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Automorphisms on K3 surfaces

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Contents

Chapter 1

Introduction

The history of K3 surfaces is long and rich. They obtained their name in honor of three geometers Kummer, Kähler and Kodaira, and the mountain K2 in Kashmir. They were the topic of researching for a long period in the world of mathematics and physics, and even today they are still a popular and interesting object of research.

The simplest way to describe these surfaces is to say that they are connected complex surfaces with trivial canonical bundle and irregularity $q = 0$. The easiest examples of algebraic K3 surfaces are smooth quartics surfaces in $\mathbb{P}^3$ and it is known that K3 surfaces are Kähler manifolds.

In my thesis I will give several approaches to K3 surfaces and I will describe special subgroup of automorphisms which are of positive entropy and free on K3 surfaces. In the first chapter I gave short review about holonomy groups .Using the Berger's theorem for classification of the Riemannian holonomy groups I showed that K3 surfaces are in the intersection between Calabi-Yau 2 manifolds and hyper-kähler manifolds.

In the next chapter , K3 surfaces , using the standard geometrical and topological techniques, one showed some very important properties about K3 surfaces. There I explicitly "draw" the Hodge diagram and showed that first homology group for K3 surfaces is trivial. Also, I showed that $H^2(X,\mathbb{Z}) = \mathbb{Z}^{22}$ and it is even, non-degenerate lattice with intersection form of signature (3,19); isomorphic to the latice $U^3 \oplus 2E_8(-1)$. Also I wrote about cones and divisors on K3 surfaces and presented several results such as Local and Global Torelli Theorem for K3 surfaces. In the book of [3] is shown that all compact complex surfaces with even first Betti's number have the Hodge decomposition for $H^1(X,\mathbb{C})$ and $H^2(X,\mathbb{C})$. This fact is used a lot of through this chapter, before we conclude fact that all K3 surfaces are Kählerian.

The last chapter is inspired by the paper of Oguiso [15]: Free Automorphisms of Positive Entropy on Smooth Kähler surfaces. It is divided in three parts, where the first 2 parts are proofs of both directions of Ougiso's main theorem, while third part is my own result. The starting point for this chapter was Cantant's theorem:

Theorem 1.0.1. *Let S be a smooth compact Kähler surface admitting an automorphism of positive entropy. Then S is bimeromorphic to either $\mathbb{P}^2$, to a 2-dimensional complex torus, an Enriques surface or a K3 surface.*

By eliminating case by case we proved that the only K"ahleriann surfaces admitting a free automorphism of positive entropy are the K3 surface.

In third part I showed how to construct surface with these desired properties and in result I got that this surface is a complete intersection of four divisors with class (1,1) . Also it is showed how to construct automorphism on that surfaces such that it is free and of positive entropy.

Chapter 2

Motivation

This chapter does not concern topic of my thesis but I found higly interesting to give different approach to K3 surfaces, by reading book of Dominic D.Joyce "Compact complex manifold with special holonomy group" [7].

In this chapter I will introduce the notion of holonomy group and show how it can be used in classification of certain surfaces.

Let X be a n-manifold and let TX, T^*X denote the tangent and cotangent bundles, respectively. The k-th exterior product of T^*X is denoted $\bigwedge^k T^*X$, it is a vector bundle whose fibres have dimension

$$\dim \left(\bigwedge^n TX_x \right) = \binom{n}{k}.$$

The space of its smooth global sections will be denoted $\mathcal{C}^\infty(T^*X)$; the elements of this space are called the smooth k-forms on X.

Definition 2.0.1. Let X be a manifold and let E be a vector bundle on X. A connection ∇ on E is a linear map

$$\mathcal{C}^\infty(T^*X) \to \mathcal{C}^\infty(E \otimes T^*X)$$

with the additional requirement

$$\nabla(f\sigma) = (df) \otimes \sigma + f(\nabla\sigma)$$

where f is a smooth function on X and σ is a smooth section of E.

Definition 2.0.2. If v is a tangent vector field on X (i.e. a smooth section of the bundle TX), then we can define the covariant derivative along v as the map

$$\nabla_v \colon \mathcal{C}^\infty(E) \to \mathcal{C}^\infty(E)$$

given by

$$\nabla_v(\sigma) = \nabla_v(\sigma) := (\nabla\sigma)(v).$$

Definition 2.0.3. The curvature F_∇ of a connection ∇ on a vector bundle E is a 2-form on X with values in $E \otimes E^*$, that is $F_\nabla \in \mathcal{C}^\infty((E \otimes E^*) \otimes \bigwedge^2 T^*X) = \Omega^2(E \otimes E^*)$ satisfying:

$$F_\nabla(v, w,)(\sigma) = \nabla_v\nabla_w\sigma - \nabla_w\nabla_v\sigma - \nabla_{[v,w]}\sigma$$

where v, w are vector fields on X and σ is a section of E.

Let $\alpha\colon [0,1] \to X$ be a smooth curve on a manifold X, then using the pull-back along α we get a vector bundle $\alpha^*(E)$ on $[0,1]$; also, we can pull-back ∇ along α, producing a connection on $\alpha^*(E)$, which we denote as $\alpha^*\nabla$.

A connection ∇ on E defines a notion of parallel transport on E along a curve α:

Definition 2.0.4. A section σ of E along α is said to be parallel if $\nabla_{\dot\alpha(t)}\sigma = 0$. A section σ of $\alpha^*(E)$ is said to be parallel if $\alpha^*\nabla_{\dot\alpha(t)}(\sigma(t)) = 0$ for all $t \in [0,1]$.

If we consider a parallel smooth section σ of $\alpha^*(E)$ over $[0,1]$, in such a way that $\sigma(t) \in E_{\alpha(t)}$ for $t \in [0,1]$, as equation $\alpha^*\nabla_{\dot\alpha(t)}(\sigma(t)) = 0$ is an ordinary differential equation of first order in $\sigma(t)$, so there is unique smooth solution $\sigma(t)$ such that $\sigma(0) = e$ for some $e \in E_{\alpha(0)}$.

Definition 2.0.5. Let $\alpha\colon [0,1] \to X$ be a smooth curve on a manifold X, such that $\alpha(0) = x$, $\alpha(1) = y$ where $x, y \in X$. Then for each $e \in E_x$ there is exactly one smooth section $\sigma(t)$ of $\alpha^*(E)$ which is parallel and such that $\sigma(0) = e$. Define $P_\alpha(e) = s(1)$. Then $P_\alpha\colon E_x \to E_y$ is a well defined linear map called parallel transport map.

Now we can define the holonomy group of a connection on a bundle E and give some interesting properties of this object. Let x be a point on the manifold X. A loop $\alpha\colon [0,1] \to X$ based at x is a piecewise smooth path with $\alpha(0) = \alpha(1) = x$. Parallel transport $P_\alpha\colon E_x \to E_x$ is an invertible linear map, so P_α lies in $\mathrm{GL}(E_x)$, the group of invertible linear transformations of E_x.

Definition 2.0.6. The holonomy group of a connection ∇ on a bundle E on the manifold X at a point $x \in X$ is the group defined as

$$\mathrm{Hol}_x(\nabla) = \{P_\alpha\colon E_x \to E_x \mid \alpha \text{ loop based at } x\} \subset \mathrm{GL}(E_x).$$

Elementary checks show that $\mathrm{Hol}_x(\nabla)$ is indeed a subgroup :

$$(P_\alpha)^{-1} = P_{\alpha^{-1}}, \qquad P_{\beta\alpha} = P_\beta \circ P_\alpha. \tag{2.1}$$

Let X be a connected manifold. We will show that the holonomy group $\mathrm{Hol}_x(\nabla)$ is independent of the choice of base point x. Let x, y be two points on X. There is a piece-wise smooth path $\gamma\colon [0,1] \to X$ such that $\gamma(0) = x$, $\gamma(1) = y$ and $P_\gamma\colon E_x \to E_y$. Let α be a loop based at x; then the loop $\gamma\alpha\gamma^{-1}$ is a loop based at y, and

$$P_{\gamma\alpha\gamma^{-1}} = P_\gamma P_\alpha P_\gamma^{-1}.$$

Hence if $P_\alpha \in Hol_x(\nabla)$ then $P_\gamma \circ P_\alpha \circ P_\gamma^{-1} \in \mathrm{Hol}_y(\nabla)$. Thus:

$$P_\gamma \mathrm{Hol}_x(\nabla) P_\gamma^{-1} = \mathrm{Hol}_y(\nabla). \tag{2.2}$$

So this shows that the holonomy group is independent of the base point up to the conjugation. If E is a bundle of rank k, then $E_x \cong \mathbb{R}^k$ and thus $\mathrm{GL}(E_x) \cong \mathrm{GL}(k,\mathbb{R})$ and $\mathrm{Hol}_x(\nabla)$ can be regarded as a subgroup H of $\mathrm{GL}(k,\mathbb{R})$.The holonomy group is a subgroup of $\mathrm{GL}(k,\mathbb{R})$, defined up to conjugation since aHa^{-1} is a subgroup of $\mathrm{GL}(k,\mathbb{R})$ for $a \in \mathrm{GL}(k,\mathbb{R})$. This implies that equation 2.2 shows that $Hol_x(\nabla)$ and $Hol_y(\nabla)$ yield the same subgroup of $\mathrm{GL}(k,\mathbb{R})$.

Remark 2.0.1. Beacause of this we will omit the subscript x in $\mathrm{Hol}_x(\nabla)$ and write $\mathrm{Hol}(\nabla)$.

Let $(U,(x^1,\ldots,x^n))$ be a local chart of X. For each point $x\in U$ the tangent vectors

$$\frac{\partial}{\partial x^1},\ldots,\frac{\partial}{\partial x^n}$$

do form a basis for T_xU. Hence, any vector field v on U may be uniquely written

$$v=\sum_{a=1}^{n} v^a\frac{\partial}{\partial x^a}$$

for some smooth functions $v^1,\ldots,v^n\colon U\to\mathbb{R}$. Similarly, at each point $x\in U$, $dx^1,\ldots,dx^n$ form a basis for T_x^*U. Hence any 1-form α on U may be uniquely written as

$$\alpha=\sum_{b=1}^{n}\alpha_b dx^b$$

for smooth functions $\alpha_1,\ldots,\alpha_n\colon U\to\mathbb{R}$. We denote v by v^a and α by α^b where a and b runs from 1 to n.

Definition 2.0.7. A tensor T on X is a smooth section of the bundle

$$\bigotimes^k TX\otimes\bigwedge^l T^*X.$$

So as in the above notations, we have

$$T=\sum_{\substack{1\le a_i\le n,1\le i\le k\\1\le b_j\le n,1\le j\le l}} T^{a_1,\ldots,a_k}_{b_1,\ldots,b_l}\frac{\partial}{\partial x^{a_1}}\otimes\ldots\otimes\frac{\partial}{\partial x^{a_k}}\otimes(dx_1^{b_1}\wedge\ldots\wedge dx^{b_l})$$

Remark 2.0.2. A connection ∇ on TX induces connections on all vector bundles of tensors on X, such as $\bigotimes TX\otimes\bigotimes T^*X$. All of these induced connections on tensors we will note by ∇.

Let ∇ be a connection on the tangent bundle TX of X. Then there is a unique tensor $T=T^a_{bc}$ in $\mathcal{C}^\infty(TX\otimes\bigwedge^2T^*X)$ called torsion of ∇ satisfying

$$T(v\wedge w)=\nabla_v w-\nabla_v w-[v,w]$$

for all $v,w\in\mathcal{C}^\infty(TX)$. A connection ∇ with zero torsion is called torsion free.

Definition 2.0.8. Let ∇ be a connection on TX. Then from Remark 2.0.2 ∇ extends to connections on all the tensor bundles $\bigotimes^k TX\otimes\bigotimes^l T^*X$. We say that a tensor S on X is a constant(flat) if $\nabla S=0$.

Theorem 2.0.2. *Let X be a manifold and ∇ a connection on TX. Fix $x\in X$ and let $H=Hol_X(\nabla)$. Then H is a subgroup of $GL(T_xX)$. Let E be the vector bundle $\bigotimes^k TX\otimes\bigwedge^l T^*X$ over X. Then the connection ∇ on TX induces a connection on E and H has a natural representation on the fibre E_x of E at x. Suppose that $S\in\mathcal{C}^\infty(E)$ is a constant tensor. Then $S_{|x}$ is fixed by the action of H on E_x. Conversely, if $S_x\in E_x$ is fixed by the action of H, then there exists a unique tensor $S\in\mathcal{C}^\infty(E)$ such that $\nabla S=0$ and $S_{|x}=S_x$.*

Corollary 2.0.1. *Let X be a manifold and ∇ a connection on TX. Fix $x\in X$. Define $G\subset GL(T_X)$ to be the subgroup of $GL(T_xX)$ that fixes $S_{|x}$ for all constant tensors S on X. Then $Hol(\nabla)$ is a subgroup of G.*

By this theorem and corollary we have the principle that given a manifold X, a connection ∇ on TX with the holonomy group $\mathrm{Hol}(\nabla)$ determines the constant tensors on X, and the constant tensors on X usually determine the holonomy group $\mathrm{Hol}(\nabla)$. Therefore, studying the holonomy of a connection and studying its constant tensors come down to the same thing.

We will now define holonomy groups of a Riemannian metric, or Riemannian holonomy groups which have stronger properties than holonomy groups of connections on arbitrary vector bundles.

Theorem 2.0.3. *Let X be a manifold and g a Riemannian metric on X. Then there exists a unique torsion free connection ∇ on TX with $\nabla g = 0$, called the Levi-Civita connection.*

Proof. Suppose first of all that ∇ is a torsion free connection on TX with $\nabla g = 0$. Let $u, v, w \in \mathcal{C}^\infty(TX)$ be vector fields on X. Then $g(v, w)$ is a smooth function on X and so u acts on $g(v, w)$ to give another smooth function $ug(v, w)$ on X. Since $\nabla g = 0$, using the properties of connections we find that

$$u \cdot g(v, w) = g(\nabla_u v, w) + g(v, \nabla_u w)$$

Combining this with similar expressions for $vg(, w)$ and $wg(u, v)$ we obtain:

$$\begin{aligned} v \cdot g(u, w) &= g(\nabla_v u, w) + g(u, \nabla_v w) \\ w \cdot g(u, v) &= g(\nabla_w u, v) + g(u, \nabla_w v) \\ u \cdot g(v, w) + v \cdot g(u, w) - w \cdot g(u, v) &= g \end{aligned}$$

Hence:

$$g(\nabla_u v, w) + g(v, \nabla_u w) + g(\nabla_v u, w) + g(v, \nabla_u w) - g(\nabla_w u, v) - g(u, \nabla w v)$$

is equal to

$$g(\nabla_u v + \nabla_v u, w) + g(\nabla_v w - \nabla_w v, u) + g(\nabla_u w - \nabla_w u, v)$$

Now if we use the fact that $\nabla_u v - \nabla_v u = [u, v]$ this becomes equal to

$$g(2\nabla_u v - [u, v], w) + g([u, w], u) + g([u, w], v)$$

(this is satisfied as ∇ is torsion free, i.e. $\nabla_v w - \nabla_w v - [v, w] = 0$). And now we have

$$2g(\nabla_u v, w) = u \cdot g(v, w) + v \cdot g(u, w) - w \cdot g(u, v) + g([u, v], w) - g([v, w], u) - g([u, w], v)$$

For fixed u, v there will be unique vector field $\nabla_u v$ which satisfies this equation for all $w \in \mathcal{C}^\infty(TM)$. This defines ∇ uniquely and it turns out that ∇ is a torsion free connection with $\nabla g = 0$. $\square$

First of all we should note that g is a constant tensor as $\nabla g = 0$, so g is invariant under $\mathrm{Hol}(g)$ by Theorem 2.0.2. That is, $\mathrm{Hol}_x(g)$ lies in the subgroup of $\mathrm{GL}(T_x M)$ which preserves $g_{|x}$. This subgroup is isomorphic to $O(n)$. Thus $\mathrm{Hol}_x(g)$ may be regarded as subgroup of $O(n)$ defined up to conjugation, and it is then independent of $x \in X$, so we will write it as $\mathrm{Hol}(g)$.

Definition 2.0.9. A Riemannian manifold (M, g) is said to be a symmetric space if for every point $p \in X$ there exists an isometry $s_p \colon X \to X$ that is an involution (that is, s_p^2 is the identity), such that p is an isolated fixed point of s_p.

Examples of these manifolds are $\mathbb{R}^n$, sphere $\mathcal{S}^n$, projective spaces $\mathbb{CP}^m$ with the Fubini-Study metric and so on...

Definition 2.0.10. A Riemannian manifold (X, g) is called locally symmetric if every point has an open neighbourhood isometric to an open set in a symmetric space and nonsymmetric if it is not locally symmetric.

The preceding definitions and results lead us to the most important theorem of this chapter. It is a theorem due to Berger. This theorem gives us the classification of Riemannian holonomy groups.

Let (X, g) and (Y, h) be a two Riemannian manifolds and let $X \times Y$ be their product manifold. Recall that the product metric $g \times h$ is defined on $X \times Y$ by

$$g \times h|_{(x,y)} = g|_x + h|_y, \quad \forall x \in X,\ y \in Y$$

We call $(X \times Y, g \times h)$ a Riemannian product.

Remark 2.0.3. We can express this more nicely by saying that the forgetful functor $\mathcal{U} \colon \mathbf{RMan} \to \mathbf{Man}$ from the category of Riemannian manifolds to the category of Manifolds creates products.

Definition 2.0.11. A Riemannian manifold (X', g') is said to be locally irreducible if every pair has neighbourhood isomorphic to a Riemannian product $(X \times Y, g \times h)$, otherwise it is irreducible.

Theorem 2.0.4 (Berger)**.** *Suppose X is a simply connected manifold of dimension n and that g is a Riemannian metric on X, which is irreducible and non-symmetric. Then there are the following mutually exclusive alternatives:*

1. *$Hol(g) = SO(n)$;*
2. *$n = 2m$ with $m \geq 2$ and $Hol(g) = U(m)$ in $SO(2m)$;*
3. *$n = 2m$ with $m \geq 2$ and $Hol(g) = SU(m)$ in $SO(2m)$;*
4. *$n = 4m$ with $m \geq 2$ and $Hol(g) = Sp(m)$ in $SO(4m)$;*
5. *$n = 4m$ with $m \geq 2$ and $Hol(g) = Sp(m)Sp(1)$ in $SO(4m)$;*
6. *$n = 7$ and $Hol(g) = G_2$ in $SO(7)$;*
7. *$n = 8$ and $Hol(g) = Spin(7)$ in $SO(8)$.*

Remark 2.0.4.

1. $\mathrm{SO}(n)$ is the holonomy group of the generic Riemannian metric ;
2. A Riemannian metric g such that $\mathrm{Hol}(g) \subset U(m)$ are called Kähler metrics. Kähler metrics form a natural class of metrics on a complex manifold, and the generic Kähler metric on a given complex manifold has holonomy group equal to $U(m)$;

3. Metrics g such that $\mathrm{Hol}(g) = \mathrm{SU}(m)$ are called Calabi-Yau metrics. Since $\mathrm{SU}(m)$ is a subgroup of $U(m)$, all Calabi-Yau metrics are Kähler;

4. Metrics g with $\mathrm{Hol}(g) = \mathrm{Sp}(m)$ are called hyper-Kähler metrics;

5. Metrics g with holonomy group $\mathrm{Sp}(m)\mathrm{Sp}(1)$ for $m \geq 2$ are called quaternionic Kähler;

6. The holonomy groups G_2 and $\mathrm{Spin}(7)$ are called the exceptional holonomy groups.

The Kähler holonomy groups are $U(m)$, $\mathrm{SU}(m)$ and $\mathrm{Sp}(m)$. Any Riemannian manifold with one of these holonomy groups is a Kähler manifold and thus a complex manifold.

The main object of this thesis will be the study of $K3$ surface (complex manifolds of dimension 2), which are in fact the lowest dimensional examples of complex manifolds which are both Calabi-Yau and compact hyper-Kähler, since $\mathrm{SU}(2) = \mathrm{Sp}(1)$.

Chapter 3

$K3$ surfaces

In this chapter I will try to give some properties about $K3$ surfaces. I will use very popular theorems and methods in the world of geometry and try to glue them together in order to get one compact picture of these very nice and interesting objects in geometry.

3.1 Definition and numerical characteristics

Definition 3.1.1. A $K3$ surface is a connected compact complex surface X (i.e. $\dim_{\mathbb{C}}(X) = 2$, or, equivalently, $\dim_{\mathbb{R}}(X) = 4$) such that its first Betti's number is $b_1(X) = 0$ and such that it has trivial canonical divisor, $K_X = 0$.

Let's explain a little bit the second part of the definition. As we know the canonical bundle for a complex manifold X of dimension n is defined to be

$$\bigwedge^n T^*X$$

, where T^*X is the holomorphic cotangent bundle. The sections of this bundle are the forms of type $(n, 0)$. As $K_X = 0$ then we have that $\bigwedge^2 T^*X$ is trivial bundle and so there is the holomorphic 2-form ω_X which is nowhere zero.

This holomorphic form which is non-zero everywhere is unique up to a multiplication by a scalar $c \in \mathbb{C}^*$. So, from this fact, we get that $h^2(\mathcal{O}_X) = h^0(K_X) = 1$, i.e.

$$h^{2,0}(X) = 1 = h^{0,2}(X)$$

Remark 3.1.1. The number h^k represent the dimension of $H^k(X, \mathcal{O}_X)$; in fact these are the Hodge numbers $h^{0,k} = \dim\, H^k(X, \mathcal{O}_X)$. By Poincare duality we have that

$$b_0 = b_4 = 1$$
$$b_3 = b_1,$$

where $b_i = \dim H^i(X, \mathbb{R})$ is i-th Betti's number. Also $h^{i,j} = \dim\, H^j(X, \Omega^i)$ $(i, j = 0, 2)$ and by Serre duality we have $h^{i,j} = h^{2-i,2-j}$ and $h^{0,0} = h^{2,2} = 1$. We use notations

$$q = h^{0,1}\dim\, H^1(X, \mathcal{O}_X), \text{we call q irregularity and}$$
$$p_g = h^{0,2} = \dim\, H^2(X, \mathcal{O}_X) = \dim\, H^0(X, \Omega^2) \text{ we call geometrical genus.}$$

Also, c_1 and c_2 represent first and second Chern classes. We will use some well known formulas from general theory of surfaces in order to obtain some numerical results about K3 surfaces . So if X is compact complex surface we have :

1. *The Noether Formula*
$$p_g - q + 1 = \frac{1}{12}(c_1^2 + c_2)$$
2. *The signature formula of Hirzebruch*
$$b^+ + b^- = \frac{1}{3}(c_1^2 - 2c_2)$$
where b^+ and b^- reprsent the signature (b^+, b^-) of the intersection form
$$H^2(X, \mathbb{R}) \times H^2(X, \mathbb{R}) \to H^4(X, \mathbb{R}) \cong \mathbb{R}$$
and satisfies $b_2 = b^+ + b^-$.
3. *The Gauss-Bonnet formula*
$$c_2 = \sum_i (-1)^i b_i = 2 - 2b_1 + b_2,$$
$$c_2 = \sum_{i,j} (-1)^{i+j} h^{i,j} = 2 - 2q + h^{1,1} + 2p_g - 2h^{1,0}.$$

Now we have:
$$12p_g - 12q + 12 = c_1^2 + c_2,$$
$$3b^+ - 3b^- = c_1^2 - 2c_2;$$

and from these two equalities it follows
$$12p_g - 12q + 12 - 3b^+ + 3b^- = 3c_2,$$
$$4p_g - 4q + 4 - b^+ + b^- = c_2.$$

From the Gauss-Bonnet formula we have
$$c_2 = 2 - 2b_1 + b_2$$
so we deduce
$$4p_g - 4q + 4 - b^+ + b^- = 2 - 2b_1 + b_2, \tag{3.1}$$
and so
$$\begin{aligned}(b^+ - 2p_g) + (2q - b_1) &= \left(\frac{1}{2}b_2 - b_1\right) + \frac{1}{2}(b^+ - b^-) - 2(p_g - q) + 2 \\ &= \left(\frac{1}{2}c_2 - 1\right) + \frac{1}{6}(c_1^2 - 2c_2) - \frac{1}{6}(c_1^2 + c_2) + 2 \\ &= \frac{1}{2}c_2 - 1 + \frac{1}{6}c_1^2 - \frac{1}{6}c_2 - \frac{1}{6}c_1^2 - \frac{1}{6}c_2 + 2 = 1\end{aligned}$$
we can find that
$$2h^{1,0} \leq b_1, \quad b_1 - h^{1,0} \leq q, \quad 2p_g \leq b^+$$
so from our result we get that the possible solutions are

1. $b^+ = 2p_g$ and $b_1 = 2q - 1$;

2. $b^+ = 2p_g + 1$ and $b_1 = 2q$.

So we can conclude that

1. If b_1 is even then

$$b_1 = 2q$$
$$b^+ = 2p_g + 1$$

and from (3.1) and from the Gauss-Bonnet formula we deduce

$$4p_g - 4q + 4 - b^+ + b^- = 2 - 2q + h^{1,1} + 2p_g - 2h^{1,0}$$
$$4p_g - 4q + 4 - 2p_g - 1 + b^- = 2 - 2g + h^{1,1} + 2p_g - 2h^{1,0}$$
$$h^{1,1} - 2h^{1,0} = b^- + 1 - 2q$$

which implys

$$h^{1,1} = b^- + 1, \quad h^{1,0} = 2.$$

Also we have that

$$c_1^2 + 8q + b^- = 10p_g + 9.$$

2. b_1 is odd: $b_1 = 2q - 1$ and $b^+ = 2p_g$. Similarly we get $h^{1,0} = q - 1$, $h^{1,1} = b^-$ and $c_1^2 + 8q + b^- = 10p_g + 8$.

As consequence we see that

$$b_1 = h^{1,0} + h^{0,1},$$
$$b_2 = h^{2,0} + h^{1,1} + h^{0,2}.$$

Definition 3.1.2. Let X be any compact complex manifold. Since there is a pairing $H^0(X, m_1K_X) \otimes H^0(X, m_2K_X) \to H^0(X, (m_1m_2)K_X)$ we can make the direct sum

$$\mathbb{C} \oplus \sum_{m \geq 1} H^0(X, mK_X)$$

into a commutative ring $R(X)$ with unit element. This ring is called the canonical ring of X. $R(X)$ has a finite degree of transcendence, say tr.deg$R(X)$ over $\mathbb{C}$. We can define Kodaira dimension Kod(X) of X as follows

$$\mathrm{Kod}(X) = \begin{cases} -\infty & \text{if } R(X) \cong \mathbb{C} \\ \mathrm{tr.deg}(R(X)) - 1 & \text{otherwise} \end{cases}$$

By Kodaira general classification of surfaces by means of 2 invariants (the first Betti's number b_1 and the Kodaira dimension $\mathcal{K}$) we obtain a division in six big classes:

1. b_1 is even and $\mathcal{K} = -\infty$;

2. b_1 is even and $\mathcal{K} = 0$;

3. b_1 is even and $\mathcal{K} = 1$;

4. $\mathcal{H} = 2$ (this implies b_1 even);
5. b_1 odd, $\mathcal{K} \geq 0$;
6. b_1 odd, $\mathcal{K} = -\infty$.

K_3 surfaces are situated in the second group, with b_1 even and $\mathcal{K} = 0$.

Since b_1 vanishes for K_3 surfaces we have that $q = 0$. The Noether's formula yields

$$\chi(\mathcal{O}_X) = \frac{c_1^2 + c_2}{12} = \frac{K_X \cdot K_X) + c_2}{12}$$

where $\chi(\mathcal{O}_X)$ is the holomorphic Euler characteristic. As K_X is trivial we have $K_X \cdot K_X = 0$ and so

$$\chi(\mathcal{O}_X) = \frac{c_2}{12}$$

while

$$\chi(\mathcal{O}_X) = h^{0,0} - h^{0,1} + h^{0,2}.$$

By definition $h^{0,0} = 1$ and $h^{0,1} = q = 0$ and from $h^{2,0} = 1 = h^{0,2}$ it follows

$$\chi(\mathcal{O}_X) = 2.$$

Thus

$$c_2 = 2 \cdot 12 = 24,$$

c_2 is the Euler topological characteristic and now we have that

$$b_2 = 24 - 2 = 22.$$

From the previous remark we see that $b^+ = 2p_g + 1$, but we know that $\chi(\mathcal{O}_X) = 1 - q + p_g$ and we have also $p_g = 1$. And now we get that $b^+ = 3$ and $b^- = 19$. This leads us to the table which represents the dimension of $H^{p,q}(X)$ for $p, q = 0, 1, 2$.

p \ q	0	1	2
0	1	0	1
1	0	20	0
2	1	0	1

Here we will give very important fact, in order avoid confusion about whether a surface is Kählerian or not. In Chapter IV.2 inside the book [3] it is explained that the Hodge decomposition

$$H^r(X, \mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}(X)$$

holds always in case $k = 2$ and it holds for $k = 1$ if $b_1(X)$ is even.

Theorem 3.1.1. *$H_1(X, \mathbb{Z}) = 0$ if X is $K3$.*

Proof. As consequence of universal coefficients theorem we have that $H_i(X, \mathbb{Z}) \cong \mathbb{Z}^{b_i(X)} \oplus T_i$, where T_i is torsion part. As we have that $H_1(X, \mathbb{R}) = 0$ since $b_1 = 0$ then we have that $H_1(X, \mathbb{Z}) \cong T_1$. So we get that $H_1(X, \mathbb{Z})$ is a torsion group. Let's suppose that $H_1(X, \mathbb{Z})$ has a torsion element of order n. Then by the "unbranched covering trick" (pg. 43 [3]) we have that X has an unbranched covering of order n. So we have $\pi\colon \widetilde{X} \to X$. Using the complex analytic structure on X we can construct an analytic structure on $\widetilde{X}$. The surface $\widetilde{X}$ will be smooth and compact. As we know that X admits holomorphic 2-form ω nowhere 0 on X, then also $\pi^*\omega$ will be a nowhere zero holomorphic 2-form on $\widetilde{X}$, so by this we have that $K_{\widetilde{X}} = 0$. By Noether's formula, for $\widetilde{X}$ we get

$$\chi_{\widetilde{X}}(0) = 1 + p_g(\widetilde{X}) - q(\widetilde{X})$$

As $K_{\widetilde{X}} = 0$ we see that $p_g(\widetilde{X}) = 1$, so we get

$$\chi_{\widetilde{X}}(0) = 2 - q(\widetilde{X}).$$

Also we know that $e(\widetilde{X})$, the topological Euler characteristic of $\widetilde{X}$, is in fact $e(\widetilde{X}) = ne(X)$, also the Noether's formula implies again

$$12\chi_{\widetilde{X}}(0) = K_{\widetilde{X}} + e(\widetilde{X}),$$

and so

$$\begin{aligned} 12(2 - q(\widetilde{X}) &= 0 + 24n = 24n, \\ 2 - q(\widetilde{X}) &= 2n, \\ q(\widetilde{X}) &= 2 - 2n. \end{aligned}$$

By this we get that the only solution is $n = 1$, $q(\widetilde{X}) = 0$, and we see that $H_1(X, \mathbb{Z})$ is torsion free. $\square$

Corollary 3.1.1. *We have that* $H^1(X,\mathbb{Z}) = H_1(X,\mathbb{Z}) = H_3(X,\mathbb{Z}) = H^3(X,\mathbb{Z}) = 0$ *and* $H^2(X,\mathbb{Z}) = H_2(X,\mathbb{Z}) = \mathbb{Z}^{22}$.

Proof. We know that $H^1(X,\mathbb{Z}) = H_1(X,\mathbb{Z}) = 0$. Using Poincare duality we get

$$\begin{aligned} H^1(X,\mathbb{Z}) &= H_3(X,\mathbb{Z}) = 0, \\ H_1(X,\mathbb{Z}) &= H^3(X,\mathbb{Z}) = 0, \\ H^2(X,\mathbb{Z}) &= H_2(X,\mathbb{Z}). \end{aligned}$$

As $H^2(X,\mathbb{Z})$ is a free abelian group and $b_2 = 22$, then $H^2(X,\mathbb{Z}) \cong \mathbb{Z}^{22}$. $\square$

Let's recall some basic facts from geometry: if X is a complex manifold and L, M are line bundles on X, then if

$$\begin{aligned} L &\leftrightarrow \{(U_i, g_{ij}\colon U_i \cap U_j \to \mathbb{C}^* \text{ hol. map})\} \\ M &\leftrightarrow \{(U_i, h_{ij}\colon U_i \cap U_j \to \mathbb{C}^* \text{ hol. map})\} \end{aligned}$$

then we define the relation $L \cong M$, isomorphism of line bundles, if and only if there are holomorphic maps $\varphi_i\colon U_i \to \mathbb{C}^*$ such that

$$g_{ij} = \varphi_i h_{ij} \varphi_j^{-1}$$

The Picard group of X is defined to be

$$\mathrm{Pic}(X) := \{\text{line bundles on } X\}/\cong.$$

It is not hard to prove that it satisfies all the properties of a group.

The exponential sequence

$$0 \to \underline{\mathbb{Z}} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0$$

gives us the long exact sequence

$$0 \to H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X^*) \xrightarrow{\delta} H^2(X, \mathbb{Z})$$

It is well known that $H^1(X, \mathcal{O}_X^*) \cong \mathrm{Pic}(X)$ and we define

$$\mathrm{Pic}^0(X) := H^1(X, \mathcal{O}_X)/H^1(X, \mathbb{Z})$$

and

$$\mathrm{NS}(X) := \mathrm{Im}(\delta) \subseteq H^2(X, \mathbb{Z})$$

$\mathrm{NS}(X)$ is the Neron-Severi group of X and we have the exact sequence

$$0 \to \mathrm{Pic}^0(X) \to \mathrm{Pic}(X) \to \mathrm{NS}(X) \to 0$$

Moreover, we have that

$$\mathrm{NS}(X) \cong \mathbb{Z}^\rho \oplus T$$

where T is a finite group and the number ρ, i.e. the rank of $\mathrm{NS}(X)$, is called the Picard number of X.

$\mathrm{NS}(X)$ is a free abelian group endowedwith a symmetric non-degenerate pairing

$$\langle \cdot, \cdot \rangle \colon \mathrm{NS}(X) \times \mathrm{NS}(X) \to \mathbb{Z}$$

Theorem 3.1.2 (Signature Theorem)**.** *Let X be a compact surface. Then the cup product on $H^2(X, \mathbb{R})$ restricted to $H^{1,1}(X)$ is non degenerate of type $(1, h^{1,1} - 1)$ if $b_1(X)$ is even and of type $(0, h^{1,1})$ and of type $(0, h^{1,1})$ if $b_1(X)$ is odd.*

Proof. Let us consider the space

$$(H^{2,0}(X) \oplus H^{0,2}(X)) \cap H^2(X, \mathbb{R})$$

This is a $2p_g$ dimensional subspace of $H^2(X, \mathbb{R})$ (as $\dim(H^{2,0}(X)) = p_g$). On this subspace the intersection product form is positive definite. Using the Hodge decomposition we see that the orthogonal complement of our space $H^2(X, \mathbb{R})$ is nothing but $H^{1,1}(X)$. And now using our remark 2.1.1 we get that for b_1 even we have $b^+ = h^{1,1} - 1$ and, for b_1 odd, $b^+ = h^{1,1}$. □

Theorem 3.1.3 (Lefschetz theorem on $(1,1)$-classes)**.** *Let X be a compact surface. Then the image of Pic(X), the Neron-Severi group, is*

$$H^{1,1}(X) \cap H^2_{DR}(X, \mathbb{Z})$$

In other words, an element of $H^2(X, \mathbb{C})$ is in the image of Pic(X) if and only if it is integral and can be represented by a real closed $(1,1)$-form.

Theorem 3.1.4. *For a $K3$ surface X, the intersection pairing $\langle \cdot, \cdot \rangle$ on NS(X) is even, non-degenerate and of signature $(1, \rho - 1)$.*

Proof. It is non-degenerate by definition of $\mathrm{NS}(X)$ for any surface. It is even because by Riemann-Roch formula we have

$$\chi(X,L) = \frac{L \cdot L}{2} + 2$$

and so $L \cdot L = 2\chi(X,L) - 4$, and this is even (in fact by Wu's formula we have that intersection-product form on $H^2(X,\mathbb{Z})$ is even). As we know that

$$\mathrm{NS}(X) = H^{1,1}(X) \cap H^2(X,\mathbb{R})$$

then we have

$$H^2(X,\mathbb{R}) = (H^{1,1}(X) \cap H^2(X,\mathbb{R})) \oplus \mathbb{R}\alpha \oplus \mathbb{R}\beta$$

with

$$\alpha = \mathrm{Re}(\omega), \quad \beta = \mathrm{Im}(\omega),$$

where ω is the generator of $H^0(X,\Omega_X^2)$. As $\langle \alpha, \beta \rangle = 0$ and

$$\langle \alpha, \alpha \rangle = \langle \beta, \beta \rangle = \frac{1}{2}\langle \omega, \omega \rangle = 0$$

and the signature of $\langle \cdot, \cdot \rangle$ on $H^2(X,\mathbb{R})$ is $(3,19)$, we have that the signature of $\langle \cdot, \cdot \rangle$ on $\mathrm{NS}(X)$ will be $(1, \rho - 1)$. □

$H^2(X,\mathbb{Z})$ is a free abelian group of rank 22. It is known that $H^2(X,\mathbb{Z}$ is an unimodular lattice which is isomorphic to the lattice

$$(-E_8) \oplus (-E_8) \oplus U \oplus U \oplus U$$

What are U and E_8? U denotes the hyperbolic plane, that is U is a free $\mathbb{Z}$-module of rank 2 whise bilinear form has matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

This is an even lattice and

$$U(-m) \cong U(m)$$

for any m.

E_8 denotes the unique even unimodular positive definite lattice of rank (, the bilinear form on E_8 is given by the matrix

$$\begin{pmatrix} 2 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & -1 & & & \\ & & -1 & 2 & 0 & & & \\ & & -1 & 0 & 2 & -1 & & \\ & & & & -1 & 2 & -1 & \\ & & & & & -1 & 2 & -1 \\ & & & & & & -1 & 2 \end{pmatrix}$$

3.2 Divisors and cones on K3 surfaces

In this section we are going to describe a little bit more the geometry of K3 surfaces. We will use divisors and cones to give results which describes curves , linear systems of divisors and projectivity of K3 surfaces.

First of all I would like to define ample divisors and to give some properties of ample divisors.

Let X be complex compact surface. We say that a line bundle:
$\xi \in H^1(X, \mathcal{O}^*) = Pic(X)$, is very ample if $\varphi_\xi : X \to P^M$ defined as
$\varphi_\xi(x) = (s_0(x), s_1(x), ..., s_M(x))$,where $\{s_0, s_1, ..., s_M\}$ represents the basis for $H^0(X, \xi)$, is an embedding. If $\xi = [D]$ for some $D \in Div(X)$ then we say that divisor D is very ample.

For divisor $D \in Div(X)$ we say that D is ample divisor if exists positive number m such that mD is very ample. We can give some well known properties for ample divisors. For exemple ,we have that :
1.If divisors H and H' are ample divisors, then $H + H'$ is also ample divisor.

2.For divisor $D \in Div(X)$ exists the number $m >> 0$ such that $mH + D$ is very ample.

3.The Kodaira Vanishing Theorem :
If $D \in Div(X)$ and D is ample divisor then we have that :
$H^q(X, K_X + D) = 0, \forall q > 0.$

4.The Nakai-Moishezon criterion:
We have that for divisor $D \in Div(X)$ is satisfied next:
D is ample if and only if : $D \cdot C > 0$ for every irreducible curve C and $D^2 > 0$.

Now we will introduce one notation which will be used a lot of trough this section. With $N(X)$ we will denote : $H^2(X, \mathbb{Z}) \cap H^{1,1}(X)$. Rank of this group will be r. The intersection form on $N(X)$ can be extended to one bilinear symmetric non-degenerate form on $\mathbb{R}^r$. We can associate to it a quadratic form q, which, by Sylvester's Theorem will have canonical form in some basis $\mathbf{e}_1, \ldots, \mathbf{e}_r$ of $\mathbb{R}^r$. By the Hodge Index Theorem this form q is of signature(1,r-1) and its matrix is

$$\begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & -1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & -1 \end{pmatrix}$$

Hence, if $x \in N(X)$, then

$$q\left(\sum_{i=1}^{r} x_i \mathbf{e}_i, \sum_{i=1}^{r} x_i \mathbf{e}_i\right) = x_1^2 - x_2^2 - \ldots - x_r^2$$

Set

$$\Omega := \{x \in N(X) \mid q(x) > 0$$

We call Ω the positive cone of X.

This cone Ω consists of two connected components and each of these two components is convex. The equation is

$$x_2^2 + \ldots + x_r^2 - x_1^2 < 0 \iff x_1^2 > x_2^2 + \ldots + x_r^2$$

From this we see that these components does not contain the vertex.

Definition 3.2.1. The upper positive cone is by definition

$$\Omega_X^+ := \{x \in \Omega \mid x_1 > 0\}$$

Theorem 3.2.1. *Every ample divisor of X has class in Ω_X^+.*

Proof. Let take H' ample , so it is $H' \cdot H' = (H')^2 > 0$ and thus class of $H' \in \Omega$. Let us choose ample divisor H such that class of H in N(X) has first coordinate $x_0 > 0$. So H belongs to Ω_X^+ .

If we suppose that $H' \notin \Omega_X^+$ then we have that the points of the segment $\overline{HH'}$ are of the form $D := \lambda H + (1 - \lambda)H'$ as λ ranges over $[0, 1]$. Now

$$q(D, D) = \lambda^2 H^2 + 2\lambda(1 - \lambda)HH' + (1 - \lambda^2)(H')^2$$

Since every term of this expression is bigger than 0, then

$$q(D, D) \geq 0$$

which is a contradiction. Thus $H' \in \Omega_X^+$. □

Definition 3.2.2. The ample cone of X is the convex cone $\mathcal{A}(X) \subset \mathbb{R}_+$ generated by classes in $N(X)$ of ample divisors. In other words, if $h \in \mathcal{A}(X)$, then

$$h = \sum_{k \geq 0} \lambda_k x_k$$

where $\lambda_k \geq 0$ and where x_k is the class of an ample divisor $H_k \in \mathrm{Div}(X)$.

Remark 3.2.1. $\mathcal{A}(X) \subseteq \Omega_X^+$.

Theorem 3.2.2. *$\mathcal{A}(X)$ is open.*

Proof. Let $H \in \mathrm{Div}(X)$ be an ample divisor. There is a family of divisors $\{D_1, \ldots, D_r\}$ such that their classes $\{d_1, \ldots, d_r\}$ in $N(X)$ form a basis. By the property 2 of ample divisors for all $i = 1, \ldots, r$ there will be $n_i > 0$ such that $\pm D_i + n_i H$ is very ample. Thus

$$H \pm \frac{1}{n_i} D_i \in \mathcal{A}(X)$$

So we have an open neighbourhood of H which is contained in $\mathcal{A}(X)$ and so H is an interior point of $\mathcal{A}(X)$. If this is satisfied for H then it is satisfied for λH, $\lambda \in \mathbb{R}$, and if also H' is an interior point of $\mathcal{A}(X)$, then so it will be $\lambda H + \mu H'$. Then we deduce that for all $h \in \mathcal{A}(X)$,

$$h = \sum \lambda_k H_k$$

with $\lambda_k \geq 0$ and H_k ample, we win that h is an interior point of $\mathcal{A}(X)$. □

Definition 3.2.3. The Nef cone of X is by definition

$$\mathcal{A}'(X) := \{x \in \overline{\Omega_X^+} \mid x \cdot C \geq 0 \,\forall C \text{ irreducible curve in } X\}$$

Remark 3.2.2. Let $D \in \mathrm{Div}(X)$ be a divisor. Then

$$\{x \in N(X) \mid x \cdot D = 0\}$$

is a hyperplane through the origin in $N(X)$ and $\{x \in N(X) \mid x \cdot D \geq 0\}$ is a half-plane which is closed in $N(X)$. With respect to the Sylvester basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_r\}$ we have

$$D = \sum \alpha_i \mathbf{e}_i, \quad x = \sum x_i \mathbf{e}_i$$

and

$$D \cdot x = \alpha_1 x_1 - \alpha_2 x_2 - \ldots - \alpha_r x_r$$

Considering this and the definition of $\mathcal{A}'(X)$ we get that if $x \in \mathcal{A}'(X)$, then $\lambda x \in \mathcal{A}'(X)$ for all $\lambda \in \mathbb{R}_+$, so $\mathcal{A}'(X)$ is a cone.

Proposition 3.2.1. $\mathcal{A}'(X) = \overline{\mathcal{A}(X)}$*, the closure being taken in* $N(X)$*.*

Proof. We have indeed

$$\mathcal{A}'(X) = \overline{\Omega^+} \cap \left(\bigcap_{\substack{C \subset X \\ \text{irreducible}}} \{x \in N(X) \mid x \cdot C \geq 0\} \right)$$

so we obtain that $\mathcal{A}'(X)$ is closed. Also, it is $\mathcal{A}(X) \subseteq \mathcal{A}'(X)$, so that $\overline{\mathcal{A}(X)} \subseteq \mathcal{A}'(X)$. Now we will prove that $\mathcal{A}'(X) \subseteq \overline{\mathcal{A}(X)}$. Let us prove that for every $h \in \mathcal{A}'(X)$ there is a sequence in $\mathcal{A}(X)$ which is convergent to h. $\mathcal{A}(X)$ is contained in $N(X)$ and there are divisors $H_1, \ldots, H_r$ in X with classes $h_1, \ldots, h_r$ in $N(X)$ which form a basis for $N(X)$. If $h \in \mathcal{A}'(X)$, then for all $n \geq 1$ we set

$$P_n := \left\{ h + \sum t_i h_i \mid 0 \leq t_i \leq \frac{1}{n} \right\}$$

As $\mathbb{Q}$ is dense in $\mathbb{R}$ there is $\widetilde{h_n} \in P_n$ with rational coordinates.

For a certain $m = m(n)$ we will have $m\widetilde{h_n} = D$, $D \in \mathrm{Div}(X)$. Then

$$D^2 = m^2 \widetilde{h_n}^2 = m^2(h^2 + 2t_i h h_i + \sum t_i t_j h_i h_j) > 0$$

because every term is bigger than 0 (recall that h_i are ample), and for C irreducible curve in X we have

$$D \cdot C = C \cdot (m\widetilde{h_n}) = m(Ch + \sum t_i C \cdot h_i) > 0$$

hence D is ample and so

$$\frac{1}{m} D \in N(X)$$

As

$$\lim \widetilde{h_n} = h$$

then we see that $h \in \overline{\mathcal{A}(X)}$, i.e. $\mathcal{A}'(X) \subseteq \overline{\mathcal{A}(X)}$. $\square$

Let's make some nice connection with curves on a $K3$ surface X and the cone $\mathcal{A}'(X)$. If C is a curve on the surface X, then by the adjunction formula we have

$$K_C = [K_X \otimes \mathcal{O}_C(C)]|_C$$

As we know that in the case of $K3$ surfaces the canonical divisor K_X is null, we obtain $K_C = \mathcal{O}_C(C)$. This implies

$$\deg K_C = \deg \mathcal{O}_C(C)$$

and since

$$\deg K_C = 2g(C) - 2$$
$$\deg \mathcal{O}_C(C) = \langle G(C), G(C) \rangle = C \cdot C$$

we get

$$2g(C) - 2 = C \cdot C$$

Consider the short exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(C) \to \mathcal{O}_C(C) \to 0 \tag{3.2}$$

The integer $\dim H^1(C, \mathcal{O}_C(C)) = h^1(C, \mathcal{O}_C(C))$ is called arithmetic genus and denoted by p_a. Sequence 3.2 we can rewrite as

$$0 \to \mathcal{O}_X(-C) \to \mathcal{O}_X \to \mathcal{O}_C \to 0$$

By Riemann - Roch we get

$$\chi(-C) = \chi(\mathcal{O}_X) + \frac{1}{2}(C^2 + C \cdot K_X)$$

and so

$$C^2 + C \cdot K_X = 2\chi(-C) - 2\chi(\mathcal{O}_X)$$

Then, as we know that $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_C) + \chi(\mathcal{O}_X(-C))$ we get, by simplifying notation $\chi(\mathcal{O}_X(-C) = \chi(-C)$, that

$$C^2 + C \cdot K_X = 2\chi(\mathcal{O}_X) - 2\chi(\mathcal{O}_C) - 2\chi(\mathcal{O}_X)$$

and

$$C^2 + C \cdot K_X = 2\chi(\mathcal{O}_C)$$

As $\chi(\mathcal{O}_C) = h^0(\mathcal{O}_C) - h^1(\mathcal{O}_C) + h^2(\mathcal{O}_C)$ and $h^2(\mathcal{O}_C) = 0$ ($\dim C < 2$), $h^0(\mathcal{O}_C) = 1$ and $h^1(\mathcal{O}_C) = p_a$. We get that

$$C^2 + C \cdot K_X = -2(1 - p_a) = 2p_a - 2$$

As in case of $K3$ surfaces $K_X = 0$ then we get that $C^2 = 2p_a - 2$, also as we know that $C^2 = 2g - 2$, then $p_a = g$.

If $C \cdot C = 0$, then $p_a = 1$ and by definition of $\mathcal{A}'(X)$ we can conclude that all irreducible curves of arithmetic genus 1 are on $\mathcal{A}'(X)$. As $C \cdot C = 2p_a(C) - 2$, now we can a little bit analyse effective divisors on $K3$:

1. if $p_a(C) = 0$ then $C \cdot C = -2$ and all divisors obtained by linear combination of these kind of curves are not in positive cone and these curves are rational smooth curves;

2. there is no curves D on $K3$ surface X with $C^2 < -2$, it is obvious by fact that $p_a(C) \geq 0$;

3. if $p_a(C) > 0$, all divisors obtained by linear combination of this kind of curves are nef, because $C \cdot C \geq 0$.

Definition 3.2.4. A divisor D on a surface X is said to be:

1. nef: if $D^2 \geq 0$ and $D \cdot C \geq 0$ for each irreducible curve C on X;

2. big and nef (or pseudo-ample): $D^2 > 0$ and $D \cdot C \geq 0$ for each irreducible curve C on$X$.

A fixed component of a linear system $|L|$ is an effective divisor D^* on X such that $D = D' + D^*$ for any $D \in |L|$ where D' is an effective divisor. When D runs through $|L|$ the divisors D' form a linear system $|L'|$ of the same dimension as $|L|$.

If X is a K3 surface with line bundle L such that $L^2 \geq 0$ the conidtion $L \cdot C \geq 0$ for each irreducible curve C on X is equivalent to the condition $L \cdot \delta \geq 0$ for each irreducible (-2)-curve δ on X (cfr. [3], Proposition 3.7).

Also, some well known result from Saint-Donat paper, *Projective Models of K3 surfaces* [16], are:

1.If H is an effective divisor on X, the intersection of H with each curve C is non negative except when C is a component of H and C is a (-2)-curve.

2. If the linear system $|H|$ does not have fixed components and if $H^2 > 0$, then the generic element in $|H|$ is smooth and irreducible and H is a pseudo-ample divisor.

3. The fixed components of a linear system on K3 surfaces are always (-2)-curves and Saint-Donat proves that a linear system on a K3 surface has no base points outside its fixed components.

Theorem 3.2.3. *Let Γ be a divisor on X, a K3 surface. We will suppose that $\Gamma^2 \geq -2$, then Γ or $-\Gamma$ is effective. Moreover if for some L, nef divisor, we have $L \cdot \Gamma > 0$, then it is Γ which is effective.*

Proof. By Kodaira vanishing theorem we have that

$$H^2(X, \mathcal{O}_X(\Gamma)) \cong H^0(X, \mathcal{O}_X(K_X - \Gamma))$$

As we know that $K_X = 0$ we get that

$$H^2(X, \mathcal{O}_X(\Gamma)) \cong H^0(X, \mathcal{O}_X(-\Gamma))$$

simplifying notation we get $h^2(\Gamma) = h^0(-\Gamma)$. By Riemann-roch we get

$$\begin{aligned} h^0(\Gamma) + h^0(-\Gamma) &\geq h^0(\Gamma) + h^0(-\Gamma) - h^1(\Gamma) - \chi(\Gamma) \\ &= \chi(\mathcal{O}_X) + \frac{\Gamma(K_X + \Gamma)}{2} \\ &= 1 - g + p_g + \frac{\Gamma \cdot \Gamma}{2} = 2 + \frac{\Gamma^2}{2} \geq 1 \end{aligned}$$

So by this Γ or $-\Gamma$ is effective. If $L \cdot \Gamma > 0$ for some nef divisor L, then $-\Gamma$ cannot be effective. ☐

Theorem 3.2.4. *Let X be a smooth projective K3 surface, let L be a nef and big line bundle on X (i.e. pseudo-ample), and suppose there is an effective divisor D such that $L \cdot D = 1$, $D^2 = 0$. then $|L|$ has a fixed component.*

Proof. We will consider the divisor $L - gD$. From genus formula we get that $L^2 = 2g - 2$. Then

$$(L - gD)^2 = L^2 - 2gLD + g^2D^2 \quad (L \cdot D = 1,\ D^2 = 0)$$
$$2g - 2 - 2g = -2$$

while

$$L \cdot (L - gD) = 2g - 2 - g = g - 2$$

L is nef and if

1. $g \geq 3$ then $L \cdot (L - gD) > 0$ and by previous lemma we get that $L - gD$ is effective.

2. $g = 2$ then we have that $L - 2D$ or $2D - L$ is effective. If $2D - L$ is effective, let $\widetilde{D} = L - D$. Then we have $L \cdot \widetilde{D} = 1$, $(\widetilde{D})^2 = 0$ and $L - 2\widetilde{D} = 2D - L$ is effective. Thus replacing D by $\widetilde{D}$ we get that $L - gD$ is effective. As $h^0(D) + h^0(-D) \geq 2 + \frac{D^2}{2} = 2$ and $h^0(-D) = 0$ we get $h^0(D) \geq 2$. Now we can conclude that $h^0(gD) \geq g + 1$. also we know that $h^0(L) = 2 + \frac{L^2}{2} = 2 + \frac{2g-2}{2} = g + 1$. . From :

$$0 \to \mathcal{O}_X(gD) \to \mathcal{O}_X((L - gD) + gD) \to \mathcal{O}_{L-gD}((L - gD) + gD) \to 0$$

we have the injection $0 \to H^0(gD) \to H^0(L)$ and $\dim(H^0(gD) \geq \dim(H^0(L))$ and so $L - gD$ has base points and it is fixed component of linear system |L|.

□

Let's suppose that we have on some smooth surface X a collection of n smooth rational curves $C_1, \ldots, C_n$ such that $C_i^2 = -2$, $\bigcup C_i$ is connected and intersection matrix is negative definite [1] .

Then there is a contraction map $\pi \colon X \to \overline{X}$ such that $\pi(\bigcup C_i) = P$, P is point and $\pi|_{X \setminus \bigcup C_i} \colon X \setminus \bigcup C_i \to \overline{X} \setminus \{P\}$ is an isomorphism. The point $P \in \overline{X}$ is a rational double point.

Theorem 3.2.5. *Let X be a smooth K3 surface, let $|L|$ be a nef and big base point-free linear system on X and suppose there is an effective curve D such that $L \cdot D = 0$, $D^2 = -2$. Then every irreducible component D_i of D satisfies $L \cdot D_i = 0$, $D_i^2 = -2$. TMoreover if $C_1, \ldots, C_n$ is maximal connected set of irreducible curves such that $L \cdot C_i = 0$, $C_i^2 = -2$, then there is a contraction $\pi \colon X \to \overline{X}$ of $\bigcup C_i$ to a rational double point, and the map $\varphi_{|L|}$ factors through π.*

Proof. Let $D = \sum n_i D_i$ with $n_i > 0$. Since $L \cdot D = 0$, we have that $0 = \sum n_i L \cdot D_i$, but as L is nef we get that for all i, $L \cdot D_i \geq 0$, as $0 = \sum n_i L \cdot D_i$ implies $L \cdot D_i = 0$ for all i. If we recall Hodge Algebraic Index Theorem ,

[1] A matrix $M_{n \times n}$ is negative definite if $x^t M x < 0$ for all n-dimensional vector x.

which states the following: *Let D, E be divisors with rationa coefficients on the algebraic surface X (projective compact surface). If $D^2 > 0$ and $D \cdot E = 0$, then $E^2 \leq 0$.* Then by this theorem we get, since $L \cdot D_i = 0$, that $D_i^2 < 0$. So as D_i is on K3 and $D_i^2 < 0$, we have that $D_i^2 = -2$ (there is no (-1)-curve on a K3 surface) and D_i is smooth rational curve. These curves can be contracted to a rational double points.

Let $C_1, \ldots, C_n$ be a maximal connected set of curves as before. Also, we will suppose that $L^2 \geq 4$, that $C = \sum n_i C_i$ satisfies $C \cdot C_i \leq 0$ for all i, and that $C^2 = -2$. Let us consider the linear system $|L - C|$. Then

$$(L - C)^2 = L^2 - 2LC + C^2 = L^2 - 2$$
$$L \cdot (L - C) = 2g - 2$$

By previous theorem we get that $L - C$ is effective. We will suppose that $L - C$ is not nef.

Let Γ be an irreducible curve such that $(L - C) \cdot \Gamma < 0$. If $\Gamma^2 \geq 0$, then $|\Gamma|$ moves (by Saint-Donat and the introductory part we have that Γ is big and nef), so that $(L - C) \cdot \Gamma$ cannot be negative since $L - C$ is effective. So we have that $\Gamma^2 = -2$. We have $C \cdot \Gamma > L \cdot \Gamma \geq 0$, since $(L - C) \cdot \Gamma < 0$ and L is nef, then Γ cannot be component of C because all components of C have negative intersection with C and so we get that $L \cdot \Gamma > 0$ (not $L \cdot \Gamma = 0$ because Γ is not component of C).

Let $x = L \cdot \Gamma$ and $y = C \cdot \Gamma$, and we know that $L^2 = 2g - 2$. We have that $0 < x < y$ and also next matrix of intersection:

$\cdot$	L	C	Γ
L	$2g - 2$	0	x
C	0	-2	y
Γ	x	y	-2

By Hodge index theorem we must have that the determinant of this matrix is ≥ 0.

So we have

$$0 \leq (2g - 2)(4 - y^2) + 2x^2 \leq (2g - 2)(4 - y^2) + 2y^2$$

and so

$$y^2 < 4\left(\frac{g - 1}{g - 2}\right) \leq 8 \quad (g > 2)$$

i.e. $y = 2$ and $x = 1$.

So we have $(C + \Gamma)^2 = 0$ and $L \cdot (C + \Gamma) = 1$, by previous lemma we have that $(C + \Gamma)$ is fixed component of $|L|$, which is not possible because we supposed that $L^2 \geq 4$ (by Saint Donat it is very ample).

So we got contradiction and then $|L - C|$ is nef. By Riemann-Roch:

$$h^0(L - C) = \frac{1}{2}(L - C)^2 + 2 = \frac{1}{2}L^2 + 2 - 1 = h^0(L) - 1$$

We see that C impose just one condition on $H^0(L)$ so $\rho_{|L|}(C)$ must be a point, so $\varphi_{|L|}$ factors through the contraction.

In the case where $L^2 = 2$, we will get:

$$h^0(L-C) \geq \frac{1}{2}(L-C)^2 + 2 = 2$$

As C is not fixed component of L we have that

$$2 \leq h^0(L-C) < h^0(L) = 3$$

and so $h^0(L-C) = h^0(L) - 1$, and we get the same result as before. □

Now, one theorem from Morrison paper: *On K3 surfaces with big Picard number* ([Mor]):

Theorem 3.2.6. *Let L be nef and big linear system on a K3 surface.*

1. *$|L|$ has base points if and only if there is a divisor D such that $L \cdot D = 1$, $D^2 = 0$;*
2. *in the case of no base points:*
   (a) *if $\pi \colon X \to \overline{X}$ denotes the contraction of all effective curves C with $L{\cdot}C = 0$, $C^2 = -2$ to rational double points, then $\varphi_{|L|}$ factors through π;*
   (b) *the induced map $\overline{\varphi} \colon \overline{X} \to \mathbb{P}^g$ has degree 2 if and only if either $L^2 =$ " ,or $L \sim 2D$ for divisor D with $D^2 = 2$ or there is a divisor D with $L \cdot D = 2$, $D^2 = 0$. Otherwise, $\overline{\varphi} \colon \overline{X} \to \mathbb{P}^g$ is an embedding.*

And now as conclusion of this part we will prove the following theorem:

Theorem 3.2.7. *If $|L|$ is a nef and big linear system on a K3 surface, then $|3L|$ induces an embedding of X into the projective space.*

Proof. By previous theorem, if we suppose that this is not true then either $2 = (3L)^2 = gL$, which is impossible, or $3L \sim 2D$ and $D^2 = 2$, so $L^2 = \frac{4}{9}D^2$ and $L^2 = \frac{8}{9}$, also impossible, or there is some D with $D^2 = 0$, and $3L \cdot D = 1$ or $3L \cdot D = 2$, both cases impossible. By elimination we proved what we wanted. □

3.3 Weyl Group and Torelli Theorem for $K3$ surfaces

Definition 3.3.1 (Kähler Cone)**.** We will define the Kähler cone for compact complex surface X as :

$$C_X := \{x \in N(X) \mid x^2 > 0 \text{ and } x \cdot d > 0 \text{ for all effective divisors}\}$$

C_X is a convex subcone of the positive cone. But also we should notice very important facts about Kähler cone:

1. The Kähler cone is the ample cone. This is provided by Nakai-Moishizon criterion which states that: for divisor $D \in \mathrm{Div}(X)$ we have that D is ample if and only if $D \cdot C > 0$ for all irreducible curves C on X and $D^2 > 0$.

2. The Kähler cone contains all Kähler classes, because in general if X has Khaler metric with cohomology class κ, then is $\kappa \cdot D > 0$ for all effective divisors. Also necessary condition which must be satisfied in order that class be Kähler is that $\kappa^2 > 0$. So by this all Kähler classes lie in Kähler cone.

Now, let's define $\Delta(X)$ as

$$\Delta(X) := \{\delta \in N(X) \mid \delta^2 = -2\}.$$

Divisors with this kind of class will be called nodal divisors. And also we will define $\Delta^+(X)$ as

$$\Delta^+(X) := \{\delta \in \Delta(X) \text{ and } \delta \text{ is the class of an effective divisor}\}$$

If we fix a Kähler class κ on X, then all effective divisors $x \in N(X)$ which satisfy $x^2 > 0$ are in the same component of the cone $\Omega(X)$ where κ belongs.

Definition 3.3.2. Let X be a complex manifold and G be a subgroup of the group of automorphism of X. Then we say that

1. G acts properly discontinuously on X if for every two compact sets K_1, K_2 from X we have that

$$\{g \in G \mid g(K_1) \cap K_2 \neq \emptyset\}$$

is finite.

2. G acts without fix points on X if for all $g \in G$, $g \neq \mathrm{id}_X$, g doesn't have fixed points.

Remark 3.3.1. If 1. and 2. are satisfied then

$$X/G = \{\text{orbits of } G \text{ in } X\}$$

is a complex manifold and $\pi\colon X \to X/G$ is holomorphic, surjective and locally biholomorphic.A nice example of a such variety is complex torus, i.e. $X = \mathbb{C}^n$, $\Gamma \subset \mathbb{C}^n$ lattice of rank $2n$.

Definition 3.3.3. If X, X' are surfaces, an isomorphism of $\mathbb{Z}$-modules $H^2(X,\mathbb{Z}) \to H^2(X',\mathbb{Z})$ is called a Hodge isometry if

1. It preserves the cup product (i.e. it is an isometry);
2. its a $\mathbb{C}$-linear extension $H^2(X,\mathbb{C}) \to H^2(X',\mathbb{C})$ preserves the Hodge-decomposition.

If X, X' are moreover Kähler surfaces, a Hodge isometry is called effective if it preserves the positive cone.

Definition 3.3.4. For $\delta \in \Delta(X)$ we will define reflection in δ to be the mapping: $S_\delta\colon H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{Z})$ defined as

$$S_\delta(x) = x + (x \cdot \delta)\delta$$

Theorem 3.3.1. *The reflection in δ, S_δ, is a Hodge isometry.*

Proof. 1. It preserves intersection form since

$$\begin{aligned}(x+(x\cdot\delta)\delta)^2 &= x^2 + x(x\cdot\delta)\delta + (x\cdot\delta)\delta x + (x\cdot\delta)\delta(x\cdot\delta)\delta \\ &= x^2 + (x\cdot\delta)^2 + (x\cdot\delta)^2 + (x\cdot\delta)^2\delta^2 = \qquad (\delta^2=-2, \delta\in\Delta(X)) \\ &= x^2 + 2(x\cdot\delta)^2 - 2(x\cdot\delta)^2 = x^2\end{aligned}$$

2. It preserves the Hodge decomposition because: $\delta \in N(X)$, so $\delta \in H^{1,1}(X)$ implies $H^{2,0} \oplus H^{0,2} \in \delta^{\perp}$.

□

Definition 3.3.5. The Weyl group of X is defined to be the subgroup $W(X)$ of $\mathrm{Aut}(H^2(X,\mathbb{Z})$ generated by $\{S_\delta \mid \delta \in \Delta\}$.

$W(X)$ is a subgroup of those automorphisms which preserve the intersection form and Hodge decomposition.

Theorem 3.3.2. *$W(X)$ is a discrete group which acts properly discontinuously on $\Omega(X)$.*

Proof. $\Omega(X) \cong \Omega^1(X) \times \mathbb{R}^+$, where with $\Omega^1(X)$ we denote

$$\Omega^1(X) = \{x \in N(X) \mid x^2 = 1\}$$

and $\mathbb{R}^+$ denotes the positive reals. So now we will prove our result by showing that $W(X)$ acts properly discontinuously on $\Omega^1(X)$.

$\mathrm{Aut}(N(X)) \cong O(1, r-1)$, this group acts transitively on $\Omega^1(X)$, and the stabilizer of point is compact group isomorphic to $O(r-1)$. $W(X)$ is a subgroup of $\mathrm{Aut}(H^2(X,\mathbb{Z}))$,so it is discrete in $\mathrm{Aut}(H^2(X,\mathbb{R}))$ and hence also in the subgroup $\mathrm{Aut}(N(X))$ in which it lies.The action of $W(X)$ on $O(1,r-1)$ is properly discontinuous, which implies that the induced action on $\Omega^1(X) \cong O(1,r-1)/O(r-1)$ is also properly discontinuous. □

Theorem 3.3.3. *If a discrete group W acts properly discontinuously on a space $\Omega(X)$ and if S is a subset of W then*

$$F = \bigcup_{s\in S} \{x \mid s(x) = x\}$$

is closed in $\Omega(X)$.

Proof. For $y \in X \setminus F$ let

$$W_y := \{w \in W \mid w(y) = y\}$$

be the stabilizer. We have $W_y \cap S = \emptyset$. Since the action is properly discontinuous, there is neighbourhood U of y such that $WU \cap U = \emptyset$ for all $w \in W \setminus W_y$ and in particular for all $w \in S$. Hence, there is no point of U fixed by any element of S, so $U \subset X \setminus F$. □

Corollary 3.3.1. $\bigcup_{\delta\in\Delta(X)} \delta^{\perp}$ *is closed in $\Omega(X)$.*

Proof. $\delta^{\perp}$ is the fixed locus of the reflections S_δ, and by applying previous theorem we will obtain the result. □

Definition 3.3.6. The hyperplane $\delta^{\perp}$ will be called *walls* in $\Omega(X)$. Connected component of $\Omega(X) \setminus \bigcup_{\delta \in \Delta(X)} \delta^{\perp}$ will be called *chambers* of $\Omega(X)$.

Remark 3.3.2. It is clear that chambers are open sets of $\Omega(X)$.

Theorem 3.3.4. *The group $W(X) \times \{\pm 1\}$ acts transitively on the set of chambers of $\Omega(X)$.*

Proof. Since ± 1 interchanges the two connected components upper and down component of positive cone, and since $W(X)$ preserves them because

$$x \cdot S_{\delta}(x) = x^2 + (x \cdot \delta)(x \cdot \delta) > 0$$

we have to check the transitivity of the action of $W(X)$ on the chambers in one of the components of $C(X)$.

Let $x, y \in \Omega(X)$ be such that x and y belongs to the same component of $\Omega(X)$, i.e. $x \cdot y > 0$, and let $x \cdot \delta \neq 0$ and $y \cdot \delta \neq 0$ for all $\delta \in \Delta(X)$. We want to prove that for at least one $w \in W(X)$, $w(x)$ and y are in the same connected component of

$$\Omega(X) \setminus \bigcup_{\delta \in \Delta(X)} \delta^{\perp}$$

Let us take x and let l be such that $x^2 = l$. Then for any $a \in \mathbb{R}$ the set

$$\{z \in \Omega(X) \mid 0 \leq y \cdot z \leq a, z^2 = l\}$$

is compact.

As we proved action of $W(X)$ on $C(X)$ is properly discontinuous so it follows that

$$\{w \in W(X) \mid y \cdot w(x) \leq a\}$$

is a finite set. Note that $0 \leq y \cdot w(x)$ and $w(x)^2 = l$. First is from fact that $w(x)$ preserves components of cone, second is from fact that $(S_{\delta}(x))^2 = x^2 = l$. From this we can conclude that the function $z \mapsto y \cdot z$ on the orbit Wx of x attains its minimum at a point $z_0 = w_0 x$. Then for all $\delta \in \Delta$ we will have

$$y \cdot w_{\delta}(w_0 x) \geq y \cdot w_0 x$$

because

$$y \cdot (z_0 + (\delta \cdot z_0)\delta) \geq y \cdot z_0$$

Now we have that $y \cdot z_0 + y(\delta \cdot z_0) \cdot \delta \geq y \cdot z_0$ and so $(\delta \cdot z_0)(\delta \cdot y) \geq 0$. So z_0 and y are on the same side of every wall because if $(\delta \cdot z_0) > 0$ then $(y \cdot \delta) > 0$ and conversely. And so both belong to the same connected component of

$$\Omega(X) \setminus \bigcup_{\delta \in \Delta(X)} \delta^{\perp}.$$

□

In this part I will give some general facts and useful results about period maps and space of period and then we will see how we will use them in order to get nice results for concrete case, i.e. case of $K3$ surface.

Let X be a compact connected, smooth complex surface. As we know, we have the Hodge decomposition for X:

$$H^2(X,\mathbb{C}) = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X).$$

The signature of the intersection form on $H^{1,1}(X)$ will be $(1, h^{1,1}-1)$ if $b_1(X)$ is even (which is the case for $K3$ surfaces). So from now on we will suppose that $b_1(X)$ is even. We will denote geometric genus p_g of X with $h = h^{2,0} = \dim H^{2,0}(X)$ and $H^2(X,\mathbb{C})$ with H^2.

Definition 3.3.7. The space of periods Ω of the surface X is the subspace of the grassmannian $\mathrm{Gr}(H^2,h)$ of h-planes of the space H^2 such that every $\mu \in \Omega$ is isotropic for the intersection bilinear form and for each $x \in \mu$, $x \neq 0$ we have $x \cdot \overline{x} > 0$. In other words we have

$$\Omega := \{\mu \in \mathrm{Gr}(H^2,h) \mid \mu \text{ is isotropic for the bilinear form and} \\ \text{for all } x \in \mu,\ x \neq 0 \text{ we have } x \cdot \overline{x} > 0\}$$

Remark 3.3.3. Given $\mu \in \Omega$ we can define Hodge decomposition by $H^{2,0} = \mu$, $H^{0,2} = \overline{\mu}$ so we get $H^{1,1} = (\mu \oplus \overline{\mu}^{\perp})$. From now we will consider that Hodge structure is ordered by μ .

Theorem 3.3.5. *Let μ be point of $\Omega \subset Gr(H^2,h)$. The tangent space at μ of the Grassmannian $Gr(H^2,h)$ is given by*

$$T_\mu Gr(H^2,h) = Hom_{\mathbb{C}}(\mu, H^2/\mu)$$

and the tangent space at μ of the space of period Ω is given by

$$T_\mu \Omega = Hom_{\mathbb{C}}(H^{2,0}, H^{1,1}).$$

Proof. Let γ be curve on $\mathrm{Gr}(H^2,h)$ defined as $\gamma\colon (\varepsilon,\varepsilon) \to \mathrm{Gr}(H^2,h)$ and $h(0) = 0$ and $\dot{\gamma}(0) = \overrightarrow{p}$. Let $\overrightarrow{p}$ be the tangent vector at μ determined by curve $\gamma(u)$.

To every point $x \in \mu$ we associate vector

$$\left.\frac{d}{du}\widetilde{\gamma}(u)\right|_{u=0}$$

of H^2, where $\widetilde{\gamma}$ is lifting of γ passing through the point x. This vector is defined by $x \pmod{\mu}$. Then

$$T_\mu \mathrm{Gr}(H^2,h) = \mathrm{Hom}_{\mathbb{C}}(\mu, H^2/\mu).$$

Let's consider a complex curve $C(t)$ on $\mathrm{Gr}(H^2,h)$ where $t = u + iv$, complex number near to 0, such that $C(0) = \mu$. If we take $\widetilde{C}(t)$ to be the lifting of $C(t)$ passing through x, where $x \in \mu$, we will get homomorphism,

$$\mu \ni x \mapsto \left.\frac{\partial}{\partial t}\widetilde{C}(t)\right|_{t=0} \pmod{\mu} \tag{3.3}$$

in real case we can write down this as

$$\mu \ni x \mapsto \frac{1}{2}\left.\left(\frac{\partial C}{\partial u} - i\frac{\partial C}{\partial v}\right)\right|_{(0,0)}$$

So we can deduce from here that $C(t)$ is holomorphic in 0 if and only if

$$\frac{\partial}{\partial \bar{t}}\widetilde{C}(t)\bigg|_{t=0}$$

belongs to μ. So our homomorphism 3.3 can be identified with

$$\mu \ni x \mapsto \frac{\partial C}{\partial u}\bigg|_{u=0}.$$

The space of periods Ω is an open of the subvariety $\mathrm{Gr}(H^2, h)$ defined by the equation $\langle x, x\rangle = 0$. tangent space at μ to Ω is identified with $T_\mu\Omega = \mathrm{Hom}_{\mathbb{C}}(\mu, H^2/\mu)$. From the remark we have

$$\mu^\perp = \mu \oplus H^{1,1} \Rightarrow \mu^\perp/\mu = H^{1,1}$$

and so

$$T_\mu\Omega \cong \mathrm{Hom}_{\mathbb{C}}(H^{2,0}, H^{1,1}).$$

$\square$

Now as we told let's consider the period space for $K3$ surface X. As we know on $H^2(X,\mathbb{Z})$ intersection product defines a bilinear form which is unimodular, even and of signature $(3, 19)$.

$H^2(X,\mathbb{Z})$ as $\mathbb{Z}$-module is isomorphic to the lattice $L = (-E_8)^2 \oplus H^3$ which we call $K3$-lattice.

For every $K3$ surface there exists an isomorphism $\alpha\colon H^2(X,\mathbb{Z}) \to L$ preserving bilinear forms. Choice of such an isomorphism is called marking of X. By using the marking $\alpha\colon H^2(X,\mathbb{Z}) \to L$ we get a Hodge structure on L. We want to consider Hodge decomposition on the $L_{\mathbb{C}} = L \otimes \mathbb{C} = H^{2,0} \oplus H^{1,1} \oplus H^{0,2}$.

Period space for $K3$ lattice L will be subspace of Grassmannian $\mathrm{Gr}(1, L_{\mathbb{C}})$. In this case grassmannian is a projective space of dimension 21 and so period space is

$$\Omega := \{[w] \in \mathbb{P}(L \otimes \mathbb{C}) \mid w \cdot w = 0, w \cdot \overline{w} > 0\}$$

with associated Hodge structure:

$$H^{2,0} = \mathbb{C}w, \quad H^{0,2} = \mathbb{C}\overline{w}, \quad H^{1,1} = \langle w, \overline{w}\rangle^\perp$$

So by this we see that Ω is open subset in a quadric in $\mathbb{P}^{21}$.

Now I will briefly explain notions of deformation of surface and local Torelli theorem. This part is a short review of Gandelion exposition "Theoreme Local de Torelli pour K3 surfaces" in the Asterisque Seminaire directed by Beauville ([Be]).

I will mention just some big important results without details.

Definition 3.3.8. The analytic family of complex manifolds is given by

1. a space of parameters S, which is complex manifold of dimension n;

2. a compact complex manifold $\mathcal{H}$ of dimension $m + n$;

3. a holomorphic map p of maximal rank, where

$$p\colon \mathcal{H} \to S$$

and every fibre $f^{-1}(s)$, $s \in S$, is a compact complex manifold of dimension n.

Example 3.3.1. If we take $S = \mathcal{H}$, the Poincare half-plane and $\mathcal{H} = \mathbb{C} \times H/\sim$ where $\sim$ is defined as

$$(z, \tau) \sim (z + a + b\tau, \tau)$$

if $a, b \in \mathbb{Z}$, then we get a family of complex tori of dimension 1, $T\tau$.

We say that real manifold X_0 has an complex structure X if there is splitting of tangent bundle

$$T_{\mathbb{C}} X_0 = T^{1,0} X_0 \oplus T^{0,1} X_0 \quad \text{and} \quad T^{1,0} X_0 = \overline{T^{0,1} X_0}.$$

We say that X_0 is integrable if it is induced by a complex structure, i.e. in neighbourhood of all points of X_0 we can find system of n linearly independent functions ξ^α, $\alpha = 1, \ldots, n$ such that ξ^α constructs linearly independent system in that neighbourhoods.

Definition 3.3.9. A deformation of X_0 is a analytic family of almost complex integrable structures.

Let now Θ_X be the sheaf of holomorphic vector fields.The Kodaira - Spencer map is defined to be the map

$$\rho\colon T_0 S \to H^1(X, \Theta_X)$$

by sending a vector $v \in T_0 S$ to form which determine element of $H^1(X, \Theta_X)$ (Dolbeault cohomology), explicitly for chart system $\{{}_n\xi_i^\alpha\}$ and vector

$$v = \sum_a v^a \frac{\partial}{\partial t^a}$$

we have

$$\rho(v) = \sum_{\alpha=1}^{n} d'' \left(\sum v^a \frac{\partial}{\partial t^a} \xi_i^\alpha \right) \cdot \frac{\partial}{\partial z^a}$$

where $\{z^\alpha(x)\}$ is a system of charts on X. We will need very nice result of Nirenberg and Spencer, it is theorem:

Theorem 3.3.6 (Nirenberg - Spencer)**.** *If X_0 is complex compact manifold such that $H^2(X_0, \Theta_X) = 0$, then there exists analytic deformation of X_0 with parametrizing space an open subset of $H^1(X_0, \Theta_X)$ such that Kodaira - Spencer map is identity.*

Definition 3.3.10 (Period map)**.** Let S be a space of parameters forthe deformation of complex manifold X.To point s of S we associate complex structure $X(s)$ of the variety X_0 and also the Hodge structure $H^2(X, \mathbb{C})$. This Hodge structure lies in Ω and we will note it as $p(S)$ and we will call p, period map associated to deformation.

Now if we return to the case of $K3$ surfaces we will have next: Θ_X is isomorphic to the sheaf of holomorphic 1-forms Ω_X^1 by contraction with nowhere vanishing holomorphic 2-form ω. So we have

$$H^2(\Theta_X) = H^2(\Omega_X^1) = 0$$

So by the previous theorem there is a smooth local deformation of X, i.e. $\pi\colon X \to S$ with smooth fibres such that $\pi^{-1}(0) = X$ and Kodaira-Spencer map is an isomorphism.

If we choose a marking $\alpha\colon H^2(X,\mathbb{Z}) \to L$ we will extend this as

$$\alpha_s\colon H^2(\pi^{-1}(s),\mathbb{Z}) \to L, \quad s \in S.$$

We now have a period mapping of the family $S \to \Omega$, which sends s to the Hodge structure on L given by $\alpha(H^2(\pi^{-1}(s),\mathbb{Z}))$. Now we can state local Torelli theorem for $K3$ surfaces:

Theorem 3.3.7. *The differential of the period map $S \to \Omega$ is an isomorphism.*

Proof. Consider the following diagram:

$$\begin{tikzcd} T_0 S \arrow[rr, "\text{differential}", "\text{of period map}"'] \arrow[d, "\cong", "\text{Kodaira - Spencer}"'] & & T_\mu \Omega \arrow[d, "\cong"] \\ H^1(\theta_X) \arrow[rr, "\cong"] & & \mathrm{Hom}(H^{2,0}(X), H^{1,1}(X)) \end{tikzcd}$$

The natural map $H^1(\theta_X) \to \mathrm{Hom}(H^{2,0}(X), H^{1,1}(X))$ is given by the mapping

$$H^1(\theta_X) \otimes H^0(\Omega_X^2) \to H^1(\Omega_X^1)$$

induced by contraction of a vector field with a 2-form to produce 1-form. In case of $K3$ surfaces, this is an isomorphism. $\square$

Main consequence of the local Torelli theorem is the set of Hodge structures corresponding to $K3$-surfaces is an open set in the 20-dimensional complex manifold Ω.

This theorem, together with Kummer surface is main tool for proving Global Torelli theorem. We will now state Global Torelli theorem and we will give important consequences of that theorem. Techniques for proving this theorem are derived and explicitly shown in Asterisque Seminar: *Geometrie des surfaces K3.*

Theorem 3.3.8 (Global Torelli Theorem)**.** *Let X and X' be Kähler K3 surfaces and suppose that there exists an effective Hodge isometry*

$$\phi\colon H^2(X,\mathbb{Z}) \to H^2(X',\mathbb{Z})$$

Then $\phi = f^*$, with $f\colon X \to X'$ biholomorphic.

Theorem 3.3.9 (Weak Torelli Theorem)**.** *Let X, X' be Kähler K3 surfaces and suppose that there exists a Hodge isometry $H^2(X',\mathbb{Z}) \to H^2(X,\mathbb{Z})$. Then X and X' are isomorphic.*

We define the Kähler chamber to be

$$V^+(X) = \{x \in N(X) \mid x^2 > 0,\ x \cdot \kappa > 0 \text{ and for all } \delta \in \Delta^+(X)\ x \cdot \delta > 0\}$$

where κ is any Kähler class on X.

The reflections generating the group W operate properly discontinuously on $V^+(X)$ as we proved in Theorem 3.3.3 and $V^+(X)$ is the set of ample divisors on X. Let's prove this:

Theorem 3.3.10. *Let X be an algebraic K3 surface. Then any element D from $V^+(X)$ is the class of an ample divisor.*

Proof. By Riemann-Roch D or $-D$ is an effective divisor. Since D is from $V^+(X)$ it must that D is effective. By Nakai - Moisezon criterion for ampleness it is sufficient to show that for effective irreducible divisors on X, say E, is $D \cdot E > 0$.

1. If $E^2 < 0$ then $E^2 = -2$ and this implies that $E \cdot D > 0$, by definition of $V^+(X)$.
2. If $E^2 \geq 0$ then $E \cdot D \geq 0$. By the Hodge Index Theorem we may take a basis $(D_1, \ldots, D_s) \in N(X)$, with $D_1 = D$ and $D_i^2 < 0$ $(i = 2, \ldots, s)$ and $D_i \cdot D_j = 0$ $(i \neq j)$. Write: $E = \alpha_1 D_1 + \ldots + \alpha_s D_s$ $(\alpha_i \in \mathbb{Z})$. If $E \cdot D = 0$, then $\alpha_1 D_1^2 + \alpha_2 D_2 \cdot D_1 + \ldots + \alpha_s D_s \cdot D_1 = 0$, and so $\alpha_1 D_1^2 = 0$. We get $\alpha_1 = 0$ and hence $E^2 < 0$. This gives us contradiction, so $E \cdot D > 0$.

□

On the end of this chapter we state a theorem which remained a conjecture for a lot of time:

Theorem 3.3.11. *Every K3 surface is Kähler. Moreover the set of Kähler classes is exactly $V^+(X)$.*

Chapter 4

Free Automorphisms of Positive Entropy on Smooth Kähler surfaces

In the paper of Keiji Oguiso [15] is proved that there is a projective K3 surface admitting a fixed-point-free automorphism of positive entropy and that no smooth compact Kähler surface other than projective K3 surfaces and their blow-ups admit such an automorphism.

Now let us introduce some important notion and definitions important for our work. Let M be a complex compact Kähler manifold. Let g be a biholomorphic automorphism of M.

Definition 4.0.11. The first dynamical degree of g, denoted $d_1(g)$, is the maximum of absolute values of eigenvalues of the $\mathbb{C}$-linear extension $g^*\colon H^2(M,\mathbb{Z}) \to H^2(M,\mathbb{Z})$.

Definition 4.0.12. We say that g is of *positive entropy* if $d_1(g) > 1$. We say that g is of *null entropy* if $d_1(g) = 1$.

Remark 4.0.4. Last definition is the result of the research contained in the papers of Gromov-Yomdin [11], Dinh-Sibony [8] and Friedland [9].

1. As g^* is an automorphism for $H^2_{DR}(M,\mathbb{Z})$, then $\det(g^*|_{H^2(X,\mathbb{Z})}) = \pm 1$. As

$$|\det g^*| = \left| \pm \prod_{i=1}^{b_2} \lambda_i \right|$$

then $1 = \prod_{i=1}^{b_2} |\lambda_i|$, then we have:

   (a) if there exists i such that $|\lambda_i| < 1$, then there exists j such that $|\lambda_j| > 1$ and g is of positive entropy;

   (b) if for all i, $|x_i| = 1$, then g is of null entropy.

2. $(g^*)^k$ has eigenvalues λ_i^k, where λ_i are the eigenvalues for g^*. If $|\lambda_i| > 1$ and we have $|\lambda_i|^k = 1$, then $k = 0$ and so g has infinite order.

Definition 4.0.13. Let $M^g := \{x \in M \mid g(x) = x\}$. Map g is said to be *free* if $M^g = \emptyset$.

Remark 4.0.5. Non-trivial translation on a complex torus is free, but it is of null entropy.

Our first goal will be to prove this theorem, whose proof will be divided in the following sections.

Theorem 4.0.12. *Let S be a smooth complex compact Kähler surface admitting a free automorphism of positive entropy. Then S is birational to a projective K3 surface of Picard number greater than 1, and conversely there is a projective K3 surface of Picard number 2 with a free automorphism of positive entropy.*

4.1 Part I

In this section we will prove the first part of the theorem, i.e. we will consider compact Kähler surface admitting an automorphism of positive entropy. We will show that this surface is bimeromorphic to either $\mathbb{P}^2$, to a 2-dimensional copmlex torus, to an Enriques surface or to a K3 surface. And the by elimination we will prove that the only surfaces which admit free and of positive entropy are the K3 surfaces.

4.1.1 Topological and Holomorphic Lefschetz Number

This part will be strongly based on the *Principles of Algebraic Geometry* of Griffiths and Harris [10].

Let $g\colon M \to M$ be a $\mathcal{C}^\infty$ map of a compact oriented manifold M of dimension n. Let p be a fixed point of g, i.e. $p \in M$ and $g(p) = p$. It's clear that p correspond to a point of intersection of the graph $\Gamma_g \subset M \times M$ and the diagonal $\Delta \subset M \times M$, where

$$\Delta = \{(x, x) \mid x \in M\}$$
$$\Gamma_g = \{(x, g(x)) \mid x \in M\}$$

The intersection number $\#(\Gamma_g \cdot \Delta)_{M\times M}$ depends only on the homology classes of Γ_g and Δ in $M \times M$.

Definition 4.1.1. A point $p \in M$ which is fixed for the map g is said to be non-degenerate if it is an isolated fixed point and the Jacobian matrix

$$J_g(p)\colon T_p(M) \to T_p(M)$$

satisfies $\det(J_g - I) \neq 0$.

In this case, we define $i_g(p)$, the index of g at p to be

$$i_g(p) = \operatorname{sgn}(\det(J_g(p) - I))$$

Let $\widetilde{g}\colon M \to \Gamma_g$ be defined as

$$p \mapsto (p, g(p))$$

and $\Delta\colon M \to \Delta$ be defined as

$$p \mapsto (p,p)$$

Also, two more maps π_1 and π_2, the projections, are defined as

$$\begin{tikzcd} & M \times M \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ M & & M \end{tikzcd}$$

We will denote with η_Δ a cohomology class from $H^n(M \times M)$ of the diagonal $\Delta \subset M \times M$.

For a fixed q, consider a collection $\{\psi_{\mu,q}\}$ of closed q-forms on M which form a basis for $H^q_{DR}(M)$ and let $\{\psi^*_{\mu,n+q}\}$ be $n-q$ forms representing the dual basis for $H^{n-q}_{DR}(M)$. Hence we have

$$\int_X \psi_{\mu,q} \wedge \psi^*_{\nu,n-q} = \begin{cases} 1 & \text{if } \mu = \nu \\ 0 & \text{if } \mu \neq \nu \end{cases} \tag{4.1}$$

By the Künneth formula, a basis of $H^k_{DR}(M \times M)$ will be represented by

$$\{\varphi_{\mu,\nu,p,q} = \pi_1^*\psi_{\mu,p} \wedge \pi_2^*\psi^*_{\nu,q}\}_{p+q=k}$$

The dual basis of $H^{2n-k}_{DR}(M \times M)$ is represented by

$$\{\varphi^*_{\mu,\nu,n-p,n-q} = (-1)^{2(p+q)}\pi_1^*\psi^*_{\mu,n-p} \wedge \pi_2^*\psi_{\nu,n-q}\}_{p+q=k}$$

And by direct computations:

$$\int_{M\times M} \varphi_{\mu,\nu,p,q} \wedge \varphi^*_{\mu',\nu',n-p',n-q'} = \delta_{\mu,\mu'} \cdot \delta_{\nu,\nu'} \cdot \delta_{p,p'} \cdot \delta_{q,q'}$$

The Poincaré dual η_Δ of the homology class of the diagonal $\Delta \subset M \times M$ is the represented by the form

$$\varphi_\Delta = \sum_{p,\mu,\nu} c_{p,\mu,\nu}\varphi_{\mu,\nu,p,n-p}$$

and we have

$$c_{p,\mu,\nu} = \int_\Delta \varphi^*_{\mu,\nu,n-p,p} = (-1)^{n-p}\delta_{\mu,\nu}$$

and now we obtain that η_Δ is represented by

$$\varphi_\Delta = \sum_{p,\mu}(-1)^{n-p}\varphi_{\mu,\mu,p,n-p}$$

The coordinates of $(p,p) \in M \times M$, using the local coordinates on $x_1, \dots, x_n$ on M will be

$$y_i = \pi_1^* x_i \text{ and } z_i = \pi_2^* x_i$$

by this we have that an oriented basis for

$$T_{(p,p)}(\Delta) \subset T_{(p,p)}(M \times M)$$

is given by

$$\Delta_*\left(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}\right)=\left(\frac{\partial}{\partial y_1}+\frac{\partial}{\partial z_1},\dots,\frac{\partial}{\partial y_n}+\frac{\partial}{\partial z_n}\right)$$

where $\Delta\colon x\mapsto(x,x)$ is the diagonal map). For $T_{(p,p)}(\Gamma_g)$ we have basis

$$\widetilde{g}_*\left(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}\right)=\left(\frac{\partial}{\partial y_1}+\sum\frac{\partial g_i}{\partial x_1}\cdot\frac{\partial}{\partial z_n},\dots,\frac{\partial}{\partial y_n}\sum\frac{\partial g_i}{\partial x_n}\cdot\frac{\partial}{\partial z_n}\right)$$

and now we have

$$\left(\Delta_*\left(\frac{\partial}{\partial x_1}\right),\dots,\Delta_*\left(\frac{\partial}{\partial x_n}\right),\widetilde{g}_*\left(\frac{\partial}{\partial x_1}\right),\dots\widetilde{g}_*\left(\frac{\partial}{\partial x_n}\right)\right)$$

which is in fact obtained from the standard basis for $T_{(p,p)}(M\times M)$

$$\left(\frac{\partial}{\partial y_1},\dots,\frac{\partial}{\partial y_n},\frac{\partial}{\partial z_1},\dots,\frac{\partial}{\partial z_n}\right)$$

by the matrix

$$\begin{pmatrix} I_n & I_n\\ I_n & J_g(p)\end{pmatrix}$$

We know that Γ_g and Δ intersect transversely at (p,p) exactly when

$$\det\begin{pmatrix} I_n & I_n\\ I_n & J_g(p)\end{pmatrix}=\det(J_g(p)-I)\neq 0,$$

In fact, when p is non-degenerate point of g. So the index of g at p, $i_g(p)$ is the intersection number of Δ with Γ_g at p. So we can conclude that

$$\sum_{g(p)=p} i_g(p)=\#(\Delta\cdot\Gamma_g)_{M\times M}$$

Now

$$\begin{aligned}\#(\Delta\cdot\Gamma_g)&=\int_{\Gamma_g}\varphi_\Delta\\ &=\sum_p(-1)^{n-p}\int_{\Gamma_g}\sum_\mu \pi_1^*\psi_{\mu,p}\wedge\pi_2^*\psi_{\mu,n-p}\end{aligned}$$

(it is obvious that $\widetilde{g}^*\pi_2^*=g^*$)

$$\begin{aligned}&=\sum_p(-1)^{n-p}\int_M\sum_\mu\psi_{\mu,p}\wedge g^*\psi_{\mu,n-p}= \qquad \text{(by (4.1))}\\ &=\sum_p(-1)^{n-p}\mathrm{tr}(g^*|_{H^{n-p}_{DR}(M)})\\ &=\sum_p(-1)^{p}\mathrm{tr}(g^*|_{H^{p}_{DR}(M)})\end{aligned}$$

thus
$$T(M,g)=\sum_p(-1)^p\mathrm{tr}(g^*|_{H^p_{DR}(M)})$$
is the topological Lefschetz number.

It is clear from here that if g has no fixed points then
$$T(M,g)=\sum_{g(p)=p}i_g(p)=0.$$

In the case when X is a compact complex Kähler manifold of dimension n and $g\colon X\to X$ a holomorphic map, then g acts not only on the De Rham cohomology of X, but also on the Dolbeault cohomology groups as well.

So we will start with the computation of the Dolbeault cohomology class of the diagonal $\Delta\subset X\times X$. For p and q we have $\{\psi_{p,q,\mu}\}$ be a basis of the (p,q)-forms on $H^{P,q}_{\overline{\partial}}(X)$ and $\{\psi_{n-p,n-q,\mu}\}$ be a $\overline{\partial}$-closed forms representing the dual basis for $H^{n-p,n-q}_{\overline{\partial}}(X)$ under the pairing
$$H^{p,q}_{\overline{\partial}}(X)\otimes H^{n-p,n-q}_{\overline{\partial}}(X)\to\mathbb{C}$$
given by
$$\psi\otimes\varphi\mapsto\int_X\psi\wedge\varphi$$
The Künneth formula provides a basis for $H^{n,n}_{\overline{\partial}}(X\times X)$:
$$\{\varphi_{p,q,\mu,\nu}=\pi_1^*\psi_{p,q,\mu}\wedge\pi_2^*\psi_{n-p,n-q,\nu}\}$$
and the dual basis for $H^{n,n}(X\times X)$ is represented by
$$\{\varphi^*_{n-p,n-q,\mu,\nu}=\pi_1^*\psi^*_{n-p,n-q,\mu}\wedge\pi_2^*\psi_{p,q,\nu}\}$$
the Dolbeault class η_Δ of the diagonal is
$$\varphi_\Delta=\sum_{p,q,\mu}(-1)^{p+q}\varphi_{p,q,\mu,\mu}\quad .$$
Now let $g\colon X\to X$ be holomorphic map with isolated non-degenerate fixed points and
$$\Gamma_g=\{(p,g(p)),\ p\in X\}.$$
We have to know that for holomorphic map g we have $i_g(p)\geq 0$.

Then we have
$$\begin{aligned}\#(\Delta\cdot\Gamma_g)&=\int_{\Gamma_g}\varphi_\Delta\\&=\sum(-1)^{p+q}\int_{\Gamma_g}\pi_1^*\psi_{p,q,\mu}\wedge\pi_2^*\psi^*_{n-p,n-q,\mu}\\&=\sum(-1)^{p+q}\mathrm{tr}(g^*|_{H^{p+q}_{\overline{\partial}}(X)})\end{aligned}$$
If $p=0$ we get
$$H(X,g)=\sum_q(-1)^q\mathrm{tr}(g^*|_{H^{0,q}_{\overline{\partial}}(X)})$$
and we call it Holomorphic Lefschetz number. So we have that $H(X,g)=0$ if g has no fixed points.

4.1.2 Cantat's result

In the paper [4] by Cantat it is shown the next theorem:

Theorem 4.1.1. *(Cantant's theorem) Let S be a smooth compact Kähler surface admitting an automorphism of positive entropy. Then S is bimeromorphic to either $\mathbb{P}^2$, to a 2-dimensional complex torus, an Enriques surface or a K3 surface.*

Now, if X is a smooth compact Kähler surface admitting an automorphism of positive entropy, we will show that that automorphism will be free just in case when S is a K3 surface.

Theorem 4.1.2. *Assume that X is birational to either $\mathbb{P}^2$ or to an Enriques surface. If $g \in Aut(X)$ is a biholomorphic morphism then $X^g \neq \emptyset$.*

Proof. In the case when $X = \mathbb{P}^2$ we have

$$\Omega^2_{\mathbb{P}^2} \cong \mathcal{O}(K_{\mathbb{P}^2}) \cong \mathcal{O}(-3).$$

As we know that

$$H^0(\mathbb{P}^n, \mathcal{O}(d)) = \begin{cases} S_d & \text{if } d \geq 0 \\ 0 & \text{if } d < 0 \end{cases}$$

where S_d is the space of homogeneous polynomial in $n+1$ variables of degree d, we can conclude that $H^0(\mathbb{P}^2, \mathcal{O}(-3)) = 0$, so $H^0(\mathbb{P}^2, \Omega^2_{\mathbb{P}^2}) = 0$ and so $H^{2,0}(\mathbb{P}^2) = 0$, as by Serre duality we have that

$$\overline{H^{2,0}(\mathbb{P}^2)} = H^{0,2}(\mathbb{P}^2) \Rightarrow H^{0,2}(\mathbb{P}^2) = 0.$$

So $H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) = 0$. Using the De Rham cohomology for $\mathbb{P}^2$ we have that $H^1_{DR}(\mathbb{P}^2) = 0$, so that $H^1(\mathbb{P}^2, \mathbb{C}) = 0$ and by Hodge decomposition we have

$$H^{0,1}(\mathbb{P}^2) = \overline{H^{1,0}(\mathbb{P}^2)} = 0$$

so $H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) = 0$. We know by maximal principle theorem that

$$H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) = \mathbb{C}.$$

The holomorphic Lefschetz formua

$$H(\mathbb{P}^2, g) = \sum_{k=0}^{2} (-1)^k \mathrm{tr}(g^*|_{H^k(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2})})$$

implies that

$$H(\mathbb{P}^2, g) = \mathrm{tr}(g^*|_{H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2})})$$

as $\det g^* = \pm 1$ and $g^*|_{H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2})} : \mathbb{C} \to \mathbb{C}$ we have that $H(\mathbb{P}^2, g) = 1 \neq 0$, so X^g and g is not free.

The same will be done in case of Enriques surface since by definition of that surface we have that $p_g = g = 0$. □

So by this theorem we have that $\mathbb{P}^2$ and Enriques surfaces do not admit free automorphisms.

Before I continue with theorems which are necessary for proving main result, I would like first to point out some facts and well-known results from bimeromorphic geometry.

Definition 4.1.2. Let S be a smooth complex surface and $p \in S$ a point on S. Then there exists a surface $\widetilde{S}$ together with a bimeromorphic morphism $\sigma\colon \widetilde{S} \to S$ such that

1. $\sigma^{-1}(p) = E$, where $E \cong \mathbb{P}^1$;
2. $\sigma|_{\widetilde{S}\setminus E}\colon \widetilde{S} \setminus E \cong S \setminus \{p\}$ is a biholomorphism.

We say that σ is a blow up and that S is blown up in p. We will write $\widehat{S} = \mathrm{Bl}_p(S)$ and we will call this the blow-up of S. The curve E is called the exceptional curve of σ at p.

I will recall some facts about blow-ups, without proof (which can be found in [17]).

1. The cohomology groups of $\widetilde{S}$ with coefficients in $\mathbb{Z}$ satisfy

$$H^q(\widetilde{S}, \mathbb{Z}) \cong \begin{cases} H^q(S, \mathbb{Z}) & \text{if } q \neq 2 \\ H^2(S, \mathbb{Z}) \oplus E & \text{if } q = 2 \end{cases}$$

   where E is the lattice generated by the class of the exceptional curve.

2. $\sigma\colon \widetilde{S} \to S$ induces:
   (a) $\sigma^*\colon \mathcal{M}(S) \to \mathcal{M}(\widetilde{S})$ by the assignment $f \mapsto f \circ \sigma$

$$\begin{tikzcd} \widetilde{S} \arrow[d] \arrow[dr, "f\circ\sigma=\sigma^*(f)"] & \\ S \arrow[r] & \mathbb{P}^1 \end{tikzcd}$$

   (b) $\sigma^*\colon \mathrm{Div}(S) \to \mathrm{Div}(\widetilde{S})$;
   (c) $\sigma^*\colon \mathrm{Pic}(S) \to \mathrm{Pic}(\widetilde{S})$;

3. let C be an irreducible curve in S and $p \in C$, with $\mathrm{mult}_p(C) = r$. Let $\widetilde{S} = \mathrm{Bl}_p(S)$. then
   (a) the *proper transformation* of C by σ will be

$$\widetilde{C} = \overline{\sigma^{-1}(C \setminus \{p\})}$$

   the closure being taken in $\widetilde{S}$;
   (b) the *total transformation* of C by σ will be

$$\sigma^* C = \widetilde{C} + rE$$

4. $\mathrm{Pic}(\widetilde{S}) \cong \mathrm{Pic}(S) \oplus \mathbb{Z}$, this isomorphism is performed by

$$([D], n) \mapsto [\sigma^* D + nE]$$

5. for all $D, D' \in \mathrm{Div}(S)$ we have:

(a) $\sigma^* D \cdot \sigma^* D' = D \cdot D'$;
(b) $\sigma^* D \cdot E = 0$;
(c) $E^2 = -1$;

6. $K_{\widetilde{S}} = \sigma^* K_S + E$;

7. $e(\widetilde{S}) = e(S) + 1$, because

$$b_i(\widetilde{S}) = \begin{cases} b_i(S) & \text{if } i \neq 2, \\ b_i(S) + 1 & \text{if } i = 2. \end{cases}$$

8. $K^2_{\widetilde{S}} = (\sigma^* K_S + E)^2 = (\sigma^* K_S)^2 + 2\sigma^* K_S E + E^2 = K^{-}_S 1$;

9. $K^2_{\widetilde{S}} + e(\widetilde{S}) = K^2_S + e(S)$;

10. $\chi(\mathcal{O}_{\widetilde{S}}) = \chi(\mathcal{O}_S)$.

Let S be a surface and E be an irreducible curve on S. Then E is (-1) curve if one of the next three condition is satisfied:

1. $E \cong \mathbb{P}^1$, so $E^2 = -1$; or

2. $E^2 = E \cdot K_S = -1$; or

3. $E^2 < 0$, $E \cdot K < 0$.

Theorem 4.1.3 (Criterium of Castelnuovo)**.** *If $E \subset S$ is (-1) curve then E is exceptional curve.*

Theorem 4.1.4 (Structure of Bimeromorphic Morphism)**.** *Let S and S' be two surfaces. Every bimeromorphic morphism between S and S' will be composition of a finite number of blow-ups.*

Definition 4.1.3. A smooth surface is called minimal, if it does not contain any (-1) -curve.

Remark 4.1.1. If $\overline{S}$ is a minimal model for S, then there exists a natural morphism $\pi\colon S \to \overline{S}$, it is the composition of finitely many blow-downs, obviously contracting exceptional curves. In other words, S is a blow-up of $\widetilde{S}$.

Nice properties for minimal models are

Theorem 4.1.5. *Every compact non-singular surface S has a minimal model.*

Theorem 4.1.6. *If S is a compact connected surface with Kod$(S) \geq 0$, then all minimal models are isomorphic.*

These two theorems are taken from the book [3], Chapter II.

Now I will present my own remarks about nef surfaces (surfaces with nef canonical bundle) and birational geometry. If S is a nef surface (i.e. for all irreducible curve C on S, $K \cdot C \geq 0$), then on S there is no (-1)-curves. Why? If we suppose that E is an irreducible curve on S with canonical divisor K such that $E^2 = -1$, then from the genus formula we get that

$$2g - 2 = E \cdot (E + K)$$

as $E \cong \mathbb{P}^1$, $g(E) = 0$ and $E^2 = -1$. So we get $-2 = E^2 + E \cdot K$ and $E \cdot K = -1$, which gives contradiction since K is a nef divisor.

Also if we now continue with S nef surface and do blow-up of S in a point p we get $S' = \mathrm{Bl}_p(S)$. Let C be a curve on S such that $p \in C$ and C has multiplicity r in p. The proper transformation $\widetilde{C}$ of C on S' will be $\widetilde{C} = \sigma^*(C) - rE$, where E is the exceptional curve corresponding to p. Let K' be the canonical divisor of S', then we have

$$K'C' = (\sigma^* K + E)(\sigma^* C - rE) = KC - rE^2 = KC + r$$

since $r > 0$, $KC > 0$, we get $K'C' \geq 0$, but what will be $K'E$? Since

$$E(K' + E) = 2g(E) - 2$$

we get

$$EK' + E^2 = -2$$

hence $EK' = -1$, so K' is not nef.

Now if we blow-up S' in a point q on S' we will get $S'' = \mathrm{Bl}_q(S')$. All proper deformation C'' of curves in S' will satisfy same as curve C' from above, $K''C'' \geq 0$, except curve E', the exceptional curve in S'' corresponding to q, and maybe E'', the proper transformation of E. We use: $E''K'' = EK' + r$ and facts that

1. $r = 1$ if $q \in E$. If $q \in E$, then E' is the only curve in S'' with $K''E' < 0$;
2. $r = 0$ if $q \notin E$. In this case we have two disjoint curves E', E'' in S'' with $K''E' < 0$ and $K''E'' < 0$, and both are (-1)-curves.

Proceeding in this way it is clear that after n blow-ups of S we will get at most n curves with negative intersection with canonical divisors, and all these curves are disjoint and (-1)-curves.

Now let's turn to our main tasks:

Theorem 4.1.7. *Assume that X is bimeromorphic to either a 2-dimensional torus or to a K3 surface. Let $\overline{X}$ be the minimal model of X and $\pi\colon X \to \overline{X}$ be the naturally induced morphism. Then g descends to an automorphism $\overline{g}$ of $\overline{X}$. Moreover:*

1. *$\overline{g}$ is of positive entropy if and only if g is of positive entropy;*
2. *$\overline{g}$ is free if and only if g is free.*

Proof. As X in both cases has non-negative Kodaira dimension then we will have by previous theorem that X has a unique minimal model. So by this, the automorphism of X will descend to an automorphism on $\overline{X}$. Now we are going to explain why positive entropy of $\overline{g}$ of g implies positive entropy of another one.

Now if we suppose that g is free

$$\overline{X} \setminus \bigcup \{p_i\} \cong X \setminus \bigcup E_i$$

where E_i is the exceptional line corresponding to p_i, so if on X there are no fixed points, then there are no on $\overline{X}$ either.

Conversely, if $\overline{g}$ has fixed point, let it be p. Then $\pi^{-1}(p)$ is either a point on X, say Q, or it is an exceptional curve obteined in first blow up $E \cong \mathbb{P}^1$. In the first case we will have $g(Q) = Q$ since $g(\pi^{-1}(P)) = \pi^{-1}(\overline{g}(P))$ and it is fixed point for g; in second case we will have that $g(E) = E$, and $H(E, g) = H(\mathbb{P}^1, g) = \mathrm{tr}(g^*|_{H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})}) + \mathrm{tr}(g^*|_{H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})}) = 1+0$, (since $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = \mathbb{C}$, $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = 0$) and g will have at least one fixed point. So g is free if and only if it is $\overline{g}$.

From Fact 1., from previous part we have decomposition

$$H^2(X, \mathbb{Z}) = \pi^* H^2(\overline{X}, \mathbb{Z}) \oplus \mathcal{E}$$

where $\mathcal{E}$ is the lattice generated by exceptional divisors:

As the bilinear form restricted on $\mathcal{E}$ is always negative (by the Hodge index theorem we have that this bilinear form is negative) then $\mathcal{E}$ is negative definite. We know that

$$g^*(\mathcal{E}) = \mathcal{E}$$

And by this we get that absolute values of $g^*|_{\mathcal{E}}$ are 1. So by this we get that $d_1(g) > 1$ if and only if $d_1(\overline{g}) > 1$ (exceptional divisors do not play any role for positive entropy).

□

Theorem 4.1.8. *Let's suppose that X is bimeromorphic to a 2-dimensional complex torus. Let's suppose that X has a free automorphism g. Then g is of null entropy.*

Proof. We can suppose that X is itself minimal, i.e. X is 2-dimensional complex torus. Global coordinates of the universal cover $\mathbb{C}^2$ of X will be $z = (z_1, z_2)$.

$$\begin{tikzcd} \mathbb{C}^2 \arrow[r, "g^*"] \arrow[d, "\pi"'] \arrow[dr, "\overline{g}"] & \mathbb{C}^2 \arrow[d, "\pi"] \\ \mathbb{C}^2/\Gamma \arrow[r, "g"] & \mathbb{C}^2/\Gamma \end{tikzcd}$$

This diagram commutes by lifting property for covering spaces.

$$\overline{g} = g \circ \pi \colon \mathbb{C}^2 \to X$$
$$\overline{g}(z + \lambda) = \overline{g}(z), \quad \lambda \in \Gamma$$

By this we get that $g^* \colon \mathbb{C}^2 \to \mathbb{C}^2$ is such that

$$g^*(z + \lambda) = g^*(z) + w_\lambda, \quad w_\lambda \in \Lambda, z = (z_1, z_2) \in \mathbb{C}^2$$

For every i we have

$$\frac{\partial}{\partial z_i} g^*(z + \lambda) = \frac{\partial}{\partial z_i} g^*(z)$$

and we can write

$$g^*(z) = Az + b$$

where $A \in \mathrm{GL}_2(\mathbb{Z})$ and $b \in \mathbb{C}^2$. Let α, β be eigenvalues of A, then we have:

$$g^* \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \alpha z_1 + b_1 \\ \beta z_2 + b_2 \end{pmatrix}$$

As we know that for a torus $X = \mathbb{C}^2/\Gamma$, $H^0(X, \Omega^1_X) = \langle dz_1, dz_2 \rangle$, we have:

$$g^*dz_1 = d(\alpha z_1 + b_1) = \alpha dz_1$$
$$g^*dz_2 = d(\beta z_2 + b_2) = \beta dz_2$$

So α, β are eigenvalues of the action of g on $H^0(X, \Omega^1_X)$. Fact that $H^0(X, \Omega^2_X) = \bigwedge^2 H^0(X, \Omega^1_X)$ and Hodge duality give

$$H(X, g) = 1 - (\overline{\alpha} + \overline{\beta}) + \overline{\alpha}\overline{\beta} = (1 - \overline{\alpha})(1 - \overline{\beta})$$

As $H(X, g) = 0$ since g is free, so we get $\alpha = 1$ or $\beta = 1$. Now we use fact that

$$H^1(X, \mathbb{Z}) \otimes \mathbb{C} = H^0(X, \Omega^1_X) \oplus \overline{H^0(X, \Omega^1_X)}$$

by Hodge decomposition and eigenvalues of the $\mathbb{C}$-linear extension of $H^1(X, \mathbb{Z})$ will be $\alpha, \beta, \overline{\alpha}$ and $\overline{\beta}$. By fact that g^* is automorphism we have

$$\alpha\beta\overline{\alpha}\overline{\beta} = \pm 1.$$

Hence if α or β is 1, then $|\alpha| = 1$ and $|\beta| = 1$. As $H^2(X, \mathbb{Z}) = \bigwedge^2 H^1(X, \mathbb{Z})$ we have that eigenvalues of $g^*|_{H^2(X,\mathbb{Z})}$ are of absolute value 1, hence $d_1(g) = 1$, so g is of null entropy. □

Theorem 4.1.9. *(Nikulin's theorem) Let X be compact hyperkahler manifold, that is smooth simply-connected compact Kähler manifold with an everywhere nondegenerate holomorphic 2-form ω_X such that $H^0(X, \Omega^2_X) = \mathbb{C}\omega_X$. Assume that X admits a bimeromorphic automorphism g such that $g^*\omega_X = \zeta_n \omega_X$, where ζ_n is a root of unity. Then X is projective.*

Proof. Proof of this theorem can be found in the paper of Nikulin [13]. □

Theorem 4.1.10. *Let X be bimeromorphic to a K3 surface $\overline{X}$. Let ω_X be a generator of $H^{2,0}(X)$. Assume furthermore that X has a free automorphism g. Then $g^*\omega_X = -\omega_X$ and X is projective. Moreover $\rho(\overline{X}) \geq 2$, where $\rho(\overline{X})$ is the Picard number of $\overline{X}$.*

Proof. As X is bimeromorphic to $\overline{X}$, where $\overline{X}$ is a K3 surface then we have that $H^{0,1}(X) = 0$ and $\dim(H^{2,0}(X)) = 1$, so $H^0(X, \Omega^2_X) = \mathbb{C}\omega_X$. Now $H^{2,0}(X) = \overline{H^{0,2}(X)}$ by Serre duality, so

$$H^{0,2}(X) = H^2(X, \mathcal{O}_X) = \mathbb{C}\overline{\omega_X}$$

As g is a free automorphism we have that

$$\begin{aligned} 0 = H(X, g) &= 1 + (-1) \cdot \mathrm{tr}(g^*|_{H^{0,1}(X)} + \mathrm{tr}(g^*|_{H^{0,2}(X)}) \\ &= 1 + \mathrm{tr}(g^*|_{H^{0,2}(X)}) \end{aligned} \tag{4.2}$$

$H^{0,2}(X) \cong \mathbb{C}\overline{\omega_X}$, $g^* \colon \mathbb{C}\overline{\omega_X} \to \mathbb{C}\overline{\omega_X}$ where $\overline{\omega_X}$ is generator, then the matrix of g^* will be one number and $\det g^* = \pm 1$, so that number, in fact $\mathrm{tr}(g^*)$, will be 1 or -1.

From equation (4.2) we have

$$\mathrm{tr}(g^*|_{H^{0,2}(X)} = -1$$

so we can conclude that $g^*\omega_X = -\omega_X$. This the implies that $\overline{g}^*\omega_{\overline{X}} = -\omega_{\overline{X}}$. Now by Nikulin theorem we can conclude that $\overline{X}$ is projective.

As g is free, by theorem 3.1.9 $\overline{g}$ is also free, so we have that the topological Lefschetz number is $T(\overline{S}, \overline{g}) = 0$.

Let $T(\overline{X})$ be the trascendental lattice of $\overline{X}$, we know that

$$\begin{aligned} T(\overline{X}) &= \{x \in H^2(\overline{X}, \mathbb{Z}) \mid x \in NS^{\perp}(\overline{X})\} \\ &= \{x \in H^2(\overline{X}, \mathbb{Z}) \mid (x, y) = 0 \, \forall y \in NS(\overline{X}\}. \end{aligned}$$

So by this it is not hard to notice that $T(\overline{X}) \cap NS(\overline{X}) = \{0\}$. Also we have that :

$$H^2(\overline{X}, \mathbb{C}) = \ (T(\overline{X}) \otimes \mathbb{C}) \oplus (NS(\overline{X}) \otimes \mathbb{C})$$

and

$$T(\overline{X}) \oplus NS(\overline{X}) \subseteq H^2(\overline{X}, \mathbb{Z})$$

is finite subgroup of finite index of $H^2(\overline{X}, \mathbb{Z})$. We know that the Picard number ρ of $\overline{X}$ is the rank of $NS(\overline{X})$ and also we know that $H^2(\overline{X}, \mathbb{Z})$ has rank 22, so rank $T(\overline{X}) = 22 - \rho$. As we proved that $\overline{X}$ is projective then by Kodaira theorem about Kähler manifolds, which says: a Kähler manifold X is projective if and only if it admits integral Kähler classe, we know that $\rho(\overline{X}) \geq 1$.

By topological Lefschetz formula we have that

$$\begin{aligned} 0 &= T(\overline{X}, \overline{g}) \\ &= 1 + (-1)^1 \mathrm{tr}(\overline{g}^*|_{H^1(\overline{X},\mathbb{Z})}) + (-1)^2 \mathrm{tr}(\overline{g}^*|_{H^2(X,\mathbb{Z})}) + (-1)^3 \mathrm{tr}(\overline{g}^*|_{H^3(X,\mathbb{Z})}) + (-1)^4 \mathrm{tr}(\overline{g}^*|_{H^4(X,\mathbb{Z})}) \\ &= 2 + \mathrm{tr}(\overline{g}^*|_{H^2(X,\mathbb{Z})}) \quad (\text{as } b_1 = b_3 = 0, b_0 = b_4 = 1) \\ &= 2 + \mathrm{tr}(\overline{g}^*|_{NS(\overline{X})} + \mathrm{tr}(\overline{g}^*|_{T(\overline{X})} \end{aligned}$$

As $\omega_{\overline{X}} \in H^{2,0}(\overline{X}, \mathbb{C})$ and

$$H^2(\overline{X}, \mathbb{C}) = H^{2,0}(\overline{X}, \mathbb{C}) \oplus H^{1,1}(\overline{X}, \mathbb{C}) \oplus H^{0,2}(\overline{X}, \mathbb{C})$$

and $NS(\overline{X}) \subseteq H^{1,1}(\overline{X}, \mathbb{C})$ it follows that $\omega_{\overline{X}} \notin NS(\overline{X})$. As

$$(\overline{g}^*\omega_{\overline{X}}, \overline{g}^*t) = (\omega_X, t)$$

for $t \in T(\overline{X})$ we have

$$(-\omega_X, \overline{g}^*t) = (\omega_X, t)$$

and so $-\omega_X \cdot \overline{g}^*t = \omega_X \cdot t$, so that $\omega_X \cdot (\overline{g}^*t + t) = 0$. Thus

$$\overline{g}^*t + t \in NS(\overline{X}).$$

As $t \in T$, $\overline{g}^* \in T$, we get $\overline{g}^*t + t \in T$, $\overline{g}^*t + t = 0$, i.e. $g^*t = -t$.

So we have that $\overline{g}^*|_{T(\overline{X})} = -\mathrm{id}|_{T(\overline{X}}$.

This gives us that $\mathrm{tr}(\overline{g}^*|_{T(\overline{X})}) = -\mathrm{rank}(T(\overline{X})) = -(22 - \rho(\overline{X})$. If we suppose now that $\rho(\overline{X}) = 1$ then $\mathrm{tr}(\overline{g}^*|_{NS(\overline{X}}) = 1$, and we have:

$$T(\overline{X}, g) = 2 + 1 - 2(22 - 1) = -20 \neq 0$$

so $\rho(\overline{X}) \neq 1$, i.e. $\rho(\overline{X}) \geq 2$.

□

The first part of our main theorem says if X is smooth complex compact Kähler surface admitting a free automorphism of positive entropy then X is birational to projective K3 surface of Picard number ≥ 2. This part is a consequence of Cantant theorem and the previous three theorems, which directly led us to the fact that X is birational to a projective K3 surface of Picard number ≥ 2.

4.2 Part II

Let X be a K3 surface of Picard number 2 with a free automorphism g. From theorem from I Part we have that

$$g^*|_{T(X)} = -\mathrm{id}_{T(X)}.$$

As g is free we have that $H(X,g) = 0$, and so

$$0 = H(X,g) = 2 + \mathrm{tr}(g^*|_{NS(X)}) + (-1) \cdot (22-2)$$

i.e. $\mathrm{tr}(g^*|_{NS(X)}) = 18$. Moreover $g^*|_{NS(X)} \in \mathrm{GL}_2(\mathbb{Z})$, so that $\det(g^*|_{NS(X)}) = \pm 1$. So characteristic polynomial of g will be

$$\varphi(t) = t^2 - 18t + 1$$

and eigenvalues of $g^*|_{NS(X)}$ are the roots of this polynomial, i.e.

$$t_{1,2} = \frac{18 \pm \sqrt{18^2 - 4}}{2} = 9 \pm 4\sqrt{5}$$

The number

$$\eta = \frac{\sqrt{5}+1}{2}$$

we call the golden number and $\eta^{\pm 6} = t_{1,2}$, so eigenvalues of $g^*|_{NS(X)}$ are η^6 and η^{-6}. One of this eingenvalues is bigger than 1, so g is of positive entropy.

This introduction and golden number η will serve as starting points for the proof of the second part of our theorem.

The minimal polynomial of η over $\mathbb{Z}$ is $t^2 - t - 1$ and we denote

$$N := \mathbb{Z}[\eta] = \{a + b\eta \mid a, b \in Z\} \cong \mathbb{Z}^2$$

The unit group of N will be

$$\mathbb{Z}[\eta]^* = \{u \in \mathbb{Z}[\eta] \mid \exists v \in \mathbb{Z}[\eta] \text{ such that } uv = 1\}$$

Example 4.2.1. $\eta^2 = \eta + 1$ (η zero of $t^2 - t - 1$). Then $\eta(\eta - 1) = 1$, so $\eta \in \mathbb{Z}[\eta]^\times$ and $1, \eta, \eta^2, \ldots$ are all units, so $\mathbb{Z}[\eta]^\times$ is infinite.

Theorem 4.2.1. *Let $\{a_n\}_{n \geq 0}$ be the Fibonacci sequence:*

$$\begin{cases} a_{n+2} = a_{n+1} + a_n \\ a_0 = 0 \\ a_1 = 1 \end{cases}$$

Then $\eta^n = a_n\eta + a_{n-1}$ for each positive number n. For instance

$$\begin{aligned}\eta^3 &= 2\eta + 1\\ \eta^4 &= 3\eta + 2\\ \eta^5 &= 5\eta + 3\\ \eta^6 &= 8\eta + 5\\ \eta^7 &= 13\eta + 8\end{aligned}$$

Proof. We will do this by induction on n. If $n = 1$, it is trival. For $n = 2$ we have $\eta^2 = a_2\eta + a_1$, where $a_2 = a_0 + a_1 = 1$, so $\eta^2 = \eta + 1$, which is true.

Assume the thesis for n. Then $a_{n+1} = a_n + a_{n-1}$ and so

$$\eta^n = a_n\eta + a_{n-1}$$

by multiplying by η we get

$$\begin{aligned}\eta^{n+1} &= a_n\eta^2 + a_{n-1}\eta = a_n(\eta + 1) + a_{n-1}\eta\\ &= \eta(a_n + a_{n-1}) + a_n = a_{n+1}\eta + a_n\end{aligned}$$

□

As we said that $N \cong \mathbb{Z}^2$ we will take for basis of N to be $\{e_1, e_2\}$, where $e_1 = 1$ and $e_2 = \eta$, so we have that $N = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$. As $\eta \in \mathbb{Z}[\eta]^\times$ then we will have that

$$\eta^n : N \to N$$

$\eta^n(p(\eta)) = \eta^n p(\eta)$ (where $p(\eta)$ is polinimial in η) is an automorphism of the $\mathbb{Z}$-module N.

Theorem 4.2.2. *Let n be a positive integer.*

1. *The eigenvalues of the automorphism η^{2n} on N are η^{2n} and η^{-2n}.*
2. *the characteristic polynomial of η^{2n} on N is $t^2 - (a_{2n} + 2a_{2n-1})t + 1$, where $\{a_n\}_{n\geq 0}$ is the Fibonacci sequence. For instance, the characteristic polynomials of η^2, η^4, η^6 are $t^2 - 3t + 1$ $t^2 - 7t + 1$, $t^2 - 18t + 1$ respectively.*

Proof. 1. As characteristic polynomial for automorphism $\eta: N \to N$ is $t^2 - (-\eta + \eta')t - \eta\eta'$, on the other side minimal polynomial for η is $t^2 - t - 1$, so we have that $\eta' = \eta^{-1}$ and $\eta' + \eta = -1$, then $\eta' = 1 - \eta$, so

$$\eta' = \frac{1 - \sqrt{5}}{2}$$

It is easy to conclude that eigenvalues for η^{2n} will be η^{2n} and η^{-2n}.

2. By previous theorem we have that $\eta^{2n} = a_{2n}\eta + a_{2n-1}$. Taking the Galois conjugate $\eta \to -\eta^{-1}$ we have

$$\frac{1}{\eta^{2n}} = a_{2n} \cdot \left(-\frac{1}{\eta}\right) + a_{2n-1}$$

Now we have

$$\eta^{2n} + \frac{1}{\eta^{2n}} = a_{2n}\left(\eta - \frac{1}{\eta}\right) + 2a_{2n-1}$$
$$= a_{2n}(\eta + 1 - \eta) + 2a_{2n-1} = a_{2n} + 2a_{2n-1}$$

As

$$\mathrm{tr}(\eta^{2n}) = \frac{1}{\eta^{2n}} + \eta^{2n} = a_{2n} + 2a_{2n-1}$$

and

$$\det \eta^{2n} = \eta^{2n} \cdot \frac{1}{\eta^{2n}} = 1$$

we have that the characteristic polynomial of η^{2n} is

$$t^2 - (a_{2n} + 2_{2n-1})t + 1$$

□

If $b\colon N \times N \to \mathbb{Z}$ is a $\mathbb{Z}$-valued symmetric bilinear form, the matrix Sb of that form is

$$p = b(e_1, e_1$$
$$g = b(e_1, e_2)$$
$$r = b(e_2, e_2)$$
$$S_b := \begin{pmatrix} p & g \\ g & r \end{pmatrix}$$

We say that b is hyperbolic if the matrix S_b has signature $(1,1)$. Then (N,b) is hyperbolic lattice if b is hyperbolic on N.

Theorem 4.2.3. *There is an embedding*

$$N^*\colon \mathit{Hom}_{\mathbb{Z}}(N, \mathbb{Z}) \hookrightarrow N \otimes \mathbb{Q}$$

Proof. We will prove that for every $f \in N^*$ there is $y \in N \otimes \mathbb{Q}$ such that $f(x) = (x,y)$ for all $x \in N$. A basis of N is $\{e_1, e_2\}$. Then we have $f(e_j) = a_j \in \mathbb{Z}$ for $j = 1, 2$. Then this gives the map : $f : N^* \to N \otimes \mathbb{Q}$

$$f\left(\sum n_i e_i\right) = \sum n_i a_i, \quad n_i \in \mathbb{Z}$$

We want $y \in N \otimes \mathbb{Q} \simeq \mathbb{Q}^2$ such that

$${}^t e_j S_b y = a_j, \quad j = 1, 2$$

If we take

$$y = S_b^{-1}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

then we will have

$${}^t e_j S_b s_b^{-1}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = a_j, \quad j = 1, 2$$

By tis we proved existence of an embedding. □

Definition 4.2.1. N^*/N is called the discriminant group of N.

Definition 4.2.2. An automorphism $f\colon N \to N$ is an isometry of the lattice (N, b) if $b(f(x), f(y)) = b(x, y)$ for all $x, y \in N$.

Remark 4.2.1. An isometry naturally induces an automorphism of the discriminant group N^*/N.

Proposition 4.2.1. *Assume that (N, b) is an even hypeerbolic lattice and that η^2 is an isometry of (N, b). Then:*

1. *the matrix S_b is of the following form:*

$$S_b := \begin{pmatrix} 2q & q \\ g & -2q \end{pmatrix}$$

   where $q \neq 0$ and q is an integer;

2. *Under 1. the discriminant group N^*/N satisfies*

$$N^*/N = \left\langle \frac{e_2}{q} \right\rangle \oplus \left\langle \frac{e_1 - 2e_2}{5q} \right\rangle \simeq \mathbb{Z}/q\mathbb{Z} \oplus \mathbb{Z}/5q\mathbb{Z}$$

3. *Under 1. b does not represent 0, i.e. there is no $x \in N$ such that $b(x, x) = \pm 2$ if and only if $q \neq \pm 1$.*

4. *Under 1., η^6 acts on the discriminant group N^*/N as $id_{N^*/N}$ if and only if q is one of $\{\pm 1, \pm 2\}$.*

Proof. 1. We take

$$\begin{aligned} 2p &= b(e_1, e_1) \\ q &= b(e_1, e_2) \\ 2r &= b(e_2, e_2) \end{aligned}$$

We have

$$\begin{aligned} \eta^2(e_1) &= \eta^2 = \eta + 1 = e_1 + e_2 \\ \eta^2(e_2) &= \eta^3 = 2\eta + 1 = e_1 + 2e_2 \end{aligned}$$

η^2 is an isometry if and only if

$$\begin{aligned} b(e_1, e_1) &= b(e_1 + e_2, e_1 + e_2) \\ b(e_1, e_2) &= b(e_1 + e_2, e_1 + 2e_e) \\ b(e_2, e_2) &= b(2_1 + 2e_2, 9 + 2e_2) \end{aligned}$$

These lead us to

$$\begin{aligned} p &= 2p + 2q + 2r \\ q &= 2p + 3g + 4r \\ 2r &= 2p + 4g + 8r \end{aligned}$$

so $r = -q$ and $p = q$, S_b is hyperbolic if $q \neq 0$.

2. As $N^* = S_b^{-1}\mathbb{Z}^2$ we have that

$$S_b = \begin{pmatrix} 2q & q \\ q & -2q \end{pmatrix}, \quad S_b^{-1} = -\frac{1}{5q^2}\begin{pmatrix} -2q & -q \\ -q & 2q \end{pmatrix} = \frac{1}{5g}\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$$

N^* is generated by

$$S_b^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{5q}\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{2e_1 + e_2}{5q}$$
$$S_b^{-1}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{5q}\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{e_1 - 2e_2}{5q}$$

Now notice that

$$\frac{2e_1 + e_2}{5q} - \frac{2(e_1 - 2e_2)}{5q} = \frac{e_2}{q}$$
$$b\left(e_1, \frac{e_2}{q}\right) = 1$$
$$b\left(e_2, \frac{e_2}{q}\right) = -2$$
$$b\left(e_1, \frac{e_1 - 2e_2}{5q}\right) = 0$$

and

$$b\left(e_2, \frac{e_1 - 2e_2}{5q}\right) = 1$$

so that

$$\frac{e_1 - 2e_2}{5q}, \frac{e_2}{q}$$

are from N^*/N. Thus

$$\left\langle \frac{e_1 - 2e_2}{5q} \right\rangle \cap \left\langle \frac{e_2}{q} \right\rangle = 0$$

As $|N^*/N| = |\det S_b|$ we get that

$$|N^*/N| = \left\langle \frac{e_2}{q} \right\rangle \oplus \left\langle \frac{e_1 - 2e_2}{5q} \right\rangle \simeq \mathbb{Z}/q \oplus \mathbb{Z}/5q$$

3. If we take $x = ae_1 + be_2$ $(a, b \in \mathbb{Z})$ we get

$$b(x, x) = 2g(a^2 + ab - b^2)$$

If $x \neq 0$ the it is not possible for $b(x, x)$ to be 0 or ± 2 for $q \neq \pm 1$.

4. From previous Lemma we know that

$$\eta^6 = 7\eta + 5, \quad \eta^7 = 13\eta + 8$$
$$\eta^6(e_1) = 5e_1 + 8e_2, \quad \eta^8(e_2) = 8e_1 + 13e_3$$
$$\eta^6\left(\frac{e_2}{q}\right) = \frac{8e_1 + 13e_2}{q}$$
$$\eta^6\left(\frac{e_1 - 2e_2}{5q}\right) = \frac{1}{8q}(\eta^6(e_1) - 2\eta^6(e_2)) = -\frac{11a + 18e_2}{5q}$$

So if $\eta^6_{N^*N} = -\mathrm{id}_{N^*/N}$ then it must be

$$\frac{82_1 + 13e_2}{q} = -\frac{e_2}{q} \Rightarrow \frac{8e_1 + 14e_2}{q} = 0$$

and

$$\frac{e_1 - 2e_2}{5q} = \frac{11e_1 + 10e_2}{5q} \Rightarrow \frac{10e_1 + 20e_2}{5q} = 0 \Rightarrow \frac{e_1 + 4e_2}{q} = 0$$

As

$$\frac{8e_1 + 4e_2}{q} = 0 \pmod{N}, \quad \frac{2e_1 + 4e_2}{q} = 0 \pmod{N}$$

it must be that $\frac{8e_1+4e_2}{q}$ and $\frac{2e_1+4e_2}{q}$ are from N. In other words $q \mid 2$, so by this q is ± 1, ± 2.

□

Theorem 4.2.4. *Let (N, b) be the lattice as before given by matrix*

$$S_b = \begin{pmatrix} 2q & q \\ g & 2-q \end{pmatrix}$$

In the case when $q = 2$, we get

$$S_b = \begin{pmatrix} 4 & 2 \\ 2 & -4 \end{pmatrix}.$$

In a case when $q = 2$ we have

1. *(N, b) is an even hyperbolic lattice which represents neither 0 nor ± 2.*
2. *η^6 is an isometry of (N, b) such that the characteristic polynomial is $t^2 - 18t + 1$ and the induced action on the discriminant group N^*/N is $-id_{N*/N}$.*

Proof. This is a particular case of the previous theorem, so the proof is based on theorem before. □

And now we will state and prove the most important theorem from II part. It is the second part of theorem 4.0.12.

Theorem 4.2.5. *There exists a projective K3 surface X of Picard number $\rho(X) = 2$ such that $NS(X) = \mathbb{Z}h_1 \oplus \mathbb{Z}h_2$, where*

$$((h_i \cdot h_j)) = \begin{pmatrix} 4 & 2 \\ 2 & -4 \end{pmatrix}$$

Any such K3 surface X admits a free automorphism g of positive entropy.

Proof. The matrix

$$\begin{pmatrix} 4 & 2 \\ 2 & -4 \end{pmatrix}$$

is even hyperbolic matrix of rank 2. By Morrison paper: on K3 surfaces with large Picard number and corollary 2.9 from that paper (more precisely, that corollary says that if $\rho \leq 10$ then every even lattice of signature $(1, \rho - 1)$ occurs as the Néron - Severi group of some algebraic K3 surface). Now, we are going to construct an automorphism g of X with desired properties. By the last theorem we get that $NS(X) \simeq (N, b)$, and that an isomorphism we will call φ. First of all let $f := \varphi^{-1}\eta^6\varphi$, f is an isometry of $NS(X)$. It is true because:

$$\begin{aligned}(f(x), f(y)) &= (\varphi^{-1} \circ \eta^6 \circ \varphi(x), \varphi^{-1} \circ \eta^6 \circ \varphi(y)\\ &= (\varphi \circ \varphi^{-1} \circ \eta^6 \circ \varphi(x), \eta^6 \circ \varphi(y))\\ &= (\eta^6 \circ \varphi(x), \eta^6 \circ \varphi(y)) = (\varphi(x), \varphi(y))\\ &= (\varphi^{-1} \circ \varphi(x), y) = (x, y)\end{aligned}$$

So f is an isometry of $NS(X)$. The eigenvalues of f are the same as eigenvalues of η^6, so that they are η^6 and η^{-6}, by Theorem 4.2.2.

Both eigenvalues of f are positive, so f preserves component $\Omega^+(X)$ of the positive cone $\Omega(X)$. As we know that ample divisors are all in $\Omega^+(X)$ and $NS(X)$ does not represent -2, the ample cone of X, $\mathcal{A}(X)$ coincides with the cone $\Omega^+(X)$. So we can conclude that f preserves the ample cone.

By theorem 3.2.4 we have that η^6 acts on the discriminant group N^*/N as $-\mathrm{id}_{N^*/N}$ and so then the isometry f will act on discriminant group $NS(X)^*/NS(X)$ as $-\mathrm{id}_{NS(X)^*/NS(X)}$.

$T(X)$ is transcendental lattice of X and we can conclude that $-\mathrm{id}_{T(X)}$ acts on the discriminant group $T(X)^*/T(X)$ as $-\mathrm{id}_{T(X)^*/T(X)}$.

Again, by Nikulin result from paper *Integer symmetric bilinear forms and some of their geometric applications*, [14], more precisely by proposition 1.6.1 from that paper we can conclude that isometry $(f, -\mathrm{id}_{T(X)})$ of $NS(X) \oplus T(X)$ extends to an isometry, say $\widetilde{f}$, of $H^2(X, \mathbb{Z})$.

By construction of $\widetilde{f}$ we can notice that it preserves the Hodge decomposition of $H^2(X, \mathbb{Z})$, and as we observed above, it preserves the ample cone.

So it preserves Hodge decomposition, and preserves positive cones and respective sets of effective classes, so $\widetilde{f}$ is an effective Hodge isometry.

By the global Torelli theorem for K3 surfaces, there is an automorphism g of X such that $g^*|_{H^2(X,\mathbb{Z})} = \widetilde{f}$. One useful well-known fact is that:
$NS(X) = T(X)^\perp$ and so

$$H^2(X, \mathbb{Z}) \supseteq NS(X) \oplus T(X).$$

By construction we have that $f = \widetilde{f}|_{NS(X)}$ and also we have that one of the eigenvalues of f is η^6, so as $\eta^6 > 1$, the g is of positive entropy.

the next step is to prove that g has no fixed points, i.e. that g is free.

As $g \neq \mathrm{id}_X$, the set of fixed points for g, X^g , consists of finitely many curves and at most finitely many points.

Let us suppose first that C is a curve from X^g. So we have $g(C) = C$. Then we have that the class $[C]$ in $H^2(X, \mathbb{Z})$ will be eigenvector of $g^*|_{NS(X)}$, with eigenvalues 1. But this is not possible since eigevalues of $g^*|_{NS(X)} = f$ are η^6 and η^{-6}, so in X^g there are no curves.

Next possible situation is that X^g consists of finitely many points, so for example n points counted with multiplicities. By Lefschetz fixed point formula, we have

$$n = T(X, g) = 2 + \mathrm{tr}(g^*|_{NS(X)}) + \mathrm{tr}(g^*|_{T(X)}).$$

As $g^*|_{NS(X)} = f$ and $\mathrm{tr}(f) = \mathrm{tr}(\eta^6|_N)$, by previous theorem we get $\mathrm{tr}(f) = 18$ and $\mathrm{tr}(g^*|_{NS(X)}) = 18$. Also we have that $g^*|_{T(X)} = -\mathrm{id}_{T(X)}$, as $\rho(X) = 2$, then $\mathrm{rank}(T(X)) = 20$ and we get that $\mathrm{tr}(g^*|_{T(X)}) = -20$. Finally we can conclude that g is free since

$$n = T(X, g) = 2 + 18 - 20 = 0.$$

So we proved that X admits automorphism g which is free of positive entropy.

□

4.3 Part III

In this part I tried ,using result of Oguiso, to show explicitly how we can obtain surface and automorphism on that surface with properties described in theorem 3.2.5 .

Let X be a K3 surface with $NS(X) = \mathbb{Z}h_1 \oplus \mathbb{Z}h_2$ such that

$$(h_i \cdot h_j)_{i,j} = \begin{pmatrix} 4 & 2 \\ 2 & -4 \end{pmatrix}$$

By Theorem 3.2.4 we have that there are no divisors with self-intersection ± 2 and 0. We have that for any $ax + by \in NS(X)$ it holds:

$$(ax + by)^2 = 4(a^2 + ab - b^2)$$

Let us take $D \in \Omega^+ \setminus \{0\}$. Then $D^2 > 0$ and by Saint-Donat [16] we have an embedding $\varphi_D \colon X \hookrightarrow \mathbb{P}^N$.

Let D be of type $(1, 0)$ and we have $D^2 = 4$. A curve in $|D|$ is defined as $D_H = H \cap X$, where H is the hyperplane in $\mathbb{P}^3$, and will have class $(1, 0)$. This curve D_H is irreducible. Let us prove this. If we suppose that $D_H = C + C'$ and if C has class (a, b) and C' has class (c, d), thus we have that $a + c = 1$ and so $a = 0$ or $c = 0$. Hence, we get that one class is zero class and thus, D_H is irreducible.

Let D' be curve on X of type $(1, 1)$. Then we get: $(D')^2 = 4(1 + 1 - 1) = 4$. We want to see how is obtained curve D'. For sure we know that $D' \neq X \cap H$, since D' is of type $(1, 1)$ and $X \cap H$ is of type $(1, 0)$. What is the degree of D'?

$$D' \cdot H = D' \cdot D = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 4 & 2 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 6$$

Is $D' \subseteq Q \cap X$, where Q is quadric and Q is of type $2H$, so with class $(2, 0)$? If we suppose that $D' \subseteq Q \cap X$ we have $Q \cap X = D' \cup D''$ and D' has class $(1, 1)$ and D'' has class $(1, -1)$. But as $(D'')^2 = -4 < 0$ we have that D'' is not from Ω^+ and by this we have that D' is not obtained as intersection of quadric Q and surface X.

Now we will investigate intersection of the cubic surface R and surface X. The cubic R has class $3H = (3, 0)$. We will suppose that $D' \subset R \cap X$. Thus we have $R \cap X = D' + D''$, and D' is of class $(1, 1)$ and so D'' will have class $(2, -1)$. As $(D'')^2 = 4$ then we have that $D'' \in \Omega^+$. What is the degree of D''?

$$D'' \cdot H = \begin{pmatrix} 2 & -1 \end{pmatrix} \begin{pmatrix} 4 & 2 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 6$$

Let us prove the existence of cubics $R \in |3H|$ such that $R = D' + D$". We have exact sequence:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-X + 3H) \to \mathcal{O}_{\mathbb{P}^3}(3H) \to \mathcal{O}_X(3H) \to 0$$

and the first map is $f \mapsto f \cdot G$, where G is homogeneous polynomial defines X . We get a long exact sequence

$$H^0(\mathcal{O}_{\mathbb{P}^3}(3H)) \to H^0(\mathcal{O}_X(3H)) \to H^1(\mathcal{O}_{\mathbb{P}^3}(3H - X)).$$

But $H^1(\mathcal{O}_{\mathbb{P}^3}(3H - X)) = H^1(\mathcal{O}_{\mathbb{P}^3}((3 - d)H) = 0$, where d is the degree of X. So we get

$$H^0(\mathcal{O}_{\mathbb{P}^3}(3H)) \xrightarrow{\varphi} H^0(\mathcal{O}_X(3H))) \longrightarrow 0$$

and so φ is surjective. This means that for every $s \in H^0(\mathcal{O}_X(3H))$ there is polynomial function F of degree 3 (of course it is an element of $H^0(\mathcal{O}_{\mathbb{P}^3}(3H))$ such that $(s = 0) \subseteq X$ is given by $(F = 0) \cap X$. Since $D' + D''$ has class $(3, 0)$ we have that $D' + D'' \in |3H|$ on X and so there exists a cubic surface R such that $R \cap X = D' + D''$.

Let D and D' be divisors with corresponding class $(1, 0)$ and $(1, 1)$. Map $\varphi_D \times \varphi_{D'}$ will define an embedding of X into $\mathbb{P}^3 \times \mathbb{P}^3$:

$$\varphi_D \times \varphi_{D'} \colon X \hookrightarrow \mathbb{P}^3 \times \mathbb{P}^3$$

Now we will use Segre's embedding of $\mathbb{P}^3 \times \mathbb{P}^3$ into $\mathbb{P}^{4\cdot 4-1} = \mathbb{P}^{15}$ defined as

$$((x_0 \colon \ldots : x_3), (y_0 \colon \cdot \colon y_3)) \mapsto (\ldots : x_i y_i \colon \ldots)$$

We denote $x_i y_j = z_{ij}$. So we define embedding of X into $\mathbb{P}^{15}$ using z_{ij}. We have a diagram

$$\begin{tikzcd} & \mathbb{P}^3 \times \mathbb{P}^3 \arrow[dr] & \\ X \arrow[ur] \arrow[rr, "\varphi_{D+D'}"'] & & \mathbb{P}^{15} \end{tikzcd}$$

But the dimension of space of section which defines embedding $\varphi_{D+D'}$ is $h^0(D + D') = \frac{20}{2} + 2 = 12$ (by Riemann-Roch), so we have that the projective space where we can embed X is of dimension $12 - 1 = 11$. Hence , we can conclude that:$15 - 11 = 4$ linear equations of type $(1, 1)$ in $\mathbb{P}^{15}$ define X. In fact X is a complete intersection of 4 divisors of type $(1, 1)$ in $\mathbb{P}^3 \times \mathbb{P}^3$.

We will now show that the surface obtained as complete intersection of 4 divisors with classes (1,1) is K3 surface with properties which are desired in the begining of III part.

Let D be divisor on X. We will define cycle class of the divisor D as $[D] = c_1(D) \in H^2(X, \mathbb{Z}) \hookrightarrow H^2_{DR}(X)$.

Remark 4.3.1. Let $Z \subseteq X$ have codimension k in the n-dimensional complex manifold X. then

$$\int_Z (H_{DR}(X)^{2(n-k)})^{\text{dual}} \cong H^{2k}_{DR}(X)$$

the last isomorphism is provided by Poincare's theorem. So we can represent $\int_Z$ as $[v]$ such that for all $[\omega] \in H_{DR}^{2(n-r)}(X)$ we have $\int_Z \omega = \int_X v \wedge \omega$. If Y is complete intersection of two divisors D_1, D_2 then we have $[Y] = [D_1] \wedge [D_2]$.

If we return now to our case we have that X is intersection of 4 divisors of class $(1, 1)$. So we have that

$$[D_1 \cap D_2 \cap D_3 \cap D_4] = [D_1] \wedge [D_2] \wedge [D_3] \wedge [D_4]$$

and $D_i \subseteq \mathbb{P}^3 \times \mathbb{P}^3$ and we know that

$$H^i_{DR}(\mathbb{P}^3) = \begin{cases} 0 & i \text{ odd} \\ \mathbb{R} & i \text{ even} \end{cases}$$

By Künneth formula we have that

$$\begin{aligned} H^2_{DR}(\mathbb{P}^3 \times \mathbb{P}^3) &= H^0(\mathbb{P}^3) \otimes H^2(\mathbb{P}^3) + H^1(\mathbb{P}^3) \otimes H^1(\mathbb{P}^3) + H^2(\mathbb{P}^3) \otimes H^0(\mathbb{P}^3) \\ &= \mathbb{R}[\omega_{FS,2}] \oplus \mathbb{R}[\omega_{FS,1}] \end{aligned}$$

where ω_{FS} is the Fubini- Study form.

If $Y \subseteq \mathbb{P}^3 \times \mathbb{P}^3$ is a manifold of codimenison 1, then we have that $[Y] = ax + by$ where $x = \pi_1^* \omega_{FS}$ and $y = \pi_2^* \omega_{FS}$ since we know that $\mathrm{Pic}(\mathbb{P}^3 \times \mathbb{P}^3) = \mathrm{Pic}(\mathbb{P}^3) \oplus \mathrm{Pic}(\mathbb{P}^3)$ $(a, b \in \mathbb{Z})$. We say that Y has type (a, b).

Again in our case we have 4 divisors D_i of type $(1, 1)$ and then

$$\begin{aligned} [D_1] \wedge [D_2] \wedge [D_3] \wedge [D_4] &= (x + y) \wedge \ldots \wedge (x + y) \\ &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 \end{aligned}$$

$x^4 = (\pi_1^* \omega_{FS})^4 = \pi_1^*(\omega_{FS}^4) \in H^8_{DR}(\mathbb{P}^3)$, since $\dim_{\mathbb{R}}(\mathbb{P}^3) = 6$ we have that $H^8_{DR}(\mathbb{P}^3) = 0$ and $x^4 = 0$. Same is for y^4. X is complete intersection of divisors D_1, D_2, D_3, D_4 and

$$\begin{aligned} [X] &= [D_1] \wedge [D_2] \wedge [D_3] \wedge [D_4] \\ &= 4x^3y + 6x^2y^2 + 4xy^3 \in H^8_{DR}(\mathbb{P}^3 \times \mathbb{P}^3). \end{aligned}$$

We have an embedding $i \colon X \hookrightarrow \mathbb{P}^3 \times \mathbb{P}^3$ and we naturally have $i^* \colon H^*_{DR}(\mathbb{P}^3 \times \mathbb{P}^3) \to H^*_{DR}(X)$. Let $x|_X$ and $y|_X$ be classes in $H^2_{DR}(X)$. Our aim is to find what are $(x|_X)^2$ and $(y|_X)^2$. We have that

$$(x|_X)^2 = (i^* x)^2 = i^*(x^2)$$

We claim that

$$i^*(x^2) = [x^2] \wedge [X] \in H^{12}_{DR}(\mathbb{P}^3 \times \mathbb{P}^3)$$

and

$$\begin{aligned} [x^2] \wedge [X] &= [x] \wedge [x] \wedge [D_1] \wedge [D_2] \wedge [D_3] \wedge [D_4] \\ &= x^2 \cdot (4x^3y + 6x^2y^2 + 4xy^3) = 4x^3y^3 = 4 \cdot 1 = 4 \end{aligned}$$

Similarly $[y^2] \wedge [X] = 4$ and $[xy] \wedge [X] = 6$. So intersection matrix on X will be

$$\begin{pmatrix} 4 & 6 \\ 6 & 4 \end{pmatrix} \sim \begin{pmatrix} 4 & 2 \\ 2 & -4 \end{pmatrix}$$

(since in basis (1,0), (0,1) intersection matrix will have this form) . And it is exactely surface with desired properties. So our surface is obtained as complete intersection of 4 divisors with classes $(1, 1)$.

How to find an automorphism g on the surface X, as before, such that automorphism is of positive entropy and free?

Motivation step:

Divisors $D' \sim (1,1)$, $D'' \sim (2,3)$, $D^* \sim (5,8)$ are such that $(D')^2 = 4$, $(D'')^2 = 4$, and $(D^*)^2 = 4$. All of them define embeddings $\varphi_{|D'|}, \varphi_{|D''|}, \varphi_{|D^*|}$ into projective space. Divisor D^* with class $(5,8)$ in $NS(X)$ correspond to the element η^6 in N because $\eta^6 = 8\eta + 5$. This element η^6, using the isomorphism φ between N and $NS(X)$, defines automorphism g with desired properties.

Let us try to write explicitly automorphism g. First of all let us consider divisor $D' \sim (1,1)$ on X. That divisor, as it is shown before, is obtained by intersection X with some cubic surface R. So $R \cap X = D' + D_0$, where $D_0 \sim (2,-1)$. D_0 is curve on X and it is defined by global section t from $H^0(\mathcal{O}_X(D_0))$ such that $(t = 0) = D_0$. We will now construct $\varphi_{(1,1)}$.

We know that $\mathcal{O}_X(D') \xrightarrow{\cdot t} \mathcal{O}_X(D' + D_0) \cong \mathcal{O}_X(3)$, where $\mathcal{O}_X(3)$ represents all cubics. Let us now see what is space of all cubics containing D_0. We will denote that space with I_0. In fact, we have next situation: $R \in I_0$ is defined by $(F \equiv 0)$, so we have that $X \cap (F \equiv 0) = D_0 + D'_F$, where $D'_F \in |D'|$, and it is valid for all $R \in I_0$. So $F|_X = s \cdot t$, where $(s = 0)$ defines D_0 and $(t = 0)$ defines D'_F.

And so we have map:

$$F|_X = s \cdot t \mapsto t \in H^0(D')$$

where as we said s is a section which is zero on D_0. This map is isomorphism between I_0 and $H^0(D')$, since:

- injectivity: if we take $R_F = D_0 + D'_F$ and $R_G = D_0 + D'_G$ such that $D'_F, D'_G \in |D'|$ and such that $D'_F \neq D'_G$ then

$$\begin{aligned} F|_X &= s \cdot t_1 \mapsto t_1 \\ G|_X &= s \cdot t_2 \mapsto t_2 \end{aligned}$$

  and so we have $t_1 \neq t_2$, where t_1 and t_2 belong to $H^0(D')$.

- surjectivity: $\forall t \in H^0(D')$ is such that $(t = 0) = \widetilde{D}$ and $\widetilde{D} \in |D'|$. Hence $\widetilde{D} \sim (1,1)$ and $\widetilde{D} + D_0 \sim (3,0)$ will be the cubic which contains D_0 .Thus our map is surjective.

Then we have that $I_0 \cong H^0(D')$. By Riemann-Roch we have

$$h^0(D') = 2 + \frac{(D')^2}{2} = 2 + \frac{4}{2} = 4,$$

so $I_0 \cong \mathbb{C}^4$. Space I_0 we can write down as $I_0 = \langle R_0, R_1, R_2, R_3 \rangle$. As we know that $\varphi_{|D'|}$ is an embedding into $\mathbb{P}^3$ of X we have that $\varphi_{|D'|} \colon X \hookrightarrow X_1 \subseteq \mathbb{P}^3$ defined by

$$x \mapsto (R_0(x) \colon \dots \colon R_3(x))$$

. If it happens that $R_i(x)$ is 0 for all i on D' then we just take some other divisor $D_1 \in |D'|$ and everything will fits. The embedding $\varphi_{|D'|} \colon X \hookrightarrow X_1$ we will call $\varphi_{(1,1)}$.

Let us now consider all cubics on X_1 cut out by divisor D_1', i.e. all cubics of class $3 \cdot (1,1) = (3,3)$. We can write them as $(3,3) = (2,3) + (1,0)$. So now we will consider all cubics on X_1 which contains $\varphi_{1,1}(D)$, where D is the hyperplane curve defined by $X \cap H$, $D \sim (1,0)$. These cubics are making space which is isomorphic to $H^0(\mathcal{O}_X(D''))$,as we proved before, and $D'' \sim (2,3)$. In this way we defined base of section on X_1 which provide embedding $\varphi_{|D''|} \colon X_1 \hookrightarrow X_2$. This embedding we will call $\varphi_{2,3}$.

Now we will consider cubics on X_2 cut out by divisor $D'' \sim (2,3)$. So we have that $3 \cdot (2,3) = (6,9) = (1,1) + (5,8)$. These cubics all contains $\varphi_{23}(D')$. Divisor with class $(5,8)$ we will denote as D^*. Space of cubics on X_2 contains $\varphi_{2,3}(D')$ is isomorphic to the space $H^0(\mathcal{O}_X(D^*))$. This space gives bases of space of sections which define an embedding $\varphi_{|D^*|} \colon X_2 \hookrightarrow X_3 \subseteq \mathbb{P}^3$, and as we get used to, we will denote it with $\varphi_{5,8}$. On the end we have $\varphi_{5,8} \colon X_2 \hookrightarrow X_3 \subseteq \mathbb{P}^3$. Explicitly we can write that g is:

$$g \colon S \xrightarrow{\varphi_{1,1}} S_1 \xrightarrow{\varphi_{2,3}} S_2 \xrightarrow{\varphi_{5,8}} S_3 \xrightarrow{M} S$$

M is a linear map and $MS_3 = S$. This linear map exists since $g^*(1,0) = (5,8)$ and explicitly if $\{t_0, \ldots, t_3\}$ is a basis of $H^0(D^*)$ and $\{s_0, \ldots, s_3\}$ is a basis for $H^0(D)$ then $g^* s_i = \sum a_{ij} t_j$ and $[a_{ij}]_{i,j}$ will represent matrix M and it will give our linear transformation M between isomorphic quartic surfaces in $\mathbb{P}^3$.

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