Amœbas and Coamœbas in Dimension 2

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Contents

Chapter 1

Tropical Mathematics

The purpose of this chapter is to introduce the basic tropical[1] notions which are necessary prerequisites. It is not meant to be a detailed summary of tropical algebraic geometry. For more details on the topic, see [**MS09**], [**RGST03**] and [**IMS07**], for example.

1.1 Tropical "Forests"

In this section we introduce the tropical arithmetic and different settings in which it takes place.

1.1.1 The Tropical Semiring

Let $\mathcal{T} := \mathbb{R} \cup \{-\infty\}$ and define $\forall\, a, b \in \mathcal{T}$

$$\begin{aligned} a \boxplus b &:= \max(a, b) \\ a \boxtimes b &:= a + b \end{aligned} \tag{1.1}$$

$(\mathcal{T}, \boxplus, \boxtimes)$ satisfies the axioms of a *semiring* with identity elements being $-\infty$ for tropical addition and 0 for tropical multiplication. Indeed, $\forall\, a \in \mathcal{T}$

$$\begin{aligned} a \boxplus -\infty = -\infty \boxplus a = \max(-\infty, a) = a \\ a \boxtimes 0 = 0 \boxtimes a = 0 + a = a \end{aligned} \tag{1.2}$$

Note that $\forall\, a \in \mathcal{T}$ $a \boxplus a = a$ and that if $a \in \mathbb{R}$, there is no $b \in \mathcal{T}$ such that $a \boxplus b = -\infty$. In other words, addition is idempotent and only $-\infty$ has an additive inverse, namely itself. However, every nonzero element in $\mathcal{T}$ has a multiplicative inverse:

$$\forall\, a \in \mathcal{T} \backslash \{-\infty\}\ \exists\, a^{-1} := -a \ \text{ s.t. } \ a \boxtimes a^{-1} = a + (-a) = 0$$

This is why this structure is sometimes also called a *semifield*.

[1] The term "tropical" was apparently first used by several French mathematicians to honour Imre Simon, a Brazilian mathematician who was one of the pioneers of tropical algebra ([**SS09**]).

Tropical semiring $(\mathcal{T}, \boxplus, \boxtimes)$ can be viewed as a limit of a continous deformation of the semiring $(\mathbb{R}_{\geq 0}, +, \times)$. Namely, for any $h > 0$ define $\mathcal{T}_h := \mathbb{R} \cup \{-\infty\}$ and

$$\begin{aligned} a \boxplus_h b &:= h \log(e^{a/h} + e^{b/h}) \\ a \boxtimes_h b &:= a + b \end{aligned} \tag{1.3}$$

Semirings $\mathbb{R}_{\geq 0}$ and $\mathcal{T}_h$ are isomorphic via $\phi : \mathbb{R}_{\geq 0} \to \mathcal{T}_h,\ t \mapsto h \log t$. As h approaches 0, $\mathcal{T}_h$ degenerates into $\mathcal{T}$. This is known as *Litvinov-Maslov dequantization*[2].

Attempts have been made to resolve the issues of the idempotency and lack of additive inversion. We shall only briefly cover some of them in the following subsections.

1.1.2 The Extended Tropical Semiring

This approach is due to Izhakian (see [**Izh08**] and [**Izh09**]). We begin by by gluing two copies $\mathcal{T} := \mathbb{R} \cup \{-\infty\}$ and $\mathcal{T}' := \mathbb{R}' \cup \{-\infty\}$ along $\{-\infty\}$. We denote the resulting set as $\mathcal{T}^\star$. This set is then equipped with a total order, extending the order from $\mathbb{R}$, in the following way:

$$\begin{aligned} &\forall\, x \in \mathcal{T}^\star \quad -\infty \prec x \\ &\forall\, a, b \in \mathbb{R} \quad a \leq b \;\Rightarrow\; a \prec b,\ a \prec b',\ a' \prec b,\ a' \prec b' \\ &\forall\, a \in \mathbb{R} \quad a \prec a' \end{aligned} \tag{1.4}$$

and with tropical operations, extending those from $\mathcal{T}$, in the following way:

$$\begin{aligned} &\forall\, x, y \in \mathcal{T}^\star \quad x \boxplus y = \begin{cases} \max_\prec(x, y) & \text{if } x \neq y \\ x' & \text{if } x = y \neq -\infty \end{cases} \\ &\forall\, a, b \in \mathbb{R} \quad a' \boxtimes b = a \boxtimes b' = a' \boxtimes b' = (a + b)' \end{aligned} \tag{1.5}$$

$(\mathcal{T}^\star, \boxplus, \boxtimes)$ is called the *extended tropical semiring* as it extends $(\mathcal{T}, \boxplus, \boxtimes)$. However, it is not idempotent as $\forall\, a \in \mathbb{R} \;\; a \boxplus a = a'$.

1.1.3 Tropical Hyperfields

These structures allow us to define a weak notion of a tropical additive inverse (but at a high price).

Definition 1.1 A set X with a multivalued operation $(x, y) \mapsto x + y$ is called a *(commutative) hypergroup* if:

(1) the operation $+$ is associative and commutative;

(2) $\exists\, 0 \in X$ s.t. $\forall\, x \in X \;\; 0 + x = x$;

(3) $\forall\, x \in X \;\; \exists! \; -x \in X$ s.t. $0 \in x + (-x)$.

[2] See [**Viro10**] and references therein.

Example 1.1 Define addition on $\mathbb{R}_{\geq 0}$ as

$$a \triangledown b := [|a-b|, a+b]$$

In other words, the sum of a and b is the set of all c such that there exists a triangle with sides a, b and c. In the ultrametric case, define addition as

$$a \triangledown b := \begin{cases} \max(a,b) & \text{if } a \neq b \\ [0,a] & \text{if } a = b \end{cases}$$

Both additions are commutative and associative in the sense that

$$(a \triangledown b) \triangledown c = \bigcup_{\alpha \in a \triangledown b} \{\alpha \triangledown c\} = \bigcup_{\beta \in b \triangledown c} \{a \triangledown \beta\} = a \triangledown (b \triangledown c).$$

Furthermore, 0 serves as the neutral element and every element is its own unique inverse, i.e.

$$\forall\, a \in \mathbb{R}_{\geq 0} \;\; 0 \in a \triangledown a$$

Thus, $(\mathbb{R}_{\geq 0}, \triangledown)$ is a hypergroup. Along with standard multiplication (which is distributive over $\triangledown$) $\mathbb{R}_{\geq 0}$ obtains a structure that is known as a *hyperfield*. We can then apply the extended logarithm map $\log : \mathbb{R}_{\geq 0} \to \mathcal{T} := \mathbb{R} \cup \{-\infty\}$ to obtain a *tropical hyperfield* $(\mathcal{T}, \widetilde{\boxplus}, \boxtimes)$.

1.1.4 The Complex Tropical Hyperfield

It is not immediate how one could carry the ideas of subsection 1.1.1 to $\mathbb{C}$. One may attempt to singularize $(\mathbb{C}, +, \times)$ via the Litvinov-Maslov process. Namely, $\forall\, h > 0$ and $\forall\, z, w \in \mathbb{C}$ let

$$\begin{aligned} z \boxplus_h w &:= S_h^{-1}(S_h(z) + S_h(w)) \\ z \boxtimes_h w &:= z + w \end{aligned}$$

where $S_h : \mathbb{C} \leftrightarrow \mathbb{C}$ is defined by

$$z \mapsto \begin{cases} |z|^{\frac{1}{h}} \dfrac{z}{|z|} & \text{if } z \neq 0 \\ 0, & \text{if } z = 0 \end{cases}$$

Let $z \boxplus_0 w := \lim\limits_{h \mapsto 0+} (z \boxplus_h w)$. This defines a tropical addition on $\mathbb{C}$ and one can check that

$$z \boxplus_0 w = \begin{cases} z & \text{if } |z| > |w| \\ w & \text{if } |z| < |w| \\ 0 & \text{if } z = -w \\ |z| \dfrac{z+w}{|z+w|} & \text{if } |z| = |w| \text{ and } z \neq -w \end{cases}$$

The problem, however, is that this addition is not continous or associative. To overcome the issue, we again introduce the "hyper"-structures. Let $z, w \in \mathbb{C}$ and define

$$z \widetilde{\boxplus} w := \begin{cases} \{z\} & \text{if } |z| > |w| \\ \{w\} & \text{if } |w| < |w| \\ \{re^{\theta i} \,:\, \theta \in [\alpha, \beta]\} & \text{if } z = re^{\alpha i},\, w = re^{\beta i},\, \beta - \alpha < \pi \\ \{\zeta \in \mathbb{C} \,:\, |\zeta| \leq |z|\} & \text{if } z = -w \end{cases}$$

The only part that is not immediately clear in showing that $(\mathbb{C}, \widetilde{\boxplus})$ is a hypergroup is associativity. We will not reproduce the proof here (see Lemma 2.B. in [**Viro10**]). Restricting this operation to the reals makes $(\mathbb{R}, \widetilde{\boxplus})$ a hypergroup as well. With the standard single-valued tropical multiplication

$$z \boxtimes w := z + w,$$

$(\mathbb{C}, \widetilde{\boxplus}, \boxtimes)$ and $(\mathbb{R}, \widetilde{\boxplus}, \boxtimes)$ become *tropical hyperfields.*

Tropical algebra and geometry can be studied in these different settings with the hope of generalizing more results from standard algebra and geometry. Certain results such as *Bézout's Theorem* (see [**St02**]), *Degree-Genus Formula*, *Riemann-Roch Theorem* (see [**Mikh06**]) and *Group Law of Cubics* have been established for tropical curves. We shall only be concerned with the tropical semiring $(\mathcal{T}, \boxplus, \boxtimes)$ as defined in subsection 1.1.1.

1.2 Tropical Polynomials

A tropical polynomial in n variables is a formal sum of the form

$$\mathfrak{p} := \bigboxplus_{\alpha \in \mathbb{N}^n} c_\alpha \boxtimes x_1^{\boxed{\alpha_1}} \boxtimes \cdots \boxtimes x_n^{\boxed{\alpha_n}} \tag{1.6}$$

where $c_\alpha \in \mathcal{T}$ and all but finitely many equal $-\infty$ and $x_i^{\boxed{\alpha_i}} := \overbrace{x_i \boxtimes \cdots \boxtimes x_i}^{\alpha_i \text{ times}}$. Putting $x = (x_1, \ldots, x_n)$, we can also use the alternative notation $c_\alpha \boxtimes x^{\boxed{\alpha}}$ for monomials. To each polynomial $\mathfrak{p}$ as in (1.6) one associates a *tropical polynomial function*

$$p : \mathcal{T}^n \to \mathcal{T},\ x \mapsto \bigboxplus_{\alpha \in \mathbb{N}^n} c_\alpha \boxtimes x^{\boxed{\alpha}} = \max_{\alpha \in \mathbb{N}^n}(c_\alpha + \langle \alpha, x \rangle) \tag{1.7}$$

As with the standard definitions, our definitions can be extended to those of Laurent polynomials and Laurent polynomial functions. We can allow $\alpha \in \mathbb{Z}^n$ provided that we never try to evaluate at $-\infty$. A *tropical Laurent polynomial function* is a function of the form

$$p : (\mathcal{T} \backslash \{-\infty\})^n = \mathbb{R}^n \to \mathcal{T},\ x \mapsto \max_{\alpha \in \mathbb{Z}^n}(c_\alpha + \langle \alpha, x \rangle) \tag{1.8}$$

and a *tropical Laurent polynomial* is a formal sum

$$\mathfrak{p} := \bigboxplus_{\alpha \in \mathbb{Z}^n} c_\alpha \boxtimes x^{\boxed{\alpha}}. \tag{1.9}$$

Note that p is a convex function. Every tropical (Laurent) polynomial function is not representable in a unique way as (1.8). However, it admits a unique representation of this form such that the coefficients c_α are maximal, i.e. one can attach to it a tropical polynomial. Conversely, given a tropical polynomial, it admits a unique evaluation function as in (1.8), where the coefficients c_α are maximal.

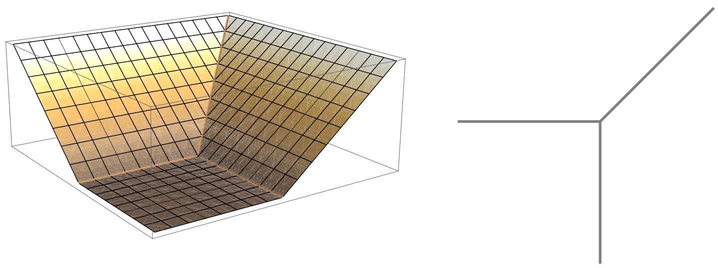
1.3 Tropical Hypersurfaces

The notion of a zero-locus from standard geometry cannot be generalized to the tropical setting because for any Laurent polynomial function p there is no $x \in \mathbb{R}^n$ such that $p(x) = -\infty$ unless p is identically $-\infty$. We must, therefore, take a different approach.

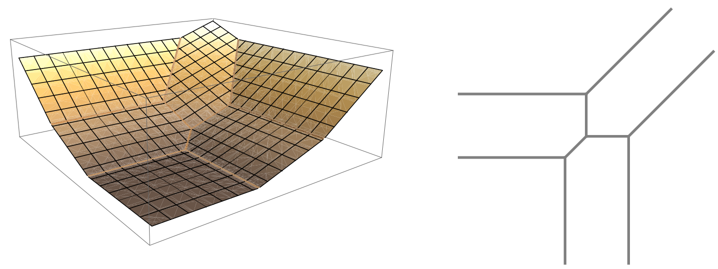
Definition 1.2 Let p be as in (1.8). The *tropical hypersurface* $\mathscr{V}_p^{\text{trop}}$ associated to p is the set of points at which p is not differentiable, i.e. the subset of $\mathbb{R}^n$ of those points at which the maximum is achieved for at least two distinct affine functions $c_\alpha + \langle \alpha, x \rangle$.

Of course, this definition is the same for all polynomials that have the same "evaluation" so it makes sense to define the hypersurface $\mathscr{V}_{\mathfrak{p}}^{\text{trop}}$ of a tropical Laurent polynomial $\mathfrak{p}$ as the hypersurface of its evaluation p.

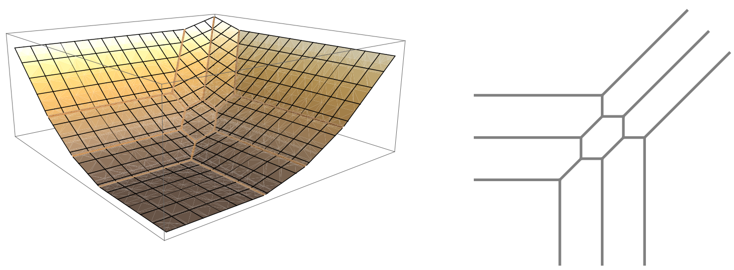
Note: If p is a monomial then $\mathscr{V}_p^{\text{trop}} = \emptyset$.



(a) Linear

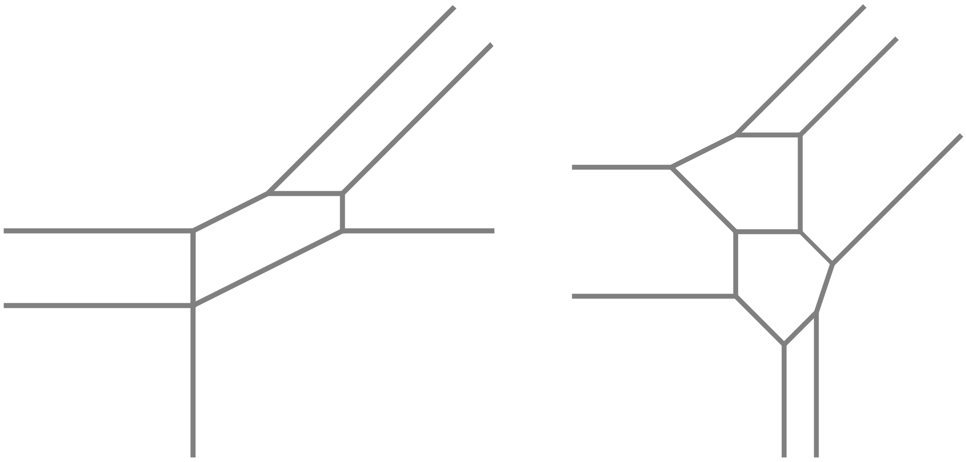


(b) Quadratic



(c) Cubic

Figure 1.1: Graphs of some tropical polynomial functions p and the corresponding curves $\mathscr{V}_p^{\text{trop}}$. The coefficients in these examples are symmetric and chosen so that every affine function is represented.



(a) Cubic (b) Quartic

Figure 1.2: Two more examples of tropical curves

1.4 Legendre-Fenchel Transform

Definition 1.3 Let $f : \mathbb{R}^n \to [-\infty, \infty]$ be an arbitrary function. Its *Legendre-Fenchel transform* (a.k.a. *the convex conjugate*) is the following function, defined on the dual space $(\mathbb{R}^n)^* \cong \mathbb{R}^n$ of $\mathbb{R}^n$:

$$\xi \in \mathbb{R}^n \mapsto \check{f}(\xi) := \sup_{x \in \mathbb{R}^n} (\langle \xi, x \rangle - f(x)) \in [-\infty, \infty] \tag{1.10}$$

It is immediate that $f \leq g$ implies $\check{f} \geq \check{g}$ and therefore $\check{\check{f}} \leq \check{\check{g}}$. Furthermore, we have $\check{\check{f}} \leq f$, with equality holding if and only if one of the following is true:

(1) f is a proper[3], lower semi-continuous[4], convex function

(2) $f \equiv -\infty$

(3) $f \equiv \infty$

In convex analysis, this is known as the *Fenchel-Moreau theorem* (see [**BL06**]). Legendre-Fenchel transform plays an important role for us because of the following theorem.

Theorem 1.1 *Let p be a tropical Laurent polynomial function. Then p can be represented as*

$$p(x) = \max_{\alpha \in \mathbb{Z}^n} (-\check{p}(\alpha) + \langle \alpha, x \rangle) \tag{1.11}$$

Moreover, the coefficients $-\check{p}(\alpha)$ are maximal over all possible representations of p.

Proof Assume $p \not\equiv -\infty$ and let

$$p(x) = \max_{\alpha \in \mathbb{Z}^n} (c_\alpha + \langle \alpha, x \rangle) \tag{1.12}$$

be any representation of p. Define $f : \mathbb{R}^n \to [-\infty, \infty]$ as

$$f(x) = \begin{cases} -c_x & \text{if } x \in \mathbb{Z}^n \\ \infty & \text{if } x \in \mathbb{R}^n \setminus \mathbb{Z}^n \end{cases}$$

[3]i.e. f is not identically ∞ and never takes value $-\infty$.

[4]i.e. $\{f > \alpha\}$ is open for every α.

so that p is the Legendre-Fenchel transform of f. Since p clearly satisfies condition (1) of the Fenchel-Moreau theorem, we have $p = \check{\check{p}} = \check{f}$ and $\check{p} = \check{\check{\check{p}}} = \check{\check{f}}$, i.e.

$$\begin{aligned}
\forall\, x \in \mathbb{R}^n \;\; p(x) = \check{\check{p}}(x) &:= \sup_{\xi\in\mathbb{R}^n} (\langle x, \xi\rangle - \check{p}(\xi)) \\
&= \sup_{\xi\in\mathbb{R}^n} (\langle x, \xi\rangle - \sup_{\eta\in\mathbb{R}^n} (\langle \xi, \eta\rangle - p(\eta))) \\
&= \sup_{\xi\in\mathbb{R}^n} (\langle x, \xi\rangle - \sup_{\eta\in\mathbb{R}^n} (\langle \xi, \eta\rangle - \max_{\alpha\in\mathbb{Z}^n}(\langle \alpha, \eta\rangle + c_\alpha))) \\
&= \sup_{\xi\in\mathbb{R}^n} (\langle x, \xi\rangle - \sup_{K_\alpha}(\sup_{\eta\in K_\alpha} (\langle \xi, \eta\rangle - \langle \alpha, \eta\rangle - c_\alpha))) \\
&= \sup_{\xi\in\mathbb{R}^n} (\langle x, \xi\rangle - \sup_{K_\alpha}(\sup_{\eta\in K_\alpha} (\langle \xi - \alpha, \eta\rangle - c_\alpha)))
\end{aligned} \tag{1.13}$$

Here K_α denotes the subset of $\mathbb{R}^n$ in which $p(\eta) = \langle \alpha, \eta\rangle + c_\alpha$. (We ignore those α for which $c_\alpha = -\infty$). If $\xi \neq \alpha \in \mathbb{Z}^n$, there is always a component K_α such that $\sup_{\eta\in K_\alpha}(\langle \xi - \alpha, \eta\rangle - c_\alpha)) = \infty$ (see figure 1.3). This leads to a contradiction as $p(x) = -\infty$ is impossible. We may, therefore, rewrite (1.13) with $\xi \in \mathbb{Z}^n$, i.e. (1.11) holds.

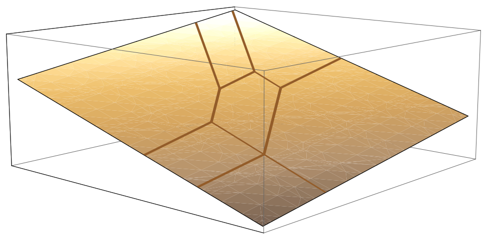


Figure 1.3: The affine plane $\langle \xi - \alpha, \eta\rangle - c_\alpha$ in $\mathbb{R}^n \times \mathbb{R}$. If it is not horizontal, i.e. $\xi = \alpha$, the second coordinate is clearly unbounded.

Moreover, we have $\forall\, \alpha \in \mathbb{Z}^n \;\; -\check{p}(\alpha) = -\check{\check{\check{f}}}(\alpha) \geq -f(\alpha) = c_\alpha$.

□

This implies that there is a bijective correspondence between tropical Laurent polynomials and tropical Laurent polynomial functions with maximal coefficients.

Chapter 2

Amœbas and Coamœbas

Before we move on to our main objects of interest, we shall introduce some definitions and notations.

$\mathbb{T}^n$ shall denote the standard complex torus $(\mathbb{C}^*)^n$ and $\mathbb{P}^{n-1}$ shall denote the standard complex projective space.

Definition 2.1 A *Laurent polynomial* in $\mathbb{T}^n$ is an expression of the form

$$f(z_1, \dots, z_n) = \sum_{\alpha \in \mathbb{Z}^n} c_\alpha z^\alpha$$

where $c_\alpha \in \mathbb{C}$ with $c_\alpha = 0$ for all but finitely many α and, as before, $z^\alpha := z_1^{\alpha_1} \dots z_n^{\alpha_n}$.

Definition 2.2 The *support* of f is the set $A_f := \{\alpha \in \mathbb{Z}^n \, : \, c_\alpha \neq 0\}$.

Definition 2.3 The *Newton polytope* Δ_f of f is the convex hull of the support of f.

Remark 2.1 Let $\mathrm{int}\Delta_f$ denote the interior of $\Delta_f \subset \mathbb{R}^n$. We will usually assume that $\mathrm{int}\Delta_f$ is non-empty. If it is not, f may be reduced to the case of a Laurent polynomial in less than n variables[1]. As with other polytopes, we can view Δ_f as an intersection of finitely many halfspaces:

$$\Delta_f = \bigcap_{k=1}^{N} \{x \in \mathbb{R}^n \, : \, \langle \mu_k, x \rangle \geq \nu_k\} \tag{2.1}$$

where $\nu_k \in \mathbb{Z}$ and $\mu_k \in \mathbb{Z}^n$ are primitive vectors normal to facets[2] of Δ_f and facing inwards. If Γ is a face of Δ_f, we shall denote its relative interior[3] by $\mathrm{relint}(\Gamma)$ and by f_Γ we shall denote the Laurent polynomial consisting of those monomials of f that correspond to the face Γ, i.e.

$$f_\Gamma = \sum_{\alpha \in \Gamma \cap A_f} c_\alpha z^\alpha \tag{2.2}$$

[1]In this case the powers are linearly dependent so we can write the general term as $z_1^{\alpha_1} \dots z_{n-1}^{\alpha_{n-1}} z_n^{c_1\alpha_1 + \dots + c_{n-1}\alpha_{n-1}} = (z_1 z_n^{c_1})^{\alpha_1} \dots (z_{n-1} z_n^{c_{n-1}})^{\alpha_{n-1}}$

[2]i.e. $(n-1)$-dimensional faces.

[3]i.e. interior of Γ viewed as a subset of the lowest dimensional hyperplane that contains it.

Definition 2.4 Let $A \subset \mathbb{R}^n$ and $a \in A$. The *normal cone* of A at a is a convex cone defined as

$$\mathcal{C}_a = \{y \in \mathbb{R}^n \,:\, \forall\, x \in A \;\; \langle y, x - a\rangle \leq 0\} \tag{2.3}$$

Equivalently, we can describe the normal cone by saying that a non-zero vector y belongs to $\mathcal{C}_a$ if and only if the following two conditions hold:

(1) y is a normal to a hyperplane that supports[4] A at a;

(2) y and A are on the opposite sides of that hyperplane.

Remark 2.2 We will be interested in normal cones of Δ_f. Notice that if Γ is a face of Δ_f and $\xi \in \Gamma$, the cone $\mathcal{C}_\xi$ is of dimension $n - \dim \Gamma$. In particular, $\mathcal{C}_\xi$ has a non-empty interior (in $\mathbb{R}^n$) precisely when ξ is a vertex of Δ_f, and it equals $\{0\}$ when $\xi \in \text{int}\Delta_f$. Also notice that $\mathcal{C}_\xi = \{x \in \mathbb{R}^n \,:\, \langle \xi, x\rangle = \max\limits_{\alpha \in \Delta_f} \langle \alpha, x\rangle\}$.

Definition 2.5 Let $r \geq 2$. A set $\{e^{x_1}, \ldots, e^{x_r}\} \subset \mathbb{R}_{>0}$ is called *lopsided* if for some j we have $e^{x_j} > \sum_{k \neq j} e^{x_k}$. If, particularly, for some $c \geq r-1$ we have $e^{x_j} > c \max_{k \neq j} e^{x_k}$, the set is said to be *c-superlopsided.* If $c = r-1$ the set is just said to be *superlopsided.*

Proposition 2.1 *$\{e^{x_1}, \ldots, e^{x_r}\} \subset \mathbb{R}_{>0}$ is **not** lopsided if and only if there exist $\theta_1, \ldots \theta_r \in \mathbb{R}$ such that $\sum_{j=1}^{r} e^{x_j + i\theta_j} = 0$.*

Proof
1) Assume $\{e^{x_1}, \ldots, e^{x_r}\}$ is not lopsided. Then $e^{x_r} \leq \sum_{j=1}^{r-1} e^{x_j}$. The triangle inequality implies $|\sum_{j=1}^{r-1} e^{x_j + i\theta_j}| \leq \sum_{j=1}^{r-1} e^{x_j}, \quad \forall\, (\theta_1, \ldots, \theta_{r-1})$. Since $|\sum_{j=1}^{r-1} e^{x_j+i\theta_j}|$ is continuous, it achieves value e^{x_r} for some $(\theta_1, \ldots, \theta_{r-1})$. Hence $\exists\, \theta_r$ such that $e^{x_r + i\theta_r} = -\sum\limits_{j=1}^{r-1} e^{x_j + i\theta_j}$.
2) Assume $\{e^{x_1}, \ldots, e^{x_r}\}$ is lopsided. Without loss of generality, assume $e^{x_r} > \sum_{j=1}^{r-1} e^{x_j}$. Then $\sum_{j=1}^{r} e^{x_j + i\theta_j} = 0$ is impossible because by triangle inequality we have $e^{x_r} = |\sum_{j=1}^{r-1} e^{x_j+i\theta_j}| \leq \sum_{j=1}^{r-1} e^{x_j} < e^{x_r}$.

□

Remark 2.3 The definitions of the support, Newton polytope and lopsidedness have an analogous definition in the tropical setting.

2.1 The Hypersurface Case

2.1.1 Amœbas

Definition 2.6 The *amœba* $\mathscr{A}_f$ of a Laurent polynomial f is the image of the zero set $\mathscr{V}_f = \{z \in \mathbb{T}^n \,:\, f(z) = 0\}$ under the map $\text{Log} : \mathbb{T}^n \to \mathbb{R}^n$ given by

$$(z_1, \ldots, z_n) \mapsto (\log|z_1|, \ldots, \log|z_n|)$$

Note that this is a slight abuse of notation as the amœba depends on the hypersurface $\mathscr{V}_f$, not on f. For example, $\mathscr{A}_{f^2} = \mathscr{A}_f$. Amœbas first appeared in [**GKZ94**] and were so named because of their typical shape (see examples below).

[4] A hyperplane H is said to support a set A if A lies entirely in one of the two closed half-spaces determined by H and has at least one point on H.

Remark 2.4 $\mathscr{A}_f$ is a closed set.

Example 2.1 One-dimensional amœbas are discrete sets and, as such, not of great interest to us.

Example 2.2 (Amœba of a line) Let $f(z,w) = z+w-1$ and consider $\mathscr{V}_f = \{(z,w) \in \mathbb{T}^2 : f(z,w) = 0\}$. A point $(z,w) \in \mathbb{T}^n$ belongs to $\mathscr{V}_f$ if and only if $\{1, |z|, |w|\}$ is not lopsided (cf. Proposition 2.1), i.e.

$$1 \le |z| + |w|, \quad |z| \le 1 + |w|, \quad |w| \le 1 + |z| \tag{2.4}$$

This implies that $\mathscr{A}_f$ is the image of $\{(u,v) \in \mathbb{R}^2_{>0} : 1 \le u+v,\ u \le 1+v,\ v \le 1+u\}$ under the map $(u,v) \mapsto (\log u, \log v)$ (see figures 2.1 and 2.2 below).

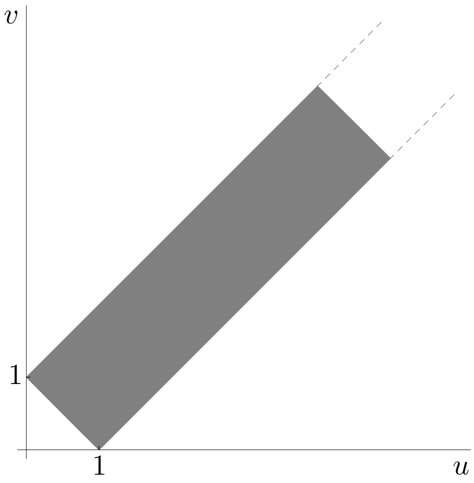


Figure 2.1: Image of $\mathscr{V}_f$ under the "norm" map $(z,w) \mapsto (|z|, |w|)$

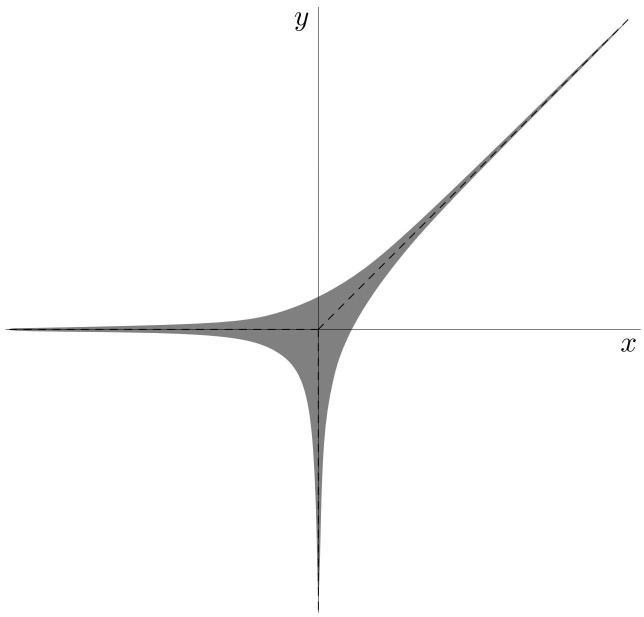


Figure 2.2: Amœba $\mathscr{A}_f$ of the line $\mathscr{V}_f$

Recall that the genus of a smooth projective curve of degree d is $(d-1)(d-2)/2$ and notice that when we take the Fermat curve $\mathscr{V}_g$ where $g(z,w) = z^d + w^d - 1$, the amœba $\mathscr{A}_g$ is a dilation of the amœba $\mathscr{A}_f$ considered above. This shows that some topological information is lost when moving to the world of amœbas.

Example 2.3 Let $f(z,w) = z^3w^4 + z^5 + 40z^3w^2 + z^3w + 80z^2w + 1$. Δ_f and $\mathscr{A}_f$ are shown in figure 2.3. Note that $\mathscr{A}_f$ has bounded components. We will come back to this example from time to time as it will provide us with relevant illustrations.

Consider a Laurent polynomial f. Rational function $1/f$ can be developed as $\sum\limits_{\alpha \in \mathbb{Z}^n} c'_\alpha z^\alpha$ where

$$c'_\alpha = \frac{1}{(2\pi i)^n} \int\limits_{\mathrm{Log}^{-1}(x)} \frac{dz_1 \wedge \cdots \wedge dz_n}{f(z) z_1^{\alpha_1+1} \ldots z_n^{\alpha_n+1}} = \frac{1}{(2\pi)^n} \int_{[0,2\pi]^n} \frac{e^{\langle \alpha, x+i\theta \rangle}}{f(e^{x+i\theta})}\, d\theta, \tag{2.5}$$

$x = (x_1, \ldots, x_n) \in \mathbb{R}^n \backslash \mathscr{A}_f$ and $e^z := (e^{z_1}, \ldots, e^{z_n})$. Coefficients c'_α do not depend on the connected component of $\mathbb{R}^n \backslash \mathscr{A}_f$ because if x and y are in the same component, $\mathrm{Log}^{-1}(x)$ and $\mathrm{Log}^{-1}(y)$ are homologous. Moreover, we have the following theorem.

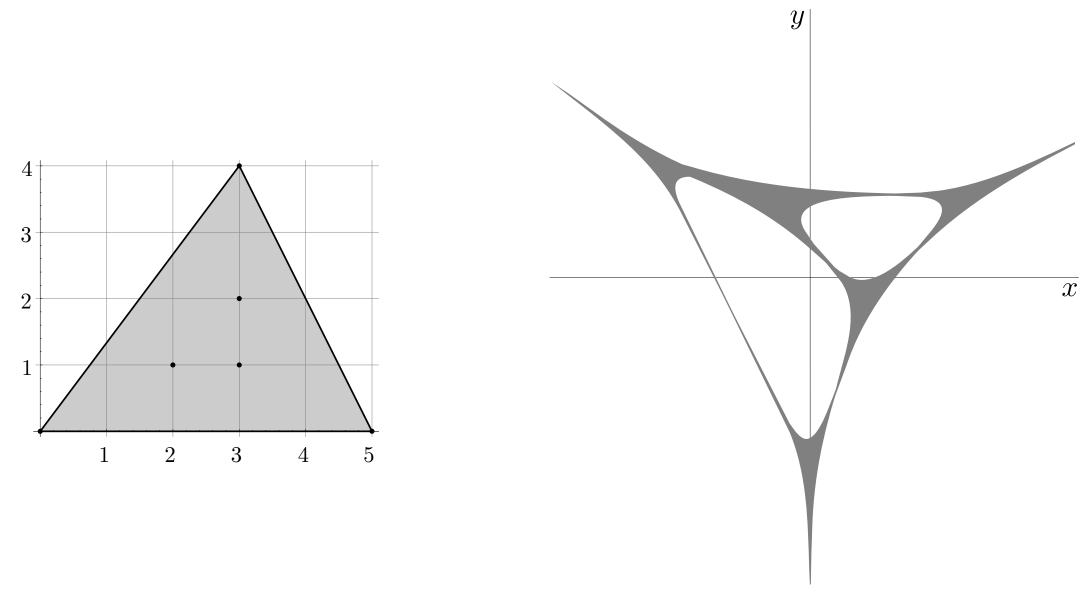


Figure 2.3: Newton polytope and amœba of $f(z,w) = z^3w^4 + z^5 + 40z^3w^2 + z^3w + 80z^2w + 1$

Theorem 2.1 [**GKZ94**] *Connected components of* $\mathbb{R}^n \backslash \mathscr{A}_f$ *are convex and in bijective correspondence with distinct Laurent expansions of* $1/f$ *(near the origin).*

Proof This follows from the following facts:

(1) $\mathbb{R}^n \backslash \mathscr{A}_f$ is open and so are its connected components.

(2) The domain of convergence of a Laurent series $f \in \mathbb{C}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$ is a *logarithmically convex*[5] *complete Reinhardt domain*[6].

(3) On any Reinhardt domain there is exactly one convergent Laurent series.

For more on this topic, see [**Horm90a**], [**KK83**], [**Kr92**].

□

We shall study components of the complement of the amœba in more detail in section 2.1.4.

2.1.2 Coamœbas

Definition 2.7 The *coamœba* $\mathscr{A}'_f$ of a Laurent polynomial f is the image of the zero set $\mathscr{V}_f = \{z \in \mathbb{T}^n : f(z) = 0\}$ under the map $\mathrm{Arg} : \mathbb{T}^n \to (\mathbb{S}^1)^n$ ($\mathbb{S}^1 := \mathbb{R}/2\pi\mathbb{Z}$) given by

$$(z_1, \ldots, z_n) \mapsto (\arg z_1, \ldots, \arg z_n)$$

We may also view the coamœba as the corresponding periodic subset of $\mathbb{R}^n$.

Example 2.4 (Coamœba of a line) Let $f(z,w) = z + w - 1$. Writing $z = r_1 e^{i\theta_1}$ and $w = r_2 e^{i\theta_2}$, we can describe the coamœba $\mathscr{A}'_f$ as the set of those (θ_1, θ_2) in the fundamental polygon $(-\pi, \pi]^2$ such that

$$r_1 e^{i\theta_1} + r_2 e^{i\theta_2} = 1 \tag{2.6}$$

[5] A set $S \subset \mathbb{C}^n$ is said to be logarithmically convex if $\mathrm{Log}(S) \subset \mathbb{R}^n$ is convex.

[6] A set $S \subset \mathbb{C}^n$ is said to be a *Reinhardt domain* if it is an open connected set such that $\forall\, \theta \in \mathbb{R}^n \;\; (z_1, \ldots, z_n) \in S \Rightarrow (e^{i\theta_1} z_1, \ldots, e^{i\theta_n} z_n) \in S$. Note that some authors do not include connectedness in the definition. S is said to be a *complete* Reinhardt domain if $\forall\, \mu \in \mathbb{C}^n$ s.t. $\forall\, j \in \{1, \ldots, n\} \;\; |\mu_j| \leq 1 \;\wedge\; (z_1, \ldots, z_n) \in S \Rightarrow (\mu_1 z_1, \ldots, \mu_n z_n) \in S$.

has a solution $(r_1, r_2) \in \mathbb{R}^2_{>0}$. We can rewrite (2.6) as

$$\begin{aligned} r_1 \cos\theta_1 + r_2 \cos\theta_2 &= 1 \\ r_1 \sin\theta_1 + r_2 \sin\theta_2 &= 0 \end{aligned} \tag{2.7}$$

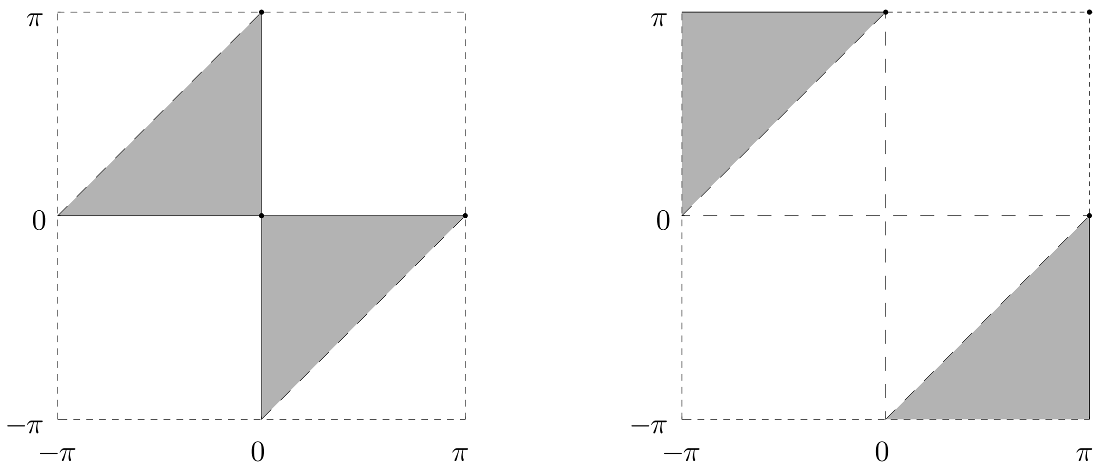
Clearly there is a solution when $(\theta_1, \theta_2) = (0, 0)$ and this is the only point in $(-\pi, \pi]$ for which the second equation in (2.7) is trivial. If $(\theta_1, \theta_2) \neq (0, 0)$, a solution exists if and only if

$$0 \neq \det \begin{bmatrix} \cos\theta_1 & \cos\theta_2 \\ \sin\theta_1 & \sin\theta_2 \end{bmatrix} = \cos\theta_1 \sin\theta_2 - \cos\theta_2 \sin\theta_1$$

and the solution can be described in terms of (θ_1, θ_2) as

$$\begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \frac{1}{\cos\theta_1 \sin\theta_2 - \cos\theta_2 \sin\theta_1} \begin{bmatrix} \sin\theta_2 \\ -\sin\theta_1 \end{bmatrix}$$

For $(\theta_1, \theta_2) \in (-\pi, \pi]^2$, $\cos\theta_1 \sin\theta_2 - \cos\theta_2 \sin\theta_1 \neq 0$ precisely when $\theta_1 = \theta_2$ or $\theta_1 = \theta_2 \pm \pi$. Simple considerations of the signs of trigonometric functions outside of these three lines tell us when r_1 and r_2 are positive and give us the full description of the coamœba. (cf. figure 2.4a).



(a) $f = z + w - 1$

(b) $f = z + w + 1$

Figure 2.4: Coamœbas of lines, represented in $(-\pi, \pi]^2$

We similarly obtain the coamœba of $f(z, w) = z + w + 1$ (cf. figure 2.4b). Notice that they differ, unlike the corresponding amœbas. Also, if we let $\mathscr{A}'_g$ denote the coamœba of the Fermat curve $\mathscr{V}_g$ where $g(z, w) = z^d + w^d - 1$, we notice that $(\theta_1, \theta_2) \in \mathscr{A}'_f$ if and only if $(d\theta_1, d\theta_2) \in \mathscr{A}'_g$.

Remark 2.5 As the above examples show, coamœbas are not closed in general.

Even though amœbas and coamœbas seem to behave quite differently, it turns out that in the case of coamœbas there is a statement analogous to Theorem 2.1 and we shall establish it in the following subsection (see [**NP10**]). We shall first lift the coamœba in $\mathbb{R}^n$ and, since it need not be closed, take its closure. In other words, we are interested in the components of the open set $\mathbb{R}^n \backslash \pi^{-1}(\overline{\mathscr{A}'_f})$ where $\pi : \mathbb{R}^n \to (\mathbb{S}^1)^n$ denotes the canonical projection.

The Mellin Transform

Definition 2.8 *The (generalized) Mellin transform* of a rational function $1/f$ is given by

$$M_{1/f}(s) := \int_{\mathbb{R}^n_{>0}} \frac{z^s}{f(z)} \frac{dz_1 \wedge \cdots \wedge dz_n}{z_1 \ldots z_n} = \int_{\mathbb{R}^n} \frac{e^{\langle s,x \rangle}}{f(e^x)} dx_1 \wedge \cdots \wedge dx_n \tag{2.8}$$

Definition 2.9 A Laurent polynomial f is said to be *completely non-vanishing* on a set X if for every face Γ of Δ_f the Laurent polynomial f_Γ (as defined in (2.2)) does not vanish on X.

Theorem 2.2 *If f is completely non-vanishing on $\mathbb{R}^n_{>0}$ then the integral* (2.8) *converges and defines an analytic function on the tube domain* $\text{int}\Delta_f + i\mathbb{R}^n$.

Proof Let $\sigma \in \text{int}\Delta_f$ and $s \in \sigma + i\mathbb{R}^n$. It suffices to show that $\exists\, c, k > 0$ such that

$$|f(e^x)e^{-\langle s,x \rangle}| = |f(e^x)|e^{-\langle \sigma,x \rangle} \geq ce^{k|x|} \tag{2.9}$$

when $x \in \mathbb{R}^n$ is far away from the origin. We proceed by induction on n. If $n = 1$, we have $\Delta_f = [\alpha, \beta]$ and $f(z) = c_\alpha z^\alpha + \cdots + c_\beta z^\beta$. Notice that for $\sigma \in (\alpha, \beta)$ we have

$$\begin{aligned} \lim_{x\to\infty} \frac{|f(e^x)|e^{-\sigma x}}{|c_\beta|e^{(\beta-\sigma)x}} &= \lim_{x\to\infty} \frac{|c_\alpha e^{(\alpha-\sigma)x} + \cdots + c_\beta e^{(\beta-\sigma)x}|}{|c_\beta|e^{(\beta-\sigma)x}} \\ &= \lim_{x\to\infty} |\frac{c_\alpha}{c_\beta} e^{(\alpha-\beta)x} + \cdots + 1| \\ &= 1 \end{aligned} \tag{2.10}$$

Therefore, if x is sufficiently large and positive, we have

$$|f(e^x)|e^{-\sigma x} \leq \frac{1}{2}|c_\beta|e^{(\beta-\sigma)|x|}$$

and, similarly, if x is sufficiently large and negative, we have

$$|f(e^x)|e^{-\sigma x} \leq \frac{1}{2}|c_\alpha|e^{(\sigma-\alpha)|x|}.$$

Suppose that (2.9) holds for dimensions $\leq n-1$ and consider a Laurent polynomial in n variables. For every face Γ of Δ_f with $0 \leq \dim\Gamma \leq n-1$, we can express any $\sigma \in \text{int}\Delta_f$ as

$$\sigma = \lambda\sigma_\Gamma + (1-\lambda)\tau_\Gamma$$

where $\sigma_\Gamma \in \text{relint}(\Gamma)$[7] and $\tau_\Gamma \in \text{relint}(\text{conv}((A_f \backslash \Gamma))$[8] . Fix one such point σ_Γ for each face Γ and consider the polytope

$$\Delta_\Gamma := \text{conv}((A_f \backslash \Gamma) \cup \sigma_\Gamma).$$

Notice that $\sigma \in \Delta_\Gamma$ for all faces Γ and that when Γ is a vertex, we have $\Delta_\Gamma = \Delta_f$. Let $\widetilde{\mathcal{C}}_\Gamma$ denote the normal cone (recall Definition 2.4) to Δ_Γ at σ_Γ translated by σ_Γ, i.e.

$$\widetilde{\mathcal{C}}_\Gamma = \{x \in \mathbb{R}^n \,:\, \forall\, \xi \in \Delta_f \;\; \langle \xi - \sigma_\Gamma, x - \sigma_\Gamma \rangle \leq 0\} \tag{2.11}$$

[7] relint(_) denotes the relative interior.
[8] conv(_) denotes the convex hull.

All $\widetilde{\mathcal{C}}_\Gamma$ are n-dimensional because σ_Γ is a vertex of Δ_Γ. Furthermore, these cones cover almost all of $\mathbb{R}^n$ in the sense that $\mathbb{R}^n\backslash(\cup_\Gamma\widetilde{\mathcal{C}}_\Gamma)$ is bounded. Let $\mathcal{C}_\Gamma$ denote a smaller closed convex cone with vertex σ_Γ such that $\mathcal{C}_\Gamma\backslash\{\sigma_\Gamma\}\subset \mathrm{int}(\widetilde{\mathcal{C}}_\Gamma)$ and such that $\mathbb{R}^n\backslash(\cup_\Gamma\mathcal{C}_\Gamma)$ is still bounded.

Observe that it suffices to prove (2.9) for $x\in\mathcal{C}_\Gamma\backslash B(0,R)$ where $B(0,R)$ is a ball of large radius. Since f_Γ depends on less than n variables (recall Remark 2.1) and $\sigma_\Gamma\in\mathrm{relint}(\Delta_{f_\Gamma})$, we deduce from the induction hypothesis that there are constants $c_\Gamma>0$ such that

$$|f_\Gamma(e^x)e^{-\langle\sigma_\Gamma,x\rangle}|\geq c_\Gamma$$

Let $g_\Gamma=\sum_{\alpha\in A_f\backslash\Gamma}c_\alpha z^\alpha$ so that $f=f_\Gamma+g_\Gamma$. We may now write

$$f(e^x)e^{-\langle\sigma,x\rangle}=e^{\langle\sigma_\Gamma-\sigma,x\rangle}(f_\Gamma(e^x)e^{-\langle\sigma_\Gamma,x\rangle}+g_\Gamma(e^x)e^{-\langle\sigma_\Gamma,x\rangle}). \tag{2.12}$$

Let $x\in\mathcal{C}_\Gamma$ and $y=x-\sigma_\Gamma$. We have $e^{\langle\sigma_\Gamma-\sigma,x\rangle}\geq c_0e^{k|y|}$ where $c_0=e^{\langle\sigma_\Gamma-\sigma,\sigma_\Gamma\rangle}$ and

$$k=\min\{\langle\sigma_\Gamma-\sigma,y\rangle\ :\ |y|=1,\ \sigma_\Gamma+y\in\mathcal{C}_\Gamma\}.$$

We may assume $|x|>|\sigma_\Gamma|$ (since $x\notin B(0,R)$) so that $|x|-|\sigma_\Gamma|\geq|x-\sigma_\Gamma|=y$ and

$$e^{\langle\sigma_\Gamma-\sigma,x\rangle}\geq c_1e^{k|x|}$$

where $c_1=c_0e^{-k|\sigma_\Gamma|}$.
It now remains to bound the second factor on the right in (2.12) by a positive constant. By induction hypothesis, we have

$$|f_\Gamma(e^x)e^{-\langle\sigma_\Gamma,x\rangle}|\geq c_\Gamma$$

so it is enough to show that the second term $g_\Gamma(e^x)e^{-\langle\sigma_\Gamma,x\rangle}$ remains sufficiently small (e.g. $<\frac{c_\Gamma}{2}$). We have

$$g_\Gamma(e^x)=\sum_{\alpha\in A_f\backslash\Gamma}c_\alpha e^{\langle\alpha,x\rangle}\sum_{\alpha\in A_f\backslash\Gamma}\widetilde{c}_\alpha e^{\langle\alpha,y\rangle}.$$

Since $\alpha\in\Delta_\Gamma$, we have a positive constant

$$k_\alpha=\min\{\langle\sigma_\Gamma-\alpha,y\rangle\ :\ |y|=1,\ \sigma_\Gamma+y\in\mathcal{C}_\Gamma\}.$$

Hence

$$|c_\alpha e^{\langle\alpha,x\rangle}|=|\widetilde{c}_\alpha e^{\langle\sigma_\Gamma-\alpha,y\rangle}|\leq|c_\alpha|e^{-k_\alpha|y|}.$$

This implies that for a large enough R' we have

$$\forall\,x\in\mathcal{C}_\Gamma\ \ |\sigma+x|\geq R'\Rightarrow|g_\Gamma(e^x)e^{-\langle\sigma_\Gamma,x\rangle}|<\frac{c_\Gamma}{2}$$

and hence

$$|f(e^x)e^{-\langle\sigma_\Gamma,x\rangle}|\geq\frac{c_\Gamma}{2}.$$

Finally, we conclude that for sufficiently large R we have

$$\forall\,x\in\mathcal{C}_\Gamma\backslash B(0,R)\ \ |f(e^x)e^{-\langle\sigma,x\rangle}|\geq ce^{k|x|}$$

with $c=\frac{c_1c_\Gamma}{2}$.

□

It turns out that (2.8) can be extended to a meromorphic function on $\mathbb{C}^n$ in an interesting way. We shall omit the proof of this (see Theorem 2 in [**NP10**]).

Theorem 2.3 *If f is completely non-vanishing on $\mathbb{R}^n_{>0}$ and Δ_f has a non-empty interior, the Mellin transform $M_{1/f}(s)$ can be meromorphically extended to $\mathbb{C}^n$ as*

$$M_{1/f}(s) = \Phi(s) \prod_{k=1}^{n} \Gamma(\langle \mu_k, s\rangle - \nu_k) \tag{2.13}$$

where μ_k and ν_k are as in (2.1) *and $\Phi(s)$ is an entire function.*

Theorem 2.4 *For any $\theta \in \mathbb{R}^n \backslash \overline{\pi^{-1}(\mathscr{A}'_f)}$, the Laurent polynomial f is completely non-vanishing on* $\mathrm{Arg}^{-1}(\theta)$.

Proof For a given θ we can consider $f_\theta(z) := f(e^{i\theta_1}z_1, \dots, e^{i\theta_n}z_n)$. We have $\pi^{-1}(\mathscr{A}'_{f_\theta}) + \theta = \pi^{-1}(\mathscr{A}'_f)$ so that $0 \in \pi^{-1}(\mathscr{A}'_{f_\theta})$ if and only if $\theta \in \pi^{-1}(\mathscr{A}'_f)$ and that f_θ is completely non-vanishing on $\mathrm{Arg}^{-1}(0)$ if and only if f is completely non-vanishing on $\mathrm{Arg}^{-1}(\theta)$. Therefore it suffices to consider the case $\theta = 0$. Note that $\mathrm{Arg}^{-1}(0) = \mathbb{R}^n_{>0}$.

Assume that f is not completely non-vanishing on $\mathrm{Arg}^{-1}(0)$ so that $0 \in \pi^{-1}(\mathscr{A}'_{f_\Gamma})$ for some face Γ, i.e. $\exists\, x_0 \in \mathbb{R}^n$ s.t. $f_\Gamma(e^{x_0}) = 0$. We need to show that $0 \in \overline{\pi^{-1}(\mathscr{A}'_f)}$. Case $\Gamma = \Delta_f$ is trivial, so we will consider faces such that $\dim \Gamma \le n-1$. Let $\mu \in \mathbb{Z}^n$ and $m \in \mathbb{Z}$ be such that $\langle \mu, \alpha\rangle = m$ for $\alpha \in \Gamma$ and $\langle \mu, \alpha \rangle < m$ for $\alpha \in \Delta_f \backslash \Gamma$ and let $g_\Gamma = f - f_\Gamma$ as before. We have

$$f_\Gamma(e^{x_0 - t\mu}) = \sum_{\alpha\in\Gamma} c_\alpha e^{\langle x_0, \alpha\rangle - t\langle \mu, \alpha\rangle} = e^{-tm} \sum_{\alpha \in \Gamma} c_\alpha c^{\langle x_0, \alpha \rangle} = 0$$

and

$$g_\Gamma(e^{x_0 - t\mu}) = \sum_{\alpha \in \Delta_f \backslash \Gamma} c_\alpha e^{\langle x_0, \alpha\rangle - t\langle\mu, \alpha\rangle} = \sum_{\alpha\in\Delta_f\backslash\Gamma} \widehat{c}_\alpha e^{-tk_\alpha}$$

where $\widehat{c}_\alpha = c_\alpha e^{\langle x_0, \alpha\rangle}$ and $k_\alpha = \langle \mu, \alpha\rangle - m > 0$. Now fix $\varepsilon > 0$ and let $D := D(x_0, \varepsilon)$ be a disk of radius ε centered at x_0 contained in a complex line on which the function $w \mapsto f_\Gamma(e^w)$ is not identically zero. Since $f_\Gamma(e^w)$ is not zero on the boundary ∂D, we have $|f_\Gamma(e^w)| \ge \delta > 0$ for $w \in \partial D$. This implies that $|f_\Gamma(e^w)| \ge \delta e^{-tm}$ for $w \in \partial D - t\mu$ for some large positive t. For a sufficiently large t, we can ensure that on $\partial D - t\mu$ we have $|g_\Gamma(e^w)| < |f_\Gamma(e^w)|$. By Rouché's theorem, $f(e^w) = f_\Gamma(e^w) + g_\Gamma(e^w)$ has a zero w_ε in $D - t\mu$. Therefore e^{w_ε} belongs to $\mathscr{V}_f$. Since $|\mathrm{Arg}(e^{w_\varepsilon})| = |\mathrm{Im}(w_\varepsilon)| < \varepsilon$ and ε was arbitrarily chosen, we conclude $0 \in \overline{\pi^{-1}(\mathscr{A}'_f)}$.

$\square$

Theorems 2.2 and 2.4 allow us to define the θ-*directional* Mellin transform

$$M^\theta_{1/f}(s) := \frac{1}{(2\pi i)^n} \int_{\mathrm{Arg}^{-1}(\theta)} \frac{z^s}{f(z)} \frac{dz}{z} = \frac{1}{(2\pi i)^n} \int_{\mathbb{R}^n} \frac{e^{\langle s, x+i\theta\rangle}}{f(e^{x+i\theta})}\, dx. \tag{2.14}$$

for any $\theta \in \mathbb{R}^n \backslash \overline{\pi^{-1}(\mathscr{A}'_f)}$ and $s \in \mathrm{int}\Delta_f$.

Lemma 2.5 *Integral* (2.14) *does not depend on the choice of θ as long as θ remains in the same connected component of $\mathbb{R}^n \backslash \overline{\pi^{-1}(\mathscr{A}'_f)}$. We shall denote it by $M^C_{1/f}(s)$ and call it the C-*directional Mellin transform *where C is the connected component that contains θ.*

Proof We begin by considering the one-dimensional case. By Theorems 2.2 and 2.4, we know that the directional Mellin transform

$$M^{\theta}_{1/f}(s) := \int_{\arg^{-1}(\theta)} \frac{z^s}{f(z)} \frac{dz}{z}$$

converges for any $\theta \in \mathbb{R}^n \backslash \pi^{-1}(\overline{\mathscr{A}'_f})$ and $s \in \text{int}\Delta_f$. In dimension one, the coamœba $\mathscr{A}'_f$ is a discrete set and for any $\theta \in \mathbb{R}^n \backslash \pi^{-1}(\mathscr{A}'_f)$, $\arg^{-1}(\theta)$ is the ray $\{re^{i\theta} : r \in \mathbb{R}_{>0}\}$. We claim that for small enough $|\Delta\theta|$

$$\int_{\arg^{-1}(\theta)} \frac{z^s}{f(z)} \frac{dz}{z} = \int_{\arg^{-1}(\theta+\Delta\theta)} \frac{z^s}{f(z)} \frac{dz}{z}.$$

To show this, we integrate along a closed path γ composed of three curves: the line that connects the origin and a point $Re^{i\theta}$, the arc that connects $Re^{i\theta}$ and $Re^{i(\theta+\Delta\theta)}$ and the line that connects $Re^{i(\theta+\Delta\theta)}$ and the origin (see figure 2.5).

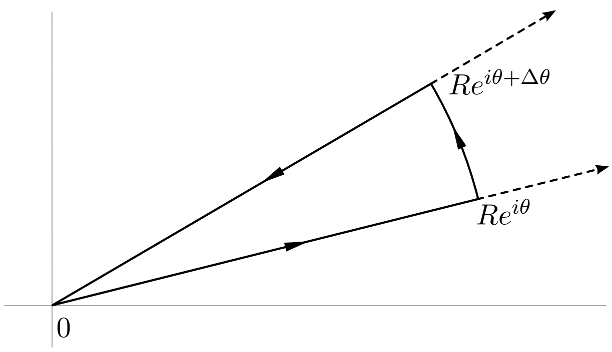


Figure 2.5: The integration path γ

When $|\Delta\theta|$ is small enough, f has no zeroes with argument between θ and $\theta + \Delta\theta$, i.e. γ does not encircle any zeroes. Moreover, as $R \to \infty$, the integral along the arc approaches zero because the integrand rapidly approaches zero when $|z| \to \infty$. By Cauchy's integral theorem, the integrals along the two infinite rays are equal. This implies that the Mellin transform does not change if we change the direction of integration by $\Delta\theta$ as long as θ and $\theta + \Delta\theta$ are in the same connected component.

To extend this argument to the n-dimensional case, we must take $\theta \in \mathbb{R}^n \backslash \pi^{-1}(\overline{\mathscr{A}'_f})$ since the coamœba need not be closed. We connect θ and $\theta + \Delta\theta$ by a piecewise linear path such that for any segment only one coordinate of θ is changed. The claim now follows by successively applying the one-variable argument.

□

The following theorem grants us a nice analogy to Theorem 2.1.

Theorem 2.6 (Mellin Inversion Formula) *For every connected component C of $\mathbb{R}^n \backslash \pi^{-1}(\overline{\mathscr{A}'_f})$, $1/f$ may be represented as the following integral*

$$\frac{1}{f(z)} = \int_{\sigma+i\mathbb{R}^n} M^{C}_{1/f}(s) z^{-s}\, ds \tag{2.15}$$

which converges for all $z \in \text{Arg}^{-1}(C)$. Here σ denotes an arbitrary point in $\text{int}\Delta_f$ *and $M^{C}_{1/f}(s)$ denotes the C-directional Mellin transform as in* (2.14) *and Lemma 2.5.*

Proof It suffices to show that for all $s \in \sigma + i\mathbb{R}^n$ such that $\sigma \in \mathrm{int}\Delta_f$, the function

$$x \mapsto \frac{e^{\langle s, x+i\theta\rangle}}{f(e^{x+i\theta})}$$

is in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ of rapidly decreasing functions[9] and the result will follow from the Fourier inversion formula (see Theorem 7.1.5 in [**Horm90b**])[10].

Assume, without loss of generality, that $\theta = 0$. From the proof of Theorem 2.2 we know that

$$\left|\frac{e^{\langle s,x\rangle}}{f(e^x)}\right| = \frac{e^{\langle \sigma,x\rangle}}{|f(e^x)|}$$

is an exponentially decreasing function. We need to show that the same holds for all of its partial derivatives. We have

$$\frac{\partial}{\partial x_k}\left(\frac{e^{\langle \sigma,x\rangle}}{f(e^x)}\right) = \frac{\sigma_k e^{\langle \sigma,x\rangle} f(e^x) - e^{\langle \sigma,x\rangle} f'_k(e^x) e^{x_k}}{f(e^x)^2} = \frac{\sigma_k e^{\langle \sigma,x\rangle}}{f(e^x)} - \frac{e^{\langle \sigma+e_k,x\rangle} f'_k(e^x)}{f(e^x)^2} \tag{2.16}$$

where f'_k denotes $\partial f/\partial z_k$. The first term on the right hand side is a constant multiple of the original function and the second term is of the form

$$\sum_{\alpha \in A_f} \frac{\alpha_k c_\alpha e^{\langle \sigma+\alpha,x\rangle}}{f(e^x)^2}.$$

The Newton polytope of f^2 is $\Delta_{f^2} = 2\Delta_f$, so $\forall\, \alpha \in A_f \quad \sigma + \alpha \in \mathrm{int}\Delta_{f^2}$. This implies that the derivative (2.16) is a sum of finitely many terms, each of which satisfies the conditions of Theorem 2.2. By induction, all derivatives of $e^{\langle s,x\rangle}/f(e^x)$ are exponentially decreasing.

□

Corollary 2.7 *The connected components of* $\mathbb{R}^n \backslash \pi^{-1}(\overline{\mathscr{A}'_f})$ *are convex (and in bijective correspondence with distinct representations of* $1/f$ *as in* (2.15)*).*

Proof Let C be a connected component of $\mathbb{R}^n \backslash \pi^{-1}(\overline{\mathscr{A}'_f})$. Consider the following two functions defined on the tube domain $C + i\mathbb{R}^n$:

$$\begin{aligned} \theta - ix &\mapsto \frac{1}{f(e^{i(\theta - ix)})} = \frac{1}{f(e^{x+i\theta})} \\ \theta - ix &\mapsto \int_{\sigma + i\mathbb{R}^n} M^C_{1/f}(s) e^{-\langle x+i\theta, s\rangle}\, ds \end{aligned} \tag{2.17}$$

By Bochner's theorem (see [**Boch38**]), they can be analitically extended to the convex hull $\mathrm{conv}(C) + i\mathbb{R}^n$. By the analytic continuation principle, equality (2.15) extends as well. As $\mathrm{conv}(C)$ is open and $C + i\mathbb{R}^n$ is the maximal open set on which we can define $1/f(e^{x+i\theta})$, it must be that $\mathrm{conv}(C) \subseteq C$, i.e. C is convex. The rest of the statement follows from the preceding theorems.

□

Remark 2.6 The role of Bochner's result here is analogous to the role of Abel's result[11] in proving convexity in Theorem 2.1.

[9] $\mathcal{S}(\mathbb{R}^n) := \{f \in C^\infty(\mathbb{R}^n) \,:\, \forall\, \alpha, \beta \, \sup_{x \in \mathbb{R}^n} |x^\alpha \partial^\beta f(x)| < \infty\}$ where α and β are multi-indices.

[10] The Fourier transform is an automorphism of $\mathcal{S}(\mathbb{R}^n)$ with inverse given by the Fourier inversion formula.

[11] If a series $\sum_\alpha c_\alpha z^\alpha$ converges near two points ζ and η in $\mathbb{T}^n$, it converges on $\{z \in \mathbb{T}^n \,:\, \forall\, j \;\; |\zeta_j| \leq |z_j| \leq |\eta_j|\}$.

2.1.3 Amœbas and Lopsidedness

Recall that Proposition 2.1 implies that in the special case of a linear polynomial $f(z) = \sum_{j=0}^{n} c_j z_j$ where $c_j \neq 0$, $x \in \mathscr{A}_f \Leftrightarrow \{|c_0|, |c_1|e^{x_1}, \ldots, |c_n|e^{x_n}\}$ is not lopsided. Let $f(z) = \sum_\alpha c_\alpha z^\alpha$ be any Laurent polynomial. It makes sense to ask whether we have

$$\mathscr{A}_f = \left\{x \in \mathbb{R}^n \,:\, \{|c_\alpha|e^{\langle\alpha,x\rangle}\}_{\alpha\in A_f} \text{ is not lopsided}\right\} \tag{2.18}$$

By Proposition 2.1, we always have inclusion from left to right. However, we need not have equality as the following example illustrates.

Example 2.5 Consider a one-variable polynomial $f(z) = c_0 + c_1 z + \cdots + c_{d-1} z^{d-1} + z^d$. Suppose $\xi_1, \ldots, \xi_d$ are its roots, indexed so that $|\xi_1| \leq \cdots \leq |\xi_d|$. Let $a_j := \log|\xi_j|$. Let $x \in \mathbb{R}^n \backslash \mathscr{A}_f$. If (2.18) is true, we expect $\{|c_0|, |c_1|e^x, \ldots, |c_{d-1}|e^{(d-1)x}, e^{dx}\}$ to be lopsided. If $x \in (-\infty, a_1)$ is very large and negative, $|c_0|$ will dominate other elements. Similarly, if $x \in (a_d, \infty)$ is very large and positive, e^{dx} will dominate. However, when x is close to an a_j (i.e. the amœba), $\{|c_0|, |c_1|e^x, \ldots, |c_{d-1}|e^{(d-1)x}, e^{dx}\}$ need not be lopsided. It is an easy exercise to choose suitable coefficients to demonstrate this.

However, it turns out that for any $x \in \mathbb{R}^n \backslash \mathscr{A}_f$, we can find a Laurent polynomial $g(z) = \sum_{\alpha\in A_g} c_{x,\alpha} z^\alpha$ (which depends on x) such that $\mathscr{A}_f = \mathscr{A}_g$ and $\{|c_{x,\alpha}|e^{\langle\alpha,x\rangle}\}_{\alpha\in A_g}$ is lopsided. We start with the one variable case. The idea is that if $f(z) = \prod_j (z - \xi_j)$ and assuming $|\xi_1| > \cdots > |\xi_d| > 0$, we consider $g(z) = \prod_j (z^k - \xi_j^k)$. When k is large, we have

$$g(z) \approx (z^d)^k - (\xi_1 z^{d-1})^k + \cdots \pm (\xi_1 \ldots \xi_{d-1} z)^k \mp (\xi_1 \ldots \xi_d)^k.$$

Suppose $|\xi_{l+1}| < z < |\xi_l|$. When $n \to \infty$, we have $g(z)/(\xi_1 \ldots \xi_{d-l} z^{d-l})^k \to 1$. Therefore one term of $g(z)$ dominates others for sufficiently large k. Formally, this is rewritten in the following two technical lemmas whose proofs we omit (see [**Purb06**] for details).

Lemma 2.8 *Let $0 \leq \beta_0 < \beta_1 \leq \infty$, $\gamma \in [\sqrt{\beta_0/\beta_1}, 1)$ and $k \in \mathbb{Z}_{>0}$. If a polynomial $f(z) = \sum_{j=0}^{d} c_j z^{kj}$ ($c_j \neq 0$, $d \in \mathbb{Z}_{>0}$) has no roots in $\{z \in \mathbb{C} : \beta_0 < |z| < \beta_1\}$ then there is an l such that $\forall\, z_0 \in \{z \in \mathbb{C} : \gamma^{-1}\beta_0 \leq |z| \leq \gamma\beta_1\}$ we have*

$$\frac{|c_j {z_0}^{kj}|}{|c_l {z_0}^{kl}|} < \frac{\sum_{m\geq|d-j-l|} \frac{(d\gamma^n)^m}{m!}}{2 - e^{d\gamma^n}} \tag{2.19}$$

For a Laurent polynomial $f(z) = \sum_{j=d_1}^{d_2} c_j z^{kj}$ with $d_1 < 0 \leq d_2$ and $c_j \neq 0$ that satifies the same condition, (2.19) *holds with* $d = d_2 - d_1$.

We omit the proof but note that l can be determined. Namely, let $f_k(z) = \prod_{j=0}^{d} (z^k - \xi_j^k)$ where $|\xi_1| \geq \cdots \geq |\xi_d|$ and let $\xi_0 := \infty$ and $\xi_{d+1} := 0$. As $f(z)$ has no roots in $\{z : \beta_0 < |z| < \beta_1\}$, we have $|\xi_{l+1}| \leq \beta_0 < \beta_1 \leq |\xi_l|$ for some l ($0 \leq l \leq d$). This l satisfies (2.19). From this, the following result is obtained.

Lemma 2.9 *Let $0 \leq \beta_0 < \beta_1 \leq \infty$, $\gamma \in [\sqrt{\beta_0/\beta_1}, 1)$, $k \in \mathbb{Z}_{>0}$ and $c_0, D_0, c_1, D_1 \in \mathbb{Z}_{>0}$. Let $\{f_k(z)\}_k$ be a family of polynomials (resp. Laurent polynomials) such that*

(1) f_k has no roots in $\{z : \beta_0 < |z| < \beta_1\}$

(2) all terms of f_k are of the form $c_{k,j}z^{kj}$, $j \in \mathbb{Z}$

(3) $\deg(f_k) \le c_0 k^{D_0+1}$ (resp. $\operatorname{maxdeg}(f_k) - \operatorname{mindeg}(f_k) \le c_0 k^{D_0+1}$)

If k is large enough that

$$k \log \gamma^{-1} \ge (D_0 + D_1)\log(k) + \log(8/3c_0c_1),$$

then $\{|c_{k,j}|e^{jx}\}_j$ is $c_1k^{D_1}$-superlopsided for all $x \in [\log(\gamma^{-1}\beta_0), \log(\gamma\beta_1)]$.

Theorem 2.10 [**Purb06**] *Let $f \in \mathbb{C}[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$ and for $k \in \mathbb{Z}_{>0}$ define*

$$f_k(z) = \prod_{l_1=0}^{k-1} \cdots \prod_{l_n=0}^{k-1} f(e^{2\pi i\, l_1/k} z_1, \dots, e^{2\pi i\, l_n/k} z_n) = \sum_{\alpha \in A_{f_k}} c_{k,\alpha} z^{\alpha}. \tag{2.20}$$

Then $\mathscr{A}_f = \mathscr{A}_{f_k}$ and for any $x \in \mathbb{R}^n \backslash \mathscr{A}_f$ there is a $k(x) \in \mathbb{Z}_{>0}$ such that for $k \ge k(x)$ the set $\{|c_{k,\alpha}|e^{\langle \alpha, x\rangle}\}_{\alpha \in A_{f_k}}$ is lopsided.

The idea behind the proof is reducing the claim to the one variable case by fixing all but one variable of f_k, applying the lemma to find a dominant term and showing that this implies that f_k has a dominant term as a Laurent polynomial in n variables. To continue, we will need several important facts.

Proposition 2.2 $\mathscr{A}_f = \mathscr{A}_{f_k}$

Proof Since f_k is a product of terms $f_{l_1,\dots,l_n}(z) := f(e^{2\pi i\, l_1/k} z_1, \dots, e^{2\pi i\, l_n/k} z_n)$ and $|e^{2\pi i\, l_1/k}| = \dots = |e^{2\pi i\, l_n/k}| = 1$, we have $\mathscr{A}_{f_{l_1,\dots,l_n}} = \mathscr{A}_f$ and hence $\mathscr{A}_{f_k} = \cup \mathscr{A}_{f_{l_1,\dots,l_n}} = \mathscr{A}_f$.

□

Proposition 2.3 *All terms of f_k are of the form $cz^{k\alpha} := cz_1^{k\alpha_1} \dots z_n^{k\alpha_n}$.*

Proof f_k is clearly invariant under action of the cyclic group of roots of $z^k - 1$ given by $(u_1, \dots, u_n)(z_1, \dots, z_n) := (u_1z_1, \dots, u_nz_n)$ and, therefore, so are all of its monomials. This is only possible if they are of the form $cz^{k\alpha}$.

□

Definition 2.10 If Δ is a polytope, its *Ehrhart polynomial* $E_\Delta(t)$ is defined as $\operatorname{card}(t\Delta \cap \mathbb{Z}^n)$.

Its coefficients are generally not known, however, it can be bounded from above in terms of the volume of Δ. See [**BM85**].

Proposition 2.4 *Suppose $d = d(\Delta_f)$ is an upper bound for $\operatorname{card}(\mathbb{Z}^n \cap t\Delta_f)/t^n$. Then f_k has at most $dk^{n(n-1)}$ terms.*

Proof Note that $\Delta_{f_k} = k^n \Delta_f$. By Proposition 2.3, the number of terms of f_k is at most the number of lattice points in $\frac{1}{k}\Delta_{f_k} = k^{n-1}\Delta_f$ which is bounded by $dk^{n(n-1)}$.

□

Lemma 2.11 *Let $f(z) = \sum_{\alpha \in A_f} c_\alpha z^\alpha$ be a Laurent polynomial and let $x \in \mathbb{R}^n \backslash \mathscr{A}_f$. If $\forall\, \zeta \in \operatorname{Log}^{-1}(x)$ we have $|f(\zeta)| < M$ then $\forall\, \beta$ $|c_\beta \zeta^\beta| < M$.*

Proof Integrating $M > |f(\zeta)|$ over $\mathrm{Log}^{-1}(x)$ gives

$$\begin{aligned} M &\geq \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \left| \sum_\alpha c_\alpha e^{\langle \alpha, x+i\theta \rangle} \right| d\theta_1 \ldots d\theta_n \\ &\geq \left| \frac{1}{(2\pi i)^n} \int_{|z_1|=1} \cdots \int_{|z_n|=1} \sum_\alpha \frac{c_\alpha e^{\langle \alpha, x \rangle} z^\alpha}{z^\beta} \frac{dz_1 \ldots dz_n}{z_1 \ldots z_n} \right| \\ &= |c_\beta e^{\langle \beta, x \rangle}| \\ &= |c_\beta \zeta^\beta| \end{aligned}$$

□

Proof (of Theorem 2.10) Let $x \in \mathbb{R}^n \backslash \mathscr{A}_f = \mathbb{R}^n \backslash \mathscr{A}_{f_k}$ and suppose that its distance from the amœba is at least $\varepsilon = \varepsilon(x) > 0$. Let $d = d(\Delta_f) > 0$ be an upper bound for $E_{\Delta_f}(t)/t^n$ as above and let $c = \max_j (\max(\pi_j(\Delta_f)) - \min(\pi_j(\Delta_f))) > 0$ where $\pi_j : \mathbb{R}^n \to \mathbb{R}$ denotes the projection map $x \mapsto x_j$. We claim that if

$$k\varepsilon > (n^2 - 1)\log(k) + \log(\frac{16}{3cd}), \tag{2.21}$$

the set $\{|c_{k,\alpha}| e^{\langle \alpha, x \rangle}\}_{\alpha \in A_{f_k}}$ is $dk^{n(n-1)}$-superlopsided.

Let $\zeta \in \mathrm{Log}^{-1}(x)$ and let

$$f_{j,k,\zeta}(z) := f_k(\zeta_1, \ldots, \zeta_{j-1}, z, \zeta_{j+1}, \ldots, \zeta_n) = \sum_{\alpha_j \in \pi_j(A_{f_k})} c_{k,\alpha_j,\zeta} z^{k\alpha_j}. \tag{2.22}$$

Note that

$$\begin{aligned} \mathrm{maxdeg}(f_{j,k,\zeta}) - \mathrm{mindeg}(f_{j,k,\zeta}) &= \max \pi_j(\Delta_{f_k}) - \min \pi_j(\Delta_{f_k}) \\ &= k^n(\max \pi_j(\Delta_f) - \min \pi_j(\Delta_f)). \end{aligned}$$

$f_{j,k,\zeta}$ has no roots in the annulus $\{z \in \mathbb{C} : e^{x_j - \varepsilon} < |z| < e^{x_j + \varepsilon}\}$ because otherwise the distance between x and $\mathscr{A}_{f_k}$ would be less than ε. We now apply Lemma 2.9 to $f_{j,k,\zeta}$ with $\gamma = e^{-\varepsilon}$, $c_0 = c$, $D_0 = n-1$, $c_1 = 2d$ and $D_1 = n(n-1)$. If k satisfies (2.21), it holds that

$$\{|c_{k,\alpha_j,\zeta}| e^{k\alpha_j x_j}\}_{\alpha_j \in \pi_j(A_{f_k})} \text{ is } 2dk^{n(n-1)}\text{−superlopsided.} \tag{2.23}$$

As $\mathrm{Log}^{-1}(x)$ is connected, this does not depend on the choice of ζ. Let ν_j denote the index of the dominating term and let $\nu = (\nu_1, \ldots, \nu_n)$ so that we can write f_k as

$$f_k(z) = c_{k,\nu} z^{k\nu} + \sum_{\substack{\alpha \in A_{f_k} \\ \alpha \neq \nu}} c_{k,\alpha} z^{k\alpha}.$$

Let $M = |c_{k,\nu}\zeta^{k\nu}|$ and $\mu = \max_{\alpha \neq \nu} |c_{k,\nu}\zeta^{k\nu}|$. Since f_k has at most $dk^{n(n-1)}$ terms, we are done if we show that $M > dk^{n(n-1)}\mu$. For a fixed l, the z^{kl}-term of $f_{j,k,\zeta}$ is

$$\sum_{\substack{\alpha \in A_{f_k} \\ \alpha_j = l}} c_{k,\alpha_j,\zeta} z^{k\alpha_j} = \sum_{\substack{\alpha \in A_{f_k} \\ \alpha_j = l}} c_{k,\alpha} \zeta_1^{k\alpha_1} \cdots \zeta_{j-1}^{k\alpha_{j-1}} \zeta_{j+1}^{k\alpha_{j+1}} \cdots \zeta_n^{k\alpha_n} z^{k\alpha_j} \tag{2.24}$$

For $l \neq \nu_j$, by (2.23), we have

$$2dk^{n(n-1)} \left| \sum_{\substack{\alpha \in A_{f_k} \\ \alpha_j = l}} c_{k,\alpha} \zeta^{k\alpha} \right| < \left| \sum_{\substack{\alpha \in A_{f_k} \\ \alpha_j = \nu_j}} c_{k,\alpha} \zeta^{k\alpha} \right| \leq \sum_{\substack{\alpha \in A_{f_k} \\ \alpha_j = \nu_j}} \left| c_{k,\alpha} \zeta^{k\alpha} \right| \leq M + \mu dk^{n(n-1)}$$

Since this does not depend on ζ, by Lemma 2.11,

$$2dk^{n(n-1)} |c_{k,\alpha} \zeta^{k\alpha}| < M + \mu dk^{n(n-1)}$$

holds for all $\alpha \in A_{f_k}$ such that $\alpha_j = l$. As this holds for every j, we have

$$2dk^{n(n-1)} |c_{k,\alpha} \zeta^{k\alpha}| < M + \mu dk^{n(n-1)}$$

whenever $\alpha \neq \nu$. In particular, $\mu = |c_{k,\alpha} \zeta^{k\alpha}|$ for some $\alpha \neq \nu$. Hence

$$2dk^{n(n-1)} \mu < M + \mu dk^{n(n-1)}$$

i.e.

$$M > 2dk^{n(n-1)} \mu$$

as required.

□

Remark 2.7 Note that the lower bound for k depends only on ε and Δ_f.

Remark 2.8 If $d' > d$, we can ensure $d'k^{n(n-1)}$-superlopsidedness by the same argument.

Corollary 2.12 *For a Laurent polynomial f, the set of all x such that $\{|c_\alpha| e^{\langle \alpha, x \rangle}\}_{\alpha \in A_f}$ is **not** (super)lopsided converges uniformly to $\mathscr{A}_f$.*

Proof This follows from the proof above. If $x \notin \mathscr{A}_f = \mathscr{A}_{f_k}$, we have

$$\varepsilon < ((n^2 - 1) \log(k) + \log(16/3cd))/k$$

which converges to 0 as $k \to \infty$.

□

2.1.4 Complement Components

In this section, which is based on [**FPT00**], we shall deal with the problem of finding components of the complement of a hypersurface amœba $\mathscr{A}_f$. Let, as before, $f(z) = \sum_{\alpha \in A_f} c_\alpha z^\alpha$. The first thing to notice is that slightly changing coefficients c_α does not suddenly decrease the number of components of $\mathbb{R}^n \backslash \mathscr{A}_f$. In other words:

Proposition 2.5 *The map $(c_\alpha) \mapsto$ card{components of $\mathbb{R}^n \backslash \mathscr{A}_f$} is lower semi-continuous.*

Proof Fix a coefficient vector (c_α) and fix a point x_C for every connected component C of $\mathbb{R}^n\backslash\mathscr{A}_f$. If we replace (c_α) with a vector $(\widetilde{c}_\alpha)$ which lies in a sufficiently small ball $B((c_\alpha),\varepsilon)$ (and defines a new Laurent polynomial $\widetilde{f}$), the points x_C remain in $\mathbb{R}^n\backslash\mathscr{A}_{\widetilde{f}}$ by continuity. Thus we only need to show that they do not lie in the same component. This follows from Theorem 2.1 and the fact that coefficients in the development of $1/f$ (see (2.5)) depend continuously on coefficients c_α of f. □

Let C be a connected component of amœba $\mathbb{R}^n\backslash\mathscr{A}_f$ and let $x \in C$. Choose $z = (e^{x_1+i\theta_1},\dots,e^{x_n+i\theta_n}) \in \mathrm{Log}^{-1}(x)$. For every $1 \le j \le n$, fix all arguments θ_k with $k \neq j$ and consider the following loop

$$[0,2\pi] \ni \theta_j \mapsto f(e^{x_1+i\theta_1},\dots,e^{x_n+i\theta_n}).$$

We know by the classical argument principle that

$$\frac{1}{2\pi i}\int_{|z_j|=e^{x_j}} \frac{f'_j(z)}{f(z)}\,dz_j \tag{2.25}$$

where $f'_j = \partial f/\partial z_j$, is always an integer. Moreover, it depends continuously on x and θ which implies that it does not depend on θ and that it is constant on the connected component C. Thus we may define the following.

Definition 2.11 The *order* of a connected component C of $\mathbb{R}^n\backslash\mathscr{A}_f$ is defined as $\nu = (\nu_1,\dots,\nu_n) \in \mathbb{Z}^n$ where

$$\nu_j := \frac{1}{(2\pi i)^n}\int_{\mathrm{Log}^{-1}(x)} \frac{z_j f'_j(z)}{f(z)}\frac{dz_1\wedge\cdots\wedge dz_n}{z_1\dots z_n}. \tag{2.26}$$

To justify this, simply rewrite (2.26) as

$$\begin{aligned}
\nu_j &= \frac{1}{(2\pi)^n}\int_{[0,2\pi]^n} \frac{e^{x_j+i\theta_j} f'_j(e^{x_1+i\theta_1},\dots,e^{x_n+i\theta_n})}{f(e^{x_1+i\theta_1},\dots,e^{x_n+i\theta_n})}\,d\theta_1\dots d\theta_n \\
&= \frac{1}{(2\pi)^n}\int_{[0,2\pi]} \frac{e^{x_j+i\theta_j} f'_j(e^{x_1+i\theta_1},\dots,e^{x_n+i\theta_n})}{f(e^{x_1+i\theta_1},\dots,e^{x_n+i\theta_n})}\,d\theta_j \int_{[0,2\pi]^{n-1}} \prod_{k\neq j} d\theta_k \\
&= \frac{1}{2\pi}\int_{[0,2\pi]} \frac{e^{x_j+i\theta_j} f'_j(e^{x_1+i\theta_1},\dots,e^{x_n+i\theta_n})}{f(e^{x_1+i\theta_1},\dots,e^{x_n+i\theta_n})}\,d\theta_j
\end{aligned}$$

which is precisely (2.25).

Remark 2.9 The order depends on the Laurent polynomial f, not its amœba $\mathscr{A}_f$. For example, it changes if we replace f by f^m for some $m \ge 2$.

Lemma 2.13 *Let f be a Laurent polynomial, C a connected component of $\mathbb{R}^n\backslash\mathscr{A}_f$, $x \in C$ and ν the order of C. For any $\mu \in \mathbb{Z}^n\backslash\{0\}$, $\langle\mu,\nu\rangle$ equals the number of zero-poles*[12] *of the one-variable Laurent polynomial*

$$w \mapsto f(z_1 w^{\mu_1},\dots,z_n w^{\mu_n})$$

inside the unit disc $|w| < 1$ where $z = (z_1,\dots,z_n)$ is an arbitrary point in $\mathrm{Log}^{-1}(x)$.

[12] i.e. the number of zeroes minus the number of poles counted with multiplicities. Note that when f is a Laurent polynomial, the origin is the only pole and it is of order $-\mathrm{mindeg} f$.

Proof We know by the classical argument principle that the number of zero-poles of $f(zw^\mu)$ in the unit disc is given by

$$\frac{1}{2\pi i}\int_{|w|=1}\frac{df(zw^\mu)}{f(zw^\mu)}. \tag{2.27}$$

Image of the unit circle $|w|=1$ under $f(zw^\mu)$ is a loop contained in $\mathrm{Log}^{-1}(x)$ which is homologous to $\mu_1\gamma_1+\cdots+\mu_n\gamma_n$ where $\gamma_j : [0,2\pi] \ni t \mapsto (z_1,\ldots,z_{j-1},z_je^{it},z_{j+1},\ldots,z_n)$ (see [**Rud69**]). Therefore, we can rewrite (2.27) as

$$\begin{aligned}\int_{|w|=1}\frac{df(zw^\mu)}{f(zw^\mu)} &= \sum_{j=1}^n \mu_j \int_{|\zeta_j|=e^{x_j}}\frac{f_j'(\zeta)}{f(\zeta)}\,d\zeta_j\\ &= 2\pi i\sum_{j=1}^n \mu_j\nu_j\\ &= 2\pi i\langle\mu,\nu\rangle\end{aligned}$$

which completes the proof.

□

Proposition 2.6 *The order of any connected component of* $\mathbb{R}^n\backslash\mathscr{A}_f$ *is a point in* Δ_f.

Proof It suffices to show that $\langle\mu,\nu\rangle \le \max_{\alpha\in\Delta_f}\langle\mu,\alpha\rangle$ for any $\mu\in\mathbb{Z}^n\backslash\{0\}$. By Lemma 2.13, $\langle\mu,\nu\rangle$ equals the number of zero-poles of the one-variable Laurent polynomial $w\mapsto f(zw^\mu)$ in the unit disc. This number is bounded by the the top degree of $f(zw^\mu)$ ($\mathrm{maxdeg} f(zw^\mu)$) which is precisely $\max_{\alpha\in\Delta_f}\langle\alpha,\mu\rangle$.

□

Proposition 2.7 *Two connected components of* $\mathbb{R}^n\backslash\mathscr{A}_f$ *cannot have the same order.*

Proof Let C and C' be two distinct connected components of $\mathbb{R}^n\backslash\mathscr{A}_f$ and ν and ν' their respective orders. Fix two points $x\in C\cap\mathbb{Q}^n$ and $x'\in C'\cap\mathbb{Q}^n$. Note that the segment $[x,x']$ intersects $\mathscr{A}_f$. Let $x'-x=t\mu$ for some $t\in\mathbb{Q}_{>0}$ and $\mu\in\mathbb{Z}^n$. We will show that $\langle\mu,\nu'\rangle > \langle\mu,\nu\rangle$. By Lemma 2.13, we know that these two numbers are equal to the number of zero-poles in the unit disc of $w\mapsto f(z'w^\mu)$ and $w\mapsto f(zw^\mu)$ respectively, where $z'\in\mathrm{Log}^{-1}(x')$ and $z\in\mathrm{Log}^{-1}(x)$. Note that $z_j'/z_j = e^{t\mu_j}$ and hence $z'w^\mu = z(e^tw)^\mu$. This means that we can interpret $\langle\mu,\nu'\rangle$ as the number of zero-poles of $f(zw^\mu)$ inside the larger disc $|w|=e^t$. $f(zw^\mu)$ must have an additional zero in the annulus $1<|w|<e^t$, otherwise $[x,x']$ would not intersect $\mathscr{A}_f$.

□

We are, therefore, justified in indexing the component by its order (and vice versa).

Proposition 2.8 *Let* C *be a component of* $\mathbb{R}^n\backslash\mathscr{A}_f$ *and* $\nu\in\Delta_f$ *its order. Then the normal cone* $\mathcal{C}_\nu$ *is the recession cone of* C*, i.e.* $C+\mathcal{C}_\nu\subset C$ *and and no larger affine convex cone is in* C.

Proof Let $x\in C$ and $\mu\in\mathbb{Z}^n\backslash\{0\}$. We need to show that the ray $x+\mu\mathbb{R}_{>0}$ does not intersect the amœba $\mathscr{A}_f$ if and only if $\langle\mu,\nu\rangle = \max_{\alpha\in\Delta_f}\langle\mu,\alpha\rangle$ (recall Remark 2.2). Lemma 2.13 implies that the ray does not intersect $\mathscr{A}_f$ precisely when $w\mapsto f(zw^\mu)$ has all of its zeroes inside the unit disc. The claim now follows from the Fundamental Theorem of Algebra because the top degree of $f(zw^\mu)$ is $\max_{\alpha\in\Delta_f}\langle\mu,\alpha\rangle$ and $\langle\mu,\nu\rangle$ counts zero-poles inside the unit disc.

□

Proposition 2.9 *Let $\nu \in \Delta_f \cap \mathbb{Z}^n$, C a connected component of $\mathbb{R}^n \backslash \mathscr{A}_f$ and $z \in \mathrm{Log}^{-1}(C)$. If $|c_\nu z^\nu| > |\sum_{\alpha \in A_f \backslash \{\nu\}} c_\alpha z^\alpha|$ then ν is the order of C.*

Proof Let ν_C denote the order of C. We show $\nu_C = \nu$ by showing that $\langle \mu, \nu_C \rangle = \langle \mu, \nu \rangle$ holds for every $\mu \in \mathbb{Z}^n \backslash \{0\}$. By Lemma 2.13, $\langle \mu, \nu_C \rangle$ counts zero-poles of $w \mapsto f(zw^\mu)$ inside the unit disc. As $\langle \mu, \nu \rangle$ does the same for $w \mapsto c_\nu z^\nu w^{\langle \mu, \nu \rangle}$, the claim follows from Rouché's Theorem.

□

Suppose ν is a vertex of Δ_f. We can write

$$f(z) = c_\nu z^\nu \left(1 + \sum_{\alpha \in A_f \backslash \{\nu\}} c_\alpha c_\nu^{-1} z^{\alpha - \nu} \right) = c_\nu z^\nu (1 + g(z)). \tag{2.28}$$

and, using the geometric series, construct the Laurent expansion

$$\frac{1}{f(z)} = c_\nu^{-1} z^{-\nu} (1 - g(z) + g^2(z) - \dots) \tag{2.29}$$

Proposition 2.10 *There exists $y \in \mathcal{C}_\nu$ such that (2.29) converges absolutely for any $z \in \mathrm{Log}^{-1}(y + \mathcal{C}_\nu)$. In particular, $f(z) \neq 0$ for such z.*

Proof Series (2.29) will converge absolutely for any z such that $|g(z)| < 1$. We have

$$|g(z)| = \left| \sum_{\alpha \in A_f \backslash \{\nu\}} c_\alpha c_\nu^{-1} z^{\alpha - \nu} \right| \leq \sum_{\alpha \in A_f \backslash \{\nu\}} |c_\alpha c_\nu^{-1}| e^{\langle \alpha - \nu, x \rangle}$$

where $x = \mathrm{Log}(z)$. If we choose $y \in \mathcal{C}_\nu$ such that $\langle y, \alpha - \nu \rangle \ll 0$, we will have $|g(z)| < 1$ whenever $x \in y + \mathcal{C}_\nu$.

□

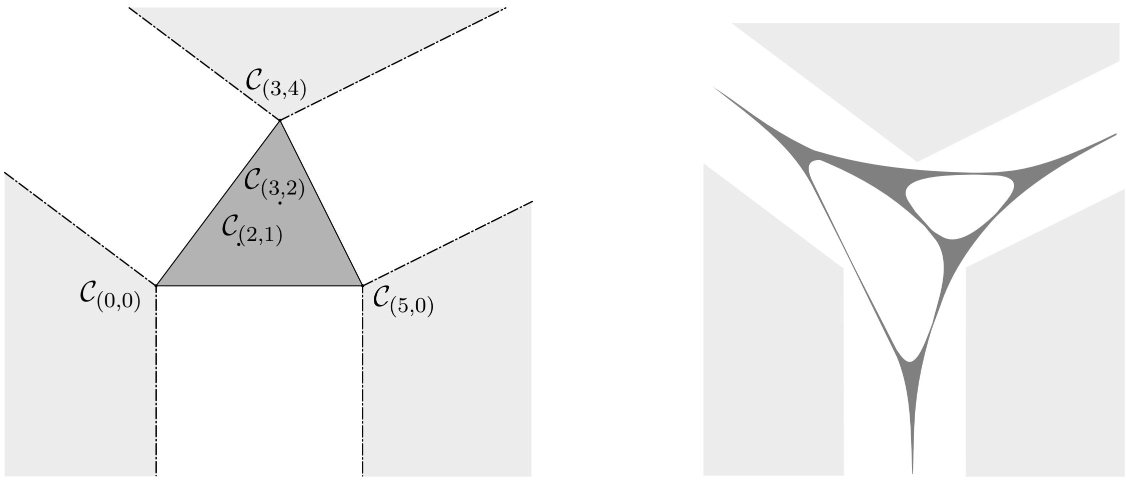


Figure 2.6: Recession cones for $f(z, w) = z^3 w^4 + z^5 + 40 z^3 w^2 + z^3 w + 80 z^2 w + 1$

Figure 2.6 illustrates (translated) normal cones of Δ_f that correspond to components of $\mathbb{R}^n \backslash \mathscr{A}_f$ where f is as in Example 2.3. The two degenerate cones correspond to the two bounded components, while the three full-dimensional cones correspond to the three vertices. The correspondence between certain lattice points in Δ_f and components of $\mathbb{R}^n \backslash \mathscr{A}_f$ suggests a kind of duality. We shall explore this in the next chapter.

Theorem 2.14 [**FPT00**] *The number of connected components of* $\mathbb{R}^n \backslash \mathscr{A}_f$ *is at least equal to the number of vertices of* Δ_f *and at most equal to* $\text{card}(\Delta_f \cap \mathbb{Z}^n)$*.*

Proof The lower bound follows immediately from Propositions 2.9 and 2.10. The upper bound follows immediately from Propositions 2.6 and 2.7.

□

Proposition 2.11 [**Rull00**] *For any lattice polytope* Δ *and any* $\mathcal{N} \subset \Delta_f \cap \mathbb{Z}^n$*, it is possible to construct a Laurent polynomial* f *such that* $\Delta_f = \Delta$ *and* $\mathcal{N}$ *is the set of orders of connected components of* $\mathbb{R}^n \backslash \mathscr{A}_f$*.*

Corollary 2.15 *Both bounds in Theorem 2.14 can be achieved.*

Definition 2.12 Amœbas achieving the lower bound are called *solid* while the amœbas achieving the upper bound are called *full.*

Definition 2.13 A Laurent polynomial f is called *maximally sparse* if its support A_f equals the set of vertices of its Newton polytope Δ_f.

Conjecture 1 (Passare-Rullgård) *Maximally sparse polynomials have solid amœbas.*

2.2 More General Case

It is natural to extend the definitions 2.6 and 2.7 to include more than just hypersurfaces.

Definition 2.14 Let $I \subset \mathbb{C}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$ be a proper ideal and $\mathscr{V}_I := \{z \in \mathbb{T}^n : \forall\, f \in I \;\; f(z) = 0\}$ its zero set. The *amœba* $\mathscr{A}_I$ (resp. coamœba $\mathscr{A}'_I$) of the ideal I is defined as $\mathscr{A}_I := \text{Log}(\mathscr{V}_I)$ (resp. $\mathscr{A}'_I := \text{Arg}(\mathscr{V}_I)$).

Remark 2.10 Let $f_1, \ldots, f_k$ be a finite set of generators of I. Recall that $\mathscr{V}_I = \cap_{j=1}^k \mathscr{V}_{f_j}$ but note that $\mathscr{A}_I \neq \cap_{j=1}^k \mathscr{A}_{f_j}$ and $\mathscr{A}'_I \neq \cap_{j=1}^k \mathscr{A}'_{f_j}$ in general[13]. Again, it would be more accurate to index the amœba and the coamœba by the algebraic set, however the following proposition justifies our notation.

Proposition 2.12 *With notations as above, the following holds:*

$$\mathscr{A}_I = \bigcap_{f \in I} \mathscr{A}_f, \quad \mathscr{A}'_I = \bigcap_{f \in I} \mathscr{A}'_f \tag{2.30}$$

Proof
1) As $\forall\, f \in I \;\; \mathscr{V}_I \subset \mathscr{V}_f$, we have $\forall\, f \in I \;\; \mathscr{A}_I \subset \mathscr{A}_f,\; \mathscr{A}'_I \subset \mathscr{A}'_f$ and therefore inclusions

$$\mathscr{A}_I \subset \bigcap_{f \in I} \mathscr{A}_f, \quad \mathscr{A}'_I \subset \bigcap_{f \in I} \mathscr{A}'_f$$

[13]The dimension need not decrease when we take intersection $\mathscr{A}_f \cap \mathscr{A}_g$ but it does decrease when we take $\mathscr{V}_f \cap \mathscr{V}_g$ (and therefore its image under Log or Arg). E.g. consider what happens when $\dim \mathscr{V}_{f_j} < n/2$.

2) For a Laurent polynomial $f(z) = \sum_\alpha c_\alpha z^\alpha$ define $\widetilde{f}(z) := \sum_\alpha \overline{c_\alpha} z^\alpha$ where $\overline{c_\alpha}$ denotes the complex conjugate of c_α. For any $x \in \mathbb{R}^n \backslash \mathscr{A}_I$ define

$$f_x(z) := \sum_{j=1}^{k} \widetilde{f}_j(e^{2x_1} z_1^{-1}, \ldots, e^{2x_n} z_n^{-1}) f_j(z_1, \ldots, z_n) \in I \tag{2.31}$$

If $z = (e^{x_1+i\theta_1}, \ldots, e^{x_n+i\theta_n}) \in \mathrm{Log}^{-1}(x)$, we have

$$\begin{aligned} f_x(z) &= \sum_{j=1}^{k} \widetilde{f}_j(\overline{z}_1, \ldots, \overline{z}_n) f_j(z_1, \ldots, z_n) \\ &= \sum_{j=1}^{k} \overline{f_j(z_1, \ldots, z_n)} f_j(z_1, \ldots, z_n) \\ &= \sum_{j=1}^{k} |f_j(z)|^2 > 0 \end{aligned}$$

Therefore $x \in \mathbb{R}^n \backslash \mathscr{A}_{f_x}$ and hence $\mathbb{R}^n \backslash \mathscr{A}_I \subset \bigcup_{f \in I} (\mathbb{R}^n \backslash \mathscr{A}_f)$. Taking complements gives $\bigcap_{f \in I} \mathscr{A}_f \subset \mathscr{A}_I$. Analogously, for any $\theta \in (\mathbb{S}^1)^n \backslash \mathscr{A}'_f$ define

$$f_\theta(z) := \sum_{j=1}^{k} \widetilde{f}_j(e^{-2i\theta_1} z_1, \ldots, e^{-2i\theta_n} z_n) f(z_1, \ldots, z_n) \in I. \tag{2.32}$$

If $z = (e^{x_1+i\theta_1}, \ldots, e^{x_n+i\theta_n}) \in \mathrm{Arg}^{-1}(\theta)$, we have $f_\theta(z) = \sum_{j=1}^{k} |f_j(z)|^2 > 0$ and therefore $(\mathbb{S}^1)^n \backslash \mathscr{A}'_I \subset \bigcup_{f \in I} ((\mathbb{S}^1)^n \backslash \mathscr{A}'_f)$. Taking complements gives $\bigcap_{f \in I} \mathscr{A}'_f \subset \mathscr{A}'_I$.

□

Remark 2.11 This also holds for ideals of $\mathbb{C}[Z_1, ..., Z_n]$. We can choose suitable monomials $m_1(z)$ and $m_2(z)$ such that $m_1(z)\widetilde{f}_j(e^{2x_1} z_1^{-1}, \ldots, e^{2x_n} z_n^{-1})$ and $m_2(z) \cdot \widetilde{f}_j(e^{-2i\theta_1} z_1, \ldots, e^{-2i\theta_n} z_n)$ are polynomials and use

$$f_x(z) = \sum_{j=1}^{k} m_1(z) \widetilde{f}_j(e^{2x_1} z_1^{-1}, \ldots, e^{2x_n} z_n^{-1}) f_j(z_1, \ldots, z_n)$$

and

$$f_\theta(z) = \sum_{j=1}^{k} m_2(z) \widetilde{f}_j(e^{-2i\theta_1} z_1, \ldots, e^{-2i\theta_n} z_n) f_j(z_1, \ldots, z_n)$$

in (2.31) and (2.32) respectively.

Theorem 2.16 *Let $I \subset \mathbb{C}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$ be a proper ideal. Then $x \in \mathbb{R}^n$ is in $\mathscr{A}_I$ if and only if $\{|c_\alpha| e^{\langle \alpha, x \rangle}\}_{\alpha \in A_f}$ is not (super)lopsided for any $f \in I$.*

Proof

1) If $x \in \mathscr{A}_I$, we cannot have lopsidedness because $\forall\, f \in I \;\; x \in \mathscr{A}_f$.

2) If $x \notin \mathscr{A}_I$ then $x \notin \mathscr{A}_{f_x}$. Now apply Theorem 2.10 to f_x.

□

Chapter 3

From Amœbas to Tropical Geometry

3.1 The Ronkin Function

Definition 3.1 Let f be a holomorphic function and $\Omega \subset \mathbb{R}^n$ a connected open set. The *Ronkin function* (see [**Ronk74**]) $R_f : \Omega \to \mathbb{R}^n$ is defined in the following way

$$\begin{aligned} R_f(x) :&= \frac{1}{(2\pi i)^n} \int_{\mathrm{Log}^{-1}(x)} \log|f(z_1, \ldots, z_n)| \frac{dz_1 \wedge \cdots \wedge dz_n}{z_1 \ldots z_n} \\ &= \frac{1}{(2\pi)^n} \int_{[0,2\pi]^n} \log|f(e^{x_1+i\theta_1}, \ldots, e^{x_n+i\theta_n})|\, d\theta_1 \ldots d\theta_n \end{aligned} \tag{3.1}$$

It can be viewed as a generalization of the function that appears in Jensen's formula[1].

Theorem 3.1 [**Ronk01**] *Let f be a Laurent polynomial. Then R_f is convex and affine on connected components of $\mathbb{R}^n \backslash \mathscr{A}_f$. If C is a component of order ν_C, we have $\forall\, x \in C \;\; R_f(x) = \langle \nu_C, x\rangle + \tau_C$ for some constant $\tau_C \in \mathbb{R}$.*

Proof That R_f is convex follows from the fact that $\log|f|$ is plurisubharmonic (see Theorem 2.12 and Corollary 1 in [**Ronk74**]).

Let C be a connected component of $\mathbb{R}^n \backslash \mathscr{A}_f$, $x \in C$ and $z = (e^{x_1+i\theta_1}, \ldots, e^{x_n+i\theta_n}) \in \mathrm{Log}^{-1}(x)$. For $j \in \{1, \ldots, n\}$, consider

$$\begin{aligned} \frac{\partial}{\partial x_j} \log|f| &= \frac{1}{2} \frac{\partial}{\partial x_j} \log|f|^2 \\ &= \frac{1}{2} \frac{\partial}{\partial x_j} \log f\overline{f} \\ &= \frac{1}{2} \frac{\partial}{\partial x_j} \left(\log f + \log \overline{f}\right) \\ &= \frac{1}{2} \left(\left(\frac{\partial}{\partial z_j} \log f + \cancel{\frac{\partial}{\partial \overline{z_j}} \log f} \right) \frac{\partial z_j}{\partial x_j} + \left(\frac{\partial}{\partial \overline{z_j}} \log \overline{f} + \cancel{\frac{\partial}{\partial z_j} \log \overline{f}} \right) \frac{\partial \overline{z_j}}{\partial x_j} \right) \\ &= \frac{1}{2} \frac{f_j'}{f} \frac{\partial z_j}{\partial x_j} + \frac{\overline{f_j}'}{\overline{f}} \frac{\partial \overline{z_j}}{\partial x_j} \end{aligned} \tag{3.2}$$

[1] Suppose f is an analytic function in a region of $\mathbb{C}$ that contains a closed disc $\overline{D(0,r)}$, $\xi_1, \ldots, \xi_k$ are zeroes of f in the open disc $D(0,r)$ (counted with multiplicities) and $f(0) \neq 0$. Jensen's formula states that $\frac{1}{2\pi} \int_0^{2\pi} \log|f(re^{it})|\, dt = \log|f(0)| + \sum_{j=1}^k \log \frac{r}{|\xi_j|}$.

In the last step we used the fact that f is a Laurent polynomial so that $\overline{f(z)} = f(\overline{z})$. Since $z_j = e^{x_j + i\theta_j}$, we have $\partial z_j/\partial x_j = z_j$ and $\partial \overline{z_j}/\partial x_j = \overline{z_j}$. (3.2) now becomes

$$\begin{aligned} \frac{\partial}{\partial x_j} \log|f| &= \frac{1}{2}\left(\frac{f_j'}{f} z_j + \frac{\overline{f_j}'}{\overline{f}} \overline{z_j}\right) \\ &= \frac{1}{2}\left(\frac{f_j'}{f} z_j + \overline{\left(\frac{f_j'}{f} z_j\right)}\right) \\ &= \operatorname{Re}\left(\frac{f_j'}{f} z_j\right) \end{aligned} \tag{3.3}$$

This implies[2] that $\partial R_f/\partial x_j$ is precisely the real part of the integral in (2.26), i.e. equals $\nu_{C,j}$ for any $x \in C$. Hence for $x \in C$ we have $R_f(x) = \langle \nu_C, x\rangle + \tau_C$ where τ_C is some real constant.

□

For all components C of $\mathbb{R}^n \backslash \mathscr{A}_f$, let $\tau_C := R_f(x) - \langle \nu_C, x\rangle$. Consider the convex function

$$p_{R_f}(x) = \max_C(\langle \nu_C, x\rangle + \tau_C) \tag{3.4}$$

and the corresponding tropical Larent polynomial $\mathfrak{p}_{R_f} := \boxplus_C \tau_C \boxtimes x^{\boxed{\nu_C}}$. Note that $p_{R_f} \le R_f$ with equality holding on $\mathbb{R}^n \backslash \mathscr{A}_f$.

Definition 3.2 The *spine* $\mathscr{S}_f$ of the amœba $\mathscr{A}_f$ is the tropical hypersurface $\mathscr{V}^{\text{trop}}_{\mathfrak{p}_{R_f}}$, i.e. the set of points in $\mathbb{R}^n$ for which p_{R_f} is not differentiable.

The spine turns out to be dual to a certain subdivision of the Newton polytope. We make this precise in what follows (see [**PR04**]).

3.2 Convex Subdivisions

Definition 3.3 Let $A \subset \mathbb{R}^n$ be a convex set. A collection $\mathfrak{T}$ of non-empty closed convex subsets (called cells) of A is called a *convex subdivision* of A if it satisfies the following conditions.

(1) $\bigcup_{\gamma\in\mathfrak{T}} \gamma = A$;

(2) If $\gamma, \delta \in \mathfrak{T}$ are such that $\gamma \cap \delta \neq \emptyset$ then $\gamma \cap \delta \in \mathfrak{T}$

(3) If $\gamma \in \mathfrak{T}$ and $\delta \subset \gamma$ then $\delta \in \mathfrak{T}$ if and only if δ is a face of γ

By a face of γ we mean a set $\{x \in \gamma : \langle \xi, x\rangle = \sup_{y\in\gamma}\langle \xi, y\rangle\}$ for some $\xi \in \mathbb{R}^n$. We say that $\mathfrak{T}$ is *polytopal* if all $\gamma \in \mathfrak{T}$ are polytopes.

A face δ of γ has a strictly lower dimension than γ which implies that any chain $\gamma_1 \supset \gamma_2 \supset \dots$ in $\mathfrak{T}$ stabilizes and that intersection of any collection of sets in $\mathfrak{T}$ is also in $\mathfrak{T}$.

To any pair $\delta \subset \gamma$ in $\mathfrak{T}$ we shall associate a convex cone, defined as

$$\mathcal{C}(\delta, \gamma) = \{t(x - y) \,:\, x \in \gamma,\ y \in \delta,\ t \ge 0\} \tag{3.5}$$

[2] $\frac{\partial}{\partial x}\int F\,d\eta = \int \frac{\partial}{\partial x} F\,d\eta$ when integrating over a set that does not depend on x.

Definition 3.4 Let $A, B \subset \mathbb{R}^n$ be two convex sets and $\mathfrak{T}$ and $\mathfrak{T}'$ be some convex subdivisions of A and B respectively. $\mathfrak{T}$ and $\mathfrak{T}'$ are called *dual* (to each other) if there is a bijection $\mathfrak{T} \to \mathfrak{T}'$, $\gamma \mapsto \gamma^*$ such that:

(1) $\forall\, \gamma, \delta \in \mathfrak{T} \;\; \delta \subset \gamma \Leftrightarrow \gamma^* \subset \delta^*$;

(2) If $\delta \subset \gamma$ then cones $\mathcal{C}(\delta, \gamma)$ and $\mathcal{C}(\gamma^*, \delta^*)$ are polar[3] (to each other).

Notice from the definition of our associated cone in (3.5) that $\mathcal{C}(\gamma, \gamma)$ is just the linear subspace spanned by γ (after translating it to the origin). The second condition above implies that γ and γ^* are orthogonal and that $\dim \gamma + \dim \gamma^* = n$.

We will show that the function p_{R_f} as defined in (3.4) determines a convex subdivision of $\mathbb{R}^n$ while its Legendre-Fenchel transform $\check{p}_{p_{R_f}}(\xi) = \sup_{x \in \mathbb{R}^n}(\langle \xi, x \rangle - p_{R_f}(x))$ (recall Section 1.4) determines a dual subdivision of its Newton polytope. Note that $\Delta_f = \Delta_{p_{R_f}}$ because, as we have seen in the previous chapter, there is a unique component of $\mathbb{R}^n \backslash \mathscr{A}_f$ for each vertex of Δ_f and Δ_f is the convex hull of its vertices.

Lemma 3.2 *The set of* $\xi \in (\mathbb{R}^n)^* \cong \mathbb{R}^n$ *for which the function*

$$\mathbb{R}^n \ni x \mapsto p_{R_f}(x) - \langle \xi, x \rangle \tag{3.6}$$

is bounded from below equals Δ_f.

Proof
1) Suppose $\xi \in \Delta_f$. Then ξ can be written as a convex combination of vertices of Δ_f, i.e. $\xi = \sum_{\sigma \in V_f} t_\sigma \sigma$ where V_f denotes the set of vertices and $t_\sigma \geq 0$ are constants such that $\sum_\sigma t_\sigma = 1$. For every $\sigma \in V_f$ we have

$$p_{R_f}(x) - \langle \sigma, x \rangle \geq \tau_{C_\sigma}$$

where C_σ denotes the connected component of $\mathbb{R}^n \backslash \mathscr{A}_f$ of order σ. This implies that

$$\begin{aligned} p_{R_f}(x) - \langle \xi, x \rangle &= p_{R_f}(x) - \sum_{\sigma \in V_f} t_\sigma \langle \sigma, x \rangle \\ &= \sum_{\sigma \in V_f} t_\sigma p_{R_f}(x) - \sum_{\sigma \in V_f} t_\sigma \langle \sigma, x \rangle \\ &= \sum_{\sigma \in V_f} t_\sigma (p_{R_f}(x) - \langle \sigma, x \rangle) \\ &\geq \sum_{\sigma \in V_f} t_\sigma \tau_{C_\sigma} \end{aligned} \tag{3.7}$$

[3]If $\mathcal{C}$ is a convex cone, its *dual cone* $\mathcal{C}^*$ is defined as $\{y \in \mathbb{R}^n : \forall\, x \in \mathcal{C} \;\; \langle y, x \rangle \geq 0\}$. (This can be defined for any set, not just a convex cone.) When $\mathcal{C}$ is closed, we have $\mathcal{C}^{**} = \mathcal{C}$. The cone $-\mathcal{C}^*$ is called *the polar cone* of $\mathcal{C}$.

2) Suppose $\xi \notin \Delta_f$ and let σ be a vertex of Δ_f such that $\xi \notin \sigma - \mathcal{C}_\sigma^*$ where $\mathcal{C}_\sigma^*$ denotes the dual cone of $\mathcal{C}_\sigma$. $\sigma - \mathcal{C}_\sigma^*$ is the smallest closed affine convex cone with vertex at σ that contains Δ_f. Because $\xi - \sigma$ is strictly outside of the cone $-\mathcal{C}_\sigma^*$, it follows that the hyperplane orthogonal to $\xi - \sigma$ intersects the interior of $\mathcal{C}_\sigma$ (see figure 3.1). This means that we can find a point $x' \in \mathcal{C}_\sigma$ such that

$$\langle \xi - \sigma, x' \rangle > 0$$

If $x \in C_\sigma$ then $x + tx' \in C_\sigma$ for any $t > 0$ by Proposition 2.8. We therefore have

$$\begin{aligned} p_{R_f}(x + tx') - \langle \xi, x + tx' \rangle &= -\langle \xi - \sigma, x + tx' \rangle + \tau_{C_\sigma} \\ &= -t\langle \xi - \sigma, x' \rangle - \langle \xi - \sigma, x \rangle + \tau_{C_\sigma} \to -\infty \end{aligned}$$

when $t \to \infty$, i.e. (3.6) is not bounded from below.

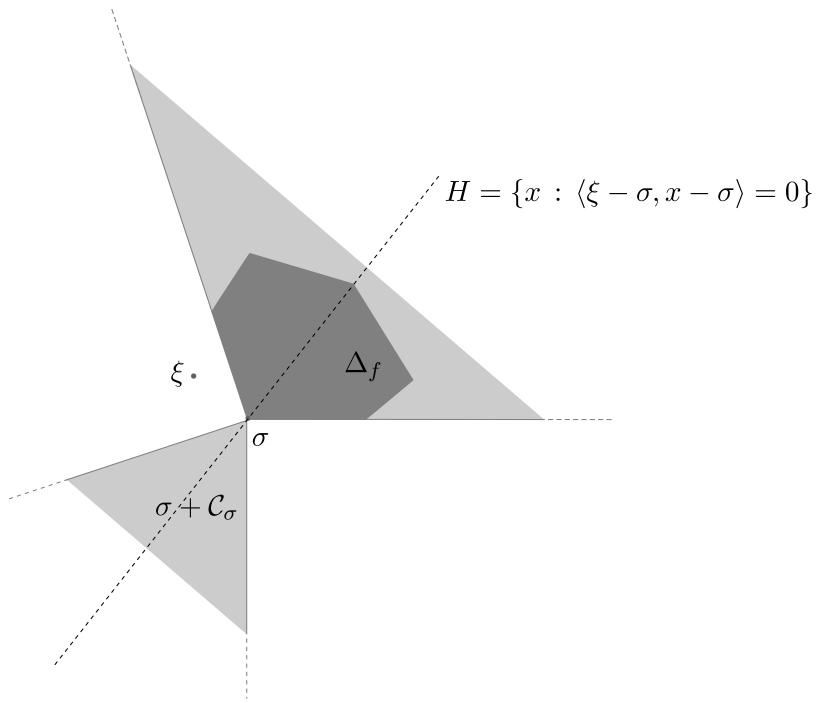


Figure 3.1: Position of $\xi \notin \Delta_f$ with respect to Δ_f

□

Hence we may define a function $H : \Delta_f \times \mathbb{R}^n \to \mathbb{R}$ in the following way.

$$\begin{aligned} H(\xi, x) &:= p_{R_f}(x) + \check{p}_{R_f}(\xi) - \langle \xi, x \rangle \\ &= \sup_{y \in \mathbb{R}^n} (\langle \xi, y \rangle - p_{R_f}(y)) - (\langle \xi, x \rangle - p_{R_f}(x)) \end{aligned} \tag{3.8}$$

Remark 3.1 This function is non-negative and convex in each argument when the other argument is fixed. Moreover, for $x, y \in \mathbb{R}^n$ and $\xi, \eta \in \Delta_f$ we have

$$\langle \xi - \eta, x - y \rangle = -H(\xi, x) + H(\xi, y) + H(\eta, x) - H(\eta, y) \tag{3.9}$$

Proposition 3.1 *For any $x \in \mathbb{R}^n$ there is a point $\xi = \xi_x \in \Delta_f$ such that $H(\xi, x) = 0$.*

Proof By Theorem 1.1, we can write $p_{R_f}(x) = \max_C(\langle \nu_C, x\rangle - \check{p}_{R_f}(\nu_C))$. If $x \in \mathbb{R}^n$ is fixed, we have $p_{R_f}(x) = \langle \nu_C, x\rangle - \check{p}_{R_f}(\nu_C)$ for some C.

□

Lemma 3.3 *For any $\xi \in \Delta_f$ the function* (3.6) *achieves its infimum at some $x \in \mathbb{R}^n$.*

Proof If $\xi \in \mathrm{int}\Delta_f$ we know by the argument above that (3.6) is bounded from below. If we let $||x|| \to \infty$, we have $p_{R_f}(x) - \langle \xi, x\rangle \to \infty$. This is because in every direction of x, for $||x||$ sufficiently large, there is a $\nu_C \notin \mathrm{int}\Delta_f$, i.e. $\mathcal{C}_\nu$ and C_ν are unbounded (recall Remark 2.2 and Proposition 2.8), such that $p_{R_f}(x) = \langle \nu_C, x\rangle + \tau_C$ dominates $\langle \xi, x\rangle$. Hence we can find a (sufficiently large) compact set on which the function (3.6) is bounded from below which implies that it achieves its infimum.

On the other hand, if $\xi \in \mathrm{relint}(\Gamma)$ where $\Gamma = \{\xi \in \Delta_f \,:\, \langle \xi, y\rangle = \max_{\alpha\in\Delta_f}\langle \alpha, y\rangle\}$ is a face of Δ_f and $y \in \mathbb{R}^n\backslash\{0\}$, we consider the truncated Laurent polynomial f_Γ (recall (2.2)). We have

$$p_{R_{f_\Gamma}}(x) = \max_{C\,:\,\nu_C\in\Gamma}\langle \nu_C, x\rangle + \tau_C$$

Clearly $p_{R_f} \geq p_{R_{f_\Gamma}}$. By the arguments above, the function $x \mapsto p_{R_{f_\Gamma}}(x) - \langle \xi, x\rangle$ is bounded from below and achieves its infimum at some point x_0 and hence at all points $x_0 + ty$ for $t \in \mathbb{R}$. There are at most finitely many $\nu_C \notin \Gamma$ for which $\langle \nu_C, x_0\rangle + \tau_C > p_{R_{f_\Gamma}}(x_0)$. For any such ν_C we have

$$p_{R_{f_\Gamma}}(x_0 + ty) - \langle \nu_C, x_0 + ty\rangle \to \infty$$

as $t \to \infty$. Hence for sufficiently large t we have $p_{R_{f_\Gamma}}(x_0 + ty) = p_{R_f}(x_0 + ty)$. Since $p_{R_f}(x) - \langle \xi, x\rangle \geq p_{R_{f_\Gamma}}(x) - \langle \xi, x\rangle$, this implies that $p_{R_f}(x) - \langle \xi, x\rangle$ achieves its infimum at $x_0 + ty$ for sufficiently large t.

If $\xi = \nu_C$ is a vertex of Δ_f, then $p_{R_f}(x) - \langle \xi, x\rangle$ achieves its infimum in the subset of $\mathbb{R}^n$ for which $p_{R_f}(x) = \langle \nu_C, x\rangle + \tau_C$, i.e. $p_{R_f}(x) - \langle \xi, x\rangle = \tau_C$.

□

Corollary 3.4 *For every $\xi \in \Delta_f$ there is a point $x = x_\xi \in \mathbb{R}^n$ such that $H(\xi, x) = 0$.*

Lemma 3.5 *If $x, y \in \mathbb{R}^n$, there is an $\epsilon > 0$ such that $[0, \epsilon] \ni t \mapsto H(\xi, x + ty)$ is a linear function. If $\xi \in \Delta_f$ and $\eta \in \mathbb{R}^n$ are such that $\xi + t\eta \in \Delta_f$ for small $t > 0$, then there is an $\epsilon > 0$ such that $[0, \epsilon] \ni t \mapsto H(\xi + t\eta, x)$ is a linear function.*

Proof Let $\mathcal{N}$ denote the set of orders of components of $\mathbb{R}^n\backslash\mathscr{A}_f$. For $x \in \mathbb{R}^n$, let $A = \{\nu_C \,:\, p_{R_f}(x) = \langle \nu_C, x\rangle + \tau_C\}$, $B = \mathcal{N}\backslash A$, and

$$p_A(x) := \max_{\nu_C\in A}(\langle \nu_C, x\rangle + \tau_C), \qquad p_B(x) := \max_{\nu_C\in B}(\langle \nu_C, x\rangle + \tau_C).$$

Clearly $p_{R_f}(x) = p_A(x) > p_B(x)$ and $p_{R_f}(y) = p_A(y)$ when y is in a neighbourhood of x. Moreover, we have

$$\begin{aligned} p_A(x + ty) &= \max_{\nu_C\in A}(\langle \nu_C, x + ty\rangle + \tau_C) \\ &= \max_{\nu_C\in A}(\langle \nu_C, x\rangle + \tau_C + t\langle \nu_C, y\rangle) \\ &= p_{R_f}(x) + t\max_{\nu_C\in A}\langle \nu_C, y\rangle \end{aligned}$$

To prove the second assertion, suppose $\xi \in \Delta_f$ and $\eta \in \mathbb{R}^n$. Let $x' \in \mathbb{R}^n$ be such that $H(\xi, x') = p_{R_f}(x') + \check{p}_{R_f}(\xi) - \langle \xi, x' \rangle = 0$ and $\langle \eta, x' \rangle$ as large as possible. Note that

$$\begin{aligned}\check{p}_{R_f}(\xi + t\eta) &\geq \langle \xi + t\eta, x' \rangle - p_{R_f}(x') \\ &= \check{p}_{R_f}(\xi) + t\langle \eta, x' \rangle.\end{aligned}$$

Since $p_{R_f}(x) \geq p_A(x)$ and the Legendre-Fenchel transform reverses inequalities, we also have

$$\begin{aligned}\check{p}_{R_f}(\xi + t\eta) &\leq \check{p}_A(\xi + t\eta) \\ &= \sup_{x \in \mathbb{R}^n} (\langle \xi + t\eta, x \rangle - p_A(x)) \\ &= \sup_{x \in \mathbb{R}^n} (\langle \xi, x \rangle - p_A(x)) + t\langle \eta, x' \rangle \\ &= \sup_{x \in \mathbb{R}^n} (\langle \xi, x \rangle - p_{R_f}(x)) + t\langle \eta, x' \rangle \\ &= \check{p}_{R_f}(\xi) + t\langle \eta, x' \rangle.\end{aligned}$$

□

Let $\mathfrak{T}$ denote the collection of all sets $K_\xi = \{x \in \mathbb{R}^n \,:\, H(\xi, x) = 0\}$.
Let $\mathfrak{T}'$ denote the collection of all sets $\kappa_x = \{\xi \in \Delta_f \,:\, H(\xi, x) = 0\}$.

Theorem 3.6 *$\mathfrak{T}$ and $\mathfrak{T}'$ as defined above are dual polytopal convex subdivisions of $\mathbb{R}^n$ and Δ_f respectively, with the correspondence given by*

$$\begin{aligned}K \mapsto K^* &= \bigcap_{x \in K} \kappa_x = \{\xi \in \Delta_f \,:\, \forall\, x \in K \;\; H(\xi, x) = 0\}; \\ \kappa \mapsto \kappa^* &= \bigcap_{\xi \in \kappa} K_\xi = \{x \in \mathbb{R}^n \,:\, \forall\, \xi \in \kappa \;\; H(\xi, x) = 0\}\end{aligned} \tag{3.10}$$

Proof It follows easily from non-negativity and convexity in each variable of $H(\xi, x)$ that K_ξ and κ_x are convex sets. Corollary 3.4 and Proposition 3.1 imply that they are non-empty and that

$$\bigcup_{\xi \in \Delta_f} K_\xi = \mathbb{R}^n, \qquad \bigcup_{x \in \mathbb{R}^n} \kappa_x = \Delta_f$$

Hence $\mathfrak{T}$ and $\mathfrak{T}'$ satisfy condition (1) of Definition 3.3.

If $K_{\xi_1} \cap K_{\xi_2} \neq \emptyset$, we claim that $K_{\xi_1} \cap K_{\xi_2} = K_{(\xi_1+\xi_2)/2}$. Suppose $x \in K_{(\xi_1+\xi_2)/2}$. Then $H(\xi_1, x) = H(\xi_2, x) = 0$ and by non-negativity and convexity (in ξ), it follows that $H(\frac{1}{2}(\xi_1 + \xi_2), x) = 0$, i.e. $x \in K_{(\xi_1+\xi_2)/2}$. Conversely, suppose $x \in K_{(\xi_1+\xi_2)/2}$ and let $y \in K_{\xi_1} \cap K_{\xi_2}$. Then $H(\frac{1}{2}(\xi_1 + \xi_2), x) = 0$ and, by the preceding argument, $H(\xi_1, y) = H(\xi_2, y) = 0 = H(\frac{1}{2}(\xi_1 + \xi_2), y)$. Applying (3.9) to $\xi = \frac{1}{2}(\xi_1 + \xi_2)$ and $\eta = \xi_1$ (and $\eta = \xi_2$) gives $\frac{1}{2}\langle \xi_1 - \xi_2, x - y \rangle = -H(\xi_1, x) = H(\xi_2, x)$. H is non-negative and convex so $H(\xi_1, x) = H(\xi_2, x) = 0$, i.e. $x \in K_{\xi_1} \cap K_{\xi_2}$. Hence $\mathfrak{T}$ satisfies condition (2) of Definition 3.3. The proof is analogous for $\mathfrak{T}'$.

If $K_{\xi_1} \subset K_{\xi_2}$, let $\eta = \xi_1 - \xi_2$ and consider

$$L = \{x \in K_{\xi_2} \,:\, \langle \eta, x \rangle = \sup_{y \in K_{\xi_2}} \langle \eta, y \rangle\}$$

which is a face of K_{ξ_2}. We claim that $K_{\xi_1} = L$. Let $x \in K_{\xi_1}$ and $y \in K_{\xi_2}$. By (3.9), it follows that $\langle \eta, x - y \rangle = H(\xi_1, y) \geq 0$, i.e. $x \in L$. If $x \in L$ and $y \in K_{\xi_1} \subset L$, we have $\langle \eta, x - y \rangle = 0 = -H(\xi_1, x)$, i.e. $x \in K_{\xi_1}$. Hence $K_{\xi_1} = L$.

Conversely, let $\xi_2 \in \Delta_f$ and let $L = \{x \in K_{\xi_2} : \langle \eta, x \rangle = \sup_{y \in K_{\xi_2}} \langle \eta, y \rangle\}$ be a face of K_{ξ_2}. By Lemma 3.5, there is an $\epsilon > 0$ such that $[0, \epsilon] \ni t \mapsto H(\xi_2 + t\eta, x)$ is linear. Let $\xi_1 = \xi_2 + \frac{\epsilon}{2}\eta$. We claim that $L = K_{\xi_1}$.

For $x \notin K_{\xi_2}$, we have $H(\xi_2, x) > 0$. Lemma 3.5 implies

$$H(\xi_1, x) = H(\xi_2, x) + \frac{\epsilon}{2} \underbrace{\langle \eta, x' - x \rangle}_{\geq 0} > 0$$

because x' is any point in L.

For $x \in K_{\xi_2} \backslash L$ and $y \in L$ we have $H(\xi_2, x) = H(\xi_2, y) = 0$ and, by Lemma 3.5, we also have

$$H(\xi_1, x) = H(\xi_2, x) + \frac{\epsilon}{2} \underbrace{\langle \eta, y - x \rangle}_{>0} > 0$$

because $x \notin L$ and $y \in L$.

Finally, if $x, y \in L$, we have $H(\xi_2, x) = H(\xi_2, y) = 0$ and Lemma 3.5 gives

$$H(\xi_1, x) - H(\xi_1, y) = H(\xi_2, x) - H(\xi_2, y) - \frac{\epsilon}{2} \underbrace{\langle \eta, x - y \rangle}_{0} = 0$$

because $x, y \in L$. This implies that $H(\xi_1, x)$ is constant for $x \in L$ and positive for $x \notin L$. By Corollary 3.4, this function attains a zero for some x which means that $H(\xi_1, x) = 0$ for all $x \in L$. Hence $\mathfrak{T}$ satisfies condition (3) of Definition 3.3. The proof for $\mathfrak{T}'$ is again analogous.

We now show that $\mathfrak{T}$ and $\mathfrak{T}'$ are dual. Recall the definition of $K \mapsto K^*$ in (3.9). It is clear that

(1) $\forall\, K_1, K_2 \in \mathfrak{T} \;\; K_1 \subset K_2 \;\Rightarrow\; K_2^* \subset K_1^*$;

(2) $\forall\, \xi \in \Delta_f \;\; \xi \in (K_\xi)^*$

(3) $\forall\, K \in \mathfrak{T} \;\; K \subset K^{**}$

and that analogous statements hold for $\mathfrak{T}'$. This implies that $K \mapsto K^*$ defines a bijection between $\mathfrak{T}$ and $\mathfrak{T}$ that reverses inclusions. Hence condition (1) of Definition 3.4 is satisfied and it only remains to show that whenever $K_1 \subset K_2$, the cones $\mathcal{C}(K_1, K_2)$ and $\mathcal{C}(K_2^*, K_1^*)$ are polar[3].

Let $K_1 \subset K_2$ in $\mathfrak{T}$ and let $x \in K_2$, $y \in K_1$, $\xi \in K_2^*$ and $\eta \in K_1^*$. We have

$$\langle \xi - \eta, x - y \rangle = -H(\xi, x) + H(\xi, y) + H(\eta, x) - H(\eta, y) = H(\eta, x) \geq 0$$

which implies $\mathcal{C}(K_2^*, K_1^*) \subset -\mathcal{C}(K_1, K_2)^*$. To prove the opposite inclusion, let $K_2 = K_\xi$, $K_1 \subset K_2$ and $\eta \in -\mathcal{C}(K_1, K_2)^*$. As in Lemma 3.5, let $\epsilon > 0$ be such that $[0, \epsilon] \ni t \mapsto H(\xi + t\eta, x)$ is linear. We claim that $\xi + \frac{\epsilon}{2}\eta \in K_1^*$. This would imply

$\eta = \frac{2}{\epsilon}(\xi + \frac{\epsilon}{2}\eta - \xi) \in \mathcal{C}(K_2^*, K_1^*)$. For all $x \notin K_2$ we have $H(\xi, x) > 0$ and, as before, by Lemma 3.5, we have $H(\xi + \frac{\epsilon}{2}\eta, x) > 0$. By Corollary 3.4, $H(\xi + \frac{\epsilon}{2}\eta, x) = 0$ for some x. Such x is in $K_2 = K_\xi$ by the preceding argument. Let $y \in K_1$. Since $\eta \in -\mathcal{C}(K_1, K_2)^*$, we have

$$0 \geq \frac{\epsilon}{2}\langle \eta, x - y\rangle = H(\xi + \frac{\epsilon}{2}\eta, y) - H(\xi + \frac{\epsilon}{2}\eta, x)$$

Hence $H(\xi + \frac{\epsilon}{2}\eta, y) \leq H(\xi + \frac{\epsilon}{2}\eta, x) = 0$ and by non-negativity of H, $H(\xi + \frac{\epsilon}{2}\eta, y) = 0$ for all $y \in K_1$, i.e. $\xi + \frac{\epsilon}{2}\eta \in K_1^*$, as wanted. This concludes the proof.

□

Remark 3.2 There is an interesting way to interpret the subdivision $\mathfrak{T}'$ of Δ_f. Namely, if $p_{R_f}(x) = \max_C(\langle \nu_C, x\rangle + \tau_C)$, consider the following subset of $\mathbb{R}^n \times \mathbb{R}$:

$$\operatorname{conv}\{(\nu_C, a) \,:\, a \leq -\check{p}_{R_f}(\nu_C)\}$$

It is an unbounded polytope contained inside the tube $\Delta_f \times \mathbb{R}$. Projecting its bounded faces via $(\xi, x) \mapsto \xi$ to Δ_f gives rise to a polytopal convex subdivision of Δ_f which, in fact, equals $\mathfrak{T}'$. For example, let κ be a full-dimensional face whose set of vertices is $\{(\alpha_j, -\check{p}_{R_f}(\alpha_j))\}_j$ where $\alpha_j \in \{\nu_C\}_C$ and $j \in \{1, \ldots, k\}$. Let $(x_\kappa, 1)$ be a normal to κ. We have

$$\begin{aligned}
\kappa &= \{\xi \in \Delta_f \,:\, \forall\, C \;\; \forall\, j \;\; -\check{p}_{R_f}(\alpha_j) - \langle \xi - \alpha_j, x_\kappa\rangle \geq -\check{p}_{R_f}(\nu_C) + \langle \nu_C - \alpha_j, x_\kappa\rangle\}\\
&= \{\xi \in \Delta_f \,:\, \forall\, j \;\; -\check{p}_{R_f}(\alpha_j) + \langle \xi, x_\kappa\rangle \geq \sup_C(\langle \nu_C, x_\kappa\rangle - \check{p}_{R_f}(\nu_C))\}\\
&= \{\xi \in \Delta_f \,:\, \forall\, j \;\; \check{p}_{R_f}(\alpha_j) + \check{p}_{R_f}(x_\kappa) \leq \langle \xi, x_\kappa\rangle\}\\
&= \{\xi \in \Delta_f \,:\, \check{p}_{R_f}(\xi) + \check{p}_{R_f}(x_\kappa) \leq \langle \xi, x_\kappa\rangle\}\\
&= \{\xi \in \Delta_f \,:\, H(\xi, x_\kappa) = 0\}
\end{aligned}$$

Repeating the same argument for all faces leads to the desired conclusion.

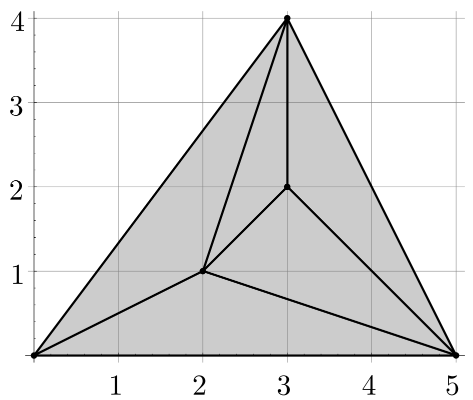


Figure 3.2: Convex subdivision $\mathfrak{T}'$ of Δ_f for f from Example 2.3

Theorem 3.7 [**PR04**] *Let f be a Laurent polynomial and $\mathfrak{T}$ and $\mathfrak{T}'$ the dual convex subdivisions of $\mathbb{R}^n$ and Δ_f respectively, as defined above. Then*

(1) The spine $\mathscr{S}_f$ is the union of all cells in $\mathfrak{T}$ of dimension less than n. Moreover, $\mathscr{S}_f \subset \mathscr{A}_f$.

(2) For any connected component C of $\mathbb{R}^n \backslash \mathscr{A}_f$, the cell dual to $\{\nu_C\}$ contains C.

(3) $\mathscr{S}_f$ is a deformation retract of $\mathscr{A}_f$.

Proof Let $K_C = \{x : p_{R_f}(x) = \langle \nu_C, x\rangle + \tau_C\}$. Then $C \subset K_C$. Since the spine is, by definition, the union of the boundaries of the sets K_C, it follows that $\mathscr{S}_f \subset \mathscr{A}_f$. Moreover, it is readily seen that the sets K_C are precisely the full-dimensional cells in $\mathfrak{T}$ and that their dual points are $\{\nu_C\}$. Thus the lower-dimensional cells are the faces of K_C and their union is, therefore, equal the union of their boundaries, i.e. the spine. This shows (1) and (2).

To prove (3), we construct the deformation retraction. For every connected component C of $\mathbb{R}^n \backslash \mathscr{A}_f$, take a point $x_C \in C$ and consider the set of all segments from x_C to the boundary of K_C. We are done if we show that the union of all such segments contains the amœba.

Let x_C be one such point and let $y \in \mathbb{R}^n \backslash \{0\}$. Suppose that the half-line $\{x + ty : t \geq 0\}$ does not intersect the boundary of K_C. We claim that this implies that it does not intersect the amœba either. Indeed, if $x + ty \in K_C$ for all $t \geq 0$, this means that

$$\forall\, C' \;\; \langle \nu_C, x + ty\rangle > \langle \nu_{C'}, x + ty\rangle.$$

For $t > 0$, we may divide by t and let $t \to \infty$, which gives

$$\forall\, C' \;\; \langle \nu_C, ty\rangle \geq \langle \nu_{C'}, y\rangle$$

which in turn gives

$$\langle \nu_C, y\rangle \geq \max_{\xi \in \Delta_f} \langle \xi, y\rangle.$$

Let $z_C \in \operatorname{Log}^{-1}(x_C)$ and let $\mu \in \mathbb{Z}^n \backslash \{0\}$ be such that

$$\langle \nu_C, \mu\rangle \geq \max_{\xi \in \Delta_f} \langle \xi, \mu\rangle. \tag{3.11}$$

By Lemma 2.13, we know that $-\langle \nu_C, \mu\rangle$ counts the zero-poles of $w \mapsto f(z_C w^{-\mu})$ in the unit disc. Hence the function $w \mapsto w^{\langle \nu_C, \mu\rangle} f(z_C w^{-\mu})$ is a polynomial and has no zeroes in the unit disc. Applying the maximum principle to its inverse implies that

$$\forall\, t \geq 0 \;\; \frac{1}{\min\limits_{|w|=1} (e^{\langle \nu_C, t\mu\rangle} |f(z_C e^{t\mu} w^{-\mu})|)} \leq \frac{1}{\min\limits_{|w|=1} |f(z_C w^{-\mu})|}$$

i.e.

$$\forall\, t \geq 0 \;\; \min_{|w|=1} |f(z_C e^{t\mu} w^{-\mu})| \geq e^{\nu_C, t\mu} \min_{|w|=1} |f(z_C w^{-\mu})|.$$

Letting z_C vary over $\operatorname{Log}^{-1}(x_C)$ gives

$$\forall\, t \geq 0 \;\; \min_{z \in \operatorname{Log}^{-1}(x_C + t\mu)} |f(z)| \geq e^{\langle \nu_C, t\mu\rangle} \min_{z \in \operatorname{Log}^{-1}(x_C)} |f(z)| > 0.$$

This remains true for all $\mu \in \mathbb{Q}^n$ that satisfy (3.11) (because it holds for all $t \geq 0$) and, by continuity, extends to $\mu \in \mathbb{R}^n$. In particular, this is true for $\mu = y$ which implies that $x + ty$ does not hit the amœba.

□

Figures 3.3 and 3.4 illustrate these theorems for f as in Example 2.3.

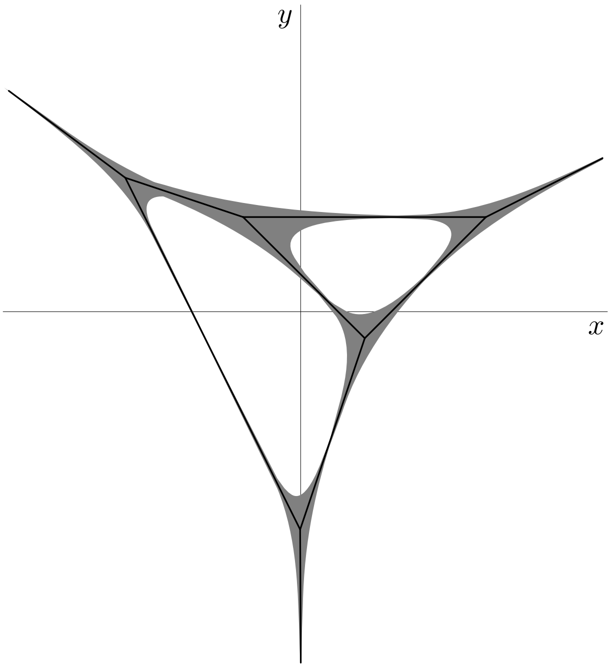


Figure 3.3: Amœba $\mathscr{A}_f$ and its spine $\mathscr{S}_f$

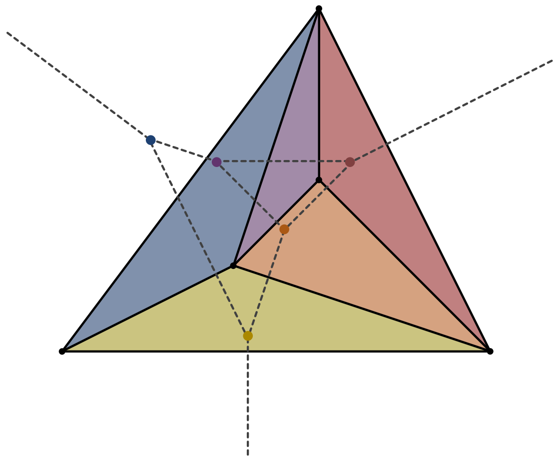


Figure 3.4: Subdivisions of $\mathbb{R}^2$ and Δ_f superimposed

3.3 Spine Approximation

Let f be a Laurent polynomial and recall our definition of the function f_k from Theorem 2.10. Note that

$$\log|f_k(z)| = \sum_{l_1=0}^{k-1} \cdots \sum_{l_n=0}^{k-1} \log|f(e^{2\pi i\, l_1/k} z_1, \ldots, e^{2\pi i\, l_n/k} z_n)|.$$

and that we may interpret $\frac{1}{k^n}\log|f_k(z)|$ as a Riemann sum for the integral (3.1). This is because we are, in fact, partitioning the segment $[0, 2\pi]$ into k equal parts of length $2\pi/k$ in every of the n variables. This corresponds to partitioning the polycircle $\mathrm{Log}^{-1}(x) \ni z$ by multiplying each of the n coordinates by each of the k roots of unity. When $\log|f_k(z)|$ is not bounded, i.e. when $x = \mathrm{Log}(z) \in \mathbb{R}^n \backslash \mathscr{A}_f$, we have

$$\lim_{k\to\infty} \frac{1}{k^n} \log|f_k(z)| = R_f(x).$$

Fix x in a connected component C_ν of $\mathbb{R}^n \backslash \mathscr{A}_f$. By Remark 2.8, for any ε we can find a large enough k such that $f_k(z) = \sum_{\alpha \in A_{f_k}} c_{k,\alpha} z^\alpha$ satisfies

$$|c_{k,\beta}| e^{\langle \beta, x\rangle} > \frac{1}{\varepsilon} \sum_{\alpha \in A_{f_k} \backslash \{\beta\}} |c_{k,\alpha}| e^{\langle \alpha, x\rangle}$$

In fact, we have $\beta = k^n \nu_C$, as the following lemma shows.

Lemma 3.8 *Let $f_k(z) = \sum_{\alpha \in A_{f_k}} c_{k,\alpha} z^\alpha$ be as in Theorem 2.10, if $x \in C_\nu \subset \mathbb{R}^n \backslash \mathscr{A}_f$. If $\{|c_{k,\alpha}| e^{\langle \alpha, x\rangle}\}$ is lopsided, the dominant term has exponent $k^n \nu_C$.*

Proof Let $\zeta \in \mathrm{Log}^{-1}(x)$ and $f_{j,k,\zeta}$ as in (2.22). The dominant term of

$$f_{j,k,\zeta}(z) = \prod_{l_1=0}^{k-1} \cdots \prod_{l_n=0}^{k-1} f(e^{2\pi i l_1/k}\zeta_1, \dots, e^{2\pi i l_{j-1}/k}\zeta_{j-1}, e^{2\pi i l_j/k} z, e^{2\pi i l_{j+1}/k}\zeta_{j+1}, \dots, e^{2\pi i l_n/k}\zeta_n) \tag{3.12}$$

is the dominant term of f_k with fixed $z_l = \zeta_l$ for $l \neq j$. Since (3.12) is a product of k^n terms, each of which has $\nu_{C,j}$ zero-poles in $\{z : |z| < e^{x_j}\}$, we have that $f_{j,k,\zeta}$ has $k^n \nu_{C,j}$ zero-poles in $\{z : |z| < e^{x_j}\}$ and this is equal to the exponent of its dominating term. Hence the dominating term of f_k has exponent $k^n \nu_C$.

□

Thus we have

$$\log |c_{k,\beta} z^\beta| + \log|1-\varepsilon| \le \log|f_k(z)| \log |c_{k,\beta} z^\beta| + \log|1+\varepsilon|.$$

Letting $k \to \infty$, we deduce that

$$\frac{1}{k^n} \log |c_{k,\beta} z^\beta| = \frac{1}{k^n} \log |c_{k,\beta}| + \langle \nu_C, x\rangle$$

converges to $R_f(x)$ and we may use this to approximate the spine.

Proposition 3.2 *Let $r_k(x) := \max(\log|c_{k,\nu_C}| + \langle k^n \nu_C, x\rangle)$. Then*

(1) $\mathscr{V}^{trop}_{r_k} \subset \left\{ x : \{|c_{k,\alpha}| e^{\langle \alpha, x\rangle}\}_{\alpha \in A_{f_k}} \text{ is not lopsided} \right\}$;

(2) $\lim_{k\to\infty} \mathscr{V}^{trop}_{\mathfrak{r}_k} = \mathscr{S}_f$.

Proof (1) follows from the fact that the maximum cannot be achieved for two different affine functions when one term dominates.

To show (2), note that $\frac{1}{k^n} \log |c_{k,\beta}| + \langle \nu_C, x\rangle - R_f(x)$ is a constant that approaches zero as $k \to \infty$. Let C_1 and C_2 be two connected components of $\mathbb{R}^n \backslash \mathscr{A}_f$ and consider the equations

$$\langle \nu_{C_1}, x\rangle + \tau_{C_1} = \langle \nu_{C_2}, x\rangle + \tau_{C_2}$$

and

$$\langle k^n \nu_{C_1}, x\rangle + \log|c_{k,\nu_{C_1}}| = \langle k^n \nu_{C_2}, x\rangle + \log|c_{k,\nu_{C_2}}|.$$

They define two hyperplanes in $\mathbb{R}^n$ that we shall denote by H and H_k respectively. They are parallel and their distance is less than ε times a constant that depends on C_1 and C_2. By Theorem 2.14, the number of components is bounded by $\mathrm{card}(\Delta_f \cap \mathbb{Z}^n)$, i.e. there are finitely many choices and we can decrease the distance across all components by taking k sufficiently large.

□

Chapter 4

Compactification and Contours

In this chapter we give a brief overview of some additional aspects of amœbas which are of importance.

4.1 Toric Compactification

Let $f(z) = \sum_{\alpha \in A_f} c_\alpha z^\alpha$ be a Laurent polynomial and, as before, let Δ_f be its Newton polytope and $\mathscr{V}_f \subset \mathbb{T}^n$ the corresponding hypersurface. Consider the collection of cones dual to faces of Δ_f, also known as the *dual fan* of Δ_f and denoted by Σ_f. Note that these cones are strongly convex rational polyhedral cones. To Σ_f one associates a toric variety[1] $\mathcal{X}_{\Sigma_f}$. Fan Σ_f can be *refined* by subdivisions to obtain a new fan Σ'_f which is simplicial[2] and regular[3]. Let Σ_f denote such a refinement and let $\mathcal{X}_{\Sigma'_f}$ denote the toric variety associated to it. $\mathcal{X}_{\Sigma'_f}$ is a resolution of singularities of $\mathcal{X}_{\Sigma_f}$ and there is a proper birational surjective morphism $\pi : \mathcal{X}_{\Sigma'_f} \to \mathcal{X}_{\Sigma_f}$. Let $\overline{\mathscr{V}_f}$ denote the Zariski closure of $\mathscr{V}_f$ in $\mathcal{X}_{\Sigma'_f}$. This grants us a compact setting in which the information about the amœba is preserved. In this setting, the role of the Log map is played by the so-called *moment map.*

Definition 4.1 The *compactified amœba* $\widetilde{\mathscr{A}_f}$ of f is the image of $\overline{\mathscr{V}_f}$ under the *moment map* $\mu_f : \mathcal{X}_{\Sigma'_f} \to \Delta_f$ given by

$$\mathbb{T}^n \ni z \mapsto \frac{\sum_{\alpha \in \Delta_f \cap \mathbb{Z}^n} |z^\alpha| \alpha}{\sum_{\alpha \in \Delta_f \cap \mathbb{Z}^n} |z^\alpha|}.$$

That compactified amœbas behave well is reflected in the following facts (see Theorems 1.11 and 1.12 in chapter 6 of [**GKZ94**]):

(1) μ_f is surjective.

[1] i.e. a normal algebraic variety that contains $\mathbb{T}^n$ as an open dense subset, such that the action of $\mathbb{T}^n$ on itself extends to the whole variety. See [**Fult93**] and Ch. 5 of [**GKZ94**].

[2] i.e. for every k, every k-dimensional cone of the fan is generated by k linearly independent vectors.

[3] i.e. every n-dimensional cone of the fan is generated by some $\eta_1, \ldots, \eta_n \in \mathbb{Z}^n$ which have coprime coordinates and $\det[\eta_1, \ldots, \eta_n] = \pm 1$. In other words, every cone of the fan is generated by a subset of a basis of the lattice $\mathbb{Z}^n$.

(2) For any face Γ of Δ_f, $\mu_f^{-1}(\Gamma)$ is the closure of the orbit that corresponds to Γ. The orbit itself is $\mu_f^{-1}(\text{int}\Gamma)$.

(3) $\mu_f^{-1}(\Gamma)$ is a point if and only if $\Gamma = \{\gamma\}$ is a vertex.

(4) For any vertex γ, there is a neighbourhood of γ that does not intersect $\widetilde{\mathscr{A}_f}$.

(5) Vertices lie in different connected components of $\Delta_f \backslash \widetilde{\mathscr{A}_f}$.

(6) If Γ is an edge, $\widetilde{\mathscr{A}_f} \cap \Gamma \neq \emptyset$.

This is illustrated in figure 4.1.

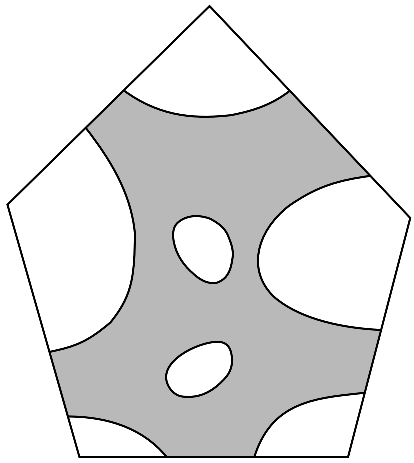


Figure 4.1: Typical shape of a 2-dimensional compactified amœba

4.2 Logarithmic Gauß Map and Contours

Definition 4.2 Let $f \in \mathbb{C}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$ be a Laurent polynomial with no multiple factors. The *logarithmic Gauß map* is the rational map $\gamma_f : \mathscr{V}_f \to \mathbb{P}^{n-1}$ given by

$$\mathscr{V}_f^{\text{reg}} \ni z_0 \mapsto [z_{0,1} \frac{\partial f}{\partial z_1}(z_{0,1}, \ldots, z_{0,n}) : \cdots : z_{0,n} \frac{\partial f}{\partial z_n}(z_{0,1}, \ldots, z_{0,n})]$$

Geometric interpretation of γ_f is as follows. Let $z_0 \in \mathscr{V}_f^{\text{reg}}$ and let $U \subset \mathbb{T}^n$ be a neighbourhood of z_0. Choose a branch of $\log_{|U} : U \to \mathbb{C}^n$, given by

$$(z_1, \ldots, z_n) \mapsto (\log(z_1), \ldots, \log(z_n)),$$

and apply the standard Gauß map on $\log(U \cap \mathscr{V}_f)$, i.e. associate to z_0 a normal to the tangent hyperplane $T_{\log z_0}(\log(U \cap \mathscr{V}_f))$ of $\log(U \cap \mathscr{V}_f)$ at $\log(z_0)$. The direction of the normal corresponds to the point $\gamma_f(z_0)$ and does not depend on the choice of the branch of log.

Definition 4.3 *Contour* $\mathscr{C}_I$ of an amœba $\mathscr{A}_I$, where $I \subset \mathbb{C}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$ is a proper ideal, is defined as the set of critical values of $\text{Log}_{|\mathscr{V}_I} : \mathbb{T}^n \supset \mathscr{V}_I \to \mathbb{R}^n$, i.e. the image of its critical points.

Remark 4.1 We always have $\text{Log}(\mathscr{V}_I{}^{\text{sing}}) \subset \mathscr{C}_I$.

The contour can give us information about the amœba $\mathscr{A}_I$ because the boundary of the amœba is contained in the contour, i.e. $\partial\mathscr{A}_I \subset \mathscr{C}_I$. The contour may also include points from the interior of the amœba.

Proposition 4.1 [**Mikh00**] *Let f be a Laurent polynomial with no multiple factors. Then $\mathscr{C}_f = \text{Log}(\gamma_f^{-1}(\mathbb{P}^{n-1}\mathbb{R}))$.*

Proof Let $z_0 \in \mathscr{V}_f{}^{\text{reg}}$, let U be a neighbourhood of z_0 and choose a branch of $\log_{|U}$. Note that $\text{Log}_{|\mathscr{V}_f}$ is the composition of $\log_{|U\cap\mathscr{V}_f}$ and the projection $\text{Re} : \mathbb{C}^n \to \mathbb{R}^n$. Hence z_0 is a critical point if $d\text{Re} : T_{\log z_0}(\log(U \cap \mathscr{V}_f)) \to \mathbb{R}^n$ is not surjective. The normal direction to the tangent hyperplane $T_{\log z_0}(\log \mathscr{V}_f)$ can be represented by some vector $\widetilde{\gamma}_f(z_0) \in \mathbb{C}^n\backslash\{0\}$. Hence we can write

$$T_{\log z_0}(\log(U \cap \mathscr{V}_f)) = \{x + i\theta \in \mathbb{C}^n \ : \ \langle\widetilde{\gamma}_f(z_0), x + i\theta\rangle = 0\}.$$

If $\widetilde{\gamma}_f(z_0)$ is real, the projection $d\text{Re}$ is not surjective. If $\widetilde{\gamma}_f(z_0)$ is not real, let $\widetilde{\gamma}_f(z_0) = a + ib$. We can consider $\langle\widetilde{\gamma}_f(z_0), x + i\theta\rangle = 0$ as a system of linear equations with fixed x, i.e. we can write

$$\langle a + ib, x + i\theta\rangle = \langle a, x\rangle + i\langle a, \theta\rangle + i\langle b, x\rangle - \langle b, \theta\rangle = 0$$

which is equivalent to

$$\begin{aligned} \langle a, \theta\rangle &= -\langle b, x\rangle \\ \langle b, \theta\rangle &= \langle a, x\rangle \end{aligned}$$

and solve for θ. Thus z_0 is not a critical point of $\mathscr{V}_f$.

□

Corollary 4.1 *Let f be a Laurent polynomial with no multiple factors and with **real** coefficients. Then*

$$\text{Log}(\mathscr{V}_f \cap \mathbb{R}^n) \subset \mathscr{C}_f$$

Proof Points in $\mathscr{V}_f{}^{\text{sing}}$ map to $\mathscr{C}_f$ under Log, so points in $\mathscr{V}_f{}^{\text{sing}} \cap \mathbb{R}^n$ are just a special case. Points in $\mathscr{V}_f{}^{\text{reg}} \cap \mathbb{R}^n$ map to $\mathbb{P}^{n-1}\mathbb{R}$ which implies they are critical (by the proof above).

□

To determine the critical points, we consider the following system of equations:

$$\begin{gathered} f(z) = 0; \\ \left(z_1\frac{\partial f}{\partial z_1}(z) : \cdots : z_n\frac{\partial f}{\partial z_n}(z)\right) = (\lambda_1 : \cdots : \lambda_n) \in \mathbb{P}^{n-1}\mathbb{R} \end{gathered} \tag{4.1}$$

By a theorem of Kouchnirenko [**Kou76**], the number of solutions to a system of n polynomial equations in n variables, in which all polynomials have the same Newton polytope Δ_f, is generically equal to $n!\text{Vol}(\Delta_f)$. As is stated more precisely in the following proposition, this will typically be the degree of the logarithmic Gauß map. When $\mathscr{V}_f$ is singular, the degree will be lower because a singular point will satisfy (4.1) for all $\lambda \in \mathbb{P}^{n-1}\mathbb{R}$.

Proposition 4.2 [**Mikh00**] *Let f be a Laurent polynomial with no multiple factors and suppose that f and all $z_j\partial f/\partial z_j$ have the same Newton polytope Δ_f. If $\overline{\mathscr{V}_f} \subset \mathcal{X}_{\Sigma'_f}$ intersects transversally all orbits corresponding to cones of Σ'_f, the logarithmic Gauß map γ_f can be extended to a dominant rational map $\overline{\gamma}_f : \overline{\mathscr{V}_f} \to \mathbb{P}^{n-1}$ with $\deg\overline{\gamma}_f = n!\text{Vol}(\Delta_f)$.*

The case where we can extend γ_f so that $\deg \widetilde{\gamma}_f = 1$ is particularly interesting. In this case, the contour $\mathscr{C}_f$ can be parametrized by composing $\widetilde{\gamma}_f^{-1} : \mathbb{P}^{n-1}\mathbb{R} \to \mathscr{V}_f$ with Log. This also leads naturally to A-discriminants and hypergeometric functions (see [**Kapr91**], [**GZK89**] and [**PST04**]) which is the context in which the concept of the amœba originally appeared in [**GKZ94**].

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