Master thesis in Mathematics

On Drinfel′d's Non-Archimedean Upper Half-Plane

Soumya Bhattacharya

Advisor: Prof. Marco Garuti

September 2008

Università degli studi di Padova

আমার বাবা, মা, তাতাই আর ঈশিতাকে

Introduction

Let K be a local field and C the completion of the algebraic closure of K. The non-archimedean upper-half plane over K is defined as $\Omega := \mathbb{P}^1(C) \setminus \mathbb{P}^1(K)$. (cf. [3])

In section 6 of [8], Drinfel'd cites three analogs of the complex upper half-plane in the non-archimedean setting, viz., Ω as an *analytic analog*, the Bruhat-Tits tree on the equivalence class of lattices as a *homogeneous analog* and the space of norms as a *topological analog*. Also, in analogy with the action of $GL_2^+(\mathbb{R})$ on the complex upper-half plane by Möbius transformation (cf. chapter 2 in [5]), an action of the group $PGL_2(K)$ is defined on each of these non-archimedean analogs and an archimedean metric is defined on the space of norms. Then the space of norms is identified with the Bruhat-Tits tree via an isomorphism and the reduction map is defined from Ω to the Bruhat-Tits tree in order to give a relation between these analogs. This map commutes with the actions of $PGL_2(K)$ and enables us to imagine intuitively Ω as the boundary of a tubular neighbourhood of the Bruhat-Tits tree with the reduction map corresponding to the projection onto the tree. (cf. subsection 2.5 in [3])
The simplest way to describe this map is to adopt the point of view of Goldman and Iwahori (cf. [12]) i.e. to send the point $[x, y] \in \Omega$ to the equivalence class of the norm $\| \ \|$ on K^2, which is defined as $\|(a, b)\| = |ax + by| \ \ \forall\, (a, b) \in K^2$, where $| \ |$ denotes the norm on C, obtained as the unique extension of the norm on K.
Finally, using the fibers of the reduction map, we obtain a rigid analytic structure on Ω in the following way:

The pre-image of any archimedean closed ball on the space of norms centred at the equivalence class of an integral norm turns out to be a connected affinoid subset of $\mathbb{P}^1(C)$. Moreover, if we consider a set of concentric closed balls on the space of norms centred at the equivalence class of an integral norm, their corresponding pre-images give an admissible covering of Ω by an increasing sequence of connected affinoid subsets of $\mathbb{P}^1(C)$. This defines a Grothendieck topology on Ω and thus, a rigid analytic structure on Ω is obtained. (cf. proposition 6.1 in [8], prposition 1.2.5 in [6] and proposition 2.4(b) in [13])

The space Ω supports a theory of analytic functions, differential forms and it also has interesting deRham and étale cohomologies. (cf. [3], [7] and [16])

The existence of the reduction map implies many important connections between Ω and the Bruhat-Tits tree. One of the most striking result among them is the theorem of Schneider and Stuhler, which states that, the deRham chohomology group $H^n_{DR}(\Omega, K)$ is isomorphic with the space of harmonic functions on the Bruhat-Tits tree for K. (cf. [16])

The rigid analytic structure on Ω lies at the heart of the theory of p-adic uniformization of Shimura curves (cf. [3] and chapter 2 of [5]). If Γ is a discrete group of $GL_2(K)$, then the quotient space $\Gamma\backslash\Omega$, may also be given a natural rigid analytic structure and the natural map $\Omega \to \Gamma\backslash\Omega$ is a morphism of rigid analytic spaces (cf. [8] or proposition 2.3(d) in [13]). In most of the times, The quotient space $\Gamma\backslash\Omega$ turns out to be an algebraic curve defined over K, by a non-archimedean analog of the GAGA theorem (cf. [11]). And amazingly, the curves constructed in this way are often the same Shimura curves which arise out of the complex upper half-plane. (cf. [19] for an explicit example)

In this thesis, we focus on the relation between the archimedean topology on the geometric realization of the Bruhat-Tits tree and the rigid analytic structure on Ω via the fibers of the reduction map.

In chapter 1, we construct an explicit isomorphism between the Bruhat-Tits tree and the space of non-archimedean norms on K^2. Section 1 and 2 develop the necessary material for this, by introducing the tree (cf. figure 1.1 on page 17) and its geometric realization as well as proving the discreteness of the non-archimedean norms on finite dimensional K-vector spaces and describing a natural map from the geometric realization of the tree to the space of norms. Then we provide a map in the opposite direction, which is shown to be the inverse of the former map. Thus establishing the isomorphism between the geometric realization of the tree and the space of norms in section 2, in the next section we proceed to investigate the effects of this isomorphism on the natural metric of the geometric realization of the tree given by the length of the unique path joining any two points on it. Indeed, the knowledge of the relation between the metric on the geometric realization of the tree and the corresponding metric on the space of norms turns out to be very useful in the next chapter.

Our goal in chapter 2 is to see the connection between the different geometric structures on the geometric realization of the Bruhat-Tits tree and on Ω, using the fibers of the reduction map. In section 1, we define the reduction map from Ω to the geometric realization of the tree, via the space of injective K-linear forms on K^2 and the space of non-archimedean norms on K^2 and then we give a precise description of its image in the tree. In section 2, we introduce the action of the group $PGL_2(K)$ on the tree, on the space of norms and on Ω. Next, we show that the metric on the tree is invariant under the $PGL_2(K)$-action, whereas the reduction map is $PGL_2(K)$-equivariant, i.e. the map commutes with the group action. Then in the final section, we define the balls on the projective space $\mathbb{P}^1(C)$ and prove that $PGL_2(K)$ sends the balls to balls in $\mathbb{P}^1(C)$. Using these facts along with the transitivity of $PGL_2(K)$-action on the tree, the situation becomes a lot simplified, i.e. considering only the finite subtrees of the Bruhat-Tits tree around a specific vertex, we are able to draw general conclusions. With these efficient tools in hand, we finally prove that the pre-image of the geometric realization of any finite subtree of the Bruhat-Tits tree is a connected affinoid subset of $\mathbb{P}^1(C)$. Particularly, when this finite subtree is a closed ball around some vertex of the tree, we have some more informations about the structure of this connected affinoid subset (cf. proposition 10 on page 59). We also study the effects of varying the radius of this closed ball in the tree, on the corresponding affinoid subset of $\mathbb{P}^1(C)$ and obtain a nice conclusion. (cf. remark 20 on page 60, figure 2.1 on page 61 and figure 2.2 on page 62)

Acknowledgements

I would like to thank Prof. Marco Garuti for his careful reading of the manuscript and also both Prof. Marco Garuti and Prof. Maurizio Cailotto for many helpful discussions on the topic of this thesis.
I am grateful to Prof. Adrian Iovita for giving me a very nice introductory lecture on the Bruhat-Tits tree, prescribing many useful references and suffering some of my talks on basic rigid geometry.

I am also thankful to the ALGANT committee for giving me an opportunity to learn Mathematics in two celebrated European universities as well as to the Erasmus Mundus authority for the generosity of the scholarship.

Thanks to all my friends in Padova, for their unusual cordiality and also for helping me learn Latex, Metapost and above all, a basketful of nice Italian words.

And finally I wish to thank my father, my mother, my little brother and little Ms. Ichitdengut for supporting me throughout the time of preparation of this thesis and for making my life so beautiful with their affection and love.

Contents

Chapter 1

The Bruhat-Tits Tree

Notation 1.

K	: *A local field of characteristic* 0 *with a discrete valuation* v_K *and a finite residue field* $k \cong \mathbb{F}_q$ *where,* $q = p^r$ *for some prime* $p \in \mathbb{Z}$ *and* $r \in \mathbb{Z}_{\geq 1}$.
$\mathcal{O}$	: *The ring of integers of* K.
π	: *A uniformizing parameter of* $\mathcal{O}$.
$\mid\ \mid_K$	: *The norm on* K*, given by* $\|x\|_K := \dfrac{1}{q^{v_K(x)}}$.
C	: *The completion of the algebraic closure of* K.
$\mid\ \mid$	: *The unique norm on* C*, which extends that of* K.
v	: *The valuation on* C*, given by* $v(x) := \log_q \|x\|$.
Ω	: $\mathbb{P}^1(C) \setminus \mathbb{P}^1(K)$.

Remark 1. $\#\, k < \infty \;\Rightarrow\; [K : \mathbb{Q}_p] < \infty$.
Therefore, C and $\mathbb{C}_p$ can be identified.

1.1 The Tree

Definition 1. *A lattice* $L \subseteq K^2$ *is a free* $\mathcal{O}$*-submodule of rank* 2.

Definition 2. *Two lattices* L *and* L' *are equivalent, written* $L \sim L'$*, if* $\exists\ \lambda \in K^*$ *such that* $L' = \lambda L$*. We denote the equivalence class of the lattice* L *by* $[L]$ *and we denote the set of the equivalence classes of lattices by* S*. i.e.,*

$$[L] := \{\ L' \mid L \sim L'\ \}.$$

$$S := \{\ [L] \mid L \subseteq K^2 \text{ is a lattice }\}.$$

Definition 3. *X is the graph with the set of its vertices, $V(X) = S$ such that given $s_1, s_2 \in S$, we have $\{s_1, s_2\} \in E(X)$, the set of the edges of X, iff $s_1 = [L_1]$ and $s_2 = [L_2]$ with*

$$\pi L_1 \subsetneq L_2 \subsetneq L_1.$$

Remark 2. *Note that, the edges of X are not directed, as*
$\pi L_1 \subsetneq L_2 \subsetneq L_1 \quad \Leftrightarrow \quad L_2 \subsetneq L_1 \subsetneq \frac{1}{\pi} L_2 \quad \Leftrightarrow \quad \pi(\frac{1}{\pi} L_2) \subsetneq L_1 \subsetneq (\frac{1}{\pi} L_2)$
and we have, $(\frac{1}{\pi} L_2) \sim L_2$.

Definition 4.
i) A tree is a graph in which between any two vertices there is exactly one path.
ii) A homogeneous tree of degree d is a tree in which each vertex belongs to exactly d edges.

Proposition 1. *The graph X is a homogeneous tree of degree $q + 1$.*

To prove this proposition, we need the following lemmas.

Lemma 1. *If $L \subseteq K^2$ is a lattice, then $L/\pi^n L \cong \mathcal{O}/\pi^n \mathcal{O} \oplus \mathcal{O}/\pi^n \mathcal{O} \ \forall \ n \in \mathbb{Z}_{\geq 0}$. In particular, $L/\pi L \cong k^2$.*

Proof. Since, for any 2 pair of modules, $M_1' \subseteq M_1$ and $M_2' \subseteq M_2$, we have, $(M_1 \oplus M_2)/(M_1' \oplus M_2') \cong (M_1/M_1') \oplus (M_2/M_2')$. □

Lemma 2. *If $L_1, L_2, L_3, L_4 \subseteq K^2$ be lattices such that*

$$\pi L_1 \subseteq L_2 \subseteq L_4 \quad \textit{and} \quad \pi L_1 \subseteq L_3 \subseteq L_4 \ ,$$

then, $L_2/\pi L_1 = L_3/\pi L_1 \ \Leftrightarrow \ L_2 = L_3$.

Proof. $L_2/\pi L_1 = L_3/\pi L_1$
$\Leftrightarrow L_2 = L_3 + \pi L_1 = L_3$. □

Lemma 3. *If $L_1, L_2, L_3 \subseteq K^2$ be lattices such that*

$$\pi L_1 \subsetneq L_2 \subsetneq L_1 \quad \textit{and} \quad \pi L_1 \subsetneq L_3 \subsetneq L_1 \ ,$$

then, $L_2 \sim L_3 \ \Leftrightarrow \ L_2 = L_3$.

Proof. Suppose, if possible, $L_2 \sim L_3$ and $L_2 \neq L_3$.

Let $L \subseteq K^2$ be any lattice.
Then $\pi L_1 \subsetneq L \subsetneq L_1$
$\Leftrightarrow \{0\} \subsetneq L/\pi L_1 \subsetneq L_1/\pi L_1$
$\Leftrightarrow L/\pi L_1$ is a 1-dimensional subspace of $L_1/\pi L_1$.

Let $x, y \in L$ and let $\bar{x}, \bar{y} \in L/\pi L$ be their images such that
$\langle \bar{x} \rangle = L_2/\pi L_1$ and $\langle \bar{y} \rangle = L_3/\pi L_1$.
Now, $L_2 \neq L_3$
$\Leftrightarrow \langle \bar{x} \rangle = L_2/\pi L_1 \neq L_3/\pi L_1 = \langle \bar{y} \rangle$ (cf. lemma 2)
$\Rightarrow \langle \bar{x}, \bar{y} \rangle = L_1/\pi L_1$ (cf. lemma 1)
$\Leftrightarrow \langle x, y \rangle = L_1$ (cf. Nakayama's lemma)
$\Rightarrow \langle x, \pi y \rangle = L_2$ and $\langle \pi x, y \rangle = L_3$. (cf. lemma 2)

Again, $L_2 \sim L_3$
$\Rightarrow \exists\ \lambda \in K^*$ such that $L_3 = \lambda L_2$.
$\Rightarrow$ either $L_2 \subseteq \pi L_3 \subsetneq L_2$ or $L_3 \subseteq \pi L_2 \subsetneq L_3$. (as $L_2 = \langle x, \pi y \rangle$ and $L_3 = \langle \pi x, y \rangle$)
Thus, we get a contradiction.
Hence, $L_2 = L_3$. □

Lemma 4. *Let $L \subseteq K^2$ be a lattice. For any 1-dimensional subspace $N \subsetneq L/\pi L$, $\exists$ a lattice $L' \subseteq K^2$ such that $\pi L \subsetneq L' \subsetneq L$ and $L'/\pi L = N$.*

Proof. Let $x, y \in L$ and let $\bar{x}, \bar{y} \in L/\pi L$ be their images such that
$\langle \bar{x} \rangle = N$ and $\bar{y} \notin N$.
$\Rightarrow \langle \bar{x}, \bar{y} \rangle = L/\pi L$ (cf. lemma 1)
$\Leftrightarrow \langle x, y \rangle = L$. (cf. Nakayama's lemma)

Let $L' := \langle x, \pi y \rangle$.
$\Rightarrow \pi L \subsetneq L' \subsetneq L$ and $L'/\pi L = N$. (cf. lemma 2) □

Proof of the proposition. Lemma 3 implies that the edges leaving a given vertex $s = [L]$ correspond to the distinct lattices L', satisfying the relation

$$\pi L \subsetneq L' \subsetneq L$$

and by lemma 1, 2 and 4, these are in bijection with the 1-dimensional subspaces of k^2, i.e., with $\mathbb{P}^1(k) = k \cup \{\infty\}$.
Hence, there are $\#\ \mathbb{P}^1(k) = q + 1$ edges leaving any vertex $s \in X$.

Again, if X is not a tree, a cycle in X can be represented as a chain of lattices :

$$L_0 \subsetneq L_1 \subsetneq \ldots \subsetneq L_n \quad \text{for some } n \in \mathbb{Z}_{\geq 3}$$

such that,

i) $\pi L_{i+1} \subsetneq L_i \subsetneq L_{i+1}\ \forall\ i \in \mathbb{Z}_{[0,n-1]}$.
ii) $L_0 = \pi^m L_n$ for some $m \in \mathbb{Z}_{[2,n-1]}$.
iii) $L_i \not\sim L_j$ *if* $i, j \notin \{0, n\}$ and $i \neq j$.

Considering the exact sequences,

$$0 \longrightarrow L_i/L_{i-1} \longrightarrow L_n/L_{i-1} \longrightarrow L_n/L_i \longrightarrow 0 \quad \forall\, i \in \mathbb{Z}_{[1,n]},$$

starting with $i = n$, by induction and repeated applications of proposition 6.9 of [2], we get, L_n/L_i has length $(n-i)$ $\forall\, i \in [0,n]$.
Since L_n/L_0 is not cyclic, hence $\exists\, i_0 \in \mathbb{Z}_{[0,n-1]}$ such that

$$i_0 := \max \{\, i \mid L_n/L_i \text{ is cyclic but } L_n/L_{i-1} \text{ is not}\}$$

$\Rightarrow L_n/L_{i_0}$ is a cyclic $\mathcal{O}$-module of length $n - i_0$
$\Rightarrow L_n/L_{i_0} \cong \mathcal{O}/\pi^{n-i_0}\mathcal{O}$
$\Rightarrow \pi^{n-i_0-1}L_n \nsubseteq L_{i_0}$
$\Rightarrow \pi^{n-i_0-1}L_n \nsubseteq L_{i_0-1}$ (as $L_{i_0-1} \subseteq L_i$)

Also, $\exists\, r_1, r_2 \in \mathbb{Z}_{\geq 1}$, such that $L_n/L_{i_0-1} = \mathcal{O}/\pi^{r_1}\mathcal{O} \oplus \mathcal{O}/\pi^{r_2}\mathcal{O}$, where $r_2 \geq r_1$.
As π^{n-i_0-1} does not annihilate all the elements of L_n/L_{i_0-1}, hence $r_2 \geq n - i_0$.
Again, $r_1 + r_2 =$ length of $L_n/L_{i_0-1} = n - i_0 + 1$, where $r_1, r_2 \in \mathbb{Z}_{\geq 1}$
$\Rightarrow r_1 = 1$ and $r_2 = n - i_0$.
$\Rightarrow L_n/L_{i_0-1} \cong \mathcal{O}/\pi\mathcal{O} \oplus \mathcal{O}/\pi^{n-i_0}\mathcal{O}$
Hence, considering the image of a generator of L_{i_0}/L_{i_0-1} in the inclusion $L_{i_0}/L_{i_0-1} \hookrightarrow L_n/L_{i_0-1}$, we see, the exact sequence in the 1st row of the following diagram splits.

$$\begin{tikzcd}
0 \arrow[r] & L_{i_0}/L_{i_0-1} \arrow[r] \arrow[d, hook] & L_n/L_{i_0-1} \arrow[r] \arrow[d, equal] & L_n/L_{i_0} \arrow[r] & 0 \\
0 \arrow[r] & L_{i_0+1}/L_{i_0-1} \arrow[r] & L_n/L_{i_0-1} \arrow[r] & L_n/L_{i_0+1} \arrow[r] & 0
\end{tikzcd}$$

$\Rightarrow L_n/L_{i_0-1} = L_{i_0}/L_{i_0-1} \oplus L_n/L_{i_0}$.
Again, $L_{i_0}/L_{i_0-1} \subsetneq L_{i_0+1}/L_{i_0-1} \subseteq L_{i_0}/L_{i_0-1} \oplus L_n/L_{i_0}$.
$\Rightarrow L_{i_0+1}/L_{i_0-1}$ contains a non-cyclic submodule.

And length of L_{i_0+1}/L_{i_0-1}
=(length of L_n/L_{i_0-1})$-$(length of L_n/L_{i_0+1})
$= 2$

$\Rightarrow L_{i_0+1}/L_{i_0-1}$ is a non-cyclic, length 2 $\mathcal{O}$-module
$\Rightarrow L_{i_0-1} = \pi L_{i_0+1}$, which contradicts (iii).

Hence, X is a tree. $\square$

Remark 3. *The tree X is called the Bruhat-Tits tree.*

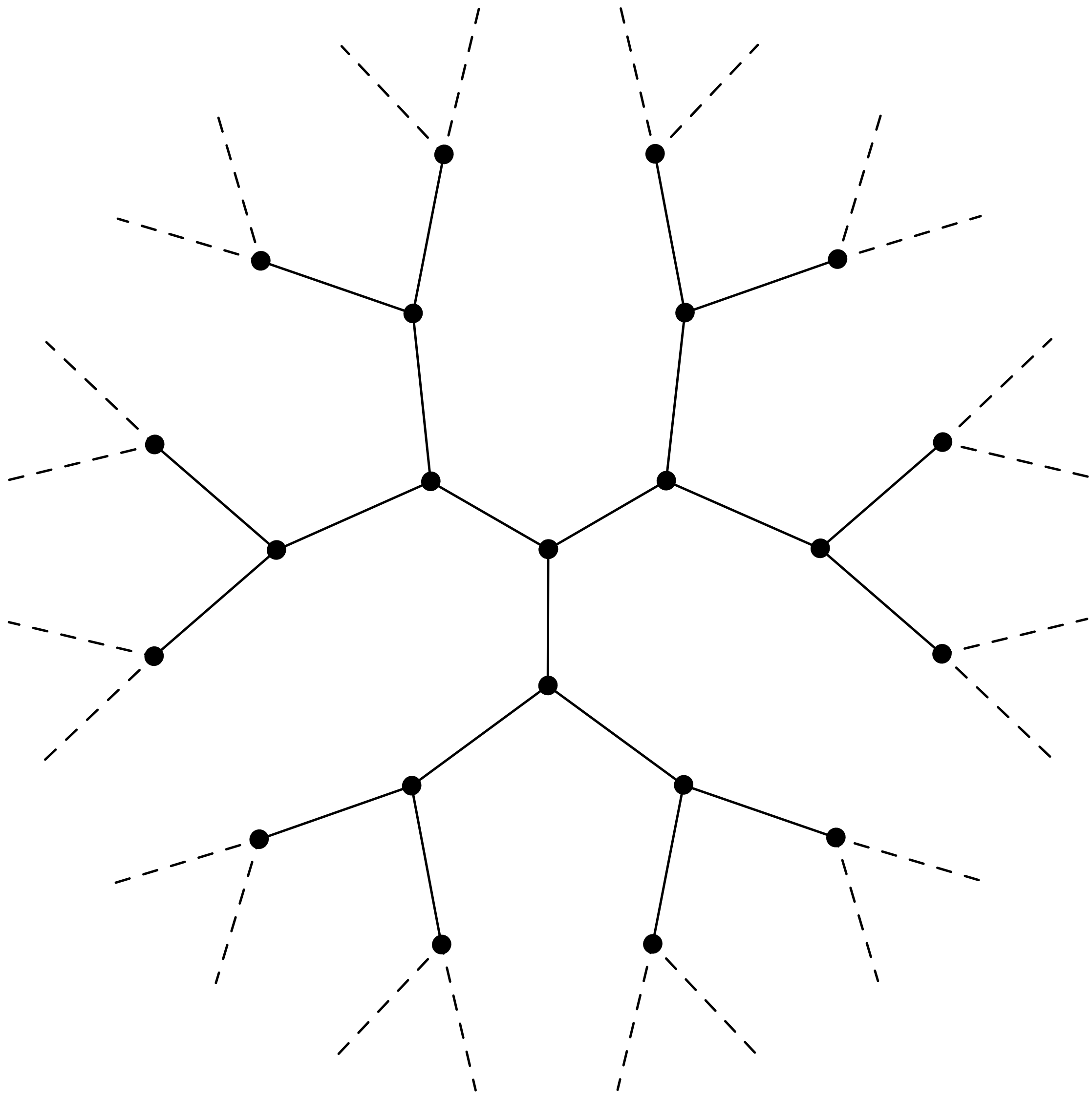

Figure 1.1: The Bruhat-Tits tree for $K = \mathbb{Q}_2$

Definition 5. $X_{\mathbb{R}} := \{(1-t)s_1 + ts_2 \mid \{s_1, s_2\} \in E(X) \text{ and } t \in [0,1]\}$.

Remark 4. *$X_{\mathbb{R}}$ is called the geometric realization of the Bruhat-Tits tree. Note that, as X is a tree, hence, also for any two points in $X_{\mathbb{R}}$, there is a unique path joining them.*

Definition 6. $\forall\ x_1, x_2 \in X_{\mathbb{R}}$,

$$\sigma(x_1, x_2) := \text{ The unique path joining } x_1 \text{ and } x_2 \text{ in } X_{\mathbb{R}}.$$

Definition 7. $X_{\mathbb{Q}} := \{(1-t)s_1 + ts_2 \mid \{s_1, s_2\} \in E(X) \text{ and } t \in \mathbb{Q} \cap [0,1]\}$.

1.2 The Space of Non-Archimedean Norms

Definition 8. *A non-archimedean norm on a vector space V over K is a function $\| \ \| : V \to \mathbb{R}_{\geq 0}$ such that, $\forall\ x, y \in V$ and $\forall\ a \in K$,*

i) $\|x\| = 0$ *iff* $x = 0$.
ii) $\|ax\| = |a|_K \ \|x\|$.
iii) $\|x + y\| \leq \max\{\|x\|, \|y\|\}$, *where equality holds if* $\|x\| \neq \|y\|$.

Lemma 5. *Given a non-archimedean norm, $\| \ \|$ on an n-dimensional vector space V over K, there exists a unique n-tuple $(a_1, \ldots, a_n) \in \ [1, q)^n$, where $a_1 \leq \ldots \leq a_n$, such that*

$$\|V\| = \bigcup_{i=1}^{n} a_i |K|_K = \begin{cases} a_1 |K|_K & \text{if} \quad a_1 = \ldots = a_n. \\ \bigsqcup_{i=1}^{d} a_i |K|_K & \text{if} \quad a_1 < \ldots < a_d = \ldots = a_n. \end{cases}$$

Proof. There are 2 possible cases:

Case 1: $\left(\frac{\|v_1\|}{\|v_2\|} \in |K|_K \ \forall\ v_1, v_2 \in V \text{ such that } v_2 \neq 0\right)$

Let $v \in V \setminus \{0\}$ be such that $\|v\| \in [q^n, q^{n+1})$ for some $n \in \mathbb{Z}$.
Let $v' := \pi^n v$ and let $a := \|v'\|$. Then $\|V\| = \|v'\| |K|_K = a|K|_K$.

Again, $a \in [1, q)$
$\Rightarrow q^m a \in [q^m, q^{m+1}) \ \ \forall\ m \in \mathbb{Z}$
$\Rightarrow a \in [1, q)$ is the unique element such that $\|V\| = a|K|_K$.

Case 2. $\left(\ \exists\ v_1, v_2 \in V \setminus \{0\} \text{ such that } \frac{\|v_1\|}{\|v_2\|} \notin |K|_K\right)$

Let $w_1, \ldots, w_d \in V \setminus \{0\}$ be such that $\frac{\|w_i\|}{\|w_j\|} \notin |K|_K \ \ \forall\ i, j \in \{1, \ldots, d\}$.
$\Rightarrow \forall\ \alpha_1, \ldots, \alpha_d \in K$, we have,
$\|\alpha_1 w_1 + \ldots + \alpha_d w_d\| = \max\ \{|\alpha_1|_K \|w_1\|, \ |\alpha_d|_K \|w_d\|\}$
$\Rightarrow K w_1 \oplus \ldots \oplus K w_d \subseteq V$
$\Rightarrow d \leq \dim V$

Hence, we can choose the set $W := \{w_1, \ldots, w_d\} \subseteq V \setminus \{0\}$ to have the maximum possible cardinality.
i.e., if $W' \subseteq V$ be some other set such that $\frac{\|w_i'\|}{\|w_j'\|} \notin |K|_K \ \ \forall\ w_i', w_j' \in W'$, then

$$d \geq \#\ W' .$$

$\Rightarrow \forall\ v \in V,\ \ \exists\ i \in \{1, \ldots, d\}$ such that $\frac{\|v\|}{\|w_i\|} \in |K|_K$
$\Rightarrow \|V\| = \sqcup_{w \in W}\ \|w\|\ |K|_K$.

Let $\forall\ i \in \{1, \ldots, d\},\ \ v_i' := \pi^{n_i} w_i$ for some suitable $n_i \in \mathbb{Z}$ such that $\|v_i'\| \in [1, q)$.
Let $\tau : \{1, \ldots, d\} \to \{1, \ldots, d\}$ be a permutation such that

$$\|v_{\tau(1)}'\| < \ldots < \|v_{\tau(d)}'\|.$$

$$\text{Let}\ \ a_i := \begin{cases} \|v_{\tau(i)}'\| & \text{if} \quad 1 \leq i \leq d-1. \\ \\ \|v_{\tau(d)}'\| & \text{if} \qquad d \leq i \leq n. \end{cases}$$

Then $\|V\| = \sqcup_{w \in W}\ \|w\|\ |K|_K = a_1 |K|_K \sqcup \ldots \sqcup\ a_d |K|_K$.
Again, $(a_1, \ldots, a_n) \in [1, q)^n$
$\Rightarrow (q^m a_1, \ldots, q^m a_n) \in [q^m, q^{m+1})\ \ \forall\ m \in \mathbb{Z}$
$\Rightarrow (a_1, \ldots, a_n) \in [1, q)^n$, is the unique element such that $a_1 < \ldots < a_d = \ldots = a_n$ and $\|V\| = a_1 |K|_K \sqcup \ldots \sqcup\ a_d |K|_K$. □

Corollary 1. *Given a non-archimedean norm,* $\|\ \|$ *on a* 2*-dimensional vector space* V *over* K, $\exists$ *a unique pair* $(a, b) \in\ [1, q) \times [1, q)$, *where* $a \leq b$, *such that*

$$\|V\| = a|K|_K \cup b|K|_K = \begin{cases} a|K|_K & \text{if} \quad a = b\ . \\ \\ a|K|_K \sqcup b|K|_K & \text{if} \quad a \neq b\ . \end{cases}$$

Definition 9. *Two non-archimedean norms* $\|\ \|_1$ *and* $\|\ \|_2$ *on a* K*-vector space* V *are equivalent, written* $\|\ \|_1 \sim \|\ \|_2$, *if* $\exists\ c \in \mathbb{R}_{>0}$ *such that* $\|\ \|_1 = c\|\ \|_2$.

$$[\|\ \|] := \{\ \|\ \|'\ |\ \|\ \| \sim \|\ \|'\ \}\ .$$

$$Y\ \ := \{\ [\|\ \|]\ |\ \|\ \|\ \textit{is a non-archimedean norm on}\ K^2\}\ .$$

Remark 5. *If we have had taken a weaker criterion of equivalence of norms, viz.,'two norms are equivalent, if they give the same topology on* V*', then, even if we have had considered all the norms (i.e., which are not necessarily non-archimedean) on a finite dimensional* K*-vector space* V*, the set of equivalence classes would have been a singleton set.* (*Theorem* 5.2.1 *of* [14])

Convention.
From now on, by a 'norm', we shall mean a 'non-archimedean norm'.

Definition 10. *For any lattice, $L = \mathcal{O}e_1 \oplus \mathcal{O}e_2 \subseteq K^2$, the norm $\| \ \|_L$ is defined as :* $\forall\ v = a_1e_1 + a_2e_2 \in K^2$,

$$\|v\|_L := \max \ \{|a_1|_K, |a_2|_K\} \ .$$

Definition 11. *For any two lattices, $L_1, L_2 \subseteq K^2$, such that $\pi L_1 \subsetneq L_2 \subsetneq L_1$, by Elementary divisors theorem, $\exists$ a basis (e_1, e_2) of L_1 such that $(e_1, \pi e_2)$ is a basis of L_2. We fix some (e_1, e_2) as above. Then $\forall\ t \in [0, 1]$, the norm $\| \ \|_{(L_1,L_2,t)}$ is defined as :* $\forall\ v = a_1e_1 + a_2e_2 \in K^2$,

$$\|v\|_{(L_1,L_2,t)} := \max \ \{|a_1|_K, q^t|a_2|_K\} \ .$$

Lemma 6. *The norms $\| \ \|_L$ and $\| \ \|_{(L_1,L_2,t)}$ are well-defined.*

Proof. Let $L \subseteq K^2$ be any lattice. Given any two bases (v_1, v_2) and (w_1, w_2) of L, $\forall\ v = a_1v_1 + a_2v_2 = b_1w_1 + b_2w_2 \in K^2$, we have,

$$\|v\|_L = \max \ \{|a_1|_K, |a_2|_K\} = \max_{n\in\mathbb{Z}} \ \{q^n \mid \pi^{-n}v \in L\} = \max \ \{|b_1|_K, |b_2|_K\}.$$

Hence, $\| \ \|_L$ is well-defined.

Again, $\forall\ v = a_1e_1 + a_2e_2$,

$$\|v\|_{(L_1,L_2,t)} = \max \ \{|a_1|_K, q^t|a_2|_K\} = \begin{cases} \|v\|_{L_1} & \text{if } \ \|v\|_{L_1} = \|v\|_{L_2} \\ q^t\|v\|_{L_1} = q^{t-1}\|v\|_{L_2} & \text{if } \ \|v\|_{L_1} \neq \|v\|_{L_2} \end{cases}$$

And, $\| \ \|_{L_1}$ and $\| \ \|_{L_2}$ are well-defined.
$\Rightarrow \| \ \|_{(L_1,L_2,t)}$ is well-defined. □

Remark 6. *Note that, any lattice $L \subseteq K^2$, is the unit ball w.r.t. the norm $\| \ \|_L$.*

Definition 12. *For each $\{s_1, s_2\} \in E(X)$, let us fix some lattices $L_1, L_2 \subseteq K^2$ such that $s_1 = [L_1]$, $s_2 = [L_2]$ and $\pi L_1 \subsetneq L_2 \subsetneq L_1$. The map $\phi : X_\mathbb{R} \to Y$ is defined as*

$$\phi((1-t)s_1 + ts_2) := [\| \ \|_{(L_1,L_2,t)}] \quad \forall \ \ (1-t)s_1 + ts_2 \in X_\mathbb{R} \ .$$

Lemma 7. *The map $\phi : X_\mathbb{R} \to Y$ is well-defined.*

Proof. Suppose, we have fixed some $L_1, L_2 \subseteq K^2$ such that $s_1 = [L_1]$, $s_2 = [L_2]$ and $\pi L_1 \subsetneq L_2 \subsetneq L_1$.

Now, there are 2 possible cases:

Case 1. We fix another pair $L_1', L_2' \subseteq K^2$ such that $L_1' \sim L_1,\ L_2' \sim L_2$ and $\pi L_1' \subsetneq L_2' \subsetneq L_1'$
$\Rightarrow \exists\ \lambda \in K^*$ such that $L_1' = \lambda L_1$
$\Rightarrow \pi L_1' \subsetneq \lambda L_2 \subsetneq L_1'$ and $\pi L_1' \subsetneq L_2' \subsetneq L_1'$ with $L_2' \sim \lambda L_2$
$\Rightarrow L_2' = \lambda L_2$ (cf. lemma 3)

$$\Rightarrow \|v\|_{(L_1',L_2',t)} = \begin{cases} \|v\|_{L_1'} = |\lambda|_K\ \|v\|_{L_1} & \text{if } \ \|v\|_{L_1'} = \|v\|_{L_2'} \\ \\ q^t\|v\|_{L_1'} = |\lambda|_K\ q^t\|v\|_{L_1} & \text{if } \ \|v\|_{L_1'} \neq \|v\|_{L_2'} \end{cases}$$

$\Rightarrow \|v\|_{(L_1',L_2',t)} = |\lambda|_K \|v\|_{(L_1,L_2,t)}$
$\Rightarrow \|\ \|_{(L_1',L_2',t)} \sim \|\ \|_{(L_1,L_2,t)}$.

Case 2. We fix another pair $L_1', L_2' \subseteq K^2$ such that $L_1' \sim L_1,\ L_2' \sim L_2$ and $\pi L_2' \subsetneq L_1' \subsetneq L_2'$.

$\pi L_1 \subsetneq L_2 \subsetneq L_1$
$\Rightarrow \pi L_2 \subsetneq \pi L_1 \subsetneq L_2$.
Again, $\exists\ \lambda \in K^2$ such that $L_2' = \lambda L_2$.
$\Rightarrow \pi L_2' \subsetneq \pi\lambda L_1 \subsetneq L_2'$ and $\pi L_2' \subsetneq L_1' \subsetneq L_2'$ with $L_1' \sim \pi\lambda L_1$
$\Rightarrow L_1' = \pi\lambda L_1$. (cf. lemma 3)

$$\Rightarrow \|v\|_{(L_2',L_1',t)} = \begin{cases} \|v\|_{L_1'} = q^{-1}|\lambda|_K\ \|v\|_{L_1} & \text{if } \ \|v\|_{L_2'} = \|v\|_{L_1'} \\ \\ q^{t-1}\|v\|_{L_1'} = q^{-1}|\lambda|_K\ q^t\|v\|_{L_1} & \text{if } \ \|v\|_{L_2'} \neq \|v\|_{L_1'} \end{cases}$$

$\Rightarrow \|v\|_{(L_2',L_1',t)} = q^{-1}|\lambda|_K \|v\|_{(L_1,L_2,t)}$
$\Rightarrow \|\ \|_{(L_2',L_1',t)} \sim \|\ \|_{(L_1,L_2,t)}$. □

Lemma 8. *Given any norm, $\|\ \|$ on K^2, $\forall\ \alpha \in \mathbb{R}_{>0},\ L_\alpha := \{v \in K^2 \mid \|v\| \leq \alpha\}$ is a lattice.*

Proof. By definition 5, L_α is a non-zero $\mathcal{O}$-submodule of K^2.
Let $u \in L_\alpha$ and let $v \in K^2 \setminus Ku$.
$\Rightarrow v \in K^2 \setminus \mathcal{O}u$
$\Rightarrow \exists\ a \in K$ such that $v' = av \in L_\alpha \setminus \mathcal{O}u$
$\Rightarrow L_\alpha$ is generated by more than 1 element.

Again, $L_\alpha/\pi L_\alpha \subseteq k^2$ is a k-vector space,
$\Rightarrow \dim(L_\alpha/\pi L_\alpha) \leq 2$.
$\Rightarrow L_\alpha$ is generated by less than 3 elements. (cf. Nakayama's lemma)

Hence, L_α is generated by 2 elements.
Let $L_\alpha = \langle v_1, v_2\rangle$.

If $\mathcal{O}v_1 \cap \mathcal{O}v_2 \neq \{0\}$
$\Rightarrow \exists\, i \in \{1,2\}$ such that $v_{3-i} \in \mathcal{O}v_i$
$\Rightarrow L_\alpha = \langle v_i \rangle$, which is impossible.
$\Rightarrow \mathcal{O}v_1 \cap \mathcal{O}v_2 = \{0\}$
$\Rightarrow L_\alpha = \mathcal{O}v_1 \oplus \mathcal{O}v_2$. □

Lemma 9. *Let $\{s, s'\} \in E(X)$, where $s = [L]$ and $s' = [L']$ with $\pi L \subsetneq L' \subsetneq L$. Then given the norm $\| \ \|_{(L,L',t)}$, we have,*

$$L_\alpha = \begin{cases} L' & \forall\ \alpha \in [1, q^t) \\ L & \forall\ \alpha \in [q^t, q) \end{cases} .$$

In particular,

$$L_\alpha = \begin{cases} L & \forall\ \alpha \in [1, q) \quad \text{if } t = 0 \\ L' & \forall\ \alpha \in [1, q) \quad \text{if } t = 1 \end{cases} .$$

Proof. By definition 7, we have, $L = \mathcal{O}e_1 \oplus \mathcal{O}e_2 = \{v \in K^2 \mid \max\{|a_1|_K, |a_2|_K\} \leq 1\}$.
Now, $\forall\ v = a_1e_1 + a_2e_2 \in K^2$, $\exists\ r, s \in \mathbb{Z}$ such that $|a_1|_K = q^r$ and $|a_2|_K = q^s$.
$\Rightarrow \{v \in K^2 \mid \max\{|a_1|_K, |a_2|_K\} \leq 1\} = \{v \in K^2 \mid \max\{|a_1|_K, |a_2|_K\} \leq \alpha\}\ \forall\ \alpha \in [1, q)$
$\Rightarrow L = \{v \in K^2 \mid \|v\|_L \leq \alpha\}\ \forall\ \alpha \in [1, q)$
$\Rightarrow L_\alpha = L\ \ \forall\ \alpha \in [1, q)$ if $t = 0$.

Again, we have, $L' = \mathcal{O}e_1 \oplus \mathcal{O}\pi e_2 = \{v \in K^2 \mid \max\{|a_1|_K, q|a_2|_K\} \leq 1\}$.
Also, $\forall\ v = a_1e_1 + a_2e_2 \in K^2$, $\exists\ r, s \in \mathbb{Z}$ such that $|a_1|_K = q^r$ and $q|a_2|_K = q^s$.
$\Rightarrow \{v \in K^2 \mid \max\{|a_1|_K, q|a_2|_K\} \leq 1\} = \{v \in K^2 \mid \max\{|a_1|_K, q|a_2|_K\} \leq \alpha\}\ \forall\ \alpha \in [1, q)$
$\Rightarrow L' = \{v \in K^2 \mid \|v\|_{L'} \leq \alpha\}\ \forall\ \alpha \in [1, q)$
$\Rightarrow L_\alpha = L'\ \ \forall\ \alpha \in [1, q)$ if $t = 1$.

So, the claim holds for $t = 0$ and $t = 1$. Now suppose $t \in (0, 1)$.
Let $v = a_1e_1 + a_2e_2 \in K^2$. $\exists\ r, s \in \mathbb{Z}$ such that $|a_1|_K = q^r$ and $|a_2|_K = q^s$.
As $t \in (0, 1)$, hence $r > s \Leftrightarrow r > s + t$ and $r \leq s \Leftrightarrow r < s + t$.

Case 1. $(\alpha \in [1, q^t))$

$\|v\|_{(L,L',t)} < q^t$

$\Leftrightarrow \max\{q^r, q^{s+t}\} < q^t$

$\Leftrightarrow \begin{cases} q^r < q^t & \text{if } r > s \\ q^{s+t} < q^t & \text{if } r \leq s \end{cases}$

$\Leftrightarrow \begin{cases} r \leq 0 & \text{if } r > s \\ s + 1 \leq 0 & \text{if } r \leq s \end{cases}$

$\Leftrightarrow \|v\|_{L'} \leq 1$

$\Leftrightarrow v \in L' = \{v \in K^2 \mid \max\{|a_1|_K, q|a_2|_K\} \leq \alpha\}\ \forall\ \alpha \in [1, q^t)$

$$\subseteq \{v \in K^2 \mid \max\{|a_1|_K, q^t|a_2|_K\} \leq \alpha\} \ \forall\, \alpha \in [1, q^t)$$
$$\Rightarrow L' = \{v \in K^2 \mid \ \|v\|_{(L,L',t)} \leq \alpha\} \ \ \forall\, \alpha \in [1, q^t)$$
$$\Rightarrow L_\alpha = L' \ \ \forall\, \alpha \in [1, q^t)\ .$$

Case 2. $(\alpha \in [q^t, q))$

$$\|v\|_{(L,L',t)} < q$$
$$\Leftrightarrow \ \max\{q^r, q^{s+t}\} < q$$
$$\Leftrightarrow \begin{cases} q^r < q & \text{if} \ \ r > s \\ q^{s+t} < q & \text{if} \ \ r \leq s \end{cases}$$
$$\Leftrightarrow \begin{cases} r \leq 0 & \text{if} \ \ r > s \\ s \leq 0 & \text{if} \ \ r \leq s \end{cases}$$
$$\Leftrightarrow \ \|v\|_L \leq 1$$

$$\begin{aligned} \Leftrightarrow \ v \in L &= \{v \in K^2 \mid \max\{|a_1|_K, |a_2|_K\} \leq 1\} \\ &= \{v \in K^2 \mid \max\{|a_1|_K, q^t|a_2|_K\} \leq 1\} \\ &\subseteq \{v \in K^2 \mid \max\{|a_1|_K, q^t|a_2|_K\} \leq \alpha\} \ \forall\, \alpha \geq 1 \end{aligned}$$
$$\Rightarrow \ v \in L \subseteq \{v \in K^2 \mid \max\{|a_1|_K, q^t|a_2|_K\} \leq \alpha\} \ \forall\, \alpha \in [q^t, q)$$
$$\Rightarrow L = \{v \in K^2 \mid \ \|v\|_{(L,L',t)} \leq \alpha\} \ \ \forall\, \alpha \in [q^t, q)$$
$$\Rightarrow L_\alpha = L \ \ \forall\, \alpha \in [q^t, q)\ .$$

□

Lemma 10. *Given any norm, $\|\ \|$ on K^2, let $(a, b) \in [1, q) \times [1, q)$ be the unique pair, with $a \leq b$, such that $\|K^2\| = a|K|_K \cup b|K|_K$. Then*

$$\{L_\alpha \mid \alpha \in [1, q)\} = \begin{cases} \{L_a,\ L_b\} & \textit{if} \ \ a = 1\ . \\ \{\pi L_b,\ L_a,\ L_b\} & \textit{if} \ \ a > 1\ . \end{cases}$$

Proof. $\|K^2\| = a|K|_K \cup b|K|_K$, where $a, b \in [1, q)$
$$\Rightarrow \ \|K^2\| \cap [1, q) = \{a, b\}$$
$$\Rightarrow \ \|K^2\| \cap [\tfrac{1}{q}, q) = (\|K^2\| \cap [\tfrac{1}{q}, 1)) \cup (\|K^2\| \cap [1, q)) = \{\tfrac{a}{q},\, \tfrac{b}{q},\, a,\, b\}$$

$$\Rightarrow \left\{ \begin{array}{l} L_{\frac{b}{q}} \subsetneq L_a \subsetneq L_b \\ \\ \quad \text{and} \\ \\ L_\alpha = \begin{cases} L_{\frac{b}{q}} & \text{if} \ \ \alpha \in [1, a) \\ L_a & \text{if} \ \ \alpha \in [a, b) \\ L_b & \text{if} \ \ \alpha \in [b, q) \end{cases} \end{array} \right.$$

$$\begin{aligned}
\text{Again, } L_{\frac{b}{q}} &= \{v \in K^2 \mid \|v\| \leq \frac{b}{q}\} \\
&= \{v \in K^2 \mid \|\pi^{-1} v\| \leq b\} \\
&= \pi\{v \in K^2 \mid \|v\| \leq b\} \\
&= \pi L_b
\end{aligned}$$

$$\Rightarrow \ \{L_\alpha \mid \alpha \in [1,q)\} = \begin{cases} \{L_a,\ L_b\} & \text{if} \quad a = 1\ . \\ \{\pi L_b,\ L_a,\ L_b\} & \text{if} \quad a > 1\ . \end{cases}$$ □

Definition 13. *For each $y \in Y$, we fix some $\| \ \|_y$ such that $y = [\| \ \|_y]$. By lemma 5, for $\| \ \|_y$, $\exists$ a unique $(a,b) \in [1,q) \times [1,q)$ such that $a \leq b$ and $\|K^2\| = a|K|_K \cup b|K|_K$. And by lemma 10, we have,*

$$\{[L_\alpha] \mid \alpha \in [1,q)\} = \{[L_a], [L_b]\}\ .$$

Let $t := \log_q(\frac{b}{a})$. The map $\psi : Y \to X_{\mathbb{R}}$ is defined as

$$\psi(y) := (1-t)[L_b] + t[L_a] \qquad \forall\ y \in Y$$

Lemma 11. *The map $\psi : Y \to X_{\mathbb{R}}$ is well-defined.*

Proof. Suppose, given $y \in Y$, we have fixed some $\| \ \|$ such that $y = [\| \ \|]$ and let $(a,b) \in [1,q) \times [1,q)$ be the unique pair such that $a \leq b$ and $\|K^2\| = a|K|_K \cup b|K|_K$.

Let $\| \ \|' \sim \| \ \|$ and let $(a',b') \in [1,q) \times [1,q)$ be the unique pair such that $a' \leq b'$ and $\|K^2\|' = a'|K|_K \cup b'|K|_K$.
$\Rightarrow \| \ \|' = c\| \ \|$ for some $c \in \mathbb{R}_{>0}$
$\Rightarrow a'|K|_K \cup b'|K|_K = \|K^2\|' = ca|K|_K \cup cb|K|_K$
$\Rightarrow \exists\ m,n \in \mathbb{Z}$ such that $\{a',b'\} = \{\frac{ca}{q^m}, \frac{cb}{q^n}\}$

Now, $\forall\ \alpha, \beta \in \mathbb{R}_{>0}$,
if $L_\alpha := \{v \in K^2 \mid \|v\| \leq \alpha\}$ and $L'_\beta := \{v \in K^2 \mid \|v\|' \leq \beta\}$,

$$\begin{aligned}
\text{then, } L'_{\frac{cb}{q^n}} &= \{v \in K^2 \mid \|v\|' \leq \frac{cb}{q^n}\} \\
&= \{v \in K^2 \mid \|v\| \leq \frac{b}{q^n}\} \\
&= \{v \in K^2 \mid \|\pi^{-n} v\| \leq b\} \\
&= \pi^n\{v \in K^2 \mid \|v\| \leq b\} \\
&= \pi^n L_b
\end{aligned}$$

$$\begin{aligned}
\text{and} \quad L'_{\frac{ca}{q^m}} &= \{v \in K^2 \mid \|v\|' \leq \frac{ca}{q^m}\} \\
&= \{v \in K^2 \mid \|v\| \leq \frac{a}{q^m}\} \\
&= \{v \in K^2 \mid \|\pi^{-m}v\| \leq a\} \\
&= \pi^m\{v \in K^2 \mid \|v\| \leq a\} \\
&= \pi^m L_a
\end{aligned}$$

$\Rightarrow L'_{\frac{cb}{q^n}} \sim L_b$ and $L'_{\frac{ca}{q^m}} \sim L_a$.

We have, $a' = \min\{\frac{ca}{q^m}, \frac{cb}{q^n}\}$, $b' = \max\{\frac{ca}{q^m}, \frac{cb}{q^n}\}$
and as $a, b, a', b' \in [1, q)$, hence $\frac{1}{q} < \frac{c}{q^m}, \frac{c}{q^n} < q$
$\Rightarrow q^{m-1} < c < q^{m+1}$ and $q^{n-1} < c < q^{n+1}$

Hence, there are 3 possible cases, viz., $m = n-1$, $m = n$ or $m = n+1$.

Case 1. $(m = n-1)$

$$\Rightarrow \quad (a', b') = \begin{cases} (\frac{ca}{q^{n-1}}, \frac{cb}{q^n}) \\ \quad or \\ (\frac{cb}{q^n}, \frac{ca}{q^{n-1}}) \end{cases}$$

$$\Rightarrow \quad \frac{b'}{a'} = \begin{cases} \frac{b}{qa} \\ or \\ \frac{qa}{b} \end{cases}$$

Case 2. $(m = n)$

$\Rightarrow (a', b') = (\frac{ca}{q^n}, \frac{cb}{q^n})$
$\Rightarrow \frac{b'}{a'} = \frac{b}{a}$

Case 3. $(m = n+1)$

$\Rightarrow (a', b') = (\frac{ca}{q^{n+1}}, \frac{cb}{q^n})$
$\Rightarrow \frac{b'}{a'} = \frac{qb}{a}$

Again, $a, b, a', b' \in [1, q)$
$\Rightarrow \frac{b}{a}, \frac{b'}{a'} \in [1, q)$
$\Rightarrow \frac{b}{qa}, \frac{qb}{a} \notin [1, q)$
$\Rightarrow \frac{b'}{a'} \neq \frac{b}{qa}, \frac{qb}{a}$

Hence, only the following 2 cases can occur:

$$(a', b') = \begin{cases} (\frac{ca}{q^n}, \frac{cb}{q^n}) \\ or \\ (\frac{cb}{q^n}, \frac{ca}{q^{n-1}}) \end{cases}$$

Let $t' := \log_q(\frac{b'}{a'})$.

Now if $(a', b') = (\frac{ca}{q^n}, \frac{cb}{q^n})$
$\Rightarrow L_a \sim L'_{\frac{ca}{q^n}} = L'_{a'}, \;\; L_b \sim L'_{\frac{cb}{q^n}} = L'_{b'}$ and $t = t'$.

And if $(a', b') = (\frac{cb}{q^n}, \frac{ca}{q^{n-1}})$
$\Rightarrow L_a \sim L'_{\frac{ca}{q^{n-1}}} = L'_{b'}, \;\; L_b \sim L'_{\frac{cb}{q^n}} = L'_{a'}$ and $t = 1 - t'$.

Therefore, in both cases, we have,

$$(1-t)[L_b] + t[L_a] = (1-t')[L'_{b'}] + t'[L'_{a'}]$$

$\square$

Proposition 2. $\psi \circ \phi = Id_{X_{\mathbb{R}}}$ *and* $\phi \circ \psi = Id_Y$.

Proof. Let $x := (1-t)s + ts' \in X_{\mathbb{R}}$. Let $L, L' \subseteq K^2$ be two lattices such that $s = [L], s' = [L']$ and $\pi L \subsetneq L' \subsetneq L$.
$\Rightarrow \phi(x) = [\| \; \|_{(L,\, L',\, t)}]$

Let $(a', b') \in [1, q) \times [1, q)$ be the unique pair such that $a' \leq b'$
and $\|K^2\|_{(L,\, L',\, t)} = a'|K|_K \cup b'|K|_K$. (cf. lemma 5)
$\Rightarrow (a', b') = (1, q^t), \; L_{a'} = L'$ and $L_{b'} = L$ (cf. lemma 9)
$\Rightarrow \psi \circ \phi(x) = (1-t)[L] + t[L'] = (1-t)s + ts' = x$

Conversely, let $y \in Y$ and let $\| \; \|$ be a norm on K^2 such that $y = [\| \; \|]$.
Let $(a, b) \in [1, q) \times [1, q)$ be the unique pair such that $a \leq b$
and $\|K^2\| = a|K|_K \cup b|K|_K$ (cf. lemma 5)
$\Rightarrow \psi(y) = (1-t)[L_b] + t[L_a]$, where $t := \log_q(\frac{b}{a})$
$\Rightarrow \phi \circ \psi(y) = [\| \; \|_{(L_b, L_a, t)}]$

Now, $\forall \; v \in K^2$, the following 2 cases are possible:

Case 1. $(\|v\| \in a|K|_k)$

Let $\|v\| = q^m a$ for some $m \in \mathbb{Z}$.

$$\Rightarrow \begin{cases} \|v\|_{L_a} = \max_{n\in\mathbb{Z}} \{q^n \mid \pi^n v \in L_a\} = q^m \\ \\ \|v\|_{L_b} = \max_{n\in\mathbb{Z}} \{q^n \mid \pi^n v \in L_b\} = q^m \end{cases}$$

$$\Rightarrow \quad \|v\|_{(L_b,L_a,t)} = q^m.$$

Case 2. $(\|v\| \in b|K|_k)$

Let $\|v\| = q^m b$ for some $m \in \mathbb{Z}$.

$$\Rightarrow \begin{cases} \|v\|_{L_a} = \max_{n\in\mathbb{Z}} \{q^n \mid \pi^n v \in L_a\} = q^{m-1} \\ \\ \|v\|_{L_b} = \max_{n\in\mathbb{Z}} \{q^n \mid \pi^n v \in L_b\} = q^m \end{cases}$$

$$\Rightarrow \quad \|v\|_{(L_b,L_a,t)} = q^t\|v\|_{L_b} = \tfrac{b}{a}q^m.$$

Hence, $\| \ \| = a\| \ \|_{(L_b,L_a,t)}$
$\Rightarrow \phi \circ \psi(y) = [\| \ \|_{(L_b,L_a,t)}] = [\| \ \|] = y.$ □

Corollary 2. *$X_{\mathbb{R}}$ and Y are naturally isomorphic as sets.* □

Corollary 3. *Given any norm $\| \ \|$ on K^2, there exists*
i) a unique $t \in [0, \frac{1}{2}]$
ii) a basis (e_1, e_2) of K^2
iii) $s \in [1, q)$
such that $\|a_1e_1 + a_2e_2\| = \sup \{s|a_1|, sq^t|a_2|\} \ \ \forall\, a_1, a_2 \in K^2$.

Remark 7. *A norm on K^2 is called irrational, rational or integral according as, respectiviely, the unique $t \in [0, \frac{1}{2}]$ determined by it, is irrational, rational or 0.*

1.3 A Metric on the Tree

Definition 14. *For each $y \in Y$, we fix some $\| \ \|_y$ such that $y = [\| \ \|_y]$. The map $\mu : Y \times Y \to \mathbb{R}_{\geq 0}$ is defined as*

$$\mu(y_1, y_2) = \log_q \left(\sup_{v \in K^2 \setminus \{(0,0)\}} \frac{\|v\|_{y_1}}{\|v\|_{y_2}} \right) + \log_q \left(\sup_{v \in K^2 \setminus \{(0,0)\}} \frac{\|v\|_{y_2}}{\|v\|_{y_1}} \right) \quad (1.1)$$

$$= \log_q \left(\sup_{v \in K^2 \setminus \{(0,0)\}} \frac{\|v\|_{y_1}}{\|v\|_{y_2}} \right) - \log_q \left(\inf_{v \in K^2 \setminus \{(0,0)\}} \frac{\|v\|_{y_1}}{\|v\|_{y_2}} \right) \quad (1.2)$$

Lemma 12. *the map μ is well-defined.*

Proof. Follows directly from (1.2). □

Lemma 13. *μ is a metric on Y.*

Proof. Let $y_1, y_2, y_3 \in Y$. We have,

i) $\mu(y_1, y_2) \geq 0$ where, equality holds iff $y_1 = y_2$. (cf. (1.2))
ii) $\mu(y_1, y_2) = \mu(y_2, y_1)$. (cf. (1.1))

Again,

$$\sup_{v \in K^2 \setminus \{(0,0)\}} \frac{\|v\|_{y_1}}{\|v\|_{y_3}} \leq \left(\sup_{v \in K^2 \setminus \{(0,0)\}} \frac{\|v\|_{y_1}}{\|v\|_{y_2}} \right) \left(\sup_{v \in K^2 \setminus \{(0,0)\}} \frac{\|v\|_{y_2}}{\|v\|_{y_3}} \right)$$

and

$$\inf_{v \in K^2 \setminus \{(0,0)\}} \frac{\|v\|_{y_1}}{\|v\|_{y_3}} \geq \left(\inf_{v \in K^2 \setminus \{(0,0)\}} \frac{\|v\|_{y_1}}{\|v\|_{y_2}} \right) \left(\inf_{v \in K^2 \setminus \{(0,0)\}} \frac{\|v\|_{y_2}}{\|v\|_{y_3}} \right)$$

Hence,
iii)$\mu(y_1, y_3) \leq \mu(y_1, y_2) + \mu(y_2, y_3)$. (cf. (1.2))
□

Lemma 14. *Let $\| \ \|_1$ and $\| \ \|_2$ be 2 norms on K^2 such that $c_1\|v\|_1 \leq \|v\|_2 \leq c_2\|v\|_1$ $\forall\ v \in K^2$, where $c_1, c_2 \in \mathbb{R}_{\geq 0}$. Then*

$$\log_q\left(\frac{c_1}{c_2}\right) \leq \mu([\| \ \|_1], [\| \ \|_2]) \leq \log_q\left(\frac{c_2}{c_1}\right) .$$

Proof. $c_1\|v\|_1 \leq \|v\|_2 \leq c_2\|v\|_1 \quad \forall\ v \in K^2$

$\Rightarrow \frac{1}{c_2} \leq \frac{\|v\|_1}{\|v\|_2} \leq \frac{1}{c_1}$ and $c_1 \leq \frac{\|v\|_2}{\|v\|_1} \leq c_2 \quad \forall\ v \in K^2$

$\Rightarrow \log_q(\frac{c_1}{c_2}) \leq \mu([\| \ \|_1], [\| \ \|_2]) \leq \log_q(\frac{c_2}{c_1})$. (cf. (1.1))
□

Definition 15. *The metric d on $X_{\mathbb{R}}$ is defined as*

$$d(x_1, x_2) = \mu(\phi(x_1), \phi(x_2)) \ \ \forall \ x_1, x_2 \in X_{\mathbb{R}} \ .$$

Lemma 15. *For $s \in V(X)$ and $y \in Y$, let $L \subseteq K^2$ be a lattice and $\| \ \|$ be a norm on K^2 such that $s = [L]$ and $y = [\| \ \|]$. Then*

$$d(s, \psi(y)) = \log_q \big(\sup_{v \in L \setminus \pi L} \|v\|\big) - \log_q \big(\inf_{v \in L \setminus \pi L} \|v\|\big) \ .$$

Proof. Since, $K^2 \setminus \{(0,0)\} = \sqcup_{n \in \mathbb{Z}} \pi^n (L \setminus \pi L)$ and $\|v\|_L = 1 \ \ \forall \ v \in L \setminus \pi L$, hence, the lemma follows directly from (1.2). □

Proposition 3. $\forall \ s, s' \in V(X), \quad d(s, s') = \# \ (\sigma(s, s') \cap V(X)) - 1.$

To prove this proposition, we need the following lemmas.

Lemma 16. *Let $k_1, l_1 \in \mathbb{R}$ be such that $k_1 \geq l_1$. Then*

$$\sup_{b,c \in \mathbb{R}_{>0}} \left(\frac{\max \ \{a^{k_1} b, \ a^{l_1} c\}}{\max \ \{b, \ c\}} \right) = a^{k_1} \quad \forall \ a \in \mathbb{R}_{>0}.$$

Proof. For $b \geq c, \ \max \ \{a^{k_1} b, \ a^{l_1} c\} = a^{k_1} b$

$\Rightarrow \frac{\max \ \{a^{k_1} b, \ a^{l_1} c\}}{\max \ \{b, \ c\}} = a^{k_1}.$

And for $b < c$,

$$\frac{\max \ \{a^{k_1} b, \ a^{l_1} c\}}{\max \ \{b, \ c\}} = \begin{cases} \frac{a^{k_1} b}{c} < a^{k_1} & \text{(as } b < c) \\ \quad or & \\ a^{l_1} \leq a^{k_1} & \end{cases}$$

□

Lemma 17. *Let $k_1, l_1 \in \mathbb{R}$ be such that $k_1 \geq l_1$. Then*

$$\inf_{b,c \in \mathbb{R}_{>0}} \left(\frac{\max \ \{a^{k_1} b, \ a^{l_1} c\}}{\max \ \{b, \ c\}} \right) = a^{l_1} \quad \forall \ a \in \mathbb{R}_{>0}.$$

Proof. For $b \geq c, \ \max \ \{a^{k_1} b, \ a^{l_1} c\} = a^{k_1} b$

$\Rightarrow \frac{\max \ \{a^{k_1} b, \ a^{l_1} c\}}{\max \ \{b, \ c\}} = a^{k_1} \geq a^{l_1}.$

And for $b < c$,

$$\frac{\max \ \{a^{k_1} b, \ a^{l_1} c\}}{\max \ \{b, \ c\}} = \begin{cases} \frac{a^{k_1} b}{c} > a^{l_1} & \text{if } a^{k_1} b > a^{l_1} c \\ a^{l_1} & \text{otherwise.} \end{cases}$$

□

Proof of the proposition. $\forall\ s, s' \in V(X),\ \exists$ a unique path joining them in X, which can be represented as a chain of lattices :

$$\begin{array}{ll} L_0 \subsetneq L_1 \text{ (here, } n := 1) & \text{if } \{s, s'\} \in E(X) \\ L_0 \subsetneq L_1 \subsetneq \ldots \subsetneq L_n \text{ for some } n \in \mathbb{Z}_{\geq 2}\ , & \text{otherwise.} \end{array}$$

such that,

i) $\pi L_{i+1} \subsetneq L_i \subsetneq L_{i+1}\ \forall\ i \in \mathbb{Z}_{[0,n-1]}$.

ii) $s = [L_0]$ and $s' = [L_n]$.

Now, by Elementary divisors theorem, $\exists$ a basis (e_1, e_2) of L_n such that $(\pi^k e_1, \pi^l e_2)$ is a basis of L_0, where $l \geq k$.

If $l = k$, we have, $L_0 = \pi^k L_n$, hence $s = s'$ and we are done.

Now, we assume $l > k$.

$\Rightarrow L_0 = \mathcal{O}\pi^k e_1 \oplus \mathcal{O}\pi^l e_2 \subsetneq \pi^k(\mathcal{O}e_1 \oplus \mathcal{O}e_2) = \pi^k L_n$

Let $L'_m := \mathcal{O}\pi^k e_1 \oplus \mathcal{O}\pi^{l-m} e_1\ \ \forall\ m \in \mathbb{Z}_{[0,l-k]}$.

$\Rightarrow L_0 = L'_0$ and $\pi^k L_n = L'_{l-k}$.

So, the unique path, joining s snd s' in X, can be represented as a chain of lattices:

$$\begin{array}{ll} L'_0 \subsetneq L'_{l-k} & \text{if } l - k = 1 \\ L'_0 \subsetneq L'_1 \subsetneq \ldots \subsetneq L'_{l-k} & \text{otherwise.} \end{array}$$

where, we have,

i) $\pi L'_{i+1} \subsetneq L'_i \subsetneq L'_{i+1}\ \forall\ i \in \mathbb{Z}_{[0,\ l-k-1]}$.

ii) $s = [L'_0]$ and $s' = [L'_{l-k}]$.

$\Rightarrow n = \#\ (\sigma(s, s') \cap V(X)) - 1 = l - k$

Hence,

$$\begin{aligned} d(s, s') &= \mu(\phi(s), \phi(s')) \\ &= \mu([\|\ \|_{L_0}], [\|\ \|_{L_n}]) \\ &= \log_q\left(\sup_{v\in K^2\backslash\{(0,0)\}} \frac{\|v\|_{L_0}}{\|v\|_{L_n}}\right) - \log_q\left(\inf_{v\in K^2\backslash\{(0,0)\}} \frac{\|v\|_{L_0}}{\|v\|_{L_n}}\right) \\ &= \log_q\left(\sup_{(a_1,a_2)\in K^2\backslash\{(0,0)\}} \frac{\max\ \{q^{-k}|a_1|, q^{-l}|a_2|\}}{\max\ \{|a_1|, |a_2|\}}\right) - \\ &\qquad \log_q\left(\inf_{(a_1,a_2)\in K^2\backslash\{(0,0)\}} \frac{\max\ \{q^{-k}|a_1|, q^{-l}|a_2|\}}{\max\ \{|a_1|, |a_2|\}}\right) \\ &= l - k \qquad \text{(cf. lemma 16 and 17, putting } k_1 = -k,\ \ l_1 = -l \text{ and } a = q) \\ &= \#\ (\sigma(s, s') \cap V(X)) - 1. \end{aligned}$$

□

Corollary 4. *Let, $L_0 \subsetneq L_1 \subsetneq \ldots \subsetneq L_n$ for some $n \in \mathbb{Z}_{\geq 2}$ be a chain of lattices in K^2 such that $\pi L_{i+1} \subsetneq L_i \subsetneq L_{i+1}$ $\forall\ i \in \mathbb{Z}_{[0,n-1]}$. Then $\exists$ a basis (e_1, e_2) of L_n such that*

$$L_i \ \sim\ \mathcal{O}e_1 \oplus \mathcal{O}\pi^{n-i}e_2 \quad \forall\ i \in \mathbb{Z}_{[0,n]}\ .$$

Corollary 5. $\{s, s'\} \in E(X) \ \Leftrightarrow\ d(s, s') = 1.$

Remark 8. *As $\mathcal{O}$ is a d.v.r., hence, by Elementary divisors theorem, for any 2 lattices $L' \subseteq L$ in K^2, (L/L') is finite and has at most 2 cyclic components.*

Corollary 6. *Let $L' \subseteq L$ be 2 lattices in K^2 and let $k, l \in \mathbb{Z}$ be such that $L/L' = (\mathcal{O}/\pi^k\mathcal{O}) \oplus (\mathcal{O}/\pi^l\mathcal{O})$. Then*

$$d([L], [L']) = |l - k|_{\mathbb{R}},$$

where, $|\ |_{\mathbb{R}}$ denotes the usual archimedean absolute value on $\mathbb{R}$.

Remark 9. *As any 2 vertices of X are joined by a path, hence, $\forall\ \{s, s'\} \subseteq V(X)$, $\exists$ lattices $L' \subsetneq L$ in K^2, such that $s = [L]$ and $s' = [L']$. Also note that, if necessary, after replacing L' by $\pi^{-m}L'$ for some $m \in \mathbb{Z}_{\geq 1}$, we can assume that $L' \nsubseteq \pi L$.*

Lemma 18. *If $L, L' \subseteq K^2$ be lattices, such that $L' \subsetneq L$ and $L' \nsubseteq \pi L$, then $\exists$ a unique n such that $\pi^n L \subsetneq L'$ and $\pi^{n-1}L \nsubseteq L'$, where,*

$$n = \text{ length of } (L/L') = \#\ (\sigma([L], [L']) \cap V(X)) - 1 = \text{ length of } (L'/\pi^n L).$$

Proof. Let $n :=$ length of (L/L') and let

$$L' = L_0 \subsetneq L_1 = L \qquad \text{for } n = 1$$
$$L' = L_0 \subsetneq L_1 \subsetneq \ldots \subsetneq L_n = L \qquad \text{for } n \geq 2$$

be the lifting of a composition series of (L/L') (1.3)
$\Rightarrow L_{i+1}/L_i$ are simple $\mathcal{O}$-modules
$\Rightarrow L_{i+1}/L_i \cong \mathcal{O}/\pi\mathcal{O}$ $\forall\ i \in \mathbb{Z}_{[0,n-1]}$
$\Rightarrow \pi L_{i+1} \subsetneq L_i \subsetneq L_{i+1}$ $\forall\ i \in \mathbb{Z}_{[0,n-1]}$ (1.4)
$\Rightarrow L_i/\pi L_{i+1} \cong \mathcal{O}/\pi\mathcal{O}$ $\forall\ i \in \mathbb{Z}_{[0,n-1]}$
$\Rightarrow \pi^i L_i/\pi^{i+1}L_{i+1} \cong L_i/\pi L_{i+1}$ are simple $\mathcal{O}$-modules $\forall\ i \in \mathbb{Z}_{[0,n-1]}$
$\Rightarrow$ the following is the lifting of a composition series of $(L'/\pi^n L)$ (1.5)

$$\pi L = \pi L_1 \subsetneq L_0 = L' \qquad \text{for } n = 1$$
$$\pi^n L = \pi^n L_n \subsetneq \pi^{n-1}L_{n-1} \subsetneq \ldots \subsetneq L_0 = L' \qquad \text{for } n \geq 2.$$

Now, by (1.4) we have, both of the chains of lattices (1.3) and (1.5) represent the unique path joining $[L']$ and $[L]$ in X.

$$\begin{aligned} \Rightarrow \#\ (\sigma([L],[L']) \cap V(X)) - 1 &= n \\ &= \text{length of } (L/L') \\ &= \text{length of } (L'/\pi^n L). \end{aligned} \tag{1.6}$$

Again, if possible, suppose, $\pi^{n-1}L \subseteq L'$.
$\Rightarrow n \geq 2$.
$\Rightarrow \pi^{n-1}L \subsetneq L'$ (cf. (1.4))
Now, let

$$\begin{aligned} \pi L = M_0 \subsetneq M_1 = L' &\qquad \text{for } n = 2 \\ \pi^{n-1}L = M_0 \subsetneq M_1 \subsetneq \ldots \subsetneq M_m = L' &\qquad \text{for } n \geq 3 \end{aligned}$$

be the lifting of a composition series of $(L'/\pi^{n-1}L)$. (1.7)

Then the chain of lattices (1.7) represents the unique path joining $[L']$ and $[L]$ in X.
$\Rightarrow \#\ (\sigma([L],[L']) \cap V(X)) - 1 \ = \ \text{length of } (L'/\pi^{n-1}L) = m$.
$\Rightarrow m = n$. (cf. (1.6))

But, considering the chain $\pi^n L \subsetneq \pi^{n-1}L = M_0 \subsetneq M_1 \subsetneq \ldots \subsetneq M_m = L'$ modulo $\pi^n L$, by (1.5) and proposition 6.7 of [2], we have, $m \leq n-1$.
Thus, we get a contradiction. □

Corollary 7. *Let $L \subseteq K^2$ be a lattice, $s \in V(X)$ and $d([L],s) = n$ for some $n \in \mathbb{Z}_{\geq 0}$. Then $\exists$ a unique lattice $L' \subseteq K^2$ with $[L'] = s$, which satisfies the following conditions:*
i) $L' \subseteq L$ and L' is maximal w.r.t. this property.
ii) $L' \subseteq L$ and $L' \nsubseteq \pi L$.
iii) $L' \subseteq L$ and L/L' is cyclic.
iv) $L' \subseteq L$ and $L/L' \cong \mathcal{O}/\pi^n\mathcal{O}$.
v) $L' \subseteq L$ and $L'/\pi^n L \cong \mathcal{O}/\pi^n\mathcal{O}$.

Corollary 8. *Let $L \subseteq K^2$ be a lattice and let $n \in \mathbb{Z}_{\geq 0}$. The elements of the set, $\{s \in V(X) \mid d([L],s) = n\}$ correspond bijectively to the direct factors of $L/\pi^n L$ of rank 1, i.e., to the points on the projective line $\mathbb{P}(L/\pi^n L) \cong \mathbb{P}^1(\mathcal{O}/\pi^n\mathcal{O})$. (cf. definition 38)*

Proposition 4. *For $s, s' \in V(X)$, if $\exists\ L, L' \subseteq K^2$ and $n \in \mathbb{Z}_{\geq 1}$, with $s = [L]$, $s' = [L']$, $\pi^n L \subsetneq L' \subsetneq L$ and $\pi^{n-1}L \nsubseteq L' \nsubseteq \pi L$, then*

$$d(s,s') = n.$$

Proof. $\pi^n L \subsetneq L' \subsetneq L$
$\Rightarrow \|v\|_L \leq \|v\|_{L'} \leq q^n \|v\|_L \;\; \forall\, v \in K^2$
$\Rightarrow \mu([\| \ \|_L], [\| \ \|_{L'}]) \;\leq n$ (cf. lemma 14)

Now, $\pi^n L \subsetneq L' \subsetneq L$ and $\pi^{n-1} L \nsubseteq L' \nsubseteq \pi L$
$\Rightarrow \exists$ a basis (e_1, e_2) of L such that $(e_1, \pi^n e_2)$ is a basis of L'.
(cf. Elementary divisors theorem)
$\Rightarrow \|e_1\|_{L'} = \|e_1\|_L = 1$
$\Rightarrow -\log_q \left(\inf_{v \in L \setminus \pi L} \|v\|_{L'} \right) \geq 0$

Again, $\|e_1\|_L = 1$ and $\|e_2\|_{L'} = q^n$.
$\Rightarrow \log_q \left(\sup_{v \in L \setminus \pi L} \|v\|_{L'} \right) \geq n$
$\Rightarrow d(s, s') = n.$ (cf. lemma 15)

□

Remark 10. *Note that, by remark* 9 *and lemma* 18*, we have the equivalence of proposition* 3 *and proposition* 4*.*

Lemma 19. *Let* $x = (1-t)s + ts'$ *and* $x' = (1-t')s + t's' \;\in X_{\mathbb{R}}$*, where* $t' \geq t$*. Then* $d(x', x) = t' - t$*.*

Proof. Let $L, L' \subseteq K^2$ be lattices such that $s = [L]$, $s' = [L']$ and $\pi L \subsetneq L' \subsetneq L$. By Elementary divisors theorem, $\exists$ a basis (e_1, e_2) of L such that $(e_1, \pi e_2)$ is a basis of L'.

$\Rightarrow \forall\; v = a_1 e_1 + a_2 e_2 \in K^2$,

$$\|v\|_{(L,L',t')} = \max\,\{|a_1|_K, q^{t'}|a_2|_K\} \;\text{ and }\; \|v\|_{(L,L',t)} = \max\,\{|a_1|_K, q^{t}|a_2|_K\}$$

Hence, the lemma follows directly from lemma 16, lemma 17 and (1.2). □

Corollary 9. *Let* $x = (1-t)s + ts' \in X_{\mathbb{R}}$*. Then* $d(s, x) = t$ *and* $d(s', x) = 1 - t$*.*

Corollary 10. *Let* $x = (1-t)s_1 + ts_2 \in X_{\mathbb{R}}$*. Then*

$$d(x, s_i) = |i - 1 - t|_{\mathbb{R}} \;\; \forall\; i \in \{1, 2\},$$

where, $|\ |_{\mathbb{R}}$ *denotes the usual archimedean absolute value on* $\mathbb{R}$*.*

Proposition 5. *For* $x_1, x_2 \in X_{\mathbb{R}}$, $\exists\ s_1, s_2, s'_1, s'_2 \in V(X)$ *and* $t_1, t_2 \in [0, \frac{1}{2}]$ *such that* $x_1 = (1-t_1)s_1 + t_1 s_2$ *and* $x_2 = (1-t_2)s'_1 + t_2 s'_2$.

i) If $\sigma(x_1, x_2) \cap V(X) = \emptyset$,

$$d(x_1, x_2) = \begin{cases} |d(x_1, s_i) - d(x_2, s_i)|_{\mathbb{R}} & \forall\ i \in \{1,2\} \quad \textit{if}\ \ s_1 = s'_1 \\ |d(x_1, s_i) - d(x_2, s_{3-i})|_{\mathbb{R}} & \forall\ i \in \{1,2\} \quad \textit{otherwise.} \end{cases}$$

In other words,

$$d(x_1, x_2) = \begin{cases} \Big| |i-1-t_1|_{\mathbb{R}} - |i-1-t_2|_{\mathbb{R}} \Big|_{\mathbb{R}} & \forall\ i \in \{1,2\} \quad \textit{if}\ \ s_1 = s'_1 \\ \Big| |i-1-t_1|_{\mathbb{R}} - |2-i-t_2|_{\mathbb{R}} \Big|_{\mathbb{R}} & \forall\ i \in \{1,2\} \quad \textit{otherwise} \end{cases}$$

ii) If $\sigma(x_1, x_2) \cap V(X) \neq \emptyset$,
let $\{s_i\} = \sigma(x_1, x_2) \cap \{s_1, s_2\}$ *and* $\{s'_j\} = \sigma(x_1, x_2) \cap \{s'_1, s'_2\}$, *for some* $i, j \in \{1,2\}$. *Then*

$$\begin{aligned} d(x_1, x_2) &= d(x_1, s_i) + d(s_i, s'_j) + d(s'_j, x_2) && (1.8) \\ &= \#\ (\sigma(x_1, x_2) \cap V(X)) + |i-1-t_1|_{\mathbb{R}} + |j-1-t_2|_{\mathbb{R}} - 1 && (1.9) \end{aligned}$$

where, $|\ |_{\mathbb{R}}$ *denotes the usual archimedean absolute value on* $\mathbb{R}$.

Proof. $i)$ Follows directly from lemma 19 and corollary 10.

$ii)$ $\{s_i\} = \sigma(x_1, x_2) \cap \{s_1, s_2\}$ and $\{s'_j\} = \sigma(x_1, x_2) \cap \{s'_1, s'_2\}$,
$\Rightarrow \sigma(s_i, s'_j) \subseteq \sigma(x_1, x_2)$
$\Rightarrow s_{3-i}, s'_{3-j} \notin \sigma(s_i, s'_j)$
$\Rightarrow d(s'_j, s_{3-i}) - 1 = d(s'_j, s_i) = d(s'_{3-j}, s_i) - 1$ (cf. proposition 3)
and $d(s'_j, s_{3-i}) + 1 = d(s'_{3-j}, s_{3-i}) = d(s'_{3-j}, s_i) + 1$ (cf. proposition 3)

Now , $d(s'_j, s_{3-i}) \leq d(s'_j, x_1) + d(x_1, s_{3-i})$ and $d(s'_j, x_1) \leq d(s'_j, s_i) + d(s_i, x_1)$
$\Rightarrow d(s'_j, s_{3-i}) - d(x_1, s_{3-i}) \leq d(s'_j, x_1) \leq d(s'_j, s_i) + d(s_i, x_1)$
$\Rightarrow d(s'_j, s_i) + 1 - d(x_1, s_{3-i}) \leq d(s'_j, x_1) \leq d(s'_j, s_i) + d(s_i, x_1)$
$\Rightarrow d(s'_j, s_i) + d(s_i, x_1) \leq d(s'_j, x_1) \leq d(s'_j, s_i) + d(s_i, x_1)$ (cf. lemma 18)
$\Rightarrow d(s'_j, x_1) = d(s'_j, s_i) + d(s_i, x_1)$

Again, $d(s'_{3-j}, s_{3-i}) \le d(s'_{3-j}, x_1) + d(x_1, s_{3-i})$ and $d(s'_{3-j}, x_1) \le d(s'_{3-j}, s_i) + d(s_i, x_1)$

$\Rightarrow d(s'_{3-j}, s_{3-i}) - d(x_1, s_{3-i}) \le d(s'_{3-j}, x_1) \le d(s'_{3-j}, s_i) + d(s_i, x_1)$

$\Rightarrow d(s'_{3-j}, s_i) + 1 - d(x_1, s_{3-i}) \le d(s'_{3-j}, x_1) \le d(s'_{3-j}, s_i) + d(s_i, x_1)$

$\Rightarrow d(s'_{3-j}, s_i) + d(s_i, x_1) \le d(s'_{3-j}, x_1) \le d(s'_{3-j}, s_i) + d(s_i, x_1)$ (cf. lemma 18)

$\Rightarrow d(s'_{3-j}, x_1) = d(s'_{3-j}, s_i) + d(s_i, x_1)$

Also, $d(x_1, x_2) \le d(x_1, s'_j) + d(s'_j, x_2)$ and $d(x_1, s'_{3-j}) \le d(x_1, x_2) + d(x_2, s'_{3-j})$

$\Rightarrow d(x_1, s'_{3-j}) - d(x_2, s'_{3-j}) \le d(x_1, x_2) \le d(x_1, s'_j) + d(s'_j, x_2)$

$\Rightarrow d(s'_{3-j}, s_i) + d(s_i, x_1) - d(x_2, s'_{3-j}) \le d(x_1, x_2) \le d(x_1, s_i) + d(s_i, s'_j) + d(s'_j, x_2)$

$\Rightarrow d(s_i, s'_j) + 1 + d(s_i, x_1) - d(x_2, s'_{3-j}) \le d(x_1, x_2) \le d(x_1, s_i) + d(s_i, s'_j) + d(s'_j, x_2)$

$\Rightarrow d(x_1, s_i) + d(s_i, s'_j) + d(s'_j, x_2) \le d(x_1, x_2) \le d(x_1, s_i) + d(s_i, s'_j) + d(s'_j, x_2)$

(cf. lemma 18)

$\Rightarrow d(x_1, x_2) = d(x_1, s_i) + d(s_i, s'_j) + d(s'_j, x_2)$

$\Rightarrow d(x_1, x_2) = \#\ (\sigma(x_1, x_2) \cap V(X)) + |i - 1 - t_1|_{\mathbb{R}} + |j - 1 - t_2|_{\mathbb{R}} - 1$

(cf. proposition 3 and corollary 10)

□

Corollary 11. $\forall\ x_1, x_2 \in X_{\mathbb{Q}}, \quad d(x_1, x_2) \in \mathbb{Q}$.

Corollary 12. *Let* $x_1, x_2, x_3 \in X_{\mathbb{R}}$ *such that* $x_2 \in \sigma(x_1, x_3)$.
Then $d(x_1, x_3) = d(x_1, x_2) + d(x_2, x_3)$.

Remark 11. *Note that, both corollary* 11 *and corollary* 12 *can also be obtained as trivial implications of the following corollary.*

Corollary 13. *The metric* d *is the same as the canonical archimedean distance metric on* $X_{\mathbb{R}}$, *given by the length of* $\sigma(x_1, x_2)$ $\forall\ x_1, x_2 \in X_{\mathbb{R}}$.

Remark 12. *At this point, one may ask that why we introduced the metric* d *through* μ *and* ϕ, *rather than taking corollary* 13 *as the definition. The reason is that we need the knowledge of the relation between the metric on the Bruhat-tits tree and the metric on the space of norms, which will very useful in the next chapter. So, if we have had defined the metric* d *as in corollary* 13, *we had to work backwards all the propositions in this section, in order to arrive at definition* 13 *anyway.*

Chapter 2

The Reduction Map

2.1 The Map and the Image

Definition 16. *Two K-linear morphisms,* $f, g : K^2 \to C$ *are equivalent, written* $f \sim g$*, if* $\exists\ c \in C^*$ *such that* $f = cg$.

$$[f] := \{\ g \mid f \sim g\ \}\ .$$

$$Z := \{\ [f] \mid f : K^2 \to C \text{ is an injective } K\text{-linear morphism } \}\ .$$

Definition 17. *For each* $\omega \in \Omega$*, we fix some* $c, d \in C$ *such that* $\omega = [c : d]$*.*
The map $\rho : \Omega \to Z$ *is defined as*

$$\rho(\omega) = [f_{(c,d)}] \quad \text{where,}$$

$$f_{(c,d)}(x, y) = cx + dy\ \ \forall\ (x, y) \in K^2.$$

Definition 18. *For each* $z \in Z$*, we fix some* $f : K^2 \to C$ *such that* $z = [f]$*.*
The map $\rho' : Z \to \Omega$ *is defined as*

$$\rho'(z) = [f(1, 0) : f(0, 1)]\ \ \forall\ z \in Z.$$

Lemma 20. *The maps* ρ, ρ' *are well-defined.*

Proof. Follows directly from the definitions. □

Lemma 21. $\rho' \circ \rho = Id_\Omega$*.*

Proof. Follows directly from the definitions. □

Lemma 22. $\rho \circ \rho' = Id_Z$*.*

Proof. Follows directly from the definitions. □

Corollary 14. *Ω and Z are naturally isomorphic as sets.* $\square$

Definition 19. *For each $z \in Z$, we fix some $f : K^2 \to C$ such that $z = [f]$. The map $\gamma : Z \to Y$ is defined as*

$$\gamma(z) = [\| \ \|_f] \ \ \forall \ z \in Z \ \text{such that}$$

$$\|v\|_f = |f(v)| \ \ \forall \ v \in K^2.$$

Remark 13. *The well-definedness of γ is evident from the definition of Y.*

Definition 20. *The reduction map, $\theta : \Omega \to X_{\mathbb{R}}$ is defined as*

$$\theta := \psi \circ \gamma \circ \rho.$$

Proposition 6. $\theta(\Omega) = X_{\mathbb{Q}}$.

Proof. Let $x := (1-t)s + ts' \in X_{\mathbb{R}}$ and let $L, L' \subseteq K^2$ be two lattices such that $s = [L], s' = [L']$ and $\pi L \subsetneq L' \subsetneq L$.
$\Rightarrow \phi(x) = [\| \ \|_{(L,\, L',\, t)}]$.

So, we have,
$x \in \theta(\Omega) \quad \Leftrightarrow \quad \exists \ \omega' \in \Omega$ such that $\gamma \circ \rho(\omega') = [\| \ \|_{(L,\, L',\, t)}]$.

Now, by Elementary divisors theorem, $\exists$ a basis (e_1, e_2) of L such that $(e_1, \pi e_2)$ is a basis of L'.
Let $\omega \in \Omega$ and suppose, $\rho(\omega) = [f]$, where $f(e_2) = c$.
Let $g := c^{-1} f$. (as $\rho(\omega) \in Z$, hence $c \neq 0$)
$\Rightarrow \rho(\omega) = [g]$ and $g(e_2) = 1$
$\Rightarrow g(e_1) \in C \setminus K$ (as $g \in Z$, hence $g(e_1) \notin K g(e_2)$)
Let $\zeta := g(e_1)$.
$\Rightarrow \gamma \circ \rho(\omega) = [\| \ \|]$,
where, $\|a_1 e_1 + a_2 e_2\| := |g(a_1 e_1 + a_2 e_2)| = |\zeta a_1 + a_2| \ \ \forall \ (a_1, a_2) \in K^2$

Hence,

$$\begin{aligned} x = \theta(\omega) \ &\Leftrightarrow \ \exists \ r' \in \mathbb{R}_{>0} \text{ such that } \|a_1 e_1 + a_2 e_2\|_{(L,\, L',\, t)} = r' |\zeta a_1 + a_2| \\ &\Leftrightarrow \ \sup \{|a_1|, q^t |a_2|\} = r' |\zeta a_1 + a_2| \ \ \forall \ (a_1, a_2) \in K^2 \\ &\Leftrightarrow \ \sup \{|a_1|, q^t |a_2|\} = q^t |\zeta a_1 + a_2| \ \ \forall \ (a_1, a_2) \in K^2 \\ &\Leftrightarrow \ \sup \{q^{-t}, |a|\} = |\zeta + a| \ \ \forall \ a \in K \end{aligned}$$

There are 3 possible cases:

Case 1. $(t = 0)$

$\sup \{q^{-t}, |a|\} = |\zeta + a| \;\; \forall\, a \in K$
$\Leftrightarrow |\zeta| = 1$ and $|\zeta + a| = 1 \;\; \forall\, a \in \mathcal{O} \setminus \pi\mathcal{O}$.

Case 2. $(0 < t < 1)$

$\sup \{q^{-t}, |a|\} = |\zeta + a| \;\; \forall\, a \in K$
$\Leftrightarrow |\zeta| = q^{-t}$

Case 3. $(t = 1)$

$\sup \{q^{-t}, |a|\} = |\zeta + a| \;\; \forall\, a \in K$
$\Leftrightarrow |\zeta| = q^{-1}$ and $|\zeta + a| = q^{-1} \;\; \forall\, a \in \pi\mathcal{O} \setminus \pi^2\mathcal{O}$.

Again, $\exists\, \alpha_i, \beta_i \in K$, such that $e_i = (\alpha_i, \beta_i) \;\; \forall\, i \in \{1, 2\}$

$\Rightarrow \quad \begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix} \in GL_2(K)$

$\Rightarrow \exists \begin{pmatrix} c_1 & c_2 \\ d_1 & d_2 \end{pmatrix} \in GL_2(K)$ such that $\begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix} \begin{pmatrix} c_1 & c_2 \\ d_1 & d_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Let $c := \zeta c_1 + c_2,\; d := \zeta d_1 + d_2$ and let $h(x, y) := cx + dy \;\; \forall\, (x, y) \in K^2$.

$$\begin{aligned} \Rightarrow h(a_1 e_1 + a_2 e_2) &= h(a_1(\alpha_1, \beta_1) + a_2(\alpha_2, \beta_2)) \\ &= h(a_1\alpha_1 + a_2\alpha_2, a_1\beta_1 + a_2\beta_2) \\ &= c(a_1\alpha_1 + a_2\alpha_2) + d(a_1\beta_1 + a_2\beta_2) \\ &= (\zeta c_1 + c_2)(a_1\alpha_1 + a_2\alpha_2) + (\zeta d_1 + d_2)(a_1\beta_1 + a_2\beta_2) \\ &= \zeta a_1 + a_2 \end{aligned}$$

$\Rightarrow h(x, y) = g(x, y) \;\; \forall\, (x, y) \in K^2$

$\Rightarrow \rho(\omega) = [h]$.

$\Rightarrow \omega = \rho'([h]) = [c : d]$.

Hence, from case 1, case 2 and case 3, we have,

$$\theta^{-1}(s) = \{\omega \mid \omega = [c : d] \in \Omega,\; \text{where } |\zeta| = 1 \text{ and } |\zeta + a| = 1 \;\; \forall\, a \in \mathcal{O} \setminus \pi\mathcal{O}\} \tag{2.1}$$

$$\theta^{-1}((1 - t)s + ts') = \{\omega \mid \omega = [c : d] \in \Omega,\; \text{where } |\zeta| = q^{-t}\} \;\; \forall\, t \in (0, 1) \tag{2.2}$$

$$\theta^{-1}(s') = \{\omega \mid \omega = [c : d] \in \Omega,\; \text{where } |\zeta| = q^{-1} \text{ and } |\zeta + a| = q^{-1} \;\; \forall\, a \in \pi\mathcal{O} \setminus \pi^2\mathcal{O}\} \tag{2.3}$$

where, $c := \zeta c_1 + c_2,\ d := \zeta d_1 + d_2$.

Again, $x \in \theta(\Omega) \quad \Leftrightarrow \quad \theta^{-1}(x) \neq \emptyset$.
Now, for $a \in C$ and $r' \in \mathbb{R}$, let us denote,
$D^+(a, r') := \{\xi \in C \mid |\xi - a| \leq r'\}$ and $D^-(a, r') := \{\xi \in C \mid |\xi - a| < r'\}$.
$\Rightarrow D^+(0,1) = \cup_{\alpha \in D^+(0,1)} D^-(\alpha, 1)$.

Let $\bar{k}$ denote the algebraic closure of k and let $\eta : D^+(0,1) \to \bar{k}$ denote the canonical projection modulo $D^-(0,1)$.
Let $\alpha, \beta \in D^+(0,1)$,
$\Rightarrow \begin{cases} D^-(\alpha,1) = D^-(\beta,1) & \text{if} \quad \eta(\alpha) = \eta(\beta) \\ D^-(\alpha,1) \cap D^-(\beta,1) = \emptyset & \text{if} \quad \eta(\alpha) \neq \eta(\beta) \end{cases}$
$\Rightarrow D^+(0,1) = \sqcup_{\delta \in \bar{k}} D^-(\alpha, 1)$, for some $\alpha \in \eta^{-1}(\delta)$
$\Rightarrow \exists\, \zeta \in D^+(0,1) \backslash (\sqcup_{\delta \in k} D^-(\alpha,1))$, for some $\alpha \in \eta^{-1}(\delta)$ (as $\#\, k < \infty = \#\, \bar{k}$)
$\Rightarrow |\zeta| = 1$ and $|\zeta + a| = 1 \ \ \forall\ a \in \mathcal{O} \setminus \pi\mathcal{O}$
$\Rightarrow \theta^{-1}(x) \neq \emptyset \ \ \forall\ x \in V(X)$.

Again, we have $|C| = \mathbb{Q}$ (cf. proposition 5.7.7 of [14])
$\Rightarrow$ For $t \in (0,1)$, $\exists\ \zeta \in C$ such that $|\zeta| = q^{-t} \quad \Leftrightarrow \quad t \in \mathbb{Q} \cap (0,1)$
$\Rightarrow$ For $x = (1-t)s + ts'$, where $t \in (0,1)$, $\theta^{-1}(x) \neq \emptyset \quad \Leftrightarrow \quad t \in \mathbb{Q} \cap (0,1)$

Therefore, $\theta(\Omega) = X_{\mathbb{Q}}$. □

2.2 The Action of $PGL_2(K)$

Definition 21. (*Möbius transformation*) *For each $\omega \in \Omega$, we fix some $c, d \in C$ such that $\omega = [c : d]$. Let $g := \begin{pmatrix} r_1 & r_2 \\ s_1 & s_2 \end{pmatrix} \in GL_2(K)$. g defines a function from $\Omega \to \Omega$, given by*

$$g(\omega) = [r_1 c + r_2 d : s_1 c + s_2 d]\ .$$

Lemma 23. $\forall\ g \in GL_2(K)$, *the Möbius transformation is well-defined.*

Proof. Follows directly from the definition. □

Definition 22. $Z := \{cI \mid c \in K^*\} \subseteq GL_2(K)$ *and* $PGL_2(K) := GL_2(K)/Z$.

Definition 23. *For each $\delta \in PGL_2(K)$, let us fix some $g \in GL_2(K)$ such that $\delta = \bar{g}$. The action of $PGL_2(K)$ on Ω is defined by*

$$\delta(\omega) := g(\omega) \ \ \forall\ \omega \in \Omega\ .$$

Lemma 24. *The action of $PGL_2(K)$ on Ω is well-defined.*

Proof. For the action of $GL_2(K)$ on Ω by Möbius transformation, we have,
$Z \subseteq Stab_\omega \;\; \forall\, \omega \in \Omega$
$\Rightarrow$ the action of $PGL_2(K)$ on Ω is well-defined. $\square$

Definition 24. *Let $g = \begin{pmatrix} r_1 & r_2 \\ s_1 & s_2 \end{pmatrix} \in GL_2(K)$. g defines a function from $K^2 \to K^2$, given by*

$$g(x,y) = (x,y) \begin{pmatrix} r_1 & r_2 \\ s_1 & s_2 \end{pmatrix}^{-1} \quad \forall\, (x,y) \in K^2 \;.$$

Definition 25. *Let $g \in GL_2(K)$. For any lattice, $L = \mathcal{O}e_1 \oplus \mathcal{O}e_2 \subseteq K^2$, the lattice gL is defined as*

$$gL := \mathcal{O}g(e_1) \oplus \mathcal{O}g(e_2) \;.$$

Lemma 25. *$\forall\, g \in GL_2(K)$ and for any lattice $L \subseteq K^2$, gL is well-defined.*

Proof. Given any 2 bases (v_1, v_2) and (w_1, w_2) of L, we have,

$$\begin{aligned} \mathcal{O}g(v_1) \oplus \mathcal{O}g(v_2) &= g(\mathcal{O}v_1 \oplus \mathcal{O}v_2) \\ &= \{\; g(x,y) \mid (x,y) \in L\} \\ &= g(\mathcal{O}w_1 \oplus \mathcal{O}w_2) \\ &= \mathcal{O}g(w_1) \oplus \mathcal{O}g(w_2) \;. \end{aligned}$$

Hence, gL is well-defined. $\square$

Lemma 26. *$GL_2(K)$ permutes the lattices in K^2 transitively.*

Proof. Let $L = \mathcal{O}e_1 \oplus \mathcal{O}e_2$ and $L' = \mathcal{O}v_1 \oplus \mathcal{O}v_2$ be two lattices, where $e_1, e_2, v_1, v_2 \in K^2$.
Let $e_i = (\alpha_i, \beta_i)$ and $v_i = (\gamma_i, \delta_i) \;\; \forall\, i \in \{1,2\}$.
$\Rightarrow \begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix}$ and $\begin{pmatrix} \gamma_1 & \delta_1 \\ \gamma_2 & \delta_2 \end{pmatrix} \in GL_2(K)$.
Let $g := \begin{pmatrix} \gamma_1 & \delta_1 \\ \gamma_2 & \delta_2 \end{pmatrix}^{-1} \begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix}$
Then $gL = L'$. $\square$

Lemma 27. *Let $g \in GL_2(K)$. If $L, L' \subseteq K^2$ are lattices, such that $L' \subsetneq L$, then, $gL' \subsetneq gL$.*

Proof. Otherwise, we have, $gL' = gL$
$\Rightarrow L' = g^{-1}(gL') = g^{-1}(gL) = L$. $\square$

Definition 26. *For each $\{s, s'\} \in E(X)$, let us fix some lattices $L, L' \subseteq K^2$ such that $s = [L]$, $s' = [L']$ and $\pi L \subsetneq L' \subsetneq L$. Let $g \in GL_2(K)$. g defines a function from $X_{\mathbb{R}} \to X_{\mathbb{R}}$, given by*

$$g((1-t)s + ts') := (1-t)[gL] + t[gL'] \quad \forall \ ((1-t)s + ts') \in X_{\mathbb{R}}$$

Lemma 28. *The action of $GL_2(K)$ on $X_{\mathbb{R}}$ is well-defined.*

Proof. Follows directly from lemma 25, lemma 27 and definition 2. □

Definition 27. *For each $\delta \in PGL_2(K)$, let us fix some $g \in GL_2(K)$ such that $\delta = \bar{g}$. The action of $PGL_2(K)$ on $X_{\mathbb{R}}$ is defined by*

$$\delta(x) := g(x) \ \ \forall \ x \in X_{\mathbb{R}} \ .$$

Lemma 29. *The action of $PGL_2(K)$ on $X_{\mathbb{R}}$ is well-defined.*

Proof. For the action of $GL_2(K)$ on $X_{\mathbb{R}}$, we have,
$Z \subseteq Stab_x \ \ \forall \ x \in X_{\mathbb{R}}$
$\Rightarrow$ the action of $PGL_2(K)$ on $X_{\mathbb{R}}$ is well-defined. □

Lemma 30. *$PGL_2(K)$ acts on $V(X)$ transitively.*

Proof. Follows directly from definition 25, definition 26 and lemma 26. □

Lemma 31. *$PGL_2(K)$ acts on $E(X)$ transitively.*

Proof. Let, L_1, L_1' and L_2, L_2' be lattices in K^2 such that
$\{[L_1], [L_1']\}$ and $\{[L_2], [L_2']\} \in E(X)$.
$\Rightarrow \exists \ e_1, e_2, v_1, v_2 \in K^2$ such that $L_1 = \mathcal{O}e_1 \oplus \mathcal{O}e_2$, $L_1' = \mathcal{O}e_1 \oplus \mathcal{O}\pi e_2$
and $L_2 = \mathcal{O}v_1 \oplus \mathcal{O}v_2$, $L_2' = \mathcal{O}v_1 \oplus \mathcal{O}\pi v_2$. (cf. Elementary divisors theorem)
$\exists \ g \in GL_2(K)$ such that $g(e_1) = v_1$ and $g(e_2) = v_2$ (cf. the proof of lemma 26)
$\Rightarrow g(\{[L_1], [L_1']\}) = \{[L_2], [L_2']\}$. □

Definition 28. *For each $y \in Y$, let us fix some $\| \ \|_y$ such that $y = [\| \ \|_y]$. Let $g = \begin{pmatrix} r_1 & r_2 \\ s_1 & s_2 \end{pmatrix} \in GL_2(K)$. g defines a function from $Y \to Y$, given by, $g(y) := [\| \ \|_{g(y)}]$, where,*

$$\|(x_1, x_2)\|_{g(y)} := \|g^{-1}(x_1, x_2)\|_y = \left\|(x_1, x_2)\begin{pmatrix} r_1 & r_2 \\ s_1 & s_2 \end{pmatrix}\right\|_y \quad \forall \ (x_1, x_2) \in K^2.$$

Definition 29. *For each $\delta \in PGL_2(K)$, let us fix some $g \in GL_2(K)$ such that $\delta = \bar{g}$. The action of $PGL_2(K)$ on Y is defined by*

$$\delta(y) := g(y) \quad \forall\ y \in Y\ .$$

Lemma 32. *The action of $PGL_2(K)$ on Y is well-defined.*

Proof. Follows directly from the definitions. □

Lemma 33. $\delta(\gamma \circ \rho(\omega)) = \gamma \circ \rho(\delta(\omega)) \quad \forall\ \omega \in \Omega$, *where,* $\delta \in PGL_2(K)$.

Proof. Let $g = \begin{pmatrix} r_1 & r_2 \\ s_1 & s_2 \end{pmatrix} \in GL_2(K)$ be such that $\delta = \bar{g}$ and let $\omega = [c : d]$.

$\Rightarrow g(\omega) = [r_1c + r_2d : s_1c + s_2d]$
$\Rightarrow \gamma \circ \rho(g(\omega)) = [\|\ \|]$, where,

$$\begin{aligned}
\|(x,y)\| &= |(r_1c + r_2d)x + (s_1c + s_2d)y| \quad \forall\ (x,y) \in K^2 \\
&= \left| (x,y) \begin{pmatrix} r_1 & r_2 \\ s_1 & s_2 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} \right| \\
&= \left\| (x,y) \begin{pmatrix} r_1 & r_2 \\ s_1 & s_2 \end{pmatrix} \right\|_{(\gamma \circ \rho(\omega))}
\end{aligned}$$

$\Rightarrow g(\gamma \circ \rho(\omega)) = \gamma \circ \rho(g(\omega))$
$\Rightarrow \delta(\gamma \circ \rho(\omega)) = \gamma \circ \rho(\delta(\omega))$. □

Lemma 34. $\psi(\delta(y)) = \delta(\psi(y)) \quad \forall\ y \in Y$, *where,* $\delta \in PGL_2(K)$.

Proof. Let $g = \begin{pmatrix} r_1 & r_2 \\ s_1 & s_2 \end{pmatrix} \in GL_2(K)$ be such that $\delta = \bar{g}$ and let
$x = (1-t)[L] + t[L'] \in X_{\mathbb{R}}$, where, $L, L' \subseteq K^2$ are lattices such that $\pi L \subsetneq L' \subsetneq L$.

Now, for any lattice $L_1 \subseteq K^2$, we have,
$\forall\ v \in K^2$,

$$\begin{aligned}
\|v\|_{gL_1} &= \max_{n \in \mathbb{Z}} \{q^n \mid \pi^{-n} v \in gL_1\} \\
&= \max_{n \in \mathbb{Z}} \{q^n \mid g^{-1}(\pi^{-n} v) \in L_1\} \\
&= \max_{n \in \mathbb{Z}} \{q^n \mid \pi^{-n} v \begin{pmatrix} r_1 & r_2 \\ s_1 & s_2 \end{pmatrix} \in L_1\} \\
&= \left\| v \begin{pmatrix} r_1 & r_2 \\ s_1 & s_2 \end{pmatrix} \right\|_{L_1} \\
&= \|g^{-1}(v)\|_{L_1}
\end{aligned}$$

Again, $\phi(\delta(x)) = [\| \ \|_{(gL,gL',t)}]$, where, $\forall\ v \in K^2$,

$$\|v\|_{(gL,gL',t)} = \begin{cases} \|v\|_{gL} & \text{if } \|v\|_{gL} = \|v\|_{gL'} \\ q^t\|v\|_{gL} & \text{if } \|v\|_{gL} \neq \|v\|_{gL'} \end{cases}$$

$$\Rightarrow \|v\|_{(gL,gL',t)} = \begin{cases} \|g^{-1}(v)\|_L & \text{if } \|g^{-1}(v)\|_L = \|g^{-1}(v)\|_{L'} \\ q^t\|g^{-1}(v)\|_L & \text{if } \|g^{-1}(v)\|_L \neq \|g^{-1}(v)\|_{L'} \end{cases}$$

$\Rightarrow \|v\|_{(gL,gL',t)} = \|g^{-1}(v)\|_{(L,L',t)}$

$\Rightarrow \phi(\delta(x)) = \delta(\phi(x)) \ \ \forall\ x \in X_{\mathbb{R}}$ (cf. definition 26)

$\Rightarrow \psi(\delta(y)) = \delta(\psi(y)) \ \ \forall\ y \in Y$ (cf. proposition 2).

□

Lemma 35. *Let $\delta \in PGL_2(K)$. Then $\theta(\delta(\omega)) = \delta(\theta(\omega)) \ \ \forall\ \omega \in \Omega$.*

Proof.

$$\begin{aligned} \theta(\delta(\omega)) &= \psi \circ \gamma \circ \rho(\delta(\omega)) \\ &= \psi(\delta(\gamma \circ \rho(\omega))) && \text{(cf. lemma 27)} \\ &= \delta(\psi \circ \gamma \circ \rho(\omega)) && \text{(cf. lemma 28)} \\ &= \delta(\theta(\omega)). \end{aligned}$$

□

Lemma 36. *Let $\delta \in PGL_2(K)$. Then $\mu(y_1, y_2) = \mu(\delta(y_1), \delta(y_2)) \ \ \forall\ y_1, y_2 \in Y$.*

Proof. Let $g \in GL_2(K)$ be such that $\delta = \bar{g}$.

$\Rightarrow \ g^{-1} \in GL_2(K)$

$\Rightarrow \ g^{-1} : K^2 \to K^2$ is an isomorphism.

Hence, we have,

$$\begin{aligned}
\mu(y_1, y_2) &= \log_q\left(\sup_{v\in K^2\setminus\{(0,0)\}} \frac{\|v\|_{y_1}}{\|v\|_{y_2}}\right) + \log_q\left(\sup_{v\in K^2\setminus\{(0,0)\}} \frac{\|v\|_{y_2}}{\|v\|_{y_1}}\right) \\
&= \log_q\left(\sup_{g^{-1}(v)\in K^2\setminus\{(0,0)\}} \frac{\|g^{-1}(v)\|_{y_1}}{\|g^{-1}(v)\|_{y_2}}\right) + \log_q\left(\sup_{g^{-1}(v)\in K^2\setminus\{(0,0)\}} \frac{\|g^{-1}(v)\|_{y_2}}{\|g^{-1}(v)\|_{y_1}}\right) \\
&= \mu(g(y_1), g(y_2)) \\
&= \mu(\delta(y_1), \delta(y_2))
\end{aligned}$$

□

Corollary 15. *Let $\delta \in PGL_2(K)$. Then $d(x_1, x_2) = d(\delta(x_1), \delta(x_2))\ \ \forall\ x_1, x_2 \in X_{\mathbb{R}}$.*

2.3 The Fibers of the Map

Definition 30. *∞_C denotes a symbol satisfying the following formal identities :*

$$\begin{array}{llcll}
i) & c + \infty_C & = & \infty_C & \forall\ c \in C\ . \\
ii) & c.\infty_C & = & \infty_C & \forall\ c \in C^*. \\
iii) & c/\infty_C & = & 0 & \forall\ c \in C^*.
\end{array}$$

Definition 31. *The map $\eta : \mathbb{P}^1(C) \to C \cup \{\infty_C\}$ is defined as*

$$\eta([c:d]) := \begin{cases} \frac{c}{d} & \text{if} \quad d \neq 0. \\ \infty_C & \text{if} \quad d = 0. \end{cases}$$

Lemma 37. *$\eta(\Omega) = C \setminus K$.*

Proof. Follows directly from the definition of Ω. □

Definition 32. *$\forall\ u = [c:d] \in \mathbb{P}^1(C)$, where, $c, d \in C$,*

$$u' := [d:c]\ .$$

Remark 14. *Note that, $\forall\ u \in \mathbb{P}^1(C) \setminus \{[1:0], [0:1]\}$, we have,*

$$u = [\eta(u) : 1] = [1 : \eta(u')]\ .$$

Convention. *We extend the norm* $|\ | : C \to \mathbb{R}$ *to* $|\ | : C \cup \{\infty_C\} \to \mathbb{R} \cup \{\infty\}$ *by defining* $|\infty_C| := \infty$.

Lemma 38. $\min\{|\eta(u)|, |\eta(u')|\} \leq 1 \quad \forall\, u \in \mathbb{P}^1(C)$.

Proof. Since, from the definitions, we have,

$$|\eta(u')| = \begin{cases} 0 & \text{if} \quad u = \infty_C\,. \\ \frac{1}{|\eta(u)|} & \text{if} \quad u \neq \infty_C\,. \end{cases}$$

□

Definition 33. $\nu : C \to \mathbb{R}_{\geq 0}$ *is defined as*

$$\nu(c) := \inf_{b \in K} |c - b| \quad \forall\, c \in C\,.$$

Remark 15. *Note that,* $\nu(c)$ *is just the distance from* $c \in C$ *to* $K \hookrightarrow C$.

Lemma 39. $\nu(c) = \begin{cases} |c| & \textit{if} \quad |c| \notin |K| \\ \inf_{b \in K,\ |b| = |c|} |c - b| & \textit{if} \quad |c| \in |K| \end{cases}$

Proof. Follows directly from the definition. □

Definition 34. $\tau := \nu \circ \eta|_{\Omega}$.

Definition 35. $\Lambda := \mathcal{O}(1,0) \oplus \mathcal{O}(0,1)$ *and* $\mathrm{x} := [\Lambda]$.

Lemma 40. $d(\mathrm{x}, \theta(\omega)) = d(\mathrm{x}, \theta(\omega')) \ \ \forall\, \omega \in \Omega$.

Proof. Follows directly from lemma 15 and the definitions of x and θ. □

Proposition 7. $\forall\, \omega \in \Omega$,

$$d(\mathrm{x}, \theta(\omega)) = \begin{cases} -\log_q \tau(\omega) & \textit{if} \quad |\eta(\omega)| \leq 1\ . \\ -\log_q \tau(\omega') & \textit{if} \quad |\eta(\omega')| \leq 1\ . \end{cases}$$

Proof.

Case 1. $(|\eta(\omega)| \leq 1)$

$$\Rightarrow |\eta(\omega)x + y| \leq \max\{|\eta(\omega)|\,|x|,\,|y|\} \leq 1 \ \ \forall\,(x,y) \in \Lambda \setminus \pi\Lambda.$$

Also, $(x,y) \in \Lambda \setminus \pi\Lambda \quad \Rightarrow \quad \max\{|x|,|y|\} = 1.$

If $|x| < 1$, then, we have, $|y| = 1$ and hence, $|\eta(\omega)x + y| = 1$

$\Rightarrow \sup_{(x,y)\in\Lambda\setminus\pi\Lambda} \ |\eta(\omega)x + y| = 1$ and

$$\begin{aligned}
\inf_{(x,y)\in\Lambda\setminus\pi\Lambda} |\eta(\omega)x + y| &= \inf_{y\in\mathcal{O}} |\eta(\omega) + y| \\
&= \inf_{y\in K} |\eta(\omega) + y| \qquad\qquad (\text{ as } |\eta(\omega)| \leq 1) \\
&= \nu \circ \eta(\omega) \\
&= \tau(\omega)
\end{aligned}$$

$$\Rightarrow d(\mathrm{x}, \theta(\omega)) = -\log_q \tau(\omega) \qquad\qquad (\text{cf. lemma } 15)$$

Case 2. $(|\eta(\omega')| \leq 1)$

Follows directly from Case 1 and lemma 39. □

Remark 16. *Since any other $s \in V(X)$ is x upto the action of $PGL_2(K)$, θ is $PGL_2(K)-$equivariant and d is $PGL_2(K)-$invariant by lemma 30, lemma 34 and corollary 15 respectively, hence from proposition 7 we obtain $d(s, \theta(\omega))$ $\forall\, s \in V(X)$ and $\forall\,\omega \in \Omega$.*

Corollary 16. *Let, $s \in V(X)$ and let $\delta \in PGL_2(K)$ be such that $\delta(s) = \mathrm{x}$. Then $\forall\,\omega \in \Omega$,*

$$d(s, \theta(\omega)) = \begin{cases} -\log_q \tau(\delta^{-1}(\omega)) & \text{if} \quad |\eta(\delta^{-1}(\omega))| \leq 1\ . \\[2ex] -\log_q \tau((\delta^{-1}(\omega))') & \text{if} \quad |\eta((\delta^{-1}(\omega))')| \leq 1\ . \end{cases}$$

Definition 36. $\forall\, x \in X_{\mathbb{R}}$, *and* $\forall\, a \in \mathbb{R}_{\geq 0}$,

$$\mathcal{B}(x,a) := \{y \in X_{\mathbb{R}} \mid d(x,y) \leq a\}.$$

Definition 37. $\forall\, u \in \mathbb{P}^1(C)$ *and* $\forall\, a \in \mathbb{R}_{>0}$, *the open balls in $\mathbb{P}^1(C)$ are defined as*

$$B(u,a) := \begin{cases} \{\ \xi \in \mathbb{P}^1(C) \ \mid\ |\eta(\xi) - \eta(u)| < a\ \} & \text{if} \quad u \neq [1:0]. \\[2ex] \{\ \xi \in \mathbb{P}^1(C) \ \mid\ |\eta(\xi)| > \frac{1}{a}\} & \text{if} \quad u = [1:0]. \end{cases}$$

and the closed balls in $\mathbb{P}^1(C)$ *are defined as*

$$B^+(u,a) := \begin{cases} \{\ \xi \in \mathbb{P}^1(C)\ \mid\ |\eta(\xi) - \eta(u)| \leq a\ \} & \text{if} \quad u \neq [1:0]. \\ \\ \{\ \xi \in \mathbb{P}^1(C)\ \mid\ |\eta(\xi)| \geq \frac{1}{a}\} & \text{if} \quad u = [1:0]. \end{cases}$$

Definition 38. *A connected affinoid subset of* $\mathbb{P}^1(C)$ *is the complement of a non-empty finite union of open balls.*

Definition 39. $\alpha, \beta, \mu_\lambda \in PGL_2(K)$ *are defined as*

$$\alpha := \overline{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}},\quad \beta := \overline{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}} \quad \text{and} \quad \mu_\lambda := \overline{\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}}\ \forall\ \lambda \in K^*.$$

Lemma 41. *Let* $a \in \mathbb{R}_{(0,1]}$, $u_1 \in \mathbb{P}^1(C)$ *and* $\delta \in PGL_2(K)$.
Then $\delta(B(u_1,a)) = B(u_2,b)$ *for some* $u_2 \in \mathbb{P}^1(C)$ *and* $b \in \mathbb{R}_{>0}$.
Moreover, we have, either $u_2 = \delta(u_1)$ *or* $u_2 = [1:0]$.

Proof. It is enough to consider $\delta(B_0(u,a))$, where δ is a generator of $PGL_2(K)$.
And we have, $PGL_2(K) = \langle \alpha,\ \beta,\ \{\mu_\lambda\}_{\lambda \in K^*} \rangle$. (cf. section 2 of chapter 1 of [1])

Now, there are two possible cases:

Case 1. $(u \neq [1:0])$

For $\delta = \beta$, we have,
$\beta(B(u,a)) = \{\ \beta(\xi) \in \mathbb{P}^1(C)\ \mid\ |\eta(\xi) - \eta(u)| < a\ \}$
$\Leftrightarrow \beta(B(u,a)) = \{\ \xi \in \mathbb{P}^1(C)\ \mid\ |\eta(\beta^{-1}(\xi)) - \eta(u)| < a\ \}$
$\Leftrightarrow \beta(B(u,a)) = \{\ \xi \in \mathbb{P}^1(C)\ \mid\ |\eta(\xi) - 1 - \eta(u)| < a\ \}$
$\Leftrightarrow \beta(B(u,a)) = \{\ \xi \in \mathbb{P}^1(C)\ \mid\ |\eta(\xi) - \eta(\beta(u))| < a\ \}$
$\Leftrightarrow \beta(B(u,a)) = B(\beta(u),a)$

For $\delta = \mu_\lambda$, we have,
$\mu_\lambda(B(u,a)) = \{\ (\mu_\lambda(\xi) \in \mathbb{P}^1(C)\ \mid\ |\eta(\xi) - \eta(u)| < a\ \}$
$\Leftrightarrow \mu_\lambda(B(u,a)) = \{\ \xi \in \mathbb{P}^1(C)\ \mid\ |\eta(\mu_\lambda^{-1}(\xi)) - \eta(u)| < a\ \}$
$\Leftrightarrow \mu_\lambda(B(u,a)) = \{\ \xi \in \mathbb{P}^1(C)\ \mid\ |\eta(\mu_{\lambda^{-1}}(\xi)) - \eta(u)| < a\ \}$
$\Leftrightarrow \mu_\lambda(B(u,a)) = \{\ \xi \in \mathbb{P}^1(C)\ \mid\ |\lambda^{-1}\eta(\xi) - \eta(u)| < a\ \}$
$\Leftrightarrow \mu_\lambda(B(u,a)) = \{\ \xi \in \mathbb{P}^1(C)\ \mid\ |\eta(\xi) - (\eta(\mu_\lambda(u))| < |\lambda|a\ \}$
$\Leftrightarrow \mu_\lambda(B(u,a)) = B(\mu_\lambda(u), |\lambda|a)$

For $\delta = \alpha$, we have,
$\alpha(B(u,a)) = \{\ \alpha(\xi) \in \mathbb{P}^1(C)\ \mid\ |\eta(\xi) - \eta(u)| < a\ \}$
$\Leftrightarrow \alpha(B(u,a)) = \{\ \xi \in \mathbb{P}^1(C)\ \mid\ |\eta(\alpha^{-1}(\xi)) - \eta(u)| < a\ \}$
$\Leftrightarrow \alpha(B(u,a)) = \{\ \xi \in \mathbb{P}^1(C)\ \mid\ |\eta(\xi') - \eta(u)| < a\ \}$

Now, there are two possible cases:

Subcase 1. $(|\eta(u)| < a \ \text{ or } \ |\eta(u)| = 0)$

$\Rightarrow \alpha(B(u,a)) = \{\ \xi \in \mathbb{P}^1(C) \ \mid \ |\eta(\xi')| < a\ \}$ (by ultrametric inequality)
$\Leftrightarrow \alpha(B(u,a)) = \{\ \xi \in \mathbb{P}^1(C) \ \mid \ |\eta(\xi)| > \frac{1}{a}\ \}$
$\Leftrightarrow \alpha(B(u,a)) = B([1:0],a)$

Subcase 2. $(|\eta(u)| \geq a \ \text{ and } \ |\eta(u)| \neq 0)$

$\Rightarrow \alpha(B(u,a)) = \{\ \xi \in \mathbb{P}^1(C) \ \mid \ |\eta(\xi') - \eta(u)| < a \text{ and } |\eta(\xi')| = |\eta(u)|\ \}$
(by ultrametric inequality)
$\Leftrightarrow \alpha(B(u,a)) = \{\ \xi \in \mathbb{P}^1(C) \ \mid \ \frac{|\eta(\xi')-\eta(u)|}{|\eta(\xi')||\eta(u)|} < \frac{a}{|\eta(u)|^2}\ \}$
$\Leftrightarrow \alpha(B(u,a)) = \{\ \xi \in \mathbb{P}^1(C) \ \mid \ |\eta(\xi) - \eta(u')| < \frac{a}{|\eta(u)|^2}\}$
$\Leftrightarrow \alpha(B(u,a)) = B(\alpha(u), \frac{a}{|\eta(u)|^2})$.

Case 2. $(u = [1:0])$
For $\delta = \alpha$, we have,
$\alpha(B(u,a)) = \{\ \alpha(\xi) \in \mathbb{P}^1(C) \ \mid \ |\eta(\xi)| > \frac{1}{a}\ \}$
$\Leftrightarrow \alpha(B(u,a)) = \{\ \xi \in \mathbb{P}^1(C) \ \mid \ |\eta(\alpha^{-1}(\xi))| > \frac{1}{a}\ \}$
$\Leftrightarrow \alpha(B(u,a)) = \{\ \xi \in \mathbb{P}^1(C) \ \mid \ |\eta(\xi')| > \frac{1}{a}\ \}$
$\Leftrightarrow \alpha(B(u,a)) = \{\ \xi \in \mathbb{P}^1(C) \ \mid \ |\eta(\xi)| < a\ \}$
$\Leftrightarrow \alpha(B(u,a)) = B([0:1],a)$

For $\delta = \mu_\lambda$, we have,
$\mu_\lambda(B(u,a)) = \{\ (\mu_\lambda(\xi) \in \mathbb{P}^1(C) \ \mid \ |\eta(\xi)| > \frac{1}{a}\ \}$
$\Leftrightarrow \mu_\lambda(B(u,a)) = \{\ \xi \in \mathbb{P}^1(C) \ \mid \ |\eta(\mu_\lambda^{-1}(\xi))| > \frac{1}{a}\ \}$
$\Leftrightarrow \mu_\lambda(B(u,a)) = \{\ \xi \in \mathbb{P}^1(C) \ \mid \ |\eta(\mu_{\lambda^{-1}}(\xi))| > \frac{1}{a}\ \}$
$\Leftrightarrow \mu_\lambda(B(u,a)) = \{\ \xi \in \mathbb{P}^1(C) \ \mid \ |\lambda^{-1}\eta(\xi)| > \frac{1}{a}\ \}$
$\Leftrightarrow \mu_\lambda(B(u,a)) = \{\ \xi \in \mathbb{P}^1(C) \ \mid \ |\eta(\xi)| > \frac{1}{|\lambda|a}\ \}$
$\Leftrightarrow \mu_\lambda(B(u,a)) = B(u, |\lambda|a)$

For $\delta = \beta$, we have,
$\beta(B(u,a)) = \{\ \beta(\xi) \in \mathbb{P}^1(C) \ \mid \ |\eta(\xi)| > \frac{1}{a}\ \}$
$\Leftrightarrow \beta(B(u,a)) = \{\ \xi \in \mathbb{P}^1(C) \ \mid \ |\eta(\beta^{-1}\xi)| > \frac{1}{a}\ \}$
$\Leftrightarrow \beta(B(u,a)) = \{\ \xi \in \mathbb{P}^1(C) \ \mid \ |\eta(\xi) - 1| > \frac{1}{a}\ \}$
$\Leftrightarrow \beta(B(u,a)) = \{\ \xi \in \mathbb{P}^1(C) \ \mid \ |\eta(\xi)| > \frac{1}{a}\ \}$
(by ultrametric inequality, as $a \leq 1$)
$\Leftrightarrow \beta(B(u,a)) = B(u,a)$. □

Remark 17. *If we wish to consider also the open balls of radius* > 1 *in the previous lemma, then to make sure that the open balls centred at* $[1:0]$ *remain in the set of open balls on* $\mathbb{P}^1(C)$ *after the action of any element of* $PGL_2(K)$, *it is necessary also to include the sets* $\{\xi \in \mathbb{P}^1(C) \mid |\xi - \zeta| > b\}$ $\forall\, \zeta \in \mathbb{P}^1(C) \setminus \{[1:0]\}$ *and* $\forall\, b \in \{q^s \mid s \in \mathbb{Q}\}$

in the definition of open balls centred at $[1:0]$.
And if we wish to prove a similar lemma for closed balls $B^+(u,a)$, *where* $a \leq 1$, *it is necessary to include at least the sets* $\{\xi \in \mathbb{P}^1(C) \mid |\xi - 1| \geq b\}$ $\forall\, b \in \{q^s \mid s \in \mathbb{Q}_{\geq 0}\}$ *in the definition of such closed balls centred at* $[1:0]$. (cf. section 2.1 of [9])

Corollary 17. *Let* $a_1, a_2 \in \mathbb{R}_{(0,1]}$, $u \in \mathbb{P}^1(C)$ *and* $\delta \in PGL_2(K)$.
Then

$$\frac{\textit{radius of } \delta(B(u,a_1))}{\textit{radius of } \delta(B(u,a_2))} = \frac{a_1}{a_2}\ .$$

Definition 40. $\forall\, n \in \mathbb{Z}_{\geq 0}$, *Let* $\jmath_n : \mathcal{O} \to (\mathcal{O}/\pi^{n+1}\mathcal{O})$ *denote the canonical projection.*
We fix a sequence of injective set-morphisms $\imath_n : (\mathcal{O}/\pi^{n+1}\mathcal{O}) \to \mathcal{O}$ *such that*
i) $\imath_n$ *is a section of* $\jmath_n$, *i.e.* $\jmath_n \circ \imath_n = id_{(\mathcal{O}/\pi^{n+1}\mathcal{O})}$.
ii) $\imath_n(\bar{0}) = 0$.

Below, we recall the definition of $\mathbb{P}^1$ of a ring:

Definition 41. *Let* R *be a ring with identity. Two pairs* (a,b) *and* $(c,d) \in R \times R$ *are equivalent, written* $(a,b) \sim_R (c,d)$ *if* $\exists\, \lambda \in R^*$ *such that* $(a,b) = (\lambda c, \lambda d)$.

$$\begin{aligned} [a:b] &:= \{(c,d) \mid (a,b) \sim_R (c,d)\}. \\ R_1 &:= \{[a:1] \mid a \in R\}. \\ R_2 &:= \{[1:a] \mid a \in R\}. \\ \mathbb{P}^1(R) &:= R_1 \cup R_2. \end{aligned}$$

Lemma 42. $\forall\, a \in \mathcal{O}$,
i)$\{(c,d) \in \mathcal{O} \times \mathcal{O} \mid (a,1) \sim_{\mathcal{O}} (c,d)\} = \{(c,d) \in C \times C \mid (a,1) \sim_C (c,d)\}$.
ii)$\{(c,d) \in \mathcal{O} \times \mathcal{O} \mid (1,a) \sim_{\mathcal{O}} (c,d)\} = \{(c,d) \in C \times C \mid (1,a) \sim_C (c,d)\}$.

Proof. Follows directly from the definition. □

Corollary 18. $\mathbb{P}^1(\mathcal{O}) \hookrightarrow \mathbb{P}^1(C)$.

Definition 42. $r_n : \mathbb{P}^1(\mathcal{O}) \to \mathbb{P}^1(\mathcal{O}/\pi^{n+1}\mathcal{O})$ *is defined as*

$$\begin{aligned} r_n([a:1]) &= [\jmath_n(a):1] \ \ \forall\, a \in \mathcal{O}. \\ r_n([1:a]) &= [1:\jmath_n(a)] \ \ \forall\, a \in \mathcal{O}. \end{aligned}$$

Definition 43. $s_n : \mathbb{P}^1(\mathcal{O}/\pi^{n+1}\mathcal{O}) \to \mathbb{P}^1(\mathcal{O}) \hookrightarrow \mathbb{P}^1(C)$ *is defined as*

$$\begin{aligned} s_n([b:1]) &= [\imath_n(b):1] \;\; \forall\, b \in \mathcal{O}/\pi^{n+1}\mathcal{O}. \\ s_n([1:b]) &= [1:\imath_n(b)] \;\; \forall\, b \in \mathcal{O}/\pi^{n+1}\mathcal{O}. \end{aligned}$$

Remark 18. *Note that,* $\jmath_n \circ \imath_n = id_{(\mathcal{O}/\pi^{n+1}\mathcal{O})} \quad \Rightarrow \quad r_n \circ s_n = id_{\mathbb{P}^1(\mathcal{O}/\pi^{n+1}\mathcal{O})}$.

Definition 44.
i) $\Lambda' := [\mathcal{O}(1,0) \oplus \mathcal{O}(0,\pi)]$ *and* $\mathrm{y} := [\Lambda']$.
ii) $e := \{(1-t)\mathrm{x} + t\mathrm{y} \mid t \in [0,1]\}$
ii) $e' := \{(1-t)\mathrm{x} + t\mathrm{y} \mid t \in (0,1)\}$

Lemma 43.
i) $\theta^{-1}(\mathrm{x}) = \mathbb{P}^1(C) \setminus \left(\sqcup_{\xi \in \mathbb{P}^1(\mathcal{O}/\pi\mathcal{O})} B(s_n(\xi),1)\right)$.
ii) $\theta^{-1}(\mathrm{y}) = \mathbb{P}^1(C) \setminus \left(\left(\sqcup_{\xi \in \{z \in \mathbb{P}^1(\mathcal{O}/\pi^2\mathcal{O}) \mid |\eta(s_n(z))| \le q^{-1}\}} B(s_n(\xi),q^{-1})\right) \sqcup B([1:0],q)\right)$.
iii) $\theta^{-1}(e') = \mathbb{P}^1(C) \setminus (B^+([0:1],q^{-1}) \sqcup B^+([1:0],1))$.
iv) $\theta^{-1}(e) = \mathbb{P}^1(C) \setminus$
$\left(\left(\sqcup_{\xi \in \mathbb{P}^1(\mathcal{O}/\pi\mathcal{O}) \setminus \{[\bar{0}:\bar{1}]\}} B(s_n(\xi),1)\right) \sqcup \left(\sqcup_{\xi \in \{z \in \mathbb{P}^1(\mathcal{O}/\pi^2\mathcal{O}) \mid |\eta(s_n(z))| \le q^{-1}\}} B(s_n(\xi),q^{-1})\right)\right)$.

Proof. All the assertions follow directly from (2.1), (2.2) and (2.3). □

Proposition 8. *Let $T_{\mathbb{R}}$ be the geometric realization of a finite subtree T of X. Then $\theta^{-1}(T_{\mathbb{R}})$ = complement of a finite number of disjoint open balls in $\mathbb{P}^1(C)$ each of which contains an element of $\mathbb{P}^1(K)$.*

To prove this proposition, we need the following lemmas:

Lemma 44. *Let B_1 and B_2 be two open balls in $\mathbb{P}^1(C)$. Then,*

$$\begin{aligned} &i) B_1 \cap B_2 \neq \emptyset. \\ &ii) B_i \not\subseteq B_{3-i} \;\; \forall\, i \in \{1,2\}. \end{aligned}$$

$\Leftrightarrow \exists\, a,b \in \mathbb{R}_{>0}$ *such that*

$$\begin{aligned} &i')\ ab > 1. \\ &ii')\ \{B_1,B_2\} = \{B([0:1],a), B([1:0],b)\} \end{aligned}$$

Proof. ($\Rightarrow$)
As the norm that we have used to define the open balls in $\mathbb{P}^1(C)$ is the non-archimedean norm on C, hence for $u_1,u_2 \in \mathbb{P}^1(C) \setminus \{[1:0]\}$ and $a_1,a_2 \in \mathbb{R}_{>0}$, we have, $B(u_1,a_1) \cap B(u_2,a_2) \neq \emptyset$ iff one of the balls is contained in the other.
(cf. proposition 2.3.6 of [14])

$\Rightarrow \exists\ j \in \{1, 2\}$ and $b \in \mathbb{R}_{>0}$ such that $B_j = \{[1:0], b\}$
(since, B_1 and B_2 satisfy (i) and (ii) simultaneously)
$\Rightarrow [1:0] \notin B_{3-j}$.

As the open balls in $\mathbb{P}^1(C)$ have been defined using the non-archimedean norm on C, hence, except the balls containing $[1:0]$, which have their fixed centres at $[1:0]$, any other ball can be taken as a ball centred at any of its point. (cf. proposition 2.3.6(i) of [14])

Suppose, $[0:1] \notin B_{3-j}$.
Now, $B_{3-j} \nsubseteq B_j$
$\Rightarrow B_{3-j} \cap (\ \mathbb{P}^1(C) \setminus B_j) \neq \emptyset$
$\Leftrightarrow B_{3-j} \cap B^+([0:1], \frac{1}{b}) \neq \emptyset$
$\Leftrightarrow B_{3-j} \subsetneq B^+([0:1], \frac{1}{b})$ (since, $[0:1] \notin B_{3-j}$, cf. proposition 2.3.6 of [14])
$\Leftrightarrow B_{3-j} \cap B_j = B_{3-j} \cap B([1:0], b) = \emptyset$
$\Leftrightarrow B_1 \cap B_2 = \emptyset$, which contradicts (i).

Hence, $\exists\ a$ such that $B_{3-j} = B([0:1], a)$.
Now, $ab \leq 1$
$\Rightarrow B([0:1], a) \subseteq B([0:1], \frac{1}{b}) \subseteq B^+([0:1], \frac{1}{b})$
$\Leftrightarrow B([0:1], a) \cap B([1:0], b) = \emptyset$, which contradicts (i).

Hence, $ab > 1$.

$(\Leftarrow)$
$a, b \in \mathbb{R}_{>0}$
$\Rightarrow [0:1] \notin B([1:0], b)$ and $[1:0] \notin B([0:1], a)$
$\Rightarrow B_i \nsubseteq B_{3-i} \quad \forall\ i \in \{1, 2\}$.

Again, $ab > 1$
$\Leftrightarrow (\frac{1}{b}, a) \cap \mathbb{Q} \neq \emptyset$
$\Leftrightarrow \{c \in C \mid \frac{1}{b} < |c| < a\} \neq \emptyset$ (since $|C| = \mathbb{Q}$, cf. proposition 5.7.7 of [14])
$\Leftrightarrow B_1 \cap B_2 = B([0:1], a) \cap B([1:0], b) = \{[c:1] \in \mathbb{P}^1(C) \mid \frac{1}{b} < |c| < a\} \neq \emptyset$ □

Lemma 45. *Let T_1 and T_2 be two subsets of $X_{\mathbb{R}}$ such that*

i) $T_1 \cap T_2 \cap X_{\mathbb{Q}} \neq \emptyset$.
ii) $\theta^{-1}(T_i)$ *is the complement of a finite number of disjoint open balls in* $\mathbb{P}^1(C)$, *each of which contains a point of* $\mathbb{P}^1(K)$ $\ \forall\ i \in \{1, 2\}$.

Then, $\theta^{-1}(T_1 \cup T_2)$ *is also the complement of a finite number of disjoint open balls in* $\mathbb{P}^1(C)$, *each of which contains a point of* $\mathbb{P}^1(K)$.

Proof. As θ was defined from $\Omega = \mathbb{P}^1(C) \setminus \mathbb{P}^1(K)$ to $X_{\mathbb{R}}$, hence $\mathbb{P}^1(K)$ is contained in the complement of the pre-image of any subset of $X_{\mathbb{R}}$,

Suppose if possible, $\exists\ i \in \{1,2\}$ such that the open ball containing $[0:1]$ in $(\mathbb{P}^1(C) \setminus \theta^{-1}(T_i))$ and the open ball containing $[1:0]$ in $(\mathbb{P}^1(C) \setminus \theta^{-1}(T_{3-i}))$ intersects. Let, B_1 and B_2 denote these balls respectively. Now, $\exists\ a, b \in \mathbb{R}_{>0}$ such that $B_1 = B([0:1], a)$ and $B_2 = B([1:0], b)$.

Hence, we have,
$i)\{\omega \in \mathbb{P}^1(C) \mid |\eta(\omega)| \leq \frac{1}{b}\} = B_1 \setminus B_2$
$ii)\{\omega \in \mathbb{P}^1(C) \mid \frac{1}{b} < |\eta(\omega)| < a\} = B_1 \cap B_2$
$iii)\{\omega \in \mathbb{P}^1(C) \mid |\eta(\omega)| \geq a\} = B_2 \setminus B_1$

$\Rightarrow B_1 \cup B_2 = \mathbb{P}^1(C)$.

Let, $U_i :=$ the union of the disjoint open balls in the complement of T_i in $\mathbb{P}^1(C)\ \ \forall\, i \in \{1,2\}$.
$\Rightarrow B_i \subseteq U_i\ \ \forall\, i \in \{1,2\}$.
$\Rightarrow \mathbb{P}^1(C) = B_1 \cup B_2 \subseteq U_1 \cup U_2 \subseteq \mathbb{P}^1(C)$
$\Rightarrow U_1 \cup U_2 = \mathbb{P}^1(C)$

Now,

$$\begin{aligned}
\theta^{-1}(T_1 \cap T_2) &= \theta^{-1}(T_1) \cap \theta^{-1}(T_2) \\
&= (\mathbb{P}^1(C) \setminus U_1) \cap (\mathbb{P}^1(C) \setminus U_2) \\
&= \mathbb{P}^1(C) \setminus (U_1 \cup U_2) \\
&= \emptyset
\end{aligned}$$

But $T_1 \cap T_2 \cap X_{\mathbb{Q}} \neq \emptyset$
$\Rightarrow \theta^{-1}(T_1 \cap T_2) \neq \emptyset$ (since $\theta(\Omega) = X_{\mathbb{Q}}$)
Thus, we get a contradiction.
Hence, the open ball containing $[0:1]$ in $(\mathbb{P}^1(C) \setminus \theta^{-1}(T_i))$ and the open ball containing $[1:0]$ in $(\mathbb{P}^1(C) \setminus \theta^{-1}(T_{3-i}))$ does not intersect.
Therefore, by lemma 44, two open balls, which are in the complements of $\theta^{-1}(T_1)$ and $\theta^{-1}(T_2)$ in $\mathbb{P}^1(C)$ respectively, intersects if and only if one of the balls is a subset of the other.
$\Rightarrow \theta^{-1}(T_1 \cup T_2) = \mathbb{P}^1(C) \setminus (U_1 \cap U_2)$=the complement of a finite number of disjoint open balls in $\mathbb{P}^1(C)$, each of which contains an element of $\mathbb{P}^1(K)$. $\square$

Proof of the proposition. Let $\epsilon_{\mathbb{R}} \subseteq T_{\mathbb{R}}$ be the geometric realization of an edge $\epsilon \in E(T)$. Then by lemma 31, $\exists\ \delta \in PGL_2(K)$ such that $\delta(e) = \epsilon_{\mathbb{R}}$. Hence, by lemma 35, lemma 43(iv) and lemma 41, we get, $\theta^{-1}(\epsilon_{\mathbb{R}})$ = the complement of a finite number of disjoint open balls in $\mathbb{P}^1(C)$, each of which contains an element of $\mathbb{P}^1(K)$.

Since T is a finite subtree, hence $\exists$ an ordering of the edges of T such that $E(T) = \{\epsilon_1, \ldots, \epsilon_n\}$ and $\epsilon_i \cap \epsilon_{i+1} \neq \emptyset\ \forall\ i \in \mathbb{Z}_{[1,n]}$.
Let, $T_{1j} := \cup_{i=1}^{j} \epsilon_{i\mathbb{R}} \quad \forall\ j \in \mathbb{Z}_{[1,n]}$ *and* $T_{2j} := \epsilon_{j+1\mathbb{R}} \quad \forall\ j \in \mathbb{Z}_{[1,n-1]}$.
$\Rightarrow T_{1j} \cap T_{2j} \cap X_{\mathbb{Q}} \neq \emptyset$ *and* $T_{1j+1} = T_{1j} \cup T_{2j} \quad \forall\ j \in \mathbb{Z}_{[1,n-1]}$.

Hence, applying lemma 45 for $j = 1, \ldots, n-1$ successively, we get,

$\theta^{-1}(T_{\mathbb{R}}) = \theta^{-1}(T_{1n})$ = the complement of a finite number of disjoint open balls in $\mathbb{P}^1(C)$, each of which contains an element of $\mathbb{P}^1(K)$. □

Proposition 9. *Let $n \in \mathbb{Z}_{\geq 0}$ and let $a \in [n, n+1)$. Then*

$$\theta^{-1}(\mathcal{B}(\mathrm{x}, a)) = \mathbb{P}^1(C) \setminus \left(\bigsqcup_{\xi \in \mathbb{P}^1(\mathcal{O}/\pi^{n+1}\mathcal{O})} B(s_n(\xi), r_\xi) \right)$$

where,

$$r_\xi := \begin{cases} |\eta(s_n(\xi))|^2 q^{-a} & \text{if } 1 \leq |\eta(s_n(\xi))| < \infty \\ \\ q^{-a} & \text{otherwise.} \end{cases}$$

To prove this proposition, we need the following lemmas.

Lemma 46. *Let $n \in \mathbb{Z}_{\geq 0}$ and let $a \in [n, n+1)$.*
For $\xi_1, \xi_2 \in \{z \in \mathbb{P}^1(\mathcal{O}/\pi^{n+1}\mathcal{O}) \mid |\eta(s_n(z))| \leq 1\}$, with $\xi_1 \neq \xi_2$,

$$B(s_n(\xi_1), q^{-a}) \cap B(s_n(\xi_2), q^{-a}) = \emptyset.$$

Proof. Let $\omega \in B(s_n(\xi_1), q^{-a})$ for some $\xi_1 \in \{z \in \mathbb{P}^1(\mathcal{O}/\pi^{n+1}\mathcal{O}) \mid |\eta(s_n(z))| \leq 1\}$.
Since, $|\eta(s_n(\xi_1))| \leq 1$, hence $\xi_1 = [\alpha_1 : 1]$ for some $\alpha_1 \in \mathcal{O}/\pi^{n+1}\mathcal{O}$
Let $\imath_n(\alpha_1) = \beta_1$.
Now, $\omega \in B(s_n(\xi_1), q^{-a})$

$\Leftrightarrow \omega \in B([\beta_1 : 1], q^{-a})$
$\Leftrightarrow |\eta(\omega) - \beta_1| < q^{-a}$

Suppose, if possible $\omega \in B(s_n(\xi_2), q^{-a})$ for some $\xi_2 \neq \xi_1$, where $\xi_2 \in \{z \in \mathbb{P}^1(\mathcal{O}/\pi^{n+1}\mathcal{O}) \mid |\eta(s_n(z))| \leq 1\}$.

$\Rightarrow \xi_2 = [\alpha_2 : 1]$ for some $\alpha_2 \in \mathcal{O}/\pi^{n+1}\mathcal{O}$.

Let $\imath_n(\alpha_2) = \beta_2$.
Now, $\xi_1 \neq \xi_2 \quad \Leftrightarrow \quad \alpha_1 \neq \alpha_2 \quad \Leftrightarrow \quad \jmath_n(\beta_1) \neq \jmath_n(\beta_2)$
$\Leftrightarrow \beta_1 \not\equiv \beta_2 \pmod{\pi^{n+1}\mathcal{O}}$
$\Leftrightarrow |\beta_1 - \beta_2| > q^{-(n+1)}$
$\Leftrightarrow |\beta_1 - \beta_2| \geq q^{-n} > q^{-a}$ (as $(\beta_1 - \beta_2) \in K$)
$\Rightarrow |\eta(\omega) - \beta_2| = |\beta_1 - \beta_2| > q^{-a}$ (by ultrametric inequality)
$\Rightarrow \eta(\omega) \notin B(s_n(\xi_2), q^{-a})$
Thus, we get a contradiction. □

Lemma 47. *Let $n \in \mathbb{Z}_{\geq 0}$ and let $a \in [n, n+1)$.
For $\xi_1, \xi_2 \in \{z \in \mathbb{P}^1(\mathcal{O}/\pi^{n+1}\mathcal{O}) \mid |\eta(s_n(z))| \geq 1\}$, with $\xi_1 \neq \xi_2$,*

$$B(s_n(\xi_1), r_{\xi_1}) \cap B(s_n(\xi_2), r_{\xi_2}) = \emptyset$$

where,

$$r_\xi := \begin{cases} |\eta(s_n(\xi))|^2 q^{-a} & \text{if } 1 \leq |\eta(s_n(\xi))| < \infty \\ q^{-a} & \text{otherwise.} \end{cases}$$

Proof. $\forall\ \xi \in \{z \in \mathbb{P}^1(\mathcal{O}/\pi^{n+1}\mathcal{O}) \mid |\eta(s_n(z))| \geq 1\}$,
we have, $B(s_n(\xi), r_\xi) = \alpha(B((s_n(\xi))', q^{-a})$. (cf. lemma 41)
$\Rightarrow B(s_n(\xi), r_\xi) = \alpha^{-1}(B((s_n(\xi))', q^{-a})$. (as $\alpha = \alpha^{-1}$)

Suppose, if possible, $\exists\ \omega \in \mathbb{P}^1(C)$ and $\xi_1, \xi_2 \in \{z \in \mathbb{P}^1(\mathcal{O}/\pi^{n+1}\mathcal{O}) \mid |\eta(s_n(z))| \geq 1\}$, with $\xi_1 \neq \xi_2$, such that $\omega \in B(s_n(\xi_1), r_{\xi_1}) \cap B(s_n(\xi_2), r_{\xi_2})$.
$\Rightarrow \alpha(\omega) \in \alpha(B(s_n(\xi_1), r_{\xi_1}) \cap B(s_n(\xi_2), r_{\xi_2}))$
$\Rightarrow \alpha(\omega) \in B((s_n(\xi_1))', q^{-a}) \cap B((s_n(\xi_2))', q^{-a})$

Now, $\xi_1, \xi_2 \in \{z \in \mathbb{P}^1(\mathcal{O}/\pi^{n+1}\mathcal{O}) \mid |\eta(s_n(z))| \geq 1\}$,
$\Rightarrow \xi_1 = [1 : \alpha_1]$ and $\xi_1 = [1 : \alpha_2]$ for some $\alpha_1, \alpha_2 \in \mathcal{O}/\pi^{n+1}\mathcal{O}$.
Let, $\xi_1' := [\alpha_1 : 1]$ and $\xi_2' := [\alpha_2 : 1]$.

Then $(s_n(\xi_i))' = s_n(\xi_i')$ and $|\eta(s_n(\xi_i'))| \leq 1 \ \ \forall\, i \in \{1, 2\}$.
Again, $\xi_1 \neq \xi_2$
$\Rightarrow \xi_1' \neq \xi_2'$.

Therefore, as $\alpha(\omega) \in B((s_n(\xi_1))', q^{-a}) \cap B((s_n(\xi_2))', q^{-a})$,
we get a contadiction from lemma 46.

□

Lemma 48. *Let, $\xi_1, \xi_2 \in \mathbb{P}^1(\mathcal{O}/\pi^{n+1}\mathcal{O})$ be such that $|\eta(s_n(\xi_1))| < 1$ and $|\eta(s_n(\xi_2))| > 1$. Then*

$$B(s_n(\xi_1), q^{-a}) \cap B(s_n(\xi_2), r_{\xi_2}) = \emptyset.$$

where,

$$r_{\xi_2} := \begin{cases} |\eta(s_n(\xi_2))|^2 q^{-a} & \text{if } \ 1 \leq |\eta(s_n(\xi_2))| < \infty \\ \\ q^{-a} & \text{otherwise.} \end{cases}$$

Proof. Suppose, if possible, $\exists\ \omega \in \mathbb{P}^1(C)$ and $\xi_1, \xi_2 \in \mathbb{P}^1(\mathcal{O}/\pi^{n+1}\mathcal{O})$ such that $\omega \in B(s_n(\xi_1), q^{-a}) \cap B(s_n(\xi_2), r_{\xi_2})$ where, $|\eta(s_n(\xi_1))| < 1$ and $|\eta(s_n(\xi_2))| > 1$.

$|\eta(s_n(\xi_1))| < 1$
$\Rightarrow \exists\ \beta_1 \in (\mathcal{O} \setminus \pi^{n+1}\mathcal{O}) \cup \{0\}$ such that $s_n(\xi_1) = [\beta_1 : 1]$
$\Rightarrow |\beta_1| < 1$.

Now, $\omega \in B(s_n(\xi_1), q^{-a})$
$\Leftrightarrow |\eta(\omega) - \beta_1| < q^{-a} \leq 1$
$\Rightarrow |\eta(\omega)| \leq \max\ \{|\eta(\omega) - \beta_1|, |\beta_1|\} < 1$.

We consider two possible cases for ξ_2 :

Case 1. $(s_n(\xi_2) = [1 : 0])$

Then $r_{\xi_2} = q^{-a}$
Now, $\omega \in B(s_n(\xi_1), q^{-a})$
$\Leftrightarrow |\eta(\omega)| > q^a \geq 1$

Case 2. $(s_n(\xi_2) \neq [1 : 0])$

$\Rightarrow \exists\ \beta_2 \in (\mathcal{O} \setminus \pi^{n+1}\mathcal{O})$ such that $s_n(\xi_1) = [1 : \beta_2]$
$\Rightarrow q^{-n} \leq |\beta_2| < 1$ and $r_{\xi_2} = \frac{q^{-a}}{|\beta_2|^2}$

Now, $\omega \in B\left(s_n(\xi_1), \frac{q^{-a}}{|\beta_2|^2}\right)$

$\Leftrightarrow |\eta(\omega) - \frac{1}{\beta_2}| < \frac{q^{-a}}{|\beta_2|^2}$

Also, $a \in [n, n+1)$
$\Leftrightarrow q^{-a} \leq q^{-n}$
$\Rightarrow q^{-a} \leq |\beta_2|$ (since $q^{-n} \leq |\beta_2|$)
$\Leftrightarrow \frac{q^{-a}}{|\beta_2|^2} \leq \frac{1}{|\beta_2|}$.

Hence, $|\eta(\omega) - \frac{1}{\beta_2}| < \frac{1}{|\beta_2|}$
$\Rightarrow |\eta(\omega)| = \frac{1}{|\beta_2|} > 1$ (by ultrametrc inequality)
Thus, both of the cases lead to contradictions. □

Lemma 49. *Let $n \in \mathbb{Z}_{\geq 0}$ and let $a \in [n, n+1)$.*
For $\xi_1, \xi_2 \in \mathbb{P}^1(\mathcal{O}/\pi^{n+1}\mathcal{O})$, with $\xi_1 \neq \xi_2$,

$$B(s_n(\xi_1), r_{\xi_1}) \cap B(s_n(\xi_2), r_{\xi_2}) = \emptyset$$

where,

$$r_\xi := \begin{cases} |\eta(s_n(\xi))|^2 q^{-a} & \text{if } 1 \leq |\eta(s_n(\xi))| < \infty \\ \\ q^{-a} & \text{otherwise.} \end{cases}$$

Proof. Follows directly from lemma 46, lemma 47 and lemma 48. □

Lemma 50. *Let $n \in \mathbb{Z}_{\geq 0}$, $a \in [n, n+1)$ and let $c \in C$ be such that $|c| \leq 1$.*
Then

$$\nu(c) < q^{-a} \quad \Leftrightarrow \quad [c:1] \in \sqcup_{\xi \in \{z \in \mathbb{P}^1(\mathcal{O}/\pi^{n+1}\mathcal{O}) \ | \ |s_n(z)| \leq 1\}} B(s_n(\xi), q^{-a}) .$$

Proof. $\nu(c) < q^{-a}$
$\Leftrightarrow b \in K$ be such that $|c - b| < q^{-a}$.

Now, $|c| \leq 1$
$\Rightarrow b \in \mathcal{O}$ (as otherwise, $|c - b| = |b| > 1 \geq q^{-a}$)
Let $\ell := \jmath_n(b)$
Then $|c - b| < q^{-a}$
$\Leftrightarrow |c - \imath_n(\ell)| < q^{-a}$ (by ultrametric inequality, as $|b - \imath_n(\ell)| < q^{-(n+1)} < q^{-a}$)
$\Leftrightarrow |\eta([c:1]) - \eta([\imath_n(\ell):1]| < q^{-a}$.
$\Leftrightarrow |\eta([c:1]) - \eta(s_n([\ell:1]))| < q^{-a}$.
$\Leftrightarrow [c:1] \in B(s_n([\ell:1]), q^{-a})$
□

Lemma 51. *Let $n \in \mathbb{Z}_{\geq 0}$, $a \in [n, n+1)$ and let $c \in C$ be such that $|c| \leq 1$. Then*

$$\nu(c) < q^{-a} \quad \Leftrightarrow \quad [1:c] \in \sqcup_{\xi \in \{ z \in \mathbb{P}^1(\mathcal{O}/\pi^{n+1}\mathcal{O}) \ | \ |\eta(s_n(z))| \geq 1\}} B(s_n(\xi), r_\xi) \ .$$

where,

$$r_\xi := \begin{cases} |\eta(s_n(\xi))|^2 q^{-a} & \text{if } 1 \leq |\eta(s_n(\xi))| < \infty \\ q^{-a} & \text{otherwise.} \end{cases}$$

Proof. From lemma 50 , we have,

$$\nu(c) < q^{-a} \quad \Leftrightarrow \quad [c:1] \in \sqcup_{\xi \in \{ z \in \mathbb{P}^1(\mathcal{O}/\pi^{n+1}\mathcal{O}) \ | \ |\eta(s_n(z))| \leq 1\}} B(s_n(\xi), q^{-a}) \ .$$

Now, there are two possible cases:

Case 1. $([c:1] \in B([0:1], q^{-a})$

Then $[1:c] = \alpha([c:1]) \in \alpha(B([0:1], q^{-a})) = B([1:0], q^{-a})$ (cf. lemma 41)

Case 2. $([c:1] \in B([\imath_n(\ell):1], q^{-a})$ for some $\ell \in (\mathcal{O}/\pi^{n+1}\mathcal{O}) \setminus \{\bar{0}\})$

Then $\imath_n(\ell) \in \mathcal{O} \setminus \pi^{n+1}\mathcal{O}$
$\Rightarrow |\imath_n(\ell)| \geq q^{-n} \geq q^{-a}$
$\Rightarrow [1:c] = \alpha([c:1]) \in \alpha(B([\imath_n(\ell):1], q^{-a})) = B\left([1:\imath_n(\ell)], \frac{q^{-a}}{|\imath_n(\ell)|^2}\right)$
(cf. lemma 41)
$\Rightarrow [1:c] \in B\left(s_n([1:\ell]), |\eta(s_n([1:\ell]))|^2 q^{-a}\right)$ □

Proof of the proposition. Let $\omega \in \Omega$.

Now, $\omega \in \theta^{-1}(\mathcal{B}(\mathrm{x}, a))$

$\Leftrightarrow \theta(\omega) \in \mathcal{B}(\mathrm{x}, a)$

$\Leftrightarrow d(\mathrm{x}, \theta(\omega)) \leq a$

$$\Leftrightarrow \begin{cases} -\log_q \tau(\omega) \leq a & \text{if } \ |\eta(\omega)| \leq 1 \\ -\log_q \tau(\omega') \leq a & \text{if } \ |\eta(\omega')| \leq 1 \end{cases} \qquad \text{(cf. proposition 7)}$$

$$\Leftrightarrow \begin{cases} -\log_q(\nu \circ \eta(\omega)) \leq a & \textit{if} \quad |\eta(\omega)| \leq 1 \\ -\log_q(\nu \circ \eta(\omega')) \leq a & \textit{if} \quad |\eta(\omega')| \leq 1 \end{cases}$$

$$\Leftrightarrow \begin{cases} \nu \circ \eta(\omega) \geq q^{-a} & \textit{if} \quad |\eta(\omega)| \leq 1 \\ \nu \circ \eta(\omega') \geq q^{-a} & \textit{if} \quad |\eta(\omega')| \leq 1 \end{cases}$$

$$\Leftrightarrow \begin{cases} \omega \in \mathbb{P}^1(C) \setminus \big(\sqcup_{\xi \in \{ z \in \mathbb{P}^1(\mathcal{O}/\pi^{n+1}\mathcal{O}) \ | \ |\eta(s_n(z))| \leq 1\}} B(s_n(\xi), q^{-a}) \big) & \textit{if} \quad |\eta(\omega)| \leq 1 \\ \omega \in \mathbb{P}^1(C) \setminus \big(\sqcup_{\xi \in \{ z \in \mathbb{P}^1(\mathcal{O}/\pi^{n+1}\mathcal{O}) \ | \ |\eta(s_n(z))| \geq 1\}} B(s_n(\xi), r_\xi) \big) & \textit{if} \quad |\eta(\omega')| \leq 1 \end{cases}$$

(cf. lemma 50 and lemma 51 and remark 14)

$$\Leftrightarrow \omega \in \mathbb{P}^1(C) \setminus \big(\sqcup_{\xi \in \mathbb{P}^1(\mathcal{O}/\pi^{n+1}\mathcal{O})} B(s_n(\xi), r_\xi) \big)$$ (cf. lemma 48 and lemma 49)

□

Remark 19. *Let us fix some ordering of the $q^n(q+1)$ elements of $\mathbb{P}^1(\mathcal{O}/\pi^{n+1}\mathcal{O})$. i.e. let*

$$\mathbb{P}^1(\mathcal{O}/\pi^{n+1}\mathcal{O}) = \{\xi_1, \dots, \xi_{q^n(q+1)}\}.$$

Let $\forall \ i \in \mathbb{Z}_{[1, q^n(q+1)]}$,

$$f_i(x, y_i) := \begin{cases} y_i(x - \eta(s_n(\xi_i))) - \pi^{n+1} & \textit{if} \quad |\eta(s_n(\xi_i))| \leq 1 \\ y_i(x - \eta(s_n(\xi_i))) - \pi^{n+\lceil 2v(\eta(s_n(\xi_i)))\rceil+1} & \textit{if} \quad 1 < |\eta(s_n(\xi_i))| < \infty \\ y_i - \pi^{n+1}x & \textit{otherwise} \end{cases}$$

Note that, $\mathbb{P}^1(C) \setminus \big(\sqcup_{\xi \in \mathbb{P}^1(\mathcal{O}/\pi^{n+1}\mathcal{O})} B(s_n(\xi), q^{-a}) \big)$ *is an affinoid space with*

$$K\langle x, y_1, \dots, y_{q^n(q+1)}\rangle / \langle f_1, \dots, f_{q^n(q+1)}\rangle$$

as its associated affinoid algebra.

(cf. defn. 2 of sec. 1.2, defn. 1 in sec. 1.4 of [4] and defn. 3.3.1, exmpl. 3.3.5 in [9])

Proposition 10. *Let $s \in V(X)$ and $n \in \mathbb{Z}_{\geq 0}$. Let, $a \in [n, n+1)$. Then $\exists \ u_\xi \in \mathbb{P}^1(K)$ and $b_\xi \in \mathbb{R}_{>0}$ $\forall \ \xi \in \mathbb{P}^1(\mathcal{O}/\pi^{n+1}\mathcal{O})$ such that*

$$\theta^{-1}(\mathcal{B}(s, a)) = \mathbb{P}^1(C) \setminus \big(\sqcup_{\xi \in \mathbb{P}^1(\mathcal{O}/\pi^{n+1}\mathcal{O})} B(u_\xi, b_\xi) \big).$$

Proof. Let $\delta \in PGL_2(K)$ be such that $\delta(\mathrm{x}) = s$

Now, $\omega \in \theta^{-1}\mathcal{B}(s,a)$

$\Leftrightarrow \theta(\omega) \in \mathcal{B}(s,a)$

$\Leftrightarrow d(s,\theta(\omega)) \leq a$

$\Leftrightarrow d(\mathrm{x},\theta(\delta^{-1}(\omega))) \leq a$ (as $s = \delta(\mathrm{x})$, cf. lemma 34 and corollary 15)

$\Leftrightarrow \delta^{-1}(\omega) \in \mathbb{P}^1(C) \setminus \big(\sqcup_{\xi\in\mathbb{P}^1(\mathcal{O}/\pi^{n+1}\mathcal{O})} B(s_n(\xi), r_\xi)\big)$ (cf. proposition 9)

$\Leftrightarrow \exists\ u_\xi \in \mathbb{P}^1(K)$ and $b_\xi \in \mathbb{R}_{>0}\ \forall\ \xi \in \mathbb{P}^1(\mathcal{O}/\pi^{n+1}\mathcal{O})$ such that

$\theta^{-1}(\mathcal{B}(s,a)) = \mathbb{P}^1(C) \setminus \big(\sqcup_{\xi\in\mathbb{P}^1(\mathcal{O}/\pi^{n+1}\mathcal{O})} B(u_\xi, b_\xi)\big).$ (cf. lemma 41)

□

Remark 20. *Let $n \in \mathbb{Z}_{\geq 0}$ and $s \in V(X)$. From proposition* 10, *we see that the pre-image of the archimedean closed ball $\mathcal{B}(s,b)$ of the geometric realization of the Bruhat-Tits tree, is the complement of $q^n(q+1)$ disjoint open balls in $\mathbb{P}^1(C)$ $\forall\ b \in [n,n+1)$. Now, if b increases by ε in the interval $[n,n+1)$ for some $\varepsilon \in [0,n+1-b)$, the radii of all these open balls in $\mathbb{P}^1(C)$ decrease by the factor $q^{-\varepsilon}$* (cf. proposition 9, proposition 10 and corollary 17). *When $b = n+1$, $q^{n+1}(q+1)$ new vertices of the tree enter in the closed ball $\mathcal{B}(s,b)$* (cf. corollary 8)*; whereas in $\mathbb{P}^1(C)$, each of the former $q^n(q+1)$ open balls split into q disjoint open balls of smaller radii (decreased by the factor q^{n+1-b} compared to the original radius of the corresponding former ball,* (cf. corollary 17)) *and each of these smaller balls lie in the corresponding former ball.* (cf. proposition 9)

Remark 21. *Thus, we see that the connected affinoid subsets of $\mathbb{P}^1(C)$ obtained as the pre-images of the archimedean closed balls in the geometric realization of the Bruhat-tits tree via the fibers of the reduction map, define a Grothendieck topology on Ω and hence, gives an admissible covering of this non-archimedean upper half-pane.*
Thus, a rigid analytic structure on Ω is obtained.
(cf. chapter 3 of [7] or section 6 of [8])

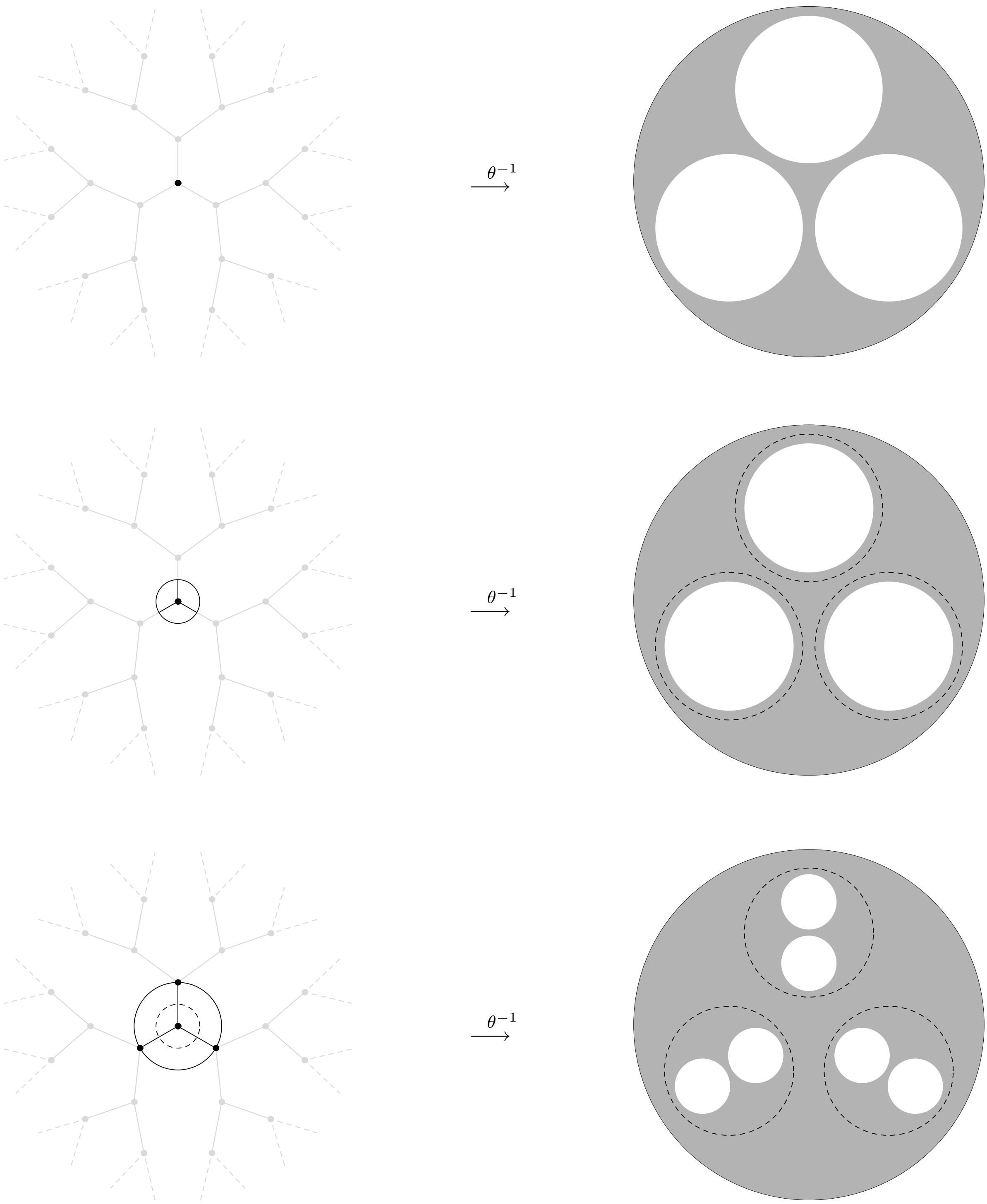

Figure 2.1: The fibers of the reduction map for $K = \mathbb{Q}_2$

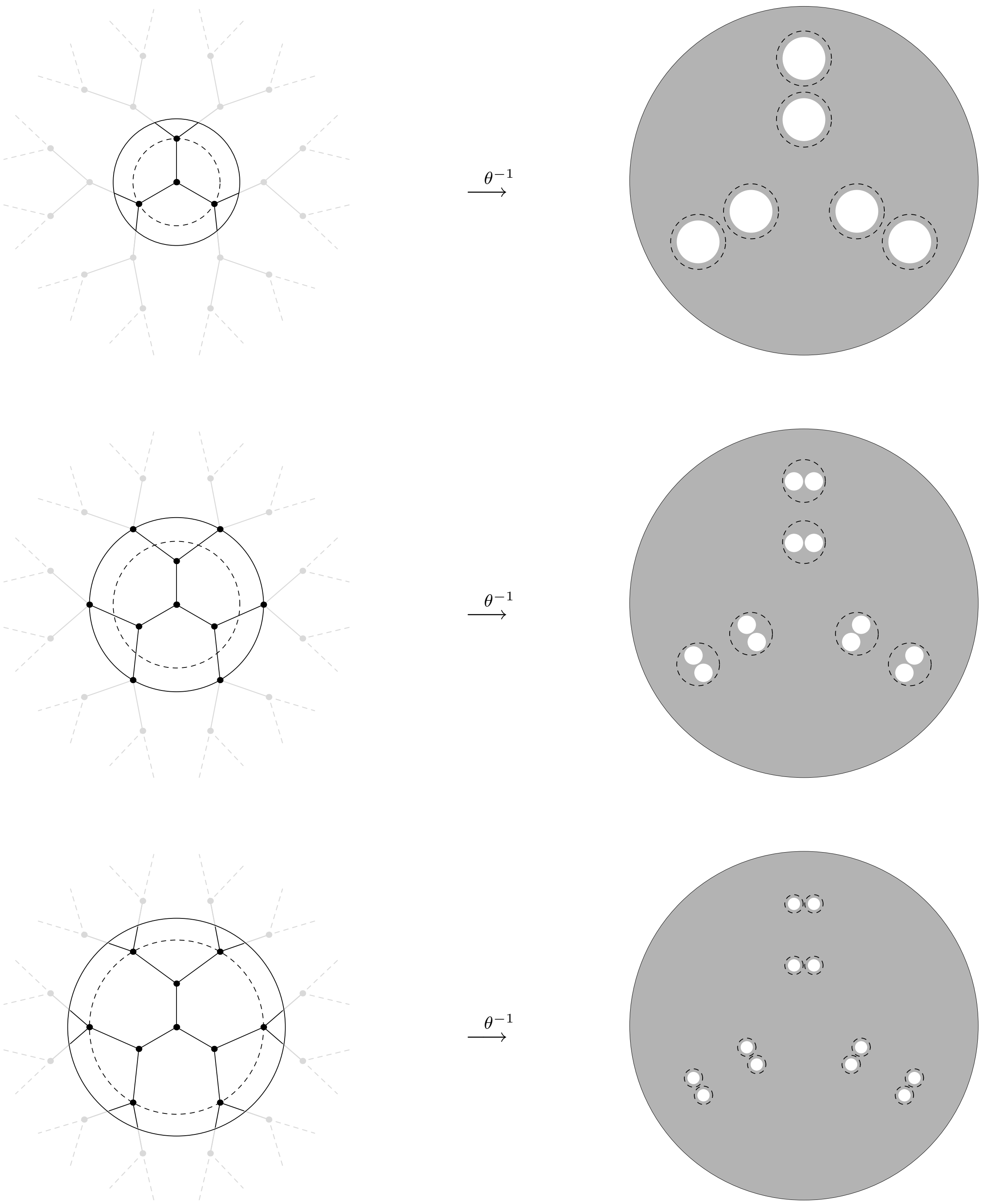

Figure 2.2: The fibers of the reduction map for $K = \mathbb{Q}_2$

Bibliography

[1] M. Artin, *Algebra*, Prentice Hall, 1991.

[2] M. F. Atiyah and I. G. Macdonald, *Introduction to Commutative Algebra*, Addison-Wesley, 1969.

[3] J.-F. Boutot et H. Carayol, *Uniformisation p-adique des courbes de Shimura: les théorèmes de Čerednik et Drinfeld*, pg. 45-158, Astérisque, No. 196-197, Sociéte Mathématique de France, 1991.

[4] S. Bosch, *Lectures on Formal and Rigid Geometry*, preprint, April 2008.

(http://wwwmath1.uni-muenster.de/sfb/about/publ/heft378.pdf)

[5] H. Darmon, *Rational points over modular elliptic curves*, CBMS Regional Conference Series in Mathematics, 101. American Mathematical Society, 2004.

(http://www.math.mcgill.ca/darmon/pub/Articles/Research/36.NSF-CBMS/chapter.pdf)

[6] S. Dasgupta and J. Teitelbaum, *The p-adic upper half plane*, A course for Arizona Winter School, 2007.

(http://math.arizona.edu/ swc/aws/07/DasguptaTeitelbaumNotesMar10.pdf)

[7] P. Deligne and D. Husemöller, *Survey of Drinfel'd Modules*, pg. 25-91, Contemporary Mathematics, Vol. 67, American Mathematical Society, 1987.

[8] V. G. Drinfel'd, *Elliptic Modules*, pg. 561-592, Math. USSR Sbornik 31, No. 4, 1974

[9] J. Frensel and M. van der Put, *Rigid analytic Geometry and Its Applications*, Progress in Mathematics, Birkhäuser, 2003.

[10] E.-U. Gekeler, *Drinfeld Modular Curves*, Lecture Notes in Mathematics 1231, Springer-Verlag, 1986.

[11] L. Gerritzen and M. van der Put, *Schottky groups and Mumford curves*, Lecture Notes in Mathematics, 817. Springer, Berlin, 1980.

[12] O. Goldman and N. Iwahori, *The Space of p-adic Norms*, pg. 137-177, Acta Mathematica, 109, 1963.

[13] D. Goss, *The Algebraist's Upper Half-Plane*, pg. 391-415, No. 3, Bulletins of the American Mathematical Society, 1980.

[14] F. Q. Gouvêa, *P-adic Numbers*, Springer-Verlag, 1993.

[15] W. Rudin, *Principles of Mathematical Analysis*, McGraw-Hill International Editions, 1976.

[16] P. Schneider and U. Stuhler, *The Cohomology of p-adic Symmetric Spaces*, pg. 47-122, Inventiones Mathematicae 105, 1991.

[17] J. P. Serre, *Local Fields*, (Translated from French original by M. J. Greenberg) Springer-Verlag, 1979.

[18] J. P. Serre, *Trees*, (Translated from French original by J. Stillwell) Springer-Verlag, 1980.

[19] J. Teitelbaum, *The Geometry of p-adic Symmetric Spaces*, Vol. 42, No. 10, Notices of the American Mathematical Society, October 1995.

(http://www.ams.org/notices/199510/teitelbaum.pdf)