as we will be interested in the subject, a quotient of an object by the action of a group on it can be defined in any category. In particular, we need the concept of quotient in the category of schemes. First of all we should introduce what is a G-invariant morphism.

Definition 5.1. Let X and Y be two S-schemes and let G be a group scheme over S with an action a of G on X. Calling pr_2 the canonical map $G \times_S X \to X$ coming from the definition of fiber product, then a morphism of schemes $f : X \to Y$ is called a G-**invariant morphism** if

$$f \circ a = f \circ pr_2.$$

In terms of diagrams, it is required that the following one commutes:

$$
\begin{array}{ccc}
G \times_S X & \xrightarrow{\ a\ } & X \\
\downarrow{pr_2} & & \downarrow{f} \\
X & \xrightarrow{\ f\ } & Y
\end{array}
$$

We are now ready to define categorical quotients, through a universal property.

Definition 5.2. Let X be an S-schemes and let G be a group scheme over S, with an action a of G on X. A **categorical quotient** of X by the action of G is a G-invariant morphism $\pi : X \to Y$ such that for any S-scheme Z with a G-invariant morphism $f : X \to Z$ there exists a unique morphism $g : Y \to Z$ such that $g \circ \pi = f$.

$$
\begin{array}{ccc}
X & \xrightarrow{\ \pi\ } & Y \\
 & {}_{f}\searrow & \downarrow{g} \\
 & & Z
\end{array}
$$

If a categorical quotient exists, it is unique up to a unique isomorphism.
A basic and useful result in Geometric Invariant Theory is the following.

Theorem 5.3. *Let k be a field and let $X = \operatorname{Spec} A$ be an affine scheme over k, and let $G = \operatorname{Spec} B$ be a reductive group scheme over k acting on X by the action a (with corresponding map $a^\sharp : A \to B \otimes A$). Then the categorical quotient of X by the action of G exists and it is (unique up to a unique homomorphism) the scheme*

$$\operatorname{Spec} A \,/\, G = \operatorname{Spec} A^B = \operatorname{Spec}(\{x \in A : a^\sharp(x) = x \otimes 1\}).$$

Proof.
See [18, Thm 1.1]. $\qquad\square$

5.3   Definition of the Cox ring

The aim of this paragraph is to present the construction of the homogeneous coordinate ring (or Cox ring) of a toric scheme defined on a base ring.
Let R be a noetherian integrally closed domain, and Δ a finite fan in the vector space $N_{\mathbb{R}}$. Let $X = X(R, \Delta)$. Now, consider the set of all edges (i.e. one dimensional cones) of Δ, call this collection $\Delta(1)$. For every $\rho \in \Delta(1)$ one can introduce a formal symbol x_ρ. The graded ring we are looking for will be the polynomial ring over these variables.

Definition 5.4. Let R be a noetherian integrally closed domain, Δ a finite fan in the vector space $N_{\mathbb{R}}$ and let X be the R-toric scheme constructed from the fan Δ.

The **homogeneous coordinate ring** (or **Cox ring**) **of the toric variety** X is defined as

$$\text{Cox}(X) = R[x_\rho : \rho \in \Delta(1)]$$

together with the degree map on monomials

$$\deg \prod_{\rho \in \Delta(1)} x_\rho^{a_\rho} = \left[\sum_{\rho \in \Delta(1)} a_\rho D_\rho \right] \in \text{Cl}(X)$$

where D_ρ is the toric divisor associated to the ray ρ, constructed as in Section 4.2.

Remark 5.1. Note that $\sum_{\rho \in \Delta(1)} a_\rho D_\rho$ is a formal finite sum of prime divisor with integer coefficient. It is hence a divisor, so it is meaningful to take its projection into the divisor class group of X.

Remark 5.2. Recall that a graded ring A is a ring that can be written as direct sum of abelian groups A_i with product satisfying $A_i \cdot A_j \subseteq A_{i+j}$ where the indices are taken in an abelian group I. In our case, we choose as set of indices the abelian group $\text{Cl}(X)$. Moreover, we already know that a polynomial ring is direct sum of (infinitely many) abelian groups isomorphic to $(R, +)$ as in the following decomposition:

$$R[x_\rho : \rho \in \Delta(1)] = R \oplus Rx_{\rho_1} \oplus \cdots \oplus Rx_{\rho_l} \oplus Rx_{\rho_1}x_{\rho_2} \oplus \dots.$$

It is possible that some monomials have the same degree (see Remark 5.3 for a necessary and sufficient condition). Putting them all together, one gets a decomposition

$$\text{Cox}(X) = \bigoplus_{[D] \in \text{Cl}(X)} \left(\bigoplus_{\deg(x^m) = [D]} Rx^m \right).$$

Calling

$$\text{Cox}(X)_{[D]} = \bigoplus_{\deg(x^m) = [D]} Rx^m$$

one effectively obtains a decomposition

$$\text{Cox}(X) = \bigoplus_{[D] \in \text{Cl}(X)} \text{Cox}(X)_{[D]}.$$

These are abelian groups, moreover if we consider two monomials $x^{\underline{a}} = \prod_{\rho \in \Delta(1)} x_\rho^{a_\rho}$ and $x^{\underline{b}} = \prod_{\rho \in \Delta(1)} x_\rho^{b_\rho}$, then:

$$\deg\left(x^{\underline{a}} \cdot x^{\underline{b}}\right) = \deg\left(x^{\underline{a}+\underline{b}}\right) = \left[\sum_{\rho \in \Delta(1)} (a+b)_\rho D_\rho \right] = \left[\sum_{\rho \in \Delta(1)} (a_\rho + b_\rho) D_\rho \right]$$

$$= \left[\sum_{\rho \in \Delta(1)} a_\rho D_\rho \right] + \left[\sum_{\rho \in \Delta(1)} b_\rho D_\rho \right] = \deg\left(x^{\underline{a}}\right) + \deg\left(x^{\underline{b}}\right).$$

Hence $\text{Cox}(X)_{[D_1]} \cdot \text{Cox}(X)_{[D_2]} \subseteq \text{Cox}(X)_{[D_1]+[D_2]}$; we have really a graduation on the ring $\text{Cox}(X)$.

Remark 5.3. Two monomials can have the same degree. It is in fact possible that

$$\left[\sum_{\rho\in\Delta(1)} a_\rho D_\rho\right] = \left[\sum_{\rho\in\Delta(1)} b_\rho D_\rho\right].$$

This happens if and only if

$$\sum_{\rho\in\Delta(1)} (a_\rho - b_\rho)D_\rho \in \mathrm{Div}_0(X).$$

Recalling the exact sequence in Theorem 4.15 (at least for algebraically closed fields) one has that the previous divisor is principal if and only if it is in the image of the morphism α, i.e. if and only if there exists $m \in M$ such that $\langle m, v_\rho \rangle = a_\rho - b_\rho$ for every $\rho \in \Delta(1)$ (here v_ρ is the minimal generator of the ray ρ).

In the following, to simplify the notation, we will avoid to put the symbol of the equivalence classes for divisors.
We will now define an interesting ideal of $\mathrm{Cox}(X)$, that will play a crucial role in the theory. Let's consider a cone σ (of any dimension) in the fan Δ. To this cone we can associate an element of $\mathrm{Cox}(X)$, that is

$$x^{\hat{\sigma}} = \prod_{\rho\in\Delta(1)-\sigma(1)} x_\rho$$

So, for any $\sigma \in \Delta$ we define a monomial in the ring $\mathrm{Cox}(X)$, taking the product of all the rays of the fan not being faces of σ. It has degree

$$\hat{\sigma} := \deg x^{\hat{\sigma}} = \deg \prod_{\rho\in\Delta(1)-\sigma(1)} x_\rho = \sum_{\rho\in\Delta(1)-\sigma(1)} D_\rho$$

These elements generate an ideal in $\mathrm{Cox}(X)$, that will turn out to be very important later.

Definition 5.5. Let X be a toric variety over R coming fro the fan Δ. The **irrelevant ideal** of the Cox ring $\mathrm{Cox}(X)$ is the ideal generated by the previous monomial, i.e.

$$B := \langle x^{\hat{\sigma}} : \sigma \in \Delta \rangle$$

Remark 5.4. We can see immediately that it is sufficient to consider the maximal cones in the fan to get the irrelevant ideal. In fact if τ is not a maximal cone of Δ, it is contained in some other cone σ. Now, by definition of fan, $\tau = \tau \cap \sigma$ is a face of σ. Then the generators of τ are a subset of the generators of σ and all the rays in τ are rays in σ. By definition of the monomial associated to a cone, clearly $x^{\hat{\sigma}}$ divides $x^{\hat{\tau}}$, hence $x^{\hat{\tau}}$ can be forgotten among the generators of the ideal B.

Example 5.6. As seen in section 2.6.3, we can see the projective space $\mathbb{P}^n_R$ as $X(R, \Delta)$, where Δ is the fan whose cones are generated by the possible subsets of $\{e_0, e_1, \ldots, e_n\}$, with the notation $e_0 = -e_1 - e_2 - \cdots - e_n$. This means that $\Delta(1)$ consists of all the ray generated by one of the vectors $e_0, e_1, \ldots, e_n$ (see Figure 5.1). By definition $\Delta(1)$ has cardinality $n + 1$, so $\mathrm{Cox}(\mathbb{P}^n_R) = R[x_0, x_1, \ldots, x_n]$. When the
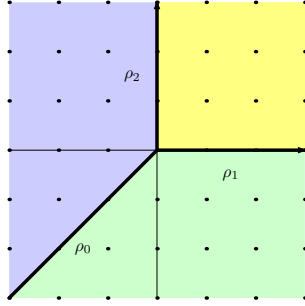
Figure 5.1: The fan in $N_{\mathbb{R}}$ for the projective space.

base ring is an algebraically closed field, by the study made in Subsection 4.5.2, one knows that $\mathrm{Cl}(\mathbb{P}_k^n) \simeq \mathbb{Z}$ and that $[D_{\rho_0}] = [D_{\rho_1}] = \cdots = [D_{\rho_n}]$. So the grading is easily given by

$$\deg\left(x_0^{a_0} x_1^{a_1} \ldots x_n^{a_n}\right) = \sum_{i=0}^{n} a_i [D_{\rho_i}] = \left(\sum_{i=0}^{n} a_i\right)[D_{\rho_0}]$$

so it is the natural grading by the isomorphism $\mathrm{Cl}(X) \to \mathbb{Z}$, $m[D_0] \mapsto m$.

To work out the irrelevant ideal, we can consider for any $i = 1, 2, \ldots, n+1$ the cone in Δ made as follows:

$$\sigma_i = \mathrm{cone}(e_0, \ldots, \hat{e}_i, \ldots, e_n)$$

where $\hat{y}$ means that we forget the generator y. Each of these cones is by definition in the fan and has corresponding monomial

$$x^{\hat{\sigma_i}} = \prod_{\rho \in \Delta(1) - \sigma(1)} x_\rho = x_{\rho_i}.$$

Since they are the monomials coming from the maximal cones in the fan, they generate the ideal

$$B = \langle x_0, x_1, \ldots, x_n \rangle$$

that is the usual irrelevant ideal for the projective space.

To resume, in the case of the projective spaces over an algebraically closed field, the cox ring is the usual homogeneous coordinate ring (with the usual graduation) and the irrelevant ideal is the usual irrelevant ideal.

5.4   Toric schemes as categorical quotients

The nice part about the Cox ring is that it comes into play when we want to express a toric scheme as a quotient. For example, we know that the projective space can be written as the quotient of an open subset of the affine space over a group. More precisely:

$$\mathbb{P}_k^n = \mathrm{Spec}(k[x_0, x_1, \ldots, x_n]) - V(\langle x_0, x_1, \ldots, x_n \rangle) \,/\, k^* = \mathrm{Spec}(\mathrm{Cox}(\mathbb{P}_k^n)) - V(B) \,/\, k^*.$$

This is in fact a property that is shared by all toric schemes coming from fans. This is one of the most interesting result of [2]; we restate it using the language of schemes.

Theorem 5.7. *Let k be an algebraically closed field. The toric scheme $X = X(k, \Delta)$ is the categorical quotient of the open set $\mathrm{Spec}(\mathrm{Cox}(X)) - V(B)$ by the action of the group $G = Hom(\mathrm{Cl}(X), k^*)$.*

Proof.
The original proof deals with the field $\mathbb{C}$ and the language of varieties. It can be found in [2, §2]. The "general" version of the statement and of the proof is readable in [14, Thm 1.2]. $\qquad\square$

Conclusions

The original aim of this thesis, as already said, was to read and understand the paper [2] by Cox. The need of understanding better and in a deeper way the theory of toric varieties required a lot of time and guided the author to a wider theory, whose first appearances are, as far as he knows, the Phd thesis of F. Rohrer (2010) and some successive papers of his (such as [21])(2013). This more general point of view has been also adopted by Q. Liu in [16].

Much has been done in this direction, but it seems that the extension of the theory of Cox rings to general base rings (and fan with not necessarily finitely many cones) requires heavier techniques, such as Geometric Invariant Theory (see the classical book by Mumford, [18]) and Sheaf Theory. In front of this lack of tools and time, the author had to surrender. In this sense he can say to have failed its original aim. But it was that same aim, that standing out far far away, brighting as a lighthouse, led him through the understanding of the subject; as somebody says, "sometimes the path is more important than the destination".

But mainly, this study concludes with some open questions for the author. The first, obvious one is: at the end, is it possible to generalize Cox's construction to any base ring? In chapter 5 we introduced the Cox ring on any base ring. The question is to understand if it is possible to work out the same quotient construction. It seems that following the proof of Cox ([2, §2] the strategy can be mimed for any algebraically closed field (with any characteristic); for general base rings the idea requires stronger efforts.

Another interesting question is the following. In Example 1.24 we saw an example of a toric scheme that cannot be realized from a fan. The reason for this lies in the fact that the scheme is not normal. So the problem is: is it true that any normal toric scheme over R is constructed from a fan? In the case we take $\mathbb{C}$ as base ring the answer is yes, as shown by Oda in [19, Thm 1.5] using Sumihiro's theorem on action of connected linear algebraic groups.

The subject requires a lot of time to be studied, and probably during its deeper studying new questions would arise. There is in principle no amount of time for which this thesis could be claimed to be finished; but, as everybody knows:

The worst thing you can do is to completely solve a problem.
Dan Kleitman

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