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Equivariant Gröbner Bases

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Table of Symbols

$\mathbb{N}$	natural numbers $\{0, 1, 2, \ldots\}$
$\mathbb{Z}$	integer numbers $\{\cdots, -2, -1, 0, 1, 2, \cdots\}$
$\sum^*$	free monoid over $\sum$
$[n]$	set of first n positive integers $\{1, 2, \cdots, n\}$
$A[x_1, x_2, \cdots]$	polynomial ring in infinitely many variables with coefficients in A
$\Pi = \mathrm{Inc}(\mathbb{N})$	monoid of strictly increasing functions on $\mathbb{N}$
$\mathrm{Sym}(\mathbb{N})$	symmetric group on $\mathbb{N}$
$\mathrm{FSym}(\mathbb{N})$	finitary subgroup of $\mathrm{Sym}(\mathbb{N})$
$\mathrm{Subs}(\mathbb{N})$	substitution monoid on $\mathbb{N}$
$\mathrm{lm}(f)$	leading monomial of polynomial f
$\mathrm{lc}(f)$	leading coefficient of polynomial f
$\mathrm{lt}(f)$	leading term of polynomial f
$S(f, g)$	$S-$polynomial of polynomials f and g

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Chapter 1

Introduction

It is well-known by Hilbert's Basis Theorem that if A is a Noetherian ring, then the ring $A[x]$ of polynomials in one variable x and coefficients from A is also Noetherian. We find by induction that the polynomial ring $R = A[x_1, x_2, \cdots, x_n]$ in finitely many variables is Noetherian. Moreover the notion of Gröbner Basis allows us to do effective computations in R/I, where I is an ideal in R, with some assumption on A.

The situation changes dramatically when one considers polynomial rings in infinitely variables. For instance, the ring $A[x_1, x_2, \cdots]$ is not Noetherian, since the ideal $(x_1, x_2, \cdots)$ does not have a finite set of generators.

However, if we have some special actions of some special monoids on the ring R, we may have finiteness. Indeed, let $X = \{x_1, x_2, \cdots\}$, and let a monoid P act on R by mean of ring homomorphisms : if $p \in P$ and $f \in R = A[x_1, x_2, \cdots, x_n]$, where $x_i \in X$, then

$$pf(x_1, x_2, \cdots, x_n) = f(px_1, \cdots, px_n)$$

This in turn gives R structure of a left module over the left *skew-monoid ring* $R * P = \{\sum_{i=1}^{m} r_i p_i : r_i \in R, p_i \in P\}$ with the multiplication given by

$$r_1p_1.r_2p_2 = r_1(p_1r_2)(p_1p_2)$$

and extended by distributivity and $A-$linearity to the whole ring. An ideal $I \subseteq R$ is called *invariant* under P (or $P-$stable) if

$$PI := \{pf : p \in P, f \in I\} \subseteq I$$

And note that invariant ideals are simply the $R * P-$submodules of R.

We study the question *whether the ring* $R = A[x_1, x_2, \cdots]$ *is* $P-$*Noetherian, which means that it has an action of* P *by ring homomorphisms and that all ascending chains of* $P-$*stable ideals stabilise after finitely many steps.*

It is shown that when $P = \mathrm{Sym}(\mathbb{N})$ is the symmetric group ([AH07]) or $P = \mathrm{Inc}(\mathbb{N})$ is the monoid of strictly increasing functions on $\mathbb{N}$ ([HS09], [D09]), the ring $R = A[x_1, x_2, \cdots]$ is $P-$Noetherian. For instance, the ideal $(x_1, x_2, \cdots)$ is $P-$stable and as $R * P-$module generated by the single polynomial x_1.

Notice that in those situations above, the monoid P acts trivially on the coefficient ring A. Hence a natural question is that *when we have a nontrivial action of a monoid* P *on the coefficient ring* A*, and when* A *is* $P-$*Noetherian, is the polynomial ring* $R = A[x_1, x_2, \cdots]$ *still* $P-$*Noetherian?* This is one of main problems that I am going to investigate in this thesis (chapter 3).

Since polynomial rings in infinitely many variables occur naturally in applications such as chemistry ([AH07]) and algebraic statistics ([HS09], [BD10]), we would like to do computations with their ideals. In case $P = \mathrm{Sym}(\mathbb{N})$, $P-$stable ideals are finitely generated as a $R*P-$submodule, and the proof of this fact can be turned into a Buchberger-type algorithm for computing with such ideals ([AH09]).

More generally, the notion of *equivariant Gröbner basis* (in [BD10]) or $P-$*Gröbner basis* (or *monoidal Gröbner basis* in [HS09]) is defined and used, where the coefficient ring $A = k$ is restricted to be a field k. Under some conditions, there exists a Buchberger-type algorithm for computing equivariant Gröbner bases of $P-$stable ideals in $k[x_1, x_2, \cdots]$ (see [BD10]).

So now, connecting with *equivariant Gröbner bases method* above ([BD10]), another question of the thesis is described as follows (chapter 4):

Let $Subs(\mathbb{N})$ be the *substitution monoid*, whose elements are infinite sequences $(\sigma_1, \sigma_2, \cdots)$ of pairwise disjoint non-empty finite subsets of $\mathbb{N}$, with multiplication defined by

$$(\sigma \circ \tau)_i = \bigcup_{j \in \tau_i} \sigma_j$$

Let $Subs_<(\mathbb{N})$ be the submonoid of all such sequences $(\sigma_1, \sigma_2, \cdots)$ satisfying

$$\max(\sigma_1) < \max(\sigma_2) < \cdots$$

Note that the full symmetric group of $\mathbb{N}$ is naturally contained in $Subs(\mathbb{N})$ and that $Inc(\mathbb{N})$ is contained in $Subs_<(\mathbb{N})$ (by taking singetons).

Now consider the polynomial ring $S = K[t; x_1, x_2, \cdots; (z_I)_{I \subseteq \mathbb{N}}]$, where I runs over all finite subsets of the natural numbers. In this ring consider the ideal $I(Y)$ generated by all elements of the form

$$z_I - t \prod_{i \in I} x_i$$

The substitution monoid acts on (monomials in) S by $\sigma t = t, \sigma x_i = \prod_{j \in \sigma_i} x_j$, and $\sigma z_I = z_{\cup_{i \in I} \sigma_i}$, and this action stabilises the ideal $I(Y)$. We will compute a $Subs_<(\mathbb{N})-$Grobner basis of $I(Y)$ with respect to the lexicographic order satisfying $t > x_i > z_I$ for all i and I and $x_{i+1} > x_i$ and $z_J > z_{J'}$ if J is lexicographically larger than J' (e.g. $\{4\} > \{2, 3\} > \{2\}$). Use this Grobner basis to compute the intersection of $I(Y)$ with $K[(z_I)_I]$.

The background of this problem is the following: the intersection of $I(Y)$ with this ring in the $z-$variables is the ideal of all polynomials vanishing on all *infinite rank-1 tensors*. This ideal is in fact known to be generated by certain $2 \times 2-$minors, and the (feasible) computation above gives a new proof of this fact. A more ambitious goal would be to do such a computation of infinite rank-2 tensors, but there the computation is probably not yet feasible (chapter 5).

My thesis is organized as the following

- Chapter 2 is devoted to introducing some background knowledge that we need for latter chapters. In this chapter, we first introduce some basic algebraic notions such as : monoids, action of a monoid, commutative Noetherian rings with some examples.

Next, we introduce the theory of $P-$ordering ([HS09], [BD10]) where P is a monoid that acts on the ring $R = A[x_1, x_2, \cdots]$ by mean of homomorphisms. That ordering is *good* in the sense that it is compatible with the monomial order in R. The notion of Gröbner basis over a general ring is then introduced in the last part of this chapter. In particular, the definition of an equivariant Gröbner basis along with the sufficient conditions for computations ([BD10]) are given.

- In chapter 3, we are going to investigate the Noetherianity of the polynomial ring $R = A[x_1, x_2, \cdots]$ under the $\mathrm{Sym}(\mathbb{N})-$actions and $\mathrm{Inc}(\mathbb{N})-$actions. In particular, we give a number of examples in which R is sometimes $\mathrm{Inc}(\mathbb{N})-$Noetherian and sometimes not $\mathrm{Inc}(\mathbb{N}-)$Noetherian.

- In chapter 4, we introduce the *infinite rank-1 tensors* problems and we give another proof with the *substitution* approach.

- In chapter 5, we introduce the *infinite rank-2 tensors* problems and two *potential* approaches that may give us a solution.

- We give a short summary in chapter 6 of this thesis. In addition, we give two open problems that we have not solved in this time.

Chapter 2

Preliminaries

2.1. Some algebraic notions

2.1.1. Action of monoids

Definition 2.1 A *monoid* is a set M together with a binary operation $\times$, that satisfies the following conditions :

- (Associativity) $a \times (b \times c) = (a \times b) \times c$ for all $a, b, c \in M$.
- (Identity element) There is an $e \in M$ such that $e \times a = a \times e = a$ for all $a \in M$.

More compactly, a monoid is a *semigroup* with an identity element. A monoid with *invertibility* (i.e. for every element $a \in M$ there is $a^{-1} \in M$ such that $a \times a^{-1} = a^{-1} \times a = e$) is a *group*.

A *submonoid* is a subset $N \subseteq M$ containing the identity element, and such that if $a, b \in N$ then $a \times b \in N$. A subset N is said to *generate* M if the set generated by N, denoted by $\langle N \rangle$, which is the intersection over all submonoids containing the elements of N, is M. Equivalently, $M = \langle N \rangle$ if and only if every element of M can be written as a finite product of elements in N. If there is a finite generating set of M, then M is said to be *finitely generated*. A monoid whose operation is commutative is called a *commutative monoid* (or, less commonly, an *abelian monoid*).

Example 2.2

- The natural numbers form a commutative monoid under addition $(\mathbb{N}, +)$ (with identity element 0), or multiplication $(\mathbb{N}, .)$ (with identity element 1).

- Given two sets M and N endowed with monoid structure, their cartesian product $M \times N$ is also a monoid. The associative operation and the identity element are defined pairwise.

- Fix a monoid M. The set of all functions from a given set to M is also a monoid. The identity element is the constant function mapping any element to the identity of M; the associative operation is defined pointwise.

- Let S be a set. The set of all functions $S \to S$ forms a monoid under function composition. The identity is just the identity function. If S is finite with n elements, the monoid of functions on S is finite with n^n elements.

- The set $\Pi = Inc(\mathbb{N})$ of strictly increasing functions on $\mathbb{N}$ is a monoid with the composition operation. The identity is just the identity map, which is also an increasing function.

- The set of all finite strings (words) over some fixed alphabet $\sum$ is a monoid with string concatenation as the operation. The empty string is the identity element. The monoid is denoted by $\sum^*$ and is called *free monoid* over $\sum$.

Definition 2.3 Let M be a monoid and a set S. A (left) action of M on S is the operation $* : M \times S \to S$ satisfying the following conditions

- $e * s = s$, for all $s \in S$.

- $a * (b * s) = (ab) * s$, for all $a, b \in M, s \in S$.

A *homomorphism* between two monoids $(M_1, *)$ and $(M_2, \centerdot)$ is a function $f : M_1 \to M_2$ such that

- $f(x * y) = f(x) \centerdot f(y)$ for all $x, y \in M$.

- $f(e_1) = e_2$

where e_1 and e_2 are the identity elements of M_1 and M_2 respectively. Monoid homomorphisms are sometimes simply called *monoid morphisms.*

Given an action of a monoid M on the set S, the *orbit* of an element $s \in S$ is the subset $Ms = \mathcal{O}_s = \{a.s | a \in M\} \subseteq S$, and the submonoid $Stab(s) = \{a \in M : a.s = a\} \subseteq M$ is defined to be the *stabilizer* of the point $s \in S$.

Example 2.4 Let $X = \{x_1, x_2, \cdots\}$ be a set of infinitely many variables and $k[X]$ be the ring of polynomials in infinitely many variables with coefficients in some field k. For every $\pi \in \Pi$, and for every $x_i \in X$ let :

$$\pi.x_i = x_{\pi(i)}$$

It is in fact an action of Π on $k[X]$ since :

- $id.x_i = x_{id(i)} = x_i$.
- For $\pi, \sigma \in \Pi$, we have :

$$\pi(\sigma.x_i) = \pi.x_{\sigma(i)} = x_{\pi\sigma(i)} = (\pi\sigma)x_i.$$

Notice that in this example, Π acts trivially on k, i.e., $\pi.a = a$ for all $\pi \in \Pi, a \in k$. This is a very important example which we will study in later sections.

2.1.2. Commutative Noetherian ring

Definition 2.5 A *commutative ring* ([AM69]) is a set with binary operations (addition and multiplication) satisfying the following conditions :

(i) A is an abelian group with respect to addition (so that A has a zero element, denoted by 0, and every $x \in A$ has an (additive) inverse, $-x$).

(ii) Multiplication is associative ($(xy)z = x(yz)$) and distributive over addition ($x(y+z) = xy + xz, (y + z)x = yx + zx$).

(iii) $xy = yx$ for all $x, y \in A$.

(iv) $\exists 1 \in A$ such that $x1 = 1x = x$ for all $x \in A$.

An (left) *ideal* of A is a subgroup I of $(A, +)$ such that $ax \in I$ for every $a \in A, x \in I$.

Definition 2.6 A *commutative Noetherian* ring A is a commutative ring satisfying one of the following equivalent conditions

(a) Every non-empty set of ideals in A has a maximal element, with respect to the inclusion ordering.

(b) Every ascending chain of ideals $I_1 \subseteq I_2 \subseteq \cdots$ in A is stationary, i.e. there exists n such that $I_n = I_{n+1} = \cdots$.

(c) Every ideal I in A is finitely generated, i.e., there are finitely many elements $x_1, \cdots, x_k \in A$ such that $I = \langle x_1, \cdots, x_k \rangle$.

Example 2.7

- The ring of integers $\mathbb{Z}$ and the ring $k[x]$ of polynomials in one variable over a field k are principal ideal domains, hence Noetherian.
- The polynomial ring $k[x_1, x_2, \cdots]$ in infinitely many variables is not Noetherian since there is a strictly increasing sequence $(x_1) \subset (x_1, x_2) \subset \cdots$ of ideals.

Theorem 2.8 (Hilbert's Basis Theorem) *If A is Noetherian, then the ring $A[x_1, x_2, \cdots, x_n]$ of polynomials in finitely many variables with coefficients in A is also Noetherian.*

Remark 2.9 As we have seen in theorem 2.8, the Hilbert's basis theorem is true only for rings of polynomials in finitely many variables, and it does not hold for rings of polynomials in infinitely many variables as in example 2.7. In later sections, we will study when $A[X] = A[x_1, x_2, \cdots]$ is Noetherian in some senses by using some *good* actions of some *good* monoids on $A[X]$ (as in example 2.4), which will be introduced in the following sections.

2.2. Theory of $P-$order relations

2.2.1. Well-partial-ordering

A *partial ordering* on a set S is a binary relation $\leq$ on S which is *reflexive*, *transitive* and *antisymmetric*. A trivial ordering on S is given by $s \leq t \Leftrightarrow s = t$ for all $s, t \in S$. We write $s < t$ if $s \leq t$ and $t \not\leq s$.

An *antichain* of S is a subset $A \subseteq S$ such that any two elements in the subset are incomparable. A *final segment* is a subset $F \subseteq S$ which is closed upwards : $s \leq t \wedge s \in F \Rightarrow t \in F$.

A parial ordered set S is said to be *well partial ordering* if (1) there are no infinite antichains and (2) there are no infinitely strictly decreasing sequences. An infinite sequence $s_1, s_2, \cdots$ in S is called *good* if $s_i \leq s_j$ for some indices $i < j$, and *bad* otherwise. We have the following characterization of well-partial-orderings as follows (see [K72], [AH07]).

Proposition 2.10 *The following are equivalent, for a partial ordered set S :*

(1) *S is well-partial-ordered.*

(2) *Every infinite sequence in S is good.*

(3) *Every infinite sequence in S contains an infinite increasing subsequence.*

(4) *Any final segment of S is finitely generated.*

(5) *$(\mathcal{F}(S), \supseteq)$, where $\mathcal{F}(S)$ is the set of final segments of S, is well-founded (i.e., the ascending chain condition holds for final segments of S).*

If $(S, \leq_S)$ and $(T, \leq_T)$ are partial ordered, then the cartesian product $S \times T$ can be turned into a partial ordered set by using the cartersian product of $\leq_S$ and $\leq_T$:

$$(s,t) \leq (s',t') :\Leftrightarrow s \leq_S s' \wedge t \leq_T t', \qquad \text{for } s, s' \in S, t, t' \in T$$

From the proposition 2.10 we easily obtain that the cartesian product of two well-partial-ordered sets is again well-partial-ordered.

Of course, a total ordering $\leq$ is well-partial-ordered if and only if it is well-founded. In this case $\leq$ is called *well-ordering.*

Definition 2.11 (The Higman partial order) Let $(S, \preceq)$ be a partially-ordered set. Let $(S_H, \preceq_H)$ be defined on the set $S_H = S^*$ of finite words of elements of S by

$$u_1 u_2 \cdots u_m \preceq_H v_1 v_2 \cdots v_n$$

if and only if there is a $\pi \in \Pi$ sending $[m]$ to $[n]$ such that $u_i \preceq v_{\pi(i)}$ for $i \in [m]$.

The main result about Higman partial orders is Higman's Lemma ([H52],[W63],[MR90]):

Lemma 2.12 (Higman's Lemma) If $(S, \preceq)$ is a well-partial-order, then the Higman partial order $(S_H, \preceq_H)$ is also well-partial-order.

Example 2.13 We may take $S = \mathbb{N}^k$, partially ordered by inequality

$$(s_1, s_2, \cdots, s_k) \preceq (t_1, t_2, \cdots, t_k) :\Leftrightarrow s_i \leq t_i \text{ for } i = 1, \cdots, k$$

which is well-partial-ordered by Dickson's Lemma ([AL94]).

A *term ordering* on monomials in polynomial ring $R = A[X] = A[x_1, x_2, \cdots]$ is a well-odering $\leq$ on the set of monomials such that

- $1 \leq x$ for all $x \in X = \{x_1, x_2, \cdots\}$, and
- $v \leq w \Rightarrow xv \leq xw$ for all monomials v, w and $x \in X = \{x_1, x_2, \cdots\}$.

2.2.2. The $P-$ordering

Let A be a commutative ring with 1, let Q be a (possibly noncommutative) monoid, and let $A[Q]$ be the semigroup ring associated to Q over A. We call the elements of Q the *monomials* of $A[Q]$. Let a monoid P act on $A[Q]$ by means of homomorphisms (with multiplication in P given by composition). Associated to $A[Q]$ and P is the *skew-monoid ring* $A[Q] * P$, which is formally the set of all linear combinations

$$A[Q] * P = \Big\{ \sum_{i=1}^{s} c_i q_i p_i : c_i \in A, q_i \in Q, p_i \in P \Big\}$$

Multiplication of monomials in the ring $A[Q] * P$ is given by

$$q_1 p_1 . q_2 p_2 = q_1 (p_1 q_2)(p_1 p_2)$$

and extended by distributivity and $A-$linearity to the whole ring. The natural (left) action of the skew-monoid ring on $A[Q]$ makes $A[Q]$ into a (left) module over $A[Q] * P$.

We say that an (left) ideal $I \subseteq A[Q]$ is $P-$*invariant* if

$$PI := \{pn : p \in P, n \in I\} = I$$

Stated another way, a $P-$invariant ideal is simply a $A[Q] * P-$submodule of $A[Q]$.

If we have a well ordering $\preccurlyeq$ of Q, we may talk about the *initial monomial* or *leading monomial* $q = lm(f)$ of any nonzero $f \in A[Q]$, which is the largest element $q \in Q$ with respect to $\preccurlyeq$ appearing with nonzero coefficient in f. We set $lm(f) = 0$ whenever $f = 0$, and also $0 \preccurlyeq q$ for all $q \in Q$.

Definition 2.14 ($P-$order) A well-ordering $\preccurlyeq$ of Q is called a $P-$*order* on $A[Q]$ if for all $q \in Q, p \in P$, and $f \in A[Q]$, we have

$$lm(qp.f) = lm(qp.lm(f))$$

i.e., P preserves the monomial order in $A[Q]$.

Remark that when $P = \{1\}$, a $P-$order is simply a term order on monomials. In next section, we will provide examples of $P-$order, in particular the *shift order.*

Lemma 2.15 Suppose that $\preccurlyeq$ is a $P-$order on $A[Q]$. Then the following hold ([HS09]):

(i) For all $q \in Q, p \in P$ and $q_1, q_2 \in Q$, we have $q_1 \prec q_2 \Rightarrow lm(qpq_1) \preceq lm(qpq_2)$.

(ii) If $lm(qpf) = lm(qpg)$ for some $q \in Q, p \in P$ and $f, g \in A[Q]$, then either $lm(f) = lm(g)$ or $qpf = qpg = 0$.

(iii) Q is left-cancellative : for all $q, q_1, q_2 \in Q$, we have $qq_1 = qq_2 \Rightarrow q_1 = q_2$.

(iv) $q_2 \preceq q_1q_2$ for all $q_1, q_2 \in Q$ (in particular, 1 is the smallest monomial).

(v) All elements of P act injectively on $A[Q]$.

(vi) For all $q \in Q$ and $p \in P$, we have $q \preceq lm(pq)$.

Proposition 2.16 (Characterization of $P-$order) *Let Q be a monoid and let P be a monoid of $A-$algebra endomorphisms of $A[Q]$. Then a well-ordering $\preceq$ of Q is a $P-$order if and only if for all $q \in Q, p \in P$, and $q_1, q_2 \in Q$, we have*

$$q_1 \prec q_2 \Rightarrow lm(qpq_1) \prec lm(qpq_2)$$

Proof. If $lm(qpq_1) = lm(qpq_2)$, since $q_1 \prec q_2$, then by (ii) of lemma 2.15 we have $qpq_1 = qpq_2 \Rightarrow pq_1 = pq_2$ by (iii), and then $q_1 = q_2$ by (v), which is a contradiction.

Conversely, suppose that $\preceq$ is a well-ordering of Q. Let $p \in P, q \in Q$ and $0 \neq f \in A[Q]$, we will prove that $lm(qpf) = lm(qp.lm(f))$. Order monomials $q_1 \prec q_2 \prec \cdots \prec q_k$ appearing in f with nonzero coefficient. By assumption, we have $lm(qpq_i) \prec lm(qpq_{i+1})$ for all i. It follows that $lm(qpf) = lm(qp.lm(f))$. $\square$

Definition 2.17 (The $P-$divisibility relation) Given monomials $q_1, q_2 \in Q$, we say that $q_1|_P q_2$ if there exists $p \in P$ and $q \in Q$ such that $q_2 = q.lm(pq_1)$. Such a p is called a *witness* of the relation $q_1|_P q_2$.

Proposition 2.18 *If $\preceq$ is a $P-$order on Q, then $P-$divisibility $|_P$ is a partial order on Q that is a coarsening of $\preceq$ (i.e., $q_1|_P q_2 \Rightarrow q_1 \preceq q_2$).*

Proof. It is clear that $|_P$ is reflexive. Assume that $q_1|_P q_2$ and $q_2|_P q_3$ for $q_1, q_2, q_3 \in Q$. Then there is $m_1, m_2 \in Q, p_1, p_2 \in P$ such that $q_2 = m_1 lm(p_1 q_1)$ and $q_3 = m_2 lm(p_2 q_2)$. We have

$$\begin{aligned} q_3 &= m_2 lm(p_2 q_2) \\ &= m_2 lm(p_2.m_1 lm(p_1 q_1)) \\ &= m_2 lm(p_2 m_1.p_2 lm(p_1 q_1)) \\ &= m_2 lm(p_2 m_1.lm(p_2 lm(p_1 q_1))) \\ &= m_2 lm(p_2 m_1.lm(p_2 p_1 q_1)) \end{aligned}$$

Since $lm(p_2 m_1.lm(p_2 p_1 q_1)) \neq 0$, it must be of the form $q.lm(p_2 p_1 q_1)$ for some $q \in Q$. Hence $q_3 = m_2 q.lm(p_2 p_1 q_1)$, which implies that $q_1|_P q_3$. So $|_P$ is transitive.

If $q_1|_P q_2$ then $q_2 = m_1 lm(p_1 q_1)$ for some $m_1 \in Q, p_1 \in P$. Then by (vi) of lemma 2.15 we have

$$q_1 \preceq lm(p_1 q_1) \preceq m_1 lm(p_1 q_1) = q_2$$

So, if we also have $q_2|_P q_1$ then by the same procedure we get $q_2 \preceq q_1$. Thus $q_1 = q_2$, which proves the antisymmetry of $|_P$. $\square$

2.2.3. The $\Pi-$ordering (Shift ordering)

Recall that

$$\Pi = Inc(\mathbb{N}) = \{\pi : \mathbb{N} \to \mathbb{N} : \pi(i) < \pi(i+1) \text{ for all } i \in \mathbb{N}\}$$

For $r \in \mathbb{N}$, let $[r] = \{1, 2, \cdots, r\}$. We consider the (linear) action of Π on $A[X_{[r]\times\mathbb{N}}]$ induced by its action on the second index of the indeterminates $X_{[r]\times\mathbb{N}}$:

$$\pi x_{i,j} := x_{i,\pi(j)}, \quad \pi \in \Pi$$

Proposition 2.19 *The column-wise lexicographic term order $x_{i,j} \preceq x_{k,l}$ if $j < l$ or (j=l and $i \leq k$) is a $\Pi-$order on $A[X_{[r]\times\mathbb{N}}]$. In addition $\Pi-$divisibility on $A[X_{[r]\times\mathbb{N}}]$ is a well-partial-order.*

Proof. Notice that every monomial in $A[X_{[r]\times\mathbb{N}}]$ is written in the form $x^u = x_1^{u_1} \cdots x_m^{u_m}$ for some $m \in \mathbb{N}$, where $x_j^{u_j} = \prod_{i\in[r]} x_{i,j}^{u_{i,j}}$. Suppose that $x^u \prec x^v$, then we can write x^u as

$$x^u = x_1^{u_1} \cdots x_m^{u_m} x_{m+1}^{v_{m+1}} \cdots x_n^{v_n}$$

for some $m \leq n$ in which $x_m^{u_m} \prec x_m^{v_m}$. For $\pi \in \Pi$, we have

$$\begin{aligned}
\pi x^u &= x_{\pi(1)}^{u_1} \cdots x_{\pi(m)}^{u_m} x_{\pi(m+1)}^{v_{m+1}} \cdots x_{\pi(n)}^{v_n} \\
\pi x^v &= x_{\pi(1)}^{v_1} \cdots x_{\pi(m)}^{v_m} x_{\pi(m+1)}^{v_{m+1}} \cdots x_{\pi(n)}^{v_n}
\end{aligned}$$

Since π is increasing so $x_{\pi(m)}^{u_m} \prec x_{\pi(m)}^{v_m}$. Hence $\pi x^u \prec \pi x^v$, which proves that $\preceq$ is a $\Pi-$order.

We now show that $\Pi-$divisibility on $A[X_{[r]\times\mathbb{N}}]$ is well-partial-ordered. Assume that $x^u|_\Pi x^v$, then there is $\pi \in \Pi$ such that $\pi x^u | x^v$, then $x_{\pi(i)}^{u_i} | x_{\pi(i)}^{v_{\pi(i)}}$ for each $i \in [m]$. By Higman's lemma applied to $\mathbb{N}^r$ with respect to partial order as in example 2.13, the $\Pi-$divisibility is well-partial-ordered. $\square$

2.2.4. The $Sym(\mathbb{N})-$ordering (symmetric cancellation ordering)

Definition 2.20 Let $Sym(\mathbb{N})$ act on monomials in $R = A[Q]$ by permutations. The symmetric cancellation ordering corresponding to $Sym(\mathbb{N})$ and a term ordering $\leq$ on R is defined by

$$v \preccurlyeq w :\Longleftrightarrow \begin{cases} v \leq w \text{ and there exists } \sigma \in Sym(\mathbb{N}) \text{ and a monomial } u \\ \text{such that } w = u\sigma v \text{ and for all } v' \leq v, \text{ we have } u\sigma v' \leq w. \end{cases}$$

Remark 2.21 Every term ordering $\leq$ is linear in the sense : $v \leq w \Leftrightarrow uv \leq uw$ for all monomials u, v, w. Hence the condition above may be written as : $v \leq w$ and there exists $\sigma \in Sym(\mathbb{N})$ such that $\sigma v | w$ and $\sigma v' \leq \sigma v$ for all $v' \leq v$. We say that σ *witnesses* $v \preccurlyeq w$.

Lemma 2.22 *The relation $\preccurlyeq$ is an ordering on monomials.*

Proof. $w \preceq w$ for all $w \in Q$, since we may choose $u = 1$ and σ to be the identity. So $\preceq$ is reflexive. Next, suppose that $u \preceq v \preceq w$, then there are $u_1, u_2 \in Q, \sigma, \tau \in Sym(\mathbb{N})$ such that $v = u_1\sigma u, w = u_2\tau v$, so $w = u_2\tau u_1\tau\sigma u$. In addition, if $v' \leq u$, then $u_1\sigma v' \leq v$, so that $u_2\tau u_1\tau\sigma v' \leq w$. This shows that $\preceq$ is transitive. Finally, if $u \preceq v$ and $v \preceq u$, then $u \leq v$ and $v \leq u$ by definition. Hence $u = v$ as desired. $\square$

We have the following result ([AH07]):

Proposition 2.23 *The ordering $\preccurlyeq$ is a well-partial-order.*

Remark 2.24 . Symmetric cancellation is not a $P-$order since it does not preserve any monomial odering. Let $A[Q] = A[X_{\mathbb{N}}]$ be the polynomial ring in infinitely many variables $X_{\mathbb{N}} = \{x_i : i \in \mathbb{N}\}$. Also let $P = Sym(\mathbb{N})$. Then there is no $P-$order on $A[X_{\mathbb{N}}]$. To see this, let $g = x_1 + x_2$, and suppose (without loss of generality) that a $P-$order makes $lm(g) = x_1$. Then if $p = (12)$, we have $lm(p.g) = lm(g) = x_1$, while $lm(p.lm(g)) = lm(p.x_1) = x_2$.

In later sections, we will show that $A[Q]$ is $\Pi-$Noetherian, with respect to the $\Pi-$order. However, although $Sym(\mathbb{N})$ is not a $P-$order on $A[Q]$, but we still can use the result of $\Pi-$order to show that $A[X]$ is $Sym(\mathbb{N})-$Noetherian.

2.3. Gröbner bases

2.3.1. Reduction of polynomials

Let $f \in R = A[Q], f \neq 0$, and let B be a set of non-zero polynomials in R. We say that f is *reducible* by B if there exists pairwise distinct $g_1, g_2, \cdots, g_m \in B, m \geq 1$, such that for each i we have $lm(g_i) \preccurlyeq lm(f)$, witnessed by some $p_i \in P$, i.e., $lm(p_i g_i) = p_i lm(g_i)$ devides $lm(f)$, and

$$lt(f) = a_1 q_1 p_1 lt(g_1) + \cdots + a_m q_m p_m lt(g_m)$$

for non-zero $a_i \in A$ and monomials $q_i \in Q$ such that $q_i p_i lm(g_i) = lm(f)$. In this case we write $f \underset{B}{\longrightarrow} h$, where

$$h = f - (a_1 q_1 p_1 lt(g_1) + \cdots + a_m q_m p_m lt(g_m))$$

and we say that f *reduces* to h by B. We say that f is *reduced* with respect to B if f is not reducible by B. By convention, the zero polynomial is reduced with respect to B. Trivially, every element of B reduces to 0.

The smallest partial-ordering on R extending the relation $\underset{B}{\longrightarrow}$ is denoted by $\overset{*}{\underset{B}{\longrightarrow}}$. If $f, g \neq 0$ and $f \underset{B}{\longrightarrow} h$, then $lm(h) \preccurlyeq lm(f)$. In particular, every chain

$$h_0 \underset{B}{\longrightarrow} h_1 \underset{B}{\longrightarrow} h_2 \underset{B}{\longrightarrow} \cdots$$

with all $h_i \in R \setminus \{0\}$ is finite (since $\preccurlyeq$ is well-founded). Hence there exists $r \in R$ such that $f \overset{*}{\underset{B}{\longrightarrow}} r$ and r is reduced with respect to B. We call such an r a *normal form* of f with respect to B.

Lemma 2.25 *Suppose that $f \overset{*}{\underset{B}{\longrightarrow}} r$. Then there exist $g_1, \cdots, g_n \in B, p_1, \cdots, p_n \in P$ and $h_1, \cdots, h_n \in R$ such that*

$$f = r + \sum_{i=1}^{n} h_i p_i g_i \quad \textit{and} \quad \max_{1 \leq i \leq n} lm(h_i p_i g_i) \preccurlyeq lm(f)$$

(In particular, $f - r \in \langle B \rangle_{A[Q]*P}$).

Proof. This is clear if $f = r$. Otherwise we have $f \underset{B}{\longrightarrow} h \overset{*}{\underset{B}{\longrightarrow}} r$ for some $h \in R$. Inductively, we may assume that there exist $g_1, \cdots, g_n \in B, p_1, \cdots, p_n \in P$ and $h_1, \cdots, h_n \in R$ such that

$$h = r + \sum_{i=1}^{n} h_i p_i g_i \quad \text{and} \quad lm(h) \succeq \max_{i \leq i \leq n} lm(h_i p_i g_i)$$

There are also $g_{n+1}, \cdots, g_{n+m} \in B, p_{n+1}, \cdots, p_{n+m} \in P, a_{n+1}, \cdots, a_{n+m} \in A$ and $q_{n+1}, \cdots, q_{n+m} \in Q$ such that $lm(q_{n+i} p_{n+i} g_{n+i}) = lm(f)$ for all i and

$$lt(f) = \sum_{i=1}^{m} a_{n+i} q_{n+i} p_{n+i} lt(g_{n+i}), \quad f = h + \sum_{i=1}^{m} a_{n+i} q_{n+i} p_{n+i} g_{n+i}$$

Hence putting $h_{n+i} := a_{n+i} q_{n+i}$ for $i = 1, \cdots, m$ we have $f = r + \sum_{j=1}^{n+m} h_j p_j g_j$ and $lm(f) \succ lm(h) \succeq lm(h_j p_j q_j)$ if $1 \leq j \leq n, lm(f) = lm(h_j p_j q_j)$ if $n < j \leq n + m$. $\square$

2.3.2. Gröbner bases

If $\preccurlyeq$ is a $P-$order, then we may compute the *initial final segment* with respect to the $P-$divisibility partial order of any subset $B \subseteq A[Q]$:

$$lm(B) = \{q \in Q : lm(g)|_P q \text{ for some } g \in B \setminus \{0\}\}$$

Moreover when $I \subseteq A[Q]$ is a $P-$invariant ideal, then it is straightforward to check that

$$lm(I) = \{lm(f) : f \in I \setminus \{0\}\}$$

Definition 2.26 A (possibly infinite) set $B \subseteq I \subseteq A[Q]$ is a $P-$Gröbner basis for a $P-$invariant ideal I (with respect to the $P-$order $\preccurlyeq$) if and only if

$$lm(I) = lm(B)$$

Additionally, in the case $A = k$ is a field, a Gröbner basis is called *minimal* if no leading monomial of an element in B is $P-$divisibility smaller than any other leading monomial of an element in B.

Proposition 2.27 *Let I be an ideal of R and B be a set of non-zero elements of I. The following are equivalent :*

(1) B is a Gröbner basis for I.

(2) Every non-zero $f \in I$ is reducible by B.

(3) Every $f \in I$ has a normal form 0. (In particular, $I = \langle B \rangle_{A[Q]*P}$)

(4) Every $f \in I$ has unique normal form 0.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) follow from the lemma above. Now suppose (4) holds. Every $f \in I \setminus \{0\}$ with $lt(f) \notin lt(B)$ is reduced with respect to B, hence it has two distinct normal forms (0 and f), a contradiction. Thus $lt(I) = lt(B)$, which implies that B is a Gröbner basis for I. □

Theorem 2.28 *Let $\preccurlyeq$ be a $P-$order. If $P-$divisibility $|_P$ is a well-partial-ordering, then every $P-$invariant ideal $I \subseteq A[Q]$ has a finite $P-$Gröbner basis with respect to $\preccurlyeq$. Moreover, if elements of P send monomials to scalar multiples of monomials, the converse holds.*

Proof. The set of monomials $lm(I)$ is a final segment with respect to $P-$divisibility, since $P-$divisibility is a well-partial-ordering, $lm(I)$ is finitely generated. Since I is $P-$invariant, these generators are leading monomials of a finite subset B of elements of I. It follows that B is a $P-$Gröbner basis.
Suppose now that elements of P send monomials to scalar multiples of monomials. Let M be any final segment of Q with respect to $|_P$, and set $I = \langle M \rangle_{A[Q]*P}$. By assumption, there is a finite set $B = \{g_1, \cdots, g_k\}$ such that

$$M \subseteq lm(B)$$

Now, each $g \in B$ has a representation of the form

$$g = \sum_{j=1}^{d} a_j q_j p_j m_j \quad a_j \in A, p_j \in P, q_j \in Q, m_j \in M$$

Since elements of P send monomials $m_j \in M$ to a scalar multiples of monomials, it follows that $lm(g) = q.lm(pm)$ for some $q \in Q, p \in P, m \in M$. Hence $m|_P lm(g)$. Thus $M = \langle lm(g_1), \cdots, lm(g_k) \rangle$ is finitely generated, which proves that $|_P$ is a well-partial-ordering. □

We immediately have the following corollary :

Corollary 2.29 *Let $\preceq$ be a $P-$order. If $P-$divisibility $|_P$ is a well-partial-ordering, then every $P-$invariant ideal $I \subseteq A[Q]$ is finitely generated over $A[Q] * P$. In other words, $A[Q]$ is a Noetherian $A[Q] * P$-module.*

2.4. Equivariant Gröbner Bases

In this section, we restrict our settings above to the case $A = k$ is a field and Q is the free commutative monoid generated by $X = \{x_1, x_2, \cdots\}$ and assume that Q is $P-$stable.

Definition 2.24 (Equivariant Gröbner Basis) Let I be a $P-$stable ideal ideal in $k[Q]$. If P is fixed, then we call a $P-$Gröbner basis B of I an *equivariant Gröbner basis* ([BD10]) (or *monoidal Gröbner basis* ([HS09])). If $P = \{1\}$, then B is an *ordinary Gröbner basis* ([AL94]).

Lemma 2.25 *If I is $P-$stable and B is a $P-$Gröbner basis of I, then $PB = \{\pi b | \pi \in P, b \in B\}$ generates the ideal I.*

Proof. If $\langle PB \rangle \neq I$, then take an $f \in I \setminus \langle PB \rangle$ with $lm(f)$ minimal. Since B is a $P-$Gröbner basis, then there exist $b \in B$ and $\pi \in P$ with $lm(\pi b) | lm(f)$. Hence $f - \frac{lt(f)}{lt(\pi b)} \pi b \in I \setminus \langle PB \rangle$ with leading term strictly smaller than $lm(f)$, a contradiction. $\square$

Example 2.26 ([BD10]) : Let $X = \{y_{ij} | i, j \in \mathbb{N}\}$, let k be a number number, and let I be the ideal of all polynomials in the y_{ij} that vanish on all $\mathbb{N} \times \mathbb{N}-$matrices y of rank at most k. Order the variables y_{ij} lexicographically by the pair (i, j), where i is the most significant index; so for instance $y_{3,5} > y_{2,6} > y_{2,4} > y_{1,10}$. The corresponding lexicographic order on monomials in the y_{ij} is a well-order. Let $P := Inc(\mathbb{N}) \times Inc(\mathbb{N})$ act on X by $(\pi, \sigma) y_{ij} = y_{\pi(i),\sigma(j)}$; this action preserves the strict order. The $P-$orbit of the determinant D of the matrix $(y_{ij})_{i,j=1,\cdots,k+1}$ consists of all $(k+1) \times (k+1)-$minors of y, which form a Gröbner basis of the ideal I. Hence, $\{D\}$ is also a $P-$Gröbner basis of I.

Definition 2.27 (Equivariant remainder) Given $f \in k[Q]$ and $B \subseteq k[Q]$, proceed as follows : if $\pi lm(b) | lm(f)$ for some $\pi \in P$ and $b \in B$, then substract the multiple $\frac{lt(f)}{lt(\pi b)} \pi b$ of πb from f. so as to lower the latter's leading monomial. Do this until no such pair (π, b) exists anymore. The resulting polynomial is called a $P-$remainder (or an *equivariant remainder*, if P is fixed) of f modulo B. This process will stop after a finite number of steps, since $\preceq$ is a well-order.

In the polynomial ring of example 2.26 the set $\{y_{12}y_{21}, y_{12}y_{23}y_{31}, y_{12}y_{23}y_{34}y_{41}, \cdots\}$ is an infinite antichain of monomials, hence the $Inc(\mathbb{N})-$stable ideal generated by it does not have a finite $Inc(\mathbb{N})-$Gröbner basis. But even in such a setting where not all $P-$stable ideals have finite $P-$Gröbner bases, ideals of interesting $P-$stable varieties may still have such bases. Hence, to have an algorithm for computing equivariant Gröbner bases, we need the following two addition assumptions :

EGB1 For all $p \in P$ and $m, m' \in Q$ we have $lcm(pm, pm') = p.lcm(m, m')$.

EGB2 For all $f, h \in k[Q]$ the set $Pf \times Ph$ is the union of a finite number of $P-$orbits (where P acts diagonally on $k[Q] \times k[Q]$), and generators of these orbits can be computed effectively.

Under all assumptions above, we have the definition of *equivariant S-polynomials* as $S-$polynomials for the *ordinary* Buchberger's definition :

Definition 2.28 (Equivariant S-polynomials) Consider two polynomials b_0, b_1 with leading monomials m_0, m_1 respectively. Let H be a set of pairs $(\sigma_0, \sigma_1) \in P \times P$ for which $Pb_0 \times Pb_1 = \bigcup_{(\sigma_0,\sigma_1)\in H}\{(\pi\sigma_0 b_0, \pi\sigma_1 b_1) | \pi \in P\}$. For every element $(\sigma_0, \sigma_1) \in H$ we consider the *ordinary S-polynomial*

$$S(\sigma_0 b_0, \sigma_1 b_1) := lc(b_1)\frac{lcm(\sigma_0 m_0, \sigma_1 m_1)}{\sigma_0 m_0}\sigma_0 b_0 - lc(b_0)\frac{lcm(\sigma_0 m_0, \sigma_1 m_1)}{\sigma_1 m_1}\sigma_1 b_1$$

The set $\{S(\sigma_0 b_0, \sigma_1 b_1) | (\sigma_0, \sigma_1) \in H\}$ is called a *complete set of equivariant S-polynomials* for b_0, b_1. It depends on the choice of H.

We then have following result (as for ordinary Buchberger Criterion) ([BD10]):

Theorem 2.29(Equivariant Buchberger Criterion) *Let B be a subset of $k[Q]$ such that for all $b_0, b_1 \in B$, there exists a complete set of $S-$polynomials each of which has 0 as a $P-$remainder modulo B. Then B is a $P-$Gröbner basis of the ideal generated by PB.*

Chapter 3

Noetherianity of the polynomial ring $R = A[x_1, x_2, \cdots]$

In this chapter, we will study the Noetherianity of polynomial ring $R = A[Q]$ under the actions of $\Pi = Inc(\mathbb{N})$ and $Sym(\mathbb{N})$.

3.1. $\Pi-$Noetherianity

For $r \in \mathbb{N}$, let $[r] = \{1, 2, \cdots, r\}$. We consider the (linear) action of Π on $A[X_{[r]\times\mathbb{N}}]$ induced by its action on the second index of the indeterminates $X_{[r]\times\mathbb{N}}$:

$$\pi x_{i,j} := x_{i,\pi(j)}, \quad \pi \in \Pi$$

By proposition 2.19, the $\Pi-$divisibility on $A[X_{[r]\times\mathbb{N}}]$ is a well-partial-ordering, hence by corollary 2.29, we have :

Theorem 3.1 *The ring* $A[X_{[r]\times\mathbb{N}}]$ *is* $\Pi-$*Noetherian.*

Remark 3.2 In the result above, we just considered the *trivial action* of Π on the ring A. A natural question is when the polynomial ring $A[X]$ is $\Pi-$Noetherian when we have a non-trivial action of Π on the ring A. We will study this question in a number of different settings.

First, let Π' be the set of all $\pi \in \Pi$ with cofinite images, and denote $|\pi| = |\mathbb{N} \setminus Im(\pi)|$. We have the following lemma :

Lemma 3.3 *The following map is a homomorphism of monoids*

$$\begin{aligned} \phi : \Pi' &\rightarrow (\mathbb{Z}, +) \\ \pi &\mapsto |\pi| \end{aligned}$$

Proof. For $\pi, \sigma \in \Pi'$, we would like to check whether $|\pi\sigma| = |\sigma| + |\pi|$. We can view $|\sigma|$ and $|\pi|$ as follows :

$$\begin{aligned} |\sigma| &= \sum_{n=0}^{\infty}(\sigma(n+1) - \sigma(n) - 1) + \sigma(0) - 1 \\ |\pi| &= \sum_{n=0}^{\infty}(\pi(n+1) - \pi(n) - 1) + \pi(0) - 1 \end{aligned}$$

We have

$$\begin{aligned} |\pi\sigma| &= \sum_{n=0}^{\infty}(\pi\sigma(n+1) - \pi\sigma(n) - 1) + \pi\sigma(0) - 1 \\ &= \sum_{n=0}^{\infty}\{(\pi\sigma(n+1) - \pi(\sigma(n+1) - 1) - 1) + (\pi(\sigma(n+1) - 1) - \pi(\sigma(n+1) - 2) - 1) + \\ &\quad + \cdots + (\pi(\sigma(n) + 1) - \pi\sigma(n) - 1) + (\sigma(n+1) - \sigma(n) - 1)\} + \pi\sigma(0) - 1 \\ &= \sum_{n=\sigma(0)}^{\infty}(\pi(n+1) - \pi(n) - 1) + \sum_{n=0}^{\infty}(\sigma(n+1) - \sigma(n) - 1) + \pi\sigma(0) - 1 \end{aligned}$$

Moreover,

$$\begin{aligned} \pi\sigma(0) - 1 &= (\pi(\sigma(0)) - \pi(\sigma(0) - 1) - 1) + \cdots + (\pi(1) - \pi(0) - 1) - 1 + \sigma(0) - 1 + \pi(0) \\ &= \sum_{n=0}^{\sigma(0)-1}(\pi(n+1) - \pi(n) - 1) + \sigma(0) - 1 + \pi(0) - 1 \end{aligned}$$

Hence we obtain

$$|\pi\sigma| = |\pi| + |\sigma|$$

as required. □

Example 3.4 Let Π' act on $A = k[z]$ by $\pi.z = z^{2^{|\pi|}}, \pi \in \Pi'$ and act on monomials in $R = A[x_1, x_2, \cdots]$ as usual. First we show that $\pi.z = z^{2^{|\pi|}}, \pi \in \Pi'$ is an action on $A = k[z]$:

- $id.z = z^{2^{|id|}} = z^{2^0} = z.$

- For $\pi, \sigma \in \Pi'$, we have

$$\begin{aligned}
(\pi\sigma)z &= z^{2^{|\pi\sigma|}} = z^{2^{|\pi|+|\sigma|}} \qquad \text{(by lemma 3.3)} \\
&= z^{2^{|\sigma|}2^{|\pi|}} = \left(z^{2^{|\sigma|}}\right)^{2^{|\pi|}} \\
&= \pi(\sigma z)
\end{aligned}$$

Hence, we have an action of Π' on $R = A[x_1, x_2, \cdots]$. Then R is not $\Pi'-$Noetherian. Indeed, consider the $\Pi'-$stable ideal

$$I = (zx_1, zx_2, \cdots)$$

If I is finitely generated, $I = (zx_1, \cdots, zx_n)$ say, then since $zx_{n+1} \in I$, then

$$zx_{n+1} = \pi.(zx_i) = (\pi z)x_{\pi(i)} = z^{2^{|\pi|}}x_{n+1}$$

for some $i \leq n$ and $\pi \in \Pi'$ such that $\pi(i) = n+1$. But then $2^{|\pi|} = 1 \Rightarrow |\pi| = 0$, hence π is the identity map, which contradicts to $\pi(i) = n+1$. So I is not finitely generated, which implies that R is not $\Pi'-$Noetherian.

Example 3.5 Now we consider the action of Π' on $A = k[z]$ by injective homomorphisms $\pi.z = z + |\pi|$. Then R now is $\Pi'-$Noetherian. To see this, we define the order on monomials, which are of the form $z^k u_i$ for the monomial u_i in $x_1, x_2, \cdots$, in R as follows

$$z^k u_i \preccurlyeq^* z^l u_j \Leftrightarrow (k \leq l \text{ and } u_i \preccurlyeq u_j)$$

Since $\leq$ in natural numbers and $\preccurlyeq$ in monomials defined as above are both well-partial-orderings, also $\preccurlyeq^*$ is a well-partial-ordering. And $\preccurlyeq^*$ is now a $\Pi'-$ordering. Since $\Pi'-$divisibility is a well-partial-ordering inspired from proposition 2.19, then R is $\Pi'-$Noetherian by corollary 2.29.

Example 3.6 Consider $A = k[z]$ the ring of polynomial in one variable z, where k is a field. Let Π act on A by $\pi.z = z$ if $\pi = id$ and $\pi.z = 0$ otherwise, and act on monomials as usual. Consider the $\Pi-$stable ideal

$$I = (zx_1, zx_2, \cdots)$$

If I is finitely generated, $I = (zx_1, \cdots, zx_n)$ say, since $zx_{n+1} \in I$, then there should exist a $\pi \in \Pi$ and $i \leq n$ such that $zx_{n+1} = \pi(zx_i) = (\pi z)(\pi x_i) = 0$ which is a contradiction. Hence in this case, R is not $\Pi-$Noetherian.

We get the same result if $id.z = z$ and $\pi.z =$ a constant α for all $\pi \neq id$. Since at this time, the stable ideal

$$I = ((z-\alpha)x_1, (z-\alpha)x_2, \cdots)$$

is not $\Pi-$finitely generated.

There is a *pointwise convergence topology* on $\Pi = Inc(\mathbb{N})$ inspired from the discrete topology on $\mathbb{N}$, namely, a neighborhood of an element $\pi \in \Pi$ is the set containing elements $\sigma \in \Pi$ that agree with π on some specified finite set of points, for instance,

$$B_r(\pi) = \{\sigma \in \Pi : \sigma|_{\{1,\cdots,r\}} = \pi|_{\{1,\cdots,r\}}\}$$

Definition 3.7 We say that Π acts on $A = k[z]$ *continuously* if there is a positive integer $m > 0$ such that for all $\sigma, \pi \in \Pi$ for which $\sigma|_{[m]} = \pi|_{[m]}$ then $\sigma.z = \pi.z$.
Note that none of the actions of Π and Π' above are continuous.

Lemma 3.8 *If $\varphi : \Pi \to M$ from Π into a monoid M is continuous with respect to the discrete topology on M and the topology of Π defined above, i.e., there is a positive integer m such that for all $\pi, \sigma \in \Pi$ for which $\pi|_{[m]} = \sigma|_{[m]}$ we have $\varphi(\pi) = \varphi(\sigma)$, then for every $\pi \in \Pi$, there is a positive integer $d_0 > 0$ such that for all $d > d_0$, $\varphi(\pi)^d$ is idempotent.*

Proof. For $\pi \in \Pi$, if $\pi(i) > m$ for all $i = 1, 2, \cdots, m$, we choose $\sigma \in \Pi$ such that

$$\sigma|_{[m]} = id|_{[m]}, \text{ and } \sigma(m+i) = \pi(m+i) \quad \text{for all } i \geq 1$$

Such an increasing function σ always exists since $\pi(m+i) > m$.

Then we have

$$\sigma\pi|_{[m]} = \pi^2|_{[m]}$$

So

$$\varphi(\sigma\pi) = \varphi(\pi^2) = \varphi(\pi)^2$$

Since $\sigma|_{[m]} = id|_{[m]}$, then $\varphi(\sigma\pi) = \varphi(\sigma)\varphi(\pi) = \varphi(id)\varphi(\pi) = \varphi(\pi)$, therefore

$$\varphi(\pi)^2 = \varphi(\pi)$$

Hence, $\varphi(\pi)$ is idempotent.

If π is not as the form above, there exist $n \leq m$ and $d \in \mathbb{N}$ such that $\pi^d(i) = i$ for $i = 1, \cdots, n$ and $\pi^d(i) > m$ for $i > n$.

Put $\delta = \pi^d$. If $n = 0$ then from the procedure above $\varphi(\pi)^d = \varphi(\delta)$ is idempotent. Otherwise, we may take $\sigma \in \Pi$ as follows

$$\sigma|_{[m]} = id_{[m]}, \text{ and } \sigma(\delta(n+i)) = \delta^2(n+i)\text{for } i \geq 1$$

Since $\delta(i+1) - \delta(i) \leq \delta^2(i+1) - \delta^2(i)$, this assures that σ can still be chosen an increasing function. Then we have

$$\delta^2|_{[m]} = \sigma\delta|_{[m]}$$

So

$$\varphi(\delta)^2 = \varphi(\delta)\varphi(\sigma) = \varphi(\delta)(\varphi(id)) = \varphi(\delta)$$

Hence $\varphi(\delta)$ is idempotent. Thus $\varphi(\pi)^d$ is also idempotent, as desired. □

Now, if Π acts continuously on $A = k[z]$, assume that $\pi.z = f(z), \sigma.z = g(z)$ for $\pi, \sigma \in \Pi$. Since $(\pi\sigma).z = \pi(\sigma.z) = g(f(z))$, then we have $deg(\pi\sigma) = deg(\pi).deg(\sigma)$, where $deg(\pi) = deg(\pi.z) = deg(f(z))$. Hence we have a continuous homomorphism from Π into the monoid $k[z]$ in which the multiplication is defined by $f \circ g(z) = g(f(z))$ for all $f, g \in k[z]$:

$$\phi : \Pi \to (k[z], \circ)$$

By lemma 3.8, for every $\pi \in \Pi$, there is $d >> 0$ such that $\phi(\pi)^d$ is idempotent.

Every idempotent element $f \in k[z]$ satisfies $f(f(z)) = f(z)$, so $\deg(f) \leq 1$, i.e., $f(z) = az + b$. Since $f(f(z)) = f(z)$, we have

$$\begin{aligned} f(z) = az + b &= f(f(z)) = a(az+b) + b \\ &= a^2 z + (a+1)b \end{aligned}$$

Thus $a^2 = a$ and $b = (a+1)b$. From $a^2 = a$ we have either $a = 1$ or $a = 0$. If $a = 1$, then from $b = (a+1)b = 2b$ we get $b = 0$, so $f(z) = z$. If $a = 0$, then $f(z) = b$, a constant.

Hence, for every $\pi \in \Pi$, either $\phi(\pi) = z$ or $\phi(\pi) = c$, a constant.

Let $\pi.z = \phi(z) = a(\pi)z + b(\pi)$ and let $I = \{\pi \in \Pi : a(\pi) = 0\}$ be a (two-sided) prime ideal of Π. Then for every $\pi \in \Pi$, we have :

$$\pi : z \longmapsto \begin{cases} b(\pi) & \text{if} \quad \pi \in I \\ z & \text{if} \quad \pi \notin I \end{cases}$$

And $R = A[x_1, x_2, \cdots]$ is always $\Pi-$Noetherian. Therefore, we have proved :

Theorem 3.9 *If Π acts on $A = k[z]$ continuously then there is a prime ideal (clopen) $I \subseteq \Pi$ and a constant $c \in k$ such that*

$$\pi : z \longmapsto \begin{cases} c & if \quad \pi \in I \\ z & if \quad \pi \notin I \end{cases}$$

And $A[x_1, x_2, \cdots]$ is always $\Pi-$Noetherian.

Remark 3.10 (Another proof of theorem 3.9) Now let Π_m be the set of all $\pi \in \Pi$ such that $\pi|_{[m]} = id|_{[m]}$. Π acts continuously on $A = k[z]$, then there is m such that for $\pi, \sigma \in \Pi$ satisfying

$$\pi|_{[m]} = \sigma|_{[m]}$$

we have

$$\sigma.z = \pi.z$$

Since $\pi|_{[m]} = id|_{[m]}$ for all $\pi \in \Pi$, Π_m acts trivially on $A = k[z]$, hence Π_m acts trivially on $A' = k[z, x_1, \cdots, x_m]$. Then

$$R = k[z][x_1, x_2, \cdots] = A'[x_{m+1}, x_{m+2}, \cdots]$$

Since A' is Π_m-Noetherian, then by [AH09], R is Π_m-Noetherian. Hence R is $\Pi-$Noetherian.

One can also prove in a similar fashion that for any continous action of Π on $A = k[z_1, z_2, \cdots, z_n]$, the polynomial ring $R = A[x_1, x_2, \cdots]$ is $\Pi-$Noetherian.

3.2. The Sym($\mathbb{N}$)$-$Noetherianity

Let Sym($\mathbb{N}$) act trivially on ring A and act on monomials in $x_1, x_2, \cdots$ by permutations. We have the following result

Theorem 3.11 *$A[X_{[r]\times\mathbb{N}}]$ is $Sym(\mathbb{N})-$Noetherian.*

Proof. Each polynomial $f \in A[X_{[r]\times\mathbb{N}}]$ depends on only finitely many column indices. Thus if $\pi \in \Pi$, there exists $\sigma \in Sym(\mathbb{N}$ such that $\sigma.f = \pi.f$. Indeed, if the largest column index increasing in f is m, then σ can be chosen to be the identity on all $i > \pi(m)$. This implies that every $Sym(\mathbb{N})-$stable ideal I is $\Pi-$stable and any $A[X_{[r]\times\mathbb{N}}] * \Pi$ generating set of I is a $A[X_{[r]\times\mathbb{N}}] * Sym(\mathbb{N})$ generating set. $\square$

When $r = 1$, this is the main result of [AH07].

However, if we consider the action of $Sym(\mathbb{N})$ on $R = A[X_{\underbrace{\mathbb{N}\times\mathbb{N}\times\cdots\times\mathbb{N}}_{k \text{ factors}}}]$ by permuting the indices simultaneously, then R is no longer $Sym(\mathbb{N})-$Noetherian for $k \geq 2$. Indeed, if we denote $R^{(k)}$ the ring $A[X_{\mathbb{N}\times\mathbb{N}\times\cdots\times\mathbb{N}}]$ in k indices, then

$$x_{u_1,\cdots,u_k,u_{k+1}} \mapsto x_{u_1,\cdots,u_k}$$

defines the surjective $A-$algebra homomorphism $\pi_k : R^{(k+1)} \to R^{(k)}$ with invariant kernel. Hence if $R^{(k+1)}$ is Sym($\mathbb{N}$)$-$Noetherian, then so is $R^{(k)}$. Hence, it is enough to do for the case $k = 2$. We state in the following proposition

Proposition 3.12 The polynomial ring $R = A[X_{\mathbb{N}\times\mathbb{N}}]$ is not $Sym(\mathbb{N})\times Sym(\mathbb{N})-$Noetherian.

Proof. It is enough to show a bad sequence of monomials in R with respect to the $Sym(\mathbb{N})-$divisibility order. For this, consider the sequence of monomials ([AH07],[JW69]):

$$\begin{array}{rcl}
s_3 & = & x_{(1,2)}x_{(3,2)}x_{(3,4)} \\
s_4 & = & x_{(1,2)}x_{(3,2)}x_{(4,3)}x_{(4,5)} \\
s_5 & = & x_{(1,2)}x_{(3,2)}x_{(4,3)}x_{(5,4)}x_{(6,7)} \\
\vdots & & \\
s_n & = & x_{(1,2)}x_{(3,2)}x_{(4,3)}\cdots x_{(n,n-1)}x_{(n,n+1)} \\
\vdots & &
\end{array}$$

For any $n < m$ and any $\sigma \in G$, the monomial σs_n does not divide s_m. Otherwise, notice that $x_{(1,2)}x_{(3,2)}$ is the only pair of indeterminates which divides s_n or s_m and has form $x_{(i,j)}x_{(l,j)}$. Therefore $\sigma(2) = 2$, and either $\sigma(1) = 1, \sigma(3) = 3$ or $\sigma(3) = 1, \sigma(1) = 3$. But since 1 does not appear as the second component j of a factor $x_{(i,j)}$ of s_m, we have $\sigma(1) = 1, \sigma(3) = 3$. Since $x_{(4,3)}$ is the only indeterminate dividing s_n or s_m of the form $x_{(i,3)}$, we get $\sigma(4) = 4$. Since x if the only indeterminate dividing s_n or s_m of the form $x_{(i,4)}$, we get $\sigma(5) = 5$, etc. So we get $\sigma(i) = i$ for all $i = 1, 2, \cdots, n$. But the only indeterminate dividing s_m of the form $x_{(n,j)}$ is $x_{(n,n-1)}$, hence the factor $\sigma x_{n,n+1} = x_{n,\sigma(n+1)}$ of σs_n does not divide s_m. This shows that $s_3, s_4, \cdots$ is a bad sequence, as required.

However, if we let $R_{\leq d}$ denote the $Sym(\mathbb{N})-$module of polynomials of degree at most d, we do have the following result ([D09])

Lemma 3.13 *The $Sym(\mathbb{N})-$module $R_{\leq d}$ is Noetherian, i.e., every $Sym(\mathbb{N})-$submodule of its is finitely generated.*

If we let $FSym(\mathbb{N}) = \bigcup_n Sym([n]) \subseteq Sym(\mathbb{N})$ be the *finitary* subgroup of $Sym(\mathbb{N})$, and if we let $R_n = A[X_{[n]}]$, and so $R_n \subseteq R_m$ naturally becomes a subring of R_m for all $n \leq m$, and hence $R = A[X_{\mathbb{N}}] = \bigcup_{n=1}^{\infty} R_n$. The group $Sym[n]$ acts on R_n naturally by permuting the indices. Furthermore, suppose that the natural embedding of $Sym([n])$ into $Sym([m])$ for $n \leq m$ is compatible with the embedding of rings $R_n \subseteq R_m$; that is , if $\sigma \in Sym[n]$ and $\hat{\sigma}$ is the corresponding element in $Sym[m]$, then $\hat{\sigma}|_{R_n} = \sigma$. Hence, we have the action of $FSym(\mathbb{N})$ on R which extends the action of each $Sym[n]$ on R_n. Then R is $FSym(\mathbb{N})$-Noetherian modulo the symmetric group. Before stating the main theorem, we introduce some notions.

Definition 3.14 For $m \geq n$, the *m-symmetrization* $L_m(B)$ for a set B of elements of R_n is the $Sym([m])-$invariant ideal of R_m given by

$$L_m(B) = \langle g : g \in B \rangle_{R_m * Sym([m])}$$

We consider the increasing chain I_0 of ideals $I_n \subseteq R_n$:

$$I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \cdots$$

simply called *chains* below. Of course, such chains will fail to stabilize since they are ideals in larger and larger rings. However, it is possible for these ideals to stabilize "up to the action of the symmetric group". We call a *symmetrization invariant chain* is one for which $L_m(I_n) \subseteq I_m$ for all $n \leq m$.

Definition 3.15 A symmetrization invariant chain of ideals *stabilizes modulo the symmetric group* (or simply stabilizes) if there exists a positive integer N such that

$$L_m(I_n) = I_m \quad \text{for all } m \geq n > N$$

We do have the following result ([AH07]):

Theorem 3.16 *Every symmetrization invariant chain stabilizes modulo the symmetric group.*

Chapter 4

Rank-1 tensors and Substitution monoids

4.1. Substitution monoids

Definition 4.1 We define the *substitution monoid* $\mathrm{Subs}(\mathbb{N})$ as follows : its elements are infinite sequences $\sigma = (\sigma_1, \sigma_2, \cdots)$ of disjoint non-empty finite subsets of $\mathbb{N}$. The product of two such sequences σ and τ is defined by

$$(\sigma \circ \tau)_i = \bigcup_{j \in \tau_i} \sigma_j$$

It makes $\mathrm{Subs}(\mathbb{N})$ a monoid :

- The identity is the infinite sequence $1 = (\{1\}, \{2\}, \cdots)$.
- The associativity : for $\sigma, \tau, \gamma \in \mathrm{Subs}(\mathbb{N})$, we have :

$$\begin{aligned}
(\sigma \circ (\tau \circ \gamma))_i &= \bigcup_{j \in (\tau \circ \gamma)_i} \sigma_j = \bigcup_{j \in \cup_{k \in \gamma_i} \tau_k} \sigma_j \\
&= \bigcup_{k \in \gamma_i} \bigcup_{j \in \tau_k} \sigma_j = \bigcup_{k \in \gamma_i} (\sigma \circ \tau)_k \\
&= ((\sigma \circ \tau) \circ \gamma)_i
\end{aligned}$$

Let $\mathrm{Subs}_<(\mathbb{N})$ be the submonoid of all such sequences $(\sigma_1, \sigma_2, \cdots)$ satisfying

$$\max(\sigma_1) < \max(\sigma_2) < \cdots$$

Note that $\mathrm{Inc}(\mathbb{N})$ is the submonoid of $\mathrm{Subs}_<(\mathbb{N})$ consisting of sequences $(\{i_1\},\{i_2\},\cdots)$ of singletons with $i_1 < i_2 < \cdots$.

4.2. Rank-1 tensors

4.2.1. The orginial rank-1 tensors problem

For a positive integer n, denote $(k^2)^n = \underbrace{k^2 \times k^2 \times \cdots \times k^2}_{n \text{ factors}}$ and $(k^2)^{\otimes n} = \underbrace{k^2 \otimes k^2 \otimes \cdots \otimes k^2}_{n \text{ factors}}$, where k is an algebraically closed field. We consider the following multi-linear mapping :

$$\begin{array}{rcl} \varphi : (k^2)^n & \to & (k^2)^{\otimes n} \\ \left(\begin{pmatrix} x_{i0} \\ x_{i1} \end{pmatrix}\right)_{i\in[n]} & \mapsto & \bigotimes_{i\in[n]} \begin{pmatrix} x_{i0} \\ x_{i1} \end{pmatrix} \end{array}$$

Choose the standard basis $\{e_{i0}, e_{i1}\}$ for the i th copy of k^2, so every element $\binom{x_{i0}}{x_{i1}} \in k^2$ can be written as

$$x_{i0}e_{i0} + x_{i1}e_{i1}$$

Hence

$$\varphi\left(\begin{pmatrix} x_{i0} \\ x_{i1} \end{pmatrix}_{i\in[n]}\right) = \bigotimes_{i\in[n]} (x_{i0} + x_{i1}e_i)$$

Let α be a map defined as follows

$$\begin{array}{rcl} \alpha : k \times k^n & \to & (k^2)^n \\ (t, x_0, x_1, \cdots, x_{n-1}) & \mapsto & \left(\begin{pmatrix} t \\ tx_0 \end{pmatrix}, \begin{pmatrix} 1 \\ x_1 \end{pmatrix}, \cdots, \begin{pmatrix} 1 \\ x_{n-1} \end{pmatrix}\right) \end{array}$$

We would like to find the image of the map $\psi = \varphi \circ \alpha : k \times k^n \to (k^2)^{\otimes n}$. We have

$$\begin{aligned} \psi(t, x_0, x_1, \cdots, x_{n-1}) &= \begin{pmatrix} t \\ tx_0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ x_1 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 1 \\ x_{n-1} \end{pmatrix} \\ &= (te_{10} + tx_0e_{11}) \otimes (e_{20} + x_1e_{21}) \otimes \cdots \otimes (e_{n0} + x_{n-1}e_{n1}) \\ &= t \sum_{s_j \in \{0,1\}} \left(\prod_{i:s_i=1} x_i \right) e_{s_1} \otimes e_{s_2} \otimes \cdots \otimes e_{s_n} \end{aligned}$$

If we let z_I, $I \subseteq [n]$, be coordinate for $(k^2)^{\otimes n}$ related to $e_{s_1} \otimes e_{s_2} \otimes \cdots \otimes e_{s_n}$ where $s_i = 1$ if $i \in I$ and $s_i = 0$ if $i \notin I$. We should take the ideal generated by all elements of the form

$$z_I - t \prod_{i\in I} x_i$$

Then by elimination theory, the intersection $I(Y)$ of this ideal with the ring $k[(z_I)_{I\subseteq[n]}]$ is exactly the ideal of the image $Y = \mathrm{Im}(\varphi)$.

It is known that the ideal $I(Y)$ is generated by certain $2 \times 2-$minors. We will approach this problem by another setting, namely, by *substitution method* which is presented in the following section.

4.2.2. The substitution approach

We rephrase our settings as follows :

Let $\mathcal{A}$ be the infinite dimensional affine space (over a field K) whose coordinates are z_I, where I runs over the finite subsets of $\mathbb{N}$. Consider the map as follows

$$\begin{array}{rcl} \phi : \mathbb{A}^1 \times \mathbb{A}^{\mathbb{N}} & \to & \mathcal{A} \\ (t, (x_i)_{i\in\mathbb{N}}) & \mapsto & (z_I)_I = (t \prod_{i\in I} x_i)_I \end{array}$$

$\mathrm{Subs}(\mathbb{N})$ acts on $\mathbb{A}^{\mathbb{N}}$ by $\sigma X_i := \prod_{j\in\sigma_i} X_j$. It is an action since :

- $1.X_i = \prod_{j\in\{i\}} X_j = X_i$.
- For $\sigma, \tau \in \mathrm{Subs}(\mathbb{N})$, we have

$$\begin{array}{rcl} (\sigma \circ \tau)X_i & = & \prod_{j\in(\sigma\circ\tau)_i} X_j = \prod_{j\in\cup_{k\in\tau_i}\sigma_k} X_j \\ & = & \prod_{k\in\tau_i}\prod_{j\in\sigma_k} X_j = \prod_{k\in\tau_i}(\sigma X_k) \\ & = & \sigma(\prod_{k\in\tau_i} X_k) = \sigma(\tau X_i) \end{array}$$

And $\mathrm{Subs}(\mathbb{N})$ acts on $\mathcal{A}$ by $\sigma Z_I = Z_{\cup_{i\in I}\sigma_i}$. Again, it is clearly an action since :

- $1.Z_I = Z_{\cup_{i\in I}1_i} = Z_{\cup_{i\in I}\{i\}} = Z_I$.
- For $\sigma, \tau \in \mathrm{Subs}(\mathbb{N})$, we have

$$\begin{array}{rcl} (\sigma \circ \tau)Z_I & = & Z_{\cup_{i\in I}(\sigma\circ\tau)_i} = Z_{\cup_{i\in I}(\cup_{j\in\tau_i}\sigma_j)} \\ & = & Z_{\cup_{(j\in\cup_{i\in I}\tau_i)}\sigma_j} = \sigma(Z_{\cup_{i\in I}\tau_i}) = \sigma(\tau Z_I) \end{array}$$

Moreover, the map ϕ is $\mathrm{Subs}(\mathbb{N})-$equivariant, since for $\sigma \in \mathrm{Subs}(\mathbb{N}), (\lambda, (x_i)_{i\in\mathbb{N}}) \in \mathbb{A}^1 \times \mathbb{A}^{\mathbb{N}}$, we have :

$$\begin{aligned}
\phi(\sigma(\lambda, (x_i)_{i\in\mathbb{N}})) &= \phi(\lambda, (\sigma x_i)_{i\in\mathbb{N}}) = \phi(\lambda, (\prod_{j\in\sigma_i} x_j)) \\
&= (\lambda \prod_{i\in I} \prod_{j\in\sigma_i} x_j)_I = \sigma(\lambda \prod_{i\in I} x_i)_I \\
&= \sigma\phi(\lambda, (x_i)_{i\in\mathbb{N}})
\end{aligned}$$

Let Y be the scheme-theoretic image of the map ϕ, and $I(Y)$ be the vanishing ideal on Y. Then $I(Y)$ is $\mathrm{Subs}_<(\mathbb{N})$-stable. We would like to know the ideal of the image of ϕ by computing the $\mathrm{Subs}_<(\mathbb{N})-$Gröbner basis of the ideal $I(Y)$ in the polynomial ring $S = K[t; x_1, x_2, \cdots; (z_J)_{J\subseteq\mathbb{N}}]$, where J runs over all finite subsets of the natural numbers, generated by all elements of the form

$$z_J - t\prod_{i\in J} x_i$$

with respect to the lexicographic order satisfying $t > x_i > z_J$ for all i and J and $x_{i+1} > x_i$ and $z_J > z_{J'}$ if J is lexicographically larger than J'. And we will use this Gröbner basis to compute the intersection of $I(Y)$ with the polynomial ring $K[(z_J)_{J\subseteq\mathbb{N}}]$. And doing this way will give us a new proof of the result in the section 4.2.1. This intersection is known as the ideal of all polynomials vanishing on all *infinite rank-1 tensors*.

We follow the method in [BD10] to compute an equivariant Gröbner basis for $I(Y)$ as follows :

- We compute the Gröbner basis for ideal $I(Y)$ in each number n of variables $x_i, i \in [n]$ from $0, 1, 2, \cdots$.
- In each step, we compute primitive generators which are those not obtained by applying the action of $\mathrm{Subs}_<(\mathbb{N})$ to generators from previous steps.
- If we are lucky, it will be stable after finitely many steps, that means from then on, no primitive generators appears.

Luckily, our computations with `Singular` ([GPS05]) stop after the fourth step. We get 18 primitive generators as follows :

At $n = 0$, there is one generator : $t - z_\emptyset$.

$\mathbf{0} \to \mathbf{1}$: There is one primitive generator, namely $x_0 z_\emptyset - z_{\{0\}}$.

$\mathbf{1} \to \mathbf{2}$: There are 3 primitive generators :

- $[1 \to 2][1] := z_{\{0,1\}} z_\emptyset - z_{\{1\}} z_{\{0\}}$.
- $[1 \to 2][2] := x_0 z_{\{1\}} - z_{\{0,1\}}$.
- $[1 \to 2][3] := x_1 z_{\{0\}} - z_{\{0,1\}}$.

$\mathbf{2} \to \mathbf{3}$: There are 5 primitive generators :

- $[2 \to 3][1] := z_{\{0,2\}} z_{\{1\}} - z_{\{2\}} z_{\{0,1\}}$.
- $[2 \to 3][2] := z_{\{1,2\}} z_{\{0\}} - z_{\{2\}} z_{\{0,1\}}$.
- $[2 \to 3][3] := z_{\{0,1,2\}} z_{\{0\}} - z_{\{0,2\}} z_{\{0,1\}}$.
- $[2 \to 3][4] := z_{\{0,1,2\}} z_{\{1\}} - z_{\{1,2\}} z_{\{0,1\}}$.
- $[2 \to 3][5] := z_{\{0,1,2\}} z_{\{2\}} - z_{\{1,2\}} z_{\{0,2\}}$.

$\mathbf{3} \to \mathbf{4}$: There are 8 primitive generators :

- $[3 \to 4][1] := z_{\{0,1,3\}} z_{\{0,2\}} - z_{\{0,3\}} z_{\{0,1,2\}}$.
- $[3 \to 4][2] := z_{\{0,1,3\}} z_{\{1,2\}} - z_{\{1,3\}} z_{\{0,1,2\}}$.
- $[3 \to 4][3] := z_{\{0,2,3\}} z_{\{0,1\}} - z_{\{0,3\}} z_{\{0,1,2\}}$.
- $[3 \to 4][4] := z_{\{0,2,3\}} z_{\{1,2\}} - z_{\{2,3\}} z_{\{0,1,2\}}$.
- $[3 \to 4][5] := z_{\{0,2,3\}} z_{\{1,3\}} - z_{\{2,3\}} z_{\{0,1,3\}}$.
- $[3 \to 4][6] := z_{\{1,2,3\}} z_{\{0,1\}} - z_{\{1,3\}} z_{\{0,1,2\}}$.
- $[3 \to 4][7] := z_{\{1,2,3\}} z_{\{0,2\}} - z_{\{2,3\}} z_{\{0,1,2\}}$.
- $[3 \to 4][8] := z_{\{1,2,3\}} z_{\{0,3\}} - z_{\{2,3\}} z_{\{0,1,3\}}$.

Set B to be the set of these 18 generators. We will prove that B is the equivariant Gröbner basis for the ideal $I(Y)$.

Firstly, we observe that the $\mathrm{Subs}_<(\mathbb{N})$-orbits of elements in B give rise to the following reduction laws :

- If one has a monomial of the form $x_i z_J$ with $i \notin J$, then we can shift it to $z_{J\cup\{i\}}$. This comes from elements $[1 \to 2][2]$ and $[1 \to 2][3]$.
- If one has a monomial of the form $z_{I\cup J} z_K$ with $I <_{lex} J, (I \cup J) \cap K = \emptyset$, then we can shift I to the second index, that means we shift $z_{I\cup J} z_K$ to $z_J z_{I\cup K}$, provided that $I \cup K <_{lex} J$. This is justified by elements $[1 \to 2][1], [2 \to 3][1], [2 \to 3][2]$.
- If one has a monomial of the form $z_{I\cup J\cup K} z_{J\cup L}$, then we shift the smaller one I to the second index, that means we transform $z_{I\cup J\cup K} z_{J\cup L}$ into $z_{J\cup K} z_{I\cup J\cup L}$. This is justified by the remaining elements of B.

In particular, if we consider $J = \emptyset$ in the third case, then we have the second case. Hence, we have only two ways of reduction laws.

Now, we consider the polynomial of the form $z_S z_T$, where $S >_{\text{lex}} T$ in general :

- If $S \cap T = \emptyset$, then $z_S z_T$ can be reduced via B unless S has only one element (by the second rule).
- If $S \cap T = J \neq \emptyset$, then it can be reduced by the third rule via B unless $S \setminus T$ has only one element.

Hence the monomial $z_S z_T$ where $S >_{\text{lex}} T$ is not reduced if and only if $S \setminus T$ has only one element. So, standard monomials are of the form $z_{S_1} z_{S_2} \cdots z_{S_k}$ where $S_1 \geq_{\text{lex}} S_2 \geq_{\text{lex}} \cdots \geq_{\text{lex}} S_k$, and $|S_i \setminus S_j| = 1$ for all $i > j$ such that $S_i >_{lex} S_j$.

We have the following lemma :

Lemma 4.2 *Let P a finite multi-set in $\mathbb{N}$, and let $k \geq max\{multiplicity\ of\ x_i : x_i \in P\ \}$. Then there is a unique standard monomial (relative ti $Subs_<(\mathbb{N})B$) $z_{S_1} z_{S_2} \cdots z_{S_k}$ satisfying*

$\cup_{i=1}^{k} S_i = P$.

For example, if $P = \{0, 1, 1, 1, 1, 2, 2, 2, 3, 3, 3, 3, 3\}$, then we have a unique standard monomial for $k = 5$:

$$z_{321}z_{321}z_{32}z_{310}z_{31} = z_{321}^2 z_{32}z_{310}z_{31}$$

If we take $k = 7$ and the same multi-set P, we have also a unique standard monomial as follows :

$$z_{32}z_{31}z_{31}z_3z_3z_{210}z_{21} = z_{32}z_{31}^2 z_3^2 z_{210}z_{21}$$

Proof. So for the chosen k, we need k variables z appearing in our monomial $z_{S_1}z_{S_2}\cdots z_{S_k}$. Assume that we index elements in P by $x_1 > x_2 > \cdots > x_m$ with n_i is the multiplicity of x_i. We create our monomial by the following steps :

- Step 1 : we distribute the copies of x_1 over $S_1, S_2, \cdots, S_{n_1}$ beginning from the left. So, we have two blocks : block 1 is $S_1 = \{x_1\}, S_2 = \{x_1\}, \cdots, S_{n_1} = \{x_1\}$ and block 2 is $S_{n_1+1} = \cdots = S_k = \emptyset$.
- Step 2 : we distribute the x_2 into the second block, from the left to the right. There are three possibilities :
  - Case 1 : $k - n_1 > n_2$, hence we divide the block 2 into 2 blocks, where the first block includes S_i's containing x_2, and the second block includes empty S_j's. So we have now three blocks.
  - Case 2 : $k - n_1 = n_2$ we distribute all the x_2's into block 2.
  - Case 3 : $k - n_1 < n_2$, hence after distributing every x_2 into the second block, we have some x_2's left. So we distribute them into the first block from the left to the right. At this time, we have 3 blocks as well : first one includes S_j's containing x_1, x_2, the second one includes S_j's containing x_1, and the last one includes S_j's containing x_2.
- Step 3 : So now we continue distributing x_3 starting at the last block first, from the left to the right. If there are some remained, we distribute them into the next rightmost block.
- We stop after step m corresponding to the distribution of x_m into our blocks.

We have to follow the procedure above to guarantee the rule of standard monomial that $|S_i \setminus S_j| = 1$ for all $i > j$. For example, if in the second step, we distribute the x_2's into the first blocks from the left to the right, then $|S_1 \setminus S_m| = 2$ at that time, which does not satisfy the condition of a standard monomial. And this makes our monomial unique. $\square$

To illustrate for the algorithm above, we take the previous example : suppose that we consider the multi-set $P = \{0, 1, 1, 1, 1, 2, 2, 2, 3, 3, 3, 3, 3\}$, with multiplicity of 3 is 5, and we want to *arrange* them into $k = 7$ *blocks*, we follow the algorithm as the following :

- Firstly, we arrange 3 in each box from the left to the right, i.e., we have

$$| \ 3 \ | \ 3 \ | \ 3 \ | \ 3 \ | \ 3 \ || \ . \ | \ . \ |$$

- Then now, we have two blocks, one contains 3 in each box, and one is empty in each box. We arrange 2 into the second block first. Since multiplicity of 2 is 3, then we return some remain 2 into the first block :

$$| \ 32 \ || \ 3 \ | \ 3 \ | \ 3 \ | \ 3 \ || \ 2 \ | \ 2 \ |$$

- So now, we have three blocks. We continue as follows :

$$| \ 32 \ || \ 31 \ | \ 31 \ || \ 3 \ | \ 3 \ || \ 21 \ | \ 21 \ |$$

- We have now 4 blocks. The last step :

$$| \ 32 \ || \ 31 \ | \ 31 \ || \ 3 \ | \ 3 \ || \ 210 \ || \ 21 \ |$$

Hence, finally, we get the element

$$z_{32}z_{31}z_{31}z_3z_3z_{210}z_{21} = z_{32}z_{31}^2z_3^2z_{210}z_{21}$$

With the result of this lemma, we see that two monomials $z_{S_1}^{e_1}z_{S_2}^{e_2}\cdots z_{S_k}^{e_k}$ and $z_{T_1}^{r_1}z_{T_2}^{r_2}\cdots z_{T_m}^{r_m}$ with $\bigcup_{i=1}^k e_iS_i = \bigcup_{j=1}^m r_jT_j$, can be reduced to a unique stardard monomial. Hence, their difference will reduce to zero.

In particular, for all $b_0, b_1 \in B$, $\sigma, \pi \in \mathrm{Sub}_<(\mathbb{N})$, the $S-$polynomial $S(\sigma b_0, \pi b_1)$ is the difference of two polynomials $z_{S_1}^{e_1}z_{S_2}^{e_2}\cdots z_{S_k}^{e_k}$ and $z_{T_1}^{r_1}z_{T_2}^{r_2}\cdots z_{T_m}^{r_m}$ with $\bigcup_{i=1}^k e_iS_i = \bigcup_{j=1}^m r_jT_j$. Since they reduce to a same standard monomial, hence their difference reduces to zero, which means that $S-$polynomial $S(\sigma b_0, \pi b_1)$ reduces to zero by B. Therefore, by theorem 2.29, we have proved that :

Theorem 4.3 $\boxed{\textit{B is an equivariant Gröbner basis for the ideal } I(Y).}$

We could compute an equivariant Gröbner basis for the ideal $I(Y)$, since $\mathrm{Subs}_<(\mathbb{N})$ has the following properties :

- The natural lexicographic order on (monomials in) the z_I is compatible with this submonoid, and so is the lexicographic order on (monomials in) the x_i, and hence so is a natural elimination order.
- For any two monomials m, m' in the x_i and z_I, we have $\gcd(\sigma m, \sigma m') = \sigma \gcd(m, m')$, which we need for the Buchberger criterion (additional condition **EGB1**).

Moreover, we have the following lemma (additional condition **EGB2**), which ensures that for any pair of polynomials, only finitely many $S-$polynomials need to be considered, so that we could in principal run the equivariant Buchberger algorithm.

Lemma 4.4 *For any two polynomials p and q in $k[t, (x_i), (z_I)]$, the set Subs$_<(\mathbb{N})p \times$ Subs$_<(\mathbb{N})q$ can be written as the union of a finite number of Subs$_<(\mathbb{N})-$orbits, relative to the diagonal action of Subs$_<(\mathbb{N})$.*

Proof. Let $\sigma', \tau' \in \mathrm{Subs}_<(\mathbb{N})$ and let m and n be the largest indices appearing in a variable of p and a variable of q, respectively. Then $\sigma' p$ and $\tau' q$ depend only on the *trunstitutions* $\sigma := (\sigma'_1, \cdots, \sigma'_m)$ and $\tau := (\tau'_1, \cdots, \tau'_n)$, respectively. We will show that there are trunstitutions $\pi, \tau'', \sigma'' \in \mathrm{Subs}_<(\mathbb{N})$ such that $\pi \circ \tau'' = \tau$ and $\pi \circ \sigma'' = \sigma$, and that the pair (τ'', σ'') only takes finitely many values as τ and σ vary.

We have that σ gives rise to a natural partition of $S := \bigcup_i \sigma_i \bigcup_j \tau_j$ into the parts $\sigma_1, \sigma_2, \cdots, \sigma_m, S \setminus \cup_i \sigma_i$ and similarly for τ. Let $\pi_1, \cdots, \pi_k$ be the parts of the coarsest common refinement of these partitions, with $\max \pi_1 < \max \pi_2 < \cdots < \max \pi_k$. Now for every $i \in [m]$ the set σ_i equals $\cup_{r \in \sigma_i''} \pi_r$ for some unique, non-empty subset σ''_i of $[k]$; define the subsets τ''_j similarly.

We claim that $\sigma'' := (\sigma''_1, \cdots, \sigma''_m)$ and $\tau'' := (\tau''_1, \cdots, \tau''_n)$ are trunstitutions. Indeed, the sets σ''_i are disjoint by construction, and $\max \sigma''_i$ is the unique r for which π_r contains $\max \sigma_i$; infact, necessarily as its maximum $\max \pi_r$. This implies that $\max \sigma''_i < \max \sigma''_{i+1}$, since $\max \sigma_i < \max \sigma_{i+1}$. The same argument applies for τ.

Furthermore, by construction $\pi := (\pi_1, \cdots, \pi_k)$ is also a trunstitution. And we have

$$(\pi \circ \sigma'')_i = \cup_{j \in \sigma''_i} \pi_j = \sigma_i$$

Hence $\pi \circ \sigma'' = \sigma$. Similarly, we have $\pi \circ \tau'' = \tau$.

Since there are $m+1$ parts for partition of σ and $n+1$ parts for partition of τ, hence k must be bounded by $(m+1)(n+1)$, and we have $\sigma''_i, \tau''_j \subseteq [k]$. Hence, there are only finitely many values for (σ'', τ''), which completes our proof. $\square$

Chapter 5

Approaches to Rank-2 tensors

For now, we consider the mapping

$$\begin{array}{rcl}\varphi : (k \times k^{\mathbb{N}}) \times (k \times k^{\mathbb{N}}) & \to & \mathcal{A} \\ ((t, (x_i)_{i\in\mathbb{N}}), (s, (y_i)_{i\in\mathbb{N}})) & \mapsto & (z_I)_{I\subset\mathbb{N}}\end{array}$$

where

$$z_I := t\prod_{i\in I} x_i + s\prod_{i\in I} y_i$$

for finite subset $I \subset \mathbb{N}$. We would like to know the image $\mathrm{Im}\varphi$, so we need to compute the ideal $J(Y)$ generated by elements in the polynomial ring $k[t, s, (x_i)_{i\in\mathbb{N}}, (y_i)_{i\in\mathbb{N}}, (z_I)_{I\subset\mathbb{N}}]$

$$z_I - t\prod_{i\in I} x_i - s\prod_{i\in I} y_i$$

The intersection $I(Y)$ of the ideal $J(Y)$ with the polynomial ring $k[(z_I)_{I\subset\mathbb{N}}]$ is exactly the ideal of the image $\mathrm{Im}\varphi$.
We have some approaches as follows

5.1. The substitution monoid approach

Let $\mathrm{Subs}_<(\mathbb{N})$ act on $k[t, s, (x_i)_{i\in\mathbb{N}}, (y_i)_{i\in\mathbb{N}}, (z_I)_{I\subset\mathbb{N}}]$ as in the chapter 4, i.e., for every $\sigma = (\sigma_1, \sigma_2, \cdots) \in \mathrm{Subs}_<(\mathbb{N})$, we have

$$\sigma.t = t, \quad \sigma.s = s$$

$$\sigma.x_i = \prod_{j \in \sigma_i} x_j, \quad \sigma.y_k = \prod_{i \in \sigma_k} y_i, \quad \text{and} \quad \sigma.z_I = z_{\cup_{i \in I} \sigma_i}$$

This makes $J(Y)$ a $\mathrm{Subs}_<(\mathbb{N})-$stable ideal. Hence, we may apply the method we did in chapter 4 to compute an *equivariant Gröbner basis* for the ideal $J(Y)$. However, it may take a lot of time for running our program by computer.

5.2. The highest weight vector approach

By reductive group theory, it is sufficient to know the highest weight vectors in the ideal $J(Y)$. Indeed, these highest weight vectors generate $J(Y)$ as a $G-$module, where $G = \bigcup_n GL_n$.

Definition 5.1 Denote $D = \{D_i, i \in \mathbb{N}\}$ the set of all *differentials* in which any one D_i of them acts on the polynomial ring $k[t, s, (x_i)_{i \in \mathbb{N}}, (y_i)_{i \in \mathbb{N}}, (z_I)_{I \subset \mathbb{N}}]$ as follows :

$$D_i.t = D_i.z = 0$$

$$D_i x_i = D_i y_i = 1, \quad \text{and} \quad D_i x_j = D_i y_j = 0 \quad \text{for } i \neq j$$

$$D_i z_I = \begin{cases} z_{I \setminus \{i\}} & \text{If } i \in I \\ 0 & \text{Otherwise} \end{cases}$$

The set D with the composition operation becomes a (commutative monoid).

Definition 5.2 (Highest weight vector) With the action of D on the polynomial ring $R = k[t, s, (x_i)_{i \in \mathbb{N}}, (y_i)_{i \in \mathbb{N}}, (z_I)_{I \subset \mathbb{N}}]$, the highest weight vector is element $\alpha \in R$ satisfying $D_i.\alpha = 0$.

Example 5.3 In the chapter 4, the highest weight vectors in the set B are $t - z_\emptyset, x_0 z_\emptyset - z_{\{0\}}, z_{\{0,1\}} z_\emptyset - z_{\{1\}} z_{\{0\}}, x_1 z_{\{0\}} - z_{\{0,1\}}$ and $z_{\{0,1,2\}} z_{\{0\}} - z_{\{0,2\}} z_{\{0,1\}}$.

We have a natural embedding

$$\begin{aligned} i : k \times k^{\mathbb{N}} \times k \quad &\hookrightarrow \quad k \times k^{\mathbb{N}} \times k \times k^{\mathbb{N}} \\ (t, (x_i)_{i \in \mathbb{N}}, s) \quad &\mapsto \quad (t, (x_i)_{i \in \mathbb{N}}, s, 0) \end{aligned}$$

And consider the following mapping

$$\begin{aligned} \phi : k \times k^{\mathbb{N}} \times k &\to \mathcal{A} \\ (t, (x_i)_{i \in \mathbb{N}}, s) &\mapsto (z_I)_{I \subset \mathbb{N}} \end{aligned}$$

where z_I is determined by

$$z_I = \begin{cases} t \prod_{i \in I} x_i & \text{If } I \neq \emptyset \\ t + s & \text{If } I = \emptyset \end{cases}$$

We have the following result

Lemma 5.4 *The $D-$invariants in the ideal $I(Im(\phi))$ are exactly the $D-$invariants in the ideal $I(Im\varphi)$.*

An equivariant Gröbner basis B for the ideal $I = I(Im(\varphi))$ can be computed exactly as in chapter 4. However, there is one problem in this approach : the set of highest weight vectors $I^{hw} = \langle B \rangle^{hw}$ is different from $\langle B^{hw} \rangle$. At present, we do not know how to efficiently compute (generators of) I^{hw}.

Chapter 6

Conclusion

In this chapter, we will give a short summary on what we have done in this thesis and some remaining open questions :

- First, we investigate the Noetherianity of the polynomial ring $R = A[x_1, x_2, \cdots]$ under the $\mathrm{Sym}(\mathbb{N})-$action and $\Pi-$action in chapter 3. Moreover, we give a number of examples in which R is $\Pi-$Noetherian and not $\Pi-$Noetherian. In particular, we give a classification of continuous actions of Π on $A = k[z]$ and prove the fact that for each of them the ring $R = A[x_1, x_2, \cdots]$ is $\Pi-$Noetherian.

  However, we still have one main problem as follows :

  Question 1 *Does there exist a continuous actions of* Π *on any* A *such that* A *is* $\Pi-$*Noetherian but the polynomial ring* $R = A[x_1, x_2, \cdots]$ *is* **NOT** $\Pi-$*Noetherian.*

- Second, we give a computational proof for the rank-1 tensors problem by computing the equivariant Gröbner basis in chapter 4. This method might work for the rank-2 tensors problem as described in chapter 5. However, we need more time for using such method in computations since it seems not to stop in a short time for some first small number of variables.

  Moreover, we give an idea to approach the rank-2 tensors problem by looking for the highest weight vectors in the ideal $I(Y)$. But we meet one problem (described in chapter 5) and I also have not enough time to investigate the SL_2-theory to deal with this problem.

Question 2 *As we said in chapter 5, we would like to start with the* $Subs_{<}(\mathbb{N})-$*stable ideal in the polynomial ring* $k[t, s, (x_i)_{i\in\mathbb{N}}, (z_I)_{I\subset\mathbb{N}}]$ *generated by the polynomials*

$$z_I - \left(t \prod_{i\in I} x_i + s \prod_{i\in I} y_i \right)$$

and then we take the intersection with the ring $k[(z_I)_{I\subset\mathbb{N}}]$. *Could we obtain the* $3\times 3-$*minors result as the* $2\times 2-$*minors result in chapter 4 ?. Precisely, we hope that the result is generated by all polynomials of the form*

$$\sum_{\pi\in S_3} sgn(\pi\ ) z_{I_1\cup J_{\pi(1)}} z_{I_2\cup J_{\pi(2)}} z_{I_3\cup J_{\pi(3)}}$$

for the subsets $I_1, I_2, I_3, J_1, J_2, J_3 \subset \mathbb{N}$ *such that* $(I_1 \cup I_2 \cup I_3) \cap (J_1 \cup J_2 \cup J_3) = \emptyset$.

In fact, this is the famous *GSS (Garcia-Stillman-Sturmfels) Conjecture* ([GSS05]) .

In hope in the near future, we may solve this question with our two potential methods as setting in chapter 5.

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