$$\operatorname{rk} K_n(\mathcal{O}_F)$$

Algebraic K-theory of Number Fields

	0	1	2	3	4	5	6	7	8	9
	1	r_1+r_2-1	0	r_2	0	r_1+r_2	0	r_2	0	r_1+r_2
(SU/SO)			$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
(SU)			0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$

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Preface

Никто не обнимет необъятного.
— Козьма Прутков

(One can't embrace the unembraceable.
— Kozma Prutkov)

One of the central topics in number theory is the study of L-functions. Probably the most well-known of these is the **Riemann zeta function**, which is defined by the series

$$\zeta(s) = \sum_{n \geqslant 1} n^{-s} = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}.$$

This is convergent for $\operatorname{Re} s > 1$, and it has analytic continuation to $\mathbb{C}$ which is holomorphic, except for a simple pole at $s = 1$. We denote the analytic continuation also by ζ. Its values at s and $1 - s$ are related by a **functional equation**

$$\zeta(1-s) = \cos\left(\frac{\pi s}{2}\right) 2\,(2\,\pi)^{-s}\,\Gamma(s)\,\zeta(s),$$

where $\Gamma(s)$ is the **gamma function** (which is $\Gamma(n) = (n-1)!$ for positive integers).

One may ask what are the values of $\zeta(n)$ at $n \in \mathbb{Z}$. For instance, one special value is

$$\zeta(0) = -\frac{1}{2}.$$

If $n = 3, 5, 7, 9, \ldots$ are positive odd numbers, then the values $\zeta(n)$ are rather mysterious; the functional equation is supposed to relate them to the values at negative even numbers $n = -2, -4, -6, -8, \ldots$, but it just tells us that

$$\zeta(-n) = 0 \text{ is a simple zero for } n \geqslant 2 \text{ even.}$$

Less mysterious are the values at $n = 2, 4, 6, 8, \ldots$ They were discovered already by Euler about 1749 (see [Ayo74] for a historical overview):

$$\begin{aligned}
\zeta(2) &= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}, \\
\zeta(4) &= 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots = \frac{\pi^4}{90}, \\
\zeta(6) &= 1 + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \cdots = \frac{\pi^6}{945}, \\
\zeta(8) &= 1 + \frac{1}{2^8} + \frac{1}{3^8} + \frac{1}{4^8} + \cdots = \frac{\pi^8}{9\,450}, \\
&\vdots
\end{aligned}$$

The pattern is more clear if we consider the corresponding values $\zeta(-1), \zeta(-3), \zeta(-5), \zeta(-7), \ldots$ These are some *rational numbers*. To explain them, introduce the **Bernoulli numbers** B_n by a generating function

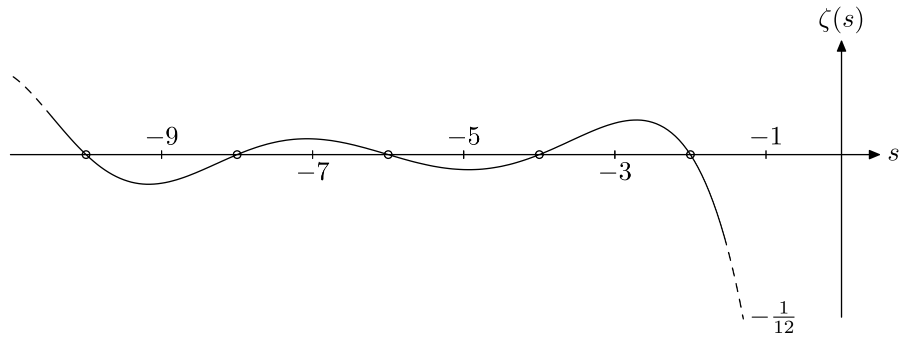
$$\frac{T}{e^T - 1} \stackrel{\text{def}}{=} \sum_{n \geqslant 0} B_n \frac{T^n}{n!} = 1 - \frac{1}{2}T + \frac{1}{6}\frac{T^2}{2!} - \frac{1}{30}\frac{T^4}{4!} + \frac{1}{42}\frac{T^6}{6!} - \frac{1}{30}\frac{T^8}{8!} + \frac{5}{66}\frac{T^{10}}{10!} - \frac{691}{2\,730}\frac{T^{12}}{12!} + \cdots$$

Then the values of ζ are related to these numbers as follows:

$$\zeta(-n) = -\frac{B_{n+1}}{n+1} \quad \text{for } n \geqslant 1 \text{ odd}.$$

This is essentially the Euler's calculation. In particular,

$$\zeta(-1) = -\frac{1}{12},\ \zeta(-3) = \frac{1}{120},\ \zeta(-5) = -\frac{1}{252},\ \zeta(-7) = \frac{1}{240},\ \zeta(-9) = -\frac{1}{132},\ \zeta(-11) = \frac{691}{32\,760},\ \ldots$$



We refer to [Neu99, Theorem VII.1.8] for a proof. Just to spice up this introduction, recall a proof of $\zeta(-1) = -\frac{1}{12}$ that one would suggest in the 18th century. If we formally differentiate the geometric series formula

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x},$$

then we get

$$1 + 2x + 3x^2 + 4x^3 + \cdots = \frac{1}{(1-x)^2}. \tag{*}$$

Now consider the sums (literally meaningless without the functional equation)

$$\begin{aligned} \zeta(-1) &\text{“=”}\ 1 + 2 + 3 + 4 + \cdots \\ 4\,\zeta(-1) &\text{“=”}\ 4 + 8 + 12 + 16 + \cdots \end{aligned}$$

$$\begin{aligned} \zeta(-1) - 4\,\zeta(-1) \text{“=”} -3\,\zeta(-1) &\text{“=”}\ 1 + (2-4) + 3 + (4-8) + \cdots \\ &\text{“=”}\ 1 - 2 + 3 - 4 + \cdots \text{“=”} \frac{1}{4}, \end{aligned}$$

where the last equality is thanks to the formula (*) with $x = -1$ (which may be considered wrong, but was used by Euler in his 1760 paper “De seriebus divergentibus”—cf. [BL76]). Therefore

$$\zeta(-1) \text{“=”}\ 1 + 2 + 3 + 4 + \cdots \text{“=”} -\frac{1}{12}.$$

The corresponding values at the positive even integers are

$$\zeta(n) = \frac{(-1)^{n/2+1}\, B_n\, (2\pi)^n}{2\, n!} \qquad \text{for } n \geqslant 2 \text{ even.}$$

Now we want to generalize the situation and consider a **number field** F, i.e. a finite algebraic extension of the field of rational numbers $\mathbb{Q}$. In F we have its **ring of integers** $\mathcal{O}_F$, which is a free $\mathbb{Z}$-module of rank $d = [F : \mathbb{Q}]$.

$$\begin{tikzcd}
F \arrow[d, dash, "d"] & \mathcal{O}_F \arrow[l, hook'] \arrow[d, dash, "d"] \\
\mathbb{Q} & \mathbb{Z} \arrow[l, hook']
\end{tikzcd}$$

By definition, the **Dedekind zeta function** of F is given by a series

$$\zeta_F(s) = L(\operatorname{Spec} \mathcal{O}_F, s) = \sum_{\mathfrak{a}} (\mathbb{N}\mathfrak{a})^{-s} = \prod_{\mathfrak{p}} \frac{1}{1 - (\mathbb{N}\mathfrak{p})^{-s}},$$

where $\mathfrak{a}$ runs through all nonzero ideals of $\mathcal{O}_F$, and $\mathfrak{p}$ runs through all prime ideals of $\mathcal{O}_F$. By $\mathbb{N}\mathfrak{a}$ we denote the **norm** of ideal. In particular, if $F = \mathbb{Q}$, then this is the same as the Riemann zeta series $\zeta(s)$ as above. Again, this is convergent for $\operatorname{Re} s > 1$, and has an analytic continuation to $\mathbb{C}$ which is holomorphic, except for a simple pole at $s = 1$. The functional equation is

$$\zeta_F(1-s) = |\Delta_F|^{s-1/2} \left(\cos \frac{\pi s}{2}\right)^{r_1+r_2} \left(\sin \frac{\pi s}{2}\right)^{r_2} \left(2\,(2\,\pi)^{-s}\,\Gamma(s)\right)^d \zeta_F(s),$$

where

- r_1 is the number of **real places**, i.e. embeddings $F \hookrightarrow \mathbb{R}$.
- r_2 is the number of **complex places**, i.e. conjugate pairs of embeddings $F \hookrightarrow \mathbb{C}$.
- $d \overset{\text{def}}{=} [F : \mathbb{Q}] = r_1 + 2\,r_2$ is the degree of F.
- Δ_F is the **discriminant** of F.

(If $F = \mathbb{Q}$, then one has $r_1 = 1$, $r_2 = 0$, $d = 1$, $\Delta_F = 1$.)

For basic facts about Dedekind zeta functions we refer to [Neu99, §VII.5].

We again want to investigate the values $\zeta_F(s)$ at points $s = -n$ with $n = 0, 1, 2, \ldots$ Looking at the functional equation, we note that these are zeros, unless $r_2 = 0$ (when the number field is **totally real**). In the latter case if $n = 0$ or $n \geqslant 1$ is odd, values $\zeta_F(-n)$ are non-zero, actually some *rational* numbers. The fact that $\zeta_F(-n) \in \mathbb{Q}$ is known as **Siegel–Klingen theorem** ([Kli62]; cf. [Neu99, VII.9.9]). There are certain ways to relate these values to some fundamental rational numbers, just as Euler related $\zeta_F(-n)$ to Bernoulli numbers. For instance, a formula of Harder [Har71, §2.2] connects the values of ζ_F, for totally real F to **Euler–Poincaré characteristic** of arithmetic groups. In case of symplectic groups $Sp_{2n}(\mathcal{O}_F)$ the formula reads

$$\chi(Sp_{2n}(\mathcal{O}_F)) = \frac{1}{2^{n\,(d-n)}} \prod_{1 \leqslant i \leqslant n} \zeta_F(1 - 2\,i).$$

Here $\chi(Sp_{2n}(\mathcal{O}_F))$ is a rational number. So by induction on i, the last formula implies that $\zeta_F(1-n)$ are rational for even n. We will not get into details and refer to [Ser71, §3.7] and [Bro74].

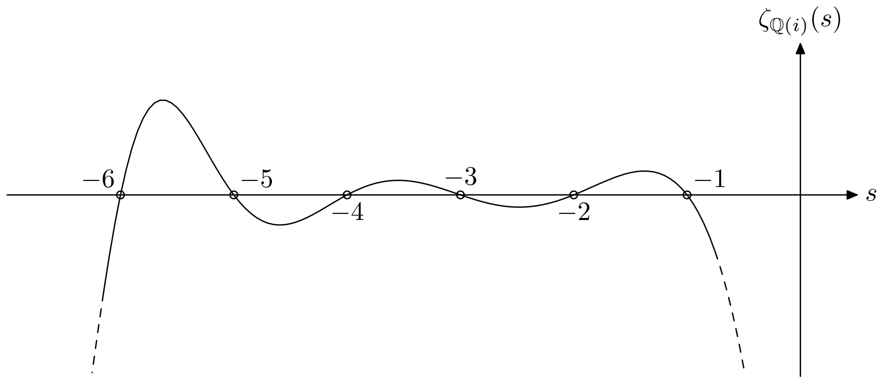
This may be seen as a manifestation of a general philosophical principle:

special values of L-functions are captured by cohomological invariants.

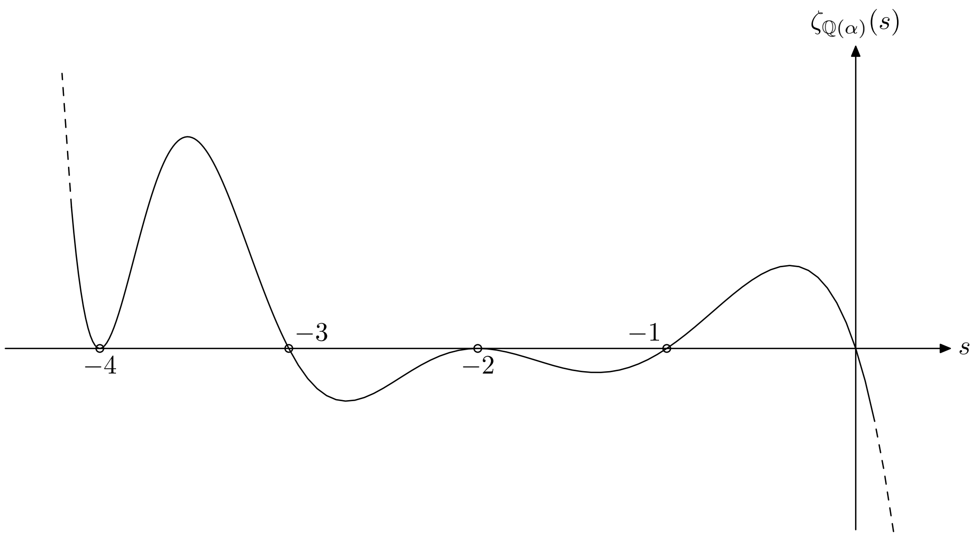
In this text we will not be too ambitious and we will look at the *zeros* $\zeta_F(s)$ at $s = -n$. This may seem trivial, but such zeros have multiplicities, depending on r_1 and r_2. Let us denote by μ_n the multiplicity of zero at $s = -n$ (if there is no zero, then $\mu_n = 0$). The functional equation, together with the fact that $\zeta_F(s)$ has no zeros for $\operatorname{Re} s > 1$ and a simple pole at $s = 1$, shows readily

$$\mu_n = \begin{cases} r_1 + r_2 - 1, & n = 0, \\ r_2, & n \geqslant 1 \text{ odd}, \\ r_1 + r_2, & n \geqslant 2 \text{ even}. \end{cases}$$

Here is an example of zeta function for $F = \mathbb{Q}(i)$. In this case $r_1 = 0$ and $r_2 = 1$, hence all negative integers are simple zeros:



If we take $F = \mathbb{Q}(\alpha)$ where α is a root of polynomial $X^3 + X + 1$, then $r_1 = r_2 = 1$, and simple zeros of $\zeta_{\mathbb{Q}(\alpha)}$ alternate with zeros of multiplicity two:



We are going to see some *cohomological account* of these multiplicities of zeros!

Recall that for a number field F one can define its **ideal class group** $\mathrm{Cl}(F)$ [Neu99, I.3]. This was studied already by Gauss, Kummer, Dedekind, and other 19th century mathematicians. It is some abelian group which vanishes if and only if $\mathcal{O}_F$ is a principal ideal domain. Moreover,

$\mathrm{Cl}(F)$ is finite.

Another basic invariant is the **group of units** $\mathcal{O}_F^\times$—the multiplicative group of invertible elements in $\mathcal{O}_F$. A remarkable theorem of Dirichlet tells that $\mathcal{O}_F^\times$ is finitely generated, it has rank exactly $r_1 + r_2 - 1$, and its torsion part is $\boldsymbol{\mu}_F$, the group of roots of unity in F:

$$\mathcal{O}_F^\times \cong \mathbb{Z}^{r_1+r_2-1} \oplus \boldsymbol{\mu}_F.$$

We will review briefly $\mathrm{Cl}(F)$ and $\mathcal{O}_F^\times$ in chapter 1.

Now the main objects of our study come into play. For any ring R (and actually any scheme, if you like) one can define a whole series of intricate algebraic invariants, named **algebraic K-groups**:

$$K_0(R), K_1(R), K_2(R), K_3(R), K_4(R), \ldots$$

These are some abelian groups. The first invariants in this list were introduced in the 50s and 60s by Grothendieck (K_0); Hyman Bass, Stephen Schanuel (K_1); and John Milnor (K_2). A brief review that fits our needs constitutes chapter 1. The general definition of $K_i(R)$ for $i \geqslant 2$ (both pretty technical and conceptual) is due to Quillen and it is the subject of chapter 2 and also appendix Q.

The only ring that interests us is $R = \mathcal{O}_F$, and in this case

$$K_0(\mathcal{O}_F) \cong \mathrm{Cl}(F) \oplus \mathbb{Z} \quad \text{and} \quad K_1(\mathcal{O}_F) \cong \mathcal{O}_F^\times.$$

So Gauss, Dirichlet, Kummer, and Dedekind were all actually studying algebraic K-theory of number fields! We note that the isomorphism $K_0(\mathcal{O}_F) \cong \mathrm{Cl}(F) \oplus \mathbb{Z}$ is pretty obvious (see § 1.1) since K_0 is really a kind of generalization of the class group. On the other hand, $K_1(\mathcal{O}_F) \cong \mathcal{O}_F^\times$ is a nontrivial theorem due to Bass, Milnor, and Serre (see § 1.2).

As for the higher K-groups $K_2(\mathcal{O}_F), K_3(\mathcal{O}_F), K_4(\mathcal{O}_F), \ldots$ for $\mathcal{O}_F$, one can think of them as of some analogues of the two basic invariants $\mathrm{Cl}(F)$ and $\mathcal{O}_F^\times$. The first important result about higher K-groups of $\mathcal{O}_F$, due to Quillen [Qui73a], is that all $K_n(\mathcal{O}_F)$ are finitely generated abelian groups. Next it is natural to ask about their ranks. Of course $\operatorname{rk} K_0(\mathcal{O}_F) = 1$ (by finiteness of the class group) and $\operatorname{rk} K_1(\mathcal{O}_F) = r_1 + r_2 - 1$ (by Dirichlet). The other ranks are much harder to get. It is a result of Garland [Gar71] that $K_2(\mathcal{O}_F)$ is a finite group, i.e. $\operatorname{rk} K_2(\mathcal{O}_F) = 0$. This was generalized by Armand Borel [Bor74] whose intricate calculation tells that the ranks of $\operatorname{rk} K_n(\mathcal{O}_F)$ are periodic, depending only on r_1 and r_2. Putting together the results of Dirichlet, Garland, and Borel, we have

$$\operatorname{rk} K_n(\mathcal{O}_F) = \begin{cases} 1, & n = 0, \\ r_1 + r_2 - 1, & n = 1, \\ 0, & n = 2i,\ i > 0 \\ r_1 + r_2, & n = 4i+1,\ i > 0, \\ r_2, & n = 4i-1,\ i > 0. \end{cases}$$

If we recall the Dirichlet's theorem proof [Neu99, §I.7], for $K_1(\mathcal{O}_F) \cong \mathcal{O}_F^\times$ it is not very difficult to see that $\mathcal{O}_F^\times$ is finitely generated, but getting the exact rank $r_1 + r_2 - 1$ requires more work. For higher K-groups this is similar: it is a very nice result that $K_n(\mathcal{O}_F)$ are finitely generated, but calculating the ranks is much harder. A detailed exposition of this is the main point of this mémoire.

As we promised, this is related to the zeta function of F; we note that these ranks are exactly the multiplicities of zeros $\zeta_F(-n)$:

n:	0	1	2	3	4	5	6	7	8	9	$\cdots$
$\operatorname{rk} K_n(\mathcal{O}_F)$:	1	$r_1 + r_2 - 1$ $= \mu_0$	0	r_2 $= \mu_1$	0	$r_1 + r_2$ $= \mu_2$	0	r_2 $= \mu_3$	0	$r_1 + r_2$ $= \mu_4$	$\cdots$

To introduce more intriguing numerology, we recall that **Bott periodicity** gives us homotopy groups of the infinite orthogonal group $O(\mathbb{R}) \stackrel{\text{def}}{=} \varinjlim O_n(\mathbb{R})$ (cf. [Bot70]). They are periodic with period eight:

n:	0	1	2	3	4	5	6	7
$\pi_n(O(\mathbb{R}))$:	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$

If we are interested only in rational homotopy, then $\pi_n(O(\mathbb{R})) \otimes \mathbb{Q}$ is periodic with period four. The same period in K-groups of $\mathcal{O}_F$ has the same nature. This will pop up during the calculation (§ 4.6).

Often one is interested in the **ring of S-integers** $\mathcal{O}_{F,S}$ for S a finite set of primes in $\mathcal{O}_F$. In this case K-groups have the same rank, and they are finitely generated as well:

$$\begin{aligned}
\operatorname{rk} K_0(\mathcal{O}_{F,S}) &= 1, \\
\operatorname{rk} K_1(\mathcal{O}_{F,S}) &= \operatorname{rk} \mathcal{O}_{F,S}^\times = |S| + r_1 + r_2 - 1, \\
\operatorname{rk} K_n(\mathcal{O}_{F,S}) &= \operatorname{rk} K_n(\mathcal{O}_F). \quad (n \geqslant 2)
\end{aligned}$$

—this is an easy consequence of the so-called “localization exact sequence”, as will be explained in corollary 2.5.7. It was also established by Borel in [Bor81] using different arguments.

Similarly, if we take the algebraic number field F itself, then

$$\begin{aligned}
K_0(F) &\cong \mathbb{Z}, \\
K_1(F) &\cong F^\times, \\
K_n(F) \otimes_{\mathbb{Z}} \mathbb{Q} &\cong K_n(\mathcal{O}_F) \otimes_{\mathbb{Z}} \mathbb{Q}. \quad (n \geqslant 2)
\end{aligned}$$

In this case, however, the groups are not finitely generated: while $K_n(F) \otimes_{\mathbb{Z}} \mathbb{Q} \cong K_n(\mathcal{O}_F) \otimes_{\mathbb{Z}} \mathbb{Q}$, there may be infinite torsion in $K_n(F)$. E.g. this is obvious already for $K_1(\mathbb{Q})$, and the infinite torsion

$$K_2(\mathbb{Q}) \cong \mathbb{Z}/2 \oplus (\mathbb{Z}/3\mathbb{Z})^\times \oplus (\mathbb{Z}/5\mathbb{Z})^\times \oplus (\mathbb{Z}/7\mathbb{Z})^\times \oplus (\mathbb{Z}/11\mathbb{Z})^\times \oplus \cdots$$

has interesting arithmetic meaning, cf. [Mil71, §11] and [BT73].

The torsion in K-groups of $\mathcal{O}_F$ or F is very important for arithmetic, but it will not be dealt here. We refer to surveys [Wei05], [Kah05], and [Gon05] for the general picture. The rest of this text examines just ranks of $K_n(\mathcal{O}_F)$. Here is a brief outline of the text.

- Chapter 1 introduces the groups $K_0(R)$, $K_1(R)$, and $K_2(R)$.
- Chapter 2 defines higher K-groups of rings via the so-called **plus-construction**. We also collect some facts from Quillen's papers [Qui73b] and [Qui73a].
- Chapter 3 reviews some rational homotopy theory and shows that in order to calculate ranks of $K_n(\mathcal{O}_F)$, it is enough to know the cohomology ring $H^\bullet(SL(\mathcal{O}_F), \mathbb{R})$.
- Chapter 4 finally gets the ranks of $K_n(\mathcal{O}_F)$, assuming certain difficult and technical result about stable cohomology of arithmetic groups.

The rest is devoted to certain steps in the direction of that “technical result”. One who is interested only in the general strategy of computing $\operatorname{rk} K_n(\mathcal{O}_F)$ may content themselves with chapters 1–4.

- Chapter 5 examines a theorem of Matsushima that involves the so-called **Matsushima's constant** $m(G(\mathbb{R}))$ that is very important for stable cohomology.
- Chapter 6 proves certain variation of Matsushima's result, due to Garland.

I tried to make the exposition as much coherent and self-contained as possible. I did my best to give motivation and explain used facts, reviewing the proofs—when they are instructive and not too technical—or providing the references. Certain constructions are both very interesting and hard to take on hearsay, so I included a long discussion of them. The tools that one would consider standard are included in the appendices. They serve to fix definitions and notation, and summarize some basic facts to be used in the main text. The additional appendix Q outlines Quillen's Q-construction, which is not crucial for the main text, although at some point we should assume results that are normally proved using that.

Some notation

Let us fix some notation for all the subsequent chapters:

- F is a number field.
- $\mathcal{O}_F$ is the ring of integers in F.
- $\boldsymbol{\mu}_F$ denotes the group of roots of unity in F.
- r_1 is the number of real places.
- r_2 is the number of complex places.
- $d \stackrel{\text{def}}{=} [F : \mathbb{Q}] = r_1 + 2\,r_2$ is the degree of F.
- Δ_F is the discriminant of F.

Letters like G, H, K will often denote Lie groups, and the corresponding Lie algebras are written in the Fraktur script like $\mathfrak{g}, \mathfrak{h}, \mathfrak{k}$.

As usual, the end of a proof is denoted by a tombstone sign ■; when there is no proof, I mark it with ☹ (unless it is something really well-known). End of an example is marked with ▲.

References

The primary sources that I used writing this text worth a separate mention: the original Borel's article is [Bor74], and there are also some surveys written by Borel himself, notably [Bor06], [Bor95], and a monograph [BW00] by Borel and Wallach.

I hope this text will be useful for someone who wants to learn about algebraic K-theory of number fields.

A note about this version

My intention was to cover all the details and preliminaries needed to calculate $\operatorname{rk} K_n(\mathcal{O}_F)$. At some point the text became quite long, so I took decision to explain only first steps towards *the technical result* (theorem 4.7.2), to avoid making all fifty pages longer. Understanding nuts and bolts of Borel's proofs is a starting point of my future PhD project suggested by Boas Erez, so I will soon post online a more detailed and lengthy version of these notes (it more resembles a book than a mémoire!).

Please send all your comments to `alexey.beshenov@math.u-bordeaux.fr`.

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— Ale

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Chapter 1

Classic algebraic K-theory: K_0, K_1, K_2

In this chapter we will review briefly the definitions of groups K_0, K_1, and K_2 of a ring. We are interested in $K_i(\mathcal{O}_F)$ for a number field F, so the main point is the following.

- $K_0(\mathcal{O}_F) \cong \mathbb{Z} \oplus \mathrm{Cl}(F)$, where $\mathrm{Cl}(F)$ is the class group of F, giving the finite torsion part of $K_0(\mathcal{O}_F)$.
- $K_1(\mathcal{O}_F) \cong \mathcal{O}_F^\times$ is the group of units of $\mathcal{O}_F$, which is isomorphic to $\mathbb{Z}^{r_1+r_2-1} \oplus \boldsymbol{\mu}_F$, according to Dirichlet's unit theorem.

It is very standard yet provides an important motivation for the rest of this text: it shows that K-groups of $\mathcal{O}_F$ are related to the arithmetic of F. Moreover, this suggests some properties of the higher K-groups, e.g. one expects $K_i(\mathcal{O}_F)$ to be finitely generated, with ranks depending on r_1 and r_2, and torsion related to the values of $\zeta_F(s)$.

Finally, we briefly review K_2, even though we will not get into details about its importance in arithmetic.

References. The classic reference for K_0, K_1, K_2 is the Milnor's book [Mil71]. A good modern textbook on algebraic K-theory is [Ros94].

1.1 K_0 of a ring

Let R be a ring. For our purposes, just to simplify things, we assume from now on that R is commutative. Recall that an R-module P is **projective** if one of the following equivalent properties holds [Wei94, §2.2]:

1. Any surjective R-module morphism $p\colon M \twoheadrightarrow P$ has a section $s\colon P \to M$ such that $p \circ s = 1_P$:

$$\begin{tikzcd} M \arrow[r, two heads, "p"] & P \arrow[l, dashed, bend left, "s"] \arrow[r] & 0 \end{tikzcd}$$

2. Any short exact sequence of R-modules

$$0 \to M \hookrightarrow N \twoheadrightarrow P \to 0$$

actually splits.

3. There is an R-module M such that the direct sum $P \oplus M$ is a free R-module.

Now consider the isomorphism classes of finitely generated projective R-modules. They form a set $\mathrm{Proj}_{\mathrm{fg}}(R)$, which can be made into a commutative monoid with addition $[P] + [Q] \stackrel{\text{def}}{=} [P \oplus Q]$ and the 0-module as the identity element. It is not a group and not even a monoid with cancellation, since in general

$$P_1 \oplus Q \cong P_2 \oplus Q \nRightarrow P_1 \cong P_2.$$

Proposition-definition 1.1.1. *Let M be a commutative monoid. Then there exists the* ***Grothendieck group*** *associated to M, which is an abelian group M^+ together with a monoid morphism $M \to M^+$ such that for any group G and a monoid morphism $M \to G$ there is a unique group morphism $M^+ \to G$ making the following diagram commute:*

$$\begin{tikzcd} M \arrow[r] \arrow[d] & M^+ \arrow[dl, dashed, "\exists!"] \\ G & \end{tikzcd}$$

The construction of M^+ is clear: we take the free abelian group on generators $[x]$ for all $x \in M$ modulo relations

$$[x] + [y] = [x + y] \quad \text{for all } x, y \in M.$$

The morphism $M \to M^+$ is given by $x \mapsto [x]$. We see that each element of M^+ can be expressed as a difference $[x] - [y]$ of two generators. By the universal property, M^+ is unique up to isomorphism, and moreover, $M \rightsquigarrow M^+$ is a functor $\mathcal{Mon} \to \mathcal{Grp}$, since for any monoid morphism $f \colon M_1 \to M_2$ one gets canonically

$$\begin{tikzcd} M_1 \arrow[r, "f"] \arrow[d] & M_2 \arrow[d] \\ M_1^+ \arrow[r, dashed, "f^+"'] & M_2^+ \end{tikzcd}$$

This functor $+\colon \mathcal{Mon} \to \mathcal{Grp}$ is left adjoint to the forgetful functor $\mathcal{Grp} \to \mathcal{Mon}$:

$$\mathrm{Hom}_{\mathcal{Grp}}(M^+, G) \cong \mathrm{Hom}_{\mathcal{Mon}}(M, G).$$

Now we are ready to define the 0-th K-group.

Definition 1.1.2. Let R be a ring. The group $K_0(R)$ is the Grothendieck group $\mathrm{Proj}_{\mathrm{fg}}(R)^+$ associated to the monoid $\mathrm{Proj}_{\mathrm{fg}}(R)$ of the isomorphism classes of finitely generated projective R-modules.

So the elements of $K_0(R)$ are $[P]$ for finitely generated projective R-modules P, with addition given by $[P] + [Q] \stackrel{\text{def}}{=} [P \oplus Q]$ and formal subtraction. We can also make $K_0(R)$ into a ring by putting $[P] \cdot [Q] \stackrel{\text{def}}{=} [P \otimes_R Q]$. The identity in this ring is the class $[R^1]$ of the free module R^1.

$K_0(R)$ is a functor, since a morphism of rings $\phi \colon R_1 \to R_2$ functorially induces a morphism of monoids $\mathrm{Proj}_{\mathrm{fg}}(R_1) \to \mathrm{Proj}_{\mathrm{fg}}(R_2)$ given by

$$[P] \mapsto [P \otimes_\phi R_2].$$

This is well-defined: if P is a finitely generated projective R_1-module, then $P \otimes_\phi R_2$ is a finitely generated projective R_2-module. It is a homomorphism since $\otimes$ commutes with $\oplus$.

Example 1.1.3. If R is a principal ideal domain, then every finitely generated projective R-module P is isomorphic to R^n for some n (as a consequence of the fact that over a principal ideal domain a submodule of a free module is free). So to each $[P] \in K_0(R)$ one can associate its **rank** $\mathrm{rk}[P] \stackrel{\text{def}}{=} n$. This is well-defined and gives a group homomorphism

$$\begin{aligned} \operatorname{rk}\colon K_0(R) &\to \mathbb{Z}, \\ [P] &\mapsto \operatorname{rk} P. \end{aligned}$$

This is an isomorphism $K_0(R) \cong \mathbb{Z}$. ▲

Definition 1.1.4. For any ring R there is a canonical morphism $i\colon \mathbb{Z} \to R$ which induces a morphism of K_0-groups $i_*\colon K_0(\mathbb{Z}) \to K_0(R)$. The **reduced K_0-group** of R is given by

$$\widetilde{K}_0(R) \stackrel{\text{def}}{=} K_0(R)/i_*(K_0(\mathbb{Z})).$$

In a sense, $\widetilde{K}_0(R)$ measures how R is far from being a principal ideal domain. Intuitively this suggests that for a Dedekind domain $\mathfrak{A}$ the group $\widetilde{K}_0(R)$ should coincide with the class group $\operatorname{Cl}(\mathfrak{A})$. Establishing this is our next goal.

K_0 of a Dedekind domain

We want to show that for a number field F the group $\widetilde{K}_0(\mathcal{O}_F)$ is exactly the class group $\operatorname{Cl}(\mathcal{O}_F)$. In fact, for any Dedekind domain $\mathfrak{A}$ one has $\widetilde{K}_0(\mathfrak{A}) \cong \operatorname{Cl}(\mathfrak{A})$. Let us briefly recall some facts about Dedekind domains [IR05, Chapter 8].

A Dedekind domain can be defined by various equivalent conditions, e.g.:

- In $\mathfrak{A}$ every nonzero ideal $I \subsetneq R$ factors uniquely into a product of maximal ideals

$$I \cong \mathfrak{m}_1^{e_1} \cdots \mathfrak{m}_n^{e_n}.$$

- $\mathfrak{A}$ is regular of dimension $\leqslant 1$, i.e. $\mathfrak{A}$ is Noetherian and for every maximal ideal $\mathfrak{m} \subset \mathfrak{A}$ the localization $\mathfrak{A}_\mathfrak{m}$ is a principal ideal domain.

Every prime ideal in $\mathfrak{A}$ is automatically maximal.

In order to identify the group $K_0(\mathfrak{A})$, we need to know what are the finitely generated projective modules over $\mathfrak{A}$.

Lemma 1.1.5. *Every finitely generated projective $\mathfrak{A}$-module M is isomorphic to a direct sum $I_1 \oplus \cdots \oplus I_n$ of ideals of $\mathfrak{A}$.*

Proof. By assumption M is a direct summand of $\mathfrak{A}^n$.

If $n = 0$, then we are done.

Assume now the lemma holds for $0, 1, \ldots, n-1$. Consider the projection to the last coordinate $p\colon \mathfrak{A}^n \to \mathfrak{A}$. If $p(M) = 0$, then M lies in a submodule $\ker p \cong \mathfrak{A}^{n-1}$, and we are done by induction. Otherwise, $I \stackrel{\text{def}}{=} p(M) \subseteq \mathfrak{A}$ is a nonzero projective ideal

$$0 \to \ker p|_M \hookrightarrow M \twoheadrightarrow p(M) \to 0$$

hence $M \cong \ker p|_M \oplus I$. Now by induction $\ker p|_M \subseteq \mathfrak{A}^{n-1}$ is a direct sum of ideals. ■

We want to relate $K_0(\mathfrak{A})$ to the class group $\operatorname{Cl}(\mathfrak{A})$. Let us recall the definitions.

Definition 1.1.6. A nonzero $\mathfrak{A}$-submodule $I \subseteq \operatorname{Frac}\mathfrak{A}$ is called a **fractional ideal** of $\mathfrak{A}$ if $aI \subseteq \mathfrak{A}$ for some $a \in \mathfrak{A}$.

A **principal fractional ideal** is given by $\frac{a}{b}\mathfrak{A}$ for some $\frac{a}{b} \in \operatorname{Frac}\mathfrak{A}$. To underline that an ideal I is not fractional, sometimes one says that it is an **integral ideal**.

Fractional ideals of $\mathfrak{A}$ form a group under multiplication with $\mathfrak{A}$ being the unit and the inverse

$$I^{-1} = \{a \in \operatorname{Frac}\mathfrak{A} \mid aI \subseteq \mathfrak{A}\}.$$

Definition 1.1.7. The **class group** of $\mathfrak{A}$ is given by

$$\operatorname{Cl}(\mathfrak{A}) \stackrel{\text{def}}{=} \frac{\text{fractional ideals}}{\text{principal fractional ideals}}.$$

Observe that $\operatorname{Cl}(\mathfrak{A})$ is isomorphic to the group of isomorphism classes of integral ideals (as $\mathfrak{A}$-modules). Indeed, any fractional ideal I is isomorphic to an integral ideal $a\,I$ for some $a \in \mathfrak{A}$. On the other hand, if $\phi\colon I \to J$ is an isomorphism of $\mathfrak{A}$-modules, then we can pick $x_0 \in I\backslash\{0\}$ and since $\phi(x_0\,x) = x_0\,\phi(x) = x\,\phi(x_0)$, we have $J = \frac{\phi(x_0)}{x_0}\,I$, meaning $[I] = [J]$ in the class group as defined above.

Lemma 1.1.8. *Any fractional ideal $I \subseteq \operatorname{Frac}\mathfrak{A}$ is a finitely generated projective $\mathfrak{A}$-module.*

Proof. If I is generated by $(x_1,\dots,x_n)$ and I^{-1} is generated by $(y_1,\dots,y_n)$ with $\sum x_i y_i = 1$, then we have a splitting

$$\begin{tikzcd} \mathfrak{A}^n \arrow[r, two heads, "p"'] & I \arrow[r] \arrow[l, bend right, dashed, "s"'] & 0 \\[-20pt] (a_1,\dots,a_n) \arrow[r, mapsto] & \sum a_i\,x_i \end{tikzcd}$$

which is given by

$$\begin{aligned} s\colon I &\to \mathfrak{A}^n, \\ b &\mapsto (b\,y_1,\dots,b\,y_n). \end{aligned}$$

■

Lemma 1.1.9. *For any two fractional ideals $I, J \subseteq \operatorname{Frac}\mathfrak{A}$ one has an $\mathfrak{A}$-module isomorphism*

$$I \oplus J \cong \mathfrak{A} \oplus IJ.$$

If I and J are two relatively prime ideals, then this is easily to be seen. We consider a map $(x,y) \mapsto x - y$. It has image $\mathfrak{A}$ and kernel consisting of pairs (x,x) with $x \in I \cap J = IJ$, and then the following short exact sequence splits since $\mathfrak{A}$ is projective:

$$0 \to I \cap J \to I \oplus J \to \mathfrak{A} \to 0$$

In general, the lemma should somehow follow from the fact that any ideal factorizes uniquely into prime ideals.

Proof. Pick a nonzero element $b \in J$ such that $b\,J^{-1}$ is an integral ideal.

Claim. *$a\,I^{-1} + b\,J^{-1} = \mathfrak{A}$ for some $a \in I$.*

We consider the factorization into prime ideals

$$b\,J^{-1} = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_k^{e_k}.$$

Now take $a_i \in I\,\mathfrak{p}_1 \cdots \widehat{\mathfrak{p}_i} \cdots \mathfrak{p}_k$ (as usual, $\widehat{\ }$ means that we omit the factor) such that $a_i \notin I\,\mathfrak{p}_1 \cdots \mathfrak{p}_k$. Then $a_i\,I^{-1} \subseteq \mathfrak{p}_j$ for each $j \neq i$ and $a_i\,I^{-1} \nsubseteq \mathfrak{p}_i$. If we take $a \stackrel{\text{def}}{=} \sum a_i$, then $a\,I^{-1} \nsubseteq \mathfrak{p}_i$ for any i, so it is coprime with $b\,J^{-1}$, as we claimed.

Thus we have $c \in I^{-1}$ and $d \in J^{-1}$ such that $ac + bd = 1$. This gives an invertible matrix

$$\begin{pmatrix} c & -b \\ d & a \end{pmatrix}.$$

We use it to define an isomorphism

$$\begin{aligned} & I \oplus J \to \mathfrak{A} \oplus IJ, \\ & (x, y) \mapsto (x, y) \cdot \begin{pmatrix} c & -b \\ d & a \end{pmatrix} = (\underbrace{c\,x + d\,y}_{\in \mathfrak{A}}, \underbrace{-b\,x + a\,y}_{\in IJ}). \end{aligned}$$

The inverse matrix gives the inverse map $\mathfrak{A} \oplus IJ \to I \oplus J$. ∎

Now we are ready to describe the finitely generated projective $\mathfrak{A}$-modules. Each of them is isomorphic to $I_1 \oplus \cdots \oplus I_n$ by lemma 1.1.5. Applying inductively lemma 1.1.9, we get that the latter is isomorphic to $\mathfrak{A}^{n-1} \oplus I_1 \cdots I_n$. So any projective $\mathfrak{A}$-module of rank n is isomorphic to $\mathfrak{A}^{n-1} \oplus I$, and the ideal I is uniquely determined up to isomorphism.

Claim. *$\mathfrak{A}^{n-1} \oplus I \cong \mathfrak{A}^{n-1} \oplus I'$ implies $I \cong I'$.*

This follows from isomorphisms $\bigwedge^n (\mathfrak{A}^{n-1} \oplus I) \cong I$:

$$\begin{tikzcd} \bigwedge^n (\mathfrak{A}^{n-1} \oplus I) \arrow[r, "\cong"] \arrow[d, "\cong"'] & \bigwedge^n (\mathfrak{A}^{n-1} \oplus I') \arrow[d, "\cong"] \\ I \arrow[r, dashed] & I' \end{tikzcd}$$

Putting all together, we have an isomorphism

$$\begin{aligned} K_0(\mathfrak{A}) & \to \mathbb{Z} \oplus \mathrm{Cl}(\mathfrak{A}), \\ [\mathfrak{A}^{n-1} \oplus I] & \mapsto (n, [I]). \end{aligned}$$

This allows to conclude $\widetilde{K}_0(\mathfrak{A}) \cong \mathrm{Cl}(\mathfrak{A})$. ∎

Remark 1.1.10. Recall that $K_0(\mathfrak{A}) \cong \mathbb{Z} \oplus \mathrm{Cl}(\mathfrak{A})$ is a ring with multiplication $[P] \cdot [Q] \stackrel{\text{def}}{=} [P \otimes_{\mathfrak{A}} Q]$.

If we think of the elements of $K_0(\mathfrak{A})$ as of formal differences $[P] - [Q]$, then $\widetilde{K}_0(\mathfrak{A})$ consists of the elements $[P] - [Q]$ with $\operatorname{rk} P = \operatorname{rk} Q = n$. Over a Dedekind domain these are $[\mathfrak{A}^{n-1} \oplus I_1] - [\mathfrak{A}^{n-1} \oplus I_2] = [I_1] - [I_2]$. We calculate the product in $\widetilde{K}_0(\mathfrak{A})$:

$$([I_1] - [I_2]) \cdot ([J_1] - [J_2]) = [I_1] \cdot [J_1] - [I_1] \cdot [J_2] - [I_2] \cdot [J_1] + [I_2] \cdot [J_2].$$

Now $[I] \cdot [J] \stackrel{\text{def}}{=} [I \otimes J] = [IJ]$, and so

$$[I_1 J_1] + [I_2 J_2] - [I_1 J_2] - [I_2 J_1] = [I_1 J_1 \oplus I_2 J_2] - [I_1 J_2 \oplus I_2 J_1].$$

Since over Dedekind domains $I \oplus J \cong \mathfrak{A}^1 \oplus (I\,J)$, remains

$$[\mathfrak{A}^1 \oplus I_1 J_1 I_2 J_2] - [\mathfrak{A}^1 \oplus I_1 J_2 I_2 J_1] = 0.$$

Hence on $\widetilde{K}_0(\mathfrak{A}) \cong \mathrm{Cl}(\mathfrak{A})$ the product is zero.

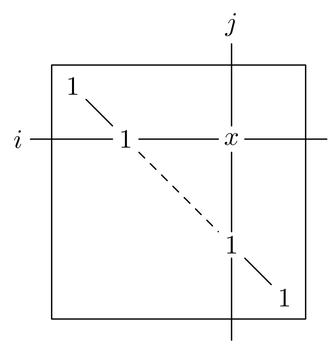
In particular, $K_0(\mathcal{O}_F) \cong \mathbb{Z} \oplus \mathrm{Cl}(F)$, so K_0 is an important arithmetic invariant. Recall that the class group $\mathrm{Cl}(F)$ of a number field is finite—this is usually shown by the celebrated Minkowski's theory [Neu99, §I.6]. From this also follows

Proposition 1.1.11. *For any n there are finitely many isomorphism classes of projective $\mathcal{O}_F$-modules of rank n.*

1.2 K_1 of a ring

Definition 1.2.1. Let R be a ring. Consider the group $GL_n(R)$ of invertible $n \times n$ matrices over R.

Denote by $e_{ij}^{(n)}(x)$ for $x \in R$ and $1 \leqslant i, j \leqslant n$, $i \neq j$ an $n \times n$ matrix having 1's one the diagonal and 0's outside, except for the position (i, j) where it has x. We call such a matrix **elementary**.



We observe that multiplying a matrix by an elementary matrix corresponds to adding to some row (or column) a multiple of another row (column).

All such matrices generate the **subgroup of elementary matrices** $E_n(R) \subset GL_n(R)$. One has embeddings

$$GL_n(R) \hookrightarrow GL_{n+1}(R),$$
$$M \mapsto \begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix},$$

and similarly $E_n(R) \hookrightarrow E_{n+1}(R)$. Under these embeddings one gets

$$GL(R) \overset{\text{def}}{=} \varinjlim_n GL_n(R), \quad E(R) \overset{\text{def}}{=} \varinjlim_n E_n(R);$$

these are just groups of arbitrarily big matrices: to multiply matrices of different size, we use the embedding $M \mapsto \begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix}$.

For a moment it may seem like working with elementary matrices is too restrictive. However, they generate a big group. The following is basically a computation with matrices, but it is a very important fact:

Claim (Whitehead's lemma)**.** *For any matrix $M \in GL_n(R)$ one has*

$$\begin{pmatrix} M & 0 \\ 0 & M^{-1} \end{pmatrix} \in E_{2n}(R).$$

Further, there are the following relations for elementary matrices:

$$e_{ij}^{(n)}(a)\, e_{ij}^{(n)}(b) = e_{ij}^{(n)}(a + b), \tag{1.1}$$

$$[e_{ij}^{(n)}(a), e_{jk}^{(n)}(b)] = e_{ik}^{(n)}(ab) \quad \text{for } i \neq k, \tag{1.2}$$

$$[e_{ij}^{(n)}(a), e_{k\ell}^{(n)}(b)] = 1 \quad \text{for } j \neq k, i \neq \ell. \tag{1.3}$$

As usual, by $[x, y]$ we denote the commutator $x y x^{-1} y^{-1}$. By $[G, G]$ we will denote the subgroup generated by all commutators $[x, y]$ with $x, y \in G$. From (1.2) one sees that $[E_n(R), E_n(R)] = E_n(R)$ for $n \geqslant 3$, and hence $[E(R), E(R)] = E(R)$. We claim that $[GL(R), GL(R)] \subseteq E(R)$, and so $[GL(R), GL(R)] = E(R)$. Indeed, for two elements $M, N \in GL_n(R)$ their commutator in $GL(R)$ becomes

$$\begin{pmatrix} [M,N] & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} M N M^{-1} N^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} M N & 0 \\ 0 & N^{-1} M^{-1} \end{pmatrix} \begin{pmatrix} M^{-1} & 0 \\ 0 & M \end{pmatrix} \begin{pmatrix} N^{-1} & 0 \\ 0 & N \end{pmatrix},$$

and by Whitehead's lemma all factors are in $E_{2n}(R)$.

So one has a *very noncommutative* group $GL(R)$ formed by arbitrarily large matrices, and its noncommutativity is measured by its commutator $E(R) = [GL(R), GL(R)]$. This suggests that one should study the abelianization of $GL(R)$:

Definition 1.2.2. For a ring R the **group** K_1 is given by

$$K_1(R) \stackrel{\text{def}}{=} GL(R)/E(R) = GL(R)^{ab} = H_1(GL(R), \mathbb{Z}).$$

We note that $GL_n(\cdot)$ is a functor $\mathcal{CRing} \to \mathcal{Grp}$, and similarly $GL(\cdot)$ is a functor $\mathcal{CRing} \to \mathcal{Grp}$. Also the abelianization is a functor $\mathcal{Grp} \to \mathcal{Ab}$ (which is left adjoint to the inclusion $\mathcal{Ab} \hookrightarrow \mathcal{Grp}$), hence K_1 is a functor from commutative rings to abelian groups.

Remark 1.2.3. K_1 was discovered in topology in the work of J.H.C. Whitehead (e.g. [Whi50]). A great exposition of topological use of K_1 is [Mil66]. In algebra, K_1 of a ring appeared first in [BS62].

By Whitehead's lemma, the product $[M] \cdot [N] = [M \cdot N]$ in $K_1(R)$ can be viewed as the "block sum" of matrices $[M] \cdot [N] = [M \oplus N]$, since $M \cdot N$ and $M \oplus N$ differ by an element of $E(R)$:

$$\begin{pmatrix} M N & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix} \underbrace{\begin{pmatrix} N & 0 \\ 0 & N^{-1} \end{pmatrix}}_{\in E(R)}.$$

Definition 1.2.4. We have the usual determinant homomorphism $\det\colon GL_n(R) \to R^\times$, and it obviously extends to a homomorphism $\det\colon GL(R) \to R^\times$, since $\det \begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix} = \det M \det N$. The kernel of this map is by definition the **special linear group** $SL(R)$. One sees that $E(R)$ lies in $SL(R)$, since all elementary matrices have determinant 1.

We put

$$SK_1(R) \stackrel{\text{def}}{=} SL(R)/E(R).$$

One has a split short exact sequence

$$0 \to SL(R) \hookrightarrow GL(R) \twoheadrightarrow R^\times \to 0$$

(the splitting is given by inclusion $R^\times = GL_1(R) \hookrightarrow GL(R)$), and there is a split short exact sequence

$$0 \to SK_1(R) \hookrightarrow K_1(R) \twoheadrightarrow R^\times \to 0$$

That is, $K_1(R) \cong SK_1(R) \oplus R^\times$. Now the question is whether $SK_1(R)$ vanishes, i.e. whether elementary matrices generate the whole $SL(R)$. In other words, given a matrix of determinant 1, can we always transform it to the identity matrix using the elementary row (or column) operations? If R is a field, then the answer is "yes" by basic linear algebra. If R is a Euclidean domain, or more generally a principal ideal domain, then the answer is "yes" [Ros94, §2.3], although it is less easy.

As in the rest of this mémoire, we are interested in the case when $R = \mathcal{O}_F$ is the ring of integers of a number field. It is not necessarily a principal ideal domain, but we will see soon that $SK_1(\mathcal{O}_F) = 0$.

Theorem 1.2.5 (Bass–Milnor–Serre). *Let $\mathcal{O}_F$ be the ring of integers in a number field F. Then*

$$K_1(\mathcal{O}_F) \cong \mathcal{O}_F^\times.$$

However, it is a subtle fact relying on the arithmetic of F.

Remark 1.2.6. In general $SK_1(R)$ does not vanish, but discussing such examples is beyond the scope of this text. For instance, for the group ring $\mathbb{Z}G$, where is G a finite abelian group, $SK_1(\mathbb{Z}G)$ vanishes "rarely"; see [ADS73, ADS85, ADOS87] and [Oli88].

Transfer map in K_1

Following [Mil71, §3 + §14], we review an additional construction that will be used below. Let R be a ring and S be its subring such that R is a finitely generated projective S-module. The inclusion $i\colon S \hookrightarrow R$ gives by functoriality a map $i_*\colon K_1(S) \to K_1(R)$, but one can also get the **transfer map** $i^*\colon K_1(R) \to K_1(S)$ going the other way.

Note that for K_0 the transfer $i^*\colon K_0(R) \to K_0(S)$ is obvious: a finitely generated projective module P over R can be viewed as such a module over S. This gives a map $[P] \mapsto [P_S]$ on the generators of K_0. By abuse of notation we will identify $[P]$ and $i^*[P]$.

First observe that $K_1(S)$ has a $K_0(S)$-module structure. Let $[P] \in K_0(S)$ be an isomorphism class of a finitely generated projective S-module. For an element $x \in K_1(S)$ we would like to define the action $[P]\cdot x$.

Since P is projective and finitely generated, one has $P \oplus Q \cong S^r$ for some S-module Q. An automorphism α of P gives an automorphism $\alpha \oplus 1_Q$ of $P \oplus Q$, which after fixing a basis of $P \oplus Q$ can be viewed as an element of $GL_r(S)$. So there is a map

$$\mathrm{Aut}(P) \hookrightarrow \mathrm{Aut}(P \oplus Q) \xrightarrow{\cong} GL_r(S) \hookrightarrow GL(S).$$

Claim. *This is well-defined up to an inner automorphism of $GL(S)$, and hence gives a well-defined homomorphism*

$$\mathrm{Aut}(P) \to K_1(S) = GL(S)^{ab}.$$

Proof. Assume that from $\alpha \in \mathrm{Aut}(P)$ we got a matrix $A \in GL(S)$ using some basis $b_1, \dots, b_r$ of $P \oplus Q$. With respect to another basis $b_1', \dots, b_s'$ the resulting matrix is $C\,A\,C^{-1} \in GL_s(S)$ for some invertible $s \times r$-matrix C.

If we replace Q with another Q' such that $P \oplus Q' \cong S^t$, then $Q \oplus S^t \cong Q' \oplus S^r$, hence a different choice of Q also alters the embedding $\mathrm{Aut}(P) \hookrightarrow GL(S)$ by an inner automorphism. ∎

Now for $[P] \in K_0(S)$ we have a map

$$\begin{tikzcd} GL_n(S) \arrow[r, "\cong"'] \arrow[rrr, dashed, bend left, "\overset{\mathrm{def}}{=} h_P"] & \mathrm{Aut}(S^n) \arrow[r] & \mathrm{Aut}(P \oplus S^n) \arrow[r] & K_1(S) \end{tikzcd}$$

$$\alpha \longmapsto 1_P \oplus \alpha$$

Observe that $h_{P\oplus P'} = h_P + h_{P'}$, so h_P depends only on the class $[P] \in K_0(S)$. Now passing to abelianization and $n \to \infty$, we get a map $K_1(S) = GL(S)^{ab} \to K_1(S)$. By definition, this is the action of $[P]$:

$$\begin{aligned} K_1(S) &\to K_1(S), \\ x &\mapsto [P]\cdot x. \end{aligned}$$

Now we define the transfer for K_1. Again, we assume that R is a finitely generated projective S-module. We pick a projective S-module Q such that $R \oplus Q \cong S^r$ is a free S-module of rank r. An element $x \in K_1(R)$ is represented by a matrix $A \in GL_n(R) \cong \mathrm{Aut}(R^n)$. Now $R^n \oplus Q^n$ is also a free S-module of rank nr. We can consider an automorphism $A \oplus 1_{Q^n} \in \mathrm{Aut}(R^n \oplus Q^n)$, represented by a matrix in $GL_{nr}(S)$. As before, this gives a map $i^{\#}\colon GL_n(R) \to GL_{nr}(S)$, which induces a well-defined morphism $i^*\colon K_1(R) \to K_1(S)$ (by the same considerations as above).

Now if we take an element $x \in K_1(S)$ and calculate $i^* i_*(x)$, then it is the same as $[R] \cdot x$, where $[R]$ is viewed as an element of $K_0(S)$ and the action is defined above.

$$\begin{tikzcd}
K_1(S) \arrow[r, "i_*"] \arrow[dr, dashed, "{[R]}"'] & K_1(R) \arrow[d, "i^*"] \\
& K_1(S)
\end{tikzcd}$$

This is really immediate from the definitions, yet it will be useful below.

Remark 1.2.7. Compare to the transfer in group cohomology [Bro94, §III.9, III.10].

Proof of $K_1(\mathcal{O}_F) \cong \mathcal{O}_F^\times$

Our goal is to show that $SK_1(\mathcal{O}_F) = 0$ for a number field F, which means that $SL(\mathcal{O}_F)$ is generated by elementary matrices. This is a very important and nontrivial result and it seems that there is no slick proof of it. A great article [BMS67] gives the solution. The exposition below is based on [Mil71, §16].

First observe that it is enough to consider SL_2:

Proposition 1.2.8 (Bass)**.** *Let $\mathfrak{A}$ be a Dedekind domain. Then every matrix in $SL(\mathfrak{A})$ can be reduced by elementary row and column operations to a matrix in $SL_2(\mathfrak{A})$. That is, $SL_2(\mathfrak{A})$ surjects to $SL(\mathfrak{A})/E(\mathfrak{A}) \overset{\text{def}}{=} SK_1(\mathfrak{A})$.*

Proof. We take a matrix $M \in SL_n(\mathfrak{A})$ for $n \geqslant 3$ and proceed by induction on n. We need to show that modulo elementary operations, M comes from $SL_{n-1}(\mathfrak{A})$. Consider the last row of the matrix:

$$M = \begin{pmatrix} * & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & * \\ x_1 & x_2 & \cdots & x_n \end{pmatrix} \in SL_n(\mathfrak{A}).$$

One should have $x_1\mathfrak{A} + \cdots + x_n\mathfrak{A} = \mathfrak{A}$, since the coefficients are relatively prime.
Case 1: If $x_1, x_2, \ldots, x_{n-1}$ generate the whole ring $\mathfrak{A}$, then we can replace x_n by 1 by elementary column operations, and then by elementary operations replace M with a matrix

$$\begin{pmatrix} M' & 0 \\ 0 & 1 \end{pmatrix}, \quad M' \in SL_{n-1}(\mathfrak{A}).$$

Case 2: If $x_2 = 0$, then by elementary column operations one can replace x_2 with 1 and proceed as in Case 1.
Case 3: If $x_2 \neq 0$, then there are finitely many maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_s$ containing $x_2, \ldots, x_{n-1}$ (*and here we use the hypothesis that $\mathfrak{A}$ is a Dedekind domain*). Assume that the first r ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ contain x_1 and the remaining ideals $\mathfrak{m}_{r+1}, \ldots, \mathfrak{m}_s$ do not contain x_1. Choose an element $y \in \mathfrak{A}$ such that

$$\begin{aligned} y &\equiv 1 \pmod{\mathfrak{m}_1, \dots, \mathfrak{m}_r}, \\ y &\equiv 0 \pmod{\mathfrak{m}_{r+1}, \dots, \mathfrak{m}_s}. \end{aligned}$$

Adding the last column multiplied by y to the first column replaces x_1 with $x_1 + x_n\, y$. Now

$$x_1 + x_n\, y,\ x_2,\ \dots,\ x_{n-1}$$

generate the whole $\mathfrak{A}$, and we can proceed as in the first case. ■

The next step is to develop some calculus for SL_2. Observe that a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(R)$ modulo $E_2(R)$ is uniquely defined by coefficients a and b. Indeed, if we have another matrix $\begin{pmatrix} a & b \\ c' & d' \end{pmatrix} \in SL_2(R)$, then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ c\,d' - c'\,d & 1 \end{pmatrix}}_{\in E_2(R)} \cdot \begin{pmatrix} a & b \\ c' & d' \end{pmatrix}.$$

If we have two elements a and b such that $a\,R + b\,R = R$, then there exist $c, d \in R$ with $a\,d - b\,c = 1$, and hence a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(R)$. This suggests the following definition:

Proposition-definition 1.2.9. *An element of $SK_1(R)$ given by a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(R)$, viewed modulo $E_2(R)$, is called a* ***Mennicke symbol*** *and denoted by $\left[\begin{smallmatrix} b \\ a \end{smallmatrix}\right]$.*

First we collect some properties:

Proposition 1.2.10. *For any $a, b \in R$ such that $aR + bR = R$ one has the following identities in $SK_1(R)$:*

1. $\left[\begin{smallmatrix} b \\ a \end{smallmatrix}\right] = \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$.
2. $\left[\begin{smallmatrix} b \\ a \end{smallmatrix}\right] = \left[\begin{smallmatrix} b+\lambda a \\ a \end{smallmatrix}\right]$ *and* $\left[\begin{smallmatrix} b \\ a \end{smallmatrix}\right] = \left[\begin{smallmatrix} b \\ a+\lambda b \end{smallmatrix}\right]$ *for all $\lambda \in R$.*
3. $\left[\begin{smallmatrix} b \\ a \end{smallmatrix}\right] \left[\begin{smallmatrix} b' \\ a \end{smallmatrix}\right] = \left[\begin{smallmatrix} b\,b' \\ a \end{smallmatrix}\right]$.
4. $\left[\begin{smallmatrix} b \\ a \end{smallmatrix}\right] = 1$ *if a or b is invertible.*

Proof. This is a calculation with matrices [Mil71, Lemma 13.2], one just routinely checks the identities modulo $E_2(R)$. ■

Now we know that Mennicke symbols generate $SK_1(\mathfrak{A})$ for a Dedekind domain $\mathfrak{A}$. The group $SL_2(\mathcal{O}_F)$ is finitely generated—it is a general property of arithmetic groups, important in the subsequent chapters—hence we know that $SK_1(\mathcal{O}_F)$ is at least finitely generated by Mennicke symbols.

Example 1.2.11. For instance [Ser73, §VII.1], the group $SL_2(\mathbb{Z})$ is generated by two elements

$$T \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S \stackrel{\text{def}}{=} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

S has order 4 and ST has order 6, and in fact $SL_2(\mathbb{Z})$ it is the "amalgamated free product" $C_4 *_{C_2} C_6$—see [Alp93] for an elementary proof.

$$\begin{tikzcd} SL_2(\mathbb{Z}) & \langle ST \rangle \arrow[l] \\ \langle S \rangle \arrow[u] & \{\pm I\} \arrow[l] \arrow[u] \arrow[ul, phantom, "\ulcorner"] \end{tikzcd}$$

▲

Now observe that for any symbol $\begin{bmatrix} b \\ a \end{bmatrix}$ we can find an integer $r > 0$ such that $b^r \equiv 1 \pmod{a}$—here we use that $\mathcal{O}_F$ is a number field!—and then by the listed properties

$$\begin{bmatrix} b \\ a \end{bmatrix}^r = \begin{bmatrix} b^r \\ a \end{bmatrix} = \begin{bmatrix} 1 + \lambda a \\ a \end{bmatrix} = \begin{bmatrix} 1 \\ a \end{bmatrix} = 1.$$

So $SK_1(\mathcal{O}_F)$ is a finitely generated torsion group, hence it is finite. We need to invoke some number theory to show that in fact $SL_1(\mathcal{O}_F)$ is trivial. Let k be a local field containing n-th roots of unity. We denote their group by $\boldsymbol{\mu}_n$. For $b \in k^\times$ consider an abelian extension $k(\sqrt[n]{b})/k$. Then the "norm residue symbol" map (cf. [Neu99, Chapter IV + V]) has form

$$\begin{aligned} k^\times &\to \operatorname{Gal}(k(\sqrt[n]{b})/k), \\ a &\mapsto (a, k(\sqrt[n]{b})/k). \end{aligned}$$

And **Hilbert symbol** [Neu99, §V.3] is a nondegenerate bilinear form

$$\left(\frac{\cdot,\cdot}{\mathfrak{p}}\right) : k^\times/(k^\times)^n \times k^\times/(k^\times)^n \to \boldsymbol{\mu}_n,$$

which is given by

$$\left(\frac{a,b}{\mathfrak{p}}\right) = \frac{(a, k(\sqrt[n]{b})/k) \cdot \sqrt[n]{b}}{\sqrt[n]{b}}.$$

Here $\mathfrak{p} = \{a \in k \mid v(a) > 0\}$ is the maximal ideal of k, and n is implicit in the notation "$\left(\frac{\cdot,\cdot}{\mathfrak{p}}\right)$".

Fact 1.2.12. *Hilbert symbol has the following properties [Neu99, Proposition V.3.2]:*

1) $\left(\frac{aa',b}{\mathfrak{p}}\right) = \left(\frac{a,b}{\mathfrak{p}}\right)\left(\frac{a',b}{\mathfrak{p}}\right)$ *and* $\left(\frac{a,bb'}{\mathfrak{p}}\right) = \left(\frac{a,b}{\mathfrak{p}}\right)\left(\frac{a,b'}{\mathfrak{p}}\right)$.

2) $\left(\frac{a,b}{\mathfrak{p}}\right) = 1$ *if and only if a is a norm from the extension* $k(\sqrt[n]{b})/k$.

3) $\left(\frac{a,b}{\mathfrak{p}}\right) = \left(\frac{b,a}{\mathfrak{p}}\right)^{-1}$.

4) $\left(\frac{a,1-a}{\mathfrak{p}}\right) = 1$ *(assuming $a \neq 1$) and* $\left(\frac{a,-a}{\mathfrak{p}}\right) = 1$.

5) *If* $\left(\frac{a,b}{\mathfrak{p}}\right) = 1$ *for all $b \in k^\times$, then $a \in (k^\times)^n$.*

If F is a number field having n-th roots of unity, then for each place $\mathfrak{p} \in M_F$ (including infinite) we can consider the completion $F_\mathfrak{p}$ and the corresponding Hilbert symbol

$$\left(\frac{\cdot,\cdot}{\mathfrak{p}}\right) : F_\mathfrak{p}^\times/(F_\mathfrak{p}^\times)^n \times F_\mathfrak{p}^\times/(F_\mathfrak{p}^\times)^n \to \boldsymbol{\mu}_n.$$

All completions are put together by the **product formula** [Neu99, Theorem VI.8.1]:

$$\prod_{\mathfrak{p}\in M_F} \left(\frac{a,b}{\mathfrak{p}}\right) = 1 \quad \text{for any } a,b \in F^\times.$$

Remark 1.2.13. For $F = \mathbb{Q}$ and $n = 2$ these are the classic Hilbert symbols [Ser73, Chapter III]

$$(\cdot,\cdot)_p\colon \mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2 \times \mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2 \to \{\pm 1\}$$

that are related to the properties of quadratic forms over $\mathbb{Q}$ [Ser73, Chapter IV]. In this case the product formula gives the quadratic reciprocity law [Neu99, VI.8.4].

The case with roots of unity. Let us assume that $\mathcal{O}_F$ has p-th roots of unity for a prime p, so that we can consider Hilbert symbols $\left(\frac{a,b}{\mathfrak{q}}\right) \in \boldsymbol{\mu}_p$. Later on we will see that this assumption is harmless and one can always pass to a field extension $F(\zeta_p)/F$. We want to show that $SK_1(\mathcal{O}_F)$ has no p-torsion. For this it is enough to prove that every Mennicke symbol $\left[{b \atop a}\right] \in SK_1(\mathcal{O}_F)$ has a p-th root, i.e. $\left[{b \atop a}\right] = \left[{b' \atop a'}\right]^p$ for some symbol $\left[{b' \atop a'}\right]$.

By Chinese remainder theorem we can find a_1 such that

$$\begin{aligned} a_1 &\equiv a \pmod{b\mathcal{O}_F}, \\ a_1 &\equiv 1 \pmod{\mathfrak{p}} \quad \text{for } \mathfrak{p} \mid p,\ \mathfrak{p} \nmid b. \end{aligned} \tag{1.4}$$

So we have $\left[{b \atop a}\right] = \left[{b \atop a_1}\right]$, where a_1 is relatively prime to p.

Claim. *Let $\mathfrak{q} \mid p$ be a prime lying over p. Then there exist u_0, w_0 in the $\mathfrak{q}$-adic completion of $\mathcal{O}_F$, such that $\left(\frac{u_0,w_0}{\mathfrak{q}}\right) \neq 1$.*

Proof. Let U be the group of units of the $\mathfrak{q}$-adic completion of $\mathcal{O}_F$. This group contains p-th roots of unity and the residue field is of characteristic p, hence $[U : U^p] \geqslant p^2$ (cf. [Lan94, §II.3, Proposition 6]). Let π be a uniformizer. Consider the subgroup

$$U_0 \stackrel{\text{def}}{=} \{u \in U \mid \left(\frac{u,\pi}{\mathfrak{q}}\right) = 1\}.$$

It has index $[U : U_0] \leqslant p$, hence there exists $u_0 \in U_0$ such that u_0 is not a p-th root of unity in the completion $F_\mathfrak{q}$ and $\left(\frac{u_0,y}{\mathfrak{q}}\right) \neq 1$ for some $y = \pi^i\, w_0$—see above property 5) of Hilbert symbols. Now $\left(\frac{u_0,w_0}{\mathfrak{q}}\right) \neq 1$. ∎

By Chinese remainder theorem we pick b_2 such that

$$b_2 \equiv b \pmod{a_1\mathcal{O}_F}, \tag{1.5}$$

$$b_2 \equiv w_0 \pmod{\mathfrak{q}^N}, \tag{1.6}$$

$$b_2 \equiv 1 \pmod{\mathfrak{p}^N} \quad \text{for } \mathfrak{p} \mid p,\ \mathfrak{p} \neq \mathfrak{q}. \tag{1.7}$$

Here N is an integer large enough so that $\frac{b_2}{w_0}$ has a p-th root in the completion $F_\mathfrak{q}$, and b_2 has a p-th root in $F_\mathfrak{p}$ for each $\mathfrak{p} \mid p$, $\mathfrak{p} \neq \mathfrak{q}$.

Claim. *Consider an "arithmetic progression" consisting of all b_2 satisfying (1.5), (1.6), (1.7). Then it contains a "prime", i.e. a number b_2 such that $b_2\,\mathcal{O}_F$ is a prime ideal. Further, this b_2 can be chosen to be positive in every real completion of F.*

This is essentially a generalized version of the Dirichlet's theorem on arithmetic progressions which is deduced from the Chebotarëv density theorem—cf. [Neu99, §VII.13].

Now by (1.6) holds (keep in mind that $\left(\frac{\cdot,\cdot}{\mathfrak{q}}\right)$ is defined on $F_{\mathfrak{q}}^\times/(F_{\mathfrak{q}}^\times)^p$, modulo p-th roots)

$$\left(\frac{u_0, b_2}{\mathfrak{q}}\right) = \left(\frac{u_0, w_0}{\mathfrak{q}}\right) \neq 1.$$

Hence for some power $u \overset{\text{def}}{=} u_0^i$ of u_0, one has

$$\left(\frac{a_1, b_2}{b_2\mathcal{O}_F}\right) \cdot \left(\frac{u, b_2}{\mathfrak{q}}\right) = 1. \tag{1.8}$$

Choose a_3 to be a "prime" (i.e. such that $a_3\mathcal{O}_F$ is prime) satisfying the congruences

$$\begin{aligned} a_3 &\equiv a_1 \pmod{b_2\mathcal{O}_F}, \\ a_3 &\equiv u \pmod{\mathfrak{q}^N}, \end{aligned} \tag{1.9}$$

with N as above. Now

$$\begin{bmatrix} b_2 \\ a_3 \end{bmatrix} \overset{(1.9)}{=} \begin{bmatrix} b_2 \\ a_1 \end{bmatrix} \overset{(1.5)}{=} \begin{bmatrix} b \\ a_1 \end{bmatrix} \overset{(1.4)}{=} \begin{bmatrix} b \\ a \end{bmatrix}.$$

For a_3 and b_2 consider the product formula:

$$\prod_{\mathfrak{p}\in M_F} \left(\frac{a_3, b_2}{\mathfrak{p}}\right) = 1.$$

- By the choice of b_2 one has $\left(\frac{a_3,b_2}{\mathfrak{p}}\right) = 1$ for $\mathfrak{p} \mid p$ and $\mathfrak{p} \neq \mathfrak{q}$, and also for infinite places.
- If $\mathfrak{r}$ is a finite prime such that $\mathfrak{r} \nmid p$, then the symbol $\left(\frac{a_3,b_2}{\mathfrak{r}}\right)$ is "tame" and $\left(\frac{a_3,b_2}{\mathfrak{r}}\right) = 1$, unless $\mathfrak{r} \mid a_3$ or $\mathfrak{r} \mid b_2$ (see [Neu99, §V.3] for calculation of tame symbols).

So from the product formula remains

$$\left(\frac{a_3, b_2}{a_3\mathcal{O}_F}\right) \cdot \left(\frac{a_3, b_2}{b_2\mathcal{O}_F}\right) \cdot \left(\frac{a_3, b_2}{\mathfrak{q}}\right) = 1.$$

For the second two symbols in this product

$$\left(\frac{a_3, b_2}{b_2\mathcal{O}_F}\right) = \left(\frac{a_1, b_2}{b_2\mathcal{O}_F}\right), \quad \left(\frac{a_3, b_2}{\mathfrak{q}}\right) = \left(\frac{u, b_2}{\mathfrak{q}}\right),$$

and using (1.8) we conclude $\left(\frac{a_3,b_2}{a_3\mathcal{O}_F}\right) = 1$, which means that b_2 is a p-th power modulo a_3, so that

$$b_2 \equiv x^p \pmod{a_3\mathcal{O}_F} \quad \text{for some } x,$$

and for Mennicke symbols it means

$$\begin{bmatrix} b \\ a \end{bmatrix} = \begin{bmatrix} b_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} x^p \\ a_3 \end{bmatrix} = \begin{bmatrix} x \\ a_3 \end{bmatrix}^p,$$

and $\left[\begin{smallmatrix} b \\ a \end{smallmatrix}\right]$ is a p-th root. This shows finally that $SK_1(\mathcal{O}_F)$ has no p-torsion whenever F contains p-th roots of unity.

The general case. To finish the proof, assume now that F has no p-th roots of unity. Then consider the extension $F(\zeta_p)/F$:

$$\begin{tikzcd} F(\zeta_p) \arrow[d, dash, "d"] & \mathcal{O}_{F(\zeta_p)} \arrow[l, hook'] \arrow[d, dash, "d"] \\ F \arrow[d, dash] & \mathcal{O}_F \arrow[l, hook'] \arrow[d, dash] \\ \mathbb{Q} & \mathbb{Z} \arrow[l, hook'] \end{tikzcd}$$

The inclusion $\mathcal{O}_F \hookrightarrow \mathcal{O}_{F(\zeta_p)}$ induces a morphism i_* and transfer map i^*, and their composition $i^* \circ i_*$ is the action of $[\mathcal{O}_{F(\zeta_p)}] \in K_0(\mathcal{O}_F)$:

$$\begin{tikzcd} K_1(\mathcal{O}_F) \arrow[r, "i_*"] \arrow[dr, dashed, "{[\mathcal{O}_{F(\zeta_p)}]}"'] & K_1(\mathcal{O}_{F(\zeta_p)}) \arrow[d, "i^*"] \\ & K_1(\mathcal{O}_F) \end{tikzcd}$$

Note that under the isomorphism $K_0(\mathcal{O}_{F(\zeta_p)}) \cong \mathbb{Z} \oplus \widetilde{K}_0(\mathcal{O}_{F(\zeta_p)})$ one has $i^*[\mathcal{O}_{F(\zeta_p)}] = d + \gamma$, where $d = [F(\zeta_p) : F] = [\mathcal{O}_{F(\zeta_p)} : \mathcal{O}_F]$. Let $\alpha \in \mathcal{O}_{F(\zeta_p)}$ be an element of order p. Then $i_*(\alpha)$ has order p in $SK_1(\mathcal{O}_{F(\zeta_p)})$, so

$$i^* i_*(\alpha) = (d + \gamma) \cdot \alpha = 0.$$

Recall that multiplication in $\widetilde{K}_0(\mathcal{O}_F)$ is trivial, thus $\gamma^2 = 0$, and

$$d^2 \cdot \alpha = (d - \gamma)\,(d + \gamma) \cdot \alpha = 0.$$

However, p does not divide d, which means that $\alpha = 0$. This completes the proof that $SK_1(\mathcal{O}_F)$ vanishes and $K_1(\mathcal{O}_F) \cong \mathcal{O}_F^\times$. ■

Structure of $K_1(\mathcal{O}_F)$

Now knowing that $K_1(\mathcal{O}_F) \cong \mathcal{O}_F^\times$, we recall what this group is.

Theorem 1.2.14 (Dirichlet unit theorem). *The group $K_1(\mathcal{O}_F) \cong \mathcal{O}_F^\times$ is finitely generated; precisely,*

$$K_1(\mathcal{O}_F) \cong \mathcal{O}_F^\times \cong \mathbb{Z}^{r_1+r_2-1} \oplus \boldsymbol{\mu}_F,$$

where

- *r_1 is the number of real embeddings $\sigma_1, \dots, \sigma_{r_1} \colon F \hookrightarrow \mathbb{R}$,*
- *r_2 is the number of conjugate pairs of complex embeddings $\sigma_{r_1+1}, \dots, \sigma_{r_2}, \overline{\sigma}_{r_1+1}, \dots, \overline{\sigma}_{r_2} \colon F \hookrightarrow \mathbb{C}$.*
- *$\boldsymbol{\mu}_F$ is the group of roots of unity in F,*

We just recall briefly that calculation of the rank starts with the **logarithmic embedding** (which is clearly a homomorphism from the multiplicative group $F^\times$ to the additive group):

$$\begin{aligned} \lambda \colon F^\times &\to \mathbb{R}^{r_1+r_2}, \\ \alpha &\mapsto (\lambda_1(\alpha), \dots, \lambda_{r_1+r_2}(\alpha)) \\ &\overset{\text{def}}{=} (\log|\sigma_1(\alpha)|, \dots, \log|\sigma_{r_1}(\alpha)|, 2\log|\sigma_{r_1+1}(\alpha)|, \dots, 2\log|\sigma_{r_1+r_2}(\alpha)|). \end{aligned}$$

For algebraic integers $\alpha \in \mathcal{O}_F^\times$ one has $N_{F/\mathbb{Q}}(\alpha) = \pm 1$, so $\sum \lambda_i(\alpha) = \log |N_{F/\mathbb{Q}}(\alpha)| = 0$, which means that the image of $\mathcal{O}_F^\times$ under λ lies in the hyperplane of codimension one

$$H \stackrel{\text{def}}{=} \{(x_1, \dots, x_{r_1+r_2}) \in \mathbb{R}^{r_1+r_2} \mid \sum x_i = 0\}.$$

It is easy to see that the image of $\mathcal{O}_F^\times$ under λ is a discrete subgroup in H, i.e. a **lattice** $\Lambda_F \stackrel{\text{def}}{=} \lambda(\mathcal{O}_F^\times)$. Indeed, if we consider a ball $B \subset H$ and the points $\lambda(\alpha) = (|\sigma_1(\alpha)|, \dots, |\sigma_{r_1+r_2}(\alpha)|) \in B$ for $\alpha \in \mathcal{O}_F^\times$, then we have a bound on $|\sigma_i(\alpha)|$, and hence some bound on the coefficients of the minimal polynomial of α (which are symmetric functions in $\sigma_i(\alpha)$). So in each ball there are finitely many points $\lambda(\alpha)$ coming from $\alpha \in \mathcal{O}_F^\times$.

The kernel of λ clearly consists of some roots of unity $\boldsymbol{\mu}_F$, since it is a subgroup of the cyclic group $F^\times$. Moreover, every root of unity is mapped to 0 because Λ_F is a free group.

Now the really hard part of the theorem is to show that the lattice $\Lambda_F \subset H$ is of the full rank $r_1 + r_2 - 1$ (see e.g. [Neu99, Theorem I.7.3], or [Jan96, p. 74–77]).

This of course can be found in any algebraic number theory textbook (e.g. [Neu99, §I.5–I.7]), and it would be embarrassing to discuss the full proof. We recall it just to note that for the higher K-groups $K_2(\mathcal{O}_F), K_3(\mathcal{O}_F), K_4(\mathcal{O}_F), \dots$ it is also *relatively* easy to show that they are finitely generated (which is made in a rather short note [Qui73a]), but calculation of their ranks is quite involved (which is the result of [Bor74]). However, these ranks also depend only on r_1 and r_2, in a simple and beautiful way.

Further we recall the **class number formula** giving the residue of zeta function $\zeta_F(s)$ at the simple pole $s = 1$ [Neu99, VII.5.11]:

$$\lim_{s\to 1} (s-1)\, \zeta_F(s) = \frac{2^{r_1}\, (2\,\pi)^{r_2}\, h_F}{\omega_F \cdot \sqrt{\Delta_F}}\, R_F,$$

where $h_F \stackrel{\text{def}}{=} \# \operatorname{Cl}(F) = \# K_0(\mathcal{O}_F)_{\text{tors}}$ is the class number, and $\omega_F \stackrel{\text{def}}{=} \# \boldsymbol{\mu}_F = \# K_1(\mathcal{O}_F)_{\text{tors}}$ is the number of roots of unity. Here R_F is the **regulator**, which is related to the volume of the lattice described above by $\operatorname{Vol} \Lambda_F = R_F \sqrt{r_1 + r_2}$.

Basically, this formula involves torsion in K_0 and K_1, and suggests that for higher K-groups one can also define regulators and get similar expressions. Using the functional equation, rewrite the class number formula for the zero at $s = 0$:

$$\lim_{s\to 0} s^{-(r_1+r_2-1)}\, \zeta_F(s) = -\frac{\# K_0(\mathcal{O}_F)_{\text{tors}}}{\# K_1(\mathcal{O}_F)_{\text{tors}}}\, R_F.$$

The **Lichtenbaum's conjecture** [Lic73] reads for $n > 0$

$$\lim_{s\to n} (n-s)^{-\mu_n}\, \zeta_F(-s) = \pm \frac{\# K_{2n}(\mathcal{O}_F)}{\# K_{2n+1}(\mathcal{O}_F)_{\text{tors}}}\, R_{F,n} \quad \text{up to a power of two},$$

where μ_n is the multiplicity of zero $\zeta_F(-n)$ (see the preface), and $R_{F,n}$ is the so-called **Borel's regulator**. The group $K_{2n}(\mathcal{O}_F)$ is finite for $n > 0$, which will be established in the subsequent chapters.

Example 1.2.15. If $F = \mathbb{Q}$, then $R_{n,\mathbb{Q}} = 1$, and for $\zeta(-1)$ we get a formula

$$\zeta(-1) = \pm \frac{\# K_2(\mathbb{Z})}{\# K_3(\mathbb{Z})_{\text{tors}}} \quad \text{up to a power of two.}$$

In fact $K_2(\mathbb{Z}) \cong \mathbb{Z}/2$ (see below) and $K_3(\mathbb{Z}) \cong \mathbb{Z}/48$, so up to a power of two, this indeed coincides with the right value $\zeta(-1) = -B_2/2 = -1/12$. ▲

This was a little digression related to the class number formula; in this text we are interested only in ranks of K-groups. We refer to [BG02], [Gon05], and [Ram89] for further discussion of regulators.

1.3 A few words about K_2

Recall that the group $E(R)$ is by definition generated by elementary matrices. They satisfy relations (1.1), (1.2), (1.3), however, depending on R, there can be other less obvious relations, and the group of elementary matrices $E(R)$ is far from being "elementary". This suggests the following

Definition 1.3.1. The **Steinberg group** $St_n(R)$ is the group generated by formal symbols $x_{ij}^{(n)}(a)$ for $1 \leqslant i, j \leqslant n$, $i \neq j$, and $a \in R$, modulo relations

$$x_{ij}^{(n)}(a)\, x_{ij}^{(n)}(b) = x_{ij}^{(n)}(a+b), \tag{1.10}$$

$$[x_{ij}^{(n)}(a), x_{jk}^{(n)}(b)] = x_{ik}^{(n)}(a\, b) \quad \text{for } i \neq k, \tag{1.11}$$

$$[x_{ij}^{(n)}(a), x_{k\ell}^{(n)}(b)] = 1 \quad \text{for } j \neq k, i \neq \ell. \tag{1.12}$$

(These are the same as (1.1), (1.2), (1.3).) The **Steinberg group** $St(R)$ is the limit

$$\varinjlim_n St_n(R),$$

given by the obvious maps $St_n(R) \to St_{n+1}(R)$. (These are not necessarily injections though!)

Obviously, St is a functor from the category of rings to the category of groups.

By the definition, there are surjections $St_n(R) \twoheadrightarrow E_n(R)$ given by $x_{ij}^{(n)}(a) \mapsto e_{ij}^{(n)}(a)$. Passing to a limit gives a surjection $St(R) \twoheadrightarrow E(R)$.

Definition 1.3.2. The group K_2 of a ring R is given by

$$K_2(R) \stackrel{\text{def}}{=} \ker(St(R) \twoheadrightarrow E(R)).$$

We do not discuss in details K_2 and its properties, in particular its rôle in arithmetic (cf. [BT73] and [Tat76]). A great reference is [Mil71], [Mag02, Part V], and the chapter on K_2 in the textbook [Ros94].

Perfect groups

Perfect groups play a major rôle in everything what follows, so we record here some basic facts about them.

Definition 1.3.3. A group P is called **perfect** if $[P,P] = P$. In other words, if

$$P/[P,P] = P^{ab} = H^1(P,\mathbb{Z}) = 0.$$

Here are some immediate properties of perfect groups:

Proposition 1.3.4. *0) If $P \leqslant G$ is a perfect subgroup, then it is contained in every subgroup of the derived series*

$$G \supseteq [G,G] \supseteq [[G,G],[G,G]] \supseteq \cdots$$

1) *The image of a perfect group under a homomorphism $f\colon P \to G$ is also a perfect group.*

2) *Any group G has a maximal perfect subgroup, the* ***perfect radical*** *$\mathfrak{P}G$, which is a characteristic subgroup of G.*

3) *If $\phi\colon G \to H$ is a homomorphism, then $\phi(\mathfrak{P}G) \leqslant \mathfrak{P}H$.*

4) *If $\phi\colon G \to H$ is a homomorphism and $\mathfrak{P}H = 1$, then $\mathfrak{P}G \leqslant \ker\phi$.*

Proof. 0) is clear from the definition.

1) is the fact that homomorphisms send commutators to commutators.

For 2) note that if P_1 and P_2 are two perfect subgroups of G, then the subgroup generated by P_1 and P_2 is perfect as well. Hence there is the maximal perfect subgroup $\mathfrak{P}G$. By 1) any automorphism $G \to G$ should send $\mathfrak{P}G$ within itself, hence $\mathfrak{P}G$ is a characteristic subgroup.

3) is a particular case of 1), and 3) implies 4). ■

Example 1.3.5. Recall that for $GL(R)$ the derived series is given by

$$[GL(R), GL(R)] = E(R), \quad [E(R), E(R)] = E(R),$$

therefore $E(R)$ is the maximal perfect subgroup of $GL(R)$. Similarly, the relation (1.11) tells that $[St(R), St(R)] = St(R)$, so the Steinberg group is also perfect. Note that $E(R)$ is the image of $St(R)$ under the surjection $St(R) \twoheadrightarrow E(R)$. ▲

Kervaire's theorem

Let us recall briefly the theory of central extensions. We will freely use some basic group cohomology—cf. [Bro94] and [Wei94, Chapter 6].

Definition 1.3.6. An **extension** of a group G by an abelian group A is a short exact sequence

$$0 \to A \to X \to G \to 1$$

An extension such that A lies in the center of X is called a **central extension**. A morphism of two extensions of G is a homomorphism $X \to Y$ giving a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] & X \arrow[r] \arrow[d] & G \arrow[r] \arrow[d, equal] & 1 \\
0 \arrow[r] & B \arrow[r] & Y \arrow[r] & G \arrow[r] & 1
\end{tikzcd}$$

An extension $0 \to A \to X \to G \to 1$ is called a **universal central extension** if for every other extension $0 \to B \to Y \to G \to 1$ there exists a unique morphism as above.

A universal central extension of G is clearly unique up to an isomorphism, since it is an initial object in the category of central extensions of G. Here is a criterion of existence:

Theorem 1.3.7. *A group G has a universal central extension if and only if G is perfect. Precisely, consider a presentation $G = F/R$ where F is a free group and $R \lhd F$ its normal subgroup:*

$$1 \to R \to F \to G \to 1$$

Then the universal central extension is given by

$$0 \to H_2(G, \mathbb{Z}) \to \frac{[F, F]}{[F, R]} \to G \to 1$$

Theorem 1.3.8. *A central extension*

$$0 \to A \to X \xrightarrow{p} G \to 1$$

is universal if and only if X is a perfect group and every central extension of X is ***trivial****, i.e. of the form*

$$0 \to B \to X \times B \to X \to 1$$

The latter two theorems are really standard. We refer to [Wei94, §6.9] for proofs.

Concerning K-theory, one has the following remarkable result:

Theorem 1.3.9 (Kervaire). *The group extension from the definition of K_2*

$$0 \to K_2(R) \to St(R) \to E(R) \to 1 \tag{1.13}$$

is a universal central extension. In particular, $K_2(R) \cong H_2(E(R), \mathbb{Z})$.

This was proved by Kervaire in [Ker70]. To establish this, first one should verify that the group extension (1.13) is central. More precisely, we have

Claim. *$K_2(R)$ is the center of $St(R)$.*

Proof. Take an element $y \in St(R)$. If it lies in the center of $St(R)$, then its image $\phi(y)$ under the map $\phi\colon St(R) \to E(R)$ should lie in the center of $E(R)$. However, we know that an $n \times n$ matrix commuting with all $n \times n$ elementary matrices should have form $\begin{pmatrix} a & & \\ & \ddots & \\ & & a \end{pmatrix}$ for some $a \in R$. This means that the center of $E(R)$ is trivial, represented by the identity matrix $\begin{pmatrix} 1 & & \\ & 1 & \\ & & \ddots \end{pmatrix}$, and therefore $Z(St(R)) \subseteq \ker \phi \overset{\text{def}}{=} K_2(R)$.

Conversely, if we start with an element $y \in St(R)$ such that $\phi(y) = 1$, we would like to see that y commutes with all the generators of $St(R)$. The element y itself is a word of generators $x_{ij}^{(n)}(a)$ for n big enough. We can take n in such a way that $i, j < n$. Now consider the subgroup P_n generated by elements $x_{1n}^{(n)}(a), x_{2n}^{(n)}(a), \ldots, x_{n-1,n}^{(n)}(a)$ for $a \in R$. This is a commutative group thanks to the relation (1.12). Each element of P_n can be written uniquely as $x_{1n}^{(n)}(a_1), x_{2n}^{(n)}(a_2), \ldots, x_{n-1,n}^{(n)}(a_{n-1})$. The image of this group in $E(R)$ is the group of matrices

$$\begin{pmatrix} 1 & & & & a_1 \\ & 1 & & & a_2 \\ & & \ddots & & \vdots \\ & & & 1 & a_{n-1} \\ & & & & 1 \end{pmatrix}$$

For $i, j < n$ we have

$$x_{ij}^{(n)}(a)\, x_{kn}^{(n)}(b)\, x_{ij}^{(n)}(-a) = \begin{cases} x_{kn}^{(n)}(b), & j \neq k, \\ x_{in}^{(n)}(ab)\, x_{kn}^{(n)}(b), & j = k. \end{cases}$$

This shows that

$$x_{ij}^{(n)}(a)\, P_n\, x_{ij}^{(n)}(a)^{-1} = x_{ij}^{(n)}(a)\, P_n\, x_{ij}^{(n)}(-a) \subset P_n \quad \text{for } i, j < n.$$

Since y is a product of $x_{ij}^{(n)}(a)$ for $i, j < n$, we have $y\, P_n\, y^{-1} \subset P_n$.

By assumption, $\phi(y) = 1$, hence for all $p \in P_n$

$$\phi(y\, p\, y^{-1}) = \phi(y)\, \phi(p)\, \phi(y^{-1}) = \phi(p),$$

and $y\, p\, y^{-1} = p$.

Now y commutes with every $x^{(n)}_{kn}(a)$ with $k < n$. By a similar argument one sees that y commutes with every $x^{(n)}_{n\ell}(a)$ with $\ell < n$. So y commutes with the commutator

$$[x^{(n)}_{kn}(a), x^{(n)}_{n\ell}(1)] = x^{(n)}_{k\ell}(a) \quad \text{where } k \neq \ell \text{ and } k, \ell < n.$$

Since n can be chosen to be arbitrarily large, this means that y commutes with all the generators of $St(R)$. ■

To finish the proof of theorem 1.3.9, we should show that the extension (1.13) is universal. According to theorem 1.3.8, this is equivalent to $St(R)$ being perfect and having only split central extensions.

Claim. *Every central extension*

$$0 \to A \to X \xrightarrow{p} St(R) \to 1$$

splits.

Proof idea. We need to find a section

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] & X \arrow[r, shift left, "p"] & St(R) \arrow[l, shift left, dashed, "s"] \arrow[r] & 1 \end{tikzcd}$$

We send an element $x_{ij}(a) \in St(R)$ to some element $s_{ij}(a) \in X$. We should choose these $s_{ij}(a)$ in such a way that they satisfy the Steinberg relations (1.10), (1.11), (1.12), so that this is a homomorphism. Further, we should take $s_{ij}(a) \in p^{-1}(x_{ij}(a))$, so that it is a section.

Since the kernel of p lies in the center of X, for any two elements $x, y \in St(R)$ it makes sense to take the commutator $[p^{-1}(x), p^{-1}(y)]$ as a well-defined element of X. One can observe [Mil71, p.49] from the commutator identities that if i, j, k, k' are distinct indices, then

$$[p^{-1}x_{ik}(a),\, p^{-1}x_{kj}(b)] = [p^{-1}x_{ik'}(1),\, p^{-1}x_{k'j}(a\,b)].$$

This shows that the map

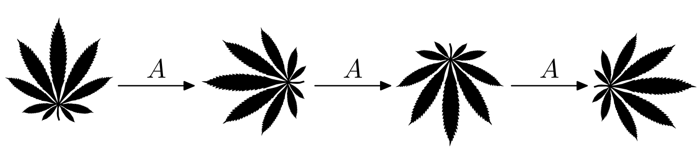
$$x_{ij}(a) \mapsto s_{ij}(a) \stackrel{\text{def}}{=} [p^{-1}x_{ik}(1),\, p^{-1}x_{kj}(a)] \quad \text{for some } k \neq i, k \neq j$$

is well-defined and does not depend on k. We see that $p(s_{ij}(a)) = [x_{ik}(1), x_{kj}(a)] = x_{ij}(a)$ by the Steinberg identity (1.11). Moreover, one can check that $s_{ij}(a)$ satisfy (1.10), (1.11), (1.12). ■

Example: $K_2(\mathbb{Z})$

To get a feeling of K_2, let us look at $K_2(\mathbb{Z})$ [Mil71, §10]. It is the kernel of $St(\mathbb{Z}) \twoheadrightarrow E(\mathbb{Z})$, where $St(\mathbb{Z})$ captures the "obvious" commutator relations (1.1), (1.2), (1.3) in $E(R)$. So $K_2(\mathbb{Z})$ should correspond to non-obvious relations between elementary matrices. In $E_2(\mathbb{Z})$ there is a matrix of order 4 defining a rotation by 90°:

$$A \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$



This gives a relation

$$(e^{(2)}_{12}(1)\, e^{(2)}_{21}(-1)\, e^{(2)}_{12}(1))^4 = 1,$$

which corresponds to a nontrivial element $(x^{(2)}_{12}(1)\, x^{(2)}_{21}(-1)\, x^{(2)}_{12}(1))^4 \in K_2(\mathbb{Z})$. One can check that it has order 2 in $K_2(\mathbb{Z})$, and in fact it *generates* $K_2(\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$:

Theorem 1.3.10. *For each $n \geqslant 3$ the group $St_n(\mathbb{Z})$ is a central extension*

$$0 \to C_n \to St_n(\mathbb{Z}) \to E_n(\mathbb{Z}) \to 1$$

where C_n is the cyclic group of order 2 generated by $(x_{12}^{(2)}(1)\, x_{21}^{(2)}(-1)\, x_{12}^{(2)}(1))^4$.

A proof can be found in [Mil71, §10]. ☹

Passing to the limit, we get $K_2(\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$, because of the universal central extension

$$0 \to K_2(\mathbb{Z}) \to St(\mathbb{Z}) \to E(\mathbb{Z}) \to 1$$

Remark 1.3.11. K-groups are *extremely* difficult to compute even for $\mathbb{Z}$. Later on we will review definitions of the higher K-groups $K_3, K_4, K_5, \ldots$ For $\mathbb{Z}$ these are the following:

n:	0	1	2	3	4	5	$\cdots$
$K_n(\mathbb{Z})$:	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/48$	0	$\mathbb{Z}$	$\cdots$
			[Mil71, §10]	[LS76]	[Rog00]	[EVGS02]	

Note that all $K_2(\mathbb{Z})$, $K_3(\mathbb{Z})$, $K_4(\mathbb{Z})$ are finite, and $K_5(\mathbb{Z})$ has rank one. We will not be able to explain the finite part, but we will see that next in this series should go some other finite groups $K_6(\mathbb{Z})$, $K_7(\mathbb{Z})$, $K_8(\mathbb{Z})$, then a group $K_9(\mathbb{Z})$ of rank one, and so on. Ranks are always periodic, with period four.

For calculation of $K_n(\mathbb{Z})$ see a survey [Wei05].

In fact for any number field F the group $K_2(\mathcal{O}_F)$ is finite. Originally this result is due to Garland [Gar71]. We will see more generally finiteness of $K_2(\mathcal{O}_F), K_4(\mathcal{O}_F), K_6(\mathcal{O}_F), \ldots$, which follows from Borel's computation [Bor74].

A definition of K_n for $n > 2$ is the subject of the next chapter.

Chapter 2

Higher algebraic K-theory of rings (plus-construction)

In this chapter we review a definition of higher K-groups of a ring via the Quillen's plus-construction.

It is worth noting that the first K-group functors K_0, K_1, K_2 as described in chapter 1 are not separate entities; they can be put together in various ways. For instance, for an ideal $I \subseteq R$ one can define **relative K-groups** $K_1(R, I)$ and $K_0(R, I)$, in such a manner that there is an exact sequence

$$? \to K_2(R) \to K_2(R/I) \to K_1(R, I) \to K_1(R) \to K_1(R/I) \to K_0(R, I) \to K_0(R) \to K_0(R/I)$$

—see [Mil71, §4 + §6] for this. Then it is natural to ask what would be "$K_2(R, I)$", and how to continue the sequence with terms $K_3, K_4, K_5, \ldots$ The key insight is that such a long exact sequence reminds the fibration long exact sequence in algebraic topology (proposition H.2.10), so one should somehow define a functor

$$\begin{array}{rcl} \mathcal{CRing} & \to & \mathcal{HCWTop}, \\ R & \rightsquigarrow & \mathbf{K}(R). \end{array}$$

from the category of (commutative) rings to the category of CW-complexes and homotopy classes of maps. Then one defines the higher K-groups by $K_i(R) \stackrel{\text{def}}{=} \pi_i(\mathbf{K}(R))$.

Now for each ideal $I \subseteq R$ the projection $p\colon R \to R/I$ induces a map $p_*\colon \mathbf{K}(R) \to \mathbf{K}(R/I)$. We consider the associated fibration (see definition H.2.8) and we *force by definition* homotopy fiber (its connected component at the base point) of such a fibration to be $\mathbf{K}(R, I)$. Then we have the desired long exact sequence

$$\cdots \to K_n(R, I) \xrightarrow{i_*} K_n(R) \xrightarrow{p_*} K_n(R/I) \xrightarrow{\partial} K_{n-1}(R, I) \to \cdots$$

A reasonable construction of $\mathbf{K}(R)$ must give $K_i(R) \cong \pi_i(\mathbf{K}(R))$, where on the left hand side are the K-groups K_0, K_1, K_2 discussed in chapter 1, and also the definition of this functor $\mathbf{K}$ on arrows should give us the classic $K_i(f)$.

One of Quillen's solutions is the following: K_i is the composition of functors

$$K_i\colon R \rightsquigarrow GL(R) \rightsquigarrow BGL(R) \rightsquigarrow BGL(R)^+ \rightsquigarrow \pi_i(BGL(R)^+).$$

Given a ring R, we consider the classifying space $BGL(R)$ of the group $GL(R)$ (cf. definition 1.2.1). Then from this space we can build another space "$BGL(R)^+$" and take its homotopy groups. Building a space $BGL(R)^+$ from $BGL(R)$ is called **plus-construction** and it is described in this chapter, together with proofs that K_i's obtained this way agree with what we saw in chapter 1.

References. A nice exposition of the plus-construction is [Ber82a], and our overview loosely follows its §§4–9.

2.1 Perfect subgroups of the fundamental group

We are going to use some basic definitions and results from algebraic topology. They are collected in appendix H, and the least standard section there is § H.4 discussing acyclic maps. In what follows, to make life easier, all spaces are tacitly assumed to have homotopy type of connected CW-complexes with finitely many cells in any given dimension. The spaces are pointed, but the base points are dropped from the notation, e.g. $\pi_n(X)$ actually means $\pi_n(X,*)$, etc.

Recall that in § 1.3 we discussed **perfect groups**, i.e. those satisfying $P/[P,P] = P^{ab} = H^1(P,\mathbb{Z}) = 0$. In particular, a homomorphic image of a perfect group is again perfect.

Proposition 2.1.1. *If $f\colon X \to Y$ is an acyclic map, then $\pi_1(Y) \cong \pi_1(X)/P$, where P is some perfect normal subgroup of $\pi_1(X)$.*

Proof. Let F be homotopy fiber of f. Consider the fibration long exact sequence

$$\pi_2(Y) \to \pi_1(F) \xrightarrow{i_*} \pi_1(X) \xrightarrow{f_*} \pi_1(Y) \to \pi_0(F)$$

The map f_* is surjective since $\pi_0(F) = 1$ (because $\widetilde{H}_0(F) = 0$). Since $\widetilde{H}_1(F) = \pi_1(F)^{ab} = 0$, the group $\pi_1(F)$ is perfect. The image of $\pi_1(F)$ under a homomorphism i_* is again a perfect group $P \stackrel{\text{def}}{=} \operatorname{im} i_*$. Finally, by exactness $\ker f_* = \operatorname{im} i_*$ we conclude $\pi_1(Y) \cong \pi_1(X)/P$. ∎

Now let us consider a pushout $Y_0 \cup_X Y_1$ in the category of topological spaces. The Seifert–van Kampen theorem tells us how the fundamental group of $Y_0 \cup_X Y_1$ is made: it is given by the "free product with amalgamation"

$$\begin{tikzcd}
Y_0 \cup_X Y_1 & Y_0 \arrow[l, "\overline{f}_1"'] \\
Y_1 \arrow[u, "\overline{f}_0"] \arrow[ur, phantom, "\ulcorner"] & X \arrow[l, "f_1"] \arrow[u, "f_0"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\pi_1(Y_0) *_{\pi_1(X)} \pi_1(Y_1) & \pi_1(Y_0) \arrow[l, "\overline{f}_{1*}"'] & \ker \overline{f}_{1*} \arrow[l, hook'] \\
\pi_1(Y_1) \arrow[u] \arrow[ur, phantom, "\ulcorner"] & \pi_1(X) \arrow[l, "f_{1*}"] \arrow[u, "f_{0*}"'] & \ker f_{1*} \arrow[l, hook']
\end{tikzcd}$$

If we assume f_1 to be an acyclic cofibration, then by proposition H.4.6 its pushout $\overline{f}_1\colon Y_0 \to Y_0 \cup_X Y_1$ is also an acyclic cofibration. By the previous proposition $\pi_1(Y_1) \cong \pi_1(X)/\ker f_{1*}$ and

$$\pi_1(Y_0 \cup_X Y_1) \cong \pi_1(Y_0)/\ker \overline{f}_{1*}.$$

Here $\ker \overline{f}_{1*}$ is the normal closure of the perfect subgroup $f_{0*} \ker f_{1*}$.

We will use later on this observation:

Proposition 2.1.2. *If $f_1\colon X \to Y_1$ is an acyclic cofibration, then the pushout $\overline{f}_1\colon Y_0 \to Y_0 \cup_X Y_1$ is also an acyclic cofibration with $\ker \overline{f}_{1*}$ the normal closure of the perfect subgroup $f_{0*} \ker f_{1*}$ of $\pi_1(Y_0)$.*

2.2 Plus-construction for a space

Given a space X, we can consider some perfect normal subgroup $P \trianglelefteq \pi_1(X)$ of the fundamental group. We would like to come up with another space X^+ such that this subgroup P is killed in $\pi_1(X^+)$. Namely, we are looking for a map $X \to X^+$ such that $\ker(\pi_1(X) \to \pi_1(X^+)) = P$. Moreover, we ask that the *homology* groups remain the same: $H_\bullet(X) = H_\bullet(X^+)$. The solution of this problem is easy: just glue in some 2-cells to kill the generators of $P \trianglelefteq \pi_1(X)$, and then glue in some 3-cells to save the second homology group untouched. This construction changes the higher *homotopy* groups $\pi_\bullet(X)$ in some very nontrivial way, and this will be the main story! Here is a precise statement:

Theorem 2.2.1 (Quillen). *Let P be a perfect normal subgroup of $\pi_1(X)$. Then there exists an acyclic cofibration $f\colon X \to X^+$ with $\ker(\pi_1(X) \xrightarrow{f_*} \pi_1(X^+)) \cong P$. If $f'\colon X \to (X^+)'$ is another acyclic cofibration with the same property, then there is a homotopy equivalence $h\colon X^+ \to (X^+)'$, making the diagram commute*

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ X^+ \arrow[rr, "h"'] & & (X^+)' \end{tikzcd}$$

Proof of existence. **First assume that $P = \pi_1(X)$ is a perfect group.** We are going to attach 2-cells to X, producing a space X', and then attach 3-cells to X', producing a space X^+ with $\pi_1(X^+) = 0$.

- For each generator $[\alpha]$ of $\pi_1(X)$ we attach a 2-cell along α. The resulting space X' has $\pi_1(X') = 0$ (by the van Kampen theorem), and there is a Hurewicz isomorphism $\pi_2(X') \xrightarrow{\cong} H_2(X')$—cf. theorem H.1.1.

  Now consider the pair long exact sequence

  $$\cdots \to H_2(X) \to H_2(X') \to H_2(X', X) \xrightarrow{\delta} H_1(X) \to \cdots$$

  Since $\pi_1(X)$ is perfect, $H_1(X) \cong \pi_1(X)^{ab} = 0$.

  By excision theorem, the group $H_2(X', X)$ is generated by the added 2-cells:

  $$H_2(X', X) \cong H_2(\bigvee_\lambda B^2, \bigvee_\lambda S^1) \cong \bigoplus_\lambda \mathbb{Z}.$$

- We chose maps $b_\lambda\colon S^2 \to X'$ such that they induce an isomorphism on homology

  $$\begin{tikzcd} \widetilde{H}_q(\bigvee S^2) \arrow[r] \arrow[rr, bend right, "\cong"'] & \widetilde{H}_q(X') \arrow[r] & H_q(X', X) \end{tikzcd}$$

  We attach 3-cells by $\bigvee b_\lambda\colon \bigvee_\lambda S^2 \to X'$ to form another connected space X^+. It still satisfies $\pi_1(X^+) = 0$.

We need to check that the inclusion $X \hookrightarrow X^+$ is acyclic. By proposition H.4.7, it is enough to establish $H_\bullet(X^+, X) = 0$:

$$\cdots \to H_{n+1}(X^+, X) \to H_n(X) \to H_n(X^+) \to H_n(X^+, X) \to \cdots$$

By 5-lemma and excision, the induced map of exact sequences of triples

$$(\bigvee B^3, \bigvee S^2, pt) \hookrightarrow (X^+, X', X)$$

gives an isomorphism $H_\bullet(\bigvee B^3, pt) \cong H_\bullet(X^+, X)$:

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(\bigvee S^2, pt) \arrow[r] \arrow[d, "\cong"] & H_n(\bigvee B^3, pt) \arrow[r] \arrow[d, "\cong"] & H_n(\bigvee B^3, \bigvee S^2) \arrow[r] \arrow[d, "\cong"] & H_{n-1}(\bigvee S^2, pt) \arrow[r] \arrow[d, "\cong"] & \cdots \\
\cdots \arrow[r] & H_n(X', X) \arrow[r] & H_n(X^+, X) \arrow[r] & H_n(X^+, X') \arrow[r] & H_{n-1}(X', X) \arrow[r] & \cdots
\end{tikzcd}$$

So $H_\bullet(X^+, X) = 0$.

Now for the general case, let $\overline{X} \to X$ be a covering with $\pi_1(\overline{X}) = P$. By the previous case, there is an acyclic cofibration $f\colon \overline{X} \to \overline{X}^+$ with $\pi_1(\overline{X}^+) = 0$. We consider the pushout of f along $\overline{X} \to X$:

$$\begin{tikzcd}
X^+ \arrow[dr, phantom, "\ulcorner"] & X \arrow[l, "\bar{f}"'] \\
\overline{X}^+ \arrow[u] & \overline{X} \arrow[l, "f"] \arrow[u]
\end{tikzcd}$$

We can apply proposition 2.1.2: we know that $\bar{f}\colon X \to X^+$ is also an acyclic cofibration, and $\ker(\pi_1(X) \xrightarrow{\bar{f}_*} \pi_1(X^+)) \cong P$. ■

Remark 2.2.2. The construction with attaching 2-cells and 3-cells goes back to Kervaire [Ker69].

The uniqueness up to homotopy is deduced from the following:

Lemma 2.2.3. *Let $f\colon X \to Y$ and $g\colon X \to Z$ be two maps with f being an acyclic cofibration. Let $\ker f_* \leqslant \ker g_*$. Then there exists a map $h\colon Y \to Z$ making the diagram commute. Moreover, any two such are homotopic.*

$$\begin{tikzcd}
& X \arrow[dl, "f"'] \arrow[dr, "g"] & \\
Y \arrow[rr, dashed, "h"'] & & Z
\end{tikzcd}
\qquad
\begin{tikzcd}
& \pi_1(X) \arrow[dl, "f_*"'] \arrow[dr, "g_*"] & \\
\pi_1(Y) \arrow[rr, dashed, "h_*"'] & & \pi_1(Z)
\end{tikzcd}$$

Proof. We can assume that g is also a cofibration by replacing it with the associated cofibration (definition H.2.8). Now consider a pushout

$$\begin{tikzcd}
Z \cup_X Y \arrow[dr, phantom, "\ulcorner"] & Z \arrow[l, "\bar{f}"'] \\
Y \arrow[u, "\bar{g}"] & X \arrow[l, "f"] \arrow[u, "g"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\pi_1(Z) *_{\pi_1(X)} \pi_1(Y) \arrow[dr, phantom, "\ulcorner"] & \pi_1(Z) \arrow[l, "\bar{f}_*"'] \\
Y \arrow[u] & X \arrow[l, "f_*"] \arrow[u, "g_*"']
\end{tikzcd}$$

Here $\ker \bar{f}_*$ is the normal closure of $g_* \ker f_*$ by proposition 2.1.2, which is trivial by the assumption. So $\bar{f}$ is a homotopy equivalence by proposition H.4.8, and so homotopy equivalence *under* X (proposition H.2.5). Let $\bar{f}^{-1}$ denote its homotopy inverse under X.

$$\begin{tikzcd}
& X \arrow[dl, "f"'] \arrow[dr, "g"] & \\
Y \arrow[rr, dashed, "h"] \arrow[dr, "\overline{g}"'] & & Z \arrow[dl, shift left, "\overline{f}"'] \\
& Z \cup_X Y \arrow[ur, shift left, "\overline{f}^{-1}"'] &
\end{tikzcd}$$

The map $h \overset{\text{def}}{=} \overline{f}^{-1} \circ \overline{g}$ is the desired homotopy, and by the universality of pushouts any map h should arise this way. ■

The main application of the plus-construction is the following. Recall from proposition 1.3.4 that any group G contains the **maximal perfect subgroup** $\mathfrak{P}G$, which is automatically normal.

Definition 2.2.4 (Plus-construction)**.** Let $P = \mathfrak{P}\pi_1(X)$ be the maximal perfect subgroup in $\pi_1(X)$. Then by virtue of theorem 2.2.1, there exists an acyclic cofibration, which we denote by $q_X \colon X \to X^+$, such that $\ker(\pi_1(X) \xrightarrow{q_{X*}} \pi_1(X^+)) \cong P$.

The plus-construction is functorial in the following sense.

Proposition 2.2.5. *Given a map $f \colon X \to Y$, there is a unique homotopy class of maps $f^+ \colon X^+ \to Y^+$ making the following diagram commute*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "q_X"'] & Y \arrow[d, "q_Y"] \\
X^+ \arrow[r, "f^+"'] & Y^+
\end{tikzcd}$$

Proof.

$$\begin{tikzcd}
& \pi_1(X) \arrow[dl, "q_{X*}"'] \arrow[dr, "f_*"] & \\
& & \pi_1(Y) \arrow[d, "q_{Y*}"] \\
\pi_1(X^+) \arrow[rr, dashed, "h_*"'] & & \pi_1(Y^+)
\end{tikzcd}$$

We have

$$f_* \ker q_{X*} = f_* \mathfrak{P}\pi_1(X) \leqslant \mathfrak{P}\pi_1(Y) = \ker q_{Y*},$$

hence $\ker q_{X*} \leqslant \ker(q_{Y*} \circ f_*)$, and we apply lemma 2.2.3. ■

Proposition 2.2.6. *For a product of two spaces one has*

$$(X \times Y)^+ = X^+ \times Y^+ \quad \textit{with} \quad q_{X \times Y} = (q_X, q_Y).$$

Proof. This follows from the properties of $\mathfrak{P}$ and π_1:

$$\mathfrak{P}\pi_1(X \times Y) \cong \mathfrak{P}(\pi_1(X) \times \pi_1(Y)) \cong \mathfrak{P}\pi_1(X) \times \mathfrak{P}\pi_1(Y).$$

■

Proposition 2.2.7. *Let $f_0 \simeq f_1 \colon X \to Y$ be homotopy equivalent maps. Then $f_0^+ \simeq f_1^+ \colon X^+ \to Y^+$ are homotopy equivalent as well.*

Proof. Consider a homotopy $h \colon X \times Y \to Y$. Applying proposition 2.2.6, we get

$$\begin{tikzcd} X \times I \arrow[r, "h"] \arrow[d, "(q_X, q_I)"'] & Y \arrow[d, "q_Y"] \\ X^+ \times I^+ \arrow[r, "h^+"'] & Y^+ \end{tikzcd}$$

∎

Now consider a fibration $F \xrightarrow{i} E \xrightarrow{p} B$. One would like to find assumptions under which the plus-construction gives again a fibration $F^+ \xrightarrow{i^+} E^+ \xrightarrow{p^+} B^+$ (i.e. so that F^+ is homotopy fiber of p^+). In this case one says that the fibration is **plus-constructive**. For a complete discussion of plus-constructive fibrations see [Ber82b], [Ber83], and [Ber82a, Chapter 4, 6, 8]. But let us sweep under the rug these technical results by citing a couple of facts to be used later.

Fact 2.2.8. *Let $F \to E \to B$ be a fibration of connected spaces. Assume that $\mathfrak{P}\pi_1(B) = 1$. Then $F^+ \to E^+ \to B^+$ is also a fibration of connected spaces.*

This is easy to show, see e.g. [Ber82a, 6.4 a)]. ☹

Fact 2.2.9. *Consider a central group extension $1 \to C \to E \to G \to 1$ where E is a perfect group. Then $BC \to BE^+ \to BG^+$ is a homotopy fibration.*

This is less easy; see for this [Ber82a, 8.4] or [Ger73b]. ☹

2.3 Homotopy groups of X^+

For a given space X, we would like to get information about homotopy groups $\pi_i(X^+)$. The idea due to Dror [Dro72], is to consider a Postnikov-like tower of spaces

$$\cdots \to X_{n+1} \to X_n \to \cdots \to X_3 \to X_2 \to X_1 = X$$

The construction is performed in such a way that each step kills more homology:

$$\widetilde{H}_i(X_n) = 0 \text{ for } i < n$$

(here and below we omit the coefficient ring $\mathbb{Z}$ in "$H_\bullet(X)$" to simplify the notation).

Consequently, taking the limit $AX = \varprojlim X_n$, one gets an acyclic space. In fact AX is homotopy fiber of the acyclic cofibration $X \to X^+$ produced by the plus-construction. This is explained in [Ber82a, §7] and [Ger73a] but we will not really need it.

Now we describe inductively what these spaces X_n are. The starting space X_2 is the covering of X having fundamental group $\pi_1(X_2) = \mathfrak{P}\pi_1(X) = H_1(X)$:

$$\begin{tikzcd} X_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & PK(H_1(X), 1) \arrow[d] \\ X \arrow[r] & K(H_1(X), 1) \end{tikzcd}$$

Similarly, $X_{n+1} \to X_n$ is the pullback of the path fibration over the Eilenberg–Mac Lane space $K(H_n(X_n), n)$:

$$\begin{tikzcd}
X_{n+1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & PK(H_n(X_n),n) \arrow[d] \\
X_n \arrow[r, "\theta_n"'] & K(H_n(X_n),n)
\end{tikzcd} \tag{2.1}$$

The morphism $\theta_n\colon X_n \to K(H_n(X_n), n)$ is given as follows. Recall that for any free chain complex $C_\bullet$ over a principal ideal domain there is a natural split short exact sequence

$$0 \to \operatorname{Ext}^1_R(H_{n-1}(C_\bullet), M) \to H^n(C_\bullet; M) \to \operatorname{Hom}(H_n(C_\bullet), M) \to 0$$

(this is the "universal coefficient theorem" [May99, §17.3]). For instance, if we take $C_\bullet = C_\bullet(X_n)$ the singular complex for X_n and $M = H_n(X)$, then by our inductive assumption $H_{n-1}(X_n) = 0$ the Ext vanishes, and remains an isomorphism

$$\operatorname{Hom}(H_n(X_n), H_n(X_n)) \cong H^n(X_n, H_n(X_n)). \tag{2.2}$$

Further, there is a natural isomorphism [May99, §22.2]

$$H^n(X_n; H_n(X_n)) \cong [X_n, K(H_n(X_n), n)], \tag{2.3}$$

where $[X_n, K(H_n(X_n), n)]$ denotes the set of homotopy classes of maps $X_n \to K(H_n(X_n), n)$. Now we can take the composition of (2.2) and (2.3):

$$\begin{tikzcd}
1_{H_n(X_n)} \arrow[dr, mapsto] & \operatorname{Hom}(H_n(X_n), H_n(X_n)) \arrow[r, "\cong"] \arrow[dr, dashed] & H^n(X_n; H_n(X_n)) \arrow[d, "\cong"] \\
& \theta_n & {[X_n, K(H_n(X_n), n)]}
\end{tikzcd}$$

The image of $1_{H_n(X_n)}$ under these maps is by definition $\theta_n\colon X_n \to K(H_n(X_n), n)$. It is defined up to homotopy. However, since X_{n+1} is, by definition, homotopy fiber of θ_n, changing θ_n within its homotopy class changes X_{n+1} within it fiber homotopy class over X_n. Hence X_{n+1} is unique up to fiber homotopy equivalence over X_n, and the construction is functorial up to fiber homotopy.

The construction is inductive and uses at each step the fact that $\widetilde{H}_i(X_n) = 0$ for $i < n$. We check it inductively. At each step there is a homotopy fibration

$$K(H_n(X_n), n-1) \to X_{n+1} \to X_n$$

We apply the Hurewicz theorem (H.1.1). The space $K(H_n(X_n), n-1)$ is $(n-2)$-connected, so

$$H_{n-1}(K(H_n(X_n), n-1)) \cong \pi_{n-1}(K(H_n(X_n), n-1)) \cong H_n(X_n).$$

Further, $\pi_n(K(H_n(X_n), n-1))$ surjects to $H_n(K(H_n(X_n), n-1))$, thus the latter is 0.

$$\widetilde{H}_i(K(H_n(X_n), n-1)) = \begin{cases} H_n(X_n), & i = n-1 \\ 0, & \text{otherwise.} \end{cases}$$

Denote $K(H_n(X_n), n-1)$ by K. We use the **Serre exact sequence** (proposition H.3.3). In this case $\widetilde{H}_i(X_n) = 0$ for $i < n$ by the induction hypothesis and $\widetilde{H}_j(K) = 0$ for $j < n-1$.

$$H_{2n-2}(K) \to \cdots \to H_n(K) \to H_n(X_{n+1}) \to H_n(X_n) \xrightarrow{\cong} H_{n-1}(K) \to \cdots$$

The last arrow is an isomorphism, hence $\widetilde{H}_n(X_{n+1}) = 0$. ■

We can apply fact 2.2.8 to homotopy fibrations $X_{n+1} \to X_n \xrightarrow{\theta_n} K(H_n(X_n), n)$ to get new fibrations

$$\begin{aligned} X_2^+ &\to X^+ \to K(\pi_1(X^+), 1), \\ X_{n+1}^+ &\to X_n^+ \to K(H_n(X_n), n) \quad \text{for } n \geqslant 2. \end{aligned}$$

Let us look at the corresponding homotopy long exact sequences.

- For $n = 1$ we have

$$\cdots \to 1 \to \pi_2(X_2^+) \xrightarrow{\cong} \pi_2(X^+) \to 1 \to \pi_1(X_2^+) \to \pi_1(X^+) \xrightarrow{\cong} \pi_1(X^+) \to 1$$

  So we deduce $\pi_1(X_2^+) = 1$, and $\pi_i(X_2^+) \cong \pi_i(X^+)$ for $i \geqslant 2$. The Hurewicz theorem gives an isomorphism $\pi_2(X_2^+) \cong H_2(X_2)$ and a surjection $\pi_3(X_2) \twoheadrightarrow H_3(X_2)$.

- For $n = 2$ we have a short exact sequence

$$\cdots \to 1 \to \pi_3(X_3^+) \xrightarrow{\cong} \pi_3(X_2^+) \to 1 \to \pi_2(X_3^+) \to \pi_2(X_2^+) \to H_2(X_2) \to \pi_1(X_3^+) \to 1$$

  Here $\pi_2(X_2^+) \to H_2(X_2)$ can be identified with the Hurewicz isomorphism as above, and we have $\pi_1(X_3^+) = \pi_2(X_3^+) = 1$. Again by Hurewicz $\pi_3(X_3^+) \cong H_3(X_3)$ and $\pi_4(X_3^+) \twoheadrightarrow H_4(X_3)$.

  For $i \geqslant 3$ one has $\pi_i(X_3^+) \cong \pi_i(X_2) \cong \pi_i(X^+)$.

- And so on...

It is clear how one proceeds by induction in this manner to conclude that for $n \geqslant 2$

$$\begin{aligned} \pi_i(X_n^+) &= \begin{cases} 0, & i < n, \\ \pi_i(X^+), & i \geqslant n; \end{cases} \\ \pi_n(X_n^+) &\cong H_n(X_n), \\ \pi_{n+1}(X_n^+) &\twoheadrightarrow H_{n+1}(X_n). \end{aligned} \tag{2.4}$$

2.4 Higher K-groups of a ring

Now we are going to apply the construction from the previous section to the classifying space $X = BG$ of a group G. In this case the calculation above gives

$$\pi_i(BG^+) = \begin{cases} G/\mathfrak{P}G, & i = 1, \\ H_i((BG)_i), & i \geqslant 2, \end{cases} \tag{2.5}$$

Take $G = GL(R)$. We have $\mathfrak{P}G = E(R)$, and hence $\pi_1(BGL(R)^+) \cong GL(R)/E(R) = K_1(R)$. Now from the definition of X_2 we see that it is the space $B\mathfrak{P}G$, hence $\pi_2(BGL(R)^+) \cong H_2(E(R), \mathbb{Z})$. We know that the latter is $K_2(R)$. This motivates the following

Definition 2.4.1. For a ring R the higher K-groups are given by

$$K_i(R) \stackrel{\text{def}}{=} \pi_i(BGL(R)^+) \quad \text{for } i > 0.$$

We would like to describe $K_3(R)$, which was not defined before. Recall that we have a group extension

$$0 \to K_2(R) \to St(R) \to E(R) \to 1$$

This is a universal central extension, hence $H_1(St(R),\mathbb{Z}) = H_2(St(R),\mathbb{Z}) = 0$. We apply fact 2.2.9 to get a homotopy fibration

$$BK_2(R) \to BSt(R)^+ \to E(R)^+$$

The fibration long exact sequence gives immediately $\pi_i(BSt(R)^+) \cong \pi_i(BE(R)^+)$ for $i \geqslant 3$. The plus-construction on $BSt(R)$ kills its fundamental group since $St(R)$ is perfect itself, so $BSt(R)^+$ is a 1-connected space. The Hurewicz theorem gives an isomorphism $\pi_2(BSt(R)^+) \cong H_2(BSt(R)^+)$. The latter is $H_2(BSt(R)) = 0$, since the plus-construction preserves homology. Again by Hurewicz we have

$$\pi_3(BE(R)^+) \cong \pi_3(BSt(R)^+) \cong H_3(St(R),\mathbb{Z}).$$

Finally, $\pi_3(BE(R)^+) \cong \pi_3(BGL(R)^+)$ by the following

Lemma 2.4.2. *One has*

$$\pi_i(BG^+) \cong \pi_i(B\mathfrak{P}G^+) \quad \textit{for } i \geqslant 2.$$

Proof. Recall that $(BG)_2$ can be identified with $B\mathfrak{P}G$ and then use (2.4). ■

We conclude that

$$K_3(R) \cong H_3(St(R),\mathbb{Z}).$$

Remark 2.4.3. For a topological approach to the theory of central extensions of a perfect group see [Ber82a, Chapter 8] and [Woj85].

The plus-construction may seem strange: we took $BGL(R)$, then modified it by gluing 2-cells and 3-cells to obtain something called $BGL(R)^+$, calculated its homotopy groups, and $\pi_1(BGL(R)^+)$ happens to be the same as $K_1(R)$ while $\pi_2(BGL(R)^+)$ is $K_2(R)$ as defined before. So why we take this particular extrapolation of lower K-groups? It all may seem puzzling at first.

From the isomorphism $\pi_n(BG^+) \cong H_n((BG)_n)$ for $n \geqslant 2$ we get a recipe of computing $K_i(R)$.

- For $i = 1$ we already saw that $K_1(R) \cong H_1(BGL(R))$.
- For $i = 2$ let $(BG)_2$ be homotopy fiber of the map $BGL(R) \to K(K_1(R), 1)$:

$$\begin{tikzcd} (BG)_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & PK(K_1(R),1) \arrow[d] \\ BGL(R) \arrow[r] & K(K_1(R),1) \end{tikzcd}$$

  Then $K_2(R) \cong H_2((BG)_2)$.

- For $i = 3$ consider homotopy fiber

$$\begin{tikzcd} (BG)_3 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & PK(K_2(R),2) \arrow[d] \\ (BG)_2 \arrow[r] & K(K_2(R),2) \end{tikzcd}$$

  And we have $K_3(R) \cong H_3((BG)_3)$.

- And so on...

One can think of the description above as of an inductive definition of higher K-groups that does not mention explicitly the plus-construction. This may look more natural than the plus-construction itself.

2.5 Quillen's results

Let us mention one complete calculation of higher K-groups (one of the few known!).

Example 2.5.1. Quillen introduced the plus-construction in order to calculate $K_i(\mathbb{F}_q)$ for finite fields $\mathbb{F}_q$ (strictly speaking, before the higher K-groups were defined). These are the following cyclic groups:

i:	0	1	2	3	4	5	6	$\cdots$
$K_i(\mathbb{F}_q)$:	$\mathbb{Z}$	$\mathbb{Z}/(q-1)$	0	$\mathbb{Z}/(q^2-1)$	0	$\mathbb{Z}/(q^3-1)$	0	$\cdots$

$$\begin{aligned} K_0(\mathbb{F}_q) &\cong \mathbb{Z}, \\ K_{2i}(\mathbb{F}_q) &= 0 \quad \text{for } i > 0, \\ K_{2i-1}(\mathbb{F}_q) &\cong \mathbb{Z}/(q^i - 1)\mathbb{Z} \quad \text{for } i > 0. \end{aligned}$$

Of course this is clear for K_0 and K_1. For K_2 of a field there is also a nice description, due to Matsumoto (see e.g. [Ros94, Theorem 4.3.15]; the original paper is [Mat69]):

For any field F the group $K_2(F)$ is the free abelian group (written multiplicatively) on symbols $\{u, v\}$ for $u, v \in F^\times$ modulo relations

a) $\{u_1 u_2, v\} = \{u_1, v\} \cdot \{u_2, v\}$ *and* $\{u, v_1 v_2\} = \{u, v_1\} \cdot \{u, v_2\}$.

b) $\{u, 1-u\} = 1$ *for* $u \neq 0$ *and* $u \neq 1$.

One sees that from these relations follow automatically

c) $\{u, -u\} = 1$. Indeed, from a) and b)

$$\{u, -u\} = \{u, 1-u\} \cdot \{u, 1-u^{-1}\}^{-1} = \{u^{-1}, 1-u^{-1}\} = 1.$$

d) $\{u, v\} = \{v, u\}^{-1}$. Indeed, from c)

$$\{u, v\} \cdot \{v, u\} = \{u, -u\} \cdot \{u, v\} \cdot \{v, u\} \cdot \{v, -v\} = \{u, -uv\} \cdot \{v, -uv\} = \{uv, -uv\} = 1.$$

Remark 2.5.2. Observe that these are the relations that e.g. Hilbert symbols satisfy (see p. 11):

a) $\left(\frac{aa', b}{\mathfrak{p}}\right) = \left(\frac{a, b}{\mathfrak{p}}\right) \left(\frac{a', b}{\mathfrak{p}}\right)$ and $\left(\frac{a, bb'}{\mathfrak{p}}\right) = \left(\frac{a, b}{\mathfrak{p}}\right) \left(\frac{a, b'}{\mathfrak{p}}\right)$.

b) $\left(\frac{a, 1-a}{\mathfrak{p}}\right) = 1$.

c) $\left(\frac{a, -a}{\mathfrak{p}}\right) = 1$.

d) $\left(\frac{a, b}{\mathfrak{p}}\right) = \left(\frac{b, a}{\mathfrak{p}}\right)^{-1}$.

Assuming the (difficult) Matsumoto's theorem, we can calculate $K_2(\mathbb{F}_q)$ for any finite field $\mathbb{F}_q$. Each element of $K_2(\mathbb{F}_q)$ is represented by a symbol $\{x,y\}$ for some $x,y \in \mathbb{F}_q^\times$. Let a be the generator of the cyclic group $\mathbb{F}_q^\times$. Then $\{x,y\} = \{a^m,a^n\}$ for some m and n. By bilinearity property a), the latter equals to $\{a,a\}^{mn}$. By property d) one has $\{x,x\}^2 = 1$ for any x. If m and n are not both odd, then $\{a,a\}^{mn} = 1$. Otherwise, we have

$$\{a^m,a^n\} = \{a,a\}^{mn} = \{a,a\}.$$

If the characteristic is 2, then $\{a,a\} = \{a,-a\} = 1$ by property c).
If the characteristic is odd, by a simple counting argument there exists a pair of odd numbers m and n such that $a^n = 1 - a^m$. Indeed, consider two sets

$$X \stackrel{\text{def}}{=} \{a^n \mid n \text{ odd}\} \quad\text{and}\quad Y \stackrel{\text{def}}{=} \{1 - a^m \mid m \text{ odd}\}.$$

Observe that $|X| = |Y| = \frac{q-1}{2}$. The set X contains all the non-squares in $\mathbb{F}_q^\times$. For the second set $1 \notin Y$, so it is not possible that in Y are only squares and $X \cap Y \neq \emptyset$. This means $\{a,a\} = \{a,a\}^{mn} = \{a^m, 1-a^m\} = 1$ by property b).

In either case, the symbols are trivial, and we conclude that $K_2(\mathbb{F}_q) = 0$.

The calculation of higher K-groups of $\mathbb{F}_q$ is more difficult. The original Quillen's paper is [Qui72], and an exposition can be found in [Ben98, vol. II, §2.9]. ▲

Now we state some important properties of K-groups $K_i(\mathcal{O}_F)$ for a number field F. The proofs are very nice and interesting, but they use an alternative definition of higher K-groups via the so-called Q-construction. Discussing this would lead us a bit too far from the main story. We just briefly mention that, starting from the category $R\text{-}\mathcal{P}roj_{fg}$ of finitely generated projective R-modules, one can build from it another category $QR\text{-}\mathcal{P}roj_{fg}$; then for the latter one can construct the **classifying space** $BQR\text{-}\mathcal{P}roj_{fg}$ (this is similar to taking the classifying space BG of a group G).

Theorem 2.5.3. *Let $R\text{-}\mathcal{P}roj_{fg}$ the the category of finitely generated projective R-modules. There is a homotopy equivalence (natural up to homotopy)*

$$BGL(R)^+ \to \Omega(BQR\text{-}\mathcal{P}roj_{fg}),$$

where Ω denotes the loop space functor (taken at the point $0 \in BQR\text{-}\mathcal{P}roj_{fg}$ coming from the zero object).

This means that $BGL(R)^+$ carries some extra structure: we can multiply loops, and this makes $BGL(R)^+$ into an H-**group**. It will be important in chapter 3. In fact, $BGL(R)^+$ is an **infinite loop space**—see [Ada78, Chapter 3] and [Ber82a, Chapter 10].

This suggests an alternative definition

$$K_i(R) \stackrel{\text{def}}{=} \pi_{i+1}(BQR\text{-}\mathcal{P}roj_{fg}),$$

which actually works for K_0—unlike the plus-construction that ignores K_0.

A brief discussion of the Q-construction is included in appendix Q. It will not be used in the main text, but it may be interesting for understanding what it is all about.

Now we list some results that are proved using the Q-construction.

Theorem 2.5.4 (Localization exact sequence)**.** *Let $\mathfrak{A}$ be a Dedekind domain with field of fractions F. Then there is a long exact sequence*

$$\cdots \to K_{i+1}(F) \to \coprod_{\mathfrak{p} \subset \mathfrak{A}} K_i(\mathfrak{A}/\mathfrak{p}) \to K_i(\mathfrak{A}) \to K_i(F) \to \cdots$$

where $\mathfrak{p}$ runs through all maximal ideals.

This is [Qui73b, Corollary p. 113]. ☹

In particular, if $\mathfrak{A} = \mathcal{O}_F$ is the ring of integers of a number field F, then $\mathcal{O}_F/\mathfrak{p}$ are finite fields. Quillen's calculation (example 2.5.1) tells that $K_i(\mathcal{O}_F/\mathfrak{p})$ are finite cyclic groups for $i > 0$. We can tensor the long exact sequence with $\mathbb{Q}$, resulting in a long exact sequence

$$\cdots \to K_{i+1}(F) \otimes \mathbb{Q} \to \coprod_{\mathfrak{p} \subset \mathcal{O}_F} \underbrace{K_i(\mathcal{O}_F/\mathfrak{p}) \otimes \mathbb{Q}}_{=0} \to K_i(\mathcal{O}_F) \otimes \mathbb{Q} \to K_i(F) \otimes \mathbb{Q} \to \cdots$$

Hence we have

Corollary 2.5.5. *Let F be a number field. Then for $i \geqslant 2$*

$$K_i(\mathcal{O}_F) \otimes \mathbb{Q} \cong K_i(F) \otimes \mathbb{Q}.$$

The following is is the main result of [Qui73a]:

Theorem 2.5.6. *Let F be a number field. The groups $K_i(\mathcal{O}_F)$ are finitely generated for all $i = 0, 1, 2, \ldots$*

Corollary 2.5.7. *Let S be a finite set of prime ideals in $\mathcal{O}_F$. Then the groups $K_i(\mathcal{O}_{F,S})$ are finitely generated. Their ranks are given by*

$$\begin{aligned} \operatorname{rk} K_0(\mathcal{O}_{F,S}) &= 1, \\ \operatorname{rk} K_1(\mathcal{O}_{F,S}) &= |S| + r_1 + r_2 - 1, \\ \operatorname{rk} K_i(\mathcal{O}_{F,S}) &= \operatorname{rk} K_i(\mathcal{O}_F). \quad (i \geqslant 2) \end{aligned}$$

Here $\mathcal{O}_{F,S}$ is the ring of S-**integers**

$$\mathcal{O}_{F,S} \stackrel{\text{def}}{=} \{x \in F \mid |x|_{\mathfrak{p}} \leqslant 0 \text{ for all } \mathfrak{p} \notin S\} \supseteq \mathcal{O}_F.$$

For $i = 0$ we know that the S-class group is finite; for $i = 1$ the structure of $\mathcal{O}_{F,S}^\times$ is given by the "Dirichlet S-unit theorem" (cf. theorem 1.2.14 and [Neu99, §I.11]):

$$\mathcal{O}_{F,S}^\times \cong \mathbb{Z}^{\#S + r_1 + r_2 - 1} \oplus \boldsymbol{\mu}_F.$$

Proof. We have the following variation of the localization exact sequence:

$$\begin{tikzcd} \cdots \arrow[r] & \coprod_{\mathfrak{p}} K_2(\mathcal{O}_F/\mathfrak{p}) \arrow[r] & K_2(\mathcal{O}_F) \arrow[r] & K_2(\mathcal{O}_{F,S}) \arrow[dll, out=0, in=180] \\ & \coprod_{\mathfrak{p}} K_1(\mathcal{O}_F/\mathfrak{p}) \arrow[r] & K_1(\mathcal{O}_F) \arrow[r] & K_1(\mathcal{O}_{F,S}) \arrow[r] & \cdots \end{tikzcd}$$

We know that $K_i(\mathcal{O}_F/\mathfrak{p})$ are finite cyclic groups for all $i > 0$ and zero for even $i > 0$ (example 2.5.1), so the maps $K_i(\mathcal{O}_F) \to K_i(\mathcal{O}_{F,S})$ have finite kernel for $i > 0$ and also finite cokernel for $i > 1$. This means that $K_i(\mathcal{O}_{F,S})$ are finitely generated. Moreover,

$$\operatorname{rk} K_i(\mathcal{O}_F) = \operatorname{rk} K_i(\mathcal{O}_{F,S}) \quad \text{for } i > 1.$$

∎

We just describe in a couple of words how Quillen proves theorem 2.5.6.

Let V be a vector space of finite dimension n. Then its proper subspaces $0 \subsetneq W \subsetneq V$ form a partially ordered set by inclusion. Any partially ordered set can be viewed as a small category with arrows

$$\mathrm{Hom}(W, W') \stackrel{\text{def}}{=} \begin{cases} *, & W \subseteq W', \\ \varnothing, & W \nsubseteq W'. \end{cases}$$

As explained in § Q.3, for a small category one can build its classifying space. In this case the simplicial set structure is clear: the p-simpleces are the chains of proper subspaces

$$0 \subsetneq W_0 \subsetneq W_1 \subsetneq \cdots \subsetneq W_p \subsetneq V.$$

Denote the geometric realization by $\boxed{V}$. We assume $\boxed{V} = \varnothing$ when $n \leqslant 1$. The following result is stated in [Sol69] and explained also in [Qui73a, Theorem 2]:

Theorem 2.5.8 (Solomon–Tits). *Let $n \geqslant 2$. The space $\boxed{V}$ has the homotopy type of a bouquet of $(n-2)$-spheres. In particular,*

$$\widetilde{H}_i(\boxed{V}; \mathbb{Z}) \cong \begin{cases} \text{a free } \mathbb{Z}\text{-module}, & i = n-2, \\ 0, & \text{otherwise}. \end{cases}$$

So the following definition makes sense

Definition 2.5.9. Let V be a vector space of dimension n. The **Steinberg module** $st(V)$ of V is the $GL(V)$-module given by the natural action of $GL(V)$ on $H_{n-2}(\boxed{V}; \mathbb{Z})$. For $n = 1$ we let $st(V)$ to be $\mathbb{Z}$ with the trivial action of $GL(V)$.

As we mentioned, $K_i(\mathcal{O}_F)$ can be defined as homotopy groups of the classifying space $BQ\mathcal{O}_F\text{-}\mathcal{P}roj_{fg}$. For brevity let us denote the category $Q\mathcal{O}_F\text{-}\mathcal{P}roj_{fg}$ simply by Q. We can consider a filtration by subcategories by the rank of projective modules

$$Q_0 \subset Q_1 \subset Q_2 \subset \cdots \subset Q = \bigcup_{n \geqslant 0} Q_n.$$

Here the category Q_0 is trivial.

The following is [Qui73a, Theorem 3]:

Theorem 2.5.10. *For $n \geqslant 1$ the inclusion $Q_{n-1} \subset Q_n$ induces a long exact sequence*

$$\cdots \to H_{i+1}(BQ_{n-1}; \mathbb{Z}) \to H_{i+1}(BQ_n; \mathbb{Z}) \to \coprod_\alpha H_{i+1-n}(GL(P_\alpha), st(V_\alpha)) \to$$
$$\to H_i(BQ_{n-1}; \mathbb{Z}) \to H_i(BQ_n; \mathbb{Z}) \to \coprod_\alpha H_{i-n}(GL(P_\alpha), st(V_\alpha)) \to \cdots$$

where P_α represent the isomorphism classes of projective $\mathcal{O}_F$-modules of rank n, and $V_\alpha \stackrel{\text{def}}{=} P_\alpha \otimes_{\mathcal{O}_F} F$. (Note that $\mathrm{rk}\, P_\alpha = \dim_F V_\alpha$.) In particular, the homology groups stabilize: the morphism

$$H_i(BQ_{n-1}; \mathbb{Z}) \to H_i(BQ_n; \mathbb{Z})$$

is surjective for $n > i$ and injective for $n > i + 1$.

Observe that α runs through a finite set—there are finitely many projective $\mathcal{O}_F$-modules of fixed rank, essentially by finiteness of the class group $\mathrm{Cl}(F)$.

Now one is ready to prove that $K_i(\mathcal{O}_F) = \pi_{i+1}(BQ)$ are finitely generated. In fact, BQ is an an H-space, in particular it is a nilpotent space, hence the condition that $\pi_i(BQ)$ is finitely generated is equivalent to $H_i(BQ;\mathbb{Z})$ being finitely generated—see [MP12, Theorem 4.5.2]. It is enough to show that $H_i(BQ_n;\mathbb{Z})$ is finitely generated for all i and n, and then we are done since $H_i(BQ;\mathbb{Z}) \cong H_i(BQ_n;\mathbb{Z})$ for n big enough. The key fact is the following:

Claim. *$H_i(GL(P), st(V))$ is finitely generated for each finitely generated projective $\mathcal{O}_F$-module P and $V \stackrel{\text{def}}{=} P \otimes_{\mathcal{O}_F} F$.*

This comes down to finiteness results for arithmetic groups that are proved in [Rag68]; namely, if Γ is an arithmetic group and M is a Γ-module finitely generated over $\mathbb{Z}$, then the group cohomology $H^i(\Gamma, M)$ is finitely generated. We refer to [Qui73a] for details on reduction.

Finally, one uses induction on n. The basic case is the trivial category Q_0:

$$H_i(BQ_0;\mathbb{Z}) = \begin{cases} \mathbb{Z}, & i = 0, \\ 0, & i > 0. \end{cases}$$

The induction step is provided by the long exact sequence from theorem 2.5.10.

Chapter 3

Rational homotopy: from $\operatorname{rk} K_\bullet(\mathcal{O}_F)$ to $\dim QH^\bullet(SL(\mathcal{O}_F), \mathbb{R})$

This chapter is devoted to reducing our problem about the ranks of $K_i(\mathcal{O}_F)$ to calculation of cohomology of $SL(\mathcal{O}_F)$. Recall the definitions from § 1.2. For a ring R we can consider the group $GL(R)$. We have $[GL(R), GL(R)] = E(R)$, and in case $R = \mathcal{O}_F$ by Bass–Milnor–Serre $E(\mathcal{O}_F) = SL(\mathcal{O}_F)$ (theorem 1.2.5). The plus-construction described in the previous chapter gives K-groups

$$K_i(\mathcal{O}_F) \overset{\text{def}}{=} \pi_i(BGL(\mathcal{O}_F)^+) \cong \pi_i(BSL(\mathcal{O}_F)^+) \quad \text{for } i \geqslant 2$$

—the last isomorphism is because $SL(\mathcal{O}_F)$ is the maximal perfect subgroup of $GL(\mathcal{O}_F)$; cf. lemma 2.4.2.

It is always easier to deal with homology instead of homotopy groups. Hurewicz homomorphism going from π_i to H_i (cf. theorem H.1.1) yields

$$\begin{aligned}
&K_i(\mathcal{O}_F) \overset{\text{def}}{=} \pi_i(BGL(\mathcal{O}_F)^+) \xrightarrow{\text{Hur.}} H_i(BGL(\mathcal{O}_F)^+;\mathbb{Z}) \xrightarrow{\cong} H_i(BGL(\mathcal{O}_F);\mathbb{Z}) \xrightarrow{\cong} H_i(GL(\mathcal{O}_F), \mathbb{Z})\\
&K_i(\mathcal{O}_F) \xrightarrow{\cong} \pi_i(BSL(\mathcal{O}_F)^+) \xrightarrow{\text{Hur.}} H_i(BSL(\mathcal{O}_F)^+;\mathbb{Z}) \xrightarrow{\cong} H_i(BSL(\mathcal{O}_F);\mathbb{Z}) \xrightarrow{\cong} H_i(SL(\mathcal{O}_F), \mathbb{Z}) \qquad \text{for } i \geqslant 2.
\end{aligned}$$

Here on the right side "$H_i(GL(\mathcal{O}_F), \mathbb{Z})$" denotes the group homology (with trivial action of $GL(\mathcal{O}_F)$ on $\mathbb{Z}$). The groups $K_i(\mathcal{O}_F)$ are finitely generated (theorem 2.5.6) and we are interested in the *ranks* of $K_i(\mathcal{O}_F)$, so we can look at the dimensions of $\mathbb{Q}$-vector spaces $\pi_i(BGL(\mathcal{O}_F)^+) \otimes_{\mathbb{Z}} \mathbb{Q}$. A classical theorem by Cartan and Serre says that if X is a *homotopy associative H-space*, then the Hurewicz homomorphism induces an injection $\pi_\bullet(X) \otimes_{\mathbb{Z}} \mathbb{Q} \hookrightarrow H_\bullet(X;\mathbb{Q})$ whose image is the subspace $PH_\bullet(X;\mathbb{Q})$ of *primitive elements* in $H_\bullet(X;\mathbb{Q})$. The rest of this chapter is devoted to explanation of this result. In our situation this means that

$$\begin{aligned}
K_i(\mathcal{O}_F) \otimes_{\mathbb{Z}} \mathbb{Q} &\cong PH_i(GL(\mathcal{O}_F), \mathbb{Q}),\\
K_i(\mathcal{O}_F) \otimes_{\mathbb{Z}} \mathbb{Q} &\cong PH_i(SL(\mathcal{O}_F), \mathbb{Q}) \quad \text{for } i \geqslant 2.
\end{aligned}$$

Example 3.0.11. For $i = 1$ we have the first homology group $H_1(GL(\mathcal{O}_F), \mathbb{Z})$, which is isomorphic to the abelianization

$$GL(\mathcal{O}_F)^{ab} \cong GL(\mathcal{O}_F)/[GL(\mathcal{O}_F), GL(\mathcal{O}_F)] \cong GL(\mathcal{O}_F)/SL(\mathcal{O}_F) \cong \mathcal{O}_F^\times.$$

The primitive elements in H_1 is the whole H_1 because of the grading reasons. We know that $K_1(\mathcal{O}_F) \cong \mathcal{O}_F^\times$, and we know that the latter has rank $r_1 + r_2 - 1$. From now on we focus of K_i with $i \geqslant 2$. ▲

The point of passing from GL to SL is that it is (psychologically) easier to work with semisimple groups instead of reductive. We also replace the coefficients with $\mathbb{R}$, since in next chapter we will use a geometric approach to the group (co)homology. We conclude that the ranks can be obtained as

$$\operatorname{rk} K_i(\mathcal{O}_F) = \dim_{\mathbb{R}} PH_i(SL(\mathcal{O}_F), \mathbb{R}) \qquad \text{for } i \geqslant 2.$$

Dually, we can take the indecomposable elements in cohomology:

$$\operatorname{rk} K_i(\mathcal{O}_F) = \dim_{\mathbb{R}} QH^i(SL(\mathcal{O}_F), \mathbb{R}) \qquad \text{for } i \geqslant 2.$$

So the key to the computation is the real cohomology of $SL(\mathcal{O}_F)$. All the hard work on this will follow in the subsequent chapters.

References. All definitions and facts about Hopf algebras come from the seminal paper by Milnor and Moore [MM65b]; there is also an appendix to [Qui69] containing a nice summary. The Cartan–Serre theorem probably appears first in [MM65b, p. 263]. A modern exposition of this is [FHT01, Chapter 16]—with a simplifying hypothesis that the space is simply connected—and [MP12, Chapter 9].

A discussion of the H-space structure on $BGL(R)^+$ can be found in [Lod76].

3.1 H-spaces

Definition 3.1.1. Let (X, e) be a pointed topological space. We say that X is an H**-space** if there is a continuous map $\mu\colon X \times X \to X$ (multiplication) such that the following diagram is homotopically commutative:

$$\begin{tikzcd} X \arrow[r, "{(1,e)}"] \arrow[dr, "\cong"'] & X \times X \arrow[d, "\mu"] & X \arrow[l, "{(e,1)}"'] \arrow[dl, "\cong"] \\ & X & \end{tikzcd}$$

$$\mu \circ (1_X, e) \simeq 1_X \simeq \mu \circ (e, 1_X).$$

We say that H is **homotopy associative** if the following diagram is homotopically commutative:

$$\begin{tikzcd} X \times X \times X \arrow[r, "\mathrm{id} \times \mu"] \arrow[d, "\mu \times \mathrm{id}"'] & X \times X \arrow[d, "\mu"] \\ X \times X \arrow[r, "\mu"'] & X \end{tikzcd}$$

$$\mu \circ (1_X \times \mu) \simeq \mu \circ (\mu \times 1_X).$$

("H" commemorates Heinz Hopf.)

Example 3.1.2. Every topological group is an H-space. For instance, the circle S^1, can be viewed as the subset of complex numbers having norm 1:

$$S^1 = \{z \in \mathbb{C} \mid |z| = 1\}.$$

So S^1 comes with a natural multiplication, making it into a Lie group, and hence a homotopy associative H-space. Similarly S^0 and S^3 arise the same way from real numbers $\mathbb{R}$ and quaternions $\mathbb{H}$. The sphere S^7 is made from octonions $\mathbb{O}$; the multiplication in $\mathbb{O}$ is non-associative, but S^7 is still an H-space. It is a famous result of Adams [Ada60] that S^0, S^1, S^3, S^7 are the only spheres carrying an H-space structure (cf. [May99, §24.6]). ▲

Example 3.1.3. A typical example of a homotopy associative H-space is the loop space $\Omega(X,*)$ of a pointed space $(X,*)$. The multiplication is the natural multiplication of loops at the base point, and the identity is the constant loop at the base point. We have mentioned in § 2.5 that $BGL(R)^+$ is a loop space, hence it is a homotopy associative H-space. ▲

One can give another description of an H-space structure on $BGL(R)^+$, coming from an explicit "direct sum" of matrices. The following "checkerboard map" is a homomorphism

$$\oplus\colon GL(R)\times GL(R)\to GL(R),$$

$$(A\oplus B)_{ij}=\begin{cases} A_{\ell m}, & i=2\ell-1 \text{ or } j=2m-1,\\ B_{\ell m}, & i=2\ell \text{ or } j=2m,\\ 0, & \text{otherwise.}\end{cases}$$

Schematically,

$$A=\begin{pmatrix} \blacksmiley & \blacksmiley & \blacksmiley & \cdots\\ \blacksmiley & \blacksmiley & \blacksmiley & \cdots\\ \blacksmiley & \blacksmiley & \blacksmiley & \cdots\\ \vdots & \vdots & \vdots & \ddots\end{pmatrix},\ B=\begin{pmatrix} \smiley & \smiley & \smiley & \cdots\\ \smiley & \smiley & \smiley & \cdots\\ \smiley & \smiley & \smiley & \cdots\\ \vdots & \vdots & \vdots & \ddots\end{pmatrix},\ A\oplus B\stackrel{\text{def}}{=}\begin{pmatrix} \blacksmiley & 0 & \blacksmiley & 0 & \blacksmiley & \cdots\\ 0 & \smiley & 0 & \smiley & 0 & \cdots\\ \blacksmiley & 0 & \blacksmiley & 0 & \blacksmiley & \cdots\\ 0 & \smiley & 0 & \smiley & 0 & \cdots\\ \blacksmiley & 0 & \blacksmiley & 0 & \blacksmiley & \cdots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots\end{pmatrix}.$$

Via the plus-construction this map $GL(R)\times GL(R)\to GL(R)$ induces a map

$$\begin{tikzcd} BGL(R)^+\times BGL(R)^+ \arrow[r, "\simeq"'] \arrow[rr, bend left, dashed] & B(GL(R)\times GL(R))^+ \arrow[r, "\oplus^+"'] & BGL(R)^+ \end{tikzcd}$$

Here the first map is some fixed homotopy equivalence, since we know that (cf. proposition 2.2.6)

$$B(GL(R)\times GL(R))^+\simeq (BGL(R)\times BGL(R))^+\simeq BGL(R)^+\times BGL(R)^+.$$

One can check that this operation makes $BGL(R)^+$ into a homotopy associative and homotopy commutative H-space. We refer to [Lod76, §1.2] for this verification.

3.2 Hopf algebras

We make a brief summary of needed theory of Hopf algebras. The main reference is a seminal paper [MM65b], and a modern and *concise* exposition is [MP12, Chapter 20, 21, 22]. The article by Milnor and Moore is written very well, so we do not reproduce any proofs that can be found there.

From now on k denotes the ground field. By $V_\bullet$ or simply V we denote a graded k-vector space

$$V_\bullet=\bigoplus_{n\geqslant 0} V_n.$$

The induced grading on tensor products is given by $(U\otimes_k V)_n=\sum_{i+j=n} U_i\otimes_k V_j$.

There is a natural **graded commutativity** isomorphism ("twisting")

$$\begin{aligned} T\colon U\otimes_k V &\to V\otimes_k U,\\ u\otimes v &\mapsto (-1)^{\deg u\cdot\deg v}\, v\otimes u.\end{aligned}$$

We denote by $V^\vee$ the dual graded vector space with $V_n^\vee\stackrel{\text{def}}{=}\operatorname{Hom}_k(V_n,k)$. Graded k-vector spaces form a "symmetric monoidal category" (cf. [ML98, Chapter XI]) in the obvious way.

We will just identify in our diagrams

$$(U \otimes_{\boldsymbol{k}} V) \otimes_{\boldsymbol{k}} W \cong U \otimes_{\boldsymbol{k}} (V \otimes_{\boldsymbol{k}} W),$$
$$\boldsymbol{k} \otimes_{\boldsymbol{k}} V \cong V \cong V \otimes_{\boldsymbol{k}} \boldsymbol{k}.$$

Definition 3.2.1. We have two dual notions of algebra and co-algebra over $\boldsymbol{k}$.

An **algebra** is
a graded vector space $A_\bullet$ coming with
a **product** $\mu\colon A \otimes_{\boldsymbol{k}} A \to A$ and
a **unit** $\eta\colon \boldsymbol{k} \to A$.

A **coalgebra** is
a graded vector space $A_\bullet$ coming with
a **coproduct** $\Delta\colon A \to A \otimes_{\boldsymbol{k}} A$ and
a **counit** $\epsilon\colon A \to \boldsymbol{k}$.

(Here and everywhere all tensor products are graded and everything is compatible with gradings.)

We require that the following diagrams commute:

$$\begin{tikzcd} A \otimes_{\boldsymbol{k}} \boldsymbol{k} \arrow[r, "\mathrm{id}\otimes\eta"] \arrow[dr, equal] & A \otimes_{\boldsymbol{k}} A \arrow[d, "\mu"] & \boldsymbol{k} \otimes_{\boldsymbol{k}} A \arrow[l, "\eta\otimes\mathrm{id}"'] \arrow[dl, equal] \\ & A & \end{tikzcd}$$

$$\begin{tikzcd} A \otimes_{\boldsymbol{k}} \boldsymbol{k} \arrow[dr, equal] & A \otimes_{\boldsymbol{k}} A \arrow[l, "\mathrm{id}\otimes\epsilon"'] \arrow[r, "\epsilon\otimes\mathrm{id}"] & \boldsymbol{k} \otimes_{\boldsymbol{k}} A \arrow[dl, equal] \\ & A \arrow[u, "\Delta"'] & \end{tikzcd}$$

Further,

it is called **associative**

it is called **coassociative**

if the following diagram commutes:

$$\begin{tikzcd} A \otimes_{\boldsymbol{k}} A \otimes_{\boldsymbol{k}} A \arrow[r, "\mathrm{id}\otimes\mu"] \arrow[d, "\mu\otimes\mathrm{id}"'] & A \otimes_{\boldsymbol{k}} A \arrow[d, "\mu"] \\ A \otimes_{\boldsymbol{k}} A \arrow[r, "\mu"'] & A \end{tikzcd}$$

$$\begin{tikzcd} A \otimes_{\boldsymbol{k}} A \otimes_{\boldsymbol{k}} A & A \otimes_{\boldsymbol{k}} A \arrow[l, "\mathrm{id}\otimes\Delta"'] \arrow[d, "\Delta"] \\ A \otimes_{\boldsymbol{k}} A \arrow[u, "\Delta\otimes\mathrm{id}"] \arrow[r, "\Delta"'] & A \end{tikzcd}$$

Moreover,

A is called **commutative**

A is called **cocommutative**

if the following diagram commutes:

$$\begin{tikzcd} & A \otimes_{\boldsymbol{k}} A \arrow[dl, "\mu"'] \arrow[dd, "T"] \\ A & \\ & A \otimes_{\boldsymbol{k}} A \arrow[ul, "\mu"] \end{tikzcd}$$

$$\begin{tikzcd} A \otimes_{\boldsymbol{k}} A \arrow[dd, "T"'] & \\ & A \arrow[ul, "\Delta"'] \arrow[dl, "\Delta"] \\ A \otimes_{\boldsymbol{k}} A & \end{tikzcd}$$

It is clear how to define the morphisms $f\colon A \to B$ in the category of algebras (coalgebras) by requiring that they preserve the structure.

$$\begin{tikzcd} A \otimes_k A \arrow[r, "\mu_A"] \arrow[d, "f\otimes f"'] & A \arrow[d, "f"] \\ B \otimes_k B \arrow[r, "\mu_B"'] & B \end{tikzcd} \qquad \begin{tikzcd} k \arrow[r, "\eta_A"] \arrow[d, "f"'] & A \arrow[d, "f"] \\ k \arrow[r, "\eta_B"'] & B \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r, "\Delta_A"] \arrow[d, "f"'] & A \otimes_k A \arrow[d, "f\otimes f"] \\ B \arrow[r, "\Delta_B"'] & B \otimes_k B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\epsilon_A"] \arrow[d, "f"'] & k \arrow[d, "f"] \\ B \arrow[r, "\epsilon_B"'] & k \end{tikzcd}$$

For two algebras A and B the product $A \otimes_k B$ has $A \otimes_k B$ as the underlying graded vector space. The unit is the obvious map $\eta_A \otimes \eta_B\colon k \to A \otimes_k B$. The product is defined by

$$A \otimes_k B \otimes_k A \otimes_k B \xrightarrow{id \otimes T \otimes id} A \otimes_k A \otimes_k B \otimes_k B \xrightarrow{\mu_A \otimes \mu_B} A \otimes_k B$$

Dually, for coproducts in coalgebras

$$A \otimes_k B \xrightarrow{\Delta_A \otimes \Delta_B} A \otimes_k A \otimes_k B \otimes_k B \xrightarrow{id \otimes T \otimes id} A \otimes_k B \otimes_k A \otimes_k B$$

Definition 3.2.2. We say that A is a **Hopf algebra** (**bialgebra**), if

1. (A, μ, η) is an associative algebra.
2. (A, Δ, ϵ) is a coassociative coalgebra.
3. $\Delta\colon A \to A \otimes_k A$ and $\epsilon\colon A \to k$ are morphisms of algebras.
4. $\mu\colon A \otimes_k A \to A$ and $\eta\colon k \to A$ are morphisms of coalgebras.

We say that A is a **quasi-Hopf algebra**, if we drop the associativity and coassociativity condition. We say that A is **connected** if $\eta\colon k \xrightarrow{\cong} A_0$ is an isomorphism (equivalently, if $\epsilon\colon A_0 \xrightarrow{\cong} k$ is an isomorphism).

Remark 3.2.3. If we just assume that $\epsilon\colon A \to k$ is a morphism of algebras and $\eta\colon k \to A$ is a morphism of coalgebras, then the fact that $\Delta\colon A \to A \otimes_k A$ and $\mu\colon A \otimes_k A \to A$ are morphisms of (co)algebras is expressed by commutativity of the following diagram:

$$\begin{tikzcd} & A \otimes_k A & A \otimes_k A \otimes_k A \otimes_k A \arrow[l, "\mu\otimes\mu"'] \\ A \arrow[ur, "\Delta"] & & \\ & A \otimes_k A \arrow[ul, "\mu"] \arrow[r, "\Delta\otimes\Delta"'] & A \otimes_k A \otimes_k A \otimes_k A \arrow[uu, "\mathrm{id}\otimes T\otimes \mathrm{id}"'] \end{tikzcd}$$

Example 3.2.4 (The only we care about)**.** Let X be a topological space. Then its homology has a natural grading

$$H_0(X; k), H_1(X; k), H_2(X; k), \ldots$$

The diagonal map $X \to X \times X$ induces a map $H_\bullet(X;k) \to H_\bullet(X \times X;k)$, and then by the Künneth formula $H_\bullet(X \times X;k) \cong H_\bullet(X;k) \otimes_k H_\bullet(X;k)$, since we work over a field. It means that there is a coproduct $\Delta\colon H_\bullet(X;k) \to H_\bullet(X;k) \otimes_k H_\bullet(X;k)$.

If we further assume that (X,e) is a homotopy associative H-space, then there is also a product $\mu\colon H_\bullet(X;k) \otimes_k H_\bullet(X;k) \to H_\bullet(X;k)$ induced by the multiplication $X \times X \to X$. The inclusion $\{e\} \hookrightarrow X$ induces a unit $\eta\colon k \to H_\bullet(X;k)$ and the projection $X \twoheadrightarrow \{e\}$ induces a counit $\epsilon\colon H_\bullet(X;k) \to k$.

With all this, for a homotopy associative H-space X the homology $H_\bullet(X;k)$ carries a cocommutative Hopf algebra structure. It is connected whenever X is connected. ▲

Assume a Hopf algebra A consists of finite dimensional spaces A_n in each degree n (note this does not mean that $\bigoplus_{n\geq 0} A_n$ is finite dimensional). Then $A^\vee$ is also a Hopf algebra in an obvious way ($\mu^*\colon A^\vee \to A^\vee \otimes_k A^\vee$ becomes a coproduct, $\eta^*\colon A^\vee \to k$ becomes a counit, etc.).

Example 3.2.5 (The only we co-care about). For $H_\bullet(X;k)$ with each $H_n(X;k)$ of finite dimension, the dual algebra is the cohomology algebra $H^\bullet(X;k)$ (where the multiplication is the usual cup-product). Indeed, recall that the cup-product

$$\smile\colon H^p(X;k) \otimes_k H^q(X;k) \to H^{p+q}(X;k)$$

is induced by the diagonal map $\Delta\colon X \to X \times X$.

If X has an H-space structure, then the multiplication $\mu\colon X \times X \to X$ induces a co-multiplication in cohomology $\mu^*\colon H^\bullet(X;k) \to H^\bullet(X;k) \otimes_k H^\bullet(X;k)$. ▲

In what follows we will work with topological spaces with each $H_n(X;k)$ having finite dimension. It is a very non-trivial fact mentioned in § 2.5 that $BGL(\mathcal{O}_F)^+$ is such a space.

Definition 3.2.6. For the counit $\epsilon\colon A \to k$ the graded subspace $IA \stackrel{\text{def}}{=} \ker \epsilon$ is called the **augmentation ideal** of A.

$$0 \to IA \hookrightarrow A \xrightarrow{\epsilon} k$$

(Note that $\epsilon \circ \eta = id_k$, hence $A \cong IA \oplus k$.)

The space of **indecomposable elements**, denoted QA, is given by the exact sequence

$$IA \otimes_k IA \xrightarrow{\mu} IA \twoheadrightarrow QA \to 0$$

Definition 3.2.7. For the unit $\eta\colon k \to A$ we denote $JA \stackrel{\text{def}}{=} \operatorname{coker} \eta$.

$$k \xrightarrow{\eta} A \twoheadrightarrow JA \to 0$$

(Note that $\epsilon \circ \eta = id_k$, hence $A \cong JA \oplus k$.)

The space of **primitive elements** PA is given by the exact sequence

$$0 \to PA \hookrightarrow JA \xrightarrow{\Delta} JA \otimes_k JA$$

Observe that actually $JA \cong IA$.

From the definitions we see that if we have a Hopf algebra $A_\bullet$ with each A_n of finite dimension, then

$$P(A^\vee) \cong (QA)^\vee \quad \text{and} \quad Q(A^\vee) \cong (PA)^\vee.$$

For the tensor product $A \otimes_k A$ we have a decomposition

$$\begin{aligned} A \otimes_k A &\cong (k \oplus JA) \otimes_k (k \oplus JA) \\ &\cong (k \otimes_k k) \oplus (JA \otimes_k k) \oplus (k \otimes_k JA) \oplus (JA \otimes_k JA). \end{aligned}$$

Further, the following diagram commutes:

$$\begin{tikzcd} A \otimes_k k & A \otimes_k A \arrow[l, "\mathrm{id} \otimes \epsilon"'] \arrow[r, "\epsilon \otimes \mathrm{id}"] & k \otimes_k A \\ & A \arrow[u, "\Delta"'] \arrow[ul, equal] \arrow[ur, equal] & \end{tikzcd}$$

$$(id \otimes \epsilon) \circ \Delta(z) = z = (\epsilon \otimes id) \circ \Delta(z).$$

So for every element $z \in JA$ the coproduct is of the form

$$\Delta(z) = z \otimes 1 + \underbrace{\sum z^{(1)} \otimes z^{(2)}}_{\in JA \otimes_k JA} + 1 \otimes z,$$

and if z is primitive, then we have

$$\Delta(z) = z \otimes 1 + 1 \otimes z.$$

We could take this as the definition:

$$PA \stackrel{\text{def}}{=} \{z \in A \mid \Delta(z) = z \otimes 1 + 1 \otimes z\}.$$

Remark 3.2.8. Note that taking indecomposable or primitive elements is consistent with tensor products:

$$I(A \otimes_k B) = (IA \otimes_k 1_B) \oplus (1_A \otimes_k IB),$$
$$P(A \otimes_k B) = (PA \otimes_k 1_B) \oplus (1_A \otimes_k PB).$$

Example 3.2.9. Consider an exterior algebra

$$A = \Lambda(x_{i_1}, x_{i_2}, x_{i_3}, \ldots)$$

over a field k, freely generated by elements $x_{i_1}, x_{i_2}, x_{i_3}, \ldots$ of degrees $i_1, i_2, i_3, \ldots$ This is anticommutative (i.e. $x \wedge y = -y \wedge x$), but if we assume that the degrees i_ℓ are odd, then it is graded commutative in the above sense (i.e. $x \wedge y = (-1)^{\deg x \cdot \deg y}\, y \wedge x$).

There are no relations between different x_{i_ℓ}, hence the space of indecomposable elements $Q^{i_\ell} A$ in degree i_ℓ is one-dimensional generated by x_{i_ℓ}. If we take tensor products of such algebras, then the dimensions of spaces $Q^{i_\ell} A$ sum up. For instance, consider

$$A = \Lambda(x_5, x_9, \ldots, x_{4i+1}, \ldots)^{\otimes r_1} \otimes_k \Lambda(x_3, x_5, \ldots, x_{2i+1}, \ldots)^{\otimes r_2}.$$

Then we have

i:	2	3	4	5	6	7	8	9	$\cdots$
$\dim_k Q^i A$:	0	r_2	0	$r_1{+}r_2$	0	r_2	0	$r_1{+}r_2$	$\cdots$

This is a rather dull example, but it will be very important for us. ▲

Now we cite some results from [MM65b] that hold for $\operatorname{char} k = 0$. The point is that for a Hopf algebra, being both algebra and co-algebra imposes severe restrictions on the structure.

Theorem 3.2.10. *Let A be a connected quasi-Hopf algebra over a field of characteristic zero. Consider the composite morphism*

$$PA \to JA \cong IA \to QA$$

Then

- *$PA \to QA$ is a monomorphism if and only if A is associative and commutative.*
- *$PA \to QA$ is an epimorphism if and only if A is coassociative and cocommutative.*
- *$PA \to QA$ is an isomorphism if and only if A is a commutative and cocommutative Hopf algebra.*

This is [MM65b, Proposition 4.17] or [MP12, Corollary 22.3.3]. ☹

For a graded vector space V we denote by $\mathcal{A}(V)$ the corresponding free commutative algebra generated by V. One has

$$\mathcal{A}(V) = \Lambda(V^-) \otimes P(V^+),$$

where $\Lambda(V^-)$ is the exterior algebra generated by the the subspace of V concentrated in odd degrees, and $P(V^+)$ is the polynomial algebra generated by the subspace concentrated in even degrees.

Theorem 3.2.11 (Leray). *Let A be a connected, commutative, and associative quasi-Hopf algebra over a field of characteristic zero. Let $\sigma\colon QA \to IA$ be a morphism of graded vector spaces such that the composition $QA \xrightarrow{\sigma} IA \xrightarrow{\pi} QA$ is the identity, where π is the quotient map. Then the morphism of algebras $f\colon \mathcal{A}(QA) \to A$ induced by σ is an isomorphism.*

This is [MM65b, Theorem 7.5] or [MP12, Theorem 22.4.1]. ☹

3.3 Rationalization of H-spaces

We are going to show the Cartan–Serre theorem. Namely, for an H-space X it characterizes its homotopy groups $\pi_\bullet(X)$ up to **rationalization**, i.e. $\pi_\bullet(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. This situation occurs very often in algebraic topology when one is interested in passing from coefficients in $\mathbb{Z}$ to coefficients in $\mathbb{Q}$, or in general to some localization of $\mathbb{Z}$—just because it is difficult to cope with the torsion part of homotopy groups. The right way to do that is to modify the topological space X itself so that the homotopy groups change from $\pi_\bullet(X)$ to $\pi_\bullet(X_{\mathbb{Q}}) = \pi_\bullet(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. We quickly summarize the needed theory following [MP12].

Given an abelian group A, we can take its **rationalization**, which is simply the $\mathbb{Q}$-vector space $A_{\mathbb{Q}} \stackrel{\text{def}}{=} A \otimes_{\mathbb{Z}} \mathbb{Q}$. There is a canonical map $A \to A_{\mathbb{Q}}$ given by $a \mapsto a \otimes 1$. This satisfies the following universal property: any morphism $f\colon A \to B$ to another $\mathbb{Q}$-vector space B factors uniquely through $A_{\mathbb{Q}}$:

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr] & & B \\ & A_{\mathbb{Q}} \arrow[ur, dashed, "\exists! \tilde{f}"'] & \end{tikzcd}$$

One would like to consider such a rationalization for *nilpotent topological spaces*.

Recall that there is a natural action of $\pi_1(X)$ on the higher homotopy groups $\pi_n(X)$. Namely, if we have a loop $\alpha\colon I \to X$ representing an element $[\alpha] \in \pi_1(X)$ and a map $f\colon (S^n, *) \to (X, *)$ representing an element $[f] \in \pi_n(X)$, then in the following diagram there exists a homotopy $S^n \times I \to X$ making it commute:

$$\begin{tikzcd}
S^n \times \{0\} \cup \{*\} \times I \arrow[r, "f \cup \alpha"] \arrow[d] & X \arrow[d] \\
S^n \times I \arrow[r] \arrow[ur, dashed, "h"] & *
\end{tikzcd}$$

That is, $h(x, 0) = f(x)$ and $h(*, t) = \alpha(t)$. The based homotopy class of $h_1 : (S^n, *) \to (X, *)$ depends only on the classes $[\alpha]$ and $[f]$, hence we can put $[\alpha] \cdot [f] \stackrel{\text{def}}{=} [h_1]$. This is the action of $\pi_1(X)$ on $\pi_n(X)$.

Definition 3.3.1. A space X is called **nilpotent** if the action of $\pi_1(X)$ on $\pi_n(X)$ is nilpotent. That is, there is a finite chain of subgroups

$$\{1\} \subset G_q \subset \cdots \subset G_2 \subset G_1 \subset G = \pi_n(X),$$

where the quotient groups G_{i-1}/G_i are abelian and the action of $\pi_1(X)$ on G_{i-1}/G_i is trivial.

Remark 3.3.2. Nilpotent spaces give the right setting for rationalization. To simplify things, some books just assume that $\pi_1(X) = 0$, however, this assumption is too severe for the applications we have in mind.

We will not need the theory of nilpotent spaces, since the only case that interests us is given by H-spaces.

Example 3.3.3. If (X, e) is an H-space, then in the diagram above we can take homotopy

$$h(x, t) = \mu(\alpha(t), f(x)).$$

We get

$$\begin{aligned}
h(x, 0) &= \mu(e, f(x)) \simeq f(x), \\
h(*, t) &= \mu(\alpha(t), e) \simeq \alpha(t), \\
h(x, 1) &= \mu(e, f(x)) \simeq f(x),
\end{aligned}$$

hence $[\alpha] \cdot [f] = [f]$, and the action of $\pi_1(X)$ on homotopy groups $\pi_n(X)$ is trivial for $n \geqslant 1$. Such a space is called **simple**. In particular, any simple space is nilpotent. In particular, the action of $\pi_1(X)$ on itself is given by conjugation, so a simple space has abelian $\pi_1(X)$. ▲

We assume from now on that all our spaces have abelian π_1. This is harmless since we have in mind only the H-space $BGL(R)^+$.

Definition 3.3.4. We say that a nilpotent space Y is **rational** if the following equivalent conditions hold:

1. The homotopy groups $\pi_n(Y)$ are $\mathbb{Q}$-vector spaces.

2. The homology groups $\widetilde{H}_n(Y; \mathbb{Z})$ are $\mathbb{Q}$-vector spaces.

Assume that X is an H-space. Consider a map $\phi : X \to X_{\mathbb{Q}}$ to a rational space $X_{\mathbb{Q}}$, which satisfies the following equivalent conditions:

1. The induced map on homotopy groups $\phi_* : \pi_n(X) \to \pi_n(X_{\mathbb{Q}})$ is a rationalization for $n \geqslant 1$.

2. The induced map on homology groups $\phi_* : \widetilde{H}_n(X; \mathbb{Z}) \to \widetilde{H}_n(X_{\mathbb{Q}}; \mathbb{Z})$ is a rationalization for $n \geqslant 1$.

3. The induced map on homology groups $\phi_* : \widetilde{H}_n(X; \mathbb{Q}) \to \widetilde{H}_n(X_{\mathbb{Q}}; \mathbb{Q})$ is an isomorphism.

A map to a rational space $\phi\colon X \to X_{\mathbb{Q}}$ with these properties is unique up to homotopy and it is called a **rationalization** of X. It satisfies the following universal property: for every map $f\colon X \to Y$ to a rational space Y there is a unique (up to homotopy) arrow $\widetilde{f}$ making the diagram commute:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\phi"'] & & Y \\ & X_{\mathbb{Q}} \arrow[ur, dashed, "\exists!\widetilde{f}"'] & \end{tikzcd}$$

To justify this definition as it is stated here, we refer to [MP12, §6.1]. ☹

Example 3.3.5. Consider the circle S^1. We would like to describe the rationalization $S^1_{\mathbb{Q}}$. First make a trivial observation that $\mathbb{Q}$ is the following direct limit. Consider a sequence of multiplication by n maps

$$\mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{3} \mathbb{Z} \xrightarrow{4} \mathbb{Z} \to \cdots$$

This obviously defines a directed system of abelian groups $A_n = \mathbb{Z}$ with maps $f_n = n\colon A_n \to A_{n+1}$, and it makes sense to consider the direct limit $\varinjlim A_n$, which is of course $\mathbb{Q}$. Similarly we can consider a sequence of maps $S^1 \to S^1$ given by $n\colon z \mapsto z^n$ (viewing S^1 as a set of complex numbers $z \in \mathbb{C}$ such that $|z| = 1$):

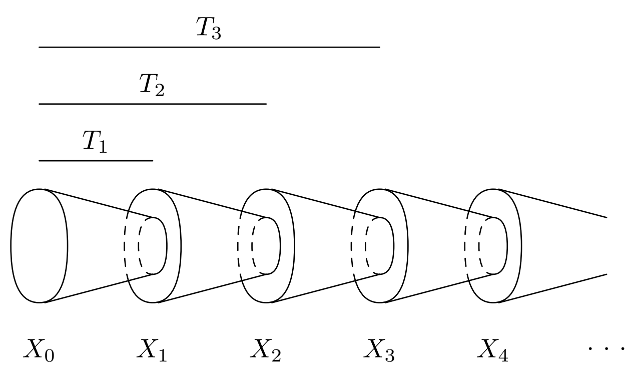
$$S^1 \xrightarrow{1} S^1 \xrightarrow{2} S^1 \xrightarrow{3} S^1 \xrightarrow{4} S^1 \to \cdots$$

On the fundamental group $\pi_1(S^1) \cong \mathbb{Z}$ this induces multiplication by n maps

$$\pi_1(S^1) \xrightarrow{1} \pi_1(S^1) \xrightarrow{2} \pi_1(S^1) \xrightarrow{3} \pi_1(S^1) \xrightarrow{4} \pi_1(S^1) \to \cdots$$

So this is the same as the sequence of maps between $\mathbb{Z}$ considered above.

Recall the "telescoping" construction for direct limit of topological spaces [May99, §14.6]: for each map $f_n\colon X_n \to X_{n+1}$ we take the mapping cylinder M_{f_n}, and we identify the copies of X_n for M_{f_n} and $M_{f_{n-1}}$. The result is a "telescope"



If X_n are CW-complexes, then it is an increasing sequence of CW-complexes

$$T_1 \subset T_2 \subset T_3 \subset \cdots$$

(T_n being the union of the first n mapping cylinders) which deformation retracts on X_n. Hence the direct limit is $\varinjlim X_n = \varinjlim T_n = \bigcup T_n$.

In our case of S^1 this telescope $\bigcup T_n$ gives some space $S^1_{\mathbb{Q}}$ together with a map $S^1 \to S^1_{\mathbb{Q}}$ (inclusion of the base of the telescope). Now we have

$$\pi_i(\varinjlim S^1) \cong \varinjlim \pi_i(S^1) \cong \begin{cases} \mathbb{Q}, & i = 1, \\ 0, & i \neq 1. \end{cases}$$

We see that the map $S^1 \to S^1_{\mathbb{Q}}$ induces rationalization of $\pi_1(S^1)$. Similarly, one could check the isomorphism $H_i(S^1;\mathbb{Q}) \cong H_i(S^1_{\mathbb{Q}};\mathbb{Q})$ using the fact that homology commutes with directed limits [May99, §14.6]:

$$H_i(\varinjlim S^1) \cong \varinjlim H_i(S^1).$$

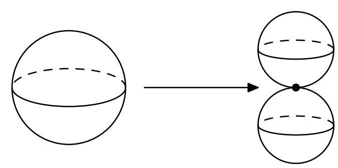
So the telescope gives the rationalization of S^1. ▲

Observe that S^1 is an Eilenberg–Mac Lane space $K(\mathbb{Z},1)$, and its rationalization $S^1_{\mathbb{Q}}$ is an Eilenberg–Mac Lane space $K(\mathbb{Q},1)$.

Theorem 3.3.6. *For any abelian group A the rationalization of an Eilenberg–Mac Lane space $K(A,n)$ is given by the map*

$$K(A,n) \to K(A \otimes_{\mathbb{Z}} \mathbb{Q}, n).$$

Observe that the crucial point in the construction of $S^1_{\mathbb{Q}}$ was the multiplication by n map $n\colon S^1 \to S^1$, i.e. the fact that S^1 is an H-space. Now let X be an arbitrary H-space with multiplication $\mu\colon X \times X \to X$. This gives a point-wise multiplication of maps $f\colon S^n \to X$, which is homotopic to the product induced by the pinch map $S^n \to S^n \vee S^n$.



It follows that the product on an H-space induces addition in $\pi_i(X)$:

$$[\mu(f,g)] = [f] + [g].$$

So the maps $\mu_n\colon X \to X$ given by

$$x \mapsto \text{“}x^n\text{”} \overset{\text{def}}{=} \underbrace{\mu(x,\mu(x,\mu(x,\cdots)))}_{n}$$

induce multiplication $[f] \mapsto n \cdot [f]$ on $\pi_i(X)$. The multiplication $\mu\colon X \times X \to X$ may not be associative, but we just put brackets in the definition as we like.

$$\begin{array}{c} X \xrightarrow{1} X \xrightarrow{\mu_2} X \xrightarrow{\mu_3} X \xrightarrow{\mu_4} X \longrightarrow \cdots \\ \pi_i(X) \xrightarrow{1} \pi_i(X) \xrightarrow{2} \pi_i(X) \xrightarrow{3} \pi_i(X) \xrightarrow{4} \pi_i(X) \to \cdots \end{array}$$

So we have the very same situation as with S^1, and we see the following

Proposition 3.3.7. *For any H-space X the described telescoping construction gives the rationalization $\phi\colon X \to X_{\mathbb{Q}}$.*

3.4 Cartan–Serre theorem

Now if X is a connected H-space, then the diagonal $\Delta\colon X \to X \times X$ induces a product on $H^\bullet(X;\mathbb{Q})$, the multiplication $\mu\colon X \times X \to X$ induces a coproduct on $H^\bullet(X;\mathbb{Q})$, and we have a commutative, associative, connected quasi-Hopf algebra (it may be not co-associative and not co-commutative, depending on the H-space).

From theorem 3.2.11 we know that A is isomorphic as an algebra to the tensor product of an exterior algebra on odd degree generators and a polynomial algebra on even degree generators. The cohomology of the Eilenberg–Mac Lane spaces $K(\mathbb{Q},n)$ gives exactly exterior and polynomial algebras:

Proposition 3.4.1. *Let $\iota_n \in H^n(K(\mathbb{Q},n);\mathbb{Q})$ denote the "fundamental class" represented by the identity map $K(\mathbb{Q},n) \to K(\mathbb{Q},n)$. The cohomology algebra $H^\bullet(K(\mathbb{Q},n);\mathbb{Q})$ is*

- *the exterior algebra $\mathbb{Q}[\iota_n]/\iota_n^2$ on ι_n,* ***if n is odd***
  (in particular, this shows that $K(\mathbb{Q},n) = S^n_{\mathbb{Q}}$),
- *the polynomial algebra $\mathbb{Q}[\iota_n]$ on ι_n,* ***if n is even.***

This is proved by induction on n using the path space fibration $K(\mathbb{Q},n) \to PK(\mathbb{Q},n+1) \to K(\mathbb{Q},n+1)$. See example H.3.4 or [tD08, §20.7]. ∎

Assume now that X is a *rational* H-space such that its homology groups

$$H_i(X;\mathbb{Z}) \cong H_i(X;\mathbb{Q})$$

are finite dimensional $\mathbb{Q}$-vector spaces. The generators in each degree can be represented by maps $f\colon X \to K(\mathbb{Q},n)$, and this gives

$$X \to \prod_n \underbrace{K(\mathbb{Q},n) \times \cdots \times K(\mathbb{Q},n)}_{\cong K(\pi_n(X),n)},$$

inducing an isomorphism on cohomology

$$H^\bullet(X;\mathbb{Q}) \xrightarrow{\cong} \bigotimes_n H^\bullet(K(\pi_n(X),n);\mathbb{Q}).$$

By our assumption that $H_i(X;\mathbb{Q})$ are finite dimensional, we can use the Künneth formula, and also we can pass to an isomorphism of homology groups

$$H_\bullet(X;\mathbb{Q}) \xrightarrow{\cong} \bigotimes_n H_\bullet(K(\pi_n(X),n);\mathbb{Q}) \cong H_\bullet(\prod_n K(\pi_n(X),n);\mathbb{Q}).$$

Now observe that both spaces X and $\prod_n K(\pi_n(X),n)$ are nilpotent and rational (cf. example 3.3.3), and we should conclude that we have a homotopy equivalence

$$X \simeq \prod_n K(\pi_n(X),n)$$

(e.g. from the universality of rationalization).

The rational homology $H_\bullet(X;\mathbb{Q})$ is a cocommutative Hopf algebra, and we look at its space of primitive elements $PH_\bullet(X;\mathbb{Q})$. Since the primitive elements are defined by the comultiplication (coming from $\Delta\colon X \to X \times X$) and they do not depend on the multiplication (coming from the H-space structure), we can replace X with the corresponding product of rational Eilenberg–Mac Lane spaces $K(\mathbb{Q},n)$.

Observe that for products of spaces we have

$$\begin{aligned} \pi_\bullet(Y \times Z) &\cong \pi_\bullet(Y) \oplus \pi_\bullet(Z), \\ PH_\bullet(Y \times Z;\mathbb{Q}) &\cong PH_\bullet(Y;\mathbb{Q}) \oplus PH_\bullet(Z;\mathbb{Q}). \end{aligned}$$

Now for an Eilenberg–Mac Lane space $K(\mathbb{Q},n)$ the Hurewicz homomorphism

$$h\colon \pi_\bullet(K(\mathbb{Q},n)) \to H_\bullet(K(\mathbb{Q},n);\mathbb{Q})$$

sends $\pi_\bullet(K(\mathbb{Q},n))$ to the subspace of primitive elements (one checks this e.g. using the calculation mentioned in proposition 3.4.1). Hence we have the following:

Theorem 3.4.2 (Cartan–Serre). *Let X be a homotopy associative H-space with finite dimensional $H_n(X;\mathbb{Q})$. Then the Hurewicz homomorphism*

$$h\colon \pi_\bullet(X_\mathbb{Q}) \cong \pi_\bullet(X) \otimes_\mathbb{Z} \mathbb{Q} \to H_\bullet(X_\mathbb{Q};\mathbb{Z}) \cong H_\bullet(X;\mathbb{Q})$$

is a monomorphism, and its image is the subspace of primitive elements.

Dually, if X is a homotopy associative and homotopy commutative H-space, then $\pi_\bullet(X) \otimes_\mathbb{Z} \mathbb{Q}$ can be identified with indecomposable elements in cohomology $H^\bullet(X;\mathbb{Q})$ (see theorem 3.2.10).

* * *

With this we say goodbye to the homotopical methods, since now we know that all the remaining difficulties are in computing real group cohomology $H^\bullet(SL(\mathcal{O}_F),\mathbb{R})$.

Chapter 4

Calculation of $\operatorname{rk} K_i(\mathcal{O}_F)$ via the stable cohomology of SL_n

Now we finally calculate the ranks $\operatorname{rk} K_i(\mathcal{O}_F)$. In the previous chapter we established

$$\operatorname{rk} K_i(\mathcal{O}_F) = \dim_{\mathbb{R}} QH^i(SL(\mathcal{O}_F), \mathbb{R}) \quad (i \geqslant 2).$$

$H^\bullet(SL(\mathcal{O}_F), \mathbb{R})$ is the cohomology ring of the infinite special linear group $SL(\mathcal{O}_F) \stackrel{\text{def}}{=} \varinjlim_n SL_n(\mathcal{O}_F)$. Here $\mathbb{R}$ is viewed as an $SL(\mathcal{O}_F)$-module with the trivial action. "Q" means that we take indecomposable elements. This suggests that one should look at cohomology for each $SL_n(\mathcal{O}_F)$ and then pass to the limit. In fact cohomology of $SL_n(\mathcal{O}_F)$ is very difficult, but it stabilizes and becomes tractable as $n \to \infty$. This chapter is supposed to explain that. We take for granted certain property of stable cohomology of arithmetic groups from [Bor74].

References. This chapter follows [Bor72] and [Bor74, §10-12].

4.1 The setting

Although SL_n is the only thing we care about, let us fix slightly more general assumptions and notation.

- Let G be a semisimple linear algebraic group defined over $\mathbb{Q}$. We will have in mind $G = SL_n/\mathbb{Q}$. In general, if a group G' defined over a number field F (e.g. $G = SL_n/F$), then we take the restriction of scalars $G = \operatorname{Res}_{F/\mathbb{Q}} G'$—see § A.2.
- The group of real points $G(\mathbb{R})$ is a Lie group, and for our purposes we assume that $G(\mathbb{R})$ is non-compact and connected. For instance, this is the case for SL_n.
- Let $\Gamma \subset G(\mathbb{R})$ be an arithmetic subgroup inside $G(\mathbb{R})$. We will have in mind $\Gamma = SL_n(\mathbb{Z})$.
- Let K be a maximal compact subgroup of $G(\mathbb{R})$—cf. [Hel01, §VI.1, VI.2]. They are all conjugate.

  For example, a maximal compact subgroup of $SL_n(\mathbb{R})$ can be identified with $SO_n(\mathbb{R})$, the subgroup of matrices that preserve the standard bilinear form on $\mathbb{R}^n$:

  $$\langle x, y\rangle \stackrel{\text{def}}{=} \sum_{1\leqslant i\leqslant n} x_i\, y_i,$$

  and have determinant 1. In other words,

  $$SO_n(\mathbb{R}) = \{A \in SL_n(\mathbb{R}) \mid A^\top A = A\, A^\top = I\}.$$

For the complexification $SL_n(\mathbb{C})$, a maximal compact subgroup can be identified with the **special unitary group** SU_n, the subgroup of complex matrices that preserve the standard Hermitian form on $\mathbb{C}^n$:

$$\langle x, y\rangle \stackrel{\text{def}}{=} \sum_{1\leqslant i\leqslant n} x_i\,\overline{y_i},$$

and have determinant 1. In other words,

$$SU_n \stackrel{\text{def}}{=} \{A \in SL_n(\mathbb{C}) \mid A^\dagger A = A\,A^\dagger = I\},$$

where $\dagger$ denotes the conjugate transpose. This group naturally contains $SO_n(\mathbb{R})$.

- The right cosets of K in $G(\mathbb{R})$ form the **symmetric space of maximal compact subgroups** $X \stackrel{\text{def}}{=} G(\mathbb{R})/K$ (recall that for any Lie group $G(\mathbb{R})$ factor by a compact subgroup $K \subset G(\mathbb{R})$ is smooth). Endowed with a $G(\mathbb{R})$-invariant Riemannian metric, it is a complete symmetric Riemannian manifold with negative curvature, diffeomorphic to Euclidean space.

- Let $G(\mathbb{R})_u$ be a maximal compact subgroup of the complexification $G(\mathbb{C})$ of $G(\mathbb{R})$, such that $G(\mathbb{R})_u \supset K$. Then $X_u \stackrel{\text{def}}{=} G(\mathbb{R})_u/K$ is called the **dual** symmetric space to X, and it is compact.

  The main example of this duality to keep in mind is the following:

$G(\mathbb{R})$	X	X_u
$SL_n(\mathbb{R})$	$SL_n(\mathbb{R})/SO_n(\mathbb{R})$	$SU_n/SO_n(\mathbb{R})$
$SL_n(\mathbb{C})$	$SL_n(\mathbb{C})/SU_n$	SU_n

- We denote by $\mathfrak{g}$ the Lie algebra of $G(\mathbb{R})$ and by $\mathfrak{k}$ the Lie algebra of K.

Example 4.1.1. Look at a group

$$SL_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = 1 \right\}.$$

$SL_2(\mathbb{R})$ acts transitively on the upper half space $\mathcal{H} \stackrel{\text{def}}{=} \{z \in \mathbb{C} \mid \operatorname{im} z > 0\}$ by Möbius transformations

$$z \mapsto \frac{az+b}{cz+d}.$$

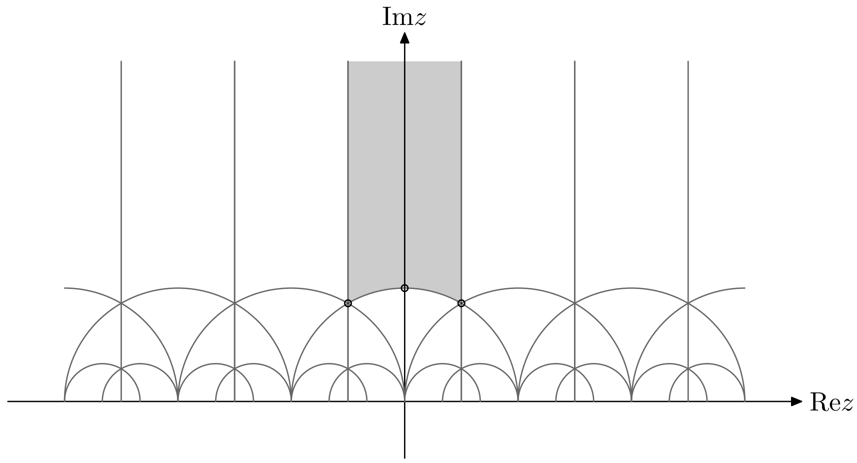
(This action is not faithful; usually one considers faithful action of $PSL_2(\mathbb{R}) \stackrel{\text{def}}{=} SL_2(\mathbb{R})/\{\pm I\}$.) The stabilizer of $i \in \mathcal{H}$ is given by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $\frac{ai+b}{ci+d} = i$, i.e. $ai + b = di - c$, that is

$$\left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid a^2 + b^2 = 1 \right\} = \left\{ \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \mid \phi \in [0, 2\pi) \right\} = SO_2(\mathbb{R}).$$

This is the “circle group”, a maximal compact subgroup in $SL_2(\mathbb{R})$. It is not a normal subgroup, but we still can consider the cosets $X = SL_2(\mathbb{R})/SO_2(\mathbb{R})$. Since $SO_2(\mathbb{R})$ is the stabilizer of i and the action is transitive, one has $X = \mathcal{H}$.

Consider now the discrete subgroup $\Gamma \stackrel{\text{def}}{=} SL_2(\mathbb{Z}) \subset SL_2(\mathbb{R})$. It naturally acts on X, and we are interested in the set $\Gamma\backslash X$. As we know [Ser73, §VII.1], a fundamental domain of the action of $SL_2(\mathbb{Z})$ on $\mathcal{H}$ can be given by

$$\{z \in \mathcal{H} \mid |z| > 1, |\operatorname{Re}(z)| < 1/2\}.$$



Note that $\Gamma\backslash X$ is not compact and it is neither a smooth manifold: the two points coming from i and $\frac{1}{2}+\frac{\sqrt{3}}{2}$ are singular; in fact it is an **orbifold** (cf. [ALR07]). The problem is that $SL_2(\mathbb{Z})$ has torsion; we will go back to this in example 4.3.3. ▲

4.2 De Rham complex

Just to fix some notation which will be used in the subsequent chapters as well, we recall de Rham cohomology of smooth manifolds.

Let M be a smooth manifold (of class $\mathscr{C}^\infty$). We denote by $\Omega^q(M)$ the space of smooth real-valued exterior differential q-forms on M. All these spaces form a graded $\mathbb{R}$-algebra with respect to exterior multiplication $\wedge$:

$$\Omega^\bullet(M) \stackrel{\text{def}}{=} \bigoplus_{q\geqslant 0} \Omega^q(M).$$

We have **de Rham differential** (also called **exterior derivative**) $d\colon \Omega^\bullet(M) \to \Omega^{\bullet+1}(M)$:

$$df \stackrel{\text{def}}{=} \text{differential of } f \quad \text{for } f \in \Omega^0(M) = \mathscr{C}^\infty(M),$$
$$d(\alpha\wedge\beta) \stackrel{\text{def}}{=} (d\alpha)\wedge\beta + (-1)^q\,\alpha\wedge(d\beta) \quad \text{for } \alpha\in\Omega^q(M).$$

These differentials form **de Rham cochain complex**

$$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \to \cdots$$

that is, $d\circ d = 0$. According to **de Rham theorem**, cohomology of the complex above is isomorphic to the usual singular cohomology:

$$H^q_{\text{dR}}(M) \stackrel{\text{def}}{=} \frac{\textbf{closed } q\textbf{-forms}}{\textbf{exact } q\textbf{-forms}} \stackrel{\text{def}}{=} \frac{\ker(\Omega^q(M) \xrightarrow{d} \Omega^{q+1}(M))}{\operatorname{im}(\Omega^{q-1}(M) \xrightarrow{d} \Omega^q(M))} \cong H^q(M;\mathbb{R}).$$

Remark 4.2.1. Let us recall the framework for de Rham theorem.

Assume that $\mathscr{F}$ is a sheaf on a smooth manifold M. An **acyclic resolution** of $\mathscr{F}$ is a long exact sequence of sheaves

$$0 \to \mathscr{F} \xrightarrow{\alpha} \mathscr{A}^0 \xrightarrow{d^0} \mathscr{A}^1 \xrightarrow{d^1} \mathscr{A}^2 \to \cdots$$

such that $H^q(M,\mathscr{A}^i) = 0$ for all $q \geqslant 1$.

Then the **abstract de Rham theorem** states that if for such an acyclic resolution one takes the complex of global sections

$$0 \to \mathcal{F}(M) \xrightarrow{\alpha_M} \mathcal{A}^0(M) \xrightarrow{d^0_M} \mathcal{A}^1(M) \xrightarrow{d^1_M} \mathcal{A}^2(M) \to \cdots$$

then $H^q(M, \mathcal{F}) \cong H^q(\mathcal{A}^\bullet(M), d^\bullet_M)$, where on the left hand side is the standard sheaf cohomology.

If $\mathcal{F} = \underline{\mathbb{R}}$ is the sheaf of locally constant functions, then de Rham complex gives an acyclic resolution

$$0 \to \underline{\mathbb{R}} \to \Omega^0 \to \Omega^1 \to \Omega^2 \to \cdots$$

Similarly, singular cohomology corresponds to another acyclic resolution of $\underline{\mathbb{R}}$

$$0 \to \underline{\mathbb{R}} \to \mathcal{S}^0 \to \mathcal{S}^1 \to \mathcal{S}^2 \to \cdots$$

($\mathcal{S}^q$ is the sheafification of the presheaf of singular q-cochains $U \mapsto S^q(U) \stackrel{\text{def}}{=} \operatorname{Hom}(\text{singular } q\text{-chains on } U, \mathbb{R})$, and the morphisms are induced by the usual simplicial differentials).

Putting together the two resolutions of $\underline{\mathbb{R}}$, we get

$$H^q(M; \mathbb{R}) \stackrel{\text{def}}{=} H^q(S^\bullet(M)) \cong H^q(M, \underline{\mathbb{R}}) \cong H^q(\Omega^\bullet(M)) \stackrel{\text{def}}{=} H^q_{\text{dR}}(M).$$

Details on this can be found e.g. in [Wel08, Chapter II] or [War83, Chapter 5].

We recall that a sheaf $\mathcal{F}$ on a manifold M is **soft** if for any closed subset $S \subset M$ the restriction $\mathcal{F}(M) \to \mathcal{F}(S)$ is surjective. Further, a sheaf $\mathcal{F}$ is **fine** if for any locally finite open cover $\{U_i\}$ of M there exists a subordinate **partition of unity**, that is a family of sheaf morphisms $\eta_i : \mathcal{F} \to \mathcal{F}$ such that

1. $\sum \eta_i = 1$.
2. $\eta_i(\mathcal{F}_x) = 0$ for all x in some neighborhood of the complement of U_i.

For instance, Ω^q are fine sheaves.

Any soft sheaf is fine [Wel08, Proposition II.3.5], and *for any fine sheaf one has $H^q(M, \mathcal{F}) = 0$ for $q \geqslant 1$ [Wel08, Theorem II.3.11]*. Hence resolution by soft or fine sheaves is acyclic.

To sum up all the above, in order to show that some cohomology theory agrees with the singular / de Rham cohomology, it is enough to show that one has a resolution of $\underline{\mathbb{R}}$ by fine sheaves.

Definition 4.2.2. We say that an $\mathbb{R}$-linear map $D : \Omega^\bullet(M) \to \Omega^\bullet(M)$ is a **derivation of degree ℓ** if it sends an element $\alpha \in \Omega^q(M)$ to an element $D(\alpha) \in \Omega^{q+\ell}(M)$, and satisfies the graded **Leibniz rule**

$$D(\alpha \wedge \beta) = D(\alpha) \wedge \beta + (-1)^{q\ell} \alpha \wedge D(\beta) \quad \text{for } \alpha \in \Omega^q(M).$$

The usual de Rham differential $d : \Omega^\bullet(M) \to \Omega^{\bullet+1}(M)$ is a derivation of degree 1.

Definition 4.2.3. A graded algebra coming with a derivation d of degree ± 1 such that $d \circ d = 0$ is called a **differential graded algebra** (or just **DG-algebra**).

So $\Omega^\bullet(M)$ with de Rham differential is a DG-algebra.

If D_1 is a derivation of degree ℓ_1 and D_2 is a derivation of degree ℓ_2, then their **graded commutator** is given by

$$[D_1, D_2] \stackrel{\text{def}}{=} D_1 \circ D_2 + (-1)^{\ell_1 \ell_2 + 1} D_2 \circ D_1.$$

Observe that $[D_1, D_2]$ is a derivation of degree $\ell_1 + \ell_2$:

$$\begin{aligned}
[D_1, D_2](\alpha \wedge \beta) &= D_1(D_2(\alpha) \wedge \beta + (-1)^{q\ell_2}\, \alpha \wedge D_2(\beta)) + \\
&\quad (-1)^{\ell_1\ell_2+1}\, D_2(D_1(\alpha) \wedge \beta + (-1)^{q\ell_1}\, \alpha \wedge D_1(\beta)) \\
&= D_1\, D_2(\alpha) \wedge \beta + (-1)^{q\ell_1+\ell_1\ell_2}\, \cancel{D_2(\alpha) \wedge D_1(\beta)} + \\
&\quad (-1)^{q\ell_2}\, \cancel{D_1(\alpha) \wedge D_2(\beta)} + (-1)^{q\,(\ell_1+\ell_2)}\, \alpha \wedge D_1 D_2(\beta) + \\
&\quad (-1)^{\ell_1\ell_2+1}\, D_2 D_1(\alpha) \wedge \beta + (-1)^{q\ell_2+1}\, \cancel{D_1(\alpha) \wedge D_2(\beta)} + \\
&\quad (-1)^{q\ell_1+\ell_1\ell_2+1}\, \cancel{D_2(\alpha) \wedge D_1(\beta)} + (-1)^{q\,(\ell_1+\ell_2)+\ell_1\ell_2+1}\, \alpha \wedge D_2 D_1(\beta) \\
&= (D_1\, D_2(\alpha) + (-1)^{\ell_1\ell_2+1}\, D_2 D_1(\alpha)) \wedge \beta \\
&\quad\; + (-1)^{q\,(\ell_1+\ell_2)}\, \alpha \wedge (D_1 D_2(\beta) + (-1)^{\ell_1\ell_2+1} D_2 D_1(\beta)) \\
&= [D_1, D_2](\alpha) \wedge \beta + (-1)^{q\,(\ell_1+\ell_2)}\, \alpha \wedge [D_1, D_2](\beta).
\end{aligned}$$

On $\Omega^\bullet(M)$ there is also a derivation of degree -1. For any vector field $X \in \Gamma(TM)$ one has the **contraction** operator $\iota_X\colon \Omega^\bullet(M) \to \Omega^{\bullet-1}(M)$:

$$\begin{aligned}
\iota_X \theta &\stackrel{\text{def}}{=} \theta(X) \quad \text{for } \theta \in \Omega^1(M), \\
\iota_X(\alpha \wedge \beta) &\stackrel{\text{def}}{=} (\iota_X \alpha) \wedge \beta + (-1)^q\, \alpha \wedge (\iota_X \beta) \quad \text{for } \alpha \in \Omega^q(M).
\end{aligned}$$

Here $\theta(X)$ is a function given by $x \mapsto \theta_x(X_x)$, where by X_x we denote the corresponding element of $T_x M$. So d is a derivation of degree $+1$ and ι_X is a derivation of degree -1, which means there is a derivation of degree 0 given by the commutator:

$$\mathcal{L}_X = [d, \iota_X] = d \circ \iota_X + \iota_X \circ d.$$

The operator on the left hand side is known as the **Lie derivative** (cf. [War83, 2.24–2.25] or [Spi99a, Chapter 5 + Exercise 7.18]), and the identity above is known as **Cartan's magic formula** (due to Élie Cartan). In particular, for a function $f \in \mathcal{C}^\infty(M)$ its Lie derivative $\mathcal{L}_X f$ is just the application of a vector field $X \in \Gamma(TM(M))$ viewed as a first order differential operator:

$$\begin{aligned}
X\colon \mathcal{C}^\infty(M) &\to \mathcal{C}^\infty(M), \\
f &\mapsto X(f).
\end{aligned}$$

This satisfies the Leibniz rule

$$X(f \cdot g) = X(f) \cdot g + f \cdot X(g).$$

For two vector fields $X, Y \in \Gamma(TM)$ one can define the Lie derivative $\mathcal{L}_X Y = [X, Y]$, which is known as **Lie bracket** [War83, 2.24-2.25], and then one can work out a formula for the Lie derivative of a differential form $\alpha \in \Omega^q(M)$:

$$\mathcal{L}_{X_0}(\alpha(X_1 \wedge \cdots \wedge X_q)) = (\mathcal{L}_{X_0}\alpha)(X_1 \wedge \cdots \wedge X_q) + \sum_{1 \leqslant i \leqslant q} \alpha(X_1 \wedge \cdots \wedge X_{i-1} \wedge [X_0, X_i] \wedge X_{i+1} \wedge \cdots \wedge X_q).$$

For instance, if $q = 1$, then this formula reads

$$\mathcal{L}_{X_0}(\alpha(X_1)) = (\mathcal{L}_{X_0}\alpha)(X_1) + \alpha[X_0, X_1].$$

One has $\mathcal{L}_{X_0}(\alpha(X_1)) = X_0 \cdot \alpha(X_1)$, and applying Cartan's magic formula to the right hand side,

$$X_0 \cdot \alpha(X_1) = d\alpha(X_0, X_1) + X_1 \cdot \alpha(X_0) + \alpha[X_0, X_1].$$

This can be written as

$$d\alpha(X_0, X_1) = X_0 \cdot \alpha(X_1) - X_1 \cdot \alpha(X_0) - \alpha[X_0, X_1].$$

Proceeding similarly by induction with Cartan's magic formula, we deduce a formula for differentials that involves Lie brackets:

$$\begin{aligned} d\alpha(X_0 \wedge \ldots \wedge X_q) = &\sum_{0 \leqslant i < j \leqslant q} (-1)^{i+j} \alpha([X_i, X_j] \wedge X_0 \wedge \ldots \wedge \widehat{X}_i \wedge \ldots \wedge \widehat{X}_j \wedge \ldots \wedge X_q) \\ &+ \sum_{0 \leqslant i \leqslant q} (-1)^i X_i \cdot \alpha(X_0 \wedge \ldots \wedge \widehat{X}_i \wedge \ldots \wedge X_q). \end{aligned} \tag{4.1}$$

4.3 Group cohomology

We recall briefly that in general for a group Γ and a Γ-module V the i-th cohomology is defined by

$$H^q(\Gamma, V) \stackrel{\text{def}}{=} \mathrm{Ext}^q_{\mathbb{Z}\Gamma}(\mathbb{Z}, V).$$

So one can start with a projective resolution of $\mathbb{Z}$ by $\mathbb{Z}\Gamma$-modules:

$$\cdots \to P_2 \to P_1 \to P_0 \to \mathbb{Z} \to 0$$

then apply to this the contravariant functor $\mathrm{Hom}_{\mathbb{Z}\Gamma}(-, V)$, and calculate $H^q(\Gamma, V) = H^q(\mathrm{Hom}_{\mathbb{Z}\Gamma}(P_\bullet, V))$. In practice one usually applies **bar-resolution** [Wei94, §6.5] that results in taking cochains

$$C^q(\Gamma; V) \stackrel{\text{def}}{=} \mathrm{Hom}_{\mathbb{Z}\Gamma}(\Gamma^{q+1}, V),$$

which is a Γ-module by means of the action $(x \cdot f)(x_0, \ldots, x_q) \stackrel{\text{def}}{=} f(x_0 \cdot x, \ldots, x_q \cdot x)$. The differentials are given by

$$df(x_0, \ldots, x_q) \stackrel{\text{def}}{=} x_0 \cdot f(x_1, \ldots, x_q) + \sum_{0 \leqslant i \leqslant q} (-1)^{i+1} f(x_0, \ldots, x_i x_{i+1}, \ldots, x_q) + (-1)^{q+1} f(x_0, \ldots, x_{q-1}). \tag{4.2}$$

(This is the so-called "non-homogeneous resolution".)

One gets an augmented cochain complex

$$0 \to V \xrightarrow{\epsilon} C^0(\Gamma; V) \xrightarrow{d} C^1(\Gamma; V) \xrightarrow{d} C^2(\Gamma; V) \to \cdots$$

where the augmentation ϵ is given by sending $v \in V$ to the function $x \mapsto x \cdot v$ on Γ.

Now $H^q(\Gamma, V) = H^q(C^\bullet(\Gamma; V), d)$.

Remark 4.3.1. We recalled the above also to make the following definition.

Assume that G is a topological group and V is a G-module with continuous action $G \to GL(V)$. Consider the augmented cochain complex as above with cochains $C^q(G; V)$ replaced by *continuous* maps. Cohomology of the resulting complex

$$H^q_{\mathrm{ct}}(G, V) \stackrel{\text{def}}{=} H^q(C^\bullet_{\mathrm{ct}}(G; V), d)$$

is called **continuous cohomology**.

Similarly, let G be a Lie group and let V be a G-module with a smooth action $G \to GL(V)$. If we replace the cochains with *differentiable* maps (of class $\mathscr{C}^\infty$), then **differentiable cohomology** is given by

$$H^q_{\mathrm{d}}(G, V) \stackrel{\text{def}}{=} H^q(C^\bullet_{\mathrm{d}}(G; V), d).$$

Since any differentiable cochain is continuous, one gets a map $H^q_{\mathrm{d}}(G, V) \to H^q_{\mathrm{ct}}(G, V)$, which is an isomorphism if V is "quasi-complete". For further discussion of continuous and differentiable cohomology we refer to [BW00, Chapter IX] and [Gui80]. We will not make use of it.

Let us recall a couple of basic properties of group cohomology [Bro94, Proposition II.10.2 and III.10.4]:

Proposition 4.3.2. *Assume that V is a vector space over a field of characteristic zero. Then*

(1) If Γ is a finite group, then $H^q(\Gamma, V) = 0$ for $q \neq 0$.

(2) If $\Gamma' \lhd \Gamma$ is a normal subgroup of finite index, then

$$H^q(\Gamma, V) \cong H^q(\Gamma', V)^{\Gamma/\Gamma'}.$$

Working with explicit formulas like (4.2) is not very insightful, so let us take a geometric approach.

Recall that we have a Lie group $G(\mathbb{R})$ and a symmetric space $X \stackrel{\text{def}}{=} G(\mathbb{R})/K$. The action of Γ on X by left translations is *proper* (given a compact set $C \subset X$, the set $\{\gamma \in \Gamma \mid C \cap \gamma \cdot C \neq \emptyset\}$ is finite). Suppose also that the action is *free*. Then $\Gamma \backslash X$ is a smooth manifold, and it is the Eilenberg–Mac Lane space $K(\Gamma, 1)$, so that

$$H^\bullet(\Gamma, \mathbb{R}) \cong H^\bullet(\Gamma \backslash X, \mathbb{R}), \tag{4.3}$$

where on the right hand side is the usual singular, or de Rham cohomology. It is a standard topological interpretation of group cohomology—cf. e.g. [Bro94, §II.4].

If Γ has torsion, then the action of Γ on X is not free, and we cannot use (4.3). But according to Selberg's lemma (proposition A.3.5), Γ contains a torsion free normal subgroup of finite index $\Gamma' \lhd \Gamma$, which it is enough for our purposes.

Example 4.3.3. Consider $\Gamma = SL_2(\mathbb{Z})$. There are two elements

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

with S of order 4 and ST of order 6, so $SL_2(\mathbb{Z})$ has torsion. However, one can find a torsion free subgroup of finite index inside $SL_2(\mathbb{Z})$. Observe that if a matrix x has finite order α, then it satisfies an equation $X^\alpha - 1 = 0$. The minimal polynomial $P(X) \in \mathbb{Q}[X]$ for x has distinct roots (the eigenvalues), and these are necessarily roots of unity. The trace of x is $\leqslant 2$.

Take any prime $p > 2$ and consider the reduction modulo p homomorphism

$$GL_2(\mathbb{Z}) \to GL_2(\mathbb{Z}/p\mathbb{Z}).$$

Its kernel $\Gamma(p) \leqslant GL_2(\mathbb{Z})$ has finite index; more precisely, we know that

$$\# GL_2(\mathbb{Z}/p\mathbb{Z}) = (p^2 - 1)\,(p^2 - p).$$

Now if $x \in \Gamma(p)$ is an element of finite order, then we know that $\operatorname{tr} x \leqslant 2$ and $\operatorname{tr} x \equiv 2 \pmod p$. But since we took $p > 2$, this means $\operatorname{tr} x = 2$. Since x is a diagonalizable matrix (the minimal polynomial has distinct roots), we must conclude $x = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

For $SL_2(\mathbb{Z})$ we take the subgroup $SL_2(\mathbb{Z}) \cap \Gamma(p)$, i.e. the kernel of $SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/p\mathbb{Z})$. It is torsion free by what we just said and it has finite index (which equals $p^3 - p$). It is known as the **principal congruence subgroup of level** p. ▲

The argument in the example of $SL_2(\mathbb{Z})$ is actually quite general. We refer to § A.3 for a full proof.

Using torsion free normal subgroups of finite index, we can deduce

Proposition 4.3.4. *One has an isomorphism*

$$H^q(\Gamma, \mathbb{R}) \cong H^q(\Omega^\bullet(X)^\Gamma), \tag{4.4}$$

where $\Omega^\bullet(X)$ is de Rham complex of X, and $\Omega^\bullet(X)^\Gamma$ is the subcomplex of Γ-invariant differential forms.

Proof. 1. If Γ is torsion-free, then we have (4.3). Using de Rham theorem (cf. remark 4.2.1) we deduce that $H^q(\Omega^\bullet(X)^\Gamma) \cong H^q(\Gamma\backslash X, \mathbb{R})$.

2. If Γ has torsion, take a torsion free normal subgroup of finite index $\Gamma' \lhd \Gamma$. The factor group Γ/Γ' acts on $H^q(\Gamma', \mathbb{R})$, and by the second part of proposition 4.3.2,

$$H^q(\Gamma, \mathbb{R}) \cong (H^q(\Gamma', \mathbb{R}))^{\Gamma/\Gamma'}.$$

We have $\Omega^\bullet(X)^\Gamma = (\Omega^\bullet(X)^{\Gamma'})^{\Gamma/\Gamma'}$. The group Γ/Γ' is finite, so by the first part of proposition 4.3.2,

$$H^q(\Omega(X)^\Gamma) \cong H^q(\Omega^\bullet(X)^{\Gamma'})^{\Gamma/\Gamma'}.$$

Hence all reduces to the torsion free case. ■

In fact the problem is that when Γ has torsion, the space $\Gamma\backslash X$ is not a smooth manifold but an orbifold. In this case we need a de Rham theorem for orbifolds. Cf. [ALR07, Chapter 2].

More generally, if $\Gamma \to GL(V)$ is a finite dimensional real or complex representation of Γ, then

$$H^\bullet(\Gamma, V) \cong H^\bullet((\Omega(X) \otimes V)^\Gamma).$$

For this see [BW00, §VII.2]. Our representations are trivial.

4.4 Lie algebra cohomology

Let $\mathfrak{g}$ be a real Lie algebra over $\Bbbk$ acting on a $\Bbbk$-vector space V. We will have in mind $\Bbbk = \mathbb{R}$ or $\mathbb{C}$. One can define cohomology $H^\bullet(\mathfrak{g}; V)$.

More precisely, let V be a $\mathfrak{g}$-**module**, i.e. a $\Bbbk$-vector space together with a morphism of Lie algebras $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$. Equivalently, a $\mathfrak{g}$-module can be viewed as a module over the ring $\mathcal{U}(\mathfrak{g})$, the universal enveloping algebra of $\mathfrak{g}$. The corresponding action of elements $x \in \mathfrak{g}$ on $v \in V$ will be denoted by $x \cdot v$.

The situation is the same as for group cohomology: one has

$$H^q(\mathfrak{g}; V) \stackrel{\text{def}}{=} \operatorname{Ext}^q_{\mathcal{U}(\mathfrak{g})}(\Bbbk, V).$$

A particular projective resolution of $\Bbbk$ by $\mathcal{U}(\mathfrak{g})$-modules gives rise to the **Chevalley–Eilenberg–Koszul complex**. It results in the following formulas. As cochains one takes

$$C^q(\mathfrak{g}; V) \stackrel{\text{def}}{=} \operatorname{Hom}_{\mathbb{R}}(\bigwedge^q \mathfrak{g}, V) = \bigwedge^q \mathfrak{g}^\vee \otimes_\Bbbk V,$$

and the differentials $d^q\colon C^q(\mathfrak{g}; V) \to C^{q+1}(\mathfrak{g}; V)$ are given by

$$\begin{aligned} d^q f(x_0 \wedge \cdots \wedge x_q) \stackrel{\text{def}}{=} & \sum_{0 \leqslant i < j \leqslant q} (-1)^{i+j} f([x_i, x_j] \wedge x_0 \wedge \cdots \wedge \widehat{x}_i \wedge \cdots \wedge \widehat{x}_j \wedge \cdots \wedge x_q) \\ & + \sum_{0 \leqslant i \leqslant q} (-1)^i x_i \cdot f(x_0 \wedge \cdots \wedge \widehat{x}_i \wedge \cdots \wedge x_q). \end{aligned} \tag{4.5}$$

In particular, the zeroth differential for $v \in V$ is

$$d^0 v(x_0) \stackrel{\text{def}}{=} x_0 \cdot v.$$

As always, $\hat{x}_i$ means that x_i is omitted. Then $d \circ d = 0$ (simply because the fancy formula for d comes from a resolution), so that we have a cochain complex

$$0 \to V \xrightarrow{d^0} C^1(\mathfrak{g}; V) \xrightarrow{d^1} C^2(\mathfrak{g}; V) \xrightarrow{d^2} C^3(\mathfrak{g}; V) \to \cdots$$

And $H^q(\mathfrak{g}; V) = H^q(C^\bullet(\mathfrak{g}; V), d)$. One can take this for a definition of cohomology.

Again, some geometric interpretation would be helpful. Observe that formula (4.5) is the same as (4.1), so the complex for Lie algebra cohomology really originates from de Rham complex. Precisely, recall that in our setting $\mathfrak{g}$ is the Lie algebra of a connected real Lie group $G(\mathbb{R})$. The group $G(\mathbb{R})$ acts on differential forms $\Omega^\bullet(G(\mathbb{R}))$ by multiplication on the left:

$$(g \cdot \alpha)_h \stackrel{\text{def}}{=} \alpha_{gh}.$$

This action is compatible with wedge products:

$$g \cdot (\alpha \wedge \beta) = (g \cdot \alpha) \wedge (g \cdot \beta) \quad \text{for } \alpha \in \Omega^q(G(\mathbb{R})),\ \beta \in \Omega^q(G(\mathbb{R})).$$

The differential forms that are stable under this action are called **left-invariant**. They form a space

$$\Omega^\bullet(G(\mathbb{R}))^{G(\mathbb{R})} \stackrel{\text{def}}{=} \{\alpha \in \Omega^\bullet(G(\mathbb{R})) \mid g \cdot \alpha = \alpha \text{ for all } g \in G\}.$$

Note that we have

$$g \cdot d\alpha = d(g \cdot \alpha) = d\alpha, \quad \text{if } \alpha \in \Omega^\bullet(G(\mathbb{R}))^{G(\mathbb{R})}.$$

So $\Omega^\bullet(G(\mathbb{R}))^{G(\mathbb{R})}$ is a subcomplex of the usual de Rham complex $(\Omega^\bullet(G(\mathbb{R})), d^\bullet)$:

$$0 \to \mathbb{R} \xrightarrow{\epsilon} \Omega^0(G(\mathbb{R}))^{G(\mathbb{R})} \xrightarrow{d} \Omega^1(G(\mathbb{R}))^{G(\mathbb{R})} \xrightarrow{d} \Omega^2(G(\mathbb{R}))^{G(\mathbb{R})} \to \cdots$$

more precisely, $(\Omega^\bullet(G(\mathbb{R}))^{G(\mathbb{R})}, d^\bullet)$ is a DG-subalgebra of de Rham DG-algebra $(\Omega^\bullet(G(\mathbb{R})), d^\bullet)$.

Remark 4.4.1. If $G(\mathbb{R})$ is compact, then by the "averaging trick" one can produce a map $\Omega^\bullet(G(\mathbb{R})) \to \Omega^\bullet(G(\mathbb{R}))^{G(\mathbb{R})}$ which is homotopic to the identity. Thus one can use left-invariant differential forms to calculate cohomology of a connected compact Lie group. However, our Lie groups are not compact.

Now the Lie algebra $\mathfrak{g}$ can be identified with the tangent space at the identity $T_e G(\mathbb{R})$ (note that any tangent vector $v \in T_e G$ extends to a left invariant vector field $g \mapsto L_g^* v$ where $L_g\colon G \to G$ is the multiplication on the left by g). Having a left invariant differential form $\alpha \in \Omega^q(G(\mathbb{R}))^{G(\mathbb{R})}$, we can evaluate it at $\bigwedge^q T_e G(\mathbb{R})$. This gives an isomorphism of graded algebras

$$\begin{gathered} \Omega^\bullet(G(\mathbb{R}))^{G(\mathbb{R})} \to \operatorname{Hom}_{\mathbb{R}}(\textstyle\bigwedge^\bullet \mathfrak{g}, \mathbb{R}), \\ \alpha \mapsto \alpha|_{\bigwedge^q T_e G(\mathbb{R})}. \end{gathered}$$

Indeed, for an element $f\colon \bigwedge^q \mathfrak{g} \to \mathbb{R}$, we can define a q-form $\alpha \in \Omega^q(G(\mathbb{R}))$ by

$$\alpha_g((X_1)_g, \ldots, (X_q)_g) = f(L_{g^{-1}*}(X_1)_g, \ldots, L_{g^{-1}*}(X_q)_g),$$

where $g \in G(\mathbb{R})$ and $X_1, \ldots, X_q$ are vector fields on $G(\mathbb{R})$.

This α is actually left invariant:

$$\begin{aligned}(L_g^*\alpha)_h((X_1)_h,\dots,(X_q)_h) &= \alpha_{gh}(L_{g*}(X_1)_h,\dots,L_{g*}(X_q)_h)\\ &= f(L_{h^{-1}*}(X_1)_h,\dots,L_{h^{-1}*}(X_q)_h)\\ &= \alpha_h((X_1)_h,\dots,(X_q)_h).\end{aligned}$$

To see that it is injective, assume that $\alpha \in \Omega^q(G(\mathbb{R}))^{G(\mathbb{R})}$ is a left-invariant form such that at the identity $\alpha|_{\bigwedge^q T_e G(\mathbb{R})} = 0$. Then at any other point $g \in G(\mathbb{R})$ we get

$$\begin{aligned}\alpha_g((X_1)_g,\dots(X_q)_g) &= (L_g^*\alpha)_e(L_{g^{-1}*}(X_1)_g,\dots,L_{g^{-1}*}(X_q)_g)\\ &= \alpha_e(L_{g^{-1}*}(X_1)_g,\dots,L_{g^{-1}*}(X_q)_g) = 0.\end{aligned}$$

Now recall the differential (4.1). If α is a left-invariant q-form and $X_0,\dots,X_q$ are left-invariant vector fields, then we have a formula

$$d\alpha(X_0\wedge\dots\wedge X_q) = \sum_{0\leqslant i<j\leqslant q} (-1)^{i+j}\,\alpha([X_i,X_j]\wedge X_0\wedge\dots\wedge \widehat{X}_i\wedge\dots\wedge\widehat{X}_j\wedge\dots\wedge X_q).$$

Similarly, on the complex $\operatorname{Hom}_{\mathbb{R}}(\bigwedge^\bullet \mathfrak{g},\mathbb{R})$ with the trivial action of $\mathfrak{g}$ on $\mathbb{R}$, there is a differential

$$df(x_0\wedge\dots\wedge x_q) = \sum_{0\leqslant i<j\leqslant q} (-1)^{i+j} f([x_i,x_j]\wedge x_0\wedge\dots\wedge\widehat{x}_i\wedge\dots\wedge\widehat{x}_j\wedge\dots\wedge x_q),$$

for $f\colon \bigwedge^q \mathfrak{g}\to\mathbb{R}$ and $x_0,\dots,x_q \in T_eG(\mathbb{R}) = \mathfrak{g}$. We have obviously a commutative diagram

$$\begin{tikzcd}
\Omega^q(G(\mathbb{R}))^{G(\mathbb{R})} \arrow[r, "d"] \arrow[d, "\cong"'] & \Omega^{q+1}(G(\mathbb{R}))^{G(\mathbb{R})} \arrow[d, "\cong"] \\
\operatorname{Hom}_{\mathbb{R}}(\bigwedge^q \mathfrak{g},\mathbb{R}) \arrow[r, "d"'] & \operatorname{Hom}_{\mathbb{R}}(\bigwedge^{q+1}\mathfrak{g},\mathbb{R})
\end{tikzcd}$$

And this leads to an isomorphism

$$H^\bullet(\Omega^\bullet(G(\mathbb{R}))^{G(\mathbb{R})}) \cong H^\bullet(\mathfrak{g},\mathbb{R}),$$

where on the right hand side is the Lie algebra isomorphism as defined above.

Remark 4.4.2. If $G(\mathbb{R})$ is compact, then we get $H^\bullet(G(\mathbb{R}),\mathbb{R}) \cong H^\bullet(\mathfrak{g},\mathbb{R})$.
As a banal example, let $G(\mathbb{R}) = \underbrace{S^1\times\cdots\times S^1}_n$ be a torus. Then the Lie algebra $\mathfrak{g}$ of $G(\mathbb{R})$ can be identified with $\mathbb{R}^n$ with the zero bracket. Hence the complex $\operatorname{Hom}_{\mathbb{R}}(\bigwedge^q\mathfrak{g},\mathbb{R})$ has zero differentials, and we obtain

$$H^q(\underbrace{S^1\times\cdots\times S^1}_n,\mathbb{R}) \cong \bigwedge^q \mathbb{R}^n.$$

$$\dim_{\mathbb{R}} H^q(\underbrace{S^1\times\cdots\times S^1}_n,\mathbb{R}) = \binom{n}{q}.$$

(The same can be deduced by induction from the Künneth formula.)

Example 4.4.3. Consider the group SU_2. By definition, it consists of all matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{C})$ such that $A^\dagger A = A A^\dagger = I$. In particular, one sees that the matrices must be of the shape $\begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix}$. All such matrices form an algebra $\mathbb{H}$ which is spanned over $\mathbb{R}$ by four matrices

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \mathbf{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{k} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$$\mathbb{H} = \mathbb{R}[\mathbf{1}] \oplus \mathbb{R}[\mathbf{i}] \oplus \mathbb{R}[\mathbf{j}] \oplus \mathbb{R}[\mathbf{k}].$$

In fact $\mathbb{H}$ is the algebra of quaternions with the usual relations

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1, \quad \mathbf{ij} = \mathbf{k}, \quad \mathbf{jk} = \mathbf{i}, \quad \mathbf{ki} = \mathbf{j}.$$

Under this identification, we see that an element $z = a\,\mathbf{1} + b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k} \in \mathbb{H}$ lies in SU_2 whenever

$$a^2 + b^2 + c^2 + d^2 = 1.$$

That is, SU_2 can be identified with the group of quaternions of norm 1, which is topologically the sphere S^3. From this it is clear that the cohomology algebra $H^\bullet(SU_2; \mathbb{R})$ is spanned by elements $1 \in H^0(S^3; \mathbb{R})$ and $x_3 \in H^3(S^3; \mathbb{R})$, with obvious cup-products

$$1 \smile 1 = 1, \quad 1 \smile x_3 = x_3 \smile 1 = 1, \quad x_3 \smile x_3 = 0.$$

That is, we get the free exterior algebra over $\mathbb{R}$ generated by one element x_3 of degree 3:

$$H^\bullet(SU_2; \mathbb{R}) \cong \Lambda(x_3).$$

Of course in what follows we are not going to calculate any Lie algebra cohomology from explicit cochains and cocycles, but let us do that just once in the easiest example of $\mathfrak{su}_2$. The algebra $\mathfrak{su}_2$ consists of matrices $A \in M_2(\mathbb{C})$ such that $\operatorname{tr} A = 0$ and $A^\dagger = -A$:

$$A = \begin{pmatrix} a & b \\ -\overline{b} & -a \end{pmatrix}.$$

Under this identification, the Lie bracket $[\cdot, \cdot]$ on $\mathfrak{su}_2$ is the usual commutator.

A convenient basis of $\mathfrak{su}_2$ over $\mathbb{R}$ is given by three matrices $u = -\frac{i}{2}\,\sigma_u$, $v = -\frac{i}{2}\,\sigma_v$, $t = -\frac{i}{2}\,\sigma_t$, where

$$\sigma_u \stackrel{\text{def}}{=} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_v \stackrel{\text{def}}{=} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_t \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The bracket in this basis is determined by

$$[u, v] = t, \quad [u, t] = -v, \quad [v, t] = u. \tag{4.6}$$

Now let us look at the complex

$$0 \to \mathbb{R} \xrightarrow{d^0} \operatorname{Hom}_{\mathbb{R}}(\mathfrak{g}, \mathbb{R}) \xrightarrow{d^1} \operatorname{Hom}_{\mathbb{R}}(\mathfrak{g} \wedge \mathfrak{g}, \mathbb{R}) \xrightarrow{d^2} \operatorname{Hom}_{\mathbb{R}}(\mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g}, \mathbb{R}) \to 0$$

$$\begin{aligned} d^0 c(x) &= 0, \\ d^1 f(x \wedge y) &= -f[x, y], \\ d^2 f(x \wedge y \wedge z) &= -f([x, y] \wedge z) + f([x, z] \wedge y) - f([y, z] \wedge x). \end{aligned}$$

- Note that $H^0(\mathfrak{g}, \mathbb{R}) = \ker d^0 = \mathbb{R}$. In general, if we have an action of $\mathfrak{g}$ on V, then H^0 is given by

$$H^0(\mathfrak{g}, V) = V^{\mathfrak{g}} \stackrel{\text{def}}{=} \{v \in V \mid x \cdot v = 0 \text{ for all } x \in \mathfrak{g}\}.$$

- Next observe that $H^1(\mathfrak{g}, \mathbb{R}) = \ker d^1 = 0$.
- From the relations (4.6) we deduce that $d^2 = 0$, and so $H^3(\mathfrak{g}, \mathbb{R}) = \ker d^3 \cong \mathbb{R}$.
- Finally, $\dim \ker d^2 = \dim \operatorname{im} d^1 = 3$, so $H^2(\mathfrak{g}, \mathbb{R}) = 0$.

So the complex gives us indeed the expected cohomology $H^\bullet(\mathfrak{su}_2, \mathbb{R}) = H^\bullet(SU_2; \mathbb{R})$. ▲

4.5 Relative Lie algebra cohomology

We are interested not in the Lie group $G(\mathbb{R})$ itself, but in the symmetric space $X = G(\mathbb{R})/K$, where K is a maximal compact subgroup in $G(\mathbb{R})$. Let $\mathfrak{k}$ be the Lie algebra of K. We want to define the relative cohomology $H^q(\mathfrak{g}, \mathfrak{k}; V)$. It is also possible to do using Ext functors of certain modules (see [BW00, Chapter I]), but for us a down to earth definition will do; we will not go into details. In addition to the differentials $d\colon C^q(\mathfrak{g}; V) \to C^{q+1}(\mathfrak{g}; V)$, for each $x \in \mathfrak{g}$ one has maps $\mathcal{L}_x\colon C^q(\mathfrak{g}; V) \to C^q(\mathfrak{g}; V)$ and $\iota_x\colon C^q(\mathfrak{g}; V) \to C^{q-1}(\mathfrak{g}; V)$ given by

$$(\mathcal{L}_x f)(x_1 \wedge \cdots \wedge x_q) = \sum_{1 \leqslant i \leqslant q} f(x_1 \wedge \cdots \wedge [x_i, x] \wedge \cdots \wedge x_q) + x \cdot f(x_1 \wedge \cdots \wedge x_q),$$

$$(\iota_x f)(x_1 \wedge \cdots \wedge x_{q-1}) = f(x \wedge x_1 \wedge \cdots \wedge x_{q-1}).$$

The three maps are related by **Cartan's magic formula**

$$\mathcal{L}_x = d \circ \iota_x + \iota_x \circ d.$$

Now take $C^q(\mathfrak{g}, \mathfrak{k}; V)$ to be the subspace of $C^q(\mathfrak{g}; V)$ given by the elements annihilated by ι_x and $\mathcal{L}_x$ for all $x \in \mathfrak{k}$:

$$C^q(\mathfrak{g}, \mathfrak{k}; V) \stackrel{\text{def}}{=} \{f \in C^q(\mathfrak{g}; V) \mid \iota_x f = 0 \text{ and } \mathcal{L}_x f = 0 \text{ for all } x \in \mathfrak{k}\} = \operatorname{Hom}_{\mathfrak{k}}(\textstyle\bigwedge^q \mathfrak{g}/\mathfrak{k}, V).$$

This gives a cochain complex

$$0 \to \mathbb{R} \xrightarrow{d} C^1(\mathfrak{g}, \mathfrak{k}; V) \xrightarrow{d} C^2(\mathfrak{g}, \mathfrak{k}; V) \xrightarrow{d} C^3(\mathfrak{g}, \mathfrak{k}; V) \to \cdots$$

And $H^q(\mathfrak{g}, \mathfrak{k}; V) \stackrel{\text{def}}{=} H^q(C^\bullet(\mathfrak{g}, \mathfrak{k}; V), d)$.

The geometric meaning of this is the following:

$$H^\bullet(\Omega^\bullet(X)^{G(\mathbb{R})}) \cong H^\bullet(\mathfrak{g}, \mathfrak{k}; \mathbb{R}), \tag{4.7}$$

i.e. this computes cohomology of the complex of $G(\mathbb{R})$-invariant differential forms on X.

The complex $\Omega^\bullet(X)^{G(\mathbb{R})}$ is very important, so we introduce a special notation:

$$I^\bullet_{G(\mathbb{R})} \stackrel{\text{def}}{=} \Omega^\bullet(X)^{G(\mathbb{R})}.$$

We are going to admit the following classical result.

Fact 4.5.1. *The differential forms in $I^q_{G(\mathbb{R})} \stackrel{\text{def}}{=} \Omega^q(X)^{G(\mathbb{R})}$ are closed (i.e. $d\alpha = 0$ for all $\alpha \in I^\bullet_{G(\mathbb{R})}$). Moreover, they are also co-closed ($\delta\alpha = 0$), and thus harmonic ($\Delta\alpha = 0$).*

This goes back to Élie Cartan, and a modern exposition can be found in [BW00, §II.3]. ☹
(The notions of co-closed and harmonic forms will be explained and applied in the next chapter.)

Since differential forms in $I^\bullet_{G(\mathbb{R})}$ are closed, (4.7) can be written simply as

$$I^\bullet_{G(\mathbb{R})} \cong H^\bullet(\mathfrak{g}, \mathfrak{k}; \mathbb{R}). \tag{4.8}$$

We note that taking the space $I^\bullet_{G(\mathbb{R})}$ is functorial. An injective $\mathbb{R}$-morphism

$$f\colon G_1 \hookrightarrow G_2.$$

induces a morphism

$$f^*\colon I^\bullet_{G_2(\mathbb{R})} \to I^\bullet_{G_1(\mathbb{R})}.$$

One of many ways to construct this is the following. In $G_2(\mathbb{R})$ we may take a maximal compact subgroup K_2 such that $K_2 \supset f(K_1)$. Then there is an inclusion

$$\underbrace{G_1(\mathbb{R})/K_1}_{X_1} \hookrightarrow \underbrace{G_2(\mathbb{R})/K_2}_{X_2},$$

and f^* may be viewed as the restriction of differential forms from X_2 to X_1. This construction does not depend on the choice of K_2, since any two maximal compact subgroups in $G_2(\mathbb{R})$ are conjugate by an inner automorphism leaving their intersection pointwise fixed.

In general, if we have a subgroup $\Gamma \subset G(\mathbb{R})$, then

$$H^\bullet(\Omega^\bullet(X)^\Gamma) \cong H^\bullet(\mathfrak{g}, \mathfrak{k}; \mathcal{C}(\Gamma\backslash G(\mathbb{R})))$$

—this is proved in [MM65a, §3]; in particular (4.7) is a special case. Then by (4.4),

$$H^\bullet(\Gamma, \mathbb{R}) \cong H^\bullet(\mathfrak{g}, \mathfrak{k}; \mathcal{C}^\infty(\Gamma\backslash G(\mathbb{R}))).$$

Now let us recall some theory for Lie algebras which will be useful also in the next chapter. For a thorough treatment we refer to the book [Hel01], or to Bourbaki [Bou60, Bou72, Bou68, Bou75].

For a Lie algebra $\mathfrak{g}$ one can consider the **adjoint representation** $\mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ given by $x \mapsto ad_x$, where

$$\begin{aligned} ad_x\colon \mathfrak{g} &\to \mathfrak{g}, \\ y &\mapsto [x, y]. \end{aligned}$$

The **Killing form** on a finite dimensional Lie algebra $\mathfrak{g}$ is the symmetric bilinear form given by

$$B_{\mathfrak{g}}(x, y) \overset{\text{def}}{=} \operatorname{tr}(ad_x \circ ad_y).$$

It is obvious that this is bilinear and symmetric, since we take a trace. Further, this form is **invariant**, in the sense that

$$B_{\mathfrak{g}}([x, y], z) = B_{\mathfrak{g}}(x, [y, z])$$

Indeed,

$$\begin{aligned} \operatorname{tr}(ad_{[x,y]} \circ ad_z) &= \operatorname{tr}(ad_x \circ ad_y \circ ad_z) - \operatorname{tr}(ad_y \circ ad_x \circ ad_z) \\ &= \operatorname{tr}(ad_x \circ ad_y \circ ad_z) - \operatorname{tr}(ad_x \circ ad_z \circ ad_y) \\ &= \operatorname{tr}(ad_x \circ ad_{[y,z]}). \end{aligned}$$

Fact 4.5.2. *If $\mathfrak{g}$ is a simple Lie algebra, then any invariant symmetric bilinear form on $\mathfrak{g}$ is a scalar multiple of the Killing form.*

Example 4.5.3. If $\mathfrak{g}$ is a subalgebra of $\mathfrak{gl}_n(\mathbb{R})$, then we see that the symmetric bilinear form given by $\langle X, Y\rangle = \operatorname{tr}(X\,Y)$ is invariant. The only problem is to find the scalar multiplier.

For instance, in $\mathfrak{sl}_n(\mathbb{R})$ we can take a matrix $X \stackrel{\text{def}}{=} e_{11} - e_{22}$. Then $X^2 = e_{11} + e_{22}$ and $\operatorname{tr}(X^2) = 2$. Now look at the adjoint action ad_X. It is given by

$$[X, e_{ij}] = [e_{11}, e_{ij}] - [e_{22}, e_{ij}] = 2\,e_{ij}.$$

Hence the Killing form is

$$B_{\mathfrak{g}}(X, X) = \operatorname{tr}(ad_X \circ ad_X) = 4\,n.$$

So the scalar multiple is $2\,n$, and $B_{\mathfrak{g}}(X, Y) = 2\,n\,\operatorname{tr}(X\,Y)$. One can work out the other examples similarly [Hel01, §III.8].

algebra :	$\mathfrak{sl}_n(\mathbb{R})$	$\mathfrak{so}_n(\mathbb{R})$	$\mathfrak{sp}_n(\mathbb{R})$
Killing form :	$2\,n\,\operatorname{tr}(X\,Y)$	$(n-2)\,\operatorname{tr}(X\,Y)$	$(2\,n+2)\,\operatorname{tr}(X\,Y)$

▲

Fact 4.5.4. *A Lie algebra $\mathfrak{g}$ is semisimple if and only if the Killing form is nondegenerate.*

Example 4.5.5. Consider the algebra $\mathfrak{sl}_n(\mathbb{R}) \subset \mathfrak{gl}_n(\mathbb{R})$ given by the $n \times n$ matrices of trace zero. It has dimension $n^2 - 1$ with a standard basis consisting of elementary matrices e_{ij} for $i \neq j$, together with matrices $e_{ii} - e_{i+1,i+1}$ for $1 \leqslant i \leqslant n-1$. In particular, for $\mathfrak{sl}_2(\mathbb{R})$ a basis is given by

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We calculate $[x, y] = h$, $[x, h] = -2\,x$, $[y, h] = 2\,y$, and the Killing form is

$B_{\mathfrak{g}}(\cdot,\cdot)$	x	y	h
x	0	4	0
y	4	0	0
h	0	0	8

We see that this is non-degenerate. ▲

Example 4.5.6. Consider the Lie algebra $\mathfrak{so}_n(\mathbb{R}) \subset \mathfrak{gl}_n(\mathbb{R})$ consisting of the skew-symmetric square matrices $n \times n$, such that $M^\top = -M$. It has dimension $\binom{n}{2} = \frac{n\,(n-1)}{2}$. The basis consists of matrices $e_{ji} - e_{ij}$ for $1 \leqslant i < j \leqslant n$. For instance, $\mathfrak{so}_3(\mathbb{R})$ has a basis

$$u = \begin{pmatrix} 0 & +1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad v = \begin{pmatrix} 0 & 0 & +1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad w = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & +1 \\ 0 & -1 & 0 \end{pmatrix}.$$

We have $[u, v] = -w$, $[u, w] = v$, $[v, w] = -u$, and the Killing form is given by

$B_{\mathfrak{g}}(\cdot,\cdot)$	u	v	w
u	-2	0	0
v	0	-2	0
w	0	0	-2

Observe that this is nondegenerate and negative definite. ▲

Fact 4.5.7. *If $G(\mathbb{R})$ is a semisimple compact Lie group and $\mathfrak{g}$ its Lie algebra, then the Killing form $B_{\mathfrak{g}}(\cdot,\cdot)$ is negative definite.*

An **involution** of a semisimple real Lie algebra $\mathfrak{g}$ is an endomorphism $\theta\colon \mathfrak{g} \to \mathfrak{g}$ such that $\theta^2 = id$. It is called a **Cartan involution** on $\mathfrak{g}$ if the bilinear form

$$B_\theta(x,y) \stackrel{\text{def}}{=} -B_{\mathfrak{g}}(x,\theta(y))$$

is symmetric and positive definite. Since θ is an involution, it has eigenvalues ± 1. We let $\mathfrak{k}$ to be the eigenspace corresponding to the eigenvalue $+1$:

$$\mathfrak{k} \stackrel{\text{def}}{=} \{x \in \mathfrak{g} \mid \theta(x) = x\},$$

and let $\mathfrak{p}$ be the eigenspace corresponding to the eigenvalue -1:

$$\mathfrak{p} \stackrel{\text{def}}{=} \{x \in \mathfrak{g} \mid \theta(x) = -x\}.$$

We have the eigenspace decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}.$$

Observe that if $x \in \mathfrak{k}$ and $y \in \mathfrak{p}$, then $[x,y] \in \mathfrak{p}$:

$$\theta[x,y] = [\theta(x),\theta(y)] = [x,-y] = -[x,y].$$

Similarly we see that

$$[\mathfrak{k},\mathfrak{k}] \subseteq \mathfrak{k}, \quad [\mathfrak{k},\mathfrak{p}] \subseteq \mathfrak{p}, \quad [\mathfrak{p},\mathfrak{p}] \subseteq \mathfrak{k}.$$

Example 4.5.8. If $\mathfrak{g}$ is a subalgebra of matrices inside $\mathfrak{gl}_n(\mathbb{R})$ and $\mathfrak{g}$ is closed under matrix transpose $x \mapsto x^\top$, then it is easy to check that $\theta\colon x \mapsto -x^\top$ is a Cartan involution. It is indeed a Lie algebra morphism, since

$$\theta[x,y] = -[x,y]^\top = -[y^\top,x^\top] = [-x^\top,-y^\top] = [\theta(x),\theta(y)].$$

Observe that θ leaves the Killing form invariant:

$$B_{\mathfrak{g}}(\theta(x),\theta(y)) = \operatorname{tr}(ad_{\theta(x)} \circ ad_{\theta(y)}) = \operatorname{tr}(\theta \circ ad_x \circ \theta^{-1} \circ \theta \circ ad_y \circ \theta^{-1}) = \operatorname{tr}(ad_x \circ ad_y) = B_{\mathfrak{g}}(x,y),$$

hence the form $B_\theta(\cdot,\cdot)$ is symmetric:

$$B_\theta(x,y) = -B_{\mathfrak{g}}(x,\theta(y)) = -B_{\mathfrak{g}}(\theta(x),\theta^2(y)) = -B_{\mathfrak{g}}(y,\theta(x)) = B_\theta(y,x).$$

$B_\theta(\cdot,\cdot)$ is positive definite:

$$B_\theta(x,x) = -\operatorname{tr}(ad_x \circ ad_{-x^\top}) = \operatorname{tr}(ad_x \circ (ad_x)^\top),$$

and the latter is positive for $x \neq 0$ (we assume that the algebra is semisimple).

So the Cartan decomposition boils down to the well-known fact that any matrix can be written as a sum of a skew-symmetric matrix $x \in \mathfrak{k}$ and a symmetric matrix $y \in \mathfrak{p}$. ▲

Example 4.5.9. More concretely, take $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{R})$ and a Cartan involution $\theta\colon x \mapsto -x^\top$. The matrices fixed by θ form a subalgebra of traceless skew-symmetric matrices, which is $\mathfrak{so}_n(\mathbb{R})$. The complementary subspace $\mathfrak{p}$ is formed by the traceless symmetric matrices. ▲

Example 4.5.10. For instance, for $\mathfrak{sl}_2(\mathbb{R})$ one has

$$\theta\colon x \mapsto -y, \quad y \mapsto -x, \quad h \mapsto -h.$$

And we calculate

$B_\theta(\cdot,\cdot)$	x	y	h
x	-4	0	0
y	0	-4	0
h	0	0	-8

Note that $\theta(x-y) = x-y$, $\theta(x+y) = -(x+y)$, and $\theta(h) = -h$. We have a decomposition of $\mathfrak{sl}_2(\mathbb{R})$ into a subalgebra of skew symmetric traceless matrices $\mathfrak{k}$ generated by $x - y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and a subspace of symmetric traceless matrices $\mathfrak{p}$ generated by $x + y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Observe that on $\mathfrak{k}$ the Killing form is negative definite:

$$B_{\mathfrak{g}}(x-y,\, x-y) = B_{\mathfrak{g}}(x,x) - 2\,B_{\mathfrak{g}}(x,y) + B_{\mathfrak{g}}(y,y) = -8.$$

On $\mathfrak{p}$ the Killing form is positive definite:

$$B_{\mathfrak{g}}(h,h) = 8, \quad B_{\mathfrak{g}}(x+y,\, x+y) = B_{\mathfrak{g}}(x,x) + 2\,B_{\mathfrak{g}}(x,y) + B_{\mathfrak{g}}(y,y) = 8.$$

▲

Now go back to the case when $\mathfrak{g}$ is the Lie algebra of a semisimple Lie group $G(\mathbb{R})$ and $\mathfrak{k}$ is the Lie algebra of its maximal compact subgroup K.

Fact 4.5.11. *To each maximal compact subgroup K is associated a Cartan involution $\theta\colon \mathfrak{g} \to \mathfrak{g}$ giving the corresponding decomposition*

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p},$$

where

$$\mathfrak{k} = \{x \in \mathfrak{g} \mid \theta(x) = x\}, \quad \mathfrak{p} \stackrel{\text{def}}{=} \{x \in \mathfrak{g} \mid \theta(x) = -x\}.$$

$$[\mathfrak{k},\mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k},\mathfrak{p}] \subset \mathfrak{p}, \quad [\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k}.$$

Further, if we assume that $G(\mathbb{R})$ is non-compact, holds equality $[\mathfrak{p},\mathfrak{p}] = \mathfrak{k}$.

As for the dual symmetric space $X_u = G(\mathbb{R})_u/K$, the Cartan decomposition for $\mathfrak{g}_u$ is given by

$$\mathfrak{g}_u = \mathfrak{k} \oplus i\mathfrak{p} \subset \mathfrak{g}_{\mathbb{C}}.$$

From this one can work out that

$$I^\bullet_{G(\mathbb{R})} \cong H^\bullet(\mathfrak{g},\mathfrak{k};\mathbb{R}) \cong H^\bullet(\mathfrak{g}_u,\mathfrak{k};\mathbb{R}) \cong H^\bullet(\Omega^\bullet(X_u)^{G(\mathbb{R})_u}).$$

But now the space X_u is compact, hence in fact $H^\bullet(\Omega^\bullet(X_u)^{G(\mathbb{R})_u}) \cong H^\bullet(X_u,\mathbb{R})$. We record this isomorphism:

$$I^\bullet_{G(\mathbb{R})} \cong H^\bullet(X_u,\mathbb{R}), \tag{4.9}$$

i.e. the space $I^\bullet_{G(\mathbb{R})}$ is the usual de Rham cohomology of the compact dual symmetric space X_u.

Example 4.5.12. Consider the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ and its subalgebra $\mathfrak{su}_2$. One can calculate the relative cohomology $H^\bullet(\mathfrak{sl}_2(\mathbb{C}),\mathfrak{su}_2;\mathbb{R})$. Recall the basis of $\mathfrak{su}_2$ was given by matrices $u = -\frac{i}{2}\,\sigma_u$, $v = -\frac{i}{2}\,\sigma_v$, $t = -\frac{i}{2}\,\sigma_t$, where

$$\sigma_u \stackrel{\text{def}}{=} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_v \stackrel{\text{def}}{=} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_t \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We can complete this to a basis of $\mathfrak{sl}_2(\mathbb{C})$ by adding $\tilde{u} = \frac{1}{2}\,\sigma_u$, $\tilde{v} = \frac{1}{2}\,\sigma_v$, $\tilde{t} = \frac{1}{2}\,\sigma_t$. Then the brackets are given by

$$\begin{aligned} &[u,v] = +t, \quad [u,t] = -v, \quad [v,t] = +u, \\ &[\tilde{u},\tilde{v}] = -t, \quad [\tilde{u},\tilde{t}] = +v, \quad [\tilde{v},\tilde{t}] = -u, \\ &[u,\tilde{v}] = +\tilde{t}, \quad [u,\tilde{t}] = -\tilde{v}, \quad [v,\tilde{t}] = +\tilde{u}. \end{aligned}$$

It is easy to see that the complex $C^\bullet(\mathfrak{sl}_2(\mathbb{C}),\mathfrak{su}_2;\mathbb{R})$ gives the same cohomology as $C^\bullet(\mathfrak{su}_2,\mathbb{R})$. ▲

Remark 4.5.13. There is an alternative interpretation, linking all to continuous cohomology already mentioned in remark 4.3.1: *let $G(\mathbb{R})$ be a connected Lie group and let K be a maximal compact subgroup of $G(\mathbb{R})$. One has*

$$H^\bullet(\mathfrak{g}, \mathfrak{k}; V) \cong H^\bullet_{\mathrm{cl}}(G(\mathbb{R}), V).$$

This is known as **van Est isomorphism**. For details we refer to [BW00, §XI.5] and [Gui80, §III.7]; the original paper is [vE55]. We will not make use of this.

4.6 Cohomology and homotopy of $SU/SO(\mathbb{R})$ and SU

In the view of (4.9), we would like to know cohomology of compact symmetric spaces X_u.

For $G(\mathbb{R}) = SL_n(\mathbb{R})$ this space is $SU_n/SO_n(\mathbb{R})$, and for $SL_n(\mathbb{C})$ this space is SU_n. In fact this is a well-known calculation. For example, in the case of SU_n one argues by induction, starting from $SU_2 \approx S^3$ and using the Leray–Serre spectral sequence for fibration (see example H.3.5)

$$SU_{n-1} \hookrightarrow SU_n \to S^{2n-1} \tag{4.10}$$

The result is

$$H^\bullet(SU_n; \mathbb{R}) \cong \Lambda(x_3, x_5, \ldots, x_{2n-1}),$$

where by $\Lambda(\ldots, x_\ell, \ldots)$ we denote the symmetric $\mathbb{R}$-algebra freely generated by elements x_ℓ of degree $\ell = 3, 5, \ldots, 2n-1$. In fact for any compact Lie group $G(\mathbb{R})$ the algebra $H^\bullet(G(\mathbb{R}); \mathbb{R})$ is given by $\Lambda(x_{2i_1+1}, \ldots, x_{2i_\ell+1})$ for some $i_1, \ldots, i_\ell$. This is a result of Hopf [MT91, Theorem IV.6.26].

As for *homotopy* groups, fibration (4.10) suggests that groups like $\pi_i(SU_n)$ are related to the homotopy groups of spheres, so their calculation is hopeless. Here is an example of calculations taken from [MT64]:

i:	3	4	5	6	7	8	9	10
$\pi_i(SU_3)$:	$\mathbb{Z}$	0	$\mathbb{Z}$	$\mathbb{Z}/6$	0	$\mathbb{Z}/12$	$\mathbb{Z}/3$	$\mathbb{Z}/30$
$\pi_i(SU_4)$:	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\mathbb{Z}/24$	$\mathbb{Z}/2$	$\mathbb{Z}/120 \oplus \mathbb{Z}/2$

i:	11	12	13	14	15	16	17	18
$\pi_i(SU_3)$:	$\mathbb{Z}/4$	$\mathbb{Z}/60$	$\mathbb{Z}/6$	$\mathbb{Z}/84 \oplus \mathbb{Z}/2$	$\mathbb{Z}/36$	$\mathbb{Z}/252 \oplus \mathbb{Z}/6$	$\mathbb{Z}/30 \oplus \mathbb{Z}/2$	$\mathbb{Z}/30 \oplus \mathbb{Z}/6$
$\pi_i(SU_4)$:	$\mathbb{Z}/4$	$\mathbb{Z}/60$	$\mathbb{Z}/4$	$\mathbb{Z}/1680 \oplus \mathbb{Z}/2$	$\mathbb{Z}/72 \oplus \mathbb{Z}/2$	$\mathbb{Z}/504 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$	$\mathbb{Z}/40 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$	$\mathbb{Z}/2520 \oplus \mathbb{Z}/12 \oplus \mathbb{Z}/2$

So higher homotopy groups of SU_n are as mysterious as those of S^n. However, we can pass to the limit $n \to \infty$:

$$SU/SO(\mathbb{R}) \stackrel{\text{def}}{=} \varinjlim_n SU_n/SO_n(\mathbb{R}),$$

$$SU \stackrel{\text{def}}{=} \varinjlim_n SU_n.$$

Then there is a nice answer, which is a part of the classical **Bott periodicity**; cf. an expository article [Bot70] by Bott himself and full proofs in Séminaire Henri Cartan, 12ième année [CDD+61]; another nice reference is [MT91].

The homotopy groups of SU and $SU/SO(\mathbb{R})$ can be obtained from the well-known calculations of $\pi_i(O(\mathbb{R}))$ and $\pi_i(BU)$ and the weak homotopy equivalences $O(\mathbb{R}) \simeq \Omega^2(SU/SO(\mathbb{R}))$ and $BU \simeq \Omega SU$—cf. [MT91, §IV.6] for this.

The answer is periodic, with period 8 (the periodic part is shaded in the table):

i:	0	1	2	3	4	5	6	7	8	9	$\cdots$
$\pi_i(O(\mathbb{R}))$:	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$			
$\pi_i(SU/SO(\mathbb{R}))$:	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	
$\pi_i(BU)$:	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$		
$\pi_i(SU)$:	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	

The cohomology rings are easier. Without Bott periodicity one obtains [MT91, §IV.3]

$$H^\bullet(SU/SO(\mathbb{R});\mathbb{R}) \cong \Lambda(x_5, x_9, \ldots, x_{4i+1}, \ldots),$$
$$H^\bullet(SU;\mathbb{R}) \cong \Lambda(x_3, x_5, \ldots, x_{2i+1}, \ldots).$$

In fact $SU/SO(\mathbb{R})$ and SU are H-spaces, so the Cartan–Serre theorem (§ 3.4) explains the relation between $\pi_\bullet(SU/SO(\mathbb{R}))\otimes\mathbb{R}$, $\pi_\bullet(SU)\otimes\mathbb{R}$ and cohomology rings $H^\bullet(SU/SO(\mathbb{R});\mathbb{R})$, $H^\bullet(SU;\mathbb{R})$.

It is interesting to know that our arithmetic investigations are related to Bott periodicity.

4.7 The morphism $j^q\colon I^q_{G(\mathbb{R})} \to H^q(\Gamma,\mathbb{R})$

Since the forms $I^\bullet_{G(\mathbb{R})}$ are closed, the inclusion $I^\bullet_{G(\mathbb{R})} \overset{\text{def}}{=} \Omega^\bullet(X)^{G(\mathbb{R})} \subset \Omega^\bullet(X)^\Gamma$ induces a homomorphism in cohomology

$$j^q\colon I^q_{G(\mathbb{R})} \to H^q(\Omega(X)^\Gamma) \cong H^q(\Gamma,\mathbb{R}).$$

Remark 4.7.1. Alternatively, by van Est theorem (remark 4.5.13) we have

$$I^\bullet_{G(\mathbb{R})} \cong H^\bullet(\mathfrak{g},\mathfrak{k},\mathbb{R}) \cong H^\bullet_{\mathrm{d}}(G(\mathbb{R})).$$

The inclusion $\Gamma \subset G(\mathbb{R})$ induces $H^\bullet(G(\mathbb{R})) \to H^\bullet(\Gamma)$, and further there is a map $H^\bullet_{\mathrm{d}}(G(\mathbb{R})) \to H^\bullet(G(\mathbb{R}))$ from the differentiable cohomology to the usual group cohomology. In this view the morphism can be interpreted as restriction $j^\bullet\colon H^\bullet_{\mathrm{d}}(G(\mathbb{R})) \to H^\bullet(\Gamma)$.

As we saw above, the spaces $I^q_{G(\mathbb{R})} = H^q(X_u,\mathbb{R})$ are known by classical computations, thus the question that interests us is for which q the morphism $j^q\colon I^q_{G(\mathbb{R})} \to H^q(\Gamma,\mathbb{R})$ is an isomorphism. The following is [Bor74, §7.5], and it is the main point for all calculations.

Theorem 4.7.2. *Let G be a semisimple linear algebraic group over $\mathbb{Q}$ and let $\Gamma \subset G(\mathbb{R})$ be an arithmetic subgroup. One can define constants $m(G(\mathbb{R}))$ and $c(G)$, such that the morphism*

$$j^q\colon (I^q_{G(\mathbb{R})})^\Gamma \to H^q(\Gamma;\mathbb{R})$$

is injective for $q \leqslant c(G)$ and surjective for $q \leqslant \min\{c(G), m(G(\mathbb{R}))\}$.

Example 4.7.3. Let $G = SL_n/\mathbb{Q}$ be the special linear group. Then both constants $m(G(\mathbb{R}))$ and $c(G)$ are arbitrarily large as $n \to \infty$, hence the theorem gives isomorphisms $(I^q_{G(\mathbb{R})})^\Gamma \cong H^q(\Gamma;\mathbb{R})$ in the stable case. ▲

We will examine the morphism j^q in the subsequent chapters. Now we would like to apply the theorem.

4.8 Final results

Remark 4.8.1. Consider a sequence of graded R-algebras $A_n = \bigoplus_j A_n^j$ with graded morphisms $f_n\colon A_{n+1} \to A_n$.

For instance, here we work with cohomology $H^\bullet(M;\mathbb{R})$ which naturally comes as a graded $\mathbb{R}$-algebra.

We are interested in stability, hence in inverse limits like $\varprojlim A_n$. But of course we want to have this limit *degree-wise*. Let us be pedantic and denote by $\varprojlim_n{}^\bullet A_n$ the inverse limit *in the graded category*. It is given by a graded R-module

$$\varprojlim_n{}^\bullet A_n = \bigoplus_j A^j, \text{ where } A^j = \varprojlim_n (A_n^j, f_n^j),$$

which has the obvious graded R-algebra structure.

In our situation A_n will be finite dimensional graded algebras over $\mathbb{R}$ or $\mathbb{C}$.

From theorem 4.7.2 we easily deduce the following:

Theorem 4.8.2. *Consider a sequence of semisimple algebraic groups $G_n/\mathbb{Q}$ and their algebraic subgroups Γ_n:*

$$\begin{aligned} f_n\colon G_n &\hookrightarrow G_{n+1}, \\ \Gamma_n &\hookrightarrow \Gamma_{n+1}. \end{aligned}$$

Here f_n are injective morphisms over $\mathbb{Q}$, such that $\Gamma_n \subset G_n(\mathbb{R})$ is mapped into $\Gamma_{n+1} \subset G_{n+1}(\mathbb{R})$. Assume the following:

1) Given any dimension q, there exists $N(q)$ such that

$$(I^q_{G_n(\mathbb{R})})^{\Gamma_n} = I^q_{G_n(\mathbb{R})} \quad \text{for all } n \geqslant N(q).$$

2) The constants $m(G_n(\mathbb{R}))$ and $c(G_n)$ tend to ∞ as $n \to \infty$.

Then

$$H^\bullet(\varinjlim \Gamma_n, \mathbb{R}) \cong \varprojlim{}^\bullet H^\bullet(\Gamma_n, \mathbb{R}) \cong \varprojlim{}^\bullet I^\bullet_{G_n(\mathbb{R})}.$$

Remark 4.8.3. If $\Gamma_n \subset G_n(\mathbb{R})^\circ$—in particular, if $G_n(\mathbb{R})$ is connected—then the condition 1) is satisfied.

The only case that interests us is $G_n' = SL_n/F$ and $G_n = \operatorname{Res}_{F/\mathbb{Q}} G_n'$. The group $G_n(\mathbb{R})$ is connected. In this case 2) is satisfied as well.

Proof. The first isomorphism

$$H^\bullet(\varinjlim \Gamma_n, \mathbb{R}) \cong \varprojlim{}^\bullet H^\bullet(\Gamma_n, \mathbb{R})$$

is just because Γ_n are arithmetic groups, and thus $H^\bullet(\Gamma_n, \mathbb{R})$ are finite dimensional $\mathbb{R}$-vector spaces; cf. theorem A.3.4. By theorem 4.7.2 and assumptions 1) and 2), we get isomorphisms

$$j_n^\bullet\colon I^\bullet_{G_n(\mathbb{R})} \xrightarrow{\cong} H^\bullet(\Gamma_n, \mathbb{R}).$$

Inclusions $G_n \hookrightarrow G_{n+1}$ induce the following commutative diagrams:

$$\begin{tikzcd} I^\bullet_{G_{n+1}(\mathbb{R})} \arrow[r] \arrow[d, "j^\bullet_{n+1}"', "\cong"] & I^\bullet_{G_n(\mathbb{R})} \arrow[d, "\cong"', "j^\bullet_n"] \\ H^\bullet(\Gamma_{n+1}, \mathbb{R}) \arrow[r] & H^\bullet(\Gamma_n, \mathbb{R}) \end{tikzcd}$$

Passing to the limit $n \to \infty$, we get

$$\varprojlim_n {}^{\bullet} H^{\bullet}(\Gamma_n, \mathbb{R}) \cong \varprojlim_n {}^{\bullet} I^{\bullet}_{G_n(\mathbb{R})}.$$

■

Example 4.8.4. Consider $G_n = SL_n/\mathbb{Q}$ and $\Gamma_n = SL_n(\mathbb{Z})$. Then

$$H^{\bullet}(\varinjlim SL_n(\mathbb{Z}), \mathbb{R}) \cong \varprojlim_n {}^{\bullet} H^{\bullet}(SL_n(\mathbb{Z}), \mathbb{R}) \cong \varprojlim_n {}^{\bullet} I^{\bullet}_{SL_n(\mathbb{R})} \cong H^{\bullet}(SU/SO(\mathbb{R}); \mathbb{R}) \cong \Lambda(x_5, x_9, \ldots, x_{4i+1}, \ldots).$$

We consider the indecomposable elements in the latter algebra and conclude that for $i \geqslant 2$

$$\dim_{\mathbb{R}} K_i(\mathbb{Z}) \otimes \mathbb{R} = \dim_{\mathbb{R}} QH^{\bullet}(\varinjlim SL_n(\mathbb{Z}); \mathbb{R}) = \begin{cases} 1, & i \equiv 1 \pmod 4, \\ 0, & \text{otherwise.} \end{cases}$$

The following table is taken from [Wei05].

n:	2	3	4	5	6	7	8	9
$K_n(\mathbb{Z})$:	$\mathbb{Z}/2$	$\mathbb{Z}/48$	0	$\mathbb{Z}$	0	$\mathbb{Z}/240$	(0?)	$\mathbb{Z} \oplus \mathbb{Z}/2$
n:	10	11	12	13	14	15	16	17
$K_n(\mathbb{Z})$:	$\mathbb{Z}/2$	$\mathbb{Z}/1008$	(0?)	$\mathbb{Z}$	0	$\mathbb{Z}/480$	(0?)	$\mathbb{Z} \oplus \mathbb{Z}/2$
n:	18	19	20	21	22	23	24	25
$K_n(\mathbb{Z})$:	$\mathbb{Z}/2$	$\mathbb{Z}/528$	(0?)	$\mathbb{Z}$	$\mathbb{Z}/691$	$\mathbb{Z}/65\,520$	(0?)	$\mathbb{Z} \oplus \mathbb{Z}/2$

(0?) — finite groups that are conjecturally zero

So we understand at least the periodicity of ranks! ▲

Now we turn to the general situation. Let F be a number field of degree $d = r_1 + 2r_2$, where r_1 is the number of real places on F and r_2 is the number of complex places on F. One has

$$F \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2}.$$

We denote by M_F^{∞} the set of all archimedean places. Consider algebraic groups $G'_n = SL_n/F$ and their arithmetic subgroups $\Gamma'_n = SL_n(\mathcal{O}_F)$. There are natural injective morphisms over F:

$$\begin{aligned} f'_n \colon G'_n &\hookrightarrow G'_{n+1}, \\ \Gamma'_n &\hookrightarrow \Gamma'_{n+1}. \end{aligned}$$

To work with algebraic groups over $\mathbb{Q}$, we take restrictions of scalars:

$$G_n \stackrel{\text{def}}{=} \operatorname{Res}_{F/\mathbb{Q}} G'_n, \quad f_n \stackrel{\text{def}}{=} \operatorname{Res}_{F/\mathbb{Q}} f'_n.$$

For each place $v \in M_F^{\infty}$ we denote by F_v the completion of F at v:

$$F_v = \begin{cases} \mathbb{R}, & v \text{ is real}, \\ \mathbb{C}, & v \text{ is complex}. \end{cases}$$

Let $G'_{n,v} \stackrel{\text{def}}{=} (G'_n)_{F_v}$ be the extension of scalars to F_v. We have

$$G_n(\mathbb{R}) = \prod_{v \in M_F^{\infty}} G'_{n,v}(F_v)$$

— cf. § A.2 for extension and restriction of scalars.

The symmetric space $X_n = G_n(\mathbb{R})/K_n$ corresponding to $G_n(\mathbb{R})$ is the product of such symmetric spaces for each $G'_{n,v}(F_v)$, and the maps $f_n\colon G_n(\mathbb{R}) \hookrightarrow G_{n+1}(\mathbb{R})$ are compatible with such decomposition. Therefore we get

$$\varprojlim{}^\bullet I^\bullet_{G_n(\mathbb{R})} \cong \bigotimes_{v\in M_F^\infty} I^\bullet_v,$$

where

$$I^\bullet_v \stackrel{\text{def}}{=} \varprojlim{}^\bullet I^\bullet_{G'_{n,v}}(F_v).$$

Precisely, in case of SL_n,

$$(G'_{n,v})(F_v) = \begin{cases} SL_n(\mathbb{R}), & v \text{ is real,} \\ SL_n(\mathbb{C}), & v \text{ is complex.} \end{cases}$$

$$G_n(\mathbb{R}) = SL_n(F \otimes_{\mathbb{Q}} \mathbb{R}) = \underbrace{SL_n(\mathbb{R}) \times \cdots \times SL_n(\mathbb{R})}_{r_1} \times \underbrace{SL_n(\mathbb{C}) \times \cdots \times SL_n(\mathbb{C})}_{r_2},$$

$$G_n(\mathbb{C}) = \underbrace{SL_n(\mathbb{C}) \times \cdots \times SL_n(\mathbb{C})}_{d}.$$

The maximal compact subgroup in $G_n(\mathbb{R})$ is

$$K_n = \underbrace{SO_n(\mathbb{R}) \times \cdots \times SO_n(\mathbb{R})}_{r_1} \times \underbrace{SU_n \times \cdots \times SU_n}_{r_2}.$$

The dual group is

$$G_{n,u} = \underbrace{SU_n \times \cdots \times SU_n}_{d}.$$

The corresponding symmetric space is

$$X_n = \underbrace{SL_n(\mathbb{R})/SO_n(\mathbb{R}) \times \cdots \times SL_n(\mathbb{R})/SO_n(\mathbb{R})}_{r_1} \times \underbrace{SL_n(\mathbb{C})/SU_n \times \cdots \times SL_n(\mathbb{C})/SU_n}_{r_2}$$

and the dual symmetric space is

$$X_{n,u} = \underbrace{SU_n/SO_n(\mathbb{R}) \times \cdots \times SU_n/SO_n(\mathbb{R})}_{r_1} \times \underbrace{SU_n \times \cdots \times SU_n}_{r_2}.$$

	X_n	$X_{n,u}$	$H^\bullet(\varinjlim X_{n,u})$
$SL_n(\mathbb{R})$	$SL_n(\mathbb{R})/SO_n(\mathbb{R})$	$SU_n/SO_n(\mathbb{R})$	$\Lambda(x_5, x_9, \ldots, x_{4i+1}, \ldots)$
$SL_n(\mathbb{C})$	$SL_n(\mathbb{C})/SU_n$	SU_n	$\Lambda(x_3, x_5, \ldots, x_{2i+1}, \ldots)$

The conditions of theorem 4.8.2 are satisfied, and we get

$$H^\bullet(\varinjlim \Gamma'_n; \mathbb{R}) \cong \varprojlim{}^\bullet H^\bullet(\Gamma'_n; \mathbb{R}) \cong \varprojlim{}^\bullet I^\bullet_{G_n(\mathbb{R})} \cong \bigotimes_{v\in M_F^\infty} I^\bullet_v.$$

$$I^\bullet_v \cong \begin{cases} \Lambda(x_5, x_9, \ldots, x_{4i+1}, \ldots), & v \text{ is real,} \\ \Lambda(x_3, x_5, \ldots, x_{2i+1}, \ldots), & v \text{ is complex.} \end{cases}$$

So the result is

$$H^\bullet(\varinjlim SL_n(\mathcal{O}_F), \mathbb{R}) \cong \Lambda(x_5, x_9, \ldots, x_{4i+1}, \ldots)^{\otimes r_1} \otimes \Lambda(x_3, x_5, \ldots, x_{2i+1}, \ldots)^{\otimes r_2}.$$

We look at the dimension of the space of indecomposable elements $QH^i(\varinjlim SL_n(\mathcal{O}_F);\mathbb{R})$:

i:	2	3	4	5	6	7	8	9	$\cdots$
$\dim_{\mathbb{R}} QH^i(SL(\mathcal{O}_F),\mathbb{R})$:	0	r_2	0	$r_1{+}r_2$	0	r_2	0	$r_1{+}r_2$	$\cdots$

Since $\operatorname{rk} K_i(\mathcal{O}_F) = \dim_{\mathbb{R}} QH^i(SL(\mathcal{O}_F),\mathbb{R})$, we are done. This is worth repeating:

Theorem 4.8.5. *Let F be a number field and $\mathcal{O}_F$ be its ring of integers. Let r_1 be the number of real places on F and let r_2 be the number of complex places on F. The ranks of K-groups $K_i(\mathcal{O}_F)$ depend only on r_1 and r_2. One has*

$$\operatorname{rk} K_0(\mathcal{O}_F) = 1, \quad \operatorname{rk} K_1(\mathcal{O}_F) = r_1 + r_2 - 1,$$

and for $i \geqslant 2$ the ranks are periodic, with period 4:

$i \pmod 4$:	0	1	2	3
$\operatorname{rk} K_i(\mathcal{O}_F)$:	0	$r_1{+}r_2$	0	r_2

The rest of this text aimed towards theorem 4.7.2.

Chapter 5

A theorem of Matsushima

Here we review a result due to Matsushima involving the **Matsushima's constant** $m(G(\mathbb{R}))$ for a semisimple Lie group $G(\mathbb{R})$. It applies to the case of a discrete subgroup $\Gamma \subset G(\mathbb{R})$ such that $\Gamma\backslash G(\mathbb{R})$ is compact. The proof in fact relies on Hodge theory for compact manifolds, which is of course a very standard material, but this chapter starts with a detailed overview, since later on we will need to adjust it to certain non-compact cases.

References. The main content of this chapter corresponds to [Bor74, §3]. For a systematic treatment we drew upon the monograph [BW00].

5.1 Harmonic forms on a compact manifold (théorie de Hodge pour les nuls)

From now on M denotes a connected, smooth (of class $\mathscr{C}^\infty$), oriented manifold. Recall from § 4.2 de Rham complex $\Omega^\bullet(M)$. We need some extra structure, so further we assume that a **Riemannian metric** is defined on M. That is, at each point $x \in M$ there is an **inner product** (= a symmetric, bilinear, positive definite map)

$$\langle \cdot, \cdot \rangle_x : T_x M \times T_x M \to \mathbb{R},$$

depending smoothly on x, which means that for all vector fields $X, Y \in \Gamma(TM)$ the map $x \mapsto \langle X_x, Y_x \rangle_x$ is smooth. Of course any smooth manifold admits a Riemannian structure, but later on its particular choice will be important.

Let us recall the definition of Laplace–Beltrami operator [Spi99c, Chapter 7, Addendum 2].

Remark 5.1.1. We begin with some linear algebra. Let V be a real vector space of dimension n coming with an inner product and orientation. By **orientation** we mean a choice of one of the two connected components of the space $\Lambda^n(V)\backslash\{0\}$. The product extends to $\Lambda^q(V)$ by

$$\langle w_1 \wedge \cdots \wedge w_q,\ v_1 \wedge \cdots \wedge v_q \rangle = \det \begin{pmatrix} \langle w_1, v_1 \rangle & \langle w_1, v_2 \rangle & \cdots & \langle w_1, v_q \rangle \\ \langle w_2, v_1 \rangle & \langle w_2, v_2 \rangle & \cdots & \langle w_2, v_q \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle w_q, v_1 \rangle & \langle w_q, v_2 \rangle & \cdots & \langle w_q, v_q \rangle \end{pmatrix} \tag{5.1}$$

and bilinearity. Then $\langle \cdot, \cdot \rangle$ extends to the whole exterior algebra $\Lambda(V)$ by letting the product of elements of different degrees to be zero. Let $e_1, \ldots, e_n$ be an orthonormal basis for V. Then an orthonormal basis of $\Lambda(V)$ is given by

$$e_{i_1} \wedge \cdots \wedge e_{i_r} \quad \text{with } 1 \leqslant i_1 < \cdots < i_r \leqslant n.$$

Now **Hodge star** is a linear map $\star\colon \Lambda^q(V) \to \Lambda^{n-q}(V)$ that can be written in this basis as

$$\begin{aligned}\star(1) &= \pm e_1 \wedge \cdots \wedge e_n,\\ \star(e_1 \wedge \cdots \wedge e_n) &= \pm 1,\\ \star(e_1 \wedge \cdots \wedge e_q) &= \pm e_{q+1} \wedge \cdots \wedge e_n.\end{aligned}$$

Here the sign "$\pm$" is determined by the orientation—one takes "$+$" whenever $e_1 \wedge \cdots \wedge e_n$ lies in the positive component of $\Lambda^n(V)\backslash\{0\}$. One easily checks that this does not depend on the choice of an orthonormal basis of V. With this definition we see

$$\star \circ \star = (-1)^{q(n-q)} \cdot id\colon \Lambda^q(V) \to \Lambda^q(V).$$

The inner product of two elements $v, w \in \Lambda^q(V)$ can be expressed as

$$\langle v, w\rangle = \star(w \wedge \star v) = \star(v \wedge \star w).$$

To wash away the sin of defining something using a particular basis, we recall an invariant definition of $\star$: there is a bilinear map

$$\{\cdot,\cdot\}\colon \Lambda^q(V) \times \Lambda^{n-q}(V) \xrightarrow{\wedge} \Lambda^n(V) \xrightarrow{\cong} \mathbb{R},$$

where the second arrow is the isomorphism defined by the inner product and orientation on V. Then one can define a map

$$A\colon \Lambda^q(V) \to (\Lambda^{n-q}(V))^\vee$$

by

$$A(\alpha)(\eta) = \{\alpha, \eta\} \quad \text{for } \alpha \in \Lambda^q(V),\ \eta \in \Lambda^{n-q}(V).$$

Now we have

$$\begin{tikzcd} \Lambda^q(V) \arrow[r, "A"] \arrow[rr, bend right, "=\star"'] & (\Lambda^{n-q}(V))^\vee \arrow[r, "\cong"] & \Lambda^{n-q}(V) \end{tikzcd}$$

where the second isomorphism is induced by the inner product on V.

For smooth manifolds the Hodge star is used as follows. The Riemannian scalar product defines dually a product on 1-forms (on the cotangent space $T^*_n M$), and hence by virtue of (5.1) an inner product $\langle\cdot,\cdot\rangle : \Omega^q(M) \times \Omega^q(M) \to \Omega^0(M)$.

So there is a Hodge star operator $\star\colon \Omega^q(M) \to \Omega^{n-q}(M)$, which satisfies

$$\star \circ \star = (-1)^{q(n-q)} \cdot id\colon \Omega^q(M) \to \Omega^q(M). \tag{5.2}$$

It is defined to be compatible with the inner product of differential forms coming from the Riemannian structure:

$$\langle \alpha, \beta\rangle = \star(\alpha \wedge \star\beta) = \star(\beta \wedge \alpha).$$

The **volume form** ω is by definition the unique positively oriented n-form having unit length, i.e. $\langle \omega, \omega\rangle = 1$. One also sees that ω is $\star 1$, the Hodge star of the constant map 1. In what follows ω denotes the volume form (one should bear in mind that in the notation "$\star$" and "ω", and for other things below, a choice of some Riemannian structure is implicit).

So we have an identity

$$\alpha \wedge \star\beta = \langle \alpha, \beta\rangle\, \omega,$$

which actually can be treated as *the definition* of Hodge star.

Using Hodge star, we can define an operator

$$\delta \overset{\text{def}}{=} (-1)^{n(q+1)+1} \star \circ\, d \circ \star\colon \Omega^q(M) \to \Omega^{q-1}(M), \tag{5.3}$$

which lowers the degree of a differential form. For 0-forms one has just $\delta f = 0$.

A form α such that $\delta\alpha = 0$ is called **co-closed.**
From identity (5.2) and definition (5.3) we deduce

$$\delta \circ \delta = 0, \quad \star \circ \delta = (-1)^q\, d \circ \star, \quad \delta \circ \star = (-1)^{q+1} \star \circ d\,. \tag{5.4}$$

$$\begin{array}{ll}
\Omega^q(M) \xrightarrow{\delta} \Omega^{q-1}(M) \xrightarrow{\star} \Omega^{n-q+1}(M) & \star \circ \delta = \\
\Omega^q(M) \xrightarrow{\star} \Omega^{n-q}(M) \xrightarrow{d} \Omega^{n-q+1}(M) & (-1)^q\, d \circ \star \\
 & \\
\Omega^q(M) \xrightarrow{\star} \Omega^{n-q}(M) \xrightarrow{\delta} \Omega^{n-q-1}(M) & \delta \circ \star = \\
\Omega^q(M) \xrightarrow{d} \Omega^{q+1}(M) \xrightarrow{\star} \Omega^{n-q-1}(M) & (-1)^{q+1} \star \circ d
\end{array}$$

E.g. for the first one,

$$\begin{aligned}
\star\, \delta\beta &= (-1)^{n\,(q+1)+1} \star \star\, d \star \beta \\
&= (-1)^{n\,(q+1)+1} (-1)^{(n-q+1)\,(q-1)}\, d \star \beta \\
&= (-1)^q\, d \star \beta.
\end{aligned}$$

Finally, **Laplace–Beltrami operator** (also called **Laplacian**) is defined by

$$\Delta \overset{\text{def}}{=} \delta \circ d + d \circ \delta \colon \Omega^q(M) \to \Omega^q(M).$$

Example 5.1.2. If $M = \mathbb{R}^n$, then on the space $\Omega^0(\mathbb{R}^n)$ of smooth functions $\mathbb{R}^n \to \mathbb{R}$ the Laplace–Beltrami operator is the usual

$$\Delta f = -\sum_{1 \leqslant i \leqslant n} \frac{\partial^2 f}{\partial x_i^2}$$

(normally it is taken with the plus sign). For instance, in $\mathbb{R}^3$

$$\begin{aligned}
\Delta f &= d \underbrace{\delta f}_{=0} + \delta d f \\
&= -\star\, d \star \left(\frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy + \frac{\partial f}{\partial z}\, dz \right) \\
&= -\star\, d \left(\frac{\partial f}{\partial x}\, dy \wedge dz - \frac{\partial f}{\partial y}\, dx \wedge dz + \frac{\partial f}{\partial z}\, dx \wedge dy \right) \\
&= -\star \left(\frac{\partial^2 f}{\partial x^2}\, dx \wedge dy \wedge dz + \frac{\partial^2 f}{\partial y^2}\, dx \wedge dy \wedge dz + \frac{\partial^2 f}{\partial z^2}\, dx \wedge dy \wedge dz \right) \\
&= -\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \right) \cdot \underbrace{\star(dx \wedge dy \wedge dz)}_{=1} \\
&= -\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \right).
\end{aligned}$$

▲

One checks easily using (5.4) that the operators $d, \delta, \star$ commute with Δ:

$$d \circ \Delta = \Delta \circ d, \quad \delta \circ \Delta = \Delta \circ \delta, \quad \star \circ \Delta = \Delta \circ \star\,.$$

Definition 5.1.3. For two q-forms $\alpha, \beta \in \Omega^q(M)$ their **Hodge inner product** (symmetric, positive definite)

$$\langle \cdot, \cdot \rangle_M : \Omega^\bullet(M) \times \Omega^\bullet(M) \to \mathbb{R}$$

is given by

$$\langle \alpha, \beta \rangle_M \overset{\text{def}}{=} \int_M \alpha \wedge \star \beta = \int_M \star(\alpha \wedge \star \beta) \star 1 = \int_M \langle \alpha_x, \beta_x \rangle_x \, \omega.$$

We extend this on $\Omega^\bullet(M)$ simply requiring that different $\Omega^q(M)$ are orthogonal. The corresponding norm of a differential form is given by

$$\|\alpha\|_M \overset{\text{def}}{=} \sqrt{\langle \alpha, \alpha \rangle_M}.$$

Definition 5.1.4. A form $\alpha \in \Omega^q(M)$ is called **square integrable** if

$$\langle \alpha, \alpha \rangle_M = \int_M \alpha \wedge \star \alpha = \int_M \|\alpha_x\|_x^2 \, \omega < \infty.$$

Similarly, $\alpha \in \Omega^q(M)$ is called **absolutely integrable** if

$$\int_M \|\alpha_x\|_x \, \omega < \infty.$$

In particular, when M is compact, all forms are integrable.

Remark 5.1.5. Observe that if we write α locally in an orthonormal basis, then $\|\alpha_x\|_x^2$ is the sum of squares of the coefficients. If we have two q-forms α and β, then the coefficients $\alpha \wedge \beta$ are products of coefficients of α and β. Hence the Cauchy–Schwarz identity gives

$$\|\alpha_x \wedge \beta_x\|_x \leqslant \|\alpha_x\|_x \cdot \|\beta_x\|_x.$$

Let now $\alpha \in \Omega^{q-1}(M)$ and $\beta \in \Omega^q(M)$. The Leibniz rule together with $d \star \beta = (-1)^q \star \delta \beta$ gives

$$d(\alpha \wedge \star \beta) = d\alpha \wedge \star \beta + (-1)^{q-1} \alpha \wedge d \star \beta = d\alpha \wedge \star \beta - \alpha \wedge \star \delta \beta.$$

Integrating this over M, we obtain

$$\int_M d(\alpha \wedge \star \beta) = \langle d\alpha, \beta \rangle_M - \langle \alpha, \delta \beta \rangle_M . \tag{5.5}$$

Remark 5.1.6. Let us recall the Stokes' formula ([War83, Theorem 4.9] or [Spi99a, Chapter 8]).
A subset $D \subset M$ of a smooth oriented n-manifold is called a **regular domain** if for each point $x \in M$ either

(a) Some open neighborhood of x is contained in M or $M \backslash D$.

(b) There is a coordinate chart (U, ϕ) centered in x such that $\phi(U \cap D) = \phi(U) \cap \overline{\mathcal{H}}^n$, where

$$\overline{\mathcal{H}}^n \overset{\text{def}}{=} \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n \geqslant 0\}.$$

The points of type (b) comprise the **boundary** ∂D.
Now if D is a regular domain and σ is an $(n-1)$-form with compact support, then

$$\int_D d\sigma = \int_{\partial D} \sigma.$$

In particular, if M is compact, then for an $(n-1)$-form σ

$$\int_M d\sigma = 0.$$

The key words here are "form with compact support". A non-compact case will be investigated in the next chapter.

Now $\alpha \wedge \star\beta$ has compact support if either α or β has compact support. In this case the Stokes' formula can be applied, and it gives $\int_M d(\alpha \wedge \star\beta) = 0$. So (5.5) implies

$$\langle d\alpha, \beta\rangle_M = \langle \alpha, \delta\beta\rangle_M \quad \text{if one of } \alpha, \beta \text{ has compact support.} \tag{5.6}$$

In particular, if M is a compact manifold, then this means that δ is adjoint to d with respect to the inner product on $\Omega^\bullet(M)$. Since $\langle\cdot,\cdot\rangle_M$ is a positive definite bilinear form, the operator δ is uniquely defined by (5.6). From this adjunction one easily sees that

$$\Delta\alpha = 0 \iff d\alpha = 0 \text{ and } \delta\alpha = 0 \quad \text{if } \alpha \text{ has compact support.} \tag{5.7}$$

Definition 5.1.7. A differential form $\alpha \in \Omega^\bullet(M)$ such that $\Delta\alpha = 0$ is called **harmonic**.

In words: *a form with compact support is harmonic if and only if it is closed and co-closed.* Indeed, this follows from

$$\begin{aligned}
\langle \Delta\alpha, \alpha\rangle_M &= \langle (\delta d + d\delta)\,\alpha, \alpha\rangle_M \\
&= \langle \delta d\alpha, \alpha\rangle_M + \langle d\delta\alpha, \alpha\rangle_M \\
&= \langle d\alpha, d\alpha\rangle_M + \langle \delta\alpha, \delta\alpha\rangle_M \\
&= \|d\alpha\|_M^2 + \|\delta\alpha\|_M^2.
\end{aligned}$$

Example 5.1.8. Recall that a function $f\colon \mathbb{R}^n \to \mathbb{R}$ is called **harmonic** if it satisfies the Laplace equation

$$\frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2} = 0.$$

Our definition generalizes this to differential forms on smooth manifolds. ▲

We denote the space of harmonic q-forms on M by

$$\mathcal{H}^q(M) \stackrel{\text{def}}{=} \{\alpha \in \Omega^q(M) \mid \Delta\alpha = 0\}.$$

The **Hodge decomposition theorem** [War83, 6.8] tells that there is an orthogonal direct sum

$$\begin{aligned}
\Omega^q(M) &= \Delta\Omega^q(M) \oplus \mathcal{H}^q(M) \quad \text{if } M \text{ is compact} \\
&= d\delta\Omega^q(M) \oplus \delta d\Omega^q(M) \oplus \mathcal{H}^q(M) \\
&= d\Omega^{q-1}(M) \oplus \delta\Omega^{q+1}(M) \oplus \mathcal{H}^q(M).
\end{aligned}$$

Recall how the Hodge decomposition implies that *for compact M each de Rham cohomology class $[\alpha] \in H^q_{\mathrm{dR}}(M;\mathbb{R})$ is represented uniquely by a harmonic form $\mathcal{H}(\alpha) \in \mathcal{H}^q(M)$.*

For a form $\alpha \in \Omega^q(M)$ with corresponding orthogonal decomposition $\alpha = \Delta G(\alpha) + \mathcal{H}(\alpha)$ with $\mathcal{H}(\alpha) \in \mathcal{H}^q(M)$ and $\Delta G(\alpha) \in \Delta\Omega^q(M) = (\mathcal{H}^q(M))^\perp$ the form $G(\alpha)$ is called the **Green operator** of α. So any q-form decomposes as

$$\alpha = d\delta G(\alpha) + \delta d G(\alpha) + \mathcal{H}(\alpha).$$

Further G commutes with d. If α is a closed form (i.e. $d\alpha = 0$), we thus get

$$\alpha = d\delta G(\alpha) + \mathcal{H}(\alpha),$$

and so α and $\mathcal{H}(\alpha)$ represent the same class in de Rham cohomology $H^q_{\mathrm{dR}}(M;\mathbb{R})$. Now assume that $\alpha_1, \alpha_2 \in \mathcal{H}^q(M)$ are two harmonic forms representing the same class in $H^q_{\mathrm{dR}}(M;\mathbb{R})$, i.e.

$$0 = d\beta + (\alpha_1 - \alpha_2)$$

for some $\beta \in \Omega^{q-1}(M)$. The forms $d\beta$ and $(\alpha_1 - \alpha_2)$ are orthogonal thanks to (5.7):

$$\langle d\beta, \alpha_1 - \alpha_2 \rangle_M = \langle \beta, \delta\alpha_1 - \delta\alpha_2 \rangle_M = \langle \beta, 0 \rangle_M = 0,$$

so we must have $d\beta = 0$ and $\alpha_1 = \alpha_2$.

Further note that since $\star$ commutes with Δ, it maps harmonic forms to harmonic forms. Having a harmonic form $\alpha \in \mathcal{H}^q(M)$ representing a nonzero cohomology class $[\alpha] \in H^q_{\mathrm{dR}}(M)$, we get a harmonic form $\star\,\alpha \in \mathcal{H}^{n-q}(M)$. Using $\star \circ \star = (-1)^{q\,(n-1)}$, we see

$$\langle \alpha, \star\,\alpha \rangle_M = \int_M \alpha \wedge \star(\star\,\alpha) = \pm \|\alpha\|_M^2 \neq 0.$$

So for each nonzero cohomology class $[\alpha] \in H^q_{\mathrm{dR}}(M)$ we have canonically a nonzero cohomology class $[\star\,\alpha] \in H^{n-q}_{\mathrm{dR}}(M)$ such that $\langle \alpha, \star\,\alpha \rangle_M \neq 0$. Since $\langle \cdot, \cdot \rangle_M$ is a nondegenerate pairing, this gives an isomorphism

$$H^q_{\mathrm{dR}}(M) \cong H^{n-q}_{\mathrm{dR}}(M)^\vee,$$

the **Poincaré duality**. Again, this works only if M is compact.

Remark 5.1.9. The most difficult thing to prove, which we left out, is the Hodge decomposition theorem. For a thorough treatment see [War83, Chapter 6].

In short, Hodge theory gives very nice results for cohomology of a compact manifold. To get some theory work in a non-compact situation, one needs an identity analogous to (5.6). This will be discussed in the next chapter.

5.2 Matsushima's constant

We go back to the particular situation of the previous chapter.

- $G(\mathbb{R})$ is a semisimple Lie group, for our purposes we can assume it is non-compact and connected. In particular, we have in mind algebraic group $G = SL_n/\mathbb{Q}$ and its group of real points $SL_n(\mathbb{R})$. More generally, we take $G' = SL_n/F$ defined over a number field F and then take its restriction $G = \mathrm{Res}_{F/\mathbb{Q}}\, G'$.

  Since in this chapter everything concerns Lie groups, we will write simply "G" instead of "$G(\mathbb{R})$".

- Γ is a discrete subgroup in G. The main example to have in mind is that of $SL_n(\mathbb{Z})$, or more generally $SL_n(\mathcal{O}_F)$.

- We denote by K a maximal compact subgroup of G.

- $X \overset{\text{def}}{=} G/K$ is the symmetric space of maximal compact subgroups.

- $B_{\mathfrak{g}}(\cdot,\cdot)$ denotes the Killing form of $\mathfrak{g}$. Since G (and hence $\mathfrak{g}$) is semisimple, we have a positive definite symmetric bilinear form on $\mathfrak{g}$ given by

$$B_\theta(x,y) \stackrel{\text{def}}{=} -B_{\mathfrak{g}}(x,\theta(y)).$$

  This gives a right invariant Riemannian metric on G, and hence a metric on $\Gamma\backslash G$.

- In everything that follows we denote $m \stackrel{\text{def}}{=} \dim G$ and $n \stackrel{\text{def}}{=} \dim X$.

- Let $\mathfrak{g}$ and $\mathfrak{k}$ be Lie algebras of G and K respectively.

- Let $\theta\colon \mathfrak{g} \to \mathfrak{g}$ be the **Cartan involution** corresponding to K. Consider the respective **Cartan decomposition**

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p},$$

  where

$$\mathfrak{k} = \{x \in \mathfrak{g} \mid \theta(x) = x\}, \quad \mathfrak{p} \stackrel{\text{def}}{=} \{x \in \mathfrak{g} \mid \theta(x) = -x\}.$$

  One has

$$[\mathfrak{k},\mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k},\mathfrak{p}] \subset \mathfrak{p}, \quad [\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k}.$$

  The composition is orthogonal with respect to the Killing form: $B_{\mathfrak{g}}(\mathfrak{k},\mathfrak{p}) = 0$. Further, since we assume that G is non-compact, holds equality $[\mathfrak{p},\mathfrak{p}] = \mathfrak{k}$.

Let $L(\cdot,\cdot)\colon \mathfrak{k} \times \mathfrak{k} \to \mathbb{R}$ be the symmetric bilinear form defined by the adjoint action of $\mathfrak{k}$ on $\mathfrak{p}$:

$$L(x,y) \stackrel{\text{def}}{=} \operatorname{tr}(ad_{\mathfrak{p},x} \circ ad_{\mathfrak{p},y}),$$

where $ad_{\mathfrak{p},x}\colon \mathfrak{p} \to \mathfrak{p}$ is the linear map on $\mathfrak{p}$ given by $z \mapsto [x,z]$. This definition makes sense because $[\mathfrak{k},\mathfrak{p}] \subset \mathfrak{p}$. One has

$$B_{\mathfrak{g}}(x,y) = B_{\mathfrak{k}}(x,y) + L(x,y) \quad \text{for } x,y \in \mathfrak{k}.$$

Note that K is compact, hence the Killing form $B_{\mathfrak{k}}(\cdot,\cdot)$ is negative definite. The eigenvalues of ad_x for $x \in \mathfrak{k}$ are purely imaginary, and $\mathfrak{k}$ acts faithfully on $\mathfrak{p}$ via the adjoint representation, hence $L(\cdot,\cdot)$ is negative nondegenerate, and we put

$$A \stackrel{\text{def}}{=} \min\{-L(x,x) \mid x \in \mathfrak{k}, B_{\mathfrak{g}}(x,x) = -1\}.$$

We have $0 < A \leqslant 1$. Let $x_1,\ldots,x_m$ be an orthonormal basis for $\mathfrak{p}$ with respect to the Killing form $B_{\mathfrak{g}}(\cdot,\cdot)$. For indices $1 \leqslant i,j,k,\ell \leqslant m$ we consider

$$R_{ijk\ell} \stackrel{\text{def}}{=} B_{\mathfrak{g}}([x_\ell,x_k],[x_j,x_i]) = B_{\mathfrak{g}}([[x_\ell,x_k],x_j],x_i). \tag{5.8}$$

It is the curvature tensor on X, with the invariant Riemannian metric given by the restriction of the Killing form on $\mathfrak{p} = T_e(X)$. In particular, it satisfies the identities (cf. [Spi99b, §4.D])

$$R_{ijk\ell} = -R_{jik\ell}, \quad R_{ijk\ell} = -R_{ij\ell k},$$
$$R_{ijk\ell} = R_{k\ell ik},$$
$$R_{ijk\ell} + R_{ik\ell j} + R_{i\ell jk} = 0 \quad \text{(“the first Bianchi identity”)}.$$

Of course these identities are immediate from the definition (5.8), and the geometric interpretation of $R_{ijk\ell}$ will not be needed in what follows.

Definition 5.2.1. For $q = 1, 2, 3, \ldots$ consider a symmetric bilinear form on $\mathfrak{p} \otimes \mathfrak{p}$ given by

$$F_{\mathfrak{g}}^{q}(\xi, \eta) \stackrel{\text{def}}{=} \frac{A}{2q} \sum_{i,j} \xi_{ij}\, \eta_{ij} + \sum_{ijk\ell} R_{ijk\ell}\, \xi_{i\ell}\, \eta_{jk}.$$

The **Matsushima's constant** is defined as

$$m(G) \stackrel{\text{def}}{=} m(\mathfrak{g}) \stackrel{\text{def}}{=} \max\{0\} \cup \{q \mid F_{\mathfrak{g}}^{q}(\xi, \xi) > 0 \text{ on } \mathfrak{p} \otimes \mathfrak{p} \backslash \{0 \otimes 0\}\}.$$

This makes sense because the form $\sum_{ijk\ell} R_{ijk\ell}\, \xi_{i\ell}\, \eta_{jk}$ is not positive definite for a trivial reason: there is some coefficient $R_{ijk\ell} < 0$, so we can set $\xi_{i\ell} = \xi_{jk} = \eta_{il} = \eta_{jk} = 1$, and the rest to zero, making sure the value $R_{ijk\ell}\, \xi_{i\ell}\, \eta_{jk} + R_{ji\ell k}\, \xi_{jk}\, \eta_{i\ell} = 2\, R_{ijk\ell}$ is negative. However, if we add to this a positive definite form $\frac{A}{2q} \sum_{i,j} \xi_{ij}\, \eta_{ij}$, then for q small enough the sum may become positive definite.

Remark 5.2.2. The definition of $m(G)$ *does* look strange, and one probably can understand it only reading the proof of theorem 5.3.1.

The constant A is relatively easy to calculate. The problem is to estimate the eigenvalues of the bilinear form $\sum_{ijk\ell} R_{ijk\ell}\, \xi_{i\ell}\, \eta_{jk}$. The constants $m(G)$ were determined case by case in [Mat62a] and [KN62].

Example 5.2.3. Consider $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$ with Cartan involution $\theta\colon x \mapsto -x^\top$. In the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the space $\mathfrak{k}$ is given by the traceless antisymmetric matrices, and $\mathfrak{p}$ by the traceless symmetric matrices.

A basis for $\mathfrak{k}$ gives $u \stackrel{\text{def}}{=} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and a basis for $\mathfrak{p}$ give $a \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $b \stackrel{\text{def}}{=} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

We see that $[u, a] = -2\, b$, $[u, b] = 2\, a$, $[a, b] = 2\, u$, hence $ad_{\mathfrak{p},u} = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$, and

$$L(u, u) = B_{\mathfrak{g}}(u, u) = tr(ad_{\mathfrak{p}u} \circ ad_{\mathfrak{p}u}) = -8.$$

Trivially $A = 1$. Next we calculate the curvature tensor $R_{ijk\ell} = B_{\mathfrak{g}}([x_\ell, x_k], [x_j, x_i])$. The values are

$$R_{uaua} = 32, \;\; R_{ubub} = 32, \;\; R_{abab} = -32$$

(and the rest are deduced from these). The quadratic form $F_{\mathfrak{g}}^{q}(\xi, \xi)$ is

$$\begin{aligned} F_{\mathfrak{g}}^{q}(\xi, \xi) = \frac{1}{2\, r}\, &(\xi_{uu}^2 + \xi_{ua}^2 + \xi_{ub}^2 + \xi_{au}^2 + \xi_{aa}^2 + \xi_{ab}^2 + \xi_{bu}^2 + \xi_{ba}^2 + \xi_{bb}^2) + \\ &64\, (-\xi_{uu}\, \xi_{aa} - \xi_{uu}\, \xi_{bb} + \xi_{ua}\, \xi_{au} + \xi_{bu}\, \xi_{ub} + \xi_{aa}\, \xi_{bb} - \xi_{ab}\, \xi_{ba}). \end{aligned}$$

This form is never positive definite. For instance, take $\xi_{ua} = -1$, $\xi_{au} = 1$, and the rest $= 0$. We have $1/r - 64 < 0$. So in this case $m(\mathfrak{g}) = 0$. ▲

Example 5.2.4. To see something less trivial, take $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{R})$. Now the dimension is 8, a base for $\mathfrak{k}$ and $\mathfrak{p}$ is given by

$$\mathfrak{k}\colon \quad u = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; v = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \; w = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},$$

$$\mathfrak{p}\colon \quad a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; b = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \; c = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; d = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \; e = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

The Killing form on $\mathfrak{sl}_3(\mathbb{R})$ is $B_{\mathfrak{g}}(x,y) = 6\,tr(x \cdot y)$, and we calculate

$B_{\mathfrak{g}}(\cdot,\cdot)$	a	b	c	d	e	u	v	w
a	12	−6	0	0	0	0	0	0
b	−6	12	0	0	0	0	0	0
c	0	0	12	0	0	0	0	0
d	0	0	0	12	0	0	0	0
e	0	0	0	0	12	0	0	0
u	0	0	0	0	0	−12	0	0
v	0	0	0	0	0	0	−12	0
w	0	0	0	0	0	0	0	−12

Further, we calculate the Killing form of $\mathfrak{k}$ and the linear form $L(\cdot,\cdot)$:

$B_{\mathfrak{k}}(\cdot,\cdot)$	u	v	w
u	−2	0	0
v	0	−2	0
w	0	0	−2

$L(\cdot,\cdot)$	u	v	w
u	−10	0	0
v	0	−10	0
w	0	0	−10

We see easily that $A = 5/6$. Since now $\mathfrak{p}$ has dimension 5, we are not going to write down explicitly the quadratic form $F^q_{\mathfrak{g}}(\xi,\xi)$. Calculations show that $m(\mathfrak{g}) = 1$. ▲

Example 5.2.5. The general formula for A obtained by Matsushima in [Mat62b, §7] is the following. Assume that $\mathfrak{g}$ and $\mathfrak{k}$ are simple Lie algebras. Then

$$A = \frac{\dim \mathfrak{p}}{2\dim \mathfrak{k}} = \frac{\dim \mathfrak{g} - \dim \mathfrak{k}}{2\dim \mathfrak{k}}.$$

In particular, for $\mathfrak{sl}_n(\mathbb{R})$ we have

$$\begin{aligned} \dim \mathfrak{g} &= \dim \mathfrak{sl}_n(\mathbb{R}) = n^2 - 1, \\ \dim \mathfrak{k} &= \dim \mathfrak{so}_n(\mathbb{R}) = \binom{n}{2} = \frac{n\,(n-1)}{2}. \end{aligned}$$

And we calculate

$$A = \frac{n+2}{2\,n}.$$

This agrees with the value 5/6 above for $\mathfrak{sl}_3(\mathbb{R})$. Other values of A for classical cases can be found in [KN62, p. 245]. In notation of Kaneyuki and Nagano, $A = 2\,b_{(\mathfrak{g},\mathfrak{k})}$. ▲

It is more difficult to see in general when the quadratic form $F^q_{\mathfrak{g}}(\xi,\xi)$ is positive definite. Such calculations also can be found in [KN62].

Example 5.2.6. The Matsushima constant for $SL_n(\mathbb{R})$ is

$$m(SL_n(\mathbb{R})) = \left\lfloor\!\left\lfloor \frac{n+2}{4} \right\rfloor\!\right\rfloor,$$

by which we denote the biggest integer strictly smaller than $(n+2)/4$

For $SL_n(\mathbb{C})$ the constant is

$$m(SL_n(\mathbb{C})) = \left\lfloor\!\left\lfloor \frac{n}{2} \right\rfloor\!\right\rfloor.$$

▲

Example 5.2.7. In the case that interests us, we take $G' = SL_n/F$ over a number field F, and then the restriction $G = \mathrm{Res}_{F/\mathbb{Q}}\, G'$. After we take real points, we obtain an identification

$$G(\mathbb{R}) \cong \underbrace{SL_n(\mathbb{R}) \times \cdots \times SL_n(\mathbb{R})}_{r_1} \times \underbrace{SL_n(\mathbb{C}) \times \cdots \times SL_n(\mathbb{C})}_{r_2}.$$

The only thing we care about is that $m(G(\mathbb{R})) \xrightarrow{n\to\infty} \infty$. ▲

5.3 Matsushima's theorem

Recall from § 4.7 that we have a morphism

$$j^q_\Gamma\colon \underbrace{I^q_G \stackrel{\text{def}}{=} \Omega^q(X)^G}_{\text{closed forms}} \to H^q(\Omega^\bullet(X)^\Gamma) \cong H^q(\Gamma\backslash X, \mathbb{R}) \cong H^q(\Gamma, \mathbb{R}).$$

A theorem of Matsushima [Mat62b, Mat62a] tells that for co-compact Γ this is an isomorphism up to degree $m(G)$:

Theorem 5.3.1. *Let Γ be a discrete subgroup of G and assume that $\Gamma\backslash G$ is compact. Then the morphism j^q_Γ is*

- *injective for all q,*
- *surjective for $q \leqslant m(G)$.*

Of course this makes sense only if the constant $m(G)$ is known. It turns out to be a relatively small number; e.g. as we mentioned above, $m(SL_n(\mathbb{R})) = \left\lfloor \frac{n+2}{4} \right\rfloor$. But if we are interested in the case $n \to \infty$, we are in business—see the previous chapter for this.

The forms I^q_G are harmonic (cf. [BW00, §II.3]), so j^q_Γ is injective by Hodge theory, under the assumption that $\Gamma\backslash G$, and hence $\Gamma\backslash X$, is compact. (The manifold $\Gamma\backslash X$ is not necessarily smooth, but we can do the same thing that we did in the previous chapter: pick a torsion free normal subgroup of finite index $\Gamma' \lhd \Gamma$ and then $H^\bullet(\Gamma, \mathbb{R}) = H^\bullet(\Gamma', \mathbb{R})^{\Gamma/\Gamma'}$, $(I^\bullet_G)^\Gamma = ((I^\bullet_G)^{\Gamma'})^{\Gamma/\Gamma'}$, $\Omega(X)^\Gamma = (\Omega(X)^{\Gamma'})^{\Gamma/\Gamma'}$.)

The nontrivial part is surjectivity, and all amounts to the following: if one has a Γ-invariant form $\eta \in (\Omega^q(X))^\Gamma$:

$$\eta = \sum_{|I|=q} \eta_I\, \omega^I \stackrel{\text{def}}{=} \sum_{1\leqslant i_1 < \cdots < i_q \leqslant n} \eta_{i_1,\ldots,i_q}\, \omega^1 \wedge \cdots \wedge \omega^q,$$

then it is G-invariant, provided $q \leqslant m(G)$:

$$y \cdot \eta = 0 \quad \text{for all } y \in \mathfrak{g}.$$

Since $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ with $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{k}$, it is enough to show the above for $y \in \mathfrak{p}$, i.e. that

$$x_i \cdot \eta_I = 0 \quad \text{for all } 1 \leqslant i \leqslant m,\ I = \{i_1, \ldots, i_q\} \subseteq \{1, \ldots, m\}$$

(recall that by $x_1, \ldots, x_m$ we denote an orthonormal basis for $\mathfrak{p}$).

The proof goes as follows. The form $F^q_\mathfrak{g}$ on $\mathfrak{p} \otimes \mathfrak{p}$ from the definition of the Matsushima's constant can be defined on $\mathfrak{p} \otimes \mathfrak{p} \otimes \mathscr{C}^\infty(\Gamma\backslash X)$ by tensoring with the scalar product $\langle f, g\rangle_{\Gamma\backslash X} \stackrel{\text{def}}{=} \int_{\Gamma\backslash X} f \cdot g\, \omega$. Then we consider an element of $\mathfrak{p} \otimes \mathscr{C}^\infty(\Gamma\backslash X)$ given in the basis $x_1, \ldots, x_m$ by

$$(x_1 \cdot \eta_I, \ldots, x_m \cdot \eta_I).$$

Using certain manipulations, one can show that

$$F^q_\mathfrak{g}(x_1 \cdot \eta_I, \ldots, x_m \cdot \eta_I) \leqslant 0,$$

which means $x_i \cdot \eta_I = 0$ since $F^q_\mathfrak{g}$ is positive definite for $q \leqslant m(G)$.

Proof. We are going to use some explicit computations with the structure constants of $\mathfrak{g}$.

Recall that we have the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. We can fix a basis $(x_i)_{1 \leqslant i \leqslant m}$ of $\mathfrak{p}$, which is orthonormal with respect to the Killing form, and a basis $(x_a)_{m+1 \leqslant a \leqslant n}$ of $\mathfrak{k}$, which is "pseudo-orthonormal", i.e. with the Kronecker δ notation,

$$B_{\mathfrak{g}}(x_i, x_j) = \delta_{ij}, \quad B_{\mathfrak{g}}(x_a, x_b) = -\delta_{ab}.$$

Now we are going to write some cumbersome formulas in the fixed bases, and in what follows the indices i, j, k, ℓ always range from 1 to m and a, b, c, d range from $m+1$ to n.

Let c_{ij}^a be the structure constants of $\mathfrak{g}$. Since $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{k}$, we get

$$[x_i, x_j] = \sum_{m < a \leqslant n} c_{ij}^a \, x_a, \quad [x_a, x_i] = \sum_j c_{a,i}^i \, x_j. \tag{5.9}$$

For a form $\eta \in \Omega^q(X)^\Gamma$ we consider an expression

$$\Phi(\eta) \stackrel{\text{def}}{=} \frac{(q-1)!}{2} \sum_{i,j,I} \| [x_i, x_j] \cdot \eta_I \|^2_{\Gamma \backslash X},$$

where

$$\|\alpha\|^2_{\Gamma \backslash X} \stackrel{\text{def}}{=} \langle \alpha, \alpha \rangle_{\Gamma \backslash X}, \quad \langle \alpha, \beta \rangle_{\Gamma \backslash X} \stackrel{\text{def}}{=} \int_{\Gamma \backslash X} \langle \alpha_x, \beta_x \rangle_x \, \omega.$$

Here and below I runs through the q element subsets of $\{1, \ldots, m\}$.

Now using (5.9) we write

$$\Phi(\eta) = \frac{(q-1)!}{2} \sum_{i,j,I} c_{ij}^a \, c_{ij}^b \, \langle x_a \cdot \eta_I, x_b \cdot \eta_I \rangle_{\Gamma \backslash X}.$$

For the bilinear form $L(\cdot, \cdot)$ on $\mathfrak{k}$ (defined in the previous section) we have

$$L(x_a, x_b) = \sum_{i,j} c_{aj}^i \, c_{bi}^j = \sum_{i,j} c_{ij}^a \, c_{ji}^b = -\sum_{i,j} c_{ij}^a \, c_{ij}^b.$$

Further note that x_a and x_b are orthogonal, and $L(x_a, x_b) = 0$ unless $a \neq b$.

Hence

$$\Phi(\eta) = -\frac{(q-1)!}{2} \sum_{a,b,I} L(x_a, x_b) \, \langle x_a \cdot \eta_I, x_b \cdot \eta_I \rangle_{\Gamma \backslash X} = -\frac{(q-1)!}{2} \sum_{a,I} L(x_a, x_a) \, \| x_a \cdot \eta_I \|^2_{\Gamma \backslash X}.$$

Now by the definition of the constant A (see the previous section) we have an inequality

$$\Phi(\eta) \geqslant \frac{A\,(q-1)!}{2} \sum_{a,I} \| x_a \cdot \eta_I \|^2_{\Gamma \backslash X}. \tag{5.10}$$

If instead of taking I running through the indices $1 \leqslant j_1 < \cdots < j_q \leqslant m$ we take all the indices $1 \leqslant j_1, \ldots, j_q \leqslant m$, then we have

$$\Phi(\eta) = \frac{1}{2\,q} \sum_{\substack{i,j \\ j_1, \ldots, j_q}} \| [x_i, x_j] \cdot \eta_{j_1, \ldots, j_q} \|^2_{\Gamma \backslash X}.$$

Using again (5.9), we write

$$\Phi(\eta) = \frac{1}{2\,q} \sum_{\substack{i,j,a \\ j_1, \ldots, j_q}} c_{ij}^a \, \left\langle x_a \cdot \eta_{j_1, \ldots, j_q}, \, [x_i, x_j] \cdot \eta_{j_1, \ldots, j_q} \right\rangle_{\Gamma \backslash X} = \frac{1}{q} \sum_{\substack{i,j,a \\ j_1, \ldots, j_q}} c_{ij}^a \, \left\langle x_a \cdot \eta_{j_1, \ldots, j_q}, \, x_i \cdot x_j \cdot \eta_{j_1, \ldots, j_q} \right\rangle_{\Gamma \backslash X} \tag{5.11}$$

(the latter since $c_{ji}^a = -c_{ij}^a$ and $[x_i, x_j] = x_i \cdot x_j - x_j \cdot x_i$).

Up to this point we just did some formal manipulations in fixed bases. Now we use the assumption that η is a Γ-invariant form, i.e. $\eta \in \Omega^q(X)^\Gamma \cong C^q(\mathfrak{g}, \mathfrak{k}; \mathcal{C}^\infty(\Gamma\backslash G))$. The action of $\mathfrak{k}$ on the latter is given by

$$x_a \cdot \eta_{j_1,\dots,j_q} = \eta([x_a, x_{j_1}], x_{j_2}, \dots, x_{j_q}) + \eta(x_{j_1}, [x_a, x_{j_2}], \dots, x_{j_q}) + \dots + \eta(x_{j_1}, x_{j_2}, \dots, [x_a, x_{i_q}]).$$

We write this as

$$x_a \cdot \eta_{j_1,\dots,j_q} = \sum_u (-1)^{u-1}\, \eta([x_a, x_{j_u}], x_{j_1}, \dots, \widehat{x}_{j_u}, \dots, x_{j_q}).$$

Now we have from (5.9) an expression $[x_a, x_{j_u}] = \sum_k c_{a,j_u}^k\, x_k$, so

$$x_a \cdot \eta_{j_1,\dots,j_q} = \sum_{u,k} (-1)^{u-1}\, c_{a,j_u}^k\, \eta(x_k, x_{j_1}, \dots, \widehat{x}_u, \dots, x_{j_q}).$$

We put this into (5.11) to obtain

$$q\, \Phi(\eta) = \sum_{\substack{i,j,k,u \\ j_1,\dots,j_q}} (-1)^{u-1} \left(\sum_a c_{ij}^a\, c_{k,j_u}^a \right) \left\langle \eta_{k,j_1,\dots,\widehat{j}_u,\dots,j_q},\, x_i \cdot x_j \cdot \eta_{j_1,\dots,j_q} \right\rangle_{\Gamma\backslash X}.$$

By assumption $\Gamma\backslash X$ is compact, hence we can use the Stokes' formula

$$\langle x \cdot f,\, g\rangle_{\Gamma\backslash X} + \langle f,\, x \cdot g\rangle_{\Gamma\backslash X} = 0$$

Hence

$$q\, \Phi(\eta) = -\sum_{\substack{i,j,k,u \\ j_1,\dots,j_q}} (-1)^{u-1} \left(\sum_a c_{ij}^a\, c_{k,j_u}^a \right) \left\langle x_i \cdot \eta_{k,j_1,\dots,\widehat{j}_u,\dots,j_q},\, x_j \cdot \eta_{j_1,\dots,j_q} \right\rangle_{\Gamma\backslash X}.$$

Now observe that from the definition of $R_{ijk\ell}$ (formula (5.8)) follows $R_{ijk\ell} = -\sum_a c_{ij}^a\, c_{k\ell}^a$, so

$$\begin{aligned} q\, \Phi(\eta) &= \sum_{\substack{i,j,k,u \\ j_1,\dots,j_q}} (-1)^{u-1} R_{ijki_u} \left\langle x_i \cdot \eta_{k,j_1,\dots,\widehat{j}_u,\dots,j_q},\, x_j \cdot \eta_{j_1,\dots,j_q} \right\rangle_{\Gamma\backslash X} \\ &= \sum_{\substack{i,j,k,u \\ j_1,\dots,j_q}} R_{ijki_u} \left\langle x_i \cdot \eta_{k,j_1,\dots,\widehat{j}_u,\dots,j_q},\, x_j \cdot \eta_{j_u,j_1,\dots,\widehat{j}_u,\dots,j_q} \right\rangle_{\Gamma\backslash X}. \end{aligned}$$

The last sum can be written as

$$q\, \Phi(\eta) = q \sum_{\substack{i,j,k,\ell \\ j_2,\dots,j_q}} R_{ijk\ell} \left\langle x_i \cdot \eta_{k,j_2,\dots,j_q},\, x_j \cdot \eta_{\ell,j_2,\dots,j_q} \right\rangle_{\Gamma\backslash X}.$$

Since $R_{ijk\ell} = -R_{ij\ell k}$, we have

$$\Phi(\eta) = -\sum_{\substack{i,j,k,\ell \\ j_2,\dots,j_q}} R_{ijk\ell} \left\langle x_i \cdot \eta_{\ell,j_2,\dots,j_q},\, x_j \cdot \eta_{k,j_2,\dots,j_q} \right\rangle_{\Gamma\backslash X}.$$

Now going back to the inequality (5.10),

$$\sum_{j_2,\dots,j_q} \left(\frac{A}{2\,q} \sum_a \|x_a \cdot \eta_I\|^2_{\Gamma\backslash X} + \sum_{i,j,k,\ell} R_{ijk\ell} \left\langle x_i \cdot \eta_{\ell,j_2,\dots,j_q},\, x_j \cdot \eta_{k,j_2,\dots,j_q} \right\rangle_{\Gamma\backslash X} \right) \leqslant 0.$$

Finally observe that in the brackets we have a form on $(\mathfrak{p} \otimes \mathfrak{p}) \otimes \mathcal{C}^\infty(\Gamma\backslash X)$, given by tensoring $F_\mathfrak{g}^q$ with the scalar product $\langle\cdot,\cdot\rangle_{\Gamma\backslash X}$ on $\mathcal{C}^\infty(\Gamma\backslash X)$. For $q \leqslant m(G)$ the form $F_\mathfrak{g}^q$ is positive definite, hence our form is positive definite as well, and we conclude

$$x_i \cdot \eta_{\ell,j_2,\ldots,j_q} = 0 \quad \text{for all } 1 \leqslant i, \ell, j_2, \ldots, j_q \leqslant m.$$

This is what we wanted to show. ∎

Hopefully, after reading this proof, the definition of Matsushima's constant becomes a bit more clear.

Chapter 6

A theorem of Garland

In this chapter we consider a theorem due to Garland [Gar71] regarding the injectivity of morphism $j^\bullet\colon I_G^\bullet \to H^\bullet(\Gamma, \mathbb{R})$.

We already reviewed in § 5.1 the classic Hodge theory. If the manifold is not compact, then it does not work, but one can still show some facts if M is a complete Riemannian manifold.

References. The discussion of square integrable forms follows [Bor74, §1–2]. The Garland's theorem is taken from [Bor74, §3].

6.1 Complete Riemannian manifolds

Let M be a smooth, oriented, connected Riemannian manifold. M has a natural metric: for two points $x, y \in M$ one puts

$$d(x,y) \stackrel{\text{def}}{=} \inf(\text{length of a piecewise smooth path joining } x \text{ and } y).$$

A Riemannian manifold M is said to be **complete** if the corresponding metric space (M, d) is complete (i.e. every Cauchy sequence in (M, d) converges). A characterization of complete Riemannian manifolds is given by Hopf–Rinow theorem [dC92, Chapter 7]. The following are equivalent:

1. M is complete as a metric space.
2. The closed and bounded sets in M are compact.
3. M is **geodesically complete**, meaning that any geodesic $\gamma(t)$ starting from a point $x \in M$ is defined for all values of the parameter $t \in \mathbb{R}$.

Recall that a continuous function $f\colon M \to \mathbb{R}$ is called **proper** if for every compact subset $K \subset \mathbb{R}$ its preimage $f^{-1}(K) \subseteq M$ is compact. The following is a useful completeness criterion [Gor73, Gor74].

Theorem 6.1.1. *A Riemannian manifold (M, g) is complete if and only if there exists a proper $\mathscr{C}^\infty$-function $\mu\colon M \to [0, \infty)$ such that $d\mu(x)$ has bounded length, i.e. for some constant $c > 0$,*

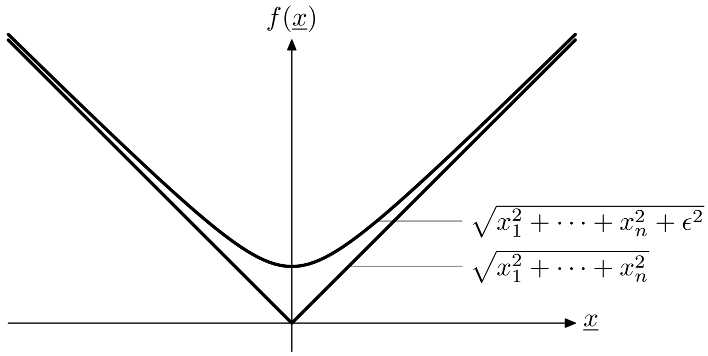
$$\|d\mu(x)\|_x \leqslant c \text{ for all } x \in M.$$

Example 6.1.2. The Euclidean space $\mathbb{R}^n$ with the canonical Riemannian structure is of course complete. For a point $\underline{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ it is natural to consider its distance to $\underline{0} = (0, \ldots, 0)$:

$$\|\underline{x}\| = \sqrt{x_1^2 + \cdots + x_n^2}.$$

This function is not smooth at $\underline{0}$. To fix this, for some $\epsilon > 0$ we replace it with

$$\mu(\underline{x}) = \sqrt{x_1^2 + \cdots + x_n^2 + \epsilon^2}.$$



We compute

$$d\mu(\underline{x}) = \frac{1}{\mu(\underline{x})}\,(x_1\,dx_1 + \cdots + x_n\,dx_n).$$

Now

$$\|d(\mu(\underline{x}))\|_x = \frac{\|\underline{x}\|}{\mu(\underline{x})} < \frac{\|\underline{x}\|}{\|\underline{x}\| + \epsilon} < 1.$$

▲

For a general proof of the theorem, we fix a point $x_0 \in M$ and consider

$$\begin{array}{rcl} r\colon M & \to & \mathbb{R}, \\ x & \mapsto & d(x_0, x). \end{array}$$

This is a continuous function, and by the triangle inequality it satisfies

$$|r(y) - r(x)| \leqslant d(x, y),$$

i.e. it is Lipschitz (with Lipschitz constant 1). This function is proper: indeed, for each $R > 0$ the set

$$\{x \in M \mid d(x_0, x) \leqslant R\}$$

is closed and bounded, hence compact (by Hopf–Rinow theorem). The function is not $\mathcal{C}^\infty$, but for every $\epsilon > 0$ there exists a $\mathcal{C}^\infty$-approximation $r_\epsilon\colon M \to \mathbb{R}$ such that

$$|r_\epsilon(x) - r(x)| < \epsilon, \tag{6.1}$$

and

$$\|dr_\epsilon(x)\|_x < 1 + \epsilon \tag{6.2}$$

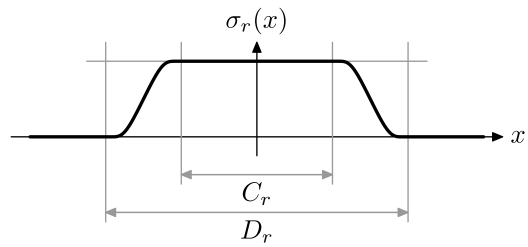
—for this see e.g. [Gaf59, §3] or [dR84, §15]. Now (6.1) means that r_ϵ is also a proper function, and (6.2) is the bound that we need.

Conversely, suppose that on M there exists a proper function μ with $\|d\mu(x)\|_x \leqslant c$. We would like to show that M is complete. Let $\gamma\colon t \mapsto \gamma(t)$ be a geodesic segment with $t \in I$ for some bounded interval $I \subset \mathbb{R}$. Assume that γ is parametrized so that $\|d\gamma/dt\| = 1$. Suppose the length of γ is finite. Then since $\|d\gamma/dt\| = 1$, the variation of $\mu \circ \gamma$ on I is bounded, and so $\operatorname{im}\gamma$ is contained in a bounded set (because μ is a proper map). But then $\operatorname{im}\gamma$ can be extended (at both ends) to a longer geodesic segment. Hence M is complete. ■

Lemma 6.1.3. *Let M be a complete Riemannian manifold. Then there exist*

- *a family of compact sets $C_r \subset D_r$ for $r > 0$ such that C_r contains the interior of $C_{r'}$ if $r > r'$ and M is the union of the C_r,*
- *a family of smooth functions $\sigma_r \colon M \to \mathbb{R}$ for $r > 0$ with values $0 \leqslant \sigma_r(x) \leqslant 1$, such that*

$$\sigma_r(x) = \begin{cases} 1, & x \in C_r, \\ 0, & x \notin D_r. \end{cases}$$



- *a constant c,*

such that

$$\|d\sigma_r(x)\|_x \leqslant c\, r^{-1} \quad \textit{for all } x \in M.$$

First let us explain why it is useful. We have the great Stokes' formula (5.6), which works for differential forms with compact support. If some form α fails to have compact support, then we can replace it with $\sigma_r \cdot \alpha$, apply Stokes to it, and then look what happens as $r \to \infty$. To analyze the case $r \to \infty$, we need the bound on $\|d\sigma_r(x)\|_x$.

Proof. To prove the lemma we recall that one can define a smooth function $m \colon [0, \infty) \to [0, 1]$ such that $m(x) = 0$ for $x \in [0, 1]$ and $f(x) = 1$ for $x \in [2, \infty)$.

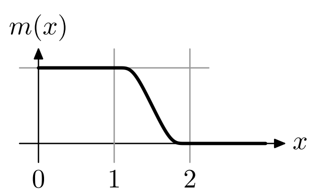
Indeed, one takes

$$\theta(x) \stackrel{\text{def}}{=} \begin{cases} 0, & x \leqslant 0, \\ e^{-1/x}, & x > 0. \end{cases}$$

And

$$m(x) \stackrel{\text{def}}{=} \frac{\theta(2-x)}{\theta(x-1) + \theta(2-x)}.$$

(Cf. the construction of "bump functions" for partitions of unity.)



We take $\sigma_r(x) \stackrel{\text{def}}{=} m(\mu(x)/r)$, where μ is given by the previous theorem, and it is clear that

$$\|d\sigma_r(x)\|_x \leqslant c'\, r^{-1} \|d\mu(x)\|_x \leqslant c\, r^{-1}.$$

■

6.2 Adjunction $\langle \alpha, \delta\beta \rangle_M = \langle d\alpha, \beta \rangle_M$ on complete manifolds

Proposition 6.2.1. *As before, let M be a connected complete Riemannian manifold. Let $\alpha \in \Omega^q(M)$ and $\beta \in \Omega^{q+1}(M)$. Assume that the functions*

$$x \mapsto \|\alpha_x\|_x \cdot \|\beta_x\|_x, \quad x \mapsto \langle (d\alpha)_x, \beta_x \rangle_x, \quad x \mapsto \langle \alpha_x, (\delta\beta)_x \rangle_x$$

are absolutely integrable on M. Then

$$\langle d\alpha, \beta \rangle_M = \langle \alpha, \delta\beta \rangle_M .$$

Proof. If one of α and β has compact support, then this is the usual Stokes' formula (5.6). If not, we replace α with $\sigma_r \cdot \alpha$ where σ_r is taken as in the lemma 6.1.3. Then $\sigma_r \cdot \alpha$ has compact support, and

$$\langle \sigma_r \cdot \alpha, \delta\beta \rangle_M = \langle d(\sigma_r \cdot \alpha), \beta \rangle_M .$$

By the Leibniz rule,

$$d(\sigma_r \cdot \alpha) = d\sigma_r \wedge \alpha + \sigma_r \cdot d\alpha.$$

We take the limit $r \to \infty$:

$$\underbrace{\lim_{r\to\infty} \langle \sigma_r \cdot \alpha, \delta\beta \rangle_M}_{=\langle \alpha, \delta\beta \rangle_M} = \lim_{r\to\infty} \langle d\sigma_r \wedge \alpha, \beta \rangle_M + \underbrace{\lim_{r\to\infty} \langle \sigma_r \cdot d\alpha, \beta \rangle_M}_{=\langle d\alpha, \beta \rangle_M} .$$

Since $\langle \sigma_r \cdot \alpha, \delta\beta \rangle_M$ tends to $\langle \alpha, \delta\beta \rangle_M$ and $\langle \sigma_r \cdot d\alpha, \beta \rangle_M$ tends to $\langle d\alpha, \beta \rangle_M$, it remains to show that

$$\lim_{r\to\infty} \langle d\sigma_r \wedge \alpha, \beta \rangle_M \stackrel{\text{def}}{=} \lim_{r\to\infty} \int_M \langle d\sigma_r(x) \wedge \alpha_x, \beta_x \rangle \; \omega = 0.$$

We apply the Cauchy–Schwarz inequality for inner products and an inequality for wedge products (remark 5.1.5):

$$| \langle d\sigma_r(x) \wedge \alpha_x, \beta_x \rangle_x | \leqslant \|d\sigma_r(x) \wedge \alpha_x\|_x \cdot \|\beta_x\|_x \leqslant \|d\sigma_r(x)\|_x \cdot \|\alpha_x\|_x \cdot \|\beta_x\|_x \leqslant c\, r^{-1} \|\alpha_x\|_x \cdot \|\beta_x\|_x.$$

Thus

$$\|(d\sigma_r \wedge \alpha, \beta)\|_M \leqslant c\, r^{-1} \int_M \|\alpha_x\|_x \cdot \|\beta_x\|_x \, \omega,$$

which tends to 0 as $r \to \infty$. ■

In particular, we have the Cauchy–Schwarz inequality

$$| \langle \alpha, \beta \rangle_M | \leqslant \|\alpha\|_M \cdot \|\beta\|_M.$$

$$\left| \int_M \langle \alpha_x, \beta_x \rangle_x \; \omega \right|^2 \leqslant \int_M \|\alpha_x\|_x^2 \, \omega \cdot \int_M \|\beta_x\|_x^2 \, \omega.$$

With this the proposition immediately implies

Corollary 6.2.2. *As before, let M be a connected complete Riemannian manifold.*

Let $\alpha \in \Omega^q(M)$ and $\beta \in \Omega^{q+1}(M)$ be differential forms such that $\alpha, d\alpha, \beta, \delta\beta$ are square integrable on M. Then

$$\langle d\alpha, \beta \rangle_M = \langle \alpha, \delta\beta \rangle_M .$$

Using the same kind of arguments as in the proof of proposition 6.2.1, one deduces the following

Proposition 6.2.3. *If α is a form on a complete Riemannian manifold, then*

$$\Delta\alpha = 0 \iff d\alpha = 0 \text{ and } \delta\alpha = 0 \quad \text{if } \alpha \text{ is square integrable.}$$

This is originally due to Andreotti and Vesentini [AV65]; we follow [dR84, §35].

Proof. We use again lemma 6.1.3 and replace α with $\alpha_r \stackrel{\text{def}}{=} \sigma_r^2 \cdot \alpha$. Now α_r has compact support, and

$$\langle d\alpha, d\alpha_r\rangle_x = \langle \delta d\alpha, \alpha_r\rangle_x . \tag{6.3}$$

By the Leibniz rule,

$$d\alpha_r = d\sigma_r^2 \wedge \alpha + \sigma_r^2 \cdot d\alpha = 2\sigma_r \cdot d\sigma_r \wedge \alpha + \sigma_r^2 \cdot d\alpha.$$

Hence

$$\langle d\alpha, d\alpha_r\rangle_x = \langle d\alpha, 2\sigma_r \cdot d\sigma_r \wedge \alpha\rangle_x + \left\langle d\alpha,\, \sigma_r^2 \cdot d\alpha\right\rangle_x . \tag{6.4}$$

Now we have $\left\langle d\alpha,\, \sigma_r^2 \cdot d\alpha\right\rangle_x = \langle \sigma_r \cdot d\alpha,\, \sigma_r \cdot d\alpha\rangle_x$ and $\langle d\alpha,\, 2\sigma_r \cdot d\sigma_r \wedge \alpha\rangle_x = \langle \sigma_r \cdot d\alpha,\, 2\,d\sigma_r \wedge \alpha\rangle_x$, so putting together (6.3) and (6.4),

$$\langle \sigma_r \cdot d\alpha,\, \sigma_r \cdot d\alpha\rangle_x = \langle \delta d\alpha, \alpha_r\rangle_x - \langle \sigma_r \cdot d\alpha,\, 2\,d\sigma_r \wedge \alpha\rangle_x . \tag{6.5}$$

Similarly, we have

$$\langle \delta\alpha, \delta\alpha_r\rangle_x = \langle d\delta\alpha, \alpha_r\rangle_x .$$

We again apply the Leibniz rule, keeping in mind the definition of operator δ:

$$\begin{aligned}
\delta\alpha_r &= \pm \star \circ d \circ \star\, \alpha_r \\
&= \pm \star \circ d(\sigma_r^2 \cdot \star\, \alpha) \\
&= \pm \star(d\sigma_r^2 \wedge \star\, \alpha + \sigma_r^2 \cdot d \star \alpha) \\
&= \pm \star(2\,\sigma_r \cdot d\sigma_r \wedge \star\, \alpha) + \sigma_r^2 \cdot \delta\alpha.
\end{aligned}$$

$$\langle \delta\alpha, \delta\alpha_r\rangle_x = \pm \langle \delta\alpha,\, \star(2\,\sigma_r \cdot d\sigma_r \wedge \star\, \alpha)\rangle_x + \left\langle \delta\alpha,\, \sigma_r^2 \cdot \delta\alpha\right\rangle_x .$$

So

$$\langle \sigma_r \cdot \delta\alpha, \sigma_r \cdot \delta\alpha\rangle_x = \langle d\delta\alpha, \alpha_r\rangle_x \pm \langle \sigma_r \cdot \delta\alpha,\, \star(2\,d\sigma_r \wedge \star\, \alpha)\rangle_x . \tag{6.6}$$

Now summing (6.5) and (6.6),

$$\|\sigma_r \cdot d\alpha\|_x^2 + \|\sigma_r \cdot \delta\alpha\|_x^2 = \langle \Delta\alpha, \alpha_r\rangle_x - \langle \sigma_r \cdot d\alpha,\, 2\,d\sigma_r \wedge \alpha\rangle_x \pm \langle \sigma_r \cdot \delta\alpha,\, \star(2\,d\sigma_r \wedge \star\, \alpha)\rangle_x . \tag{6.7}$$

If $\Delta\alpha = 0$, then $\langle \Delta\alpha, \alpha_r\rangle_x = 0$, and we will show that $d\alpha = \delta\alpha = 0$ if we show that $\|\sigma_r \cdot d\alpha\|_x^2 + \|\sigma_r \cdot \delta\alpha\|_x^2$ tends to zero as $r \to \infty$. We use the Cauchy–Schwarz inequality combined with the inequality of arithmetic and geometric means:

$$|\,\langle \eta, \zeta\rangle_x\,| \leqslant \sqrt{\langle \eta, \eta\rangle_x \cdot \langle \zeta, \zeta\rangle_x} \leqslant \frac{1}{2}\,\langle \eta, \eta\rangle_x + \frac{1}{2}\,\langle \zeta, \zeta\rangle_x .$$

$$\begin{aligned}
|\,\langle \sigma_r \cdot d\alpha,\, 2\,d\sigma_r \wedge \alpha\rangle_x\,| &\leqslant \frac{1}{2} \cdot \|\sigma_r \cdot d\alpha\|_x^2 + 2 \cdot \|d\sigma_r \wedge \alpha\|_x^2, \\
|\,\langle \sigma_r \cdot \delta\alpha,\, \star(2\,d\sigma_r \wedge \star\, \alpha)\rangle_x\,| &\leqslant \frac{1}{2} \cdot \|\sigma_r \cdot \delta\alpha\|_x^2 + 2 \cdot \|d\sigma_r \wedge \star\, \alpha\|_x^2.
\end{aligned}$$

We put these inequalities together with (6.7) and get

$$\|\sigma_r \cdot d\alpha\|_x^2 + \|\sigma_r \cdot \delta\alpha\|_x^2 \leqslant 4 \cdot \|d\sigma_r \wedge \alpha\|_x^2 + 4 \cdot \|d\sigma_r \wedge \star\,\alpha\|_x^2.$$

Now it remains to note that $\|d\sigma_r \wedge \alpha\|_x^2$ and $\|d\sigma_r \wedge \star\,\alpha\|_x^2$ are bounded by $\|d\sigma_r\|_x^2 \cdot \|\alpha\|_x^2$ (cf. remark 5.1.5). Since $\|d\sigma_r\|_x^2 \leqslant c\,r^{-2}$ for some constant c not depending on r, we conclude that $\|\sigma_r \cdot d\alpha\|_x^2 + \|\sigma_r \cdot \delta\alpha\|_x^2$ tends to zero as $r \to \infty$. ∎

6.3 Square integrable forms

We consider the following spaces:

- $\Omega^q_{(2)}(M)$ is the space of square integrable q-forms.
- $\mathcal{H}^q_{(2)}(M) \subset \Omega^q_{(2)}(M)$ is subspace of square integrable harmonic q-forms.
- $H^q_{\mathrm{dR},(2)}(M) \subset H^q_{\mathrm{dR}}(M)$ is the space of q-dimensional cohomology classes represented by square integrable forms.

Remark 6.3.1. Naturally, one has a cochain complex

$$0 \to \Omega^0_{(2)}(M) \xrightarrow{d} \Omega^1_{(2)}(M) \xrightarrow{d} \Omega^2_{(2)}(M) \to \cdots$$

and its cohomology is called L^2-**cohomology** of M. For this see a survey [Dai11].

The space $H^q_{\mathrm{dR},(2)}(M)$ should not be confused with L^2-cohomology. For instance, in the easiest example $M = \mathbb{R}^1$ it is not difficult to see that

$$\dim_{\mathbb{R}}(q\text{-th } L^2\text{-cohomology of } \mathbb{R}^1) = \begin{cases} \infty, & q = 1, \\ 0, & q \neq 1, \end{cases}$$

which differs radically from de Rham cohomology.

Indeed, $\Omega^1_{(2)}(\mathbb{R}^1)$ is a huge space, containing all 1-forms with compact support. Among them in the image of $\Omega^0_{(2)}(\mathbb{R}^1) \to \Omega^1_{(2)}(\mathbb{R}^1)$ lie just differential forms $\frac{\partial\psi}{\partial x}\,dx$ with $\psi(x)$ a square integrable function, and for them necessarily $\int_{\mathbb{R}^1} \frac{\partial\psi}{\partial x}\,dx = 0$. So we see that

$$\dim_{\mathbb{R}} \frac{\Omega^1_{(2)}(\mathbb{R}^1)}{\mathrm{im}(\Omega^0_{(2)}(\mathbb{R}^1) \to \Omega^1_{(2)}(\mathbb{R}^1))} = \infty.$$

There are natural maps

$$\mathcal{H}^q_{(2)}(M) \xrightarrow{\mu} H^q_{\mathrm{dR},(2)}(M) \xhookrightarrow{\nu} H^q_{\mathrm{dR}}(M)$$

The second map ν is just the inclusion. The first map μ is induced by the natural surjection $\Omega^q_{(2)}(M) \twoheadrightarrow H^q_{\mathrm{dR},(2)}(M)$, and actually μ itself is a surjection by a theorem of Kodaira [Kod49, §4], which says there is an orthogonal decomposition

$$\Omega^q_{(2)}(M) = \mathcal{H}^q_{(2)}(M) \oplus \overline{d\Omega^{q-1}_{\mathrm{cpt}}}(M) \oplus \overline{\delta\Omega^{q+1}_{\mathrm{cpt}}}(M).$$

Here "cpt" means "with compact support", and $\overline{\cdot}$ denotes the closure. It follows from the Kodaira decomposition that if $\alpha \in \Omega^q_{(2)}(M)$ is a closed form, i.e. $d\alpha = 0$, then $\alpha = \mathcal{H}(\alpha) + d\sigma$ for some $\mathcal{H}(\alpha) \in \mathcal{H}^q_{(2)}(M)$ and $\sigma \in \Omega^{q-1}(M)$.

If M is compact, then Hodge theory tells us that μ and ν are bijective; in general it is not true: μ is not necessarily injective (different harmonic forms may represent the same cohomology class) and ν is not necessarily surjective (not any cohomology class can be represented by a square integrable form).

Here is a weak form of injectivity for ν:

Proposition 6.3.2. *As before, let M be a complete Riemannian manifold.*

Let $\alpha \in \mathcal{H}^q_{(2)}(M)$ be an exact square integrable harmonic form such that $\alpha = d\sigma$ for some $\sigma \in \Omega^{q-1}_{(2)}(M)$ (N.B. σ being also square-integrable). Then $\alpha = 0$.

In words: *on a complete Riemannian manifold, a non-zero square integrable harmonic form is not the differential of a square integrable form.*

Proof. If α is harmonic and square integrable, then by proposition 6.2.3 one has also $\delta\alpha = 0$. So $\sigma, d\sigma, \alpha, d\sigma$ are all square integrable, and one has by corollary 6.2.2

$$\|\alpha\|^2_M = \langle\alpha,\alpha\rangle_M = \langle d\sigma, d\sigma\rangle_M = \langle\sigma,\delta\alpha\rangle_M = 0.$$

■

Remark 6.3.3. In the view of proposition 6.2.1, instead of $\sigma \in \Omega^{q-1}_{(2)}(M)$ it is enough to assume that the function $x \mapsto \|\sigma_x\|_x \cdot \|\alpha_x\|_x$ is integrable.

6.4 A Stokes' formula for complete Riemannian manifolds

Proposition 6.4.1. *Let M be a complete Riemannian manifold. Let X be a vector field on M such that $\|X_x\|_x$ is bounded and $\mathcal{L}_X(\omega) = 0$.*

Let $f\colon M \to \mathbb{R}$ be a $\mathcal{C}^1$-function such that f and Xf are absolutely integrable. Then

$$\int_M Xf\,\omega = 0.$$

Proof. The Cartan's magic formula gives

$$\mathcal{L}_X(f\,\omega) = d\,\iota_X(f\,\omega) + \underbrace{\iota_X\,d(f\,\omega)}_{=0}.$$

On the other hand, because of the assumption $\mathcal{L}_X(\omega) = 0$, we have

$$\mathcal{L}_X(f\,\omega) = \underbrace{\mathcal{L}_X f}_{=Xf}\,\omega + f\,\underbrace{\mathcal{L}_X(\omega)}_{=0} = X f\,\omega.$$

Hence $(X f)\,\omega = d\,\iota_X(f\,\omega)$. The idea is the same as already used before. If f has compact support, then we can use the Stokes' formula. Let D be a bounded open regular domain containing the support of f. Then

$$\int_M (X f)\,\omega = \int_M d(\iota_X(f\,\omega)) = \int_D d(\iota_X\,f\omega) = \int_{\partial D} \iota_X\,f\,\omega = 0.$$

Otherwise, we use lemma 6.1.3 and replace f with $\sigma_r \cdot f$, which has compact support, hence

$$0 = \int_M (X\,(\sigma_r \cdot f))\,\omega = \int_M f \cdot X(\sigma_r)\,\omega + \int_M \sigma_r \cdot X(f)\,\omega.$$

We need to show that

$$\lim_{r\to\infty} \int_M \sigma_r \cdot X(f)\,\omega = \int_M Xf\,\omega \quad \text{and} \quad \lim_{r\to\infty} \int_M f \cdot X(\sigma_r)\,\omega = 0.$$

The first is clear. For the second one, observe that by the Cauchy–Schwarz inequality

$$|X\sigma_r(x)| = |\langle X_x, d\sigma_r(x)\rangle_x| \leqslant \|X_x\|_x \cdot \|d\sigma_r(x)\|_x \leqslant \|X_x\|_x \cdot c\, r^{-1},$$

so $X\sigma_r$ is bounded on M (we assume that $\|X_x\|_x$ is bounded). Now by Cauchy–Schwarz

$$\left|\int_M f \cdot X(\sigma_r)\, \omega\right| \leqslant (\max_{x\in M} \|X_x\|_x) \cdot c\, r^{-1} \int_M |f(x)|\, \omega,$$

and the latter tends to 0 as $r \to \infty$. ∎

Corollary 6.4.2. *With the same assumptions on M and X, let $f, g\colon M \to \mathbb{R}$ be functions of class $\mathscr{C}^1$. Assume that the functions*

$$h\colon x \mapsto f(x) \cdot g(x), \quad x \mapsto Xf(x) \cdot g(x), \quad x \mapsto f(x) \cdot Xg(x)$$

are absolutely integrable on M. Then

$$\langle Xf, g\rangle_M + \langle f, Xg\rangle_M = 0.$$

Proof. We have the Leibniz rule

$$X(f \cdot g) = X(f) \cdot g + f \cdot X(g).$$

Integrating this over M, we obtain

$$\int_M (X(f \cdot g))(x)\, \omega = \int_M ((Xf)(x) \cdot g(x))\, \omega + \int_M (f(x) \cdot (Xg)(x))\, \omega.$$

But the integral on the left hand side satisfies the previous proposition, hence it is 0. ∎

Note that in the case of compact support this follows immediately from the usual Stokes' theorem, so the formula $\langle Xf, g\rangle_M + \langle f, Xg\rangle_M = 0$ can be viewed as some analogue of Stokes.

In particular, we record a special case of corollary 6.4.2:

Proposition 6.4.3. *Let M be a complete Riemannian manifold. Let X be a vector field on M such that $\|X_x\|_x$ is bounded and $\mathscr{L}_X(\omega) = 0$. Let $f, g\colon M \to \mathbb{R}$ be functions of class $\mathscr{C}^1$. Assume that f, g, Xf, Xg are all square integrable on M. Then*

$$\langle Xf, g\rangle_M + \langle f, Xg\rangle_M = 0.$$

(For this apply the Cauchy–Schwarz inequality $|\langle f, g\rangle_M| \leqslant \|f\|_M \cdot \|g\|_M$.)

6.5 Garland's theorem

Now we are ready to go back to Matsushima's theorem 5.3.1. It was proved under assumption that $\Gamma\backslash X$ is compact. Note that most of the proof consists of formal manipulations with formulas; one important point is the use of Stokes' formula

$$\langle x \cdot f, g\rangle_{\Gamma\backslash X} + \langle f, x \cdot g\rangle_{\Gamma\backslash X} = 0.$$

As we just saw above, this can be recovered if we work with square integrable forms (the other assumptions are satisfied if we take $X = G/K$ and the vector fields as in Matsushima's theorem proof).

In the proof of Matsushima's theorem we made use of Lie derivatives "$x_i \cdot \eta_I$". This is problematic, since if we assume that η_I is square-integrable, then $x_i \cdot \eta_I$ a priori is not square integrable anymore. To overcome this, one can replace η with convolution

$$\eta_\alpha = \eta * \alpha \overset{\text{def}}{=} \sum_I (\eta_I * \alpha)\, \omega^I,$$

where $\alpha \in C^\infty_{\mathrm{cpt}}(G)$ a smooth function on G with compact support, that is invariant under the action of K (recall that we work with complex $C^q(\mathfrak{g}, \mathfrak{k}; \mathscr{C}^\infty(\Gamma\backslash G))$).

Definition 6.5.1. For two smooth functions $f, g\colon G \to \mathbb{R}$ their **convolution** $f * g\colon G \to \mathbb{R}$ is given by

$$(f * g)(x) \stackrel{\text{def}}{=} \int_G f(x\,y^{-1})\,g(y)\,dy,$$

where dy is a Haar measure on G.

Now if $f \in L^2(\Gamma\backslash G)$ and $\alpha \in C^\infty_{\text{cpt}}(G)$, then $f * \alpha$ is a smooth square integrable function. Moreover, if we act on this by elements $D \in \mathcal{U}(\mathfrak{g})$, then $D \cdot (f * \alpha)$ is square integrable as well. It remains to find a sequence $\{\alpha_i\}$ such that $\eta * \alpha_i \to \eta$. This is done using "Dirac sequences" [Lan75, §I.1], [HC66, §2].

Definition 6.5.2. A **Dirac sequence** on a Lie group G is a sequence of smooth functions $\delta_n\colon G \to \mathbb{R}$ such that

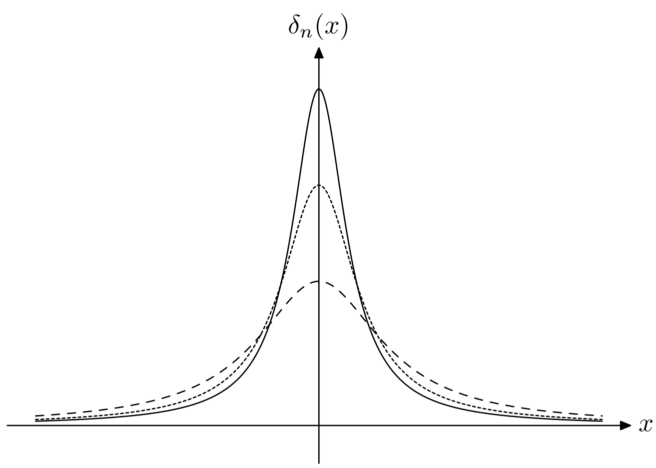
1. $\delta_n \geqslant 0$ for all n.

2. $\int_G \delta_n(x)\,dx = 1$ for all n.

3. For every neighborhood of identity $V \ni e$ and for every $\epsilon > 0$ one has

$$\int_{G\backslash V} \delta_n(x)\,dx < \epsilon$$

for all n sufficiently large.



Example 6.5.3. For instance on $G = \mathbb{R}^1$ one can take functions $\delta_n(x) \stackrel{\text{def}}{=} \frac{n}{\pi(1+n^2x^2)}$.

The first and third conditions are clear; the second condition is a calculus exercise:

$$\begin{aligned}
\int_{-\infty}^{\infty} \delta_n(x)\,dx &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{n}{1+n^2x^2}\,dx \\
&= \left[\begin{array}{c} y = nx \\ dy = n\,dx \end{array}\right] = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{1+y^2}\,dy \\
&= \frac{1}{\pi}\,\arctan y\big|_{y=-\infty}^{\infty} = \frac{1}{\pi}\,\pi = 1.
\end{aligned}$$

▲

Dirac sequences exist, and one can replace the third condition with a stronger one:

3′. For every neighborhood of identity $V \ni e$ the support of δ_n is contained in V for all n sufficiently large.

So one can make the Matsushima's argument work for square integrable forms, and the result is the following:

Theorem 6.5.4. *Let $\Gamma \subset G$ be a discrete torsion free subgroup. $\Gamma\backslash X$ is not assumed to be compact anymore. Let $q \leqslant m(G)$ and suppose that every class of $H^q(\mathfrak{g}, \mathfrak{k}; \mathcal{C}^\infty(\Gamma\backslash G))$ is representable by a square integrable form. Then $j_G^q \colon I_G^q \to H^q(\Gamma, \mathbb{R})$ is surjective.*

This is essentially due to Garland [Gar71, Theorem 3.5]. This result is crucial in Borel's original proof of theorem 4.7.2.

Appendix A

Algebraic groups

Here we collect some rudiments of the theory of linear algebraic groups that are needed in the main text. We also discuss very briefly arithmetic groups.

References. We relied mainly on the notes of J.S. Milne available at

http://jmilne.org/math/CourseNotes/ala.html

A nice survey for arithmetic groups is [Ser79].

A.1 Basic definitions

Let k be a commutative ring.

Definition A.1.1. An **affine group** G over k is a group object in the category of representable functors $k\text{-}\mathcal{A}lg \to \mathcal{S}et$. If G is represented by a finitely generated k-algebra, then it is called an **affine algebraic group**.

This means that one has a functor $G\colon k\text{-}\mathcal{A}lg \to \mathcal{S}et$ which is isomorphic to the functor $\operatorname{Hom}(\mathcal{O}(G), -)$ for some finitely generated k-algebra $\mathcal{O}(G)$ which we call the **coordinate ring** of G. Further, there is a natural transformation $m\colon G \times G \Rightarrow G$, such that for any k-algebra R the multiplication morphism

$$m(R)\colon G(R) \times G(R) \to G(R)$$

gives a group structure on $G(R)$. The latter is called the **group of R-points**.

Example A.1.2. Let G be an affine algebraic group over $\mathbb{Q}$. Then $G(\mathbb{R})$ is a Lie group. ▲

- A morphism of affine k-groups $G \to H$ is just a natural transformation of functors $G \Rightarrow H$.
- The product of affine k-groups $G \times H$ is defined as the functor $R \rightsquigarrow G(R) \times H(R)$. It is representable, since

$$\operatorname{Hom}_{k\text{-}\mathcal{A}lg}(\mathcal{O}(G), R) \times \operatorname{Hom}_{k\text{-}\mathcal{A}lg}(\mathcal{O}(H), R) \cong \operatorname{Hom}_{k\text{-}\mathcal{A}lg}(\mathcal{O}(G) \otimes_k \mathcal{O}(H), R).$$

Remark A.1.3. We recall that the Yoneda lemma tells us that the category of representable functors $k\text{-}\mathcal{Alg} \to \mathcal{Set}$ is isomorphic to the opposite category $k\text{-}\mathcal{Alg}^{\mathrm{op}}$. Recall that $k\text{-}\mathcal{Alg}^{\mathrm{op}}$ is isomorphic to the category of affine schemes over k. So **affine groups over** k are the same as group objects in the category of affine schemes over k, i.e. **affine group schemes over** k.

See [EH00, Chapter VI].

Example A.1.4. Let GL_n be the functor which sends a k-algebra R to the set of invertible $n \times n$ matrices with elements in R. In other words, $GL_n(R)$ are the matrices with determinant $\neq 0$. We see that GL_n is an affine algebraic group, since this functor is isomorphic to $\operatorname{Hom}(A, -)$ with A given by

$$A \stackrel{\text{def}}{=} \frac{k[X_{11}, X_{12}, \ldots, X_{nn}, Y]}{\det(X_{ij}) \cdot Y - 1}.$$

Here $\det(X_{ij})$ is the polynomial in n^2 variables $X_{11}, X_{12}, \ldots, X_{nn}$ given by

$$\det(X_{ij}) \stackrel{\text{def}}{=} \sum_{\sigma \in S_n} \operatorname{sign}(\sigma)\, X_{1,\sigma(1)} \cdots X_{n,\sigma(n)}.$$

The group GL_1 is usually denoted by $\mathbb{G}_m$ (**multiplicative group**), since $\mathbb{G}_m(R)$ can be identified with the multiplicative group $R^\times$. ▲

Example A.1.5. Let SL_n be the functor which sends a k-algebra R to the set matrices $n \times n$ with elements in R having determinant 1. It is an affine algebraic group represented by

$$A \stackrel{\text{def}}{=} \frac{k[X_{11}, X_{12}, \ldots, X_{nn}]}{\det(X_{ij}) - 1}.$$

▲

We say that H is an **affine subgroup** of G if H is a closed subfunctor of G such that $H(R)$ is a subgroup of $G(R)$ for all k-algebras R. The fact that H is a closed subfunctor of G means that H is representable by a quotient of $\mathcal{O}(G)$.

Example A.1.6. SL_n is an affine subgroup of GL_n. ▲

Definition A.1.7. An affine subgroup of GL_n is called a **linear algebraic group**.

A.2 Extension and restriction of scalars

Let L be an algebra over k. Then

- Starting from an affine algebraic group G over k, one can obtain an affine algebraic group G_L over L. This is called the **extension of scalars**.

  Namely, for $G \cong \operatorname{Hom}_{k\text{-}\mathcal{Alg}}(\mathcal{O}(G), -)$ we define a functor $G_L\colon L\text{-}\mathcal{Alg} \to \mathcal{Set}$ by

  $$G_L(R) \stackrel{\text{def}}{=} \operatorname{Hom}_{L\text{-}\mathcal{Alg}}(\mathcal{O}(G) \otimes_k L, R) \cong \operatorname{Hom}_{k\text{-}\mathcal{Alg}}(\mathcal{O}(G), R).$$

- Starting from an affine algebraic group G over L, one can obtain an affine algebraic group $\operatorname{Res}_{L/k} G$ over k. This is called the **restriction of scalars**.

  Namely, we define a functor $\operatorname{Res}_{L/k} G\colon k\text{-}\mathcal{Alg} \to \mathcal{Set}$ by

  $$\operatorname{Res}_{L/k}(R) \stackrel{\text{def}}{=} G(R \otimes_k L).$$

  If $\operatorname{Res}_{L/k} G$ is representable and gives an affine group, we say that the restriction of scalars exists.

  This was defined originally by André Weil in [Wei82, §1.3], and sometimes it is called **Weil restriction**.

Proposition A.2.1. *Assume that L is finitely generated and projective as a k-module. Then for any affine L-group G the restriction of scalars $\operatorname{Res}_{L/k} G$ exists.*

The functors $G \rightsquigarrow G_L$ and $G \rightsquigarrow \operatorname{Res}_{L/k} G$ are adjoint; namely, there is a natural bijection

$$\operatorname{Hom}_L(G_L, H) \cong \operatorname{Hom}_k(G, \operatorname{Res}_{L/k} H).$$

(For this see e.g. [Mil12, §V.5].)

Proposition A.2.2. *Let k'/k be a finite separable field extension and let K be a field containing all k-conjugates of k'; i.e. such that $|\operatorname{Hom}_k(k', K)| = [k' : k]$. Then*

$$(\operatorname{Res}_{k'/k} G)_K \cong \prod_{\alpha\colon k' \to K} \alpha G,$$

where αG is the affine group over K obtained by extension of scalars with respect to $\alpha\colon k' \to K$.

(Again, we refer to [Mil12, §V.5].)

Example A.2.3. For instance, if we consider $G' = SL_n/F$ over a number field F and then take its restriction $G = \operatorname{Res}_{F/\mathbb{Q}} G'$, then the real Lie group $G(\mathbb{R})$ decomposes as

$$\underbrace{SL_n(\mathbb{R}) \times \cdots \times SL_n(\mathbb{R})}_{r_1} \times \underbrace{SL_n(\mathbb{C}) \times \cdots \times SL_n(\mathbb{C})}_{r_2},$$

where r_1 is the number of real places on F and r_2 is the number of complex places on F. ▲

A.3 Arithmetic groups

Definition A.3.1. Let G be a linear algebraic group over a number field F, i.e. a subgroup of GL_n/F.

Consider the group $G_{\mathcal{O}_F} \stackrel{\text{def}}{=} G(F) \cap GL_n(\mathcal{O}_F)$. A subgroup $\Gamma \subset G(F)$ is called **arithmetic** if Γ is **commensurable** with $G_{\mathcal{O}_F}$, that is, $\Gamma \cap G_{\mathcal{O}_F}$ has finite index both in Γ and $G_{\mathcal{O}_F}$. In general, a group Γ is called **arithmetic** if it is an arithmetic subgroup in $G(F)$ for some linear algebraic group G/F. Observe that any subgroup of finite index in Γ is also an arithmetic subgroup.

Example A.3.2. $SL_n(\mathcal{O}_F)$ is an arithmetic subgroup in SL_n/F. ▲

Remark A.3.3. Let Γ be an arithmetic subgroup of a linear algebraic group $G'/F \subset GL_n/F$. Take the restriction of scalars $G \stackrel{\text{def}}{=} \operatorname{Res}_{F/\mathbb{Q}} G'$. Then it is naturally a subgroup of GL_{nd} where $d = [F : \mathbb{Q}]$. Note that under identification of $G(\mathbb{Q})$ with $G'(F)$, the subgroup $G_\mathbb{Z} \subset G'_{\mathcal{O}_F}$ is of finite index. So one does not loose anything considering arithmetic groups only for $F = \mathbb{Q}$.

Arithmetic groups enjoy various nice finiteness properties.

Theorem A.3.4. *Let Γ be an arithmetic group. Then*

1. *Γ is **finitely presented**. That is, $\Gamma \cong \langle X \mid \mathcal{R} \rangle$, where $X \subset \Gamma$ is a finite set of elements and $\mathcal{R}$ is a finite set of relations.*

2. *Any Γ-module M that is finitely generated over $\mathbb{Z}$, the cohomology groups $H^\bullet(\Gamma, M)$ are finitely generated.*

This was proved in [Rag68].

Further, we have the following useful fact:

Proposition A.3.5 (Selberg's lemma). *Let k be a field of characteristic zero. Let Γ be a finitely generated subgroup of $GL_n(k)$ (in particular, arithmetic groups satisfy these requirements). Then Γ admits a torsion free normal subgroup $\Gamma' \lhd \Gamma$ of finite index $[\Gamma : \Gamma']$.*

This was proved by Selberg in [Sel60]. It follows immediately from the following:

Proposition A.3.6. *Let A be a finitely generated integral domain of characteristic 0. Then the group $GL_n(A)$ contains a torsion free normal subgroup of finite index.*

The elementary argument below is taken from [Alp87]. In fact, in the case of $GL_n(\mathbb{Z})$ this was first observed by Minkowski.

Proof. The fraction field $K \stackrel{\text{def}}{=} \operatorname{Frac} A$ is a finite algebraic extension of degree d of a purely transcendental field $k \stackrel{\text{def}}{=} \mathbb{Q}(X_1, \ldots, X_m)$. We fix a basis of K over k. We can express the generators of A in terms of this basis, and it is clear that the coefficients lie in a finitely generated ring

$$B \stackrel{\text{def}}{=} \mathbb{Z}\left[\frac{1}{s}\right]\left[X_1, \ldots, X_m, \frac{1}{f}\right]$$

for some $s \in \mathbb{Z}$ and $f \in \mathbb{Z}[X_1, \ldots, X_m]$ (this is exactly where we need to assume that A is finitely generated).

A fixed basis of K over k gives an injective morphism $\rho\colon GL_n(K) \hookrightarrow GL_{nd}(k)$ which gives a representation $\rho\colon GL_n(A) \hookrightarrow GL_{nd}(B)$.

Now let $x \in GL_{nd}(B)$ be an element of finite order α. It satisfies the equation $X^\alpha = 1$. The minimal polynomial of x has distinct roots that are some roots of unity. The coefficients of the characteristic polynomial of x are the symmetric functions in roots of unity, hence these are algebraic integers in $k \stackrel{\text{def}}{=} \mathbb{Q}(X_1, \ldots, X_m)$. So the trace of an element of finite order in $GL_{nd}(B)$ is an integer with absolute value $\leqslant nd$. This means there are finitely many possible traces for elements of finite order; we denote the corresponding finite set by $\mathcal{T}$.

Now let p be a prime number such that

- $p \nmid s$,
- p does not divide the coefficients of f,
- p does not divide the nonzero integers of the form $t - nd$ for $t \in \mathcal{T}$.

We take $a_1, \ldots, a_m \in \overline{\mathbb{F}}_p$ so that $f(a_1, \ldots, a_m) \neq 0$. Consider a homomorphism

$$\sigma\colon A \to \overline{\mathbb{F}}_p$$

given by reduction of the coefficients modulo p and evaluation $(X_1, \ldots, X_m) \mapsto (a_1, \ldots, a_m)$.

Now $\sigma(A) = \mathbb{F}_p(a_1, \ldots, a_m)$ is a finite field, hence $\mathfrak{m} \stackrel{\text{def}}{=} \ker \sigma$ is a maximal ideal of finite index in A. We consider the induced homomorphism

$$GL_{nd}(A) \to GL_{nd}(A/\mathfrak{m}).$$

Let $\Gamma(\mathfrak{m})$ denote its kernel and let $\Gamma_0 \stackrel{\text{def}}{=} GL_{nd}(B) \cap \Gamma(\mathfrak{m})$. The latter has finite index in $GL_{nd}(B)$.

Every element of finite order $x \in \Gamma_0$ has trace $\operatorname{tr} x \in \mathcal{T}$ and $\operatorname{tr} x \equiv nd \pmod{\mathfrak{m}}$, hence $p \mid (\operatorname{tr} x - nd)$. By our choice of p it implies $\operatorname{tr} x = nd$. Since the minimal polynomial of x has distinct roots, this means that x is diagonalizable. We must conclude that $x = 1$.

So Γ_0 is a torsion free subgroup of finite index in $GL_{nd}(B)$. ■

Appendix H

Homotopy theory

Here we collect some facts from algebraic topology that are used in chapters 2 and 3. By default all spaces are assumed to be pointed, having homotopy type of connected CW-complexes, with finitely many cells in any given dimension. The base point is usually dropped from the notation.

References. For proofs of the basic facts we refer to the great J.P. May's book [May99].
The book on spectral sequences is [McC01].

H.1 Hurewicz theorem

Everyone knows the Hurewicz theorem, but it is so important that we state it for the record.

Theorem H.1.1 (Hurewicz). *There is a well-defined natural homomorphism*

$$\begin{aligned} h\colon \pi_n(X) &\to \widetilde{H}_n(X) \quad (n \geqslant 1), \\ [f] &\mapsto f_*[S^n], \end{aligned}$$

where $f\colon S^n \to X$ is a map representing a class in $\pi_n(X)$, the map $f_*\colon \widetilde{H}_n(S_n) \to \widetilde{H}_n(X)$ is the induced homomorphism of homology groups, and $[S^n]$ is the generator of $\widetilde{H}_n(S^n)$.

- *If X is a connected space, then $h\colon \pi_1(X) \to \widetilde{H}_1(X)$ is the abelianization homomorphism.*
- *If X is a $(n-1)$-connected space for $n \geqslant 2$, then $h\colon \pi_n(X) \cong \widetilde{H}_n(X)$ is an isomorphism and $h\colon \pi_{n+1}(X) \twoheadrightarrow \widetilde{H}_{n+1}(X)$ is an epimorphism.*

See [May99, §15.1] for this.

H.2 Fibrations and cofibrations

Definition H.2.1. A map $i\colon A \hookrightarrow X$ is called a **cofibration**, if given map $f\colon X \to Y$ (for any Y) and $h\colon A \times I \to Y$ such that the following diagram commutes

$$\begin{tikzcd}
A \arrow[rr, hook, "i_0"] \arrow[dd, hook, "i"'] & & A \times I \arrow[dl, "h"'] \arrow[dd, hook, "i\times 1"] \\
& Y & \\
X \arrow[ur, "f"] \arrow[rr, hook, "i_0"'] & & X \times I \arrow[ul, dashed, "\tilde{h}"']
\end{tikzcd}$$

then there exists a map $\tilde{h}\colon X \times I \to Y$.

Definition H.2.2. A map $p\colon E \twoheadrightarrow B$ is called a **fibration**, if given a map $f\colon Y \to E$ and $h\colon Y \times I \to B$ such that the following diagram commutes

$$\begin{tikzcd}
Y \arrow[r, "f"] \arrow[d, hook, "i_0"'] & E \arrow[d, two heads, "p"] \\
Y \times I \arrow[r, "h"'] \arrow[ur, dashed, "\tilde{h}"] & B
\end{tikzcd}$$

then there exists a map $\tilde{h}\colon Y \times I \to E$.

Having in mind the adjunction $\mathrm{Hom}(Y \times I, E) \cong \mathrm{Hom}(Y, E^I)$, we can draw a diagram

$$\begin{tikzcd}
B & & B^I \arrow[ll, two heads, "p_0"'] \\
& Y \arrow[ur, "h"] \arrow[dl, "f"'] \arrow[dr, dashed, "\tilde{h}"] & \\
E \arrow[uu, two heads, "p"] & & E^I \arrow[ll, two heads, "p_0"] \arrow[uu, two heads, "p^I"']
\end{tikzcd}$$

which is dual to the definition of cofibration.

Proposition H.2.3. *1. Let $i\colon A \hookrightarrow X$ be a cofibration. Then its pushout is again a cofibration.*

2. Let $p\colon E \twoheadrightarrow B$ be a fibration. Then its pullback is again a fibration.

$$\begin{tikzcd}
B \cup_A X & X \arrow[l] \\
B \arrow[u, "\bar{i}"] \arrow[ur, phantom, "\ulcorner"] & A \arrow[l] \arrow[u, "i"']
\end{tikzcd}
\qquad
\begin{tikzcd}
A \times_B E \arrow[r] \arrow[d, "\bar{p}"'] \arrow[dr, phantom, "\lrcorner"] & E \arrow[d, "p"] \\
A \arrow[r] & B
\end{tikzcd}$$

(This is deduced from abstract nonsense.)

Definition H.2.4. For a fixed topological space A, the category of spaces **under** A consists of maps $i: A \to X$, and the morphisms are commutative diagrams

$$\begin{tikzcd} & A \arrow[dl, "i"'] \arrow[dr, "j"] & \\ X \arrow[rr, "f"'] & & Y \end{tikzcd}$$

Proposition H.2.5. *If in the diagram above i and j are cofibrations and f is a homotopy equivalence, then it is actually a **cofiber homotopy equivalence**, meaning that the homotopy is given by*

$$\begin{aligned} &h: X \times I \to I, \\ &h(i(a), t) = j(a) \quad \textit{for } a \in A. \end{aligned}$$

Definition H.2.6. For a fixed topological space B, the category of spaces **over** B consists of maps $p: X \to B$, and the morphisms are commutative diagrams

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & B & \end{tikzcd}$$

Proposition H.2.7. *If in the diagram above p and q are fibrations and f is a homotopy equivalence, then it is actually a **fiber homotopy equivalence**.*

Definition H.2.8. Recall that for any continuous map $f: X \to Y$ we can take the **associated cofibration** or **fibration** as follows. Consider the **mapping cylinder** M_f and **mapping cocylinder** N_f given by

$$\begin{tikzcd} M_f \overset{\text{def}}{=} Y \cup_f (X \times I) & X \times I \arrow[l] \\ Y \arrow[u] \arrow[ur, phantom, "\ulcorner"] & X \arrow[l, "f"] \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} N_f \overset{\text{def}}{=} X \times_f PY \arrow[r] \arrow[d, "\overline{p}"'] \arrow[dr, phantom, "\lrcorner"] & PY \arrow[d, "p"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

Here by PY we denote the path space Y^I, and $p: PY \to Y$ is the path space fibration $\omega \mapsto \omega(0)$. Now f can be factorized as

$$\begin{tikzcd} X \arrow[r, "j", "\text{cofibration}"'] \arrow[rr, bend left, "f"] & M_f \arrow[r, "r"] & Y \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\nu"] \arrow[rr, bend left, "f"] & N_f \arrow[r, "\rho", "\text{fibration}"'] & Y \end{tikzcd}$$

Here r and ν are homotopy equivalences (with inverses given by $i: Y \hookrightarrow M_f$ and $\overline{p}: N_f \to X$ respectively).

$$\begin{aligned} r(y) &\overset{\text{def}}{=} y \text{ on } Y, \\ r(x, s) &\overset{\text{def}}{=} f(x) \text{ on } X \times I. \end{aligned}$$

$$\nu(x) \overset{\text{def}}{=} (x, c_{f(x)}),$$

where $c_{f(x)}$ is the constant path.

j is a cofibration:

$$j(x) \stackrel{\text{def}}{=} (x, 1).$$

ρ is a fibration:

$$\rho(x, \omega) \stackrel{\text{def}}{=} \omega(1).$$

Definition H.2.9. Given a map of pointed spaces $f\colon X \to Y$, its **homotopy cofiber** and **homotopy fiber** are given by

$$\begin{tikzcd} C_f \stackrel{\text{def}}{=} Y \cup_f CX \arrow[dr, phantom, "\ulcorner"] & CX \arrow[l] \\ Y \arrow[u] & X \arrow[l, "f"] \arrow[u, "i"'] \end{tikzcd} \qquad \begin{tikzcd} F_f \stackrel{\text{def}}{=} X \times_f PY \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & PY \arrow[d, "p"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

Here CX is the (reduced) cone over X:

$$CX \stackrel{\text{def}}{=} \frac{X \times I}{\{*\} \times I \cup X \times \{1\}}.$$

The morphism $p\colon PY \to Y$ is again the path space fibration.

Proposition H.2.10. *Let $p\colon E \to B$ be a fibration, let $* \in B$ be the base-point of B and let $F \stackrel{\text{def}}{=} p^{-1}(*)$ be a fiber. Then one has a long exact sequence*

$$\cdots \to \pi_n(F) \xrightarrow{i_*} \pi_n(E) \xrightarrow{p_*} \pi_n(B) \xrightarrow{\partial} \pi_{n-1}(F) \to \cdots \to \pi_0(F) \to \pi_0(E) \to 0$$

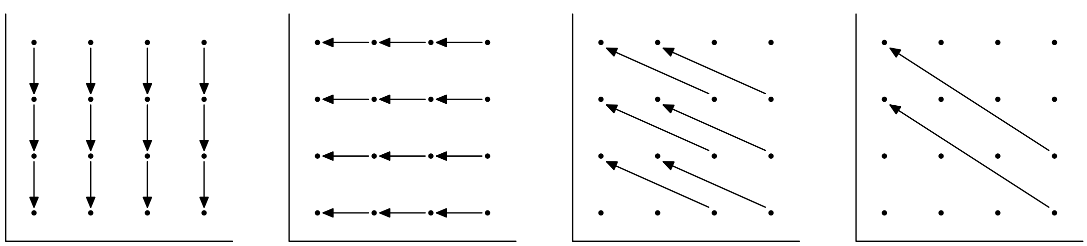
We refer to [May99, §9.3].

H.3 Leray–Serre spectral sequence

We make a brief summary of the needed facts about spectral sequences. The reference for everything is [McC01].

Recall that a (first quadrant) **homological spectral sequence** is a family of objects $E^r_{p,q}$ (where $E^r_{p,q} = 0$ unless $p, q \geqslant 0$), coming with **differentials**

$$d^r_{p,q}\colon E^r_{p,q} \to E^r_{p-r,q+r-1}.$$

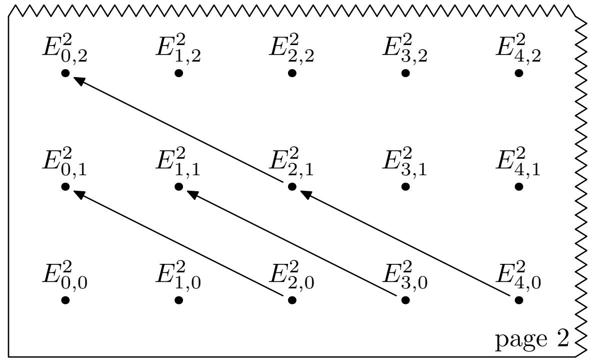


$E^0_{\bullet\bullet}$ $\qquad$ $E^1_{\bullet\bullet}$ $\qquad$ $E^2_{\bullet\bullet}$ $\qquad$ $E^3_{\bullet\bullet}$

such that $d^r \circ d^r = 0$. The object $E^{r+1}_{p,q}$ is given by the homology of $E^r_{\bullet\bullet}$ at $E^r_{p,q}$:

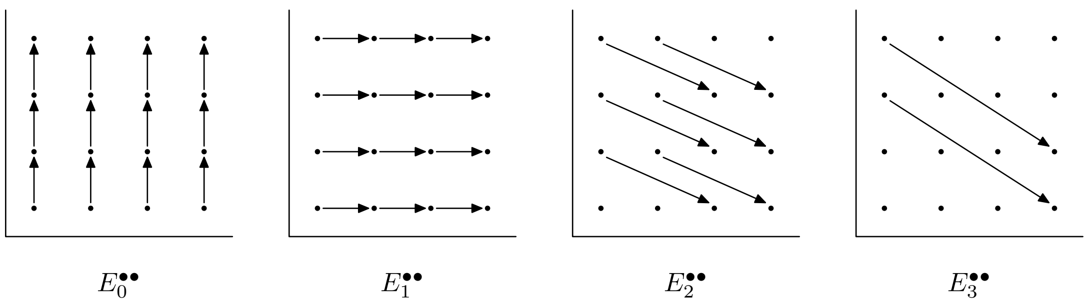
$$\cdots \to E^r_{p+r,q-r+1} \xrightarrow{d^r} E^r_{p,q} \xrightarrow{d^r} E^r_{p-r,q+r-1} \to \cdots$$

$$E^{r+1}_{p,q} \cong \frac{\ker d^r_{p,q}}{\operatorname{im} d^r_{p+r,q-r+1}}.$$



Dually, a **cohomological spectral sequence** is a family of objects $E_r^{p,q}$ (where $E_r^{p,q} = 0$ unless $p, q \geqslant 0$), coming with differentials

$$d_r^{p,q} \colon E_r^{p,q} \to E_r^{p+r,q-r+1}.$$



$$\cdots \to E_r^{p-r,q+r-1} \xrightarrow{d_r} E_r^{p,q} \xrightarrow{d_r} E_r^{p+r,q-r+1} \to \cdots$$

$$E_{r+1}^{p,q} \cong \frac{\ker d_r^{p,q}}{\operatorname{im} d_r^{p-r,q+r-1}}.$$

Suppose $F \hookrightarrow E \xrightarrow{p} B$ is a fibration, where B is path connected and F is connected.

Theorem H.3.1 (The homology Leray–Serre spectral sequence)**.** *Let G be an abelian group. There is a first quadrant spectral sequence*

$$E^2_{p,q} = H_p(B; \mathcal{H}_q(F; G)) \Rightarrow H_{p+q}(E; G).$$

Theorem H.3.2 (The cohomology Leray–Serre spectral sequence)**.** *Let R be a commutative ring. There is a first quadrant spectral sequence of algebras*

$$E_2^{p,q} = H^p(B; \mathcal{H}^q(F; R)) \Rightarrow H^{p+q}(E; R).$$

The differentials satisfy the Leibniz rule:

$$u \cdot_2 v = (-1)^{p'q}\, u \smile v \quad \textit{for } u \in E_2^{p,q}, v \in E_2^{p',q'}.$$

For both theorems see [McC01, §5.1].

From the Serre spectral sequence one can deduce the following [McC01, Example 5.D]:

Proposition H.3.3 (Serre exact sequence)**.** *Let $F \hookrightarrow E \to B$ be a fibration with B simply connected. Suppose that $H_i(B) = 0$ for $0 < i < p$ and $H_j(F) = 0$ for $0 < j < q$. There is an exact sequence*

$$H_{p+q-1}(F) \to H_{p+q-1}(E) \to H_{p+q-1}(B) \to H_{p+q-2}(F) \to \cdots \to H_1(E) \to 0$$

Example H.3.4. *Let $\iota_n \in H^n(K(\mathbb{Q}, n); \mathbb{Q})$ denote the "fundamental class" represented by the identity map $K(\mathbb{Q}, n) \to K(\mathbb{Q}, n)$.*

The cohomology algebra $H^\bullet(K(\mathbb{Q}, n); \mathbb{Q})$ is the exterior algebra on ι_n if n is odd, and the polynomial algebra on ι_n if n is even.

For $n = 1$ we have $K(\mathbb{Z}, 1) = S^1$, and the statement is trivial.
For $n = 2$ a model for $K(\mathbb{Z}, 2)$ is the infinite-dimensional complex projective space $\mathbb{CP}^\infty$, and the cohomology ring $H^\bullet(\mathbb{CP}^\infty, \mathbb{Z})$ is known to be isomorphic to $\mathbb{Z}[\iota_2]$ (see [Hat02, Theorem 3.19] and [May99, Chapter 23]).

We proceed by induction on n using the Serre spectral sequence for the path space fibration

$$K(\mathbb{Q}, n) \to PK(\mathbb{Q}, n+1) \to K(\mathbb{Q}, n+1).$$

$$E_2^{p,q} = H^p(K(\mathbb{Q}, n+1); \mathcal{H}^q(K(\mathbb{Q}, n); \mathbb{Q})) \Rightarrow H^{p+q}(PK(\mathbb{Q}, n+1); \mathbb{Q}).$$

$$0 \to E_2^{p,q} \to E_2^{p+2,q-1} \to E_2^{p+4,q-2} \to \cdots \to E_2^{p+2k,q-k} \to 0$$

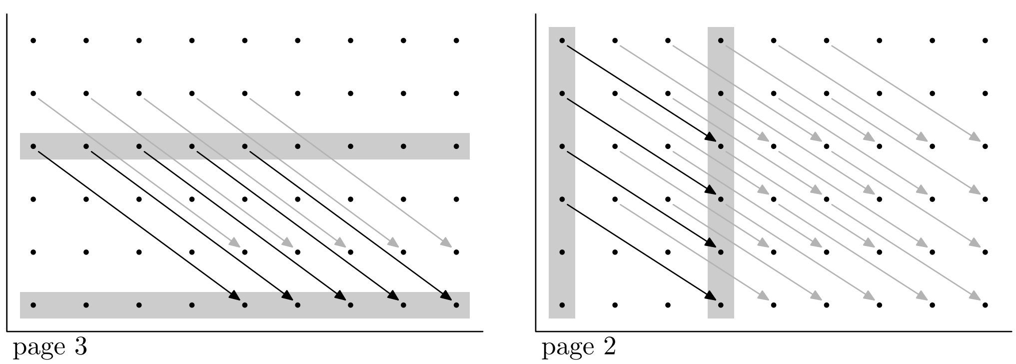
ι_n transgresses via d_{n+1} to ι_{n+1}.
If n is odd, then the Leibniz rule implies that

$$d_{n+1}(\iota_{n+1}^q\, \iota_n) = \iota_{n+1}^{q+1},$$

and the spectral sequence is concentrated in 0-th and n-th rows (the picture shows $n = 3$).
If n is even, then the Leibniz rule implies that

$$d_{n+1}(\iota_n^q) = q\, \iota_{n+1}\, \iota_n^{q-1},$$

and the spectral sequence is concentrated in 0-th and $(n+1)$-st columns (the picture shows $n = 2$).



▲

Example H.3.5. Let us compute the cohomology of SU_n. It naturally acts on $\mathbb{C}^n$. The action restricts to a *transitive* action on the unit sphere $S^{2n-1} \subset \mathbb{C}^n$. The stabilizer of a point $(0, \ldots, 0, 1) \in S^{2n-1}$ can be identified with SU_{n-1}, hence $SU_n/SU_{n-1} \cong S^{2n-1}$, and this gives a fibration

$$SU_{n-1} \hookrightarrow SU_n \to S^{2n-1}.$$

We know that $SU_2 \cong S^3$, hence the cohomology ring is $H^\bullet(SU_n; \mathbb{Q}) \cong \Lambda(x_3)$, the free exterior algebra on one element of degree three. In general

$$H^\bullet(SU_n) \cong \Lambda(x_3, x_5, \ldots, x_{2n-1}).$$

This is obtained by induction using the Leray–Serre spectral sequence—cf. [McC01, Example 5.F].

▲

H.4 Acyclic maps

Recall that a space X is called **acyclic** if $\widetilde{H}_\bullet(X) = 0$. One has the following result [Spa66, 7.5.5]:

Fact H.4.1 (Whitehead theorem). *A space X is contractible if and only if X is acyclic and it has trivial fundamental group $\pi_1(X) = 0$.*

If we drop the assumption that $\pi_1(X) = 0$, then an acyclic space X is not necessarily contractible, but we can extract some information about $\pi_1(X)$.

Proposition H.4.2. *Suppose X is acyclic. Let $\pi \stackrel{\text{def}}{=} \pi_1(X)$. Then $H_1(\pi, \mathbb{Z}) = H_2(\pi, \mathbb{Z}) = 0$.*

Proof. Consider the classifying space $B\pi$ and the fibration

$$\widetilde{X} \to X \to B\pi,$$

where $\widetilde{X}$ denotes the universal covering space of X.

We have the Leray–Serre spectral sequence

$$E^2_{p,q} = H_p(B\pi, H_q(\widetilde{X})) \Rightarrow H_{p+q}(X).$$

There is a short exact sequence

$$0 \to E^\infty_{0,1} \to H_1(X) \to E^\infty_{1,0} \to 0$$

Observe that $E^\infty_{1,0} = E^2_{1,0}$, since for $r \geqslant 2$ there are no nonzero differentials involving $E^2_{1,0}$. The only nonzero differential involving $E^r_{0,1}$ or $E^r_{2,0}$ is the knight move $d^2\colon E^r_{2,0} \to E^2_{0,1}$. We have a short exact sequence

$$0 \to E^\infty_{2,0} \to E^2_{2,0} \xrightarrow{d^2} E^2_{0,1} \to E^\infty_{0,1} \to 0$$

Putting all together, we have

$$0 \to H_2(X) \to H_2(B\pi, H_0(\widetilde{X})) \to H_0(B\pi, H_1(\widetilde{X})) \to H_1(X) \to H_1(B\pi, H_0(\widetilde{X})) \to 0$$

Because of the assumption that X is acyclic, $H_2(X) = H_1(X) = 0$. Since $\widetilde{X}$ is contractible, $H_1(\widetilde{X}) = 0$.

$$H_2(\pi, \mathbb{Z}) = H_2(B\pi, H_0(\widetilde{X})),$$
$$H_1(\pi, \mathbb{Z}) = H_1(B\pi, H_0(\widetilde{X})).$$

So the last exact sequence implies $H_1(\pi, \mathbb{Z}) = H_2(\pi, \mathbb{Z}) = 0$. ■

We will be interested in acyclic *maps*.

Definition H.4.3. A map $f\colon X \to Y$ is called **acyclic** if its homotopy fiber F_f is acyclic, i.e. $\widetilde{H}_\bullet(F_f) = 0$.

Proposition H.4.4. *Consider a pullback*

$$\begin{tikzcd} X_0 \times_Y X_1 \arrow[r, "\bar{f}_1"] \arrow[d, "\bar{f}_0"'] \arrow[dr, phantom, "\lrcorner"] & X_0 \arrow[d, "f_0"] \\ X_1 \arrow[r, "f_1"'] & Y \end{tikzcd}$$

Assume f_0 or f_1 is a fibration. Then f_i is acyclic if and only if $\bar{f}_i$ is acyclic.

Proof. Consider a commutative cube

$$\begin{tikzcd}
F_{f_1} \arrow[rr, "FP(f_1)"] \arrow[dr] \arrow[dd] & & F_{f_0} \arrow[dr] \arrow[dd] & \\
& X_0 \times_Y X_1 \arrow[rr, "\bar{f}_1"] \arrow[dd] & & X_0 \arrow[dd, "f_0"] \\
P(X_1) \arrow[rr, "P(f_1)"'] \arrow[dr, "\pi_{X_1}"'] & & P(Y) \arrow[dr, "\pi_Y"'] & \\
& X_1 \arrow[rr, "f_1"'] & & Y
\end{tikzcd}$$

π_Y is a fibration. π_{X_1} is a fibration and f_1 is a fibration, hence $f_1 \circ \pi_{X_1}$ is a fibration as well. $P(f_1)$ is a homotopy equivalence (hence a homotopy equivalence over Y), and $FP(f_1)$ is a homotopy equivalence as well. ■

Corollary H.4.5. *Consider a commutative diagram*

$$\begin{tikzcd}
E \arrow[rr, "f"] \arrow[dr, "p"'] & & E' \arrow[dl, "p'"] \\
& B &
\end{tikzcd}$$

f is acyclic if and only if the induced map $F(f)\colon F_p \to F_{p'}$ is acyclic.

Proof. Consider the cube

$$\begin{tikzcd}
F_p \arrow[rr] \arrow[dr, dashed, "F(f)"] \arrow[dd] & & P(B) \arrow[dr, equal] \arrow[dd] & \\
& F_{p'} \arrow[rr] \arrow[dd] & & P(B) \arrow[dd, "\pi_B"] \\
E \arrow[rr, "p"'] \arrow[dr, "f"'] & & B \arrow[dr, equal] & \\
& E' \arrow[rr, "p'"'] & & B
\end{tikzcd}$$

Observe that the left side of the cube is a pullback square:

$$F_p \overset{\text{def}}{=} E \times_B P(B) = (P(B) \times_B E') \times_{E'} E = F_{p'} \times_{E'} E.$$

$$\begin{tikzcd}
F_p \arrow[r, "F(f)"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & F_{p'} \arrow[d] \\
E \arrow[r, "f"'] & E'
\end{tikzcd}$$

Now $F(f)$ is acyclic if and only if f is acyclic. ■

The following is dual to proposition H.4.4:

Proposition H.4.6. *Consider a pushout*

$$\begin{tikzcd} Y_0 \cup_X Y_1 & Y_0 \arrow[l, "\bar{f}_1"'] \\ Y_1 \arrow[u, "\bar{f}_0"] & X \arrow[l, "f_1"] \arrow[u, "f_0"'] \arrow[ul, phantom, "\ulcorner"] \end{tikzcd}$$

Assume f_1 is a cofibration. Then f_i is acyclic if and only if $\bar{f}_i$ is acyclic.

Here is a characterization of acyclic maps.

Proposition H.4.7. *The following are equivalent.*

(1) $f\colon X \to Y$ is acyclic.

(2) For $\tilde{Y}$ the universal covering space of Y the induced map $\bar{f}\colon X \times_Y \tilde{Y} \to \tilde{Y}$

$$\begin{tikzcd} X \times_Y \tilde{Y} \arrow[r, "\bar{f}"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \tilde{Y} \arrow[d] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

gives an isomorphism

$$\bar{f}_*\colon H_\bullet(X \times_Y \tilde{Y}) \to H_\bullet(\tilde{Y}).$$

(3) There is an isomorphism between homology groups with local coefficients

$$f_*\colon H_\bullet(X; f^*\mathbb{Z}[\pi_1(Y)]) \to H_\bullet(Y; \{\mathbb{Z}[\pi_1(Y)]\}).$$

(4) For any local coefficient system $\mathcal{G}$ of abelian groups on Y

$$f_*\colon H_\bullet(X; f^*\mathcal{G}) \to H_\bullet(Y; \mathcal{G})$$

is an isomorphism.

Proof. For (1) $\Rightarrow$ (2), let F_f be the homotopy fiber of $f\colon X \to Y$. Then we have a homotopy fibration

$$F_f \to X \times_Y \tilde{Y} \xrightarrow{\pi_{\tilde{Y}}} \tilde{Y}$$

Applying the Serre spectral sequence

$$H_p(\tilde{Y}; \mathcal{H}_q(F_f)) \Rightarrow H_{p+q}(X \times_Y \tilde{Y}),$$

we see that if f is acyclic, then $\mathcal{H}_\bullet(F_f) = 0$, and we get an isomorphism

$$H_\bullet(X \times_Y \tilde{Y}) \xrightarrow{\cong} H_\bullet(\tilde{Y}).$$

Conversely, (2) $\Rightarrow$ (1): if we have an isomorphism as above, then we can show that $H_q(F_f) = 0$. Use induction on q. Assume it is true for $q < n$ for some $n \geqslant 2$. Then the spectral sequence gives an exact sequence

$$H_{n+1}(X \times_Y \tilde{Y}) \xrightarrow{\cong} H_{n+1}(\tilde{Y}) \to H_n(F_f) \to H_n(X \times_Y \tilde{Y}) \xrightarrow{\cong} H_n(\tilde{Y}) \to 0$$

so we should have $H_n(F_f) = 0$.

Next to get (3) $\Leftrightarrow$ (2), observe that we have a local coefficient system $\mathbb{Z}[\pi_1(Y)]$ and

$$H_\bullet(\widetilde{Y}) = H_\bullet(Y; \mathbb{Z}[\pi_1(Y)]).$$

Now

$$H_\bullet(X \times_Y \widetilde{Y}) \cong H_\bullet(\mathbb{Z}[\widetilde{X}] \otimes_{\mathbb{Z}\pi_1(X)} \mathbb{Z}[\pi_1(Y)]) = H_\bullet(X; f^*\mathbb{Z}[\pi_1(Y)]).$$

Hence f is acyclic if and only if it induces an isomorphism

$$H_\bullet(X; f^*\mathbb{Z}[\pi_1(Y)]) \cong H_\bullet(Y; \mathbb{Z}[\pi_1(Y)]).$$

We have trivially (4) $\Rightarrow$ (3). We get the less trivial implication (1) $\Rightarrow$ (4). For the fibration $F_f \xrightarrow{i} X \xrightarrow{f} Y$ consider the Serre spectral sequence with local coefficients:

$$H_p(Y; H_q(F_f; i^*f^*\mathcal{G})) \Rightarrow H_{p+q}(X; f^*\mathcal{G}).$$

But $i^*f^*\mathcal{G}$ is a trivial local coefficient system, so if we assume that $\widetilde{H}_\bullet(F_f) = 0$, then the edge homomorphism gives the desired isomorphism

$$H_\bullet(X; f^*\mathcal{G}) \cong H_\bullet(Y; \mathcal{G}).$$

■

Proposition H.4.8. *If $f\colon X \to Y$ is acyclic and $f_*\colon \pi_1(X) \to \pi_1(Y)$ is an isomorphism, then f is a homotopy equivalence.*

Proof. Consider the fibration long exact sequence

$$\cdots \to \pi_n(F_f) \to \pi_n(X) \xrightarrow{f_*} \pi_n(Y) \to \pi_{n-1}(F_f) \to \cdots \to \pi_1(F_f) \xrightarrow{f_*} \pi_1(X) \to \pi_1(Y) \to \pi_0(F_f)$$

We know that F_f is acyclic, so $\widetilde{H}_\bullet(F_f) = 0$. However, we should also have $\pi_1(F_f) = 0$, so F_f is contractible (by the Whitehead theorem), and we have isomorphisms $f_*\colon \pi_n(X) \xrightarrow{\cong} \pi_n(Y)$ for all n. This means that f is a homotopy equivalence (Whitehead). ■

Appendix Q

Quillen's Q-construction

Apart from the plus-construction (chapter 2), there is another definition of higher K-groups, which is more natural and general, and often more useful for proofs. K-groups may be defined for a category $\mathcal{C}$, e.g. the category $R\text{-}\mathcal{P}roj_{fg}$ of finitely generated projective R-modules. As in the plus-construction, the idea is to take homotopy groups of the *classifying space*, this time of a category. To obtain something interesting, instead of taking the initial category $\mathcal{C}$, one uses a modified category $Q\mathcal{C}$—the same way as in the plus-construction one takes $BGL(R)^+$ instead of $BGL(R)$. This is called **Quillen's Q-construction**.

In order to define classifying spaces, first we review simplicial sets and their geometric realization. Then we review some results from [Qui73b] and prove one of them, namely $\pi_1(BQ\mathcal{C}, 0) \cong K_0(\mathcal{C})$, just to get some feeling of the Q-construction.

References. The review of simplicial sets and their geometric realization follows [May67] and [Wei94, Chapter 8]. Definitions regarding classifying spaces of categories can be found in [Seg68]; what we call a "simplicial set" is a "semi-simplicial set" in the old terminology.

The main reference for the Q-construction is Quillen's paper [Qui73b]. The book [Sri96] has some details and background which may be useful to understand original Quillen's texts.

A definition of quotient category is from [Gab62], and a modern treatment can be found in [BK00, Chapter 6].

Q.1 K_0 of a category

In everything what follows, we will need to make sure that the classes under consideration form sets:

Definition Q.1.1. Let $\mathcal{C}$ be a category such that the isomorphism classes of its objects (the **skeleton** of $\mathcal{C}$) form a set. We say in this case that $\mathcal{C}$ is **skeletally small**.

Following Grothendieck (cf. [BS58, §4]), K_0 can be defined for any skeletally small category $\mathcal{C}$ in which the notion of short exact sequence makes sense. For this it is enough to assume that $\mathcal{C}$ is an additive category which lies in some ambient abelian category $\mathcal{A}$.

Definition Q.1.2. Let $\mathcal{C}$ be an additive category embedded as a full additive subcategory in some abelian category $\mathcal{A}$. Suppose that $\mathcal{C}$ is **closed under extensions in $\mathcal{A}$**. That is, whenever in $\mathcal{A}$ there is an exact sequence

$$0 \to A \to B \to C \to 0$$

with A and C isomorphic to objects of $\mathcal{C}$, then also B is isomorphic to an object of $\mathcal{C}$. We say in this case that $\mathcal{C}$ is an **exact category**.

A sequence $0 \to A \to B \to C \to 0$ in $\mathcal{C}$ is called **short exact** if it is short exact in the ambient abelian category $\mathcal{A}$.

Example Q.1.3. Consider the category $R\text{-}\mathcal{Proj}_{fg}$ of finitely generated projective R-modules. It is a full subcategory of the abelian category of R-modules $R\text{-}\mathcal{Mod}$, closed under extensions.

The short exact sequences in $R\text{-}\mathcal{Proj}_{fg}$ are the sequences that are split in $R\text{-}\mathcal{Mod}$:

$$0 \to P \to P \oplus Q \to Q \to 0$$

▲

Now the general definition of K_0 is the following:

Definition Q.1.4. Let $\mathcal{C}$ be a skeletally small exact category. The group $K_0(\mathcal{C})$ is the abelian group freely generated by isomorphism classes of objects in $\mathcal{C}$ modulo relations

$$[B] = [A] + [C] \quad \text{for any short exact sequence} \quad 0 \to A \to B \to C \to 0$$

Example Q.1.5. For $\mathcal{C} = R\text{-}\mathcal{Proj}_{fg}$ any short exact sequence is split, so $K_0(R\text{-}\mathcal{Proj}_{fg})$ is the same as $K_0(R)$ defined in section 1.1. ▲

Example Q.1.6. Grothendieck had in mind a generalization of the Riemann–Roch theorem, and the category $\mathcal{C}$ being $\mathcal{VB}(X)$, vector bundles on a scheme X (that is, locally free sheaves of $\mathcal{O}_X$-modules of finite rank). Since in this text we are interested only in $\operatorname{Spec} \mathcal{O}_F$, we do not deal with general K-theory of schemes. ▲

Q.2 Simplicial sets and their geometric realization

Definition Q.2.1. The **category of simpleces** Δ is given by the following data.

- The objects are finite ordered sets $\underline{n} \stackrel{\text{def}}{=} \{0 < 1 < \cdots < n\}$.
- The morphisms $f\colon \underline{m} \to \underline{n}$ are non-decreasing monotone maps; that is, $f(i) \leqslant f(j)$ for $i \leqslant j$.

One counts that in category Δ there are $\binom{m+n+1}{m+1}$ morphisms $\underline{m} \to \underline{n}$.

Definition Q.2.2. Let $\mathcal{C}$ be a category. A **simplicial object** in $\mathcal{C}$ is a presheaf with values in $\mathcal{C}$ on the category of simpleces. In other words, a simplicial object is a contravariant functor $F\colon \Delta^{\mathrm{op}} \to \mathcal{C}$. A morphism of simplicial objects is a natural transformation of functors. So the category of simplicial objects in $\mathcal{C}$ is the functor category $\mathcal{C}^{\Delta^{\mathrm{op}}}$.

In particular, a **simplicial set** is a simplicial object in the category of sets. A **simplicial space** is a simplicial object in the category of topological spaces.

Example Q.2.3. The **standard n-simplex** is a simplicial set $\Delta[n]$, which is defined as a contravariant functor $\operatorname{Hom}_\Delta(-, \underline{n})\colon \Delta^{\mathrm{op}} \to \mathcal{Set}$:

$$\underline{\ell} \rightsquigarrow \operatorname{Hom}_\Delta(\underline{\ell}, \underline{n}) = \{\text{non-decreasing maps } \underline{\ell} \to \underline{n}\}.$$

On an arrow $\underline{\ell} \to \underline{m}$ the corresponding map of sets $\operatorname{Hom}_\Delta(\underline{m}, \underline{n}) \to \operatorname{Hom}_\Delta(\underline{\ell}, \underline{n})$ is defined as usual:

$$\begin{tikzcd} \underline{\ell} \arrow[d] \arrow[dr, dashed, "\tilde{f}"] & \\ \underline{m} \arrow[r, "f"'] & \underline{n} \end{tikzcd}$$

Note that by Yoneda lemma, for a simplicial set $F\colon \Delta^{\mathrm{op}} \to \mathcal{Set}$ we have a natural isomorphism

$$F(\underline{n}) \cong \operatorname{Hom}_{\mathcal{C}^{\Delta^{\mathrm{op}}}}(\Delta[n], F).$$

▲

There is also another description of simplicial sets by "generators and relations". For each n one can define the **face maps**

$$\epsilon_i : \underline{n-1} \hookrightarrow \underline{n} = \text{the injection missing } i,$$

$$\epsilon_i(j) \stackrel{\text{def}}{=} \begin{cases} j, & \text{if } j < i, \\ j+1, & \text{if } j \geqslant i; \end{cases}$$

and **degeneracy maps**

$$\eta_i : \underline{n+1} \twoheadrightarrow \underline{n} = \text{the projection mapping two elements to } i,$$

$$\eta_i(j) \stackrel{\text{def}}{=} \begin{cases} j, & \text{if } j \leqslant i, \\ j-1, & \text{if } j > i. \end{cases}$$

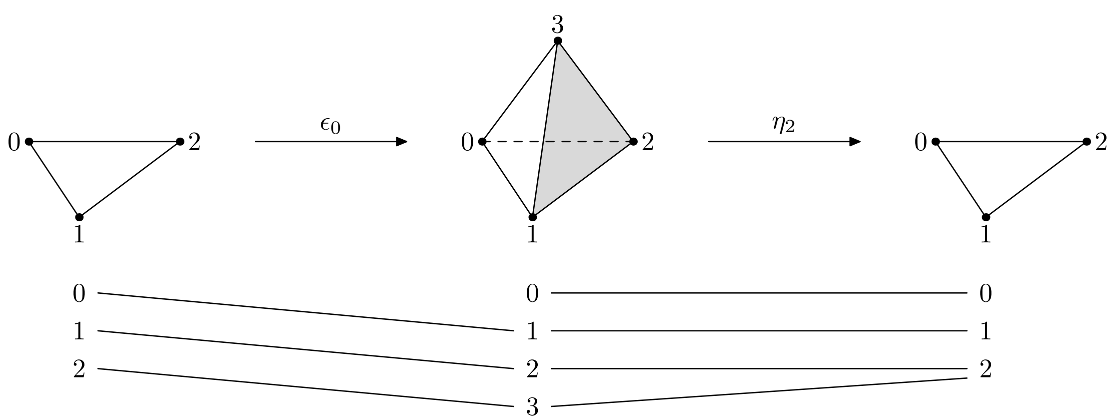
One has the following "simplicial identities":

$$\begin{aligned} \epsilon_j \circ \epsilon_i &= \epsilon_i \circ \epsilon_{j-1}, \quad \text{if } i < j, \\ \eta_j \circ \eta_i &= \eta_i \circ \eta_{j+1}, \quad \text{if } i \leqslant j, \\ \eta_j \circ \epsilon_i &= \begin{cases} \epsilon_i \circ \eta_{j-1}, & \text{if } i < j, \\ id, & \text{if } i = j \text{ or } i = j+1, \\ \epsilon_{i-1} \circ \eta_j, & \text{if } i > j+1. \end{cases} \end{aligned}$$

Example Q.2.4. The names "face map" and "degeneracy map" come from the usual simpleces in geometry. The **standard geometric n-simplex** is the set

$$\Delta^n \stackrel{\text{def}}{=} \{(t_0, \ldots, t_n) \in \mathbb{R}^n \mid \sum t_i = 1,\ 0 \leqslant t_i \leqslant 1\}.$$

Then one has obvious maps of face inclusion, and degeneration sending the vertices to the vertices an $(n-1)$-simplex:



▲

Every morphism $f : \underline{m} \to \underline{n}$ has a unique epi-monic factorization

$$\begin{tikzcd} & \underline{p} \arrow[dr, tail, "\epsilon"] & \\ \underline{m} \arrow[ur, two heads, "\eta"] \arrow[rr, "f"'] & & \underline{n} \end{tikzcd}$$

Where η and ϵ are factorized uniquely as

$$\begin{aligned}\eta &= \eta_{i_1} \cdots \eta_{i_s}, \quad 0 \leqslant i_1 < \cdots < i_t < m,\\ \epsilon &= \epsilon_{j_1} \cdots \epsilon_{j_t}, \quad 0 \leqslant j_t < \cdots < j_1 \leqslant n.\end{aligned}$$

Indeed, let $i_1 < \cdots < i_s$ be the elements of $\underline{m}$ such that $f(i) = f(i+1)$ and let $j_t < \cdots < j_1$ be the elements in $\underline{n}$ that are not in the image of f. Then for $p = m - s = n - t$ we have the factorization as above.

It follows that for a simplicial object $F\colon \Delta^{\mathrm{op}} \to \mathcal{C}$, it is enough to give the values of F on the objects $\underline{0}, \underline{1}, \underline{2}, \ldots \in \mathrm{Ob}(\Delta)$ and the values of F on arrows ϵ_i and η_i. If we denote $\partial_i \stackrel{\text{def}}{=} F(\epsilon_i)$ and $\sigma_i \stackrel{\text{def}}{=} F(\eta_i)$, then we get the following equivalent definition of a simplicial set.

Definition Q.2.5. A simplicial object F in a category $\mathcal{C}$ is given by a sequence of objects

$$F_0, F_1, F_2, \ldots \in \mathrm{Ob}(\mathcal{C})$$

together with **face operators** $\partial_i\colon F_n \to F_{n-1}$ and **degeneracy operators** $\sigma_i\colon F_n \to F_{n+1}$ for $i = 1, \ldots, n$, satisfying the following relations:

$$\begin{aligned}\partial_i \circ \partial_j &= \partial_{j-1} \circ \partial_i \quad \text{if } i < j,\\ \sigma_i \circ \sigma_j &= \sigma_{j+1} \circ \sigma_i \quad \text{if } i \leqslant j,\\ \partial_i \circ \sigma_j &= \begin{cases} \sigma_{j-1} \circ \partial_i, & \text{if } i < j,\\ id, & \text{if } i = j \text{ or } i = j+1,\\ \sigma_j \circ \partial_{i-1}, & \text{if } i > j+1.\end{cases}\end{aligned}$$

Now from a simplicial set $X\colon \Delta^{\mathrm{op}} \to \mathcal{S}et$ one can build a CW-complex $|X|$ as follows.

Definition Q.2.6. Let X be a simplicial set given by a sequence of sets $X_0, X_1, X_2, \ldots$ together with operators $\partial_i\colon X_n \to X_{n-1}$ and $\sigma_i\colon X_n \to X_{n+1}$ as above.

The **geometric realization** of X is given by

$$|X| \stackrel{\text{def}}{=} \left(\coprod_{n \geqslant 0} X_n \times \Delta^n \right) \Big/ \sim .$$

Here $\Delta^n \subset \mathbb{R}^{n+1}$ is the geometric n-simplex, and $X_n \times \Delta^n$ is the disjoint union of copies of Δ^n indexed by the elements of X_n.

The equivalence relation $\sim$ is defined as follows. For any map $f\colon \underline{m} \to \underline{n}$ look at the induced maps $f^*\colon X_n \to X_m$ (keep in mind that the functor is contravariant). Further, there are continuous maps $f_*\colon \Delta^m \to \Delta^n$ between geometric simpleces. We define them on vertices $v_0, \ldots, v_m$ by $v_i \mapsto v_{f(i)}$, and then by linearity this can be defined on all the faces of Δ^m. We identify for each $x \in X_n$ and $s \in \Delta^m$

$$(f^*(x), s) \sim (x, f_*(s)).$$

Now $|X|$ has a CW-complex structure, where the n-cells are given by elements $x \in X_n$ that are **nondegenerate**, i.e. not of the form $\sigma_i(y)$ for some $y \in X_{n-1}$.

Geometric realization enjoys certain properties one would expect from it:

- $|\cdot|$ is a functor $\mathcal{S}et^{\Delta^{\mathrm{op}}} \to \mathcal{T}op$. A morphism of simplicial sets $f\colon X \to Y$ induces a continuous map $|X| \to |Y|$. Indeed, f is a natural transformation of contravariant functors $X \Rightarrow Y\colon \Delta^{\mathrm{op}} \to \mathcal{S}et$:

$$\begin{tikzcd} \underline{n} & X_n \arrow[r, "f_n"] \arrow[d] & Y_n \arrow[d] \\ \underline{m} \arrow[u] & X_m \arrow[r, "f_m"'] & Y_m \end{tikzcd}$$

And we can define a map

$$\begin{aligned} X_n \times \Delta^n &\to Y_n \times \Delta^n, \\ (x, s) &\mapsto (f_n(x), s). \end{aligned}$$

- If X and Y are simplicial sets, then one can form a simplicial set $X \times Y$ with simpleces $X_n \times Y_n$ and the obvious maps. If $|X \times Y|$ is a CW-complex, then the natural continuous map $|X \times Y| \to |X| \times |Y|$ is a homeomorphism [May67, Theorem 14.3]. This happens e.g. when X and Y are countable, or when either $|X|$ or $|Y|$ is locally finite.

We refer to [May67, Chapter III] for proofs and further properties. Probably the most important fact, explaining the point of geometric realization, is the following.

Fact Q.2.7. *Let $Y \in \mathrm{Ob}(\mathcal{Top})$ be a topological space. The* ***singular complex*** *for Y is a simplicial set $\mathcal{S}Y \colon \Delta^{\mathrm{op}} \to \mathcal{Set}$, given by*

$$\underline{n} \rightsquigarrow \mathrm{Hom}_{\mathcal{Top}}(\Delta^n, Y) = \{\textit{continuous maps from the standard geometric } n\textit{-simplex to } Y\}.$$

Then the geometric realization functor $|\cdot| \colon \mathcal{Set}^{\Delta^{\mathrm{op}}} \to \mathcal{Top}$ is left adjoint to the singular functor $\mathcal{S} \colon \mathcal{Top} \to \mathcal{Set}^{\Delta^{\mathrm{op}}}$:

$$\mathrm{Hom}_{\mathcal{Top}}(|X|, Y) \cong \mathrm{Hom}_{\mathcal{Set}^{\Delta^{\mathrm{op}}}}(X, \mathcal{S}Y).$$

The adjunction maps are the ones that come first to mind:

$$\begin{aligned} X &\mapsto \mathcal{S}|X|, \\ X_n \ni x &\mapsto (\Delta^n \xrightarrow{s \mapsto (x,s)} X_n \times \Delta^n \xrightarrow{\sim} |X|) \in \mathcal{S}|X|_n; \end{aligned}$$

$$\begin{aligned} |\mathcal{S}Y| &\mapsto Y, \\ \mathcal{S}Y_n \times \Delta^n \ni (y, s) &\mapsto y(s) \in Y. \end{aligned}$$

Example Q.2.8. For a group G consider a simplicial set BG given by a sequence of sets $BG_0 \stackrel{\text{def}}{=} 1$, $BG_1 \stackrel{\text{def}}{=} G$, $BG_2 \stackrel{\text{def}}{=} G \times G$, $BG_3 \stackrel{\text{def}}{=} G \times G \times G$, ... Define the face and degeneracy operators by

$$\begin{aligned} \partial_i(g_1, \ldots, g_n) &\stackrel{\text{def}}{=} \begin{cases} (g_2, \ldots, g_n), & \text{if } i = 0, \\ (g_1, \ldots, g_i g_{i+1}, \ldots, g_n), & \text{if } 0 < i < n, \\ (g_1, \ldots, g_{n-1}), & \text{if } i = n; \end{cases} \\ \sigma_i(g_1, \ldots, g_n) &\stackrel{\text{def}}{=} (g_1, \ldots, g_i, 1, g_{i+1}, \ldots, g_n). \end{aligned}$$

The geometric realization $|BG|$ is an Eilenberg–Mac Lane space $K(G, 1)$. See e.g. [May99, §16.5]. ▲

Q.3 Classifying space of a category

Similarly to the last example, one can start from a small category $\mathcal{C}$ and then build a CW-complex $B\mathcal{C}$ which is called its **classifying space**. It enjoys some expected properties, e.g. equivalent categories have homotopy equivalent classifying spaces.

Definition Q.3.1. Let $\mathcal{C}$ be a small category. The **nerve** of $\mathcal{C}$, denoted by $N\mathcal{C}$, is a simplicial set constructed as follows. Consider a sequence $N\mathcal{C}_0, N\mathcal{C}_1, N\mathcal{C}_2, \ldots$, were $N\mathcal{C}_n$ is the set of diagrams of n consecutive morphisms

$$N\mathcal{C}_n \stackrel{\text{def}}{=} \{A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} A_2 \xrightarrow{f_3} \cdots \xrightarrow{f_n} A_n \mid A_i \in \mathrm{Ob}(\mathcal{C})\}.$$

Face and degeneracy operators are given by composition and by insertion of the identity morphism:

$$\partial_i(A_0 \to A_1 \to \cdots \to A_n) \stackrel{\text{def}}{=} A_0 \to A_1 \to \cdots \to A_{i-1} \xrightarrow{f_{i+1}\circ f_i} A_{i+1} \to \cdots \to A_n.$$

$$\sigma_i(A_0 \to A_1 \to \cdots \to A_n) \stackrel{\text{def}}{=} A_0 \to A_1 \to \cdots \to A_{i-1} \xrightarrow{f_i} A_i \xrightarrow{id} A_i \xrightarrow{f_{i+1}} A_{i+1} \to \cdots \to A_n.$$

Now the **classifying space** of $\mathcal{C}$ is the geometric realization of the nerve:

$$B\mathcal{C} \stackrel{\text{def}}{=} |N\mathcal{C}|.$$

It is clear that a functor between two small categories $\mathcal{C} \to \mathcal{D}$ induces a map between nerves $N\mathcal{C} \to N\mathcal{D}$, and hence a continuous map $B\mathcal{C} \to B\mathcal{D}$.

For the product of categories $\mathcal{C} \times \mathcal{D}$ one has a homeomorphism $B(\mathcal{C} \times \mathcal{D}) \cong B\mathcal{C} \times B\mathcal{D}$ under assumption that $B(\mathcal{C} \times \mathcal{D})$ is a CW-complex (cf. [May67, Theorem 14.3]).

Example Q.3.2. A group G can be viewed as a category $\mathcal{G}$ with one object $\star$ and all arrows $\mathrm{Hom}_{\mathcal{G}}(\star, \star)$ being isomorphisms. The arrows correspond to the elements of G and the composition corresponds to multiplication. In this case definition Q.3.1 gives the same as example Q.2.8, i.e. $BG \cong B\mathcal{G}$. ▲

An important property is the following.

Proposition Q.3.3. *Let $F, G\colon \mathcal{C} \to \mathcal{D}$ be functors between small categories, such that there is a natural transformation $\eta\colon F \Rightarrow G$. Then the induced maps $BF, BG\colon B\mathcal{C} \to B\mathcal{D}$ are homotopic.*

Proof. A natural transformation corresponds to a functor $H\colon \mathcal{C} \times I \to \mathcal{D}$, where I is the ordered set $\{0 < 1\}$ regarded as a category:

$$\begin{tikzcd} 0 \arrow[loop left] \arrow[r] & 1 \arrow[loop right] \end{tikzcd}$$

The correspondence is the following:

$$\begin{array}{rcl} \eta\colon F \Rightarrow G & \leftrightarrow & H\colon \mathcal{C} \times I \to \mathcal{D}, \\ F(X) & = & H(X, 0), \\ G(X) & = & H(X, 1), \\ \eta_X & = & H(id_X, 0 \to 1). \end{array}$$

Now H induces a continuous map $BH\colon B\mathcal{C} \times BI \to B\mathcal{D}$. The space $BI \cong [0,1]$ is the unit interval, hence BH gives a homotopy between BF and BG. ■

Corollary Q.3.4. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor between small categories. If F has a left adjoint or right adjoint, then BF is a homotopy equivalence.*

In particular, if $\mathcal{C}$ and $\mathcal{D}$ are equivalent categories, then there is a homotopy equivalence of spaces $B\mathcal{C} \simeq B\mathcal{D}$.

Example Q.3.5. Consider a small category $\mathcal{C}$ and the category $\star$ having one object $\star$ and one identity morphism $\star \to \star$. There exists a unique functor $F\colon \mathcal{C} \to \star$.

- If $\mathcal{C}$ has an initial object $I \in \mathrm{Ob}(\mathcal{C})$, then the functor $\star \rightsquigarrow I$ is left adjoint to F:

$$\mathrm{Hom}_{\mathcal{C}}(I, X) \cong \mathrm{Hom}_{\star}(\star, \star).$$

- If $\mathcal{C}$ has a terminal object $T \in \mathrm{Ob}(\mathcal{C})$, then the functor $\star \rightsquigarrow T$ is right adjoint to F:

$$\mathrm{Hom}_{\star}(\star, \star) \cong \mathrm{Hom}_{\mathcal{C}}(X, T).$$

This means that a small category having either initial or terminal object is **contractible**, i.e. its classifying space is homotopy equivalent to a point. ▲

Q.4 Coverings

We are going to look at the fundamental group $\pi_1(B\mathcal{C})$ of the classifying space of a category $\mathcal{C}$, and to study it, we need a notion of covering in the simplicial setting. For the usual theory of coverings of topological spaces and groupoids see [May99, Chapter 3].

Definition Q.4.1. A morphism of simplicial sets $p\colon E \to X$ is called a **covering** of X if for any commutative diagram as below in the category of simplicial sets (where $\Delta[n]$ is the standard n-simplex) there is a unique morphism $\Delta[n] \to E$ making the diagram commute:

$$\begin{tikzcd}
\Delta[0] \arrow[r] \arrow[d] & E \arrow[d, "p"] \\
\Delta[n] \arrow[r] \arrow[ur, dashed] & X
\end{tikzcd}$$

All coverings of a simplicial set X form a category $\mathcal{C}ov/X$, where the morphisms are given by commutative diagrams

$$\begin{tikzcd}
E \arrow[rr, "f"] \arrow[dr, "p"'] & & E' \arrow[dl, "p'"] \\
& X &
\end{tikzcd}$$

As one can guess, the main point of this definition is the following [GZ67, Appendix I, §3.2]:

Fact Q.4.2. *The geometric realization $p\colon |E| \to |X|$ of a simplicial covering $p\colon E \to X$ is a usual covering of a topological space.*

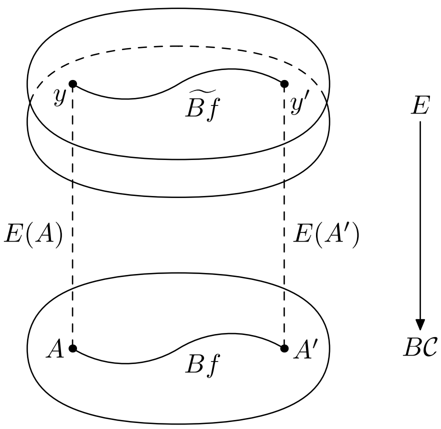
The following characterization of coverings of $B\mathcal{C}$ will be useful [Qui73b, Proposition 1]:

Theorem Q.4.3. *Let $\mathcal{C}$ be a small category. The category $\mathcal{C}ov/B\mathcal{C}$ of coverings over the classifying space of $\mathcal{C}$ is equivalent to the category of* ***morphism-inverting functors*** *$F\colon \mathcal{C} \to \mathcal{S}et$, i.e. functors taking each arrow $A \to A'$ to a bijection of sets $F(A) \to F(A')$.*

In one direction, if we have a covering $p\colon E \to B\mathcal{C}$, then for an object $A \in \mathrm{Ob}(\mathcal{C})$, which can be viewed as a point in $B\mathcal{C}$, we consider its fiber $E(A) \stackrel{\text{def}}{=} p^{-1}(A)$. A morphism $f\colon A \to A'$ in $\mathcal{C}$ determines a path $Bf\colon A \to A'$ in $B\mathcal{C}$.

Fix a point $y \in E(A)$. Then by the unique path lifting property (see e.g. [May99, §3.2]) we have a corresponding path $\widetilde{Bf}$ in E starting in y and ending at a point $y' \in E(A')$. This gives a bijection

$$
\begin{aligned}
(Bf)_* \colon E(A) &\to E(A'),\\
y &\mapsto y'.
\end{aligned}
$$



Hence each covering $p \colon E \to B\mathcal{C}$ defines a morphism-inverting functor $F_p \colon \mathcal{C} \to \mathcal{S}et$:

$$
\begin{aligned}
A &\rightsquigarrow E(A),\\
A \xrightarrow{f} A' &\rightsquigarrow E(A) \xrightarrow{(Bf)_*} E(A').
\end{aligned}
$$

Now assume we are given a morphism-inverting functor $F \colon \mathcal{C} \to \mathcal{S}et$. We need to construct a covering from F. Let $F \backslash \mathcal{C}$ denote the category of pairs (A, x) where $A \in \mathrm{Ob}(\mathcal{C})$ and $x \in F(A)$, and a morphism $(A, x) \to (A', x')$ is an arrow $f \colon A \to A'$ in $\mathcal{C}$ such that $F(f)$ maps x to x'.

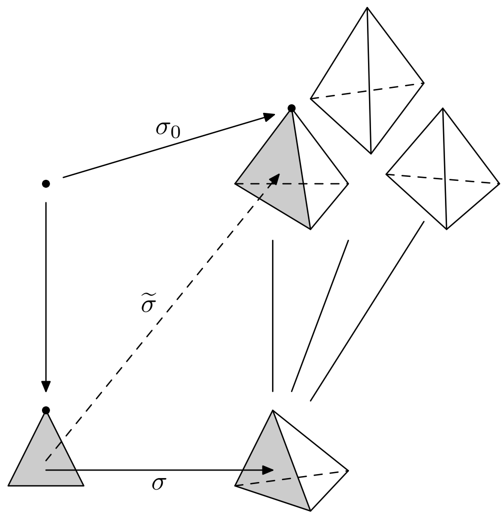
$$
\begin{aligned}
F(f) \colon F(A) &\to F(A'),\\
x &\mapsto x'.
\end{aligned}
$$

The forgetful functor $F \backslash \mathcal{C} \to \mathcal{C}$ induces a map of classifying spaces $p \colon B(F \backslash \mathcal{C}) \to B\mathcal{C}$. For $A \in \mathrm{Ob}(\mathcal{C})$ the fiber of this map over A is $F(A)$. We claim that $p \colon B(F \backslash \mathcal{C}) \to B\mathcal{C}$ is a covering. For this recall that this map comes from the corresponding morphism of nerves $N(F \backslash \mathcal{C}) \to N\mathcal{C}$. In the view of fact Q.4.2, it is enough to check that $N(F \backslash \mathcal{C}) \to N\mathcal{C}$ is a simplicial covering in the sense of definition Q.4.1. Namely, we should check that for each commutative diagram

$$
\begin{tikzcd}
\Delta[0] \arrow[r, "\sigma_0"] \arrow[d, "i"'] & N(F \backslash \mathcal{C}) \arrow[d] \\
\Delta[n] \arrow[r, "\sigma"'] \arrow[ur, dashed, "\tilde{\sigma}"] & \mathcal{C}
\end{tikzcd}
$$

there exists a unique arrow $\tilde{\sigma} \colon \Delta[n] \to N(F \backslash \mathcal{C})$ making all commute.

This amounts to checking that if we are given an n-simplex $\sigma \in N_n\mathcal{C}$ and $\sigma_0 \in N_0(F \backslash \mathcal{C})$ is a simplex lying over the i-th vertex of σ, then there is a unique simplex $\tilde{\sigma} \in N_n(F \backslash \mathcal{C})$ lying over σ and having σ_0 as its i-th vertex.



$\sigma \in N_n\mathcal{C}$ is given by a diagram in $\mathcal{C}$

$$\sigma\colon A_0 \to A_1 \to \cdots \to A_i \to \cdots \to A_n.$$

The i-th vertex of σ is the object $A_i \in \mathrm{Ob}(\mathcal{C})$. Over A_i in $N_0(F\backslash\mathcal{C})$ lie all pairs (A_i, x_i) with $x_i \in F(A_i)$. The functor F maps the diagram above to a chain of *bijections* (we assumed that F is morphism-inverting)

$$F(A_0) \leftrightarrow F(A_1) \leftrightarrow \cdots \leftrightarrow F(A_i) \leftrightarrow \cdots \leftrightarrow F(A_n).$$

Hence if we specify $x_i \in F(A_i)$, the bijections determine uniquely elements $x_0 \in F(A_0)$, $x_1 \in F(A_1)$, ..., $x_n \in F(A_n)$, and the simplex σ lifts uniquely to $\tilde{\sigma}$ given by

$$\tilde{\sigma}\colon (A_0, x_0) \to (A_1, x_1) \to \cdots \to (A_i, x_i) \to \cdots \to (A_n, x_n).$$

This finishes our check that $N(F\backslash\mathcal{C}) \to \mathcal{C}$ is a simplicial covering, hence $B(F\backslash\mathcal{C}) \to B\mathcal{C}$ is a covering.

It is immediate that the two constructions provide an equivalence of categories

$$\begin{array}{rcl} \mathcal{C}ov/B\mathcal{C} & \simeq & \boxed{\text{morphism-inverting functors } F \to \mathcal{C}} \\ p\colon E \to B\mathcal{C} & \rightsquigarrow & F_p, \\ B(F\backslash\mathcal{C}) \to B\mathcal{C} & \leftsquigarrow & F. \end{array}$$

Q.5 Exact categories

Let $\mathcal{C}$ be an exact category (definition Q.1.2). Let us write down some properties of $\mathcal{C}$ that also give an axiomatic definition of "exactness". Let $\mathcal{E}$ denote the class of sequences in $\mathcal{C}$

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0 \tag{Q.1}$$

which are exact in $\mathcal{A}$. If a morphism $i\colon A \to B$ in $\mathcal{C}$ occurs as a morphism in a short exact sequence (Q.1), then we say that it is an **admissible monomorphism**. We write in this case "$A \rightarrowtail B$". If a morphism $p\colon B \to C$ in $\mathcal{C}$ occurs as a morphism in a short exact sequence (Q.1), then we say that it is an **admissible epimorphism**. We write in this case "$B \twoheadrightarrow C$".

The class $\mathcal{E}$ satisfies the following properties:

a) Any exact sequence in $\mathcal{C}$ which is isomorphic to a sequence in $\mathcal{E}$, is in $\mathcal{E}$.

   For any $A, C \in \mathrm{Ob}(\mathcal{C})$ the "split exact" sequence

$$0 \to A \xrightarrow{(id,0)} A \oplus B \xrightarrow{pr_2} B \to 0$$

   is in $\mathcal{E}$. For any sequence (Q.1) in $\mathcal{E}$ one has $i = \ker p$ and $p = \operatorname{coker} i$ in the additive category $\mathcal{C}$.

b) The class of admissible epimorphisms is closed under composition and pullbacks (base change) and the class of admissible monomorphisms is closed under composition and pushouts (cobase change):

$$\begin{tikzcd} B' \arrow[r] \arrow[d, two heads, "\bar{p}"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, two heads, "p"] \\ C' \arrow[r] & C \end{tikzcd} \qquad \begin{tikzcd} B' \arrow[dr, phantom, "\ulcorner", very near start] & B \arrow[l] \\ A' \arrow[u, tail, "\bar{i}"] & A \arrow[l] \arrow[u, tail, "i"'] \end{tikzcd}$$

c) Let $B \to C$ be a map possessing a kernel in $\mathcal{C}$. Suppose there exists a map $B' \to B$ in $\mathcal{C}$ such that the composition $B' \to B \to C$ is an admissible epimorphism. Then $B \to C$ is an admissible epimorphism.

   Let $A \to B$ be a map possessing a cokernel in $\mathcal{C}$. Suppose there exists a map $B \to B'$ in $\mathcal{C}$ such that $A \to B \to B'$ is an admissible monomorphism. Then $A \to B$ is an admissible monomorphism.

All these properties follow easily from our assumptions on $\mathcal{C}$. For instance, for b) let $B \twoheadrightarrow C$ be an admissible epimorphism. Let $C' \to C$ be any morphism. We can take the pullback of $B \twoheadrightarrow C$ over $C' \to C$ in the category $\mathcal{A}$.

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0 \\ 0 \arrow[r] & A \arrow[r] \arrow[u, equal] & B' \arrow[r] \arrow[u] \arrow[ur, phantom, "\urcorner"] & C' \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

But $\mathcal{C}$ is closed under extensions, so B' is isomorphic to an object of $\mathcal{C}$. Hence $B' \to C'$ is an admissible epimorphism.

Definition Q.5.1 (Quillen)**.** An **exact category** $\mathcal{C}$ is an additive category $\mathcal{C}$ with a family $\mathcal{E}$ of sequences of the form (Q.1), called the **short exact sequences** in $\mathcal{C}$, such that the properties a), b), c) hold.

A functor $F\colon \mathcal{C} \to \mathcal{C}'$ between exact categories is called **exact** if it carries each short exact sequence in $\mathcal{C}$ to a short exact sequence in $\mathcal{C}'$:

$$0 \to A \to B \to C \to 0 \quad \rightsquigarrow \quad 0 \to F(A) \to F(B) \to F(C) \to 0$$

Remark Q.5.2. Just to prevent confusion, this is not the same as "exact categories" in the sense of Barr [Bar71].

Given any exact category $\mathcal{C}$ defined axiomatically as above, one can embed it in the category $\mathcal{A}$ of additive left exact contravariant functors $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{Ab}$. I.e. $\mathcal{A}$ consists of contravariant functors F that take a short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{C}$ to an exact sequence of abelian groups

$$0 \to F(C) \to F(B) \to F(A)$$

The category $\mathcal{A}$ is abelian, and $\mathcal{C} \hookrightarrow \mathcal{A}$ is given by Yoneda:

$$\begin{aligned} h\colon \mathcal{C} &\hookrightarrow \mathcal{A}, \\ C &\rightsquigarrow \operatorname{Hom}_{\mathcal{C}}(-, C). \end{aligned}$$

This embeds $\mathcal{C}$ as a full abelian subcategory of $\mathcal{A}$ closed under extensions. A sequence

$$0 \to A \to B \to C \to 0$$

is in $\mathcal{E}$ if and only if h carries it to an exact sequence in $\mathcal{A}$.

Q.6 The category $Q\mathcal{C}$

Now for an exact category $\mathcal{C}$ we define a category $Q\mathcal{C}$ as follows.

The objects in $Q\mathcal{C}$ are the same as in $\mathcal{C}$, but a morphism $X \to Y$ is a diagram of the form

$$\begin{tikzcd} & V \arrow[dl, two heads] \arrow[dr, tail] & \\ X & & Y \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r] & V' \arrow[r] & V \arrow[r, two heads] & X \arrow[r] & 0 \\ 0 \arrow[r] & V \arrow[r, tail] & Y \arrow[r] & Y' \arrow[r] & 0 \end{tikzcd}$$

where $V \twoheadrightarrow X$ is an admissible epimorphism in $\mathcal{C}$ and $V \rightarrowtail Y$ is an admissible monomorphism in $\mathcal{C}$.

Moreover, we take *isomorphism classes* of such diagrams: we identify two morphisms as above if there is an isomorphism $V \xrightarrow{\cong} V'$ making the diagram commute:

$$\begin{tikzcd} & V \arrow[dl, two heads] \arrow[dr, tail] \arrow[dd, "\cong"] & \\ X & & Y \\ & V' \arrow[ul, two heads] \arrow[ur, tail] & \end{tikzcd}$$

We assume that such isomorphism classes of diagrams form a set, so that $Q\mathcal{C}$ is a small category. The composition of two such morphisms in $Q\mathcal{C}$ is defined by taking a bicartesian square

$$\begin{tikzcd} & & V \times_Y W \arrow[dl, two heads] \arrow[dr, tail] & & \\ & V \arrow[dl, two heads] \arrow[dr, tail] & & W \arrow[dl, two heads] \arrow[dr, tail] & \\ X & & Y & & Z \end{tikzcd}$$

This indeed exists in $\mathcal{C}$, since $\mathcal{C}$ is closed under extensions, and we have a short exact sequence

$$0 \to \ker \overline{p} \to V \times_Y W \xrightarrow{\overline{p}} V \to 0$$

Observe now that $\ker(V \times_Y W \xrightarrow{\overline{p}} V) \cong \ker(W \xrightarrow{p} Y)$.

The associativity of composition is verified by the universal property of pullbacks. Finally, one can check that the composition depends only on isomorphism classes of diagrams.

Definition Q.6.1. Let $i\colon A \rightarrowtail B$ be an admissible monomorphism in $\mathcal{C}$. This gives a morphism $i_!\colon A \to B$ in $Q\mathcal{C}$ represented by a diagram

$$\begin{tikzcd} & A \arrow[dl, two heads, "id"'] \arrow[dr, tail, "i"] & \\ A & & B \end{tikzcd}$$

All morphisms of the form $i_!$ are called **injective**. Similarly, if $p\colon B \twoheadrightarrow C$ is an admissible epimorphism in $\mathcal{C}$, then we define a morphism $p^!\colon C \to B$ in $Q\mathcal{C}$:

$$\begin{tikzcd} & B \arrow[dl, two heads, "p"'] \arrow[dr, tail, "id"] & \\ C & & B \end{tikzcd}$$

All morphisms of the form $p^!$ are called **surjective**.

Remark Q.6.2. To prevent confusion, the terms "injective" and "surjective" do not imply "monomorphism in $Q\mathcal{C}$" and "epimorphism in $Q\mathcal{C}$".

By definition, every morphism $f\colon X \to Y$ in $Q\mathcal{C}$ factors uniquely (up to a unique isomorphism) into a surjection and injection $i_! \circ p^!$:

$$\begin{tikzcd} & & V \arrow[dl, "id"'] \arrow[dr, "id"] & & \\ & V \arrow[dl, two heads, "p"'] \arrow[dr, "id"] & & V \arrow[dl, "id"'] \arrow[dr, tail, "i"] & \\ X & & V & & Y \end{tikzcd}$$

On the other hand, there is also a unique factorization (up to a unique isomorphism) into an injection and surjection $\overline{p}^! \circ \overline{i}_!$ given by a bicartesian square

$$\begin{tikzcd} & & V \arrow[dl, two heads, "p"'] \arrow[dr, tail, "i"] & & \\ & X \arrow[dl, "id"'] \arrow[dr, tail, "\overline{i}"] & & Y \arrow[dl, two heads, "\overline{p}"'] \arrow[dr, "id"] & \\ X & & X *_V Y & & Y \end{tikzcd}$$

The operations $i \mapsto i_!$ and $p \mapsto p^!$ have the following properties:

a) If i and j are composable admissible monomorphisms, then

$$A \overset{i}{\rightarrowtail} B \overset{j}{\rightarrowtail} C$$

$$(j \circ i)_! = j_! \circ i_!$$

$$\begin{tikzcd} & & A \arrow[dl, two heads, "id"'] \arrow[dr, tail, "i"] & & \\ & A \arrow[dl, two heads, "id"'] \arrow[dr, tail, "i"] & & B \arrow[dl, two heads, "id"'] \arrow[dr, tail, "j"] & \\ A & & B & & C \end{tikzcd}$$

Dually, if p and q are composable admissible epimorphisms, then

$$(p \circ q)^! = q^! \circ p^!.$$

Also one has

$$(id_A)_! = (id_A)^! = id_A.$$

b) Suppose one has a bicartesian square

$$\begin{tikzcd}
Z & Y \arrow[l, two heads, "\overline{p}"'] \\
X \arrow[u, tail, "\overline{i}"] \arrow[ur, phantom, "\ulcorner\ \ulcorner"] & V \arrow[l, two heads, "p"] \arrow[u, tail, "i"']
\end{tikzcd}$$

where i and $\overline{i}$ are admissible monomorphisms, p and $\overline{p}$ are admissible epimorphisms. Then

$$i_! \circ p^! = \overline{p}^! \circ \overline{i}_!.$$

$$\begin{tikzcd}[column sep=small]
& & V \arrow[dl, "id"'] \arrow[dr, "id"] & & \\
& V \arrow[dl, two heads, "p"'] \arrow[dr, "id"] & & V \arrow[dl, "id"'] \arrow[dr, tail, "i"] & \\
X & & V & & Y
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
& & V \arrow[dl, two heads, "p"'] \arrow[dr, tail, "i"] & & \\
& X \arrow[dl, "id"'] \arrow[dr, tail, "\overline{i}"] & & Y \arrow[dl, two heads, "\overline{p}"'] \arrow[dr, "id"] & \\
X & & Z & & Y
\end{tikzcd}$$

This leads to a certain characterization of the category $Q\mathcal{C}$:

Proposition Q.6.3. *Let $\mathcal{C}$ be an exact category and let $\mathcal{D}$ be a category. Assume that the following data is given:*

- *for each object $A \in \mathrm{Ob}(\mathcal{C})$, an object $F(A) \in \mathrm{Ob}(\mathcal{D})$,*
- *for each admissible monomorphism $i\colon A \rightarrowtail B$ in $\mathcal{C}$, a morphism $i_{!!}\colon F(A) \to F(B)$ in $\mathcal{D}$,*
- *for each admissible epimorphism $p\colon B \twoheadrightarrow C$ in $\mathcal{C}$, a morphism $p^{!!}\colon F(C) \to F(B)$.*

$$\begin{array}{rcl}
\mathcal{C} & \to & \mathcal{D}, \\
A & \rightsquigarrow & F(A), \\
(i\colon A \rightarrowtail B) & \rightsquigarrow & (i_{!!}\colon F(A) \to F(B)), \\
(p\colon B \twoheadrightarrow C) & \rightsquigarrow & (p^{!!}\colon F(C) \to F(B)),
\end{array}$$

Further, require that properties a) and b) as above hold for the arrows $i_{!!}$ and $p^{!!}$ in $\mathcal{D}$, that is,

a) for admissible monomorphisms $(j \circ i)_{!!} = j_{!!} \circ i_{!!}$ and for admissible epimorphisms $(p \circ q)^{!!} = q^{!!} \circ p^{!!}$, whenever the compositions make sense.

b) suppose one has a bicartesian square

$$\begin{tikzcd} Z & Y \arrow[l, two heads, "\overline{p}"'] \\ X \arrow[u, tail, "\overline{i}"] \arrow[ur, phantom, "\ulcorner\quad\ulcorner"] & V \arrow[l, two heads, "p"] \arrow[u, tail, "i"'] \end{tikzcd}$$

where i and $\overline{i}$ are admissible monomorphisms, p and $\overline{p}$ are admissible epimorphisms. Then

$$i_{!!} \circ p^{!!} = \overline{p}^{!!} \circ \overline{i}_{!!}.$$

This data uniquely defines a functor $F\colon \mathcal{QC} \to \mathcal{D}$.

Proof. The functor F on the arrows of $\mathcal{QC}$ is given by

$$\begin{tikzcd} & V \arrow[dl, two heads, "p"'] \arrow[dr, tail, "i"] & & & \\ X & & Y & \rightsquigarrow & i_{!!} \circ p^{!!} \end{tikzcd}$$

We need to check that this depends only on the equivalence class of the diagram. Suppose we have another diagram, which is equivalent to the above via an isomorphism $\phi\colon V \to V'$.

$$\begin{tikzcd} & V \arrow[dl, two heads, "p"'] \arrow[dr, tail, "i"] \arrow[dd, "\phi"', "\cong"] & \\ X & & Y \\ & V' \arrow[ul, two heads, "p'"] \arrow[ur, tail, "i'"'] & \end{tikzcd}$$

We have $p = p' \circ \phi$ and $i = i' \circ \phi$. Since ϕ can be viewed as both admissible monomorphism and admissible epimorphism, this gives

$$p^{!!} = \phi^{!!} \circ (p')^{!!}, \quad i_{!!} = i'_{!!} \circ \phi_{!!}.$$

From a bicartesian square

$$\begin{tikzcd} V' & V' \arrow[l, two heads, "id"'] \\ V' \arrow[u, tail, "id"] \arrow[ur, phantom, "\ulcorner\quad\ulcorner"] & V \arrow[l, two heads, "\phi"] \arrow[u, tail, "\phi"'] \end{tikzcd}$$

we deduce

$$\phi_{!!} \circ \phi^{!!} = id^{!!}_{V'} \circ (id_{V'})_{!!} = id_{F(V')}.$$

And therefore

$$i'_{!!} \circ (p')^{!!} = i'_{!!} \circ \underbrace{\phi_{!!} \circ \phi^{!!}}_{id} \circ (p')^{!!} = i_{!!} \circ p^{!!}.$$

Further, we need to check that the definition of functor respects composition in $\mathcal{QC}$. A composition is represented by a bicartesian square

$$\begin{tikzcd}
& & V \times_Y W \arrow[dl, two heads, "\overline{q}"'] \arrow[dr, tail, "\overline{i}"] \arrow[dd, phantom, "\vee"] & & \\
& V \arrow[dl, two heads, "p"'] \arrow[dr, tail, "i"] & & W \arrow[dl, two heads, "q"'] \arrow[dr, tail, "j"] & \\
X & & Y & & Z
\end{tikzcd}$$

From which

$$(j \circ \overline{i})_{!!} \circ (p \circ \overline{q})^{!!} = j_{!!} \circ \overline{i}_{!!} \circ \overline{q}^{!!} \circ p^{!!} = (j_{!!} \circ q^{!!}) \circ (i_{!!} \circ p^{!!}).$$

■

In particular, an exact functor $F\colon \mathcal{C} \to \mathcal{C}'$ between exact categories induces a functor

$$\begin{array}{rcl}
\mathcal{QC} & \to & \mathcal{QC}', \\
A & \rightsquigarrow & F(A), \\
i_! & \rightsquigarrow & F(i)_!, \\
p^! & \rightsquigarrow & F(p)^!.
\end{array}$$

Proposition Q.6.4. *One has an isomorphism of categories*

$$Q(\mathcal{C}^{\mathrm{op}}) \cong \mathcal{QC},$$

such that injective arrows in $\mathcal{QC}$ correspond to surjective arrows in $\mathcal{QC}^{\mathrm{op}}$ and vice versa.

Proof. If we have a bicartesian square in $\mathcal{C}$, then we have a bicartesian square in $\mathcal{C}^{\mathrm{op}}$:

$$\begin{tikzcd}
Z & Y \arrow[l, two heads, "\overline{p}"'] \\
X \arrow[u, tail, "\overline{i}"] & V \arrow[l, two heads, "p"] \arrow[u, tail, "i"']
\end{tikzcd}
\quad \Longrightarrow \quad
\begin{tikzcd}
Z \arrow[r, tail, "\overline{p}^{\mathrm{op}}"] \arrow[d, two heads, "\overline{i}^{\mathrm{op}}"'] & Y \arrow[d, two heads, "i^{\mathrm{op}}"] \\
X \arrow[r, tail, "p^{\mathrm{op}}"'] & V
\end{tikzcd}$$

Consider a functor which is identity on objects and defined on arrows by

$$i_! \circ p^! \rightsquigarrow (\overline{p}^{\mathrm{op}})_! \circ (\overline{i}^{\mathrm{op}})^!.$$

This is full and faithful:

$$\mathrm{Hom}_{\mathcal{QC}}(X, Y) \cong \mathrm{Hom}_{\mathcal{QC}^{\mathrm{op}}}(X, Y).$$

■

Q.7 Higher K-groups via the Q-construction

The following is [Qui73b, Theorem 1, p. 102]:

Theorem Q.7.1. *Let $\mathcal{C}$ be a skeletally small exact category. Let 0 be a zero object in $\mathcal{C}$. Then there is a natural isomorphism*

$$\pi_1(B\mathcal{QC}, 0) \cong K_0(\mathcal{C}).$$

This motivates the following definition [Qui73b, p. 103]:

Definition Q.7.2. For a skeletally small exact category $\mathcal{C}$ its K-groups are given by

$$K_i(\mathcal{C}) \stackrel{\text{def}}{=} \pi_{i+1}(BQ\mathcal{C}, 0),$$

where 0 refers to the point $0 \in BQ\mathcal{C}$ corresponding to the zero object.

This is related to the K-groups of a ring defined by the plus-construction as follows.

Theorem Q.7.3. *Let R-$\mathcal{P}roj_{fg}$ the the category of finitely generated projective R-modules. There is a homotopy equivalence (natural up to homotopy)*

$$BGL(R)^+ \to \Omega(BQR\text{-}\mathcal{P}roj_{fg}, 0),$$

where Ω is the loop space functor (taken at the point 0).

Hence there is a natural isomorphism

$$K_i(R\text{-}\mathcal{P}roj_{fg}) \cong \pi_i(BGL(R)^+), \quad i \geqslant 1.$$

Remark Q.7.4. It is important that we defined $K_i(\mathcal{C})$ for any skeletally small exact category $\mathcal{C}$. E.g. for a scheme X we can take $\mathcal{C} = \mathcal{VB}(X)$, and this defines the K-groups $K_i(X)$. See [Qui73b, §7].

Discussing a proof of $BGL(R)^+ \simeq \Omega(BQR\text{-}\mathcal{P}roj_{fg})$ would lead us a bit too far. It can be found in [Ada78, Chapter 3] or [Sri96, Chapter 7]. We are going to see at least a proof of $\pi_1(BQ\mathcal{C}) \cong K_0(\mathcal{C})$ just to understand better the Q-construction. In fact, all the needed machinery was already introduced above.

According to the theorem Q.4.3, the category of covering spaces of $BQ\mathcal{C}$ is equivalent to the category of morphism-inverting functors $F\colon Q\mathcal{C} \to \mathcal{S}et$. Let us denote the latter by $\mathcal{F}$. Similarly, the category of covering spaces of $BK_0(\mathcal{C})$ is equivalent to the category of morphism-inverting functors $K_0(\mathcal{C}) \to \mathcal{S}et$, i.e. the category of $K_0(\mathcal{C})$-sets.

Recall that for a space X its fundamental group $\pi_1(X)$ can be identified with the automorphism group of the universal cover $\operatorname{Aut}(\widetilde{X})$. So $\pi_1(BQ\mathcal{C}) \cong K_0(\mathcal{C})$ will follow once we show an equivalence of categories $\mathcal{F} \simeq K_0(\mathcal{C})$-$\mathcal{S}et$.

- First observe that $\mathcal{F}$ is equivalent to its full subcategory $\mathcal{F}'$, which consists of morphism-inverting functors $F'\colon Q\mathcal{C} \to \mathcal{S}et$ such that

  $$F'(B) = F'(0) \quad \text{and} \quad F'(i_{X!}) = id_{F'(0)} \quad \text{for all } X \in \operatorname{Ob}(\mathcal{C}),$$

  where i_X denotes the admissible monomorphism $0 \rightarrowtail X$.

  Note that for an admissible monomorphism $i\colon A \rightarrowtail B$ holds $i \circ i_A = i_B$:

  $$\begin{tikzcd} 0 \arrow[r, tail, "i_A"'] \arrow[rr, bend left, dashed] & A \arrow[r, tail, "i"'] & B \end{tikzcd}$$

  From this we deduce $id_{F'(0)} = F'(i_{B!}) = F'(i_! \circ i_{A!}) = F'(i_!) \circ F'(i_{A!}) = F'(i_!)$. That is, for any admissible monomorphism $i\colon A \rightarrowtail B$ we automatically have

  $$F'(i_!) = id_{F'(0)}.$$

If we have an arbitrary morphism inverting functor $F\colon \mathcal{QC} \to \mathcal{S}et$, then we can define a functor F' in the category $\mathcal{F}'$ by

$$\begin{tikzcd} & V \arrow[dl, two heads, "p"'] \arrow[dr, tail, "i"] & \\ X & & Y \end{tikzcd} \quad \rightsquigarrow \quad F(i_{V!})^{-1} \circ F(p^!) \circ F(i_{X!})$$

$$\begin{tikzcd} & 0 \arrow[dl, two heads] \arrow[dr, tail, "i_X"] & \\ 0 & & X \end{tikzcd} \qquad \begin{tikzcd} & V \arrow[dl, two heads, "p"'] \arrow[dr, tail] & \\ X & & V \end{tikzcd} \qquad \begin{tikzcd} & 0 \arrow[dl, two heads] \arrow[dr, tail, "i_V"] & \\ 0 & & V \end{tikzcd}$$

$$F(0) \xrightarrow{F(i_{X!})} F(X) \qquad F(X) \xrightarrow{F(p^!)} F(V) \qquad F(V) \xrightarrow{F(i_{V!})^{-1}} F(0)$$

Now consider a natural transformation of functors $F' \Rightarrow F$ given by $X \mapsto F(i_{X!})$. Since $F(i_{X!})$ is the bijection in the category $\mathcal{S}et$, this gives an isomorphism $F' \cong F$. Hence any object in the category $\mathcal{F}$ is isomorphic to an object in the category $\mathcal{F}'$.

- If S is a $K_0(\mathcal{C})$-set, we define a morphism inverting functor $F_S\colon \mathcal{QC} \to \mathcal{S}et$ which belongs to the category $\mathcal{F}'$. Using proposition Q.6.3, we see that it is enough to give the following data:

$$\begin{aligned} F_S(A) &\stackrel{\text{def}}{=} S, \\ F_S(i_!) &\stackrel{\text{def}}{=} id_S, \\ F_S(p^!) &\stackrel{\text{def}}{=} \text{the action of } [\ker p] \text{ on } S. \end{aligned}$$

  Here by $[\ker p]$ we denote the class of the object $\ker p$ in $K_0(\mathcal{C})$.

- In the other direction, for any given morphism inverting functor $F\colon \mathcal{QC} \to \mathcal{S}et$ which belongs to the category $\mathcal{F}'$, we describe a natural action of $K_0(\mathcal{C})$ on $F(0)$, i.e. a morphism $K_0(\mathcal{C}) \to \operatorname{Aut}(F(0))$. For $[A] \in K_0(\mathcal{C})$ we take $F(p_A^!) \in \operatorname{Aut}(F(0))$, where p_A denotes the obvious admissible epimorphism $A \twoheadrightarrow 0$. We have to check that this is indeed a homomorphism on $K_0(\mathcal{C})$. For a short exact sequence in $\mathcal{C}$

$$0 \to A \rightarrowtail B \twoheadrightarrow C \to 0$$

  we should have

$$F(p_A^!) \circ F(p_C^!) = F(p_C^!) \circ F(p_A^!) = F(p_B^!).$$

  For this look at the bicartesian square

$$\begin{tikzcd} C \arrow[dr, phantom, "\ulcorner\ \ulcorner"] & B \arrow[l, two heads, "p"'] \\ 0 \arrow[u, tail, "i_C"] & A \arrow[l, two heads, "p_A"] \arrow[u, tail, "i"'] \end{tikzcd}$$

  From this we deduce

$$i_! \circ p_A^! = p^! \circ i_{C!}.$$

  Since $F(i_!) = F(i_{C!}) = id_{F(0)}$, we conclude that $F(p_A^!) = F(p^!)$.

Further, $p_B = p_C \circ p$:

$$\begin{tikzcd} B \arrow[r, two heads, "p"'] \arrow[rr, bend left, dashed] & C \arrow[r, two heads, "p_C"'] & 0 \end{tikzcd}$$

So we have

$$F(p_B^!) = F((p_C \circ p)^!) = F(p^! \circ p_C^!) = F(p^!) \circ F(p_C^!) = F(p_A^!) \circ F(p_C^!).$$

We claim that also $F(p_A^!) \circ F(p_C^!) = F(p_C^!) \circ F(p_A^!)$. For this in the argument above we replace B with $A \oplus C$ and consider split exact sequences

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, tail] & A \oplus C \arrow[r, two heads] & C \arrow[r] & 0 \\ 0 \arrow[r] & C \arrow[r, tail] & A \oplus C \arrow[r, two heads] & A \arrow[r] & 0 \end{tikzcd}$$

From the trivial fact $\ker p_A \cong A$, one readily sees that the constructions $S \mapsto F_S$ and $F \mapsto K_0$-set $F(0)$ are mutually inverse. This finally shows that $\pi_1(BQ\mathcal{C}, 0) \cong K_0(\mathcal{C})$. ∎

Q.8 Quotient categories

We recall what a quotient category of an abelian category is. The reference for this is [Gab62, Chapitre III]. Let us ignore set theoretical issues and from now on we denote by $\mathcal{A}$ and $\mathcal{B}$ abelian categories whose objects lie in some "universe" $\mathfrak{U}$. We have in mind only one particular example, when the categories are skeletally small.

Remark Q.8.1. Although for us it is enough to work with concrete categories, recall how in general one can use the notion of subobjects. For any object $A \in \mathrm{Ob}(\mathcal{A})$ its **subobjects** are isomorphism classes of monomorphisms $B \rightarrowtail A$. The isomorphism of subobjects is given by a diagram

$$\begin{tikzcd} B \arrow[dd, "\cong"'] \arrow[dr, tail] & \\ & A \\ B' \arrow[ur, tail] & \end{tikzcd}$$

For two subobjects $i_1 \colon A_1 \rightarrowtail A$ and $i_2 \colon A_2 \rightarrowtail A$ we say that $i_1 \subset i_2$ if there is a commutative diagram of monomorphisms

$$\begin{tikzcd} A_1 \arrow[dd, tail] \arrow[dr, tail, "i_1"] & \\ & A \\ A_2 \arrow[ur, tail, "i_2"'] & \end{tikzcd}$$

This is a partial order on the set of subobjects of A.

Definition Q.8.2. Let $\mathcal{A}$ be an abelian category and let $\mathcal{B} \subset \mathcal{A}$ be a full additive subcategory of $\mathcal{A}$ (so that the abelian group structure on Hom-sets is the same). We say that $\mathcal{B}$ is a **Serre subcategory** (sometimes called **catégorie épaisse**) if the following holds

1. Any object of $\mathcal{A}$ isomorphic to an object of $\mathcal{B}$ lies in $\mathcal{B}$.

2. $\mathcal{B}$ is closed under taking subobjects, quotients and extensions in $\mathcal{A}$. That is, if one has a short exact sequence in $\mathcal{A}$
$$0 \to A \to B \to C \to 0$$
then $B \in \mathrm{Ob}(\mathcal{B})$ if and only if $A, C \in \mathrm{Ob}(\mathcal{B})$.

Example Q.8.3. Let R be a Noetherian commutative ring and let $S \subset R$ be a multiplicative subset. Let $\mathcal{A} = R\text{-}\mathcal{M}od_{fg}$ be the category of finitely generated R-modules and let $\mathcal{B} = S\text{-}\mathcal{T}ors_{fg}$ be the full subcategory of S-**torsion modules**. In other words, R-modules M such that $s \cdot M = 0$ for some $s \in S$. Then $S\text{-}\mathcal{T}ors_{fg}$ is a Serre subcategory of $R\text{-}\mathcal{M}od_{fg}$. ▲

Definition Q.8.4. If $\mathcal{B} \subset \mathcal{A}$ is a Serre subcategory, then one can construct the **quotient category** (sometimes called **localization**) $\mathcal{A}/\mathcal{B}$ as follows. The objects of $\mathcal{A}/\mathcal{B}$ coincide with the objects of $\mathcal{A}$. If A, B are two objects, then consider their subobjects $A' \rightarrowtail A$ and $B' \rightarrowtail B$. The morphisms $i\colon A' \rightarrowtail A$ and $p\colon B \to B/B'$ induce $\mathbb{Z}$-linear maps
$$\mathrm{Hom}_{\mathcal{A}}(A, B) \to \mathrm{Hom}_{\mathcal{A}}(A', B/B').$$
Assume now $A/A' \in \mathrm{Ob}(\mathcal{B})$ and $B' \in \mathrm{Ob}(\mathcal{B})$. The abelian groups $\mathrm{Hom}_{\mathcal{A}}(A', B/B')$ form a directed system with obvious maps
$$A'' \subset A' \text{ and } B'' \subset B' \Rightarrow \mathrm{Hom}_{\mathcal{A}}(A'', B/B'') \to \mathrm{Hom}_{\mathcal{A}}(A', B/B').$$
Then one puts
$$\mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(A, B) \stackrel{\text{def}}{=} \varinjlim_{\substack{(A',B') \\ A/A',\ B' \in \mathrm{Ob}(\mathcal{B})}} \mathrm{Hom}_{\mathcal{A}}(A', B/B').$$

One checks that this gives a $\mathbb{Z}$-bilinear composition
$$\mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(A, B) \times \mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(B, C) \to \mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(A, C).$$
Then $\mathcal{A}/\mathcal{B}$ is again an additive category, and the canonical functor $T\colon \mathcal{A} \to \mathcal{A}/\mathcal{B}$ is exact. For details and proofs we refer to [Gab62, Chapitre III]. In particular, one has the following: *for a morphism $f \in \mathrm{Hom}_{\mathcal{A}}(A, B)$ the corresponding morphism $T(f) \in \mathrm{Hom}_{\mathcal{A}/\mathcal{B}}$ is an isomorphism if and only if* $\ker f$ *and* $\operatorname{coker} f$ *lie in* $\mathrm{Ob}(\mathcal{B})$.

Example Q.8.5. Consider as above $\mathcal{A} \stackrel{\text{def}}{=} R\text{-}\mathcal{M}od_{fg}$ and $\mathcal{B} \stackrel{\text{def}}{=} S\text{-}\mathcal{T}ors_{fg}$. We claim that the quotient category $\mathcal{A}/\mathcal{B}$ is equivalent to the category of finitely generated $S^{-1}R$-modules.

We have the localization functor
$$L\colon \mathcal{A} = R\text{-}\mathcal{M}od_{fg} \to S^{-1}R\text{-}\mathcal{M}od_{fg}$$
and the quotient functor
$$T\colon \mathcal{A} \to \mathcal{A}/\mathcal{B}.$$
We claim that there is an equivalence of categories $U\colon \mathcal{A}/\mathcal{B} \to S^{-1}R\text{-}\mathcal{M}od_{fg}$ such that $U \circ T$ and L are isomorphic functors.

For any $R\text{-}\mathcal{M}od_{fg}$-module M, the set $\mathrm{Hom}_{R\text{-}\mathcal{M}od_{fg}}(R, M)$ carries structure of a module over the ring $\mathrm{Hom}_{R\text{-}\mathcal{M}od_{fg}}(R, R) \cong R$ (where multiplication is given by composition), and it is naturally isomorphic to M. One has a homomorphism of commutative rings
$$R \xrightarrow{\cong} \mathrm{Hom}_{R\text{-}\mathcal{M}od_{fg}}(R, R) \xrightarrow{T_*} \mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(T(R), T(R))$$

Now let ϕ define a map $R \to \mathrm{Hom}_{R\text{-}\mathcal{M}od_{fg}}(R, R)$ that takes an element $r \in R$ to a multiplication by r map $x \mapsto r\,x$. For any $s \in S$ the map $\phi(s)\colon R \to R$ has its kernel and cokernel in $\mathcal{B}$, hence $T_* \circ \phi(s)$ is an isomorphism (invertible element in $\mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(T(R), T(R))$). Therefore by the universal property of localization, the map $T_* \circ \phi$ factors uniquely through $S^{-1}R$:

$$\begin{tikzcd}
R \arrow[rr, "T_* \circ \phi"] \arrow[dr] & & \mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(T(R), T(R)) \\
& S^{-1}R \arrow[ur, dashed, "\exists!"'] &
\end{tikzcd}$$

One checks that this is a ring isomorphism $S^{-1}R \cong \mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(T(R), T(R))$.

Now for any module $M \in \mathrm{Ob}(R\text{-}\mathcal{M}od_{fg})$ we get a module $\mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(T(R), T(M))$ over the ring $\mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(T(R), T(R)) \cong S^{-1}R$. There is an R-module homomorphism

$$M \xrightarrow{\cong} \mathrm{Hom}_{R\text{-}\mathcal{M}od_{fg}}(R, M) \xrightarrow{T_*} \mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(T(R), T(M))$$

By the universal property of localization, the map above factors uniquely through $S^{-1}M$:

$$\begin{tikzcd}
M \arrow[rr] \arrow[dr] & & \mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(T(R), T(M)) \\
& S^{-1}M = S^{-1}R \otimes_R M \arrow[ur, dashed, "\psi_M", "\exists!"'] &
\end{tikzcd}$$

One can check that ψ_M is an isomorphism of $S^{-1}R$-modules.

Now the desired functor U is given by

$$\begin{aligned}
U\colon \mathcal{A}/\mathcal{B} &\to S^{-1}R\text{-}\mathcal{M}od_{fg}, \\
T(M) &\mapsto \mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(T(R), T(M)).
\end{aligned}$$

On arrows U is given by the composition of arrows in $\mathcal{A}/\mathcal{B}$.
The morphism ψ_M gives a natural transformation of functors $\psi\colon L \Rightarrow U \circ T$ which is an isomorphism.

$$\begin{tikzcd}
M \arrow[d, "f"'] & S^{-1}M \arrow[r, "\psi_M", "\cong"'] \arrow[d] & \mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(T(R), T(M)) \arrow[d] \\
N & S^{-1}N \arrow[r, "\cong", "\psi_N"'] & \mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(T(R), T(N))
\end{tikzcd}$$

▲

Remark Q.8.6. One can show that taking the quotient category satisfies a universal property similar to the universal property of localization and work out the last example using this. See [BK00, §6.3.8 + exercise 6.3.2].

Q.9 Quillen's results

Now we mention some important results of [Qui73b]; proofs can be found in the original paper, or in [Sri96, Chapter 6]. The following is [Qui73b, §4, p. 108]:

Theorem Q.9.1 (Resolution theorem)**.** *Let $\mathcal{M}$ be an exact category and let $\mathcal{P} \subset \mathcal{M}$ be a full additive subcategory which is closed under extensions in $\mathcal{M}$, such that $\mathcal{P}$ is an exact category and $\mathcal{P} \hookrightarrow \mathcal{M}$ is an exact functor.*

1. *Assume that if*

$$0 \to M' \to M \to M'' \to 0$$

is exact in $\mathcal{M}$ and $M', M'' \in \mathrm{Ob}(\mathcal{P})$, then $M \in \mathrm{Ob}(\mathcal{P})$.

2. *Assume that for each object $M \in \mathrm{Ob}(\mathcal{M})$ there is a finite length resolution in $\mathcal{M}$*

$$0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$$

with $P_i \in \mathrm{Ob}(\mathcal{P})$ (where the resolution length n may depend on M).

Then $BQ\mathcal{P} \to BQ\mathcal{M}$ is a homotopy equivalence, hence $K_i(\mathcal{P}) \cong K_i(\mathcal{M})$.

Example Q.9.2. Let $\mathfrak{A}$ be a Dedekind domain. Then any finitely generated $\mathfrak{A}$-module $M \in \mathrm{Ob}(\mathfrak{A}\text{-}\mathcal{Mod}_{fg})$ has projective dimension $\leqslant 1$ over $\mathfrak{A}$ (cf. e.g. [Wei94, Chapter 4]), and so by the resolution theorem

$$K_i(\mathfrak{A}\text{-}\mathcal{Mod}_{fg}) \cong K_i(\mathfrak{A}\text{-}\mathcal{Proj}_{fg}) \cong K_i(\mathfrak{A}).$$

▲

The following is a corollary from the so-called "dévissage theorem" [Qui73b, Corollary 1, p. 112]:

Theorem Q.9.3. *Let $\mathcal{B}$ be a (skeletally small) abelian category such that every object $B \in \mathrm{Ob}(\mathcal{B})$ has a finite filtration by subobjects*

$$0 = B_0 \subset B_1 \subset \cdots \subset B_n = B.$$

Let $\{X_\alpha\}$ be the set of representatives of the isomorphism classes of simple objects of $\mathcal{B}$. Then

$$K_i(\mathcal{B}) \cong \coprod_\alpha K_i(D_\alpha), \quad \text{where } D_\alpha \stackrel{\text{def}}{=} \mathrm{End}(X_\alpha)^{\mathrm{op}}.$$

Example Q.9.4. Let $\mathfrak{A}$ be a Dedekind domain and let $\mathcal{B}$ be the category of finitely generated torsion $\mathfrak{A}$-modules (modules M such that $M \otimes_{\mathfrak{A}} k \cong 0$). Such modules are of the form

$$\bigoplus_{1 \leqslant j \leqslant n} \mathfrak{A}/I_j$$

for some ideals $I_j \subseteq \mathfrak{A}$ (see e.g. [IR05, §8.8]), so we deduce

$$K_i(\mathcal{B}) \cong \coprod_{\mathfrak{p} \subset \mathfrak{A}} K_i(\mathfrak{A}/\mathfrak{p}),$$

where $\mathfrak{p}$ runs through the maximal ideals. ▲

The following is [Qui73b, Theorem 5, p. 113]:

Theorem Q.9.5 (Localization theorem)**.** *Let $\mathcal{A}$ be a (skeletally small) abelian category and let $\mathcal{B}$ be its Serre subcategory. Then the natural exact functors*

$$\mathcal{B} \hookrightarrow \mathcal{A} \to \mathcal{A}/\mathcal{B}$$

induce a homotopy fibration

$$BQ\mathcal{B} \to BQ\mathcal{A} \to BQ(\mathcal{A}/\mathcal{B}),$$

and hence a long exact sequence

$$\cdots \to K_{i+1}(\mathcal{A}/\mathcal{B}) \xrightarrow{\delta} K_i(\mathcal{B}) \xrightarrow{i_*} K_i(\mathcal{A}) \xrightarrow{p_*} K_i(\mathcal{A}/\mathcal{B}) \to \cdots \to K_0(\mathcal{B}) \to K_0(\mathcal{A}) \to K_0(\mathcal{A}/\mathcal{B}) \to 0$$

Let's deduce from the cited theorems the following result [Qui73b, Corollary p. 113]:

Proposition Q.9.6. *Let $\mathfrak{A}$ be a Dedekind domain with field of fractions F. Then there is a long exact sequence*

$$\cdots \to K_{i+1}(F) \to \coprod_{\mathfrak{p}\subset\mathfrak{A}} K_i(\mathfrak{A}/\mathfrak{p}) \to K_i(\mathfrak{A}) \to K_i(F) \to \cdots$$

where $\mathfrak{p}$ runs through maximal ideals.

Proof. We apply the localization theorem to the category $\mathcal{A} \stackrel{\text{def}}{=} \mathfrak{A}\text{-}\mathcal{M}od_{fg}$ of finitely generated $\mathfrak{A}$-modules and $\mathcal{B} \stackrel{\text{def}}{=} \mathfrak{A}\text{-}\mathcal{T}ors_{fg}$ its full subcategory of finitely generated torsion $\mathfrak{A}$-modules. As we observed in example Q.9.2, one has $K_i(\mathfrak{A}\text{-}\mathcal{M}od_{fg}) \cong K_i(\mathfrak{A})$. By example Q.8.5 the localization $\mathcal{A}/\mathcal{B}$ can be identified with the category of finite dimensional F-vector spaces, hence $K_i(\mathcal{A}/\mathcal{B}) \cong K_i(F)$. Finally, by example Q.9.4 we identify $K_i(\mathcal{B})$ with $\coprod_{\mathfrak{p}\subset\mathfrak{A}} K_i(\mathfrak{A}/\mathfrak{p})$. ∎

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