Classification of Barsotti-Tate groups after Breuil-Kisin

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Introduction

Let k be a perfect field of characteristic $p > 0$ and let $W := W(k)$ be its ring of Witt vectors. We consider a totally ramified extension K of degree e of the field of fractions $K_0 := W[1/p]$. Fix a uniformizer $\pi \in K$ and denote by $E(u)$ its minimal polynomial. The aim of this mémoire is to present a classification of Barsotti-Tate groups over the ring of integers $\mathscr{O}_K$ of K, following Marc Kisin's paper *Crystalline representations and F-crystals* [**Kis**].

Fix a positive integer h. A *Barsotti-Tate group* (or *p-divisible group*) over a scheme S_0 is an inductive system $G = \{(G_n, i_n)\}_n$, such that for each n:
(1) G_n is a finite commutative group over S_0 of order p^{hn},
(2) there is an exact sequence

$$0 \to G_n \xrightarrow{i_n} G_{n+1} \xrightarrow{p^n} G_{n+1}.$$

Barsotti-Tate groups over a scheme S_0 form a category $\mathrm{BT}(S_0)$. One should keep in mind the following example as an important motivation for the study of p-divisible groups. Given an abelian scheme X, the inductive system $\{(X(p^n), X(p^n) \to X(p^{n+1}))\}_n$ gives a Barsotti-Tate group $X(p^\infty)$, where $X(p^n)$ denotes the kernel of the multiplication by p^n on X. The group $X(p^\infty)$ encodes a lot of information about the abelian scheme X and has additional structures with respect to regular formal groups, such as a notion of duality.

A first classification of Barsotti-Tate groups was given by Jean Dieudonné [**Dieu**] in the case $S_0 = \operatorname{Spec} k$. To each Barsotti-Tate group G, we can associate a *Dieudonné module*, that is, a module $\mathbb{D}(G)$ over W, endowed with a Frobenius and a Verschiebung maps. In the paper *Groupes p-divisible sur les corps locaux* [**Fon2**], Jean-Marc Fontaine generalizes the theory by Dieudonné, obtaining a classification of p-divisible groups over $\operatorname{Spec} \mathscr{O}_K$ in the case $e < p - 1$.

Moreover, Alexander Grothendieck ([**Gro1**], [**Gro2**]) suggests that there should be a *crystalline Dieudonné theory* in order to classify p-divisible groups over any base scheme. Through the theory of fundamental extensions, Grothendieck points out that the Lie algebra of a Barsotti-Tate group provides a generalization of the notion of Dieudonné module. The fundamental observation is that one can associate to a p-divisible group G a *crystal* $\mathbb{D}(G)$, that is, a sheaf on the crystalline site which satisfies some rigidity conditions. The deformation theory by William Messing ([**Mess**]) provides a classification of p-divisible groups in terms of crystals.

In his paper *Schémas en groupes et groupes des normes* [**Br1**], Christophe Breuil suggests, conjecturally, a new classification for p-divisible groups over $\operatorname{Spec} \mathscr{O}_K$, for any ramification index e. Denote by $\mathfrak{S} = W[[u]]$ the ring of formal series in the indeterminate u and equip this object with a W-semi-linear Frobenius φ. Define the category $\mathrm{BT}^{\varphi}_{/\mathfrak{S}}$ of finite free $\mathfrak{S}$-modules $\mathfrak{M}$, equipped with an injective semi-linear map $\varphi : \mathfrak{M} \to \mathfrak{M}$ such that $\mathfrak{M}/(1 \otimes \varphi)(\mathfrak{M})$ is killed by $E(u)$.

THEOREM 0.0.0.1 (Breuil-Kisin). *There is an exact contravariant functor*

$$\mathrm{BT}^{\varphi}_{/\mathfrak{S}} \longrightarrow \mathrm{BT}(\mathscr{O}_K).$$

When $p > 2$ this functor is an equivalence of categories, when $p = 2$ it is an equivalence up to isogeny.

The strategy of Kisin in order to prove this result is to describe $\mathfrak{S}$-modules in terms of other, better known objects.

Consider the natural surjection $W[u] \twoheadrightarrow \mathscr{O}_K$: this extends to a surjection $W[u]\left[\frac{E(u)^i}{i!}\right]_{i\geq 1} \twoheadrightarrow \mathscr{O}_K$, and finally to a surjection

$$S \twoheadrightarrow \mathscr{O}_K,$$

where S denotes the p-adic completion of $W[u]\left[\frac{E(u)^i}{i!}\right]_{i\geq 1}$. Denote $\mathrm{Fil}^1 S = \mathrm{Ker}(S \twoheadrightarrow \mathscr{O}_K)$ and by φ the extension of the Frobenius of W on S. We define the category $\mathrm{BT}^{\varphi}_{/S}$, of finite free S-modules $\mathcal{M}$, equipped with an S-sub-module $\mathrm{Fil}^1\mathcal{M}$ such that $\mathrm{Fil}^1 S \cdot \mathcal{M} \subset \mathrm{Fil}^1\mathcal{M}$, the quotient $\mathcal{M}/\mathrm{Fil}^1\mathcal{M}$ is a free $\mathscr{O}_K$-module and there exists a φ-semilinear map $\varphi_1 : \mathrm{Fil}^1\mathcal{M} \to \mathcal{M}$ such that $\varphi^*(\mathrm{Fil}^1\mathcal{M}) \to \mathcal{M}$ is onto.

THEOREM 0.0.0.2 (Kisin). *There is an exact contravariant functor*

$$\begin{array}{ccc} \mathrm{BT}(\mathscr{O}_K) & \longrightarrow & \mathrm{BT}^{\varphi}_{/S} \\ G & \longmapsto & \mathbb{D}(G)(S) \end{array}$$

For $p > 2$ this is an equivalence of cateogories, for $p = 2$ it is an equivalence of categories up to isogeny.

The proof uses a deformation argument and it holds consistently on the theory by Grothendieck-Messing. The map

$$\begin{array}{ccc} \mathfrak{S} & \xrightarrow{\varphi} & S \\ u & \longmapsto & u^p \end{array}$$

defines a functor

$$\begin{array}{ccc} \mathrm{BT}^{\varphi}_{/\mathfrak{S}} & \longrightarrow & \mathrm{BT}^{\varphi}_{/S} \\ \mathfrak{M} & \longmapsto & \mathfrak{M} \otimes_{\mathfrak{S},\varphi} S \end{array}$$

and hence a functor

$$\mathrm{BT}^{\varphi}_{/\mathfrak{S}} \longrightarrow \mathrm{BT}(\mathscr{O}_K).$$

In order to construct an inverse, Kisin describes the category $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$, containing $\mathrm{BT}^{\varphi}_{/\mathfrak{S}}$, of finite free $\mathfrak{S}$-modules $\mathfrak{M}$, equipped with an injective semi-linear map $\varphi : \mathfrak{M} \to \mathfrak{M}$ such that $\mathfrak{M}/(1 \otimes \varphi)(\mathfrak{M})$ is killed by a power of $E(u)$. The first step is to relate the category $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ to algebraic objects over the field K.

Theorem 0.0.0.3. *There is a fully faithful functor*

$$\mathrm{MF}_K^{\varphi,N,\mathrm{Fil}_{\geq 0},ad} \longrightarrow \mathrm{Mod}_{/\mathfrak{S}}^{\varphi,N} \otimes \mathbb{Q}_p,$$

where $\mathrm{MF}_K^{\varphi,N,\mathrm{Fil}_{\geq 0},ad}$ *denotes the category of effective, admissible* (φ, N)*-modules with a filtration over* K *(see* (4.0.5) *for the definition).*

The proof is obtained by relating effective, filtered (φ, N)-modules over K with modules over the ring $\mathscr{O} \subset K_0[[u]]$ of rigid analytic functions on the unit disk with indeterminate u. In particular, the notion of admissibility is characterized through the theory of slopes by Kedlaya ([**Ked1**] and [**Ked2**]). By restricting the result to $\mathrm{BT}_{/\mathfrak{S}}^{\varphi}$, we obtain

Proposition 0.0.0.4. *There is an equivalence of categories*

$$\{\textit{admissible } \varphi\textit{-modules of BT-type}\} \xrightarrow{\simeq} \mathrm{BT}_{/\mathfrak{S}}^{\varphi} \otimes \mathbb{Q}_p,$$

where an admissible module D *over* K_0 *is said to be of Barsotti-Tate type if* $gr^i D_K = 0$ *for* $i \notin \{0, 1\}$.

The second step is to relate $\mathrm{Mod}_{/\mathfrak{S}}^{\varphi}$ to Galois representations by constructing an analogue of Fontaine's theory of (φ, Γ)-modules [**Fon1**], where the cyclotomic extension is replaced by *Breuil's extension* $K_\infty := \bigcup_{n\geq 1} K(\sqrt[p^n]{\pi})$.

Theorem 0.0.0.5. *There is a fully faithful functor*

$$\mathrm{Mod}_{/\mathfrak{S}}^{\varphi} \otimes \mathbb{Q}_p \xrightarrow{\textit{fully faithful}} Rep_{\mathbb{Q}_p}(G_{K_\infty}),$$

where $Rep_{\mathbb{Q}_p}(G_{K_\infty})$ *denotes the category of* p*-adic representations of the absolute Galois group* $G_{K_\infty} = \mathrm{Gal}(\overline{K}/K_\infty)$.

We obtain hence also a proof of a conjecture by Breuil [**Br2**]:

Proposition 0.0.0.6. *The functor*

$$Rep_{\mathbb{Q}_p}^{cris}(G_K) \longrightarrow Rep_{\mathbb{Q}_p}(G_{K_\infty}),$$

obtained by restricting the action of G_K *to* G_{K_∞}*, is fully faithful.*

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CHAPTER 1

Divided powers and the crystalline site

1.1. Grothendieck topologies, sheaves and sites

It is often useful to have a notion of *sheaf* on a given category $\mathscr{C}$. This is a particular contravariant functor from $\mathscr{C}$ to other categories such as *Sets* or *Ab*. The fundamental notion is that of *Grothendieck topology*, through which we interpret the objects of $\mathscr{C}$ as open sets of a topological space. It is useful to keep in mind, as a motivation, the category $\mathcal{T}_X$,with objects the open sets of the topological space X and morphisms the inclusions.

DEFINITION 1.1.0.7. *Fix a category $\mathscr{C}$. A* Grothendieck topology *T on $\mathscr{C}$ consists of the following data:*

(1) *A category $\mathscr{C}(T)$,*
(2) *A set $Cov(T)$ of families $\{\phi_i : U_i \to U \mid U, U_i \in Ob(\mathscr{C})\}_i$ of morphisms in $\mathscr{C}(T)$, called* coverings *satisfying:*
  - *If ϕ is an isomorphism, $\{\phi : U \to U\} \in Cov(T)$,*
  - *If $\{U_i \to U\}_i \in Cov(T)$ and $\{V_{ij} \to U_i\}_j \in Cov(T)$ for each i, then the composed family $\{V_{ij} \to U\}_{ij} \in Cov(T)$,*
  - *If $\{U_i \to U\}_i \in Cov(T)$ and $V \to U$ is any morphism in $\mathscr{C}(T)$, then the product $U_i \times_U V$ exists $\forall i$ and $\{U_i \times_U V \to V\}_i \in Cov(T)$.*

DEFINITION 1.1.0.8. *Let T be a Grothendieck topology and $\mathscr{D}$ a category with products (we will mainly consider Sets and Ab). A* pre-sheaf *on T with values in $\mathscr{D}$ is a contravariant functor $\mathscr{F} : T \to \mathscr{D}$. A sheaf $\mathscr{F}$ is a pre-sheaf such that for $\{U_i \to U\} \in Cov(T)$, the following sequence is exact*

$$\mathscr{F}(U) \to \Pi_i \mathscr{F}(U_i) \rightrightarrows \Pi_{i,j} \mathscr{F}(U_i \times_U U_j).$$

DEFINITION 1.1.0.9. *A category $\mathscr{C}$ with a choice of a Grothendieck topology is called a* site. *The category of sheaves on a given site is called* topos.

The $fppf$ site. On the category Sch/S of schemes over S we define the $fppf$ *site (fidèlement plate de présentation finie)*. For any U a scheme over S, define the coverings as the sets of families $\{f_i : U_i \to U\}_i$ such that f_i is flat and locally of finite presentation and $\bigcup_i f_i(U_i) = U$.

1.2. Divided powers

In the following section we define and list some results on divided powers, following as a reference [**BO**], Chapter 3. Fix A a commutative ring and $I \subseteq A$ an ideal. The notion of divided powers structure on A is introduced to give mathematical meaning to the symbol $\frac{x^n}{n!}$, even when $n!$ is not invertible in A. The main result is the construction of the P.D. envelope of an ideal.

Definition 1.2.0.10. *By* divided powers on I *we mean a collection of maps* $\{\gamma_i : I \to A\}_{i\geq 0}$ *satisfying the following properties:*

1. *for* $x \in I$ *we have* $\gamma_0(x) = 1$, $\gamma_1(x) = x$ *and* $\forall i \geq 1$, $\gamma_i(x) \in I$,
2. *if* $x, y \in I$ *then* $\gamma_k(x+y) = \Sigma_{i+j=k}\gamma_i(x)\gamma_j(y)$,
3. *for* $\lambda \in A$, $\gamma_i(\lambda x) = \lambda^i \gamma_i(x)$,
4. *for* $x \in I$, $\gamma_i(x)\gamma_j(x) = \frac{(i+j)!}{i!j!}\gamma_{i+j}(x)$,
5. $\gamma_p(\gamma_q(x)) = \frac{(pq)!}{p!(q!)^p}$.

We call (I, γ) *a P.D. ideal,* (A, I, γ) *a P.D. ring and* γ *a P.D. structure on* I.

Note that by points (a) and (d), we have $n!\gamma_n(x) = \gamma_1(x)^n = x^n$, and by point (c) we have $\gamma_i(0) = 0$, for $i > 0$. These observations lead to uniqueness of divisible powers in many cases.

Definition 1.2.0.11. *Let* (A, I, γ) *and* (B, J, δ) *be P.D. rings. A* morphism *of P.D. rings* $f : (A, I, \gamma) \to (B, J, \delta)$ *is a ring homomorphism* $f : A \to B$ *such that* $f(I) \subseteq J$ *and such that* $f(\gamma_n(x)) = \delta_n(f(x))$, $\forall x \in I, n \in \mathbb{N}$. *We say that* J *is a* sub P.D. ideal *of* I *if* $\gamma(x) \in J$ *for every* $x \in J$.

Some examples. 1. Any $\mathbb{Q}$-algebra A has a (unique) natural structure of P.D. ring, given by $\gamma_n(x) = \frac{x^n}{n!}, x \in A$.

2. [*Important*] Consider a discrete valuation ring (A, π, k) of mixed characteristic $(0, p)$. It is not always true that π admits a P.D. structure (though, if it exists, it is unique). Denote by e the absolute ramification index, that is, the integer e such that $p = \pi^e u$ (u an invertible). Then (π) admits a P.D. structure if and only if $e \leq p-1$. This is true since $\gamma_n(\pi) = \pi^n/n! \in (\pi)$ for all $n \geq 1$ if and only if $v(\gamma_n(\pi)) \geq 1$, that is, if and only if $p - 1 - e \geq 0$.

3. Suppose $mA = 0$, for $m \in \mathbb{N}_{>0}$. If a P.D. structure exists on an ideal $I \triangleleft A$, then $x^n = n!\gamma_n(x) = 0$ for $n \geq m$, that is I is a nil ideal. On the other hand, if $(m-1)!$ is invertible in A and $mA = 0$, then I has a (not unique) P.D. structure given by $\gamma_n(x) = x^n/n!$ if $n < m$ and $\gamma_n(x) = 0$ for $n \geq m$. In particular if $I^2 = 0$, it has a P.D. structure, with $\gamma_n(x) = 0$ for $n \geq 2$.

1.2.1. Some results on P.D. rings.

The graded algebra $\Gamma_A(M)$. We define a functor $\Gamma_A : \mathrm{Mod_A} \to \{\text{A-algebras}\}$ such that for an A-module M, $(\Gamma_A(M), \Gamma_A^+(M))$ has a divided powers structure, where $\Gamma_A^+(M)$ denotes the augmentation ideal.

Definition 1.2.1.1. *Given an* A*-module* M*,we define* $\Gamma_A(M)$ *to be the* A*-algebra generated by elements* $x^{[n]}$ *with relations*

1. $x^{[0]} = 1$,
2. $(\lambda x)^{[n]} = \lambda^n x^{[n]}$, *for* $\lambda \in A$ *and* $x \in M$,
3. $x^{[n]}x^{[m]} = x^{[n+m]}$, *for* $x \in M$,
4. $(x+y)^{[n]} = x^{[n]} + y^{[n]} + \Sigma_{i=1}^{n-1} x^{[n-i]} + y^{[i]}$, *for* $x, y \in M$.

To see the construction in a more concrete way, one can see $\Gamma_A(M)$ as the quotient $G_A(M)/I_A(M)$ where $G_A(M)$ is the graded polynomial algebra of indeterminates $\{(x, n) | x \in M, n \in \mathbb{N}\}$ and $I_A(M)$ is an ideal realizing the relations above. This shows in particular that $\Gamma_A(M)$ is graded, since both $G_A(M)$ and $I_A(M)$ are.

In particular, $\Gamma^0_A(M) \simeq A$ and $\Gamma^1_A(M) \simeq M$. There are several compatibilities and functorial properties characterizing this object.

PROPOSITION 1.2.1.2. (1) *The functor Γ is compatible with base change $A \to A'$, that is*

$$A' \otimes_A \Gamma_A(M) \simeq \Gamma_{A'}(A' \otimes_A M).$$

(2) *If $\{M_\lambda\}$ is a direct system of A-Modules, then*

$$\varinjlim \Gamma_A(M_\lambda) \simeq \Gamma_A(\varinjlim M_\lambda).$$

(3) *For M, N A-Modules, we have*

$$\Gamma_A(M) \otimes \Gamma_A(N) \simeq \Gamma_A(M \oplus N).$$

THEOREM 1.2.1.3 (P.D. structure). *There is a unique structure of divided powers on the augmentation ideal $\Gamma^+(M) \simeq \oplus_{n\geq 1}\Gamma^n(M)$ such that $\gamma_n(x) = x^{[n]}$, for every $x \in M$ and for every $n \geq 1$.*

PROOF. [**BO**, Appendix]. □

The structure of a P.D. ring.

LEMMA 1.2.1.4. *Let (A, I, γ) be a P.D. ring, J an ideal of A. There is a (unique) P.D. structure $\overline{\gamma}$ on $\overline{I} \subseteq A/J$, together with a P.D. morphism $(A, I, \gamma) \to (A/J, \overline{I}, \overline{\gamma})$ if and only if $J \cap I$ is a sub-P.D. ideal of I in A.*

LEMMA 1.2.1.5. *Let (A, I, γ) be a P.D. ring, let $S \subseteq I$ be a subset. The ideal J generated by S is a sub-P.D. ideal if and only if for every n, $\gamma_n(s) \in J$ for every $s \in S$.*

The *direct limit* is defined in the category of P.D-rings. If $\{A_i, I_i, \gamma_i\}$ is a direct system of P.D. rings, then $\{\varinjlim A_i, \varinjlim I_i\}$ has a unique P.D. structure γ such that for every i there is a P.D. morphism $\{A_i, I_i, \gamma_i\} \to \{\varinjlim A_i, \varinjlim I_i, \gamma\}$.

PROPOSITION 1.2.1.6. *Let I, J be ideals of A with P.D. structures, respectively, γ and δ. Then IJ is always a sub-P.D. ideal of both I and J and the two P.D. structures γ and δ agree on IJ.*

COROLLARY 1.2.1.7. *In particular we have that for every n, I^n us a sub-P.D. ideal of I.*

DEFINITION 1.2.1.8. *Let (A, I, γ) be a P.D. ring and $N \geq 1$. We define $I^{[N]}$ as the ideal generated by $\{\gamma_{i_1}(x_1), \cdots, \gamma_{i_k}(x_k) | \Sigma i_j \geq N$ and $x_j \in I\}$.*

Note that $I^{[N]} \subseteq I$ is a sub-P.D. ideal and $I^{[N]} I^{[M]} \subseteq I^{[N+M]}$.

DEFINITION 1.2.1.9 (P.D. Nilpotent ideal). *I is* nilpotent *if $I^{[N]} = 0$ for some N.*

DEFINITION 1.2.1.10 (Extension of a P.D. structure). *Let (A, I, γ) be a P.D. ring and $f : A \to B$ be an A-algebra. We say that γ* extends *to B if there is a P.D. structure $(B, IB, \overline{\gamma})$ such that $(A, I, \gamma) \to (B, IB, \overline{\gamma})$ is a P.D. morphism, that is $f(\gamma_n(x)) = \overline{\gamma}_n(f(x))$ for every $x \in I, n \in \mathbb{N}$.*

Note that if the extension exists, this is unique. Moreover the definition is equivalent to the following statement: there is a P.D. structure on B with a P.D. morphism $(A, I, \gamma) \to (B, J, \delta)$, indeed IB is a sub-P.D. ideal of J.

Note that the extension does not exist in general. There is however a particular case in which it always.

Proposition 1.2.1.11. *If I is principal, γ does always extend.*

Proof. If $I = (t)$, we can define $\overline{\gamma}_n(tb) = b^n\gamma_n(t)$, and this satisfies $f(\gamma_n(at)) = f(a^n\gamma_n(t)) = f(a)^n f(\gamma_n(t)) = \overline{\gamma}(f(a)t)$, hence the conclusion. $\square$

Definition 1.2.1.12 (Compatibility). *Let (A, I, γ) be a P.D. ring and B with a P.D. structure (J, δ). We say that γ and δ are* compatible *if the following equivalent conditions hold:*

(1) *γ extends to δ and $\overline{\gamma} = \delta$ on $IB \cap J$.*
(2) *The ideal $K = IB + J$ has a (unique) P.D. structure $\overline{\delta}$ such that $(A, I, \gamma) \to (B, K, \delta)$ and $(B, J, \delta) \to (B, K, \delta)$ are P.D. morphisms.*
(3) *There is an ideal $K \supseteq IB + J$ with a P.D. structure δ' such that $(A, I, \gamma) \to (B, K', \delta')$ and $(B, J, \delta) \to (B, K', \delta')$ are P.D. morphisms.*

1.2.2. The P.D. envelope of an ideal. We fix a P.D. algebra (A, I, γ) and consider compatible P.D. structures.

Theorem 1.2.2.1. *Let (A, I, γ) be a P.D. algebra, B an A-algebra and J an ideal of B. There exists a B-algebra $\mathcal{D}_{B,\gamma}(J)$ with a P.D. ideal $(\overline{J}, [\ ])$, such that $J\mathcal{D}_{B,\gamma}(J) \subseteq \overline{J}$ and satisfying to the following* universal property*: for any B-algebra C with a P.D. ideal (K, δ) such that K contains the image of J and δ is compatible with γ there exists a unique P.D. morphism $(\mathcal{D}_{B,\gamma}, \overline{J}, [\ ]) \to (C, K, \delta)$ making the following diagram commutative:*

$$\begin{tikzcd}
 & (\mathcal{D}_{B,\gamma}, \overline{J}, [\ ]) \arrow[dr] & \\
(B, J) \arrow[ur, "\psi"] \arrow[rr] & & (C, K, \delta) \\
 & (A, I, \gamma) \arrow[ul, "f"] \arrow[ur, "g"'] &
\end{tikzcd}$$

Sketch of Proof. The proof is constructive and consists in generalizing the following first case:

Suppose $f(I) \subseteq J$, that is the map $f \circ \psi$ is P.D. compatible. We may consider the P.D. algebra $(\Gamma_B(J), \Gamma_B^+(J), [\ ])$ and the ideal $\mathcal{J}$ generated by elements of type:

(1) $\phi(x) - x$ for $x \in J$,
(2) $\phi(f(y))^{[n]} - f(\gamma_n(y))$ for $y \in I$.

The key point is that $\mathcal{J} \cap \Gamma_B^+(J)$ is a sub P.D. ideal of $\Gamma_B^+(J)$. This tells us that the image $\overline{J}$ of $\Gamma_B^+(J)$ in the quotient $\mathcal{D}_{B,\gamma} = \Gamma_B(J)/\mathcal{J}$ has P.D. structure (by abuse of notation we will denote this $[\ ]$ as well). In particular, condition (a) for $\mathcal{J}$ tells us that $J\mathcal{D} \subseteq \overline{J}$, while condition (b) tells us $[\ ]$ is compatible with γ.

The general case is proved by setting $J_1 = J + IB$, $\mathcal{D}_{B,\gamma}(J) = \mathcal{D}_{B,\gamma}(J_1)$ and $\overline{J} \subseteq \overline{J}_1$ the sub P.D. ideal generated by J. See [**BO**, Theorem 3.19] $\square$

Remark. *1. If the map of A-algebras $A \to B$ factors through some A' and γ extends to some γ', then $\mathcal{D}_{B,\gamma}(J) = \mathcal{D}_{B,\gamma'}(J)$.*

2.[Extension of scalars] *Suppose that $(A, I, \gamma) \to (A', I', \gamma')$ is a surjective P.D. morphism and $B' = A' \otimes_A B$, $J' = JB$. Then we have a canonical isomorphism*

$$A' \otimes_A \mathcal{D}_{B,\gamma}(J) \to \mathcal{D}_{B',\gamma'}(J').$$

3. If γ extends to B/J we have that $B/J \simeq \mathcal{D}_{B,\gamma}(J)/\overline{J}$.

4. If $B \to B'$ is flat, then $\mathcal{D}_{B',\gamma}(JB') \simeq B' \otimes_B \mathcal{D}_{B,\gamma}(J)$.

5. If B is a flat A-algebra, γ extends to B.

1.2.3. P.D. ringed spaces.

Definition 1.2.3.1 (P.D. sheaf). *Let X be a scheme, $\mathcal{A}$ a quasi-coherent $\mathscr{O}_X$-algebra and $\mathcal{I}$ a quasi-coherent ideal. We say that $(\mathcal{A}, \mathcal{I})$ is a* P.D. sheaf *if there is an affine covering $\{U_i\}_i$ of X such that $\{\mathcal{A}(U_i), \mathcal{I}(U_i)\}$ is a P.D. ring for every i.*

This construction is preserved by the meaningful maps of sheaves: given a map of topological spaces $f : X \to Y$, $(f_*\mathcal{A}, f_*\mathcal{I}, f_*\gamma)$ is a sheaf of P.D. rings on Y and given a P.D. sheaf on Y, $(\mathcal{B}, \mathcal{J}, \delta)$, we have that $(f^{-1}(\mathcal{B}), f^{-1}(\mathcal{J}), f^{-1}(\delta))$ is a P.D. sheaf on X.

Definition 1.2.3.2. *A* P.D. ringed space *is a pair $(X, (\mathcal{A}, \mathcal{I}, \gamma))$ where $(X, \mathcal{O}_X)$ is a ringed space and $(\mathcal{A}, \mathcal{I}, \gamma)$ is a P.D. sheaf. Given two P.D. ringed spaces $(X, (\mathcal{A}, \mathcal{I}, \gamma))$ and $(Y, (\mathcal{B}, \mathcal{J}, \delta))$ a map P.D. ringed spaces is a map of topological spaces*

$$f : X \to Y,$$

together with a map of sheaves

$$(\mathcal{B}, \mathcal{J}, \delta) \to (f_*\mathcal{A}, f_*\mathcal{I}, f_*\gamma).$$

Let (A, I, γ) be a P.D.-algebra. By Remark (1.2.2) γ extends to any flat A-algebra, therefore, for any element $f \in A$, the localization (A_f, I_f) has P.D. structure and the localization map $(A, I) \to (A_f, I_f)$ is a P.D. morphism. We obtain hence a sheaf of P.D. rings on $\operatorname{Spec} A$. Moreover, If $X = \operatorname{Spec} A$ and $\mathcal{I} \subseteq \mathcal{O}_X$ is a quasi-coherent sheaf of ideals, we have that P.D. structures on $\mathcal{I}$ correspond to P.D. structures on the global sections $H^0(X, \mathcal{I})$. Similarly, P.D. morphisms $\operatorname{Spec}(\mathcal{A}, \mathcal{I}, \gamma) \to \operatorname{Spec}(\mathcal{B}, \mathcal{J}, \delta)$ can be identified with P.D. morphisms $(B, J, \delta) \to (A, I, \gamma)$. All this follows from the fact that the functor$\tilde{}$: A*–modules* $\to$ {Quasi-Coherent sheaves of $\operatorname{Spec} A$} is an equivalence of categories.

The P.D. envelope structure extends to the schematic case.

Proposition 1.2.3.3. *Let S be a scheme with P.D. structure $(\mathcal{O}_S, \mathcal{I}, \gamma)$ and let X be an S-scheme. If $\mathcal{B}$ is a quasi-coherent $\mathcal{O}_X$-Algebra and $\mathcal{J} \subseteq \mathcal{B}$ is a quasi-coherent ideal, then $\mathcal{D}_{\mathcal{B},\gamma}(\mathcal{J})$ is a quasi-coherent $\mathcal{O}_X$-Algebra.*

Let now $(S, \mathcal{I}, \gamma)$ be a P.D. scheme and consider a closed immersion of S-schemes

$$i : X \hookrightarrow Y,$$

defined by the ideal $\mathcal{J}$ of $\mathcal{O}_Y$. We may define hence $\mathcal{D}_{\mathcal{O}_Y,\gamma}(\mathcal{J})$ and finally a scheme

$$D_{X,\gamma}(Y) = \mathrm{Spec}(\mathcal{D}_{\mathcal{O}_Y,\gamma}).$$

Moreover *if* γ extends to X, that is, to $\mathcal{O}_X$ (for example if $\mathcal{O}_X$ is flat over $\mathcal{O}_S$), it follows by Remark (1.2.2) that $\mathcal{O}_X \simeq \mathcal{D}_{\mathcal{O}_Y,\gamma}/\overline{\mathcal{J}}$. This means in particular that i factors through a closed immersion

$$j : X \hookrightarrow D_{X,\gamma},$$

with kernel $\overline{\mathcal{J}}$, a P.D. ideal (j is a *P.D. immersion*). Moreover j is universal in the following sense:

if $i' : X' \to Y'$ is a P.D. immersion which is compatible with γ and following diagram holds, we get a unique map:

$$\begin{tikzcd}
X' \arrow[r] \arrow[d, "i'"'] & X \arrow[d, "i"'] \arrow[dr, "j"] & \\
Y' \arrow[r] \arrow[rr, bend right] & Y & D_{X,\gamma}(Y) \arrow[l]
\end{tikzcd}$$

Example. Let k be a prefect field of characteristic $p > 0$ and consider the ring of Witt vectors $W = W(k)$. This admits a P.D. structure, since it is an absolutely unramified discrete valuation ring of characteristic $(0, p)$. Consider now a totally ramified extension

(1.2.1)
$$\begin{tikzcd}
\mathscr{O}_K \arrow[r, dash] \arrow[d, dash] & K \arrow[d, dash] \\
W \arrow[r, dash] & W[1/p]
\end{tikzcd}$$

and fix a uniformizer $\pi \in K$, with minimal Eisenstein polynomial $E(u) \in W[u]$ and uniformizer π. We have a closed immersion

$$Spf(\mathscr{O}_K) \hookrightarrow Spf(W[u]).$$

Take now the P.D. envelope of $W[u]$ with respect to the ideal $(E(u))$, $W[u]\Big[\frac{E(u)^i}{i!}\Big]_{i\geq 1}$ and denote by S its p-adic completion. Hence the closed immersion above factors through the diagram

$$\begin{tikzcd}
Spf(\mathscr{O}_k) \arrow[rr, hook] \arrow[dr, hook] & & Spf(W[u]) \\
& Spf(S) \arrow[ur] &
\end{tikzcd}$$

1.3. The crystalline site

P.D. schemes are the fundamental notion in order to define the *crystalline site*, which is the good environment for introducing the notion of *crystal.* The idea is to replace Zariski open stes over a scheme S with infinitesimal *thickenings* of Zariski open sets, equipped with divided powers structures. As a working assumption, we suppose that all schemes are killed by a power of a fixed prime p.

Fix a base P.D. scheme $(S, \mathcal{I}, \gamma)$ and consider a scheme X to which γ extends. This means explicitly that the $\mathcal{O}_S$-algebra $\mathcal{O}_X$ has a P.D. structure $\overline{\gamma}$ such that

$$(\mathcal{O}_S, \mathcal{I}, \gamma) \to (\mathcal{O}_X, \mathcal{I}\mathcal{O}_X, \overline{\gamma})$$

is a P.D. morphism.

The *objects* of the Crystalline site $\mathrm{Cris}(X/S)$ of X over S are pairs $(U \subseteq T, \delta)$ such that U is an open subset of X, $U \hookrightarrow T$ is a closed immersion defined by an ideal $\mathcal{J}$ and δ is a P.D. structure on $\mathcal{J}$ compatible with γ. The working assumption implies that the ideal $\mathcal{J}$ is nilpotent and therefore $U \hookrightarrow T$ is a homeomorphism. We call the object $(U \subseteq T, \delta)$ a *S-P.D. thickening of* U. The assumption "γ extends to X" tells us that given a Zariski open set U, the set of thickenings of U is never empty. Indeed, for $T = U$ and $\mathcal{J} = (0)$ we have that γ is compatible with the trivial P.D. structure.

A *morphism* $T \to T'$ of the Crystalline Site is a a commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & T \arrow[d] \\ U' \arrow[r] & T' \end{tikzcd}$$

where $U \hookrightarrow U'$ is an inclusion in the Zariski sense and $T \to T'$ is an S-P.D. morphism $(T, \mathcal{J}, \delta) \to (T', \mathcal{J}', \delta')$.

A *covering family* of an object $(U \hookrightarrow T, \delta)$ of the crystalline site is a collection of morphisms $\{T_i \to T\}_i$ such that for all $i \in I$, $T_i \to T$ is an open immersion and $\bigcup T_i = T$.

One standard situation to keep in mind is $S = S_n = \mathrm{Spec}(W(k)/p^n W(k))$, where k is a perfect field and X a scheme over k. Note that the nilpotency assumption implies that X is an S_n scheme for some n.

Example. As in the example above, we consider the field extension (1.2.1), and surjections

$$\begin{array}{ccc} W[u]/u^i & \twoheadrightarrow & \mathscr{O}_K/\pi^i \\ u & \mapsto & \pi \end{array}$$

for $1 \leq i \leq e$, where e is the ramification index of π in K. These maps have kernel $p \cdot W[u]/u^i$ equipped with divided powers. Therefore the inclusion

$$\mathrm{Spec}(\mathscr{O}_K/\pi^i) \hookrightarrow \mathrm{Spec}(W[u]/u^i)$$

is a P.D. thickening in the crystalline site $\mathrm{Cris}((\mathscr{O}_K/\pi^i)/W)$.

A sheaf of sets $\mathscr{F}$ on the crystalline site can be described as the data of: for every element $(U, T, \delta) \in \mathrm{Cris}(X/S)$, a sheaf $\mathscr{F}_{(U,T,\delta)}$ on T (where $\mathscr{F}_{(U,T,\delta)}$ denotes the evaluation of $\mathscr{F}$ at $(U \hookrightarrow T, \delta)$) and for every map in the crystalline site

$$\begin{array}{ccc} U' & \longrightarrow & T' \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ U & \longrightarrow & T \end{array}$$

a morphism of sheaves

$$v^{-1}\mathscr{F}_{(U,T,\delta)} \longrightarrow \mathscr{F}_{(U',T',\delta')},$$

where v^{-1} is the pull-back of a sheaf of sets, together with the cocycle condition.

Examples. 1. The *structural sheaf* $\mathscr{O}_{X/S}$ on $\mathrm{Cris}(X/S)$ is defined by: for every (U, T, δ),

$$(\mathscr{O}_{X/S})_{(U,T,\delta)} = \mathscr{O}_T.$$

2. The sheaf $i_*(\mathscr{O}_S)$ defined by

$$(i_*(\mathscr{O}_S))_{(U,T,\delta)} = \mathscr{O}_U.$$

3. The sheaf $\mathscr{I}_{X/S}$ defined by

$$(\mathscr{I}_{X/S})_{(U,T,\delta)} = \mathrm{Ker}(\mathscr{O}_T \to \mathscr{O}_U).$$

There is an exact sequence

$$0 \to \mathscr{I}_{X/S} \to \mathscr{O}_{X/S} \to i_*(\mathscr{O}_S) \to 0.$$

The notion of crystal was introduced by Grothendieck as a sheaf on the crystalline site, which is "rigid" and "grows", as he explains in a letter to John Tate. Even though the definition is more general, we give here the definition of crystal only for sheaves of $\mathscr{O}_{X/S}$-modules.

DEFINITION 1.3.0.4. *A* crystal *of $\mathscr{O}_T$-modules is a sheaf $\mathscr{F}$ of $\mathscr{O}_{X/S}$-modules such that for every morphism on* $\mathrm{Cris}(X/S)$ *the map of $\mathscr{O}_{T'}$-modules*

$$u^*(\mathscr{F}_{(U,T,\delta)}) \xrightarrow{\simeq} \mathscr{F}_{(U',T',\delta')},$$

is an isomorphism, where u^* is the pull-back of $\mathscr{O}_{T'}$-modules.

The structural sheaf is a crystal.

CHAPTER 2

Barsotti-Tate groups and deformation theory

2.1. Group schemes

Definition 2.1.0.5. *Fix a base scheme S. We say that $G \to S$ is an S-*group *if it has group structure as an object in the category Sch/S. Explicitly, this means that there are S-maps*

$$m : G \times_S G \to G \text{ multiplication},$$
$$i : G \to G \text{ inversion},$$
$$e : S \to G \text{ neutral element},$$

satisfying the group axioms. We say that a group scheme is commutative *if the commutativity is satisfied by these maps.*

Namely, a group scheme over S corresponds to a contravariant functor from the category of schemes over S to the category of groups.

Basic examples. 1. The *additive group scheme* $\mathbb{G}_a$, corresponding to the additive group structure underlying the affine line.

2. The *multiplicative group scheme* $\mathbb{G}_m$, corresponding to the multiplicative group structure underlying the affine line without the origin.

Remark. *We will always use commutative groups schemes and therefore all the results will refer to these (even though some of them might be true in general).*

A fundamental role in the construction of p-divisible groups is played by the *multiplication by n* map. For a group scheme $G \to S$ we denote this map

$$[n]_G : G \longrightarrow G,$$

and denote its kernel by $G(n)$.

Definition 2.1.0.6 (Cartier dual). *Let G be a commutative group. We define its dual as the group of characters*

$$G^* := \mathrm{Hom}_{GrSch/S}(G, \mathbb{G}_m).$$

Definition 2.1.0.7. *A* finite flat group scheme *is a commutative group scheme $f : G \to S$, such that the structural morphism is finite and flat and such that $f_*(\mathscr{O}_G)$ is a locally free $\mathscr{O}_S$-module of locally constant rank $r > 0$ (note that if S is noetherian this condition is always verified).*

Examples. 1. The *p^n-th roots of unity* $\mu_{p^n} = \{x \in \mathbb{G}_m \mid x^{p^n} = 1\} \subset \mathbb{G}_m$.

2. Its dual $\mathbb{Z}/p^n\mathbb{Z}$.

3. The *p^n-th roots of zero* $\alpha_{p^n} = \{x \in \mathbb{G}_a \mid x^{p^n} = 0\}$.

Immersion in the fppf topos: Recall that we may interpret the notion of group scheme *à la Yoneda*, that is, as a functor $F : Sch/S \to Groups$. Consider the $fppf$ site over S defined above. Note that for G a group scheme over S, the coverings $\{G_\alpha \to G\}_\alpha$, such that the G_α are group schemes, respect the group structure. If the functor F is representable and the map $\coprod_\alpha G_\alpha \to G$ is faithfully flat, of finite presentation and quasi-finite, then the sequence of groups

$$F(G) \to \prod_\alpha F(G_\alpha) \rightrightarrows \prod_{\beta,\gamma} F(G_\beta \times_G G_\gamma)$$

is exact. That is, an abelian group scheme G over S can be seen as a sheaf of groups on the $fppf$ site Sch/S.

2.1.1. The Frobenius and Verschiebung maps. Fix S a scheme of characteristic $p > 0$. The Frobenius on $\operatorname{Spec} \mathbb{F}_p$ induces on S a Frobenius $F_S : S \to S$, which is the identity topologically and which sends a section s to s^p. Consider now an S-scheme X; clearly we have also a Frobenius $F_X : X \to X$. We define a scheme $X^{(p)}$ through the cartesian diagram

$$\begin{tikzcd} X^{(p)} \arrow[r] \arrow[d] & X \arrow[d] \\ S \arrow[r, "F_S"'] & S \end{tikzcd}$$

Note moreover that the Frobenius morphisms $F_X : X \to X$ and $F_S : S \to S$ commute through the structural map $X \to S$. Therefore we may define a map $F_{X/S} : X \to X^{(p)}$, called the *relative Frobenius*, making the following diagram commutative

$$\begin{tikzcd} X \arrow[dr, "F_{X/S}"] \arrow[drr, bend left, "F_X"] \arrow[ddr, bend right] & & \\ & X^{(p)} \arrow[r] \arrow[d] & X \arrow[d] \\ & S \arrow[r, "F_S"'] & S \end{tikzcd}$$

Remark. *This diagram respects the structure of group schemes, that is, if X is a group scheme, then $F_{X/S} : X \to X^{(p)}$ is a morphism of group schemes.*

We define now a map $V_{X/S} : X^{(p)} \to X$, with the same functorial property described in the previous Remark. This is not trivial and it follows from a theorem by Lazard under some non-restrictive hypotheses on X. In this case we obtain commutative diagrams

$$\begin{tikzcd} X \arrow[d, "F_{X/S}"'] \arrow[dr, "{[p]_X}"] & \\ X^{(p)} \arrow[r, "V_{X/S}"'] & X \end{tikzcd} \qquad \begin{tikzcd} & X^{(p)} \arrow[dl, "{[p]_{X^{(p)}}}"'] \arrow[d, "V_{X/S}"] \\ X^{(p)} & X \arrow[l, "F_{X/S}"] \end{tikzcd}$$

2.2. Barsotti-Tate groups

Fix a base scheme S and consider a commutative $fppf$ sheaf of groups G on the site Sch/S such that $p^nG = (0)$. For the following results we will follow [**Mess**], Chapter 1.

LEMMA 2.2.0.1. *The following conditions are equivalent:*

(1) *G is a flat $\mathbb{Z}/p^n\mathbb{Z}$-module,*
(2) *for $i = 0, \dots, n-1$ we have* $\mathrm{Ker}([p]_G^{n-i}) = \mathrm{Im}([p]_G^i)$.

PROOF. [**Mess**, Lemma (1.1)]. □

The first condition tells us in particular that, with respect to the filtration "powers of p", we have

$$(\mathbb{Z}/p^n\mathbb{Z})^\bullet \otimes_{\mathbb{F}_p} (G)^0 \simeq (G)^\bullet.$$

DEFINITION 2.2.0.2 (Truncated Barsotti-Tate group). *For $n \geq 2$ a* truncated Barsotti-Tate group *of level n is an S-group such that*

(1) *G is a finite, locally-free group scheme,*
(2) *G satisfies one of the equivalent conditions of Lemma* (2.2.0.1).

We define a Barsotti-Tate group of level 1 *as a group G satisfying:*

(1) *G is finite, locally free and killed by p,*
(2) *if $S_0 = Var(p \cdot 1_S)$ and $G_0 = G \times_S S_0$,* $\mathrm{Im}(V_{G_0/S_0}) = \mathrm{Ker}(F_{G_0/S_0})$ *and* $\mathrm{Im}(F_{G_0/S_0}) = \mathrm{Ker}(V_{G_0/S_0})$, *where $F_{G_0/S_0} : G_0 \to G_0^{(p)}$ and $V_{G_0/S_0} : G_0^{(p)} \to G_0$ are the Frobenius and the Verschiebung maps respectively.*

Recall that for G a group scheme, we denote $G(p^n) = \mathrm{Ker}([p]_G^n)$. We have the following results for the groups $G(p^n)$:

LEMMA 2.2.0.3. (1) *If $G(p^n)$ is a flat $\mathbb{Z}/p^n\mathbb{Z}$-module, then it is finite, locally-free if and only if $G(p)$ is, and then consequently every $G(p^i)$ is.*
(2) *If $G(p^n)$ is finite, locally-free then $[p]_G^i : G(p^n) \to G(p^{n-i})$ is an epimorphism if and only if it is faithfully flat.*

PROOF. [**Mess**, Lemma (1.5)]. □

DEFINITION 2.2.0.4.

- *We say that G is of p-*torsion *if $\varinjlim G(p^n) = G$.*
- *We say that G is p-divisible if $p \cdot \mathrm{id}_G : G \to G$ is an epimorphism.*

DEFINITION 2.2.0.5 (Barsotti-Tate group). *We say that G is a* Barsotti-Tate group *if it satisfies the following conditions*

(1) *G is of p-torsion,*
(2) *G is p-divisible,*
(3) *$G(p)$ is a finite, locally-free group scheme.*

Denote by $\mathrm{BT}(S)$ the category of Barsotti-Tate groups over S with morphisms the homomorphisms of S-groups.

LEMMA 2.2.0.6. *Let G be a Barsotti-Tate group. For $n \geq 2$ the $G(p^n)$ are truncated Barsotti-Tate groups and we have an exact sequence*

$$0 \longrightarrow G(p^{n-i}) \longrightarrow G(p^n) \xrightarrow{[p]_{G(p^n)}^{n-i}} G(p^i) \longrightarrow 0. \tag{2.2.1}$$

Proof. Note that $G(p^n) = G(p^{n+1})(p^n)$ and that for any $0 \le i \le n$ the map $[p]_G^{n-i}$ induces an epimorphism $G(p^n) \xrightarrow{[p]_{G(p^n)}^{n-i}} G(p^i)$, hence the exact sequence (2.2.1). This tells us that the equivalent conditions of Lemma (2.2.0.1) are satisfied. Moreover, $G(1)$ is by definition finite and locally-free, hence we conclude by Lemma (2.2.0.3). $\square$

Remark. *We point out that the definition* after Tate [**Tate**] *given in the Introduction coincides with Definition* (2.2.0.5), after Grothendieck. *Indeed, suppose that there is a directed system of groups $\{G(p^n)\}$ such that*
a) the $G(p^n)$ are finite and locally-free,
b) $G(p^n) = G(p^{n+1})(p^n)$,
c) there exists a locally constant function h of S such that the rank of the fiber of $G(p^n)$ at $s \in S$ is $p^{nh(s)}$.
Then, by [**E.G.A.IV**, Criterium for flatness by fibers], *we have that $G(p^n) \xrightarrow{p^{n-i}} G(p^i)$ is faithfully flat, hence an epimorphism, that is, there is an exact sequence*

$$0 \to G(p^{n-i}) \to G(p^n) \xrightarrow{p^{n-i}} G(p^i) \to 0,$$

and hence $G := \varinjlim_n G(p^n)$ is a Barsotti-Tate group.
On the other hand, by [**GA**] *we have that, given a Barsotti-Tate group (à la Grothendieck), there exists a locally constant function h on S such that the rank of $G(p^n)$ at $s \in S$ is $p^{nh(s)}$, for every $n \ge 1$. By this consideration, together with Lemma* (2.2.0.6), *we conclude.*

Lemma 2.2.0.7 (Functoriality properties). *If $f : S' \to S$ is a morphism of schemes and $G \in \mathrm{BT}(S)$ then $f^*(G) \in \mathrm{BT}(S')$.*

Note that $\mathrm{BT}(S)$ admits a notion of duality. Consider G a Barsotti-Tate group. From the exact sequence (2.2.1) we see that the family of Cartier duals $G(p^n)^*$ of $G(p^n)$, together with maps $[p]^* : G(p^n)^* \to G(p^{n+1})^*$ give us a Barsotti-Tate group G^* of G.

Examples of p-divisible groups. 1. $\mu_{p^\infty} := \varinjlim_n \mu_{p^n}$, where μ_{p^n} denotes the group of p^n-th roots of unity.

2. Its dual $\mathbb{Q}_p/\mathbb{Z}_p = \varinjlim_n \mathbb{Z}/p^n\mathbb{Z}$.

3. Let X be an abelian scheme. If we denote by $X(p^n)$ the multiplication by p^n in X we have that $X(p^\infty) = \varinjlim_n X(p^n)$ is a p-divisible group.

Fix a field K of characteristic 0 and let $\overline{K}$ be its algebraic closure. Denote by G_K the absolute Galois group $\mathrm{Gal}(\overline{K}/K)$. The following definition is a fundamental step for linking representations of the Galois group G_K to p-divisible groups.

Definition 2.2.0.8. *Given a p- divisible group G over K, we define the* Tate module

$$T_p(G) := \varprojlim G(p^n)(\overline{K}),$$

where the limit is taken over the projective system

$$G(p^{n+1}) \xrightarrow{[p]_G} G(p^n).$$

The Tate module encodes somehow all the knowledge about the generic fiber $G \otimes K$. Note that G_K acts on the $\mathbb{Z}_p$-module $T_p(G)$.The following functoriality property is proved in [**Tate**].

Theorem 2.2.0.9 (Tate). *If G and G' are p-divisible groups over* $\operatorname{Spec} \mathscr{O}_K$. *Then there is a bijection*

$$\operatorname{Hom}_{p-div/\mathscr{O}_K}(G, G') \xrightarrow{\simeq} \operatorname{Hom}_{p-div/K}(G \otimes K, G' \otimes K).$$

Corollary 2.2.0.10. *There is a bijection*

$$\operatorname{Hom}_{p-div/\mathscr{O}_K}(G, G') \xrightarrow{\simeq} \operatorname{Hom}_{\mathbb{Z}_p[G_K]}(T_p(G), T_p(G')).$$

Remark. *Another way to see the Tate module is through the equality*

$$\varprojlim G(p^n)(\overline{K}) = \varprojlim \operatorname{Hom}(\mathbb{Z}/p^n\mathbb{Z}, G(\overline{K})) = \operatorname{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, G(\overline{K})).$$

2.3. The classification by Dieudonné

Fix k a perfect field of characteristic $p > 0$. We define $W = W(k)$ the ring of Witt vectors of k and $\sigma : W \to W$ the induced Frobenius map.

Definition 2.3.0.11. *We define the* Dieudonné ring $\mathfrak{D}$ *as the (non commutative) polynomial ring $W[F, V]$, where F and V satisfy to the relations*

$$F \cdot V = V \cdot F = p,$$
$$F \cdot \lambda = \sigma(\lambda) \cdot F,$$
$$V \cdot \lambda = \sigma^{-1}(\lambda) \cdot V.$$

Any element of $\mathfrak{D}$ can be written uniquely as a finite sum

$$a_0 + \sum_{i=1} (b_i V^n + c_i F^n), \qquad \text{for } a_0, b_i, c_i \in W.$$

Theorem 2.3.0.12 (Dieudonné 1). *There is an anti-equivalence of categories*

$$\begin{array}{ccc} \{p\text{-}gp\ sch\ /k\} & \xrightarrow{\mathbb{D}_{fin}} & \{\text{Dieudonné modules of finite } W\text{-length}\} \\ G & \longmapsto & \operatorname{Hom}_{k-GpSch}(G, W) = \varinjlim_n \operatorname{Hom}_{k-GpSch}(G, W_n) \end{array}$$

Note that $\mathbb{D}_{fin}(G)$ is a module over the Witt vectors. Moreover, the Frobenius and the Verschiebung maps on the group scheme G induce maps

$$F : \mathbb{D}_{fin}(G) \to \mathbb{D}_{fin}(G)^{(p)}, \qquad V : \mathbb{D}_{fin}(G)^{(p)} \to \mathbb{D}_{fin}(G).$$

That is, $\mathbb{D}_{fin}(G)$ has a module structure of the ring $\mathfrak{D}$; we call it a *Dieudonné module*. For the proof of the theorem see [**Dem**, p. 65].

Examples. 1.) We have $\mathbb{D}_{fin}(\mathbb{Z}/p^n\mathbb{Z}) = \mathfrak{D}/(F - 1, p^n)$. Indeed, the multiplication by p^n is clearly zero, while the Frobenius map is the identity. From the relation on $\mathfrak{D}$ it follows that $V = p$.

2.) In the dual case, we have $\mathbb{D}_{fin}(\mu_{p^n}) = \mathfrak{D}/(F - p, p^n)$.

By passing to the limit we obtain

Theorem 2.3.0.13 (Dieudonné 2).

$$\begin{array}{ccc} \mathrm{BT}(k) & \xrightarrow{\simeq} & \{W\text{-}free\ Dieudonné\ modules\} \\ G & \longmapsto & \varprojlim \mathbb{D}_{fin}(G(p^n)) \end{array}$$

The Dieudonné functor satisfies to the following properties (see [**Fon2**, chapter 3] for the proofs).

(1) The Dieudonné functor is exact and it commutes with Cartier duality.
(2) It compatible with base change, that is, if $k \to k'$ is an extension of finite fields, then $\mathbb{D}(G \times_k k') = \mathbb{D}(G) \otimes_{W(k)} W(k')$; it follows that $\mathbb{D}(G^{(p)}) = \mathbb{D}(G)^{(p)}$.
(3) $\mathbb{D}(G)/F\mathbb{D}(G)$ is naturally isomorphic to the cotangent space of G.

2.4. Formal Lie groups

In this section we establish the relation between Barsotti-Tate groups and formal Lie groups. Let S be a base scheme.

Definition 2.4.0.14. *Let X, Y be $fppf$ sheaves of groups over S such that $Y \hookrightarrow X$. We define for every $k \geq 0$ a subsheaf $\mathrm{Inf}_Y^k(X)$ of X whose $\Gamma(T, \mathrm{Inf}_Y^k(X))$ on an S-scheme T are those sections $t \in \Gamma(T, X)$ such that there exists a covering $\{T_i \to T\}_i$ and for every T_i a subscheme T_i', defined by a $(k+1)$-nilpotent ideal, such that $t_{T_i'} \in \Gamma(T_i', X)$ is an element of $\Gamma(T_i', Y)$.*

$\mathrm{Inf}_Y^k(X)$ is compatible with base change. When a sheaf X over S is provided with a section $e_X : S \to X$, that is, (X, e_X) is a pointed sheaf, we get a particular case of the definition and we write $\mathrm{Inf}^k(X) := \mathrm{Inf}_S^k(X)$.

Definition 2.4.0.15 (Formal Lie Variety). *A pointed sheaf (X, e_X) over S is said to be a* formal Lie variety *if*

(1) $X = \varinjlim_k \mathrm{inf}^k(X)$ *and the* $\mathrm{Inf}^k(X)$ *are representable for every* $k \geq 0$,
(2) $\underline{\omega}_X := e_X^* \Omega^1_{X/S} = e_X^* \Omega^1_{\mathrm{Inf}^k(X)/S}$ *is locally free of finite type,*
(3) *denoting by $gr^{inf}(X)$ the unique graded $\mathcal{O}_S$-algebra such that $gr_i^{inf}(X) = gr_i(\mathrm{Inf}^i(X))$ holds for all $i \geq 0$, we have an isomorphism $Sym(\underline{\omega}_X) \xrightarrow{\simeq} gr_i^{inf}(X)$ induced by the canonical mapping $\underline{\omega}_X \xrightarrow{\simeq} gr_1^{inf}(X)$.*

Definition 2.4.0.16 (Formal Lie group). *A* formal Lie group (G, e_G) *over S is a group in the category of formal Lie varieties.*

There is a good characterization of formal Lie groups in the case S of characteristic $p > 0$. Moreover one can extend some of the properties to the case p locally nilpotent over S and these will give us a relation between p-divisible groups and formal Lie groups.

Let S be a scheme of characteristic p and G an $fppf$ sheaf of groups over S. In this case we have a Frobenius and a Verschiebung morphisms:

$$F_{G/S} : G \to G^{(p)}, \qquad V_{G/S} : G^{(p)} \to G.$$

We denote by $G[n]$ the kernel of the n-th iterate $(F_{G/S})^n$.

Definition 2.4.0.17. *We say that G is of* $F_{G/S}$-torsion *if* $G = \varinjlim G[n]$. *We say that G is* $F_{G/S}$-divisible *if $F_{G/S}$ is surjective.*

Theorem 2.4.0.18 (Characterization of formal Lie groups in char p). *G is a formal Lie group if and only if*

(1) *G is of $F_{G/S}$-torsion,*
(2) *G is $F_{G/S}$-divisible,*
(3) *The $G[n]$ are finite and locally free S-group schemes.*

Proof. [**Mess**], chapter 2, Theorem (2.1.7). □

Suppose now p is locally nilpotent on S.

Theorem 2.4.0.19. *If G is a p-divisible group, then $\overline{G} := \varinjlim \mathrm{Inf}^k(G)$ is a formal Lie group.*

Proof. [**Mess**], chapter 2, Theorem (3.3.18). □

2.5. Grothendieck-Messing theory

In this section we associate to any p-divisible group a universal extension by a vector group. Several crystals arise from this construction and this observation leads to the deformation theory by Grothendieck and Messing. Let S be a scheme on which p is locally nilpotent.

2.5.1. Universal extensions. Let $\mathcal{L}$ be a quasi-coherent $\mathcal{O}_S$-module. We may regard this as a sheaf on the *fppf* site over S in the following sense

$$\underline{\mathcal{L}}(T) = H^0(T, f^*(\mathcal{L})),$$

where $T \xrightarrow{f} S$ is an *fppf* S-scheme. If $\mathcal{L}$ is locally-free of finite rank, we call it a *vector group over* S. In this case, it is representable by a group scheme which is locally isomorphic to a finite product $\mathbb{G}_a \times \cdots \times \mathbb{G}_a$. Recall that a group scheme G over S corresponds to a sheaf on the *fppf* site over S.

We are looking for a solution to the following two *universal problems*: let G be a finite, flat group scheme.

(1) There exists a map
$$\alpha : G \to \underline{V}(G),$$
where $\underline{V}(G)$ is a vector group over S such that given a map to any other vector group $G \to M$ there is a unique map $V(G) \to M$ making the following diagram commutative:

(a)
$$\begin{tikzcd} G \arrow[r, "\alpha"] \arrow[d] & \underline{V}(G) \\ M \arrow[ur] & \end{tikzcd}$$

(2) Assuming $\mathrm{Hom}(G,V) = (0)$ for any vector group V, there is an extension of group schemes over S

$$(\epsilon) \qquad 0 \to \underline{V}(G) \to E(G) \to G \to 0,$$

which is *universal*, that is, given any extension

$$(\epsilon') \qquad 0 \to M \to \bullet \to G \to 0$$

by another vector group, there is a unique map $\phi : \underline{V}(G) \to M$ such that $\phi_*((\epsilon)) = (\epsilon')$.

Define

$$\underline{\omega}_{G^*} := e^*\Omega^1_{G^*/S} \simeq \mathrm{Hom}_{S-schemes}(\mathbb{G}_1^*, \mathbb{G}_m),$$

where $e : S \hookrightarrow G^*$ and G^* denotes the Cartier dual.

PROPOSITION 2.5.1.1. *Let G be an abelian group scheme over S such that G^* is representable. The functor $M \to \mathrm{Hom}_{S-gr}(G,M)$ is represented by $\underline{\omega}_{G^*}$. That is, in the notation of Universal Problem* (a), *$\underline{V}(G) = \underline{\omega}_{G^*}$, where*

$$\begin{array}{rcl} \alpha : G & \to & \underline{\omega}_{G^*} \\ f & \mapsto & f^* \frac{dT}{T} \end{array}$$

where $f \in G = (G^*)^* = \mathrm{Hom}(G^*, \mathbb{G}_m)$.

PROOF. [**MM**, Prop. 1.4]. □

Note that this construction is functorial, that is, if $u : G \to H$ is a map of locally-free, finite groups, then there is a commutative diagram

$$\begin{tikzcd} G \arrow[r, "u"] \arrow[d, "\alpha_G"'] & H \arrow[d, "\alpha_H"] \\ \underline{\omega}_{G^*} \arrow[r] & \underline{\omega}_{H^*} \end{tikzcd}$$

Problem (b) is interesting not only for Barsotti-Tate groups, but we will present two 'ad hoc' solutions for this case.

PROPOSITION 2.5.1.2. *Suppose S is a scheme such that $p^N = 0$ for some N, $G \in \mathrm{BT}(S)$. There is a universal extension of G by a vector group, namely we have:*

$$\begin{tikzcd} 0 \arrow[r] & G(p^N) \arrow[r] \arrow[d, "\alpha"] & G \arrow[r, "p^N"] \arrow[d] & G \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \underline{V}(G) = \underline{\omega}_{G(p^N)^*} \arrow[r] & E(G) \arrow[r] & G \arrow[r] & 0 \end{tikzcd}$$

PROOF. [**MM**, 1.8]. □

Since of course $\underline{\omega}_{G^*} = \underline{\omega}_{G(N)^*}$ if $p^N = 0$ on S, the universal extension is

$$0 \to \underline{\omega}_{G^*} \to E(G) \to G \to 0.$$

COROLLARY 2.5.1.3. *Since Cartier duality commutes with base change, also the universal extension does. More precisely, if $f : S' \to S$ is a morphism of schemes and $G \in \mathrm{BT}(S)$, then $f^*(G) \in \mathrm{BT}(S')$. Moreover $f^*(E(G)) = E(f^*G)$.*

Moreover, it has the following functorial property: if $u : G \to H$ is a morphism of BT-groups over S, then there is a map $E(u) : E(G) \to E(H)$, which is a morphism of extensions:

$$
\begin{tikzcd}
0 \arrow[r] & \underline{V}(G) \arrow[r] \arrow[d, "\underline{V}(u)"] & E(G) \arrow[r] \arrow[d, "E(u)"] & G \arrow[r] \arrow[d, "u"] & 0 \\
0 \arrow[r] & \underline{V}(H) \arrow[r] & E(H) \arrow[r] & H \arrow[r] & 0
\end{tikzcd}
\tag{2.5.1}
$$

We apply to the construction the Lie algebra functor

$$\{\text{Lie groups}\} \to \{\text{Lie algebras}\},$$

which associates to a Lie group its Lie algebra. We dispose of a canonical Lie group.

LEMMA 2.5.1.4. *Given a universal extension $E(G)$, we have that*

$$\overline{E(G)} := \varinjlim \operatorname{Inf}^k(E(G))$$

is a formal Lie group.

DEFINITION 2.5.1.5. *We define*

$$\underline{Lie}(E(G)) := \underline{Lie}(\overline{E(G)}).$$

This is a locally-free sheaf of $\mathscr{O}_S$-Modules.

LEMMA 2.5.1.6. *We have the two following exact sequences:*

$$
\begin{array}{ccccccc}
0 \longrightarrow \overline{\underline{V}(G)} & \longrightarrow & \overline{E(G)} & \longrightarrow & \overline{G} & \longrightarrow & 0, \\
0 \longrightarrow \underline{V}(G) & \longrightarrow & \underline{Lie}(E(G)) & \longrightarrow & \underline{Lie}(G) & \longrightarrow & 0.
\end{array}
$$

PROOF. [**MM**, Chapter 4, Prop. 1.21, Prop 1.22]. □

2.5.2. The crystals associated to Barsotti-Tate groups. Let S_0 be a base scheme with p locally nilpotent. We define a full subcategory $\mathrm{BT}'(S_0)$ of $\mathrm{BT}(S_0)$ with objects those p-divisible groups G_0 over S_0 such that there exists an affine open cover $\{U_0\}$ of S_0 such that for any nilpotent immersion $U_0 \hookrightarrow U$ (this is a special element of the site) there is a BT-group G on U such that $G|_{U_0} = G_0|_{U_0}$.

We associate a crystal to a p-divisible group G in this sub-category. Since $fppf$ groups form a stack with respect to the Zariski topology, it suffices to specify the values of the going to be defined crystal on elements $(U_0 \hookrightarrow U) \in \mathrm{Cris}(S_0)$ such that U_0 is affine in S_0 and $G_0|_{U_0}$ can be lifted to U.

THEOREM 2.5.2.1. *Consider for a ring A, schemes $S = \operatorname{Spec}(A)$ and $S_0 = \operatorname{Spec}(A/I)$, where I is a P.D.-ideal of A. Consider $G, H \in \mathrm{BT}(S)$, their restrictions $G_0 = G|_{S_0}, H_0 = H|_{S_0} \in \mathrm{BT}(S_0)$ and a map*

$$u_0 : G_0 \longrightarrow H_0.$$

By diagram (2.5.1) *there is a unique morphism of extensions $E(u_0) : E(G_0) \to E(H_0)$.*

*Then there exists a unique morphism of groups (*not necessairly of extensions!*)*

$$E_S(u_0) : E(G) \longrightarrow E(H),$$

such that $E_S(u_0)$ is a lifting of $E(u_0)$. Note that there is not necessairly a map $G \to H$.

Proof. [**Mess**, Chapter 4, Thm. 2.2]. □

Corollary 2.5.2.2. *Given a third p-divisible group K in the notation above, with a map $\tilde{u_0} : H_0 \to K_0$,*

$$E_S(\tilde{u_0}) \circ E_S(u_0) = E_S(\tilde{u_0} \circ u_0)$$

Corollary 2.5.2.3. *If, in the notation above, u_0 is an isomorphism, then $E_S(u_0)$ is an isomorphism.*

Corollary 2.5.2.4. *Suppose we have a diagram*

$$\begin{tikzcd}
\mathrm{Spec}(A/I) = S_0 \arrow[r, hook, "nilpotent"] & S = \mathrm{Spec}(A) \\
\mathrm{Spec}(A'/I') = S_0' \arrow[r, hook, "nilpotent"'] \arrow[u] & S' = \mathrm{Spec}(A') \arrow[u, "f", "P.D.\text{-}morphism"']
\end{tikzcd}$$

Consider $G, H \in \mathrm{BT}(S)$ and a morphism $u_0 : G_0 \to H_0$ as in the theorem. Recall that by Corollary (2.5.1.3), universal extensions are compatible with base change. Then

$$E_{S'}(f^*(u_0)) = (E_S(u_0))_{S'}.$$

This statement can be applied in particular to morphisms on the crystalline site

$$\begin{tikzcd}
U_0 \arrow[r, hook] & U \\
V_0 \arrow[r, hook] \arrow[u, "f"] & V \arrow[u, "\overline{f}"']
\end{tikzcd}$$

Consider now a p-divisible group $G_0 \in \mathrm{BT}'(S_0)$, $(U_0 = \mathrm{Spec}(R) \hookrightarrow U) \in \mathrm{Cris}(S_0)$ such that $G_0|_{U_0}$ can be lifted to U. We define $\mathrm{E}(G_0)$ as the sheaf on $\mathrm{Cris}(S_0)$ with value $E(G)$ on $(U_0 \hookrightarrow U)$ for G a lifting of $G_0|_{U_0}$ to U.

This is a crystal according to the definition. Indeed, by the definition of $\mathrm{BT}'(S_0)$ there exists a p-divisible group G over U such that $G_0|_{U_0} = G|_{U_0}$. Consider the universal extension $E(G)$. By Corollary (2.5.2.3) this is independent of the lifting G of G_0 up to isomorphism. Indeed, given H another lifting, we have an isomorphism $G|_{U_0} \xrightarrow{\sim} H|_{U_0}$, which extends to an isomorphism $E(G) \xrightarrow{\sim} E(H)$. Moreover, suppose we are given a morphism $\overline{f} : V \to U$ on the crystaline site. Then for a lifting G_U of $G_0|_{U_0}$ and a lifting G_V of $G_0|_{V_0}$ there is an isomorphism by Corollary (2.5.2.4)

$$\overline{f}^*(E(G_U)) \xrightarrow{\simeq} E(G_V), \tag{2.5.2}$$

hence the conclusion.

Note that given a morphism on the crystalline site

$$\begin{tikzcd}
U_0 \arrow[r, hook] & U \\
V_0 \arrow[r, hook] \arrow[u, "f"] & V \arrow[u, "\overline{f}"']
\end{tikzcd}$$

and a lifting G of $G_0|_{U_0}$ to U, the functoriality of universal extensions gives us

$$\overline{f}^*(E(G)) = E(G_V) = \mathrm{E}(f^*(G_0))_{(V_0 \hookrightarrow V)},$$

that is,

$$f^*(\mathrm{E}(G_0)) = \mathrm{E}(f^*(G_0)).$$

Finally, for G_0 as above, we define

$$\overline{\mathrm{E}}(G_0)_{(U_0 \hookrightarrow U)} := \overline{\mathrm{E}(G_0)}_{(U_0 \hookrightarrow U)},$$
$$\overline{\mathbb{D}}(G_0)_{(U_0 \hookrightarrow U)} := \underline{Lie}(\overline{\mathrm{E}(G_0)}_{(U_0 \hookrightarrow U)}.$$

That is, if $S_0 \hookrightarrow S$ is a P.D.-immersion and $G_0 \in \mathrm{BT}'(S_0)$ can be lifted to a p-divisible group on S, then

$$\mathrm{E}(G_0)_{(S_0 \hookrightarrow S)} = E(G),$$
$$\overline{\mathrm{E}}(G_0)_{(S_0 \hookrightarrow S)} = \overline{E(G)},$$
$$\mathbb{D}(G_0)_{(S_0 \hookrightarrow S)} = \underline{Lie}(E(G)).$$

These equalities are meaningful since we are interested in *some particular values* of the crystals on objects on the crystalline site over S of type $S_0 \hookrightarrow S$. We therefore re-define the category $\mathrm{BT}'(S_0)$ to be the subcategory of $\mathrm{BT}(S_0)$, whose objects can be lifted to a $G \in \mathrm{BT}(S)$. We denote respectively by $\mathrm{E}(G_0)(S)$, $\overline{\mathrm{E}}(G_0)(S)$ and $\mathbb{D}(G_0)(S)$ the values of the crystals on $(S_0 \hookrightarrow S) \in \mathrm{Cris}(S_0)$.

Remark. *We have constructed the crystals under the hypothesis p locally nilpotent on the base scheme S_0. However, we can evaluate the crystal on p-adic P.D. thickenings $S_0 \hookrightarrow S$ by passing the construction to the limit. The same holds for the results in the following section.*

2.5.3. The theorem of Grothendieck-Messing. Let $(S_0 \hookrightarrow S) \in \mathrm{Cris}(S)$ as above. Denote by $\mathbb{D}(G_0)(S)$ the evaluation of the Dieudonné crystal on $(S_0 \hookrightarrow S)$.

Definition 2.5.3.1 (Admissible filtrations). *We say that a filtration* $\mathrm{Fil}^1\mathbb{D}(G_0)(S) \subseteq \mathbb{D}(G_0)(S)$ *is* admissible *if* $\mathrm{Fil}^1\mathbb{D}(G_0)(S)$ *is a locally-free vector sub-group with locally-free quotient, which reduces to $\underline{V}(G_0) \hookrightarrow \underline{Lie}(E(G_0))$ by restricting to S_0. Define the category*

$$\mathscr{C} = \{(G_0, \mathrm{Fil}^1\mathbb{D}(G_0)(S)) \mid G_0 \in \mathrm{BT}'(S_0), \mathrm{Fil}^1\mathbb{D}(G_0)(S) \subseteq \mathbb{D}(G_0)(S) \textit{ is admissible}\},$$

with morphisms the pairs (u_0, ξ), $u_0 : G_0 \to H_0$, $\xi : \mathrm{Fil}^1\mathbb{D}(G_0)(S) \to \mathrm{Fil}^1\mathbb{D}(H_0)(S)$ making the following diagram commutative

$$\begin{tikzcd}
\mathrm{Fil}^1\mathbb{D}(G_0)(S) \arrow[r, hook] \arrow[d, "\xi"'] & \mathbb{D}(G_0)(S) \arrow[d, "\mathbb{D}(u_0)_S"] \\
\mathrm{Fil}^1\mathbb{D}(H_0)(S) \arrow[r, hook] & \mathbb{D}(H_0)(S)
\end{tikzcd}$$

which of course, restricting to S_0, becomes

$$\begin{tikzcd}
\underline{V}(G_0) \arrow[r, hook] \arrow[d, "\underline{V}(u_0)"'] & \underline{Lie}(E(G_0)) \arrow[d, "\underline{Lie}(E(u_0))"] \\
\underline{V}(H_0) \arrow[r, hook] & \underline{Lie}(E(H_0))
\end{tikzcd}$$

THEOREM 2.5.3.2 (Grothendieck-Messing). *There is a contravariant equivalence of categories*

$$\begin{array}{ccc} \mathrm{BT}(S) & \xrightarrow{\mathbb{D}} & \mathscr{C} \\ G & \longmapsto & (G_0 \coloneqq S_0 \times_S G, \underline{V}(G) \hookrightarrow \underline{Lie}(E(G)) = \mathbb{D}(G_0)(S)) \end{array}$$

2.5.4. Comparison with the theory by Dieudonné. Suppose $S = \operatorname{Spec} k$ for k a perfect field of characteristic $p > 0$. The classifcation by Messing agrees with the classification by Dieudonné in this case ([**MM**], Section 15).

THEOREM 2.5.4.1. *Recall that for $S = \operatorname{Spec} k$ we are given an equivalent functor*

$$\begin{array}{ccc} \mathbb{D}^{Dieu} : \mathrm{BT}(k) & \xrightarrow{\simeq} & \{W\text{-}free\ Dieudonn\acute{e}\ modules\} \\ G & \longmapsto & \varprojlim \mathbb{D}^{Dieu}_{fin}(G(n)) \end{array}$$

There is a canonical isomorphism of functors

$$\mathbb{D} \longrightarrow \mathbb{D}^{Dieu}.$$

CHAPTER 3

The first classification

Consider a W-scheme T_0 where $p=0$ and consider a P.D. thickening in the crystalline site $\mathrm{Cris}(T/W)$

$$\begin{tikzcd} T_0 \arrow[r, hook] \arrow[d] & T \arrow[dl] \\ \mathrm{Spec}(W) & \end{tikzcd}$$

where p is locally nilpotent. Since $p=0$ on T_0, it is endowed with a Frobenius map φ from W. Recall that we can see any p-divisible group G_0 over T_0 as a sheaf over T_0 and hence we can pull-back the Frobenius and consider the map $G_0 \to \varphi^* G_0$. At the level of crystals we get by functoriality

$$\varphi^*(\mathbb{D}(G_0)) \xrightarrow{\sim} \mathbb{D}(\varphi^*(G_0)) \longrightarrow \mathbb{D}(G_0),$$

(the map on the left is an isomorphism by (2.5.2)).

Consider $G_0 \in \mathrm{BT}'(T_0)$ and a lifting $G \in \mathrm{BT}(T)$. Note that by the considerations of section (2.5.2)

$$\mathbb{D}(G_0)(T) \xrightarrow{\simeq} \mathbb{D}(G)(T),$$

where $\mathbb{D}(G)(T)$ denotes the value of the crystal in T. With this notation, we get by Lemma (2.5.1.6) an exact sequence

$$0 \longrightarrow \underline{V}(G) \longrightarrow \mathbb{D}(G)(T) \longrightarrow \underline{Lie}(G) \longrightarrow 0. \tag{3.0.3}$$

Definition 3.0.4.2. *We say that a ring A is* special *if it is a p-adically complete, separated, p-torsion free local $\mathbb{Z}_p$-algebra with residue field k, and endowed with a lifting $\varphi : A \to A$ of the Frobenius on the quotient A/pA. A map of special rings is a map of $\mathbb{Z}_p$-algebras, compatible with the action of φ.*

Lemma 3.0.4.3. *Let A be a special ring and (I,γ) a P.D. ideal. Then $\varphi(I) \subset pA$. In particular it makes sense to consider $\varphi_1 = \varphi/p$ on I, since A has no p-torsion.*

Proof. If $a \in I$, then $\varphi(a) \equiv a^p \bmod pA$, and hence $\varphi(a) \equiv \gamma(a)p! \bmod (pA)$, so $\varphi(a) \in pA$. $\square$

Consider $W[u]\Big[\frac{E(u)^i}{i!}\Big]$ the P.D. envelope of $W[u]$ with respect to $(E(u))$. There is a surjective map

$$\begin{array}{ccc} W[u]\Big[\frac{E(u)^i}{i!}\Big]_{i\geq 1} & \twoheadrightarrow & \mathscr{O}_K \\ u & \mapsto & \pi \end{array}$$

Denote by S its p-adic completion and $\mathrm{Fil}^1 S \subseteq S$ the ideal generated by all the $\frac{E(u)^i}{i!}$. The ring $(S, \mathrm{Fil}^1 S)$ is a P.D. ring. The above map induces an isomorphism

$$S/\mathrm{Fil}^1 S \xrightarrow{\simeq} \mathscr{O}_K.$$

We extend to S the Frobenius on W, by putting

$$\begin{array}{ccc} S & \xrightarrow{\varphi} & S \\ u & \longmapsto & u^p \end{array} \tag{3.0.4}$$

S is a special ring. It is by definition a p-adically complete, separated, p-torsion free $\mathbb{Z}_p$-algebra. By Lemma (3.0.4.3), we get a map $\varphi_1 = \varphi/p$ on $\mathrm{Fil}^1 S$.

Definition 3.0.4.4. *Denote by* $\mathrm{BT}^{\varphi}_{/S}$ *the category with objects the finite free* S*-modules* $\mathcal{M}$ *together with an* S*-submodule* $\mathrm{Fil}^1\mathcal{M}$ *and a* φ*-semilinear map* $\varphi_1 : \mathrm{Fil}^1\mathcal{M} \to \mathcal{M}$ *such that*

(1) $p\varphi_1 = \varphi$ *on* $\mathrm{Fil}^1\mathcal{M}$,
(2) $\mathrm{Fil}^1 S \cdot \mathcal{M} \subset \mathrm{Fil}^1\mathcal{M}$ *and the quotient* $\mathcal{M}/\mathrm{Fil}^1\mathcal{M}$ *is a free* $\mathscr{O}_K$*-module,*
(3) *the map* $\varphi^*(\mathrm{Fil}^1\mathcal{M}) \xrightarrow{1\otimes\varphi_1} \mathcal{M}$ *is surjective.*

Theorem 3.0.4.5. *The Dieudonné crystal defines an exact contravariant functor*

$$\begin{array}{ccc} \mathrm{BT}(\mathscr{O}_K) & \xrightarrow{\mathcal{M}} & \mathrm{BT}^{\varphi}_{/S} \\ G & \longmapsto & \mathbb{D}(G)(S) \end{array}$$

For $p > 2$ *this is an equivalence of categories; for* $p = 2$ *this is an equivalence of categories up to isogeny.*

In order to define the functor $\mathcal{M}$ and construct an inverse we will go through some technical lemmas.

Lemma 3.0.4.6. *Consider a surjection* $A \twoheadrightarrow A_0$ *of special rings whose kernel* $\mathrm{Fil}^1 A$ *is equipped with divided powers. This gives us in particular an element of* $\mathrm{Cris}(S_0/\mathbb{Z}_p)$*:*

$$\mathrm{Spec}(A_0) \hookrightarrow \mathrm{Spec}(A).$$

Suppose the following hypotheses are satisfied

(1) A *is* p*-torsion free and has a Frobenius* $\varphi : A \to A$ *lifting that of* A/pA,
(2) *the induced map (Lemma* (3.0.4.3)*)* $\varphi^*(\mathrm{Fil}^1 A) \xrightarrow{1\otimes\varphi/p} A$ *is surjective.*

Let G_0 *be a* p*-divisible group over* $\mathrm{Spec}(A_0)$ *and* G *a lifting of this to* $\mathrm{Spec}(A)$*. Denote by* $\mathrm{Fil}^1\mathbb{D}(G)(A)$ *the preimage of* $\underline{V}(G_0)$ *in* $\mathbb{D}(G)(A)$*: by* (3.0.3) *there is an injection* $\underline{V}(G_0) \hookrightarrow \mathbb{D}(G_0)(A_0)$ *and, following the notation of Theorem* (2.5.3.2) *there is a lifting*

$$\begin{tikzcd} \mathrm{Fil}^1(\mathbb{D}(G)(A)) \arrow[r, hook] \arrow[d] & \mathbb{D}(G)(A) \simeq \mathbb{D}(G_0)(A) \arrow[d] \\ \underline{V}(G_0) \arrow[r, hook] & \underline{Lie}(E(G_0)) \end{tikzcd}$$

The restriction of $\varphi : \mathbb{D}(G)(A) \to \mathbb{D}(G)(A)$ *to* $\mathrm{Fil}^1\mathbb{D}(G)(A)$ *is divisible by* p *and the induced map*

$$\varphi^*(\mathrm{Fil}^1\mathbb{D}(G)(A)) \xrightarrow{1\otimes\varphi/p} \mathbb{D}(G)(A)$$

is surjective.

Proof. Note first that $\varphi|_{\mathrm{Fil}^1\mathbb{D}(G)(A)}$ is divisible by p. Indeed there is an equality

$$\mathrm{Fil}^1\mathbb{D}(G)(A) = \underline{V}(G) + \mathrm{Fil}^1 A \cdot \mathbb{D}(G)(A), \tag{3.0.5}$$

(the reduction mod $\mathrm{Fil}^1 A$ on both sides is precisely $\underline{V}(G_0)$) and by the fact that $\varphi(\underline{V}(G)) \subset p\mathbb{D}(G)(A)$ we conclude.

Since A has no p-torsion, also $\mathbb{D}(G)(A)$ doesn't, hence we consider the map

$$\varphi/p : \mathrm{Fil}^1\mathbb{D}(G)(A) \to \mathbb{D}(G)(A).$$

By hypothesis (b) of the statement we get

$$A\varphi(\mathbb{D}(G)(A)) = A\varphi/p(\mathrm{Fil}^1 A)A\varphi(\mathbb{D}(G)(A)) \subset A\varphi/p(\mathrm{Fil}^1\mathbb{D}(G)(A)),$$

the last inclusion being true because of equality (3.0.5).

Therefore, the statement $\varphi/p(\mathrm{Fil}^1\mathbb{D}(G)(A))$ generates $\mathbb{D}(G)(A)$ is equivalent to the statement $\varphi/p(\mathrm{Fil}^1\mathbb{D}(G)(A) + p\mathbb{D}(G)(A))$ generates $\mathbb{D}(G)(A)$.

There is a unique φ-compatible map $A \to W(A)$ and, by functoriality, a diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & k \arrow[d] \\ W(A) \arrow[r] & W(k) \end{tikzcd}$$

Hence we get a unique map $A \to W(k)$, compatible with the Frobenius action. Define hence $H := G \otimes_A W(k)$ and $\overline{H} := H \otimes_{W(k)} k$. In this case Messing's theory coincides with the classical theory by Dieudonné by Theorem (2.5.4.1), that is, $\mathbb{D}(H)(W(k))$ is naturally, φ-compatibly isomorphic to the Dieudonné module associated to H, that is, it is given a Frobenius map φ and a Verschiebung morphism V. By restricting to $W(k)$ we get

$$\underline{V}(H) = V(\varphi/p)\underline{V}(H) \subseteq V\mathbb{D}(H)(W(k)).$$

By restricting again to k we get an inclusion $V(\varphi/p)\underline{V}(\overline{H}) \subseteq V\mathbb{D}(\overline{H})(k)$. This is in fact an isomorphism, since the two terms have the same k-dimension, being both isomorphic to $\mathbb{D}(\overline{H})(k)/\varphi\mathbb{D}(\overline{H})(k)$. Lifting the equality to the Witt vectors we get that $\mathrm{Fil}^1\mathbb{D}(G)(W) + p\mathbb{D}(G)(W) = V\mathbb{D}(G)(W)$, that is

$$\varphi/p(\mathrm{Fil}^1\mathbb{D}(G)(W) + p\mathbb{D}(G)(W)) = \mathbb{D}(G)(W),$$

since $V \cdot \varphi/p = 1$. Since $\mathbb{D}(G)(A)$ is a finitely generated module we obtain the equality we are looking for by Nakayama's lemma. $\square$

For a special ring A, we define the category $\mathcal{C}_A$ with objects finite free A-modules $\mathcal{M}$, with a Frobenius semilinear map $\varphi : \mathcal{M} \to \mathcal{M}$ and an A-submodule $\mathcal{M}_1 \subset \mathcal{M}$ such that $\varphi(\mathcal{M}_1) \subset p\mathcal{M}$ and the induced map $\varphi^*(\mathcal{M}_1) \xrightarrow{1\otimes\varphi/p} \mathcal{M}$ is surjective.

Given a map of special rings $A \to B$ there is a functor

$$\begin{array}{ccc} \mathcal{C}_A & \longrightarrow & \mathcal{C}_B \\ \mathcal{M} & \longmapsto & \mathcal{M} \otimes_A B \end{array}$$

Indeed $\mathcal{M} \otimes_A B$ inherits a Frobenius in the obvious way and we define $(\mathcal{M} \otimes_A B)_1$ as the image of $\mathcal{M}_1$ through the natural map $\mathcal{M} \to \mathcal{M} \otimes_A B$. Clearly $\varphi((\mathcal{M} \otimes_A B)_1) \subset p(\mathcal{M} \otimes_A B)$. Moreover, the right exactness of tensor product gives us the surjectivity of $\varphi^*((\mathcal{M} \otimes_A B)_1) \xrightarrow{1\otimes\varphi/p} (\mathcal{M} \otimes_A B)$.

Lemma 3.0.4.7. *Let $h : A \to B$ be a surjection of special rings with kernel J. Suppose that for $i \geq 1$, $\varphi^i(J) \subset p^{i+j_i}J$, where $\{j_i\}_{i\geq 1}$ is a sequence of integers such that $\lim_{i\to\infty} j_i = \infty$. Let $\mathcal{M}$ and $\mathcal{M}'$ be objects in the category $\mathcal{C}_A$ and $\theta_B : \mathcal{M} \otimes_A B \xrightarrow{\sim} \mathcal{M}' \otimes_A B$ be an isomorphism in $\mathcal{C}_B$. Then there exists a map $\theta : \mathcal{M} \to \mathcal{M}'$ such that $\theta \otimes_A B = \theta_B$.*

Construction of the functor $\mathcal{M}$. Let G be a p-divisible group over $\mathscr{O}_K$, then $\mathbb{D}(G)(S)$ belongs to the category $\mathrm{BT}^{\varphi}_{/S}$. Indeed there is a natural map

$$\begin{tikzcd} & S \arrow[dr] & \\ W \arrow[ur] \arrow[rr] & & \mathscr{O}_K \end{tikzcd}$$

and hence we may evaluate the crystal $\mathbb{D}(G)$ on the formal scheme $Spf(S)$. Moreover, the P.D. couple $(S, \mathrm{Fil}^1 S)$ satisfies the hypotheses of Lemma (3.0.4.6) and hence $(\mathbb{D}(G)(S), \mathrm{Fil}^1\mathbb{D}(G)(S)) \in \mathrm{BT}^{\varphi}_{/S}$. We define $\mathcal{M}(G) := \mathbb{D}(G)(S)$.

Proof of Theorem (3.0.4.5). We wish to construct a quasi-inverse

$$\mathrm{BT}^{\varphi}_{/S} \longrightarrow \mathrm{BT}(\mathscr{O}_K).$$

Fix $(\mathcal{M}, \mathrm{Fil}^1\mathcal{M}) \in \mathrm{BT}^{\varphi}_{/S}$. Note first that $\mathcal{M} \in \mathcal{C}_S$. Indeed, $\mathcal{M}$ is by definition a finite free S-module, with a Frobenius endomorphism. Moreover, by point (b) in the definition of the category $\mathrm{BT}^{\varphi}_{/S}$, there is a surjection $\varphi^*(\mathcal{M}_1) \xrightarrow{1\otimes\varphi/p} \mathcal{M}$.

For $i = 1, \ldots, e$ define the algebra $R_i := W[u]/u^i$. It has a Frobenius

$$W[u]/u^i \to W[u]/u^i,$$

induced by the Frobenius

$$\begin{array}{rcl} \varphi : W[u] & \longrightarrow & W[u] \\ u & \longmapsto & u^p \end{array}$$

By the universal property P.D. envelopes there is a diagram

$$\begin{tikzcd} & S \arrow[dr, "f_i"] & \\ W[u] \arrow[ur] \arrow[rr] & & W[u]/u^i =: R_i \end{tikzcd}$$

where f_i is the map sending $u \mapsto u$ and $\frac{(u^e)^j}{j!} \mapsto 0$, for $j \geq 1$.

Put $\mathcal{M}_i := \mathcal{M} \otimes_S R_i$. Note that $f_i : S \to R_i$ is a map of special rings and $\mathcal{M}_i \in \mathcal{C}_{R_i}$ with $\mathrm{Fil}^1\mathcal{M}_i := g_i(\mathrm{Fil}^1\mathcal{M})$, where $g_i : \mathcal{M} \to \mathcal{M}_i$ is the obvious map. In particular there is a surjective map

$$\varphi^*(\mathrm{Fil}^1\mathcal{M}_i) \twoheadrightarrow \mathcal{M}_i.$$

For every i there is a surjective map

$$\begin{array}{ccc} R_i = W[u]/u^i & \twoheadrightarrow & \mathscr{O}_K/\pi^i \\ u & \mapsto & \pi \end{array}$$

with kernel pR_i. This tells us in particular that $(\mathrm{Spec}(\mathscr{O}_K/\pi^i) \hookrightarrow \mathrm{Spec}(R_i)) \in \mathrm{Cris}(\mathrm{Spec}(\mathscr{O}_K/\pi^i)/W)$ and hence, given a p-divisible group H_i over $\mathscr{O}_K/\pi^i$, we may evaluate the Dieudonné crystal $\mathbb{D}(H_i)(R_i)$. In particular, there is a diagram

$$\begin{tikzcd} \mathrm{Fil}^1\mathbb{D}(H_i)(R_i) \arrow[r] \arrow[d] & \mathbb{D}(H_i)(R_i) \arrow[d] \\ \underline{V}(H_i) \arrow[r] & \mathbb{D}(H_i)(\mathscr{O}_K/\pi^i) \end{tikzcd} \tag{3.0.6}$$

We want to construct *inductively* p-divisible groups G_i over $\mathscr{O}_K/\pi^i$ for $i = 1, \dots, e$, such that

$$\mathbb{D}(G_i)(R_i) \xrightarrow{\simeq} \mathcal{M}_i.$$

Suppose first $i = 1$. We have that $R_1 = W[u]/u \simeq W$ and that $\mathcal{M}_1$ and $\varphi^*(\mathrm{Fil}^1\mathcal{M}_1)$ are W-modules of the same rank, that is,

$$1 \otimes \varphi_1 : \varphi^*(\mathrm{Fil}^1\mathcal{M}_1) \xrightarrow{\simeq} \mathcal{M}_1.$$

This allows us to define a Vershiebung map, namely

$$\begin{array}{ccccccc} \mathcal{M}_1 & \xrightarrow{\simeq} & \varphi^*(\mathrm{Fil}^1\mathcal{M}_1) & \xrightarrow{\varphi^*(\hookrightarrow)} & \varphi^*(\mathcal{M}_1) & \xrightarrow{\simeq} & \mathcal{M}_1 \\ & & & & a \otimes m & \longmapsto & \varphi^{-1}(a)m \end{array}$$

This makes $\mathcal{M}_1$ into a *Dieudonné module* and therefore by classical Dieudonné theory ((2.3)) it is uniquely associated to a p-divisible group G_1 over R_1.

Suppose now $2 \le i \le e$ and assume there exists G_{i-1} over $\mathscr{O}_K/\pi^{i-1}$ such that

$$\mathbb{D}(G_{i-1})(R_{i-1}) \xrightarrow{\simeq} \mathcal{M}_{i-1} \tag{3.0.7}$$

in $\mathcal{C}_{R_{i-1}}$. Note that $(\mathrm{Spec}(\mathscr{O}_K/\pi^{i-1}) \hookrightarrow \mathrm{Spec}(R_i))$ is an element of the crystalline site $\mathrm{Cris}((\mathscr{O}_K/\pi^{i-1})/W)$ as well, since there is an obvious map

$$R_i \twoheadrightarrow \mathscr{O}_K/\pi^{i-1}$$

with kernel the P.D. ideal (u^{i-1}, p). Hence we may consider the evaluation of the crystal $\mathbb{D}(G_{i-1})(R_i)$.

Put $\mathrm{Fil}^1\mathbb{D}(G_{i-1})(R_i)$ the pre-image of $\underline{V}(G_{i-1}) \subseteq \mathbb{D}(G_{i-1})(\mathscr{O}_K/\pi^{i-1})$. By Lemma (3.0.4.6), $\mathbb{D}(G_{i-1})(R_i) \in \mathcal{C}_{R_i}$ and hence the isomorphism (3.0.7), being a map of $\mathbb{Z}_p$-algebras, respecting the Frobenius map, is an isomorphism in $\mathcal{C}_{R_{i-1}}$. Hence, applying Lemma (3.0.4.7) to the obvious surjection $R_i \twoheadrightarrow R_{i-1}$, we get an isomorphism in $\mathcal{C}_{R_i}$

$$\mathbb{D}(G_{i-1})(R_i) \xrightarrow{\simeq} \mathcal{M}_i.$$

By the definition of $\mathrm{BT}^{\varphi}_{/S}$, we have that $\mathrm{Fil}^1\mathcal{M}_i$ is admissible, and hence we can apply Grothendieck-Messing's theorem to $(\mathcal{M}_i, \mathrm{Fil}^1\mathcal{M}_i)$. The natural surjection

$\mathscr{O}_K/\pi^i \twoheadrightarrow \mathscr{O}_K/\pi^{i-1}$ gives us a P.D.-immersion $\mathrm{Spec}(\mathscr{O}_K/\pi^{i-1}) \hookrightarrow \mathrm{Spec}(\mathscr{O}_K/\pi^i)$. Hence we may define a unique p-divisible group G_i over $\mathscr{O}_K/\pi^i$ lifting G_{i-1}. The structure of diagram (3.0.6) provides the map

$$\mathcal{M}_i \xrightarrow{\simeq} \mathbb{D}(G_{i-1})(R_i) \to \mathbb{D}(G_{i-1})(\mathscr{O}_K/\pi^i) \simeq \mathbb{D}(G_i)(\mathscr{O}_k/\pi^i).$$

Through this last morphism $\mathrm{Fil}^1\mathcal{M}_i$ maps to $\underline{V}(G_i)$. Hence the induction gives us an isomorphism

$$\mathbb{D}(G_{i-1})(R_i) \simeq \mathbb{D}(G_i)(R_i) \xrightarrow{\sim} \mathcal{M}_i \qquad 1 \le i \le e,$$

which is compatible with filtrations and the action of φ.

Suppose $i = e$. We have defined a p-divisible group G_e over $\mathscr{O}_K/\pi^e = \mathscr{O}_K/p$ and we have, by the considerations written above, an isomorphism in $\mathcal{C}_{R_e}$

$$\mathbb{D}(G_e)(R_e) \simeq \mathcal{M}_e. \tag{3.0.8}$$

Note that the kernel $(pS+\mathrm{Fil}^1 S)$ of the surjection $S \twoheadrightarrow \mathscr{O}_K/p$ admits divided powers, hence we may evaluate the crystal $\mathbb{D}(G_e)$ associated to G_e, which by Lemma (3.0.4.6) is an object $\mathbb{D}(G_e)(S)$ of the category $\mathcal{C}_S$. Therefore we may apply Lemma (3.0.4.7) to the surjection $S \twoheadrightarrow R_e$ and obtain from the isomorphism (5.1.1.4) in $\mathcal{C}_{R_e}$ an isomorphism in $\mathcal{C}_S$

$$\mathbb{D}(G_e)(S) \xrightarrow{\simeq} \mathcal{M}.$$

Suppose now that $p > 2$. In this case, the kernel of the surjection $\mathscr{O}_K \twoheadrightarrow \mathscr{O}_K/p$ is a P.D. ideal. By Grothendick-Messing theorem we define $G(\mathcal{M})$ as the unique lifting of G_e to $\mathscr{O}_K$ such that $\underline{V}(G) \subset \mathbb{D}(G_e)(\mathscr{O}_K) \simeq \mathbb{D}(G)(\mathscr{O}_K)$ is the image of $\mathrm{Fil}^1\mathcal{M}$ through the map

$$\mathcal{M} \xrightarrow{\simeq} \mathbb{D}(G)(S) \longrightarrow \mathbb{D}(G)(\mathscr{O}_K).$$

By this map moreover, it is obvious that $\mathcal{M} \xrightarrow{\simeq} \mathcal{M}(G(\mathcal{M}))$. In order to see that $G \xrightarrow{\simeq} G(\mathcal{M}(G))$, note that at every step of the induction we used Messing's theory, and by unicity of the p-divisible group we obtain for every $i = 1, \dots, e$

$$G_i(\mathcal{M}(G)) \simeq G \text{ modulo } \pi^i,$$

and

$$G(\mathcal{M}(G)) \simeq G,$$

and hence we conclude.

Suppose now $p = 2$. In this case the problem is that the kernel of the surjection $\mathscr{O}_K \twoheadrightarrow \mathscr{O}_K/p$ does not have divided powers. This case requires therefore a little more work. We may give to the kernel of $\mathscr{O}_K/p^2 \to \mathscr{O}_K/p$ a P.D. structure $(*)$ by putting $p^{[i]} = 0$ for $i \ge 2$. Therefore we may lift the p-divisible group G_e to a p-divisible group G_{2e} over $\mathscr{O}_K/p^2$ such that the image of $\mathrm{Fil}^1\mathcal{M}$ through the map

$$\mathcal{M} \xrightarrow{\simeq} \mathbb{D}(G_e)(S) \longrightarrow \mathbb{D}(G_e)(\mathscr{O}_K/p^2) \simeq \mathbb{D}(G_{2e})(\mathscr{O}_K/p^2)$$

is $\underline{V}(G_{2e})$. Finally we may lift G_{2e} to a p-divisible group G over $\mathscr{O}_K$ such that the image of $\mathrm{Fil}^1\mathcal{M}$ through

$$\mathcal{M} \xrightarrow{\simeq} \mathbb{D}(G_e)(S) \to \mathbb{D}(G_e)(\mathscr{O}_K/p^2) \simeq \mathbb{D}(G_{2e})(\mathscr{O}_K/p^2) \to \mathbb{D}(G_{2e})(\mathscr{O}_K) \simeq \mathbb{D}(G)(\mathscr{O}_K)$$

is equal to $\underline{V}(G_{2e})$. As in the case $p > 2$, the isomorphism $\mathcal{M}(G(\mathcal{M})) \simeq \mathcal{M}$ holds. Given G a p-divisible group over $\mathscr{O}_K$ we obtain as above an isomorphism

$$G_e(\mathcal{M}(G)) \simeq G \text{ modulo } p.$$

On the other hand though, we do not always have an isomorphism between G_{2e} and G modulo p^2. This happens because the P.D. structure $(*)$ on the kernel $p \subset \mathscr{O}_K/p^2$ is not compatible with the divided powers $(p) \subset S$. Since both G_{2e} and G mod p^2 lift G_e, by [**Katz**], we get maps between the two in both directions

$$G_{2e} \leftrightarrows G \text{ mod } p^2.$$

Moreover, since G and $G(\mathcal{M}(G))$ are both obtained by lifting the image of $\mathrm{Fil}^1\mathcal{M}$ in $\mathbb{D}(G_{2e})(\mathscr{O}_K) \simeq \mathbb{D}(G \bmod p^2)(\mathscr{O}_K)$, we obtain two maps

$$G(\mathcal{M}(G)) \leftrightarrows G.$$

Both composites are the multiplication by p^4, hence we conclude.

□

Remark. *As already pointed out before, deformation theory holds in the case p is locally nilpotent on the base scheme. We use Messing theorem at every step of the induction, however p is not nilpotent on S. In the proof we pass implicitly to the limit at every step of the induction.*

3.0.4.1. *Examples.* We would like to see how the functor

$$\begin{array}{ccc} \mathrm{BT}(\mathscr{O}_K) & \xrightarrow{\mathcal{M}} & \mathrm{BT}^{\varphi}_{/S} \\ G & \longmapsto & \mathbb{D}(G)(S) \end{array}$$

works. The universal extension of G gives us a sequence

$$0 \longrightarrow \underline{\omega}_{G^*} \longrightarrow \underline{Lie}(E(G)) \longrightarrow \underline{\omega}_G \longrightarrow 0$$

.

1. Consider $G = \mu_{p^\infty} = \varinjlim \mu_{p^n}$. We have

$$\mathcal{M}(\mu_{p^\infty}) = S, \qquad \mathrm{Fil}^1\mathcal{M}(\mu_{p^\infty}) = \mathrm{Fil}^1 S.$$

Indeed $\underline{\omega}_{G^*} = 0$ and hence $\mathbb{D}(G)(S)$ equals the evaluation in S of $\underline{\omega}_G$, that is, $\mathbb{D}(G)(S) = S\frac{dT}{T} \simeq S$. Moreover we have $\mathbb{D}(G)(S)/\mathrm{Fil}^1\mathbb{D}(G)(S) \simeq \mathscr{O}_K\frac{dT}{T}$ (evaluation of $\underline{\omega_G}$ and hence $\mathrm{Fil}^1\mathcal{M}(G) \simeq \mathrm{Fil}^1 S$, since $\mathscr{O}_K \simeq S/\mathrm{Fil}^1 S$.

2. For the dual $\mathbb{Q}_p/\mathbb{Z}_p$ we have

$$\mathcal{M}(\mathbb{Q}_p/\mathbb{Z}_p) = S, \qquad \mathrm{Fil}^1\mathcal{M}(\mathbb{Q}_p/\mathbb{Z}_p) = S.$$

CHAPTER 4

Categories of semi-linear algebra data

Let k be a perfect field, $W := W(k)$ the ring of Witt vectors, $K_0 := W[1/p]$ its field of fractions. We denote by φ the Frobenius endomorphism on W, and by extension, on K_0. We consider a totally ramified extension

$$\begin{tikzcd} \mathscr{O}_K \arrow[r, dash] \arrow[d, dash] & K \arrow[d, dash] \\ W \arrow[r, dash] & K_0 \end{tikzcd}$$

with *fixed* uniformizer π, of minimal Eisenstein polynomial $E[u] \in K_0[u]$. We fix moreover an algebraic closure $\overline{K}$ of K. Define a sequence of elements $\pi_n \in \overline{K}$ such that

$$\pi_0 = \pi \qquad \pi_{n+1}^p = \pi_n,$$

and hence a tower of fields

$$K_0 \hookrightarrow K = K_0(\pi) \hookrightarrow \cdots \hookrightarrow K_{n+1} = K(\pi_n) \hookrightarrow \cdots \hookrightarrow \overline{K}.$$

In this chapter we define several categories of semi-linear algebra data, that is, categories of algebraic objects over K. The aim is to give a description of a certain category of $W[[u]]$-modules in terms of modules over K equipped with additional structures, such as a Frobenius map, a differential operator and a filtration.

4.0.5. φ-modules over K_0.

Definition 4.0.5.1 (Étale φ-modules). *Consider a W-algebra A, together with a W-linear Frobenius φ. A φ*-module *over A is a finite free A-module M, together with a φ-semilinear endomorphism, that is, a map $\varphi_M : M \to M$, such that*

$$\varphi_M(x+y) = \varphi_M(x) + \varphi_M(y),$$
$$\varphi_M(\lambda x) = \varphi(\lambda)\varphi_M(x),$$

for $x, y \in M$ and $\lambda \in A$.

A φ-module M is said to be étale *if the A-linearization*

$$\begin{array}{ccc} \varphi^*(M) & \longrightarrow & M \\ \lambda \otimes x & \longmapsto & \lambda\varphi_M(x) \end{array}$$

is an isomorphism ($x \in M, \lambda \in A$). We will often write $\varphi_M = \varphi$. We denote by ΦM_A^{et} the category of étale φ-modules.

Definition 4.0.5.2. *A (φ, N)-module over K_0 is a K_0-vector space D, equipped with two maps*

$$\varphi, N : D \to D$$

such that

1. *D is étale,*
2. *N is a K_0-linear map,*
3. *the relation $N \circ \varphi = p(\varphi \circ N)$ holds.*

A morphism of (φ, N)-modules $\eta : D_1 \to D_2$ is a K_0-linear map commuting with N and φ. We denote by $\mathrm{Mod}^{\varphi,N}_{/K_0}$ the category of (φ, N)-modules over K_0.

The category of (φ, N)-modules is abelian and Tannakian.

DEFINITION 4.0.5.3. *A* filtered *(φ, N)-module over K is a (φ, N)-module D over K_0, together with a filtration on the K-vector sapce $D_K = K \otimes_{K_0} D$, which is decreasing, separated and exhaustive, that is, the $\mathrm{Fil}^i D_K$ satisfy*

$$\mathrm{Fil}^{i+1} D_K \subset \mathrm{Fil}^i D_K, \qquad \bigcap_{i\in\mathbb{Z}} \mathrm{Fil}^i D_K = 0, \qquad \bigcup_{i\in\mathbb{Z}} \mathrm{Fil}^i D_K = D_K.$$

A morphism of filtered (φ, N)-modules is a morphism of (φ, N)-modules $\eta : D_1 \to D_2$ such that $\eta_K : D_1 \otimes_{K_0} K \to D_2 \times_{K_0} K$ satisfies $\eta_K(\mathrm{Fil}^i D_1 \otimes K) \subset \mathrm{Fil}^i(D_2 \otimes_{K_0} K)$ $\forall i$. We denote by $\mathrm{MF}^{\varphi,N}_K$ the category of filtered (φ, N)-modules over K

This is an additive, non-abelian, Tannakian category.

Suppose now $\dim_{K_0} D < \infty$.

- Suppose first $\dim_{K_0} D = 1$, that is, $D = K_0 \cdot d$ for some non zero $d \in D$. Since φ is bijective, we have $\varphi(d) = \alpha d$, with $\alpha \in K_0 - 0$. Note that the p-adic valuation $v_p(\alpha)$ is independent of the choice of the basis $\{d\}$, hence it makes sense to give the following definition:
$$t_N(D) = v_p(\alpha).$$
- If $\dim_{K_0} D = k > 1$, then the exterior product $\bigwedge^k_{K_0} D$ has dimension 1 and hence we define $t_N(D) = t_N(\bigwedge^k_{K_0} D)$.

PROPOSITION 4.0.5.4. *If D is a (φ, N)-module such that $\dim_{K_0} D < \infty$ and φ is bijective, then N is nilpotent.*

For any finite-dimensional filtered K-vector space Δ we may give the following definition

- If $\dim_K \Delta = 1$, we define
$$t_H(\Delta) = \max\{i \in \mathbb{Z} | \mathrm{Fil}^i \Delta = \Delta\},$$
- if $\dim_K \Delta = h$, then we define
$$t_H(\Delta) = t_H(\bigwedge^h \Delta).$$

DEFINITION 4.0.5.5 (Admissible (φ, N)-modules). *A filtered (φ, N)-module is said to be* admissible *if $\dim_{K_0} D < \infty$ and*

1. *$t_H(D_K) = t_N(D)$,*
2. *for any sub-object D', $t_H(D'_K) \leq t_N(D')$.*

The category of admissible filtered (φ, N)-modules, denoted by $\mathrm{MF}^{\varphi,N,ad}_K$, is abelian, see [**Fon3**].

Definition 4.0.5.6. *A filtered* (φ, N)*-module is said to be* effective *if* $\mathrm{Fil}^0 D = D$. *We denote by* $\mathrm{MF}_K^{\varphi,N,\mathrm{Fil}_{\geq 0}}$ *the full sub-category of* MF_K^{φ} *of filtered* φ*-modules which are effective.*

4.1. A geometric interpretation of $\mathrm{MF}_K^{\varphi,N}$

Put $\mathfrak{S} = W[[u]]$. We have the following strict inclusions:

$$\mathfrak{S} = W[[u]] \hookrightarrow W[[u]][1/p] \hookrightarrow W[1/p][[u]] = K_0[[u]].$$

Define moreover $\mathfrak{S}_n = \mathfrak{S} \otimes_W K_n$ and denote by $\widehat{\mathfrak{S}_n}$ its completion at the maximal ideal $(u - \pi_n)$.

Consider the open rigid analytic disk $D[0,1)$ over K_0 with coordinate u and for I an interval in $[0,1)$ the admissible open subspace $D(I) \subset D[0,1)$. Define the rings of rigid analytic functions

$$\mathcal{O}_I = \Gamma(D(I), \mathcal{O}_{D(I)});$$

this is a K_0-subalgebra of $K_0[[u]]$. Put $\mathcal{O} = \mathcal{O}_{[0,1)}$ the ring of rigid analytic function on the disk. In particular we have

$$\mathfrak{S}[1/p] \hookrightarrow \mathcal{O} \hookrightarrow K_0[[u]].$$

4.1.0.1. *The Frobenius map.* All these objects are endowed with an action by a Frobenius map. The Frobenius map φ on W extends to a Frobenius

$$\begin{array}{rcl} \varphi : \mathfrak{S} & \longrightarrow & \mathfrak{S} \\ u & \longmapsto & u^p. \end{array}$$

Together with the regular Frobenius we may define the two following endomorphisms on $\mathfrak{S}$:

$$\begin{array}{lll} \text{the } \mathbb{Z}_p[[u]]\text{-linear map} & \varphi_W : \mathfrak{S} \to \mathfrak{S}, & x \mapsto x^p, \text{ for } x \in W, \\ \text{the } W\text{-linear map} & \varphi_{\mathfrak{S}/W} : \mathfrak{S} \to \mathfrak{S}, & u \mapsto u^p. \end{array}$$

Through these we may induce maps on $\mathcal{O}_I$:

$$\varphi_W : \mathcal{O}_I \to \mathcal{O}_I \qquad \varphi_{\mathfrak{S}/W} : \mathcal{O} \to \mathcal{O}_{p^{-1}I},$$

and consider

$$\varphi := \varphi_W \circ \varphi_{\mathfrak{S}/W} : \mathcal{O}_I \to \mathcal{O}_{p^{-1}I}.$$

In particular on the ring $\mathcal{O}$ there is a Frobenius map

$$\varphi : \mathcal{O} \to \mathcal{O}.$$

4.1.0.2. *The differential operator* N_∇.

Note. *The definitions which follow depend all on the choice of the uniformizer* π.

Since $E(u) \in \mathfrak{S}$ we have that $E(u) \in \mathcal{O}$. Define the element

$$\lambda := \prod_{n \geq 0} \varphi^n(E(u)/E(0)) \in \mathcal{O}$$

and a *derivation* on $\mathcal{O}$

$$N_\nabla := -u\lambda \frac{d}{du} : \mathcal{O} \to \mathcal{O}.$$

We have of course an induced derivation 'N_∇' on every $\mathscr{O}_I$.

The operator N_∇ satisfies the following *monodromy relation* on $\mathscr{O}$:

$$N_\nabla \circ \varphi = pE(u)/E(0)(\varphi \circ N_\nabla). \tag{4.1.1}$$

Indeed, for $\sum_{n\geq 0} a_n u^n \in \mathscr{O}$:

$$N_\nabla \circ \varphi(\sum a_n u^n) = N_\nabla(\sum \varphi(a_n)u^{pn}) = -\sum \varphi(a_n)pnu^{pn}\lambda,$$
$$\varphi \circ N_\nabla(\sum a_n u^n) = -\varphi(\sum a_n u^n E(0)/E(u)\lambda) = -\sum \varphi(a_n)nE(0)/E(u)u^{pn}\lambda.$$

Note that if we evaluate in 0 the equation (4.1.1), we get the classical relation

$$N \circ \varphi = p\varphi \circ N. \tag{4.1.2}$$

4.1.1. (φ, N_∇)-modules over $\mathscr{O}$ of finite E-height.

Definition 4.1.1.1. *A* φ-module *over $\mathscr{O}$ is a finite free $\mathscr{O}$-module $\mathcal{M}$, equipped with a φ-semilinear map $\varphi : \mathcal{M} \to \mathcal{M}$ such that the linearlization $\varphi^*(\mathcal{M}) \xrightarrow{1\otimes\varphi} \mathcal{M}$ is injective. A* (φ, N_∇)-module *over $\mathscr{O}$ is an φ-module $\mathcal{M}$ over $\mathscr{O}$, with a differential operator $N_\nabla^{\mathcal{M}}$ over N_∇. Namely, we have the relation*

$$N_\nabla^{\mathcal{M}}(fm) = N_\nabla(f)m + fN_\nabla^{\mathcal{M}}(m), \qquad \text{for } f \in \mathscr{O}, m \in \mathcal{M},$$

and φ and N_∇ are related by the formula

$$N_\nabla^{\mathcal{M}} \circ \varphi = pE(0)/E(u)(\varphi \circ N_\nabla^{\mathcal{M}}).$$

Definition 4.1.1.2. *We say that a φ-module $\mathcal{M}$ over $\mathscr{O}$ is of* finite E-height *if the cokernel of the linearization $1\otimes\varphi : \varphi^*(\mathcal{M}) \to \mathcal{M}$ is killed by some power of $E(u)$. We denote by $Mod^{\varphi}_{/\mathscr{O}}$ the category of φ-modules over $\mathscr{O}$ that are of finite E-height. In particular a (φ, N_∇)-module over $\mathscr{O}$ is of finite E-height if it is of finite E-height as φ-module. We denote by $Mod^{\varphi,N_\nabla}_{/\mathscr{O}}$ the category of such objects.*

Note. *One can see $\mathcal{M}$ as a coherent sheaf on the unitary disk. It will often be useful to study φ-modules in a neighborhood of a point of $D_{[0,1)}$.*

Theorem 4.1.1.3 (Kisin). *There is an exact, quasi-inverse equivalence of categories*

$$\mathcal{M} : \mathrm{MF}_K^{\varphi,N,\mathrm{Fil}_{\geq 0}} \xrightarrow{\simeq} Mod^{\varphi,N_\nabla}_{/\mathscr{O}}. \tag{4.1.3}$$

We show now, how the functor $\mathcal{M}$ and its quasi-inverse $\mathcal{D}$ are defined.

Define the ring $\mathscr{O}[l_u]$, where l_u is a formal variable. For every natural n there is a natural map

$$\begin{array}{ccccc} \mathfrak{S}[1/p] & \to & \mathscr{O} & \to & \widehat{\mathfrak{S}_n} \\ u & \mapsto & u & \mapsto & u \end{array}$$

extending to a map

$$\begin{array}{ccc} \mathscr{O}[l_u] & \longrightarrow & \widehat{\mathfrak{S}_n} \\ l_u & \longmapsto & \log\left[\frac{u-\pi_n}{\pi_n} + 1\right] \end{array}$$

where $\log\left[\frac{u-\pi_n}{\pi_n} + 1\right] := \sum_{i=1}^{\infty} (-1)^{i-1} i^{-1} (\frac{u-\pi_n}{\pi_n})^i \in \widehat{\mathfrak{S}_n}$.

Consider D an effective, filtered (φ, N)-module over K. We may consider the tensor $\mathscr{O}[l_u] \otimes_{K_0} D$ and define for any n a map

$$\iota_n : \mathscr{O}[l_u] \otimes_{K_0} D \xrightarrow{\varphi_W^{-n} \otimes \varphi^{-n}} \mathscr{O}[l_u] \otimes_{K_0} D \longrightarrow \widehat{\mathfrak{S}_n} \otimes_{K_0} D = \widehat{\mathfrak{S}_n} \otimes_K D_K.$$

The last equality is true, since $\mathfrak{S}_n := \mathfrak{S} \otimes_W K_n$ is a K-algebra and therefore $\widehat{\mathfrak{S}_n} \otimes_{K_0} D = \widehat{\mathfrak{S}_n} \otimes_K D_K$. We can extend ι_n to

$$\iota_n : \mathscr{O}[l_u, 1/\lambda] \longrightarrow \widehat{\mathfrak{S}_n}[1/(u - \pi_n)] \otimes_K D_K = Fr(\widehat{\mathfrak{S}_n}) \otimes_K D_K.$$

Indeed, the Frobenius on $\mathscr{O}[l_u]$ extends to $\mathscr{O}[l_u, \lambda]$ (we have $\varphi(1/\lambda) = \prod_n \varphi^{-n+1}(E(u)/E(0)) = E(u)/E(0) \cdot 1/\lambda$). Note that in $\mathscr{O}[l_u, 1/\lambda]$ has a differential operator N_∇ induced by $N_\nabla \otimes 1$. Define finally

$$\mathcal{M}(D) := \{x \in (\mathscr{O}[l_u, 1/\lambda] \otimes_{K_0} D)^{N=0} \mid \forall n, \iota_n(x) \in \mathrm{Fil}^0(\widehat{\mathfrak{S}_n}[1/(u - \pi_n)] \otimes_K D_K)\}.$$

Lemma 4.1.1.4. *The module $\mathcal{M}(D)$ has a structure of (φ, N_∇)-module. Moreover it is of finite E-height, that is, $\mathcal{M}(D) \in \mathrm{Mod}_{/\mathscr{O}}^{\varphi, N_\nabla}$.*

Proof. [**Kis**, Lemma 1.2.2]. □

Let's define now a quasi-inverse

$$\mathcal{D} : \mathrm{Mod}_{/\mathscr{O}}^{\varphi, N_\nabla} \longrightarrow \mathrm{MF}_K^{\varphi, N, \mathrm{Fil}_{\geq 0}}.$$

Consider $\mathcal{M} \in \mathrm{Mod}_{/\mathscr{O}}^{\varphi, N_\nabla}$. We define a φ-module $\mathcal{D}(\mathcal{M})$ as the object $\mathcal{M}/u\mathcal{M}$ with operator φ induced by the Frobenius on $\mathcal{M}$. This is given an operator N by reducing N_∇ modulo $u\mathcal{M}$.

Lemma 4.1.1.5. *The (φ, N)-module $\mathcal{D}(\mathcal{M})$ defined above has an effective filtration.*

4.1.2. Kedlaya's theory of slopes. We give now a characterization of admissibility in the equivalence of categories (4.1.3). Following a very original idea by L. Berger ([**Ber**]) we can describe this notion in terms of *Kedlaya's theory of slopes*. This is a generalization of the classification (*Dieudonné-Manin*) of the filtrations of finite free φ-modules over a complete discrete valuation ring with algebrically closed residue field. This classical result is no longer true under more general hypotheses and two papers by Kedlaya ([**Ked1**] and [**Ked2**]) provide a universal filtration (*slope filtration*) on étale φ-modules over the Robba ring $\mathscr{R}$.

Definition 4.1.2.1. *We define the following rings of functions:*

- *The Robba ring*
$$\mathscr{R} := \lim_{r \to 1^-} \mathscr{O}_{(r,1)},$$
with a Frobenius map induced by the Frobenius maps on the rings $\mathscr{O}_{(r,1)}$;
- *the* bounded *Robba ring*
$$\mathscr{R}^b := \lim_{r \to 1^-} \mathscr{O}^b_{(r,1)},$$
where $\mathscr{O}^b_{(r,1)}$ is the set of bounded functions[1] *in $\mathscr{O}_{(r,1)}$. The Frobenius on $\mathscr{R}$ induces a Frobenius on $\mathscr{R}_b$.*

[1]Rings of bounded functions: note that $\mathscr{O} \cap \mathscr{R}^b = \mathscr{O}^b = \mathfrak{S}[1/p]$.
The inclusion $\mathfrak{S}[1/p] \subset \mathscr{O}^b$ is clear. We have that the uniform norm is equivalent to the Gauss

We define a category $Mod^{\varphi}_{/\mathscr{R}}$ of finite free $\mathscr{R}$-modules equipped with an isomorphism

$$1 \otimes \varphi : \varphi^*(\mathcal{M}) \xrightarrow{\simeq} \mathcal{M}.$$

This is a Tannakian category. We define the category $Mod^{\varphi}_{/\mathscr{R}^b}$ analogously.

We state now the results by Kedlaya. Namely, we can associate to any object $\mathcal{M} \in Mod^{\varphi}_{/\mathscr{R}}$ a finite set of rational numbers $\{s_1, \dots, s_r\} \subset \mathbb{Q}$. Its existence is guaranteed by the following result:

THEOREM 4.1.2.2 (Kedlaya). *There exists an $\mathscr{R}$-algebra $\mathscr{R}^{alg}$ which contains $W(\overline{k})$ and has a lifting of the Frobenius on $\mathscr{R}$, such that for any $\mathcal{M} \in \mathrm{Mod}^{\varphi}_{/\mathscr{R}}$ there exists a finite extension $W(\overline{k})[1/p] \hookrightarrow E$ such that the tensor product*

$$\mathcal{M} \otimes_{\mathscr{R}} \mathscr{R}^{alg} \otimes_{W(\overline{k})[1/p]} E$$

admits a basis of eigenvectors $v_1, \dots, v_n$ for φ, with eigenvalues belonging to E. The p-adic valuations of these eigenvalues are called the slopes *of $\mathcal{M}$. If the set of slopes contains only an $s \in \mathbb{Q}$, then we say that $\mathcal{M}$ is* pure *of slope s. Moreover for any $\mathcal{M}$ there exists a canonical filtration*

$$\text{slope filtration} \qquad 0 = \mathcal{M}_0 \subset \mathcal{M}_1 \subset \cdots \subset \mathcal{M}_r = \mathcal{M}$$

by φ-stable submodules such that the quotient $\mathcal{M}_i/\mathcal{M}_{i-1}$ is finite free over $\mathscr{R}$ and pure of slope s_i, for $s_1 < \cdots < s_r$.

PROOF. [**Ked1**] □

We denote by $Mod^{\varphi,s}_{/\mathscr{R}}$ the full sub-category of modules $\mathcal{M} \in Mod^{\varphi}_{/\mathscr{R}}$ that are pure of slope $s \in \mathbb{Q}$. Analogously we define $Mod^{\varphi,s}_{/\mathscr{R}^b}$. We have the following result.

THEOREM 4.1.2.3 (Kedlaya). *There is an equivalence of categories*

$$\begin{array}{ccc} Mod^{\varphi,s}_{/\mathscr{R}^b} & \longrightarrow & Mod^{\varphi,s}_{/\mathscr{R}} \\ \mathcal{M} & \longmapsto & \mathcal{M} \otimes_{\mathscr{R}^b} \mathscr{R} \end{array}$$

PROOF. [**Ked2**] □

Define a differential operator $N_{\nabla} := -u\lambda \frac{d}{du}$ on $\mathscr{R}$.

DEFINITION 4.1.2.4. *We define the category $Mod^{\varphi,N_{\nabla}}_{/\mathscr{R}}$ of modules $\mathcal{M} \in Mod^{\varphi}_{/\mathscr{R}}$ equipped with an operator $N^{\mathcal{M}}_{\nabla} = N_{\nabla}$ over N_{∇} on $\mathscr{R}$ satisfying the relation*

$$N_{\nabla} \circ \varphi = pE(u)/E(0)(\varphi \circ N_{\nabla}).$$

We define a functor

$$\begin{array}{ccc} Mod^{\varphi,N_{\nabla}}_{/\mathscr{O}} & \longrightarrow & Mod^{\varphi}_{/\mathscr{R}} \\ \mathcal{M} & \longmapsto & \mathcal{M}_{\mathscr{R}} = \mathcal{M} \otimes_{\mathscr{O}} \mathscr{R} \end{array}$$

norm. Hence, given $f \in O^b$, we have that $\|f\|_{\infty} \leq C$ a constant and hence we may find (through the properties of the Gauss norm, a non-zero scalar λ such that $\|\lambda f\|_{Gauss} = \|\lambda f\|_{\infty} = 1$, and therefore $f \in \mathfrak{S}[1/p]$.

Remark. *In order to show that the tensor product is well defined we need to show (according to the definition of* $\mathrm{Mod}^{\varphi}_{/\mathscr{R}}$*), that the map* $\varphi^*(\mathcal{M}_{\mathscr{R}}) \to \mathcal{M}_{\mathscr{R}}$ *is an isomorphism. Note that the modules* $\mathcal{M}$ *are free and hence injectivity is preserved after tensoring. Moreover,* $E(u)$ *is a unit in* $\mathscr{R}$*. Since* $\varphi^*(\mathcal{M}) \to \mathcal{M}$ *has cokernel killed by a power of* $E(u)$ *we obtain also the surjectivity of the map in* $\mathrm{Mod}^{\varphi}_{/\mathscr{R}}$.

For any subinterval $I \subset [0,1)$ *there is a natural map*

$$\mathscr{O} \to \mathscr{O}_I \to \mathscr{R}.$$

In particular, given $\mathcal{M}$ *as in the statement, the* $\mathscr{O}_I$*-Module* $\mathcal{M}_I = \mathcal{M} \otimes_{\mathscr{O}} \mathscr{O}_I$ *is given a differential operator (that we'll call again* N_∇*) induced by* N_∇ *on* $\mathcal{M}$*. Passing to the limit we get an operator* N_∇ *on* $\mathcal{M}_{\mathscr{R}}$*. Hence the functor above extends to a functor*

$$Mod^{\varphi,N_\nabla}_{/\mathscr{O}} \longrightarrow Mod^{\varphi,N_\nabla}_{/\mathscr{R}}.$$

We show now that the slope filtration of $\mathcal{M}_{\mathscr{R}}$ is induced by the filtration on $\mathcal{M}$. In order to do this, we will state some technical lemmas. For the proofs, see [**Kis**]

Definition 4.1.2.5 (Saturated module). *Let* $\mathcal{M}$ *be a finite free* $\mathscr{R}$*-module (for example* $\mathcal{M} \in Mod^{\varphi}_{/\mathscr{R}}$*). We say that an* $\mathscr{R}$*-submodule* $\mathcal{N} \subset \mathcal{M}$ *is* saturated *if it is finitely generated and* $\mathcal{M}/\mathcal{N}$ *is torsion-free. We may define the* saturation *of* $\mathcal{N} \subset \mathcal{M}$ *as the smallest saturated* $\mathscr{R}$*-submodule* $\mathcal{N}' \subset \mathcal{M}$ *containing* $\mathcal{N}$.

Lemma 4.1.2.6. *Let* $\mathcal{M}$ *be a finite free* $\mathscr{O}$*-module equipped with a* φ*-semilinear map* $\varphi : \mathcal{M} \to \mathcal{M}$ *such that* $\varphi^*(\mathcal{M}) \to \mathcal{M}$ *is injective. Let* $\mathcal{N}_{\mathscr{R}} \subset \mathcal{M}_{\mathscr{R}}$ *be a saturated submodule which is stable under* φ*. Then there is a unique saturated submodule* $\mathcal{N}_{(0,1)} \subset \mathcal{M}_{(0,1)}$ *such that* $\mathcal{N}_{(0,1)} \otimes_{\mathscr{O}_{(0,1)}} \mathscr{R} = \mathcal{N}_{\mathscr{R}}$ *and* $\mathcal{N}_{(0,1)}$ *is* φ*-stable.*

Lemma 4.1.2.7. *Let* $\mathcal{M}$ *be a finite free* $\mathcal{O}$*-module equipped with a differential operator* ∂ *over* $-u\frac{d}{du}$*, and suppose that the operator* $N : \mathcal{M}/u \to \mathcal{M}/u\mathcal{M}$ *induced by* ∂ *is nilpotent. If* $\mathcal{N}_{(0,1)} \subset \mathcal{M}_{(0,1)}$ *is a saturated* $\mathscr{O}_{(0,1)}$*-submodule which is stable under* ∂*, then* $\mathcal{N}_{(0,1)}$ *extends uniquely to a saturated,* ∂*-stable* $\mathcal{O}$*-submodule* $\mathcal{N} \subset \mathcal{M}$.

Proof. [**Kis**, Lemma 1.3.5]. □

Proposition 4.1.2.8. *Let* $\mathcal{M} \in Mod^{\varphi,N_\nabla}_{/\mathscr{O}}$ *and* $\mathcal{M}_{\mathscr{R}} = \mathcal{M} \otimes_{\mathscr{O}} \mathscr{R}$*. Given* $\mathcal{M}_{\mathscr{R}}$*'s slope filtration*

$$0 = \mathcal{M}_{0,\mathscr{R}} \subset \mathcal{M}_{1,\mathscr{R}} \subset \cdots \subset \mathcal{M}_{r,\mathscr{R}} = \mathcal{M}_{\mathscr{R}},$$

for every $i = 0,1,\ldots,r$*, we have that* $\mathcal{M}_{i,\mathscr{R}}$ *extends uniquely to a saturated* $\mathscr{O}$*-submodule* $\mathcal{M}_i \subset \mathcal{M}$ *which is stable under* φ *and* N_∇.

Sketch of Proof. Note that for every i, the modules $\mathcal{M}_{i,\mathscr{R}}$ of the slope filtration are obviously saturated according to the definition. Hence by Lemma (4.1.2.6) we may find a saturated, φ-stable module $\mathcal{M}_{i,(0,1)} \subset \mathcal{M}_{(0,1)}$ such that $\mathcal{M}_{i,(0,1)} \otimes_{\mathscr{O}_{(0,1)}} \mathscr{R} = \mathcal{M}_{i,\mathscr{R}}$. We have in particular that $\mathcal{M}_{(0,1)}$ is N_∇-stable. Then by Lemma (4.1.2.7) it extends to a unique N_∇-stable saturated $\mathscr{O}$-Module $\mathcal{M}_i \subset \mathcal{M}$. □

Theorem 4.1.2.9 (Characterization of admissibility). *Let* D *be an effective filtered* (φ, N)*-module. Then* D *is admissible if and only if* $\mathcal{M}(D) \in Mod^{\varphi,N_\nabla}_{/\mathscr{O}}$ *is pure of slope 0 (that is if* $\mathcal{M}(D) \otimes_{\mathscr{O}} \mathscr{R}$ *is pure of slope* 0*).*

Sketch of proof. Note that the functor $\mathcal{M}$ is compatible with tensor product and preserves the rank, from which it follows that

$$\det \mathcal{M}(D) = \mathcal{M}(\det D). \tag{4.1.4}$$

We focus first on the case D of rank 1, with basis $e \in D$. Put $D_0 = (\mathscr{O}[l_u] \otimes_{K_0} D)^{N=0}$, then we have, by the definition of the functor $\mathcal{M}$

$$\mathcal{M}(D) = \lambda^{-t_H(D)} D_0.$$

Consider the eigenvalue $\alpha \in K_0 - \{0\}$ of e, with respect to φ. Recall that by definition its p-adic valuation is $t_N(D)$. We have

$$\varphi(\lambda^{-t_H(D)} e) = (E(u)/E(0))^{t_H(D)} \alpha \lambda^{-t_H(D)} e.$$

Now $E(u)$ is invertible in $\mathscr{R}$ and $E(0) \in p\mathscr{R}$ we have that the p-adic valuation of $(E(u)/E(0))^{t_H(D)}$ is $-t_H(D)$. We conclude hence that $\mathcal{M}(D)$ has slope $t_N(D) - t_H(D)$, hence the conclusion in the case D has rank 1.

Consider now the general case. By (4.1.4) and by

$$t_N(D) = t_N(\det D), \qquad t_H(D) = t_H(\det D),$$

we have that if $\mathcal{M}(D)$ has slope 0, then D is admissible. Conversely, suppose that D is admissible and consider the $\mathscr{R}$-module $\mathcal{M}(D)_{\mathscr{R}} = \mathcal{M}(D) \otimes_{\mathscr{O}} \mathscr{R}$. By Proposition (4.1.2.8) the slope filtration on $\mathcal{M}(D)_{\mathscr{R}}$ is induced by a filtration of $\mathcal{M}(D)$ by saturated (φ, N_∇)-modules over $\mathscr{O}$

$$0 = \mathcal{M}_0 \subset \mathcal{M}_1 \subset \cdots \subset \mathcal{M}_r = \mathcal{M}(D).$$

Moreover we have that $\mathcal{M}_i / \mathcal{M}_{i-1} \in \mathrm{Mod}^{\varphi, N_\nabla}_{/\mathscr{O}}$ is pure of slope s_i. By results by Kedlaya [**Ked1**] we obtain that $r = 1$ and $s_1 = s = 0$. Hence the conclusion. □

4.2. $\mathfrak{S}$-modules and the category $\mathrm{MF}_K^{\varphi, N, \mathrm{Fil}_{\geq 0}, ad}$

4.2.1. (φ, N)-modules over $\mathscr{O}$.

Definition 4.2.1.1. *A (φ, N)-module over $\mathscr{O}$ is a φ-module $\mathcal{M}$ over $\mathscr{O}$ together with a K_0-linear map*

$$N : \mathcal{M}/u\mathcal{M} \to \mathcal{M}/u\mathcal{M}$$

satisfying to the relation

$$N \circ \varphi = p(\varphi \circ N),$$

where φ is the reduction modulo $u\mathcal{M}$ of $\varphi : \mathcal{M} \to \mathcal{M}$.

We denote by $Mod^{\varphi, N}_{/\mathscr{O}}$ the category of (φ, N)-modules over $\mathscr{O}$ of finite E-height, and by $Mod^{\varphi, N, 0}_{\mathscr{O}}$ the subcategory of modules of slope 0 (that is, according to Theorem (4.1.2.9)., the subcategory corresponding to admissible, effective, filtered (φ, N)-modules over K_0).

Lemma 4.2.1.2. *We define a functor*

$$\begin{array}{ccc} \mathrm{Mod}^{\varphi, N_\nabla}_{/\mathscr{O}} & \to & \mathrm{Mod}^{\varphi, N}_{/\mathscr{O}} \\ \mathcal{M} & \mapsto & \widetilde{\mathcal{M}} \end{array}$$

by taking $\widetilde{\mathcal{M}} = \mathcal{M}$ equipped with the operator φ and taking N to be the reduction modulo $u\mathcal{M}$ of N_∇. We have the following facts:

(1) $\mathcal{M}[1/\lambda]$ *is canonically equipped with an operator* N_∇ *such that* $N_\nabla\varphi = (p/E(0))E(u)\varphi N_\nabla$ *and* $N_\nabla|_{u=0} = N$*,*
(2) *the functor* $\sim$ *is fully faithful, with essential image the modules* $\mathcal{M}$*, stable under the operator* N_∇ *on* $\mathcal{M}[1/\lambda]$*,*
(3) *any* $\mathcal{M}$ *which has* $\mathscr{O}$*-rank* 1 *is in the image of the functor .*

Proof. [**Kis**, Lemma 1.3.10]. ☐

4.2.2. (φ, N)-modules over $\mathfrak{S}$.

Definition 4.2.2.1. *A* (φ, N)*-module over* $\mathfrak{S}$ *is a finite free* $\mathfrak{S}$*-module* $\mathfrak{M}$*, equipped with a semilinear Frobenius* $\varphi : \mathfrak{M} \to \mathfrak{M}$ *and a linear endomorphism* $N : \mathfrak{M}/u\mathfrak{M} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to \mathfrak{M}/u\mathfrak{M} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$*. We say that* $\mathfrak{M}$ *is* of finite E-height *if the cokernel of* $\varphi^*(\mathfrak{M}) \to \mathfrak{M}$ *is killed by some power of* $E(u)$*. We denote by* $\mathrm{Mod}^{\varphi,N}_{/\mathfrak{S}}$ *the category of* (φ, N)*-modules over* $\mathfrak{S}$ *of finite* E*-height.*

Lemma 4.2.2.2. *The functor*

$$\begin{array}{rcl} \Theta : \mathrm{Mod}^{\varphi,N}_{/\mathfrak{S}} \otimes \mathbb{Q}_p & \longrightarrow & \mathrm{Mod}^{\varphi,N,0}_{/\mathscr{O}} \\ \mathfrak{M} & \longmapsto & \mathfrak{M} \otimes_{\mathfrak{S}} \mathscr{O} \end{array}$$

is an equivalence of Tannakian categories.

Proof. We show that Θ is fully faithful and essentially surjective. Take $\mathcal{M} \in \mathrm{Mod}^{\varphi,N,0}_{/\mathscr{O}}$ and fix an $\mathscr{O}$-basis for this module. We have maps

$$\begin{array}{ccccc} \mathrm{Mod}^{\varphi,N,0}_{/\mathscr{O}} & \longrightarrow & \mathrm{Mod}^{\varphi,0}_{/\mathscr{R}} & \xrightarrow{\simeq} & \mathrm{Mod}^{\varphi,0}_{/\mathscr{R}^b} \\ \mathcal{M} & \longmapsto & \mathcal{M}_{\mathscr{R}} = \mathcal{M} \otimes \mathscr{R} & & \\ & & \mathcal{M} \otimes_{\mathscr{R}^b} \mathscr{R} & \longleftarrow & \mathcal{M} \end{array}$$

therefore, given $\mathcal{M} \in \mathrm{Mod}^{\varphi,N,0}_{/}$ we get a module $\mathcal{M}_{\mathscr{R}^b} \in \mathrm{Mod}^{\varphi,0}_{/\mathscr{R}^b}$. We have the following diagram:

$$\begin{array}{ccc} \mathfrak{S}[1/p] & \longrightarrow & \mathscr{O} \\ \downarrow & & \downarrow \\ \mathscr{R}^b & \longrightarrow & \mathscr{R} \end{array}$$

By results of Kedlaya, we may choose a $\mathscr{R}^b$-basis for $\mathcal{M}_{\mathscr{R}^b}$ coinciding with the fixed basis for $\mathcal{M}$. Define hence $\mathcal{M}^b$ to be the $\mathfrak{S}[1/p]$-span of this basis. We have hence defined a φ-stable module of finite E-height. Since $\mathfrak{S}[1/p] = \mathscr{O} \cap \mathscr{R}^b$ we have that

$$\mathcal{M}^b = \mathcal{M} \cap \mathcal{M}^b.$$

Full faithfullness: Note that if $\mathcal{M} \in \mathrm{Mod}^{\varphi,N,0}_{/\mathscr{O}}$ comes from a $\mathfrak{S}$-module $\mathfrak{M} \in \mathrm{Mod}^{\varphi,N}_{/\mathfrak{S}}$, that is, $\mathfrak{M} \otimes \mathbb{Q}_p \in \mathrm{Mod}^{\varphi,N}_{/\mathfrak{S}} \otimes \mathbb{Q}_p$ is such that $\mathfrak{M} \otimes_{\mathfrak{S}} \mathscr{O} = \mathcal{M}$, then clearly

$$(\mathfrak{M} \otimes_{\mathfrak{S}} \mathscr{O})^b = (\mathcal{M})^b = \mathfrak{M}[1/p],$$

hence we conclude.

Essential surjectivity: Suppose $\mathcal{M} \in \mathrm{Mod}^{\varphi,N,0}_{/\mathscr{O}}$ and consider $, \mathcal{M}_{\mathscr{R}^b}$, $\mathcal{M}^b$ as above. We need to show that such a module comes from an $\mathfrak{S}$-module $\mathfrak{M} \in \mathrm{Mod}^{\varphi,N}_{/\mathfrak{S}}$. Note that, by the definition of slope, all eigenvalues of the Frobenius acting on $\mathcal{M}_{\mathscr{R}^b}$ have

p-adic valuation 0, that is, they are invertible, hence the matrix of the linearization of the Frobenius is invertible. This implies in particular, that there exists a φ-stable $\mathscr{O}_{\mathscr{R}^b}$-lattice $\mathcal{L} \subset \mathcal{M}_{\mathscr{R}^b}$, where $\mathscr{O}_{\mathscr{R}^b}$ denotes the ring of integers of $\mathscr{R}^b$. Define

$$\mathfrak{M} = \mathscr{O}_{\mathscr{R}^b} \otimes_{\mathfrak{S}} (\mathcal{M}^b \cap \mathcal{L}) \cap (\mathcal{M}^b \cap \mathcal{L})[1/p].$$

This is a finite, free φ-stable module of finite E-height. The fact that it is finite and φ-stable comes from the fact that $\mathfrak{M} \subset \mathcal{M}_{\mathscr{R}^b}$. On the other hand, for every finite $\mathfrak{S}$-module, there exists a finite free module F such that $\mathfrak{M} \subset F$, hence we can assume $\mathfrak{M} = F$. For the last assertion, note that if d is the rank of the finite free module $\mathfrak{M}$, then $\bigwedge_{\mathfrak{S}}^d \mathfrak{M}$ is an $\mathfrak{S}$-module of rank 1. Choose a basis $\{w \in \mathfrak{M} - \{0\}\}$. Then $\varphi(w) = p^r E(u)^s w$, with $r, s \geq 0$. But since $\mathcal{M}_{\mathscr{R}^b} = \mathfrak{M} \otimes_{\mathfrak{S}} \mathscr{R}^b$ and $\bigwedge_{\mathscr{R}^b}^d \mathcal{M}_{\mathscr{R}^b}$ is of pure slope 0, we conclude that $r = 0$, hence the conclusion. □

From Theorem (4.1.2.9) it follows, as a corollary, the fundamental result of this section.

Corollary 4.2.2.3. *There exists a fully faithful functor*

$$\mathrm{MF}_K^{\varphi,N,\mathrm{Fil}_{\geq 0},ad} \longrightarrow \mathrm{Mod}_{/\mathfrak{S}}^{\varphi,N} \otimes \mathbb{Q}_p.$$

Proof. We have indeed the functors

$$\mathrm{MF}_K^{\varphi,N,\mathrm{Fil}_{\geq 0},ad} \xrightarrow{\simeq} \mathrm{Mod}_{/\mathscr{O}}^{\varphi,N_\nabla,0} \xrightarrow{fullyfaithful} \mathrm{Mod}_{/\mathscr{O}}^{\varphi,N,0} \xrightarrow{\simeq} \mathrm{Mod}_{/\mathfrak{S}}^{\varphi,N} \otimes \mathbb{Q}_p.$$

□

CHAPTER 5

Integral p-adic Hodge theory

5.1. Some p-adic Hodge theory

We keep the notation of the previous sections. We consider the extensions of fields

$$\begin{tikzcd} K_0 \arrow[r, dash] \arrow[d, dash] & K \arrow[r, dash] \arrow[d, dash] & \overline{K} \arrow[d, dash] \\ W \arrow[r, dash] & \mathscr{O}_K \arrow[r, dash] & \mathscr{O}_{\overline{K}} \end{tikzcd}$$

We present now some notions on p-adic Hodge theory. We put $G_K = \mathrm{Gal}(\overline{K}/K)$. The general motivation for this theory comes from the fact that the category $Rep_{\mathbb{Q}_p}(G_K)$ of p-adic Galois representations is extremely big and very hard to study. The idea is to establish equivalences between sub-categories of $Rep_{\mathbb{Q}_p}(G_K)$ and more "treatable", well-behaved abelian categories of semi-linear algebra data.

Definition 5.1.0.4 (p-adic representations). *A* p-adic representation *of G_K, or a* p-adic Galois representation *of K, is a finite dimensional vector space V over $\mathbb{Q}_p$, together with a linear continuous action of G_K. In other words, it is a continuous linear map*

$$\rho : G_K \to GL(V).$$

We denote by $Rep_{\mathbb{Q}_p}(G_K)$ the category of p-adic representations of G_K. A $\mathbb{Z}_p$-representation of G_K is a free $\mathbb{Z}_p$-module of finite rank, together with a linear continuous action of G_K. We denote by $Rep_{\mathbb{Z}_p}(G_K)$ the category of $\mathbb{Z}_p$-representations of G_K.

Note that given $T \in Rep_{\mathbb{Z}_p}(G_K)$ we have $T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \in Rep_{\mathbb{Q}_p}(G_K)$. On the other hand, given a representation $V \in Rep_{\mathbb{Q}_p}(G_K)$ one can construct a $\mathbb{Z}_p$-representation T such that $T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = V$. Indeed, given a free sub $\mathbb{Z}_p$-module T_0 of V of full rank, we have that $g(T_0)$ is still a free sub $\mathbb{Z}_p$-module of V of full rank. Moreover, the stabilizer H of T_0 is an open subset of the profinite group G_K and hence the sum

$$T = \sum_{g \in G/H} g(T_0)$$

is finite. This is a $\mathbb{Z}_p$-representation and a basis of T over $\mathbb{Z}_p$ is also a basis of V over $\mathbb{Z}_p$. Hence the conclusion.

The idea by Fontaine to obtain sub-categories of $Rep_{\mathbb{Q}_p}(G_K)$ relies on the definition of *rings of periods*, that is, topological $\mathbb{Q}_p$-algebras, equipped with a continuous linear action of G_K. The ring of periods B might have in general additional structures compatible with the action of G_K, such as a Frobenius map, a filtration, a differential operator.

Fix B a topological $\mathbb{Q}_p$-algebra with a continuous linear action of G_K.

Definition 5.1.0.5. *A B-*representation *V of G_K is a free B-module of finite rank equipped with a semi-linear*[1] *and continuous action of G_K. If G_K acts trivially on B, we have that V is just a representation of G_K. We say that a B-representation V is* trivial *if $V \simeq B^d$ for some d, with the natural action of G_K.*

Suppose B^{G_K} is a field. If F is a subfield of B^{G_K} and V is an F-representation of G_K, then the tensor $B \otimes_F V$ is equipped with a G_K-action $g(\lambda \otimes x) = g(\lambda) \otimes g(x)$ (for $x \in V, g \in G, \lambda \in B$) and $B \otimes_F V$ is a B-representation of G_K.

Definition 5.1.0.6. *We say that V is B-*admissible *if $B \otimes_F V$ is a trivial B-representation of G_K. The category of B-admissible representations is denoted by $Rep^B_{\mathbb{Q}_p}(G_K)$.*

Given the B^{G_K}-vector space

$$D_B(V) := (B \otimes_F V)^{G_K} \tag{5.1.1}$$

we obtain a B-linear map

$$\begin{array}{rcl} \alpha_V : B \otimes_{B^{G_K}} D_B(V) & \longrightarrow & B \otimes_F V \\ \lambda \otimes x & \longmapsto & \lambda x \end{array}$$

Note that G_K acts on $B \otimes_{B^{G_K}} D_B(V)$ through $g(\lambda \otimes x) = g(\lambda) \otimes x$, for $\lambda \in B, x \in D_B(V), g \in G$.

Definition 5.1.0.7. *We say that B is is (F,G)-regular if the following conditions hold*

1. *B is a domain,*
2. *$B^{G_K} = (FrB)^{G_K}$,*
3. *every non-zero $b \in B$, such that $\forall g \in G_K$ there exists $\lambda \in F$ such that $g(b) = \lambda b$, is invertible in B.*

A field always satisfies the three conditions.

Proposition 5.1.0.8. *Suppose B is (F,G)-regular. Then for any F-representation V the map α_V is injective and it is an isomorphism if and only if V is B-admissible.*

Proof. [**Fon3**, Theorem 2.13]. $\square$

5.1.0.1. *Some rings of periods - the example of crystalline representations.* As a useful example for what follows, we define the ring of periods B_{cris}.

[1] By *semi-linear* we mean:

$$g(x_1 + x_2) = g(x_1) + g(x_2), \qquad g(\lambda x) = g(\lambda)g(x), \qquad \text{for } \lambda \in B, g \in G, x_1, x_2 \in V.$$

The ring R. Let A be a ring such that the map on the quotient

$$\varphi : A/pA \twoheadrightarrow A/pA$$

is surjective. We can associate canonically to A/pA a perfect ring, defined as

$$R(A/pA) := \varprojlim_{a \to a^p} A/pA,$$

where the limit is taken over the projective system with rings $(A/pA)_n = A/pA$ for every n and transition maps the Frobenius $\varphi : A/pA \to A/pA$. The elements of $R(A/pA)$ are the sequences $x = (x_n)_{n \in \mathbb{N}}$ such that $x_{n+1}^p = x_n$, $x_n \in A/pA$ for every n.

PROPOSITION 5.1.0.9. *The ring $R(A/pA)$ is perfect of characteristic p.*

PROOF. For every element $x \in R(A/pA)$, we have $x^p = x$ and $x^p = 0$ if and only if $x_{n+1}^p = x_n = 0$ for every n, that is $x = 0$. $\square$

Note that for every $n \in \mathbb{N}$ there is a map

$$\begin{array}{rcc} \theta_n : R(A/pA) & \to & A/pA \\ x = (x_n)_{n \in \mathbb{N}} & \mapsto & x_n \end{array}$$

Moreover an element $x = (x_n)_{n \in \mathbb{N}} \in R(A/pA)$ is a unit if and only if x_0 is a unit in A/pA.

Suppose now that A is a separated, complete ring, with respect to the p-adic topology. We have the following characterization for the ring $R(A/pA)$.

LEMMA 5.1.0.10. *The ring $R(A/pA)$ is isomorphic to the set*

$$\{(x^{(n)})_{n \in \mathbb{N}} \mid x^{(n+1)p} = x^{(n)}\}.$$

PROOF. Consider an element $x = (x_n)_{n \in \mathbb{N}}$. For each n, we can choose a lifting $\hat{x}_n \in A$ of $x_n \in A/pA$. We obtain a sequence $\hat{x} = (\hat{x}_n)_{n \in \mathbb{N}}$ with relations

$$\hat{x}_{n+1}^p \equiv \hat{x}_n \bmod pA.$$

In particular, for $m, n \in \mathbb{N}$ we have the relation

$$\hat{x}_{n+m+1}^{p^{m+1}} \equiv \hat{x}_{n+m}^{p^m} \bmod p^{m+1}A.$$

Under the hypotheses on A, for every n, this sequence converges to the limit $\lim_{m \to \infty} \hat{x}_{n+m}^{p^m}$ in A, and it doesn't depend on the choice of the lifting. This defines hence a sequence in $(x^{(n)})_{n \in \mathbb{N}} \in A$

$$x^{(n)} := \lim_{m \to \infty} \hat{x}_{n+m}^{p^m}.$$

From the definition it is clear that $x^{(n+1)p} = x^{(n)}$ and this defines hence a map

$$\begin{array}{ccc} R(A/pA) & \to & \{(x^{(n)})_{n \in \mathbb{N}} \mid x^{(n+1)p} = x^{(n)}\} \\ x & \mapsto & (x^{(n)})_{n \in \mathbb{N}} \end{array}$$

$\square$

We study now the introduced object in the case $A = \mathcal{O}_{\widehat{\overline{K}}}$. Put $R := R(\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}) = R(\mathcal{O}_{\widehat{\overline{K}}}/p\mathcal{O}_{\widehat{\overline{K}}})$.

PROPOSITION 5.1.0.11. *The ring R is a complete valuation ring and its fraction field FrR is algebrically closed.*

Proof. Define a valuation on R by

$$v_R(x) = v_p(x^{(0)}).$$

From the surjectivity of the map

$$\begin{array}{ccc} R & \twoheadrightarrow & \mathcal{O}_{\widehat{\overline{K}}} \\ (x_n)_{n\in\mathbb{N}} & \mapsto & x^{(0)} \end{array}$$

we deduce that

$$v_R(R) = \mathbb{Q}_{\geq 0} \cup \{+\infty\}.$$

Note that $v_R(x) = 0 = v_p(x^{(0)}) \Leftrightarrow x^{(0)} = 0 \Leftrightarrow x = 0$ and by multiplicativity of v_p we obtain right away the multiplicativity of v_R. Moreover, for $x \neq 0$ we have the relation $v_R(x) = v_p(x^{(0)}) = p^n v_p(x^{(n)}) < \infty$ and hence there exists a positive integer n such that $v_p(x^{(n)}) < 1$. Since by definition, $(x+y)^{(n)} \equiv x^{(n)} + y^{(n)} \bmod p$, we have

$$v_p((x+y)^{(n)}) \geq \min\{v_p(x^{(n)}), v_p(y^{(n)}), 1\} \geq \min\{v_p(x^{(n)}), v_p(y^{(n)})\}.$$

Hence v_R is a valuation. Moreover, R is complete since the topology of the inverse limit is the same as the topology induced by the valuation. Indeed by $v_R(x) \geq p^n \Leftrightarrow v_p(x^{(n)}) \geq 1 \Leftrightarrow x_n = 0$ we get

$$\{x \in R \mid v_R(x) \geq p^n\} = \mathrm{Ker}\{\theta_n : R \to \mathcal{O}_{\widehat{\overline{K}}}/p\mathcal{O}_{\widehat{\overline{K}}}\}.$$

FrR is algebrically closed: see [**Fon3**, Prop. 4.8]. □

Note that G_K acts on R and FrR in the natural way. As a significant example of element in R, take $\epsilon \in R$, such that $\epsilon^{(0)} = 1$ and $\epsilon^{(1)} \neq 1$. It is invertible in R. Moreover, by definition of the sequence $\tilde{\pi} := (\pi_n)_{n\in\mathbb{N}}$, we have that $\widetilde{\pi} \in R$.

Consider now the ring of Witt vectors $W(R)$. An element $a \in W(R)$ is by construction a sequence $a = (a_0, a_1, \dots, a_m, \dots)$ with $a_m \in R$. In particular a_m corresponds to a sequence $(a_{m,i})_{i n\in\mathbb{N}}$ such that $a_{m,i} \in \mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}$ and $a^p_{m,i+1} = a_{m,i}$, for every i. We define a map

$$\theta : W(R) \to \mathcal{O}_{\widehat{\overline{K}}} \tag{5.1.2}$$

by $\theta((a_0, a_1, \dots)) = \sum_{n\geq 0} p^n a_n^{(n)}$.

Lemma 5.1.0.12 (Structure of the map θ). *The map θ is a ring homomorphism.*

Proof. Note first that for every n there is a map

$$\begin{array}{ccc} W(R) & \longrightarrow & W_n(\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}) \\ a & \longmapsto & (a_{0,n}, a_{1,n}, \dots, a_{n-1,n}) \end{array}$$

obtained through the diagram

$$\begin{tikzcd} W(R) \arrow[r] \arrow[d, "\pi_n"'] & W_n(\mathcal{O}_{\widehat{\overline{K}}}/p\mathcal{O}_{\widehat{\overline{K}}}) \\ W_n(R) \arrow[r, "\psi^{-n}"] & W_n(R) \arrow[u, "W_n(\theta_0)"'] \end{tikzcd}$$

where π_n is the projection on the first n components and the map ψ^{-n} is

$$\begin{array}{ccc} W_n(R) & \to & W_n(R) \\ ((a_{0,i})_i, (a_{1,i})_i, \dots, (a_{n-1,i})_i) & \mapsto & ((a_{0,i+n})_i, (a_{1,i+n})_i, \dots, (a_{n-1,i+n})_i) \end{array}$$

Moreover the diagram

$$\begin{tikzcd} & W_n(\mathscr{O}_{\widehat{\overline{K}}}/p\mathscr{O}_{\widehat{\overline{K}}}) \\ W(R) \arrow[ur] \arrow[dr] & \\ & W_{n+1}(\mathscr{O}_{\widehat{\overline{K}}}/p\mathscr{O}_{\widehat{\overline{K}}}) \arrow[uu, "f_n"'] \end{tikzcd}$$

is commutative, for $f_n((a_0, a_1, \dots, a_n)) = (a_0^p, \dots, a_{n-1}^p)$. By the universal property of inverse limits, we obtain a map

$$W(R) \to \varprojlim_{f_n} W_n(\mathscr{O}_{\widehat{\overline{K}}}/p\mathscr{O}_{\widehat{\overline{K}}}),$$

which is actually an isomorphism (injectivity and surjectivity can be checked on the elements).

On the other hand, there is a natural map

$$W_{n+1}(\mathscr{O}_{\widehat{\overline{K}}}) \to \mathscr{O}_{\widehat{\overline{K}}}/p^n\mathscr{O}_{\widehat{\overline{K}}}.$$

This is obtained by composing the quotient map $\mathscr{O}_{\widehat{\overline{K}}} \to \mathscr{O}_{\widehat{\overline{K}}}/p^n\mathscr{O}_{\widehat{\overline{K}}}$ after the map

$$\begin{array}{rcl} w_{n+1} : W_{n+1}(\mathscr{O}_{\widehat{\overline{K}}}) & \to & \mathscr{O}_{\widehat{\overline{K}}} \\ (a_0, a_1, \dots, a_n) & \mapsto & a_0^{p^n} + pa_1^{p^{n-1}} + \dots + p^n a_n \end{array}$$

Moreover, the surjective map

$$\begin{array}{rcl} W_{n+1}(\mathscr{O}_{\widehat{\overline{K}}}) & \to & W_n(\mathscr{O}_{\widehat{\overline{K}}}/p\mathscr{O}_{\widehat{\overline{K}}}) \\ (a_0, \dots, a_n) & \mapsto & (\overline{a}_0, \dots, \overline{a}_{n-1}) \end{array}$$

has kernel $I = \{(pa_0, pa_1, \dots, pa_{n-1}, a_n \mid a_i \in \mathscr{O}_{\widehat{\overline{K}}}\}$. Since

$$w_{n+1}(pa_0, pa_1, \dots, pa_{n-1}, a_n) = (pa_0)^{p^n} + p(pa_1)^{p^{n-1}} + \dots + p^n a_n \in p^n\mathscr{O}_{\widehat{\overline{K}}},$$

there is a unique morphism

$$\begin{array}{rcl} \theta_n : W_n(\mathscr{O}_{\widehat{\overline{K}}}/p\mathscr{O}_{\widehat{\overline{K}}}) & \to & \mathscr{O}_{\widehat{\overline{K}}}/p^n\mathscr{O}_{\widehat{\overline{K}}} \\ (\overline{a}_0, \dots, \overline{a}_{n-1}) & \mapsto & \sum_{i=0}^{n-1} p^i a_i^{p^{n-i}} \end{array}$$

making the following diagram commutative

$$\begin{tikzcd} W_{n+1}(\mathscr{O}_{\widehat{\overline{K}}}) \arrow[r, "w_{n+1}"] \arrow[d] & \mathscr{O}_{\widehat{\overline{K}}} \arrow[d] \\ W_n(\mathscr{O}_{\widehat{\overline{K}}}/p\mathscr{O}_{\widehat{\overline{K}}}) \arrow[r, "\theta_n"] & \mathscr{O}_{\widehat{\overline{K}}}/p^n\mathscr{O}_{\widehat{\overline{K}}} \end{tikzcd}$$

Moreover, there is compatibility with the projective system $\{W_n(\mathcal{O}_{\widehat{\overline{K}}}/p\mathcal{O}_{\widehat{\overline{K}}}), f_n\}_n$

$$\begin{tikzcd} W_{n+1}(\mathcal{O}_{\widehat{\overline{K}}}/p\mathcal{O}_{\widehat{\overline{K}}}) \arrow[r, "\theta_{n+1}"] \arrow[d] & \mathcal{O}_{\widehat{\overline{K}}}/p^{n+1}\mathcal{O}_{\widehat{\overline{K}}} \arrow[d] \\ W_n(\mathcal{O}_{\widehat{\overline{K}}}/p\mathcal{O}_{\widehat{\overline{K}}}) \arrow[r, "\theta_n"] & \mathcal{O}_{\widehat{\overline{K}}}/p^n\mathcal{O}_{\widehat{\overline{K}}} \end{tikzcd}$$

Now, consider $x \in W(R)$ and its image $(x_{0,n}, x_{1,n}, \dots, x_{n-1,n}) \in W_n(\mathcal{O}_{\widehat{\overline{K}}}/p\mathcal{O}_{\widehat{\overline{K}}})$. Each $x_{i,n} \in \mathcal{O}_{\widehat{\overline{K}}}/p\mathcal{O}_{\widehat{\overline{K}}}$ corresponds to a lifting $x_i^{(n)} \in \mathcal{O}_{\widehat{\overline{K}}}$ and hence

$$\theta(x_{0,n}, x_{1,n}, \dots, x_{n-1,n}) = \sum_{i=0}^{n-1} p^i \overline{(x_i^{(n)})^{p^{n-i}}} = \sum_{i=0}^{n-1} p^i \overline{x_i^{(i)}}.$$

Therefore, by passing the morphism of projective systems θ_n to the limit we obtain the homomorphism θ. □

Consider an element $\tilde{p} \in R$ such that $\tilde{p}^{(0)} = p$ and put $\xi = [\tilde{p}] - p \in W(R)$, where $[\cdot]$ is Teichmüller map.

Proposition 5.1.0.13. *The map*

$$\theta : W(R) \to \mathcal{O}_{\widehat{\overline{K}}}$$

is surjective and its kernel is the principal ideal generated by ξ.

Proof. For any element $a \in \mathcal{O}_{\widehat{\overline{K}}}$ there exists a sequence $x \in R$ such that $x^{(0)} = a$. Consider hence $[x] \in W(R)$. We have that $\theta([x]) = x^{(0)} = a$, hence the surjectivity of the map.

To see that $\mathrm{Ker}\theta = (\xi)$, note first that $\theta(\tilde{p} - p) = \theta(\tilde{p}) - \theta(p) = p - p = 0$. To prove the result it is enough to verify that $\mathrm{Ker}\theta \subset (\xi, p)$, since $W(R)/\mathrm{Ker}\theta = \mathcal{O}_{\widehat{\overline{K}}}$ has no p-torsion and $W(R)$ is complete and p-adically separated (the topology on $W(R)$ is the product topology).For any $x = (x_0, x_1, \dots) \in \mathrm{Ker}\theta$, we have

$$0 = \theta(x) = x_0^{(0)} + p\sum_{n\geq 1} p^{n-1} x_n^{(n)},$$

and hence $v_p(x_0^{(0)}) \geq 1 = v_p(p)$ and hence $v_R(x_0) \geq 1 = v_R(\tilde{p})$. Hence there exists $b_0 \in R$ such that $x_0 = b_0\tilde{p}$. Put $b = [b_0]$, then

$$\begin{aligned} x - b\xi &= (x_0, x_1, \dots) - (b, 0, \dots) \\ &= (x_0 - b_0\tilde{p}, \dots) = (0, y_1, y_2, \dots) \\ &= p(y_1', y_2', \dots) = pW(R), \end{aligned}$$

where $(y_i')^p = y_i$. □

Consider now the ring of fractions $W(R)[1/p] = K_0 \otimes_W W(R)$. There is an identification

$$W(R)[1/p] = \bigcup_{n\geq 0} W(R)p^{-n} = \varinjlim_{n\in\mathbb{N}} W(R)p^{-n},$$

and hence, by extension of scalars on θ, we can consider the morphism of K_0-algebras

$$W(R)[1/p] \to \widehat{\overline{K}},$$

with kernel the ideal (ξ).

Definition 5.1.0.14. *We define the* $\mathrm{Ker}\theta$*-adic completion of* $W(R)[1/p]$,

$$B_{dR}^{+}=\varprojlim_{n\in\mathbb{N}} W(R)[1/p]/(\xi)^n;$$

it is a complete valuation ring, with maximal ideal $\mathrm{Ker}\theta$ *and valuation field*

$$B_{dR}=B_{dR}^{+}[1/\xi].$$

The Galois group G_K acts naturally on B_{dR}^{+} and B_{dR}. There is a natural filtration on B_{dR} indexed by $\mathbb{Z}$,

$$\mathrm{Fil}^i B_{dR}=(\xi)^i, \qquad i\in\mathbb{Z}.$$

An issue regarding the newly defined ring is that it does not admit a canonical extension of the Frobenius map on $W(R)[1/p]$, $\varphi((a_0,a_1,\dots))=(a_0^p,a_1^p,\dots)$. Indeed, $[\tilde{p}^{1/p}]+p\notin \mathrm{Ker}\theta$, that is, $[\tilde{p}^{1/p}]+p$ is invertible in B_{dR}^{+}, but on the other hand, if there was a Frobenius extension φ on B_{dR}^{+}, then one would have $\varphi\Big(\frac{1}{[\tilde{p}^{1/p}]+p}\Big)=\frac{1}{\xi}\notin B_{dR}^{+}$.

The next goal is to define a ring of periods $B_{cris}\subseteq B_{dR}$ endowed with a natural Frobenius. Considering the element $\epsilon\in R$ defined above, we have that $[\epsilon]-1\in W(R)$ belongs to $\mathrm{Ker}\theta$. Indeed $\theta([\epsilon]-1)=\epsilon^{(0)}-1=0$. Then the element $(-1)^{n+1}\frac{([\epsilon]-1)^n}{n}\in W(R)[1/p]\xi^n$ and hence

$$t:=\log([\epsilon])=\sum_{n\geq 1}(-1)^{n+1}\frac{([\epsilon]-1)^n}{n}\in B_{dR}^{+}.$$

Definition 5.1.0.15. *Consider the P.D. envelope of* $W(R)$ *with respect to* $\mathrm{Ker}\theta$, $W(R)\Big[\frac{\xi^m}{m!}\Big]_{m\geq 1}$. *We denote by* $A_{cris}\subset B_{dR}^{+}$ *its p-adic completion.*

Note that the element $t\in B_{dR}^{+}$ belongs to A_{cris}. Indeed $[\epsilon]-1$ belongs to $\mathrm{Ker}\theta$ and hence $[\epsilon]-1=b\xi$ for some $b\in W(R)$. It follows that the fraction $\frac{([\epsilon]-1)^n}{n}=(n-1)!b^n\frac{\xi^n}{n!}$ and we conclude, since $(n-1)!\to 0$ with the p-adic topology.

Definition 5.1.0.16. *Define the* G_K*-stable ring*

$$B_{cris}=A_{cris}[1/t]\subset B_{dR}.$$

Note that since $W\subset A_{cris}$, we have that $K_0\subset B_{cris}$. The Frobenius map $\varphi: W(R)\to W(R)$ extends to a Frobenius on A_{cris} and B_{cris} (see [**Fon3**] for details). Note moreover that the K_0-algebra $B_{cris}\otimes_{K_0}K$ inherits the filtration on B_{dR}. This comes from the following result.

Theorem 5.1.0.17. *The* G_K*-map*

$$K\otimes_{K_0}B_{cris}\longrightarrow B_{dR} \tag{5.1.3}$$

is an injection. Hence we can put on $K\otimes_{K_0}B_{cris}$ *the sub-space filtration*

$$\mathrm{Fil}^i B_{cris}=B_{cris}\cap \mathrm{Fil}^i B_{dR}, \qquad i\in\mathbb{Z}.$$

Proof. See [**Fon4**] □

Proposition 5.1.0.18. *The domain* B_{cris} *is* $(\mathbb{Q}_p,G_K)$*-regular. In particular* $B_{cris}^{G_K}=K_0$.

Proof. See [**Fon4**]. □

Definition 5.1.0.19 (Crystalline representations). *We say that a representation V of G_K is* crystalline *if it is B_{cris}-admissible. The set $Rep_{G_K}^{cris}$ of crystalline representations is a sub-category of $Rep_{\mathbb{Q}_p}(G_K)$.*

Recall that (5.1.1) gives us a functor

$$D_{cris} := D_{B_{cris}} : Rep_{\mathbb{Q}_p}^{cris}(G_K) \longrightarrow K_0 - Vect.$$

From (5.1.0.8) and (5.1.0.18) we get that, for V a crystalline representation, the map

$$\alpha : B_{cris} \otimes_{K_0} D_{cris}(V) \to B_{cris} \otimes_{\mathbb{Q}_p} V$$

is an isomorphism. Moreover we have the following result.

Proposition 5.1.0.20. *The natural map*

$$K \otimes_{K_0} D_{cris}(V) \to D_{dR}(V) := D_{B_{dR}}$$

is an isomorphism of filtered K-algebras. Moreover, the isomorphism α is such that its scalar extension α_K is a filtered isomorphism.

A result by Colmez and Fontaine ([**CF**]) describes precisely the relation between crystalline representations and φ-modules over K.

Theorem 5.1.0.21 (Colmez-Fontaine). *The functor D_{cris} induces an equivalence of categories*

$$Rep_{\mathbb{Q}_p}^{cris}(G_K) \longrightarrow \mathrm{MF}_{\mathrm{K}}^{\varphi,\mathrm{ad}}$$

with inverse $V_{cris}(D) = \mathrm{Fil}^0(B_{cris} \otimes_{K_0} D_K)^{\varphi=1}$, for $D \in \mathrm{MF}_{\mathrm{K}}^{\varphi,\mathrm{ad}}$. There is a contravariant version of the theorem, with functors:

$$D^*_{cris}(V) = \mathrm{Hom}_{\mathbb{Q}_p[G_K]}(V, B_{cris}), \qquad V \in Rep_{\mathbb{Q}_p}^{cris}(G_K),$$
$$V^*_{cris}(D) = \mathrm{Hom}_{\mathrm{Fil},\varphi}(D, B_{cris}), \qquad D \in \mathrm{MF}_{\mathrm{K}}^{\varphi,\mathrm{ad}}.$$

Definition 5.1.0.22. *We define the* Hodge-Tate weights *of the representation V as the integers $i \in \mathbb{Z}$ such that $gr^i D_{cris}(V)_K \neq 0$.*

5.1.0.2. *Étale φ-modules over $\mathscr{O}_{\mathcal{E}}$.* Consider $\tilde{\pi} \in R$; by taking its Teichmüller representative $[\tilde{\pi}] \in W(R)$, since $\theta([(\pi_n)]) = \pi$, we get a map, compatible with the Frobenius

$$\begin{array}{ccccc} \mathfrak{S} & \to & W(R) & \xrightarrow{\overline{\theta}} & \mathscr{O}_K \\ u & \mapsto & [\tilde{\pi}] & \mapsto & \pi, \end{array} \tag{5.1.4}$$

where $\overline{\theta}$ is the restriction of θ to the image of $\mathfrak{S}$.

Define $\mathscr{O}_{\mathcal{E}} = \widehat{\mathfrak{S}[1/u]}$, where the completion is taken with respect to the p-adic topology. This is a complete discrete valuation ring with residue field $k((u))$. The ring $\mathscr{O}_{\mathcal{E}}$ admits an endomorphism $\varphi : \mathscr{O}_{\mathcal{E}} \to \mathscr{O}_{\mathcal{E}}$ lifting the Frobenius map on $k((u))$. By fixing a separable closure $k((u))^s$ of $k((u))$, we may define the maximal unramified extension $\mathscr{O}_{\mathcal{E}^{nr}}$ of $\mathscr{O}_{\mathcal{E}}$ with residue field $k((u))^s$. There is a lifting $\varphi : \mathscr{O}_{\mathcal{E}^{nr}} \to \mathscr{O}_{\mathcal{E}^{nr}}$ of the Frobenius on $\mathscr{O}_{\mathcal{E}}$. The ring $\mathscr{O}_{\mathcal{E}}$ can be seen as a Cohen ring of the field $k((u))$. Define respectively the fraction fields $\mathcal{E} = \mathscr{O}_{\mathcal{E}}[1/p]$ and $\mathcal{E}^{nr}$ of $\mathscr{O}_{\mathcal{E}}$ and $\mathscr{O}_{\mathcal{E}^{nr}}$.

Note that the Galois group $\mathrm{Gal}(\mathcal{E}^{nr}/\mathcal{E})$ acts on the p-adic completion $\widehat{\mathcal{E}^{nr}}$ of $\mathcal{E}^{nr}$ and by construction we have an isomorphism

$$\mathrm{Gal}(\mathcal{E}^{nr}/\mathcal{E}) \simeq G_{k((u))},$$

where $G_{k((u))}$ is the absolute Galois group of $k((u))$. In his paper *Représentations p-adiques des corps locaux* [**Fon1**], J.-M. Fontaine describes the categories $Rep_{\mathbb{Z}_p}(G_{k((u))})$ and $Rep_{\mathbb{Q}_p}(G_{k((u))})$ of p-adic representations of the group $\mathrm{Gal}(\mathcal{E}^{nr}/\mathcal{E})$ establishing equivalences with categories of linear algebra data over $\mathscr{O}_{\mathcal{E}}$.

Definition 5.1.0.23 (Étale φ-Modules over $\mathscr{O}_{\mathcal{E}}$). *We say that a φ-module $\mathfrak{M}$ over $\mathscr{O}_{\mathcal{E}}$ is* étale *the $\mathscr{O}_{\mathcal{E}}$-linear map*

$$\varphi^*\mathfrak{M} \xrightarrow{\simeq} \mathfrak{M}$$

is an isomorphism. Denote by $\mathrm{Mod}^{\varphi}_{/\mathscr{O}_{\mathcal{E}}}$ the category of finite free étale φ-modules over $\mathscr{O}_{\mathcal{E}}$. Denote by $\mathrm{Mod}^{\varphi,tor}_{/\mathscr{O}_{\mathcal{E}}}$ the category of torsion étale φ-modules over $\mathscr{O}_{\mathcal{E}}$.

To any p-torsion $\mathbb{Z}_p$-representation V of $G_{k((u))}$ we can associate a p-torsion φ-module

$$D^*_{\mathcal{E}}(V) \coloneqq \mathrm{Hom}_{G_{k((u))}}(V, \mathcal{E}^{nr}/\mathscr{O}_{\mathcal{E}^{nr}}).$$

To any torsion-free representation V we may associate the torsion-free φ-module

$$D^*_{\mathcal{E}}(V) \coloneqq \mathrm{Hom}_{G_{k((u))}}(V, \mathscr{O}_{\widehat{\mathcal{E}^{nr}}}).$$

Theorem 5.1.0.24 (Fontaine). *For $V \in Rep_{\mathbb{Z}_p}(G_{k((u))})$ p-torsion (resp. torsion-free), we have that $D^*_{\mathcal{E}}(V)$ is étale and $D^*_{\mathcal{E}}$ defines a tensor-preserving anti-equivalence of categories*

$$D^*_{\mathcal{E}} : Rep_{\mathbb{Z}_p}(G_{k((u))}) \xrightarrow{\simeq} \mathrm{Mod}^{\varphi,tor}_{/\mathscr{O}_{\mathcal{E}}}, \tag{5.1.5}$$

(resp. $D^*_{\mathcal{E}} : Rep_{\mathbb{Z}_p}(G_{k((u))}) \xrightarrow{\simeq} \mathrm{Mod}^{\varphi}_{/\mathscr{O}_{\mathcal{E}}}$). The functor $D^*_{\mathcal{E}}$ has quasi-inverse

$$V^*_{\mathcal{E}}(\mathfrak{M}) \coloneqq \mathrm{Hom}_{\mathscr{O}_{\mathcal{E}},\varphi}(\mathfrak{M}, \mathcal{E}^{nr}/\mathscr{O}_{\mathcal{E}^{nr}}),$$

for $\mathfrak{M} \in \mathrm{Mod}^{\varphi,tor}_{/\mathscr{O}_{\mathcal{E}}}$ (resp. $V^*_{\mathcal{E}}(\mathfrak{M}) = \mathrm{Hom}_{\mathscr{O}_{\mathcal{E}},\varphi}(\mathfrak{M}, \mathscr{O}_{\widehat{\mathcal{E}^{nr}}})$, for $\mathfrak{M} \in \mathrm{Mod}^{\varphi}_{/\mathscr{O}_{\mathcal{E}}}$).

5.1.1. Relation between $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ and $\mathrm{Mod}^{\varphi}_{/\mathscr{O}_{\mathcal{E}}}$. Recall that $\mathscr{O}_{\mathcal{E}} \coloneqq \widehat{\mathfrak{S}[1/u]}$. Since $\tilde{\pi}$ is invertible in $W(FrR)$, we have an extension of the map (5.1.4)

$$\mathfrak{S}[1/u] \to W(FrR),$$

and hence a map

$$\widehat{\mathfrak{S}[1/u]} \to W(FrR),$$

that is, a commutative diagram

$$\begin{tikzcd} \mathfrak{S} \arrow[r] \arrow[d, dash] & W(R) \arrow[d, dash] \\ \mathscr{O}_{\mathcal{E}} \arrow[r] & W(FrR) \end{tikzcd}$$

Moreover the inclusion $\mathscr{O}_{\mathcal{E}} \hookrightarrow W(FrR)$ extends to an inclusion

$$\mathcal{E} \hookrightarrow W(FrR)[1/p],$$

that is, there is a diagram

$$\begin{tikzcd} \mathcal{E} \arrow[r] \arrow[d, no head] & W(FrR)[1/p] \arrow[d, no head] \\ \mathscr{O}_{\mathcal{E}} \arrow[r] \arrow[d] & W(FrR) \arrow[d] \\ k((u)) \arrow[r] & FrR \end{tikzcd}$$

Denote by $\mathscr{O}_{\mathcal{E}^{nr}}$ the maximal unramified extension of $\mathscr{O}_{\mathcal{E}}$ in $W(FrR)$, and define $\mathcal{E}^{nr}$ its field of fractions. This fixes a separable closure $k((u))^s$ of $k((u))$. We will also consider the p-adic completion $\mathscr{O}_{\widehat{\mathcal{E}^{nr}}} \subset \widehat{\mathcal{E}^{nr}}$. Define finally $\mathfrak{S}^{nr} = \mathscr{O}_{\mathcal{E}^{nr}} \cap W(R) \subset W(FrR)$.

Recall that $K_\infty := \bigcup_{n\geq 0} K_n$, where $K_n := K(\pi_n)$, and define $G_{K_\infty} = \mathrm{Gal}(\overline{K}/K_\infty) \subset G_K$. Note that G_{K_∞} fixes $\mathfrak{S} \subset W(R)$ and acts on $\mathfrak{S}^{nr} \subset \mathscr{O}_{\mathcal{E}^{nr}}$.

In what follows, we adapt Fontaine's theory of étale φ-modules over $\mathscr{O}_{\mathcal{E}}$ in order to describe $\mathfrak{S}$-modules in terms of Theorem (5.1.0.24). The first observation is that we can re-state the theorem in terms of the Galois group G_{K_∞}.

THEOREM 5.1.1.1. *The action of G_{K_∞} on $\mathscr{O}_{\mathcal{E}^{nr}}$ induces an isomorphism*

$$G_{K_\infty} \xrightarrow{\simeq} \mathrm{Gal}(\mathcal{E}^{nr}/\mathcal{E}).$$

For a proof see [**FW**]. It follows that there is an equivalence of categories

$$D^*_{\mathcal{E}} : Rep_{\mathbb{Z}_p}(G_{K_\infty}) \xrightarrow{\simeq} \mathrm{Mod}^{\varphi}_{/\mathscr{O}_{\mathcal{E}}}.$$

The map $\mathfrak{S} \to \mathscr{O}_{\mathcal{E}}$ is flat. To see this, note first that the localization $\mathfrak{S}_{(p)}$ identifies with the localization at (p) of $\mathfrak{S}[1/u]$, since $u \notin (p)$. Hence, by construction of $\mathscr{O}_{\mathcal{E}}$, the map above factorizes through the map $\mathfrak{S}_{(p)} \to \mathscr{O}_{\mathcal{E}}$ and this is flat. Hence there is a functor

$$\begin{array}{ccc} \mathrm{Mod}^{\varphi}_{/\mathfrak{S}} & \longrightarrow & \mathrm{Mod}^{\varphi}_{/\mathscr{O}_{\mathcal{E}}} \\ \mathfrak{M} & \longmapsto & \mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}} \end{array}$$

since $E(u)$ is invertible in $\mathscr{O}_{\mathcal{E}}$.

LEMMA 5.1.1.2. *Suppose $\mathfrak{M} \in \mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ of p-torsion,*

(1) *there is an isomorphism of $\mathbb{Z}_p[G_{K_\infty}]$-modules:*

$$V^*_{\mathfrak{S}}(\mathfrak{M}) := \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \mathfrak{S}^{nr}[1/p]/\mathfrak{S}^{nr}) \xrightarrow{\simeq} \mathrm{Hom}_{\mathscr{O}_{\mathcal{E}},\varphi}(\mathfrak{M}\otimes_{\mathfrak{S}}\mathscr{O}_{\mathcal{E}}, \mathcal{E}^{nr}/\mathscr{O}_{\mathcal{E}^{nr}}) = V^*_{\mathcal{E}}(\mathfrak{M}\otimes_{\mathfrak{S}}\mathscr{O}_{\mathcal{E}}),$$

(2) *The functor $V^*_{\mathfrak{S}}$ is exact, it commutes with tensors and if $\mathfrak{M} \simeq \oplus_{i=1}^n \mathfrak{S}/p^{n_i}\mathfrak{S}$ then $V^*_{\mathfrak{S}} \simeq \oplus_{i=1}^n \mathbb{Z}_p/p^{n_i}\mathbb{Z}_p$.*

PROOF. [**Fon1**, part B1]. □

By passing to the limit one gets

Corollary 5.1.1.3. *For $\mathfrak{M} \in \mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ there is an exact functor*

$$V^*_{\mathfrak{S}}(\mathfrak{M}) := \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \mathfrak{S}^{nr}).$$

This is a free $\mathbb{Z}_p$-module of rank $rk_{\mathfrak{S}}(\mathfrak{M})$. Moreover there is a bijection

$$\mathrm{Hom}_{\mathfrak{S}\varphi}(\mathfrak{M}, \mathfrak{S}^{nr}) \longleftrightarrow \mathrm{Hom}_{\mathcal{O}_{\mathcal{E}},\varphi}(\mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M}, \mathcal{O}_{\widehat{\mathcal{E}^{nr}}}).$$

Lemma 5.1.1.4. *Let $\mathfrak{M} \in \mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$. Then $\mathfrak{M}' := \mathrm{Hom}_{\mathbb{Z}_p[G_{K_\infty}]}(V^*_{\mathfrak{S}}(\mathfrak{M}), \mathfrak{S}^{nr})$ is a free $\mathfrak{S}$-module of rank $d = rk_{\mathfrak{S}}\mathfrak{M}$ and there is a natural injection*

$$\mathfrak{M} \hookrightarrow \mathfrak{M}'.$$

Proof. [**Fon1**, part B1]. □

Proposition 5.1.1.5. *The functor*

$$\mathrm{Mod}^{\varphi}_{/\mathfrak{S}} \longrightarrow \mathrm{Mod}^{\varphi}_{/\mathcal{O}_{\mathcal{E}}}$$

is fully faithful.

In order to prove the statement, we need the following technical results.

Lemma 5.1.1.6. *Let $h : \mathfrak{M}_1 \to \mathfrak{M}_2$ be a morphism in $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ such that*

$$\tilde{h} : \mathfrak{M}_1 \otimes \mathcal{O}_{\mathcal{E}} \xrightarrow{\simeq} \mathfrak{M}_2 \otimes \mathcal{O}_{\mathcal{E}}.$$

Then h is an isomorphism as well.

Proof. Note that if $\tilde{h}$ is an isomorphism, then $\mathfrak{M}_1$ and $\mathfrak{M}_2$ have the same rank. Moreover also the map $det(h)$ is an isomorphism, hence we may suppose that $rk(\mathfrak{M}_1) = rk(\mathfrak{M}_2) = 1$. This tells us in particular that $\mathcal{M}_1 := \mathfrak{M}_1 \otimes \mathcal{O}, \mathcal{M}_2 := \mathfrak{M}_2 \otimes \mathcal{O} \in \mathrm{Mod}^{\varphi,N}_{/\mathcal{O}}$ are elements of $\mathrm{Mod}^{\varphi,N_\nabla}_{/\mathcal{O}}$. So, by the equivalence of categories

$$\mathrm{MF}_{\mathrm{K}}^{\varphi,\mathrm{N},\mathrm{Fil}_{\geq 0}} \xrightarrow{\simeq} \mathrm{Mod}^{\varphi,\mathrm{N}_\nabla}_{/\mathcal{O}}$$

we get a non-zero map, that is, an isomorphism, of (φ, N)-modules. By the fully faithful functor

$$\mathrm{MF}_{\mathrm{K}}^{\varphi,\mathrm{N},\mathrm{Fil}_{\geq 0},\mathrm{ad}} \xrightarrow{\text{fullyfaithful}} \mathrm{Mod}^{\varphi}_{/\mathfrak{S}} \otimes \mathbb{Q}_{\mathrm{p}},$$

we get that h has to be the multiplication by some non-negative power of p. But since $\tilde{h}$ is an isomorphism, this power is zero. □

Lemma 5.1.1.7. *Let $\mathcal{M} \in \mathrm{Mod}^{\varphi}_{/\mathcal{O}_{\mathcal{E}}}$, $\mathfrak{M} \subset \mathcal{M}$ a finitely generated $\mathfrak{S}$-module, stable under φ and of finite E-height. The $\mathfrak{S}$-module*

$$F(\mathfrak{M}) := \mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M} \cap \mathfrak{M}[1/p]$$

is finite, free, it is a submodule of $\mathcal{M}$ which contains $\mathfrak{M}$ and it is an object of $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$.

Remark. *If $\mathfrak{M} \in \mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$, then $F(\mathfrak{M}) = \mathfrak{M}$.*

Proof. [**Fon1**, Prop. B1.2.4]. □

Proof of Proposition (5.1.1.5). We want to prove that for any couple $\mathfrak{M}_1, \mathfrak{M}_2 \in \mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ the map

$$\mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}_1, \mathfrak{M}_2) \longrightarrow \mathrm{Hom}_{\mathscr{O}_{\mathcal{E}},\varphi}(\mathfrak{M}_1 \otimes \mathscr{O}_{\mathcal{E}}, \mathfrak{M}_2 \otimes \mathscr{O}_{\mathcal{E}})$$

is bijective. We show first the result in the following case: $\mathfrak{M}_1, \mathfrak{M}_2 \in \mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ are such that $\mathfrak{M}_1 \otimes_{\mathfrak{S}} \mathscr{O}_{\mathcal{E}} = \mathfrak{M}_2 \otimes_{\mathfrak{S}} \mathscr{O}_{\mathcal{E}} = \mathcal{M}$ and the map $h \in \mathrm{Hom}_{\mathscr{O}_{\mathcal{E}}}(\mathcal{M}, \mathcal{M})$ is an isomorphism. Define the module $\mathfrak{M} := \mathfrak{M}_1 + \mathfrak{M}_2$. There are obvious maps

$$h_1 : \mathfrak{M}_1 \to \mathfrak{M}, \qquad h_2 : \mathfrak{M}_2 \to \mathfrak{M}.$$

By Corollary (5.1.1.3) $V^*_{\mathfrak{S}}(\mathfrak{M}_1) \simeq V^*_{\mathfrak{S}}(\mathfrak{M}_2)$ and hence by Lemma (5.1.1.4) both $\mathfrak{M}_1$ and $\mathfrak{M}_2$ inject in $\mathrm{Hom}_{\mathbb{Z}_p[G_{K_\infty}]}(V^*_{\mathfrak{S}}(\mathfrak{M}_1), \mathfrak{S}^{nr})$, which has rank $rk_{\mathscr{O}_{\mathcal{E}}}(\mathcal{M})$. In particular $\mathfrak{M} := \mathfrak{M}_1 + \mathfrak{M}_2$ is of finite type, it is φ-stable and of finite E-height (since $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are). By construction $F(\mathfrak{M}) \otimes_{\mathfrak{S}} \mathscr{O}_{\mathcal{E}} \simeq \mathcal{M}$ and hence, by Lemma (5.1.1.6), the obvious maps $\mathfrak{M}_1 \to F(\mathfrak{M})$ and $\mathfrak{M}_2 \to F(\mathfrak{M})$ are isomorphisms, hence the conclusion in this case.

In the general case consider $\mathcal{M}_1 := \mathfrak{M}_1 \otimes \mathscr{O}_{\mathcal{E}}, \mathcal{M}_2 := \mathfrak{M}_2 \otimes \mathscr{O}_{\mathcal{E}}$ and a map of $\mathscr{O}_{\mathcal{E}}$-modules

$$h : \mathcal{M}_1 \longrightarrow \mathcal{M}_2.$$

Define $\mathfrak{M}_3 := h(\mathfrak{M}_1)$ and $\mathfrak{M}'_3 := h(\mathcal{M}_1) \cap \mathfrak{M}_2$. These are φ-stable, finitely generated modules, of finite E-height[2] such that

$$F(\mathfrak{M}_3) = F(\mathfrak{M}'_3).$$

Therefore we have $h(\mathfrak{M}_3) = h(\mathfrak{M}'_3)$. We define the map $\tilde{h} : \mathfrak{M}_1 \to \mathfrak{M}_2$ to be the composition

$$\mathfrak{M}_1 \to F(\mathfrak{M}_3) \simeq F(\mathfrak{M}'_3) \to F(\mathfrak{M}_2) = \mathfrak{M}_2.$$

□

We can summarize all this through the following fully faithful diagram

$$V^*_{\mathfrak{S}} \otimes \mathbb{Q}_p : \mathrm{Mod}^{\varphi}_{/\mathfrak{S}} \otimes \mathbb{Q}_p \xrightarrow{fullyfaithful} \mathrm{Mod}^{\varphi}_{/\mathscr{O}_{\mathcal{E}}} \otimes \mathbb{Q}_p \xrightarrow{\simeq} Rep_{\mathbb{Z}_p}(G_{K_\infty}) \otimes \mathbb{Q}_p \simeq Rep_{\mathbb{Q}_p}(G_{K_\infty}).$$

We get hence a concrete description of the image of free φ-Modules over $\mathfrak{S}$ of finite E-height.

Moreover these results allow us to give a proof of the following conjecture by Breuil [**Br2**].

Corollary 5.1.1.8. *The restriction from a G_K-representation to a G_{K_∞}-representation gives us a fully faithful functor*

$$Rep^{cris}_{\mathbb{Q}_p}(G_K) \longrightarrow Rep_{\mathbb{Q}_p}(G_{K_\infty}).$$

[2] to see this for $\mathfrak{M}'_3$ note that there is an exact sequence

$$0 \to \mathfrak{M}'_3 := h(\mathcal{M}_1) \cap \mathfrak{M}_2 \to h(\mathcal{M}_1) \oplus \mathfrak{M}_2 \to \mathcal{M}_2,$$

and the map $1 \otimes \varphi$ is injective on each term of the sequence.

PROOF. Note that it is enough to prove that there is a fully faithful functor

$$Rep^{cris,+}_{\mathbb{Q}_p}(G_K) \longrightarrow Rep_{\mathbb{Q}_p}(G_{K_\infty}).$$

Indeed, one can always twist a crystalline representation so that it has positive Hodge-Tate weights. The result follows then by the properties of the functors defined in the previous sections:

$$\begin{tikzcd}
Rep^{cris,+}_{\mathbb{Q}_p}(G_K) \arrow[r, dotted] \arrow[d, "D^*_{cris}"'] & Rep_{\mathbb{Q}_p}(G_{K_\infty}) \\
\mathrm{MF}_{\mathrm{K}}^{\varphi,\mathrm{ad},\mathrm{Fil}_{\geq 0}} \arrow[d, "fullyfaithful"'] & \\
\mathrm{Mod}^{\varphi}_{/\mathfrak{S}} \otimes \mathbb{Q}_p \arrow[r, "fullyfaithful"'] & \mathrm{Mod}^{\varphi}_{/\mathscr{O}_{\mathcal{E}}} \otimes \mathbb{Q}_p \arrow[uu, "\simeq"']
\end{tikzcd}$$

□

LEMMA 5.1.1.9. *Let $\mathfrak{M} \in \mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ be a module of rank d. Recall that $V^*_{\mathfrak{S}}(\mathfrak{M})$ is a module of rank d and hence $V := V^*_{\mathfrak{S}}(\mathfrak{M}) \otimes \mathbb{Q}_p$ is a vector space of dimension d. Define $\mathcal{M} = \mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{E}$, vector space of dimension d as well. The functor*

$$\mathrm{Mod}^{\varphi}_{/\mathfrak{S}} \longrightarrow Rep_{\mathbb{Z}_p}(G_{K_\infty})$$

restricts to a bijection of free, finite, φ-stable modules

$$\mathfrak{N} \longmapsto \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{N}, \mathfrak{S}^{nr})$$

between modules $\mathfrak{N} \in \mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ such that $\mathfrak{N} \subset \mathcal{M}$, $\mathfrak{N} \otimes_{\mathfrak{S}} \mathcal{E} \simeq \mathcal{M}$, $\mathfrak{N}/\varphi^*(\mathfrak{N})$ of finite E-height and G_{K_∞}-stable $\mathbb{Z}_p$-lattices $L \subset V$.

PROOF. The map in the statement makes sense, since for $\mathfrak{N} \in \mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$, the bijection $\mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{N}, \mathfrak{S}^{nr}) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{O}_{\mathcal{E}},\varphi}(\mathfrak{N} \otimes_{\mathfrak{S}} \mathscr{O}_{\mathcal{E}}, \mathscr{O}_{\widehat{\mathcal{E}^{nr}}})$ tells us that $V^*_{\mathfrak{S}}(\mathfrak{N})$ is a G_{K_∞}-stable in V.

Moreover, the map $\mathfrak{N} \mapsto V^*_{\mathfrak{S}}(\mathfrak{N})$ is injective. To prove surjectivity, take $L \subset V$ a G_{K_∞}-stable lattice. By Fontaine's Theorem (5.1.0.24) there is a bijection

$$\{\text{étale } \varphi\text{-Modules of full rank over } \mathscr{O}_{\mathcal{E}}\ \} \xrightarrow{\sim} \{G_{K_\infty}\text{-stable lattices contained in } V\}.$$

Hence there exists a finite free $\mathscr{O}_{\mathcal{E}}$-Module $\mathcal{N} = \mathrm{Hom}_{\mathbb{Z}_p[G_{K_\infty}]}(L, \mathscr{O}_{\widehat{\mathcal{E}^{ur}}})$ corresponding to L. Define $\mathfrak{N} = \mathcal{N} \cap \mathfrak{M}[1/p] \subset \mathcal{M}$. Note that $\mathfrak{N} \otimes_{\mathfrak{S}} \mathscr{O}_{\mathcal{E}} = \mathcal{N}$ and $\mathfrak{N} \in \mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ (the proof is identical to that of Lemma (5.1.1.7)). Since $V^*_{\mathfrak{S}}(\mathfrak{N}) = V^*_{\mathscr{O}_{\mathcal{E}}}(\mathcal{N})$ we have that $\mathfrak{N}$ maps into L. □

CHAPTER 6

Breuil-Kisin's classification

The first observation in order to classify Barsotti-Tate groups in terms of $\mathfrak{S}$-modules is that there is a relation between a sub-category of $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ and $\mathrm{BT}^{\varphi}_{/S}$. Note that there is a map

$$\begin{array}{ccc} \mathfrak{S} & \xrightarrow{\varphi} & S \\ u & \longmapsto & u^p \end{array}$$

Define $\mathrm{BT}^{\varphi}_{/\mathfrak{S}}$ as the full sub-category of $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ with objects the modules $\mathfrak{M}$ of E-height at most 1. The reason for giving this definition is that we may define a functor

$$\begin{array}{ccc} \mathrm{BT}^{\varphi}_{/\mathfrak{S}} & \longrightarrow & \mathrm{BT}^{\varphi}_{/S} \\ \mathfrak{M} & \longmapsto & \mathfrak{M}\otimes_{\mathfrak{S},\varphi} S \end{array}$$

Indeed, for $\mathcal{M} = \mathfrak{M}\otimes_{\mathfrak{S},\varphi} S$ we define

$$\mathrm{Fil}^1\mathcal{M} = \{m \in \mathcal{M} \mid (1\otimes\varphi)(m) \in \mathfrak{M}\otimes_{\mathfrak{S},\varphi} \mathrm{Fil}^1 S \subset \mathfrak{M}\otimes_{\mathfrak{S},\varphi} S\},$$

in this way of course $\mathrm{Fil}^1\mathcal{M} \subset p\mathcal{M}$ and hence we may define

$$\varphi/p : \mathrm{Fil}^1\mathcal{M} \xrightarrow{1\otimes\varphi} \mathrm{Fil}^1 S\otimes_{\mathfrak{S},\varphi}\mathfrak{M} \xrightarrow{1\otimes\varphi/p} \mathfrak{M}\otimes_{\mathfrak{S},\varphi} S = \mathcal{M}.$$

The fact that $\mathfrak{M}$ has by definition E-height at most one, makes sure that this is a surjective map.

This defines in particular a functor

$$\mathrm{BT}^{\varphi}_{/\mathfrak{S}} \longrightarrow \mathrm{BT}(\mathscr{O}_K)$$

Theorem 6.0.1.10 (Kisin). *The functor above is an equivalence of categories for $p>2$ and an equivalence up to isogeny for $p=2$.*

The key results in order to do this are the equivalences of categories established in chapter (5). In terms of these we may describe the category $\mathrm{BT}^{\varphi}_{/\mathfrak{S}}$.

Definition 6.0.1.11. *We say that an admissible module D is of* BT-tipe *if $gr^i D_K = 0$ for $i \notin \{0,1\}$.*

Proposition 6.0.1.12. *Recall that we established the following fully-faithful functor*

$$\mathrm{MF}_{\mathrm{K}}^{\varphi,\mathrm{N},\mathrm{Fil}_{\geq 0},\mathrm{ad}} \longrightarrow \mathrm{Mod}^{\varphi,\mathrm{N}}_{/\mathfrak{S}}\otimes\mathbb{Q}_{\mathrm{p}}.$$

This induces an equivalence of categories

$$\{\textit{admissible } \varphi\textit{-modules of BT-type}\} \xrightarrow{\simeq} \mathrm{BT}^{\varphi}_{/\mathfrak{S}}\otimes\mathbb{Q}_p.$$

Proof. [**Kis**, Proposition 2.2.2]. □

Recall that the *Tate-module* of a p-divisible group G is defined as $T_p(G) = \mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, G \otimes_{\mathscr{O}_K} \mathscr{O}_{\overline{K}})$. Consider the map of W-algebras

$$\begin{array}{ccc} S & \longrightarrow & A_{cris} \\ u & \longmapsto & [\tilde{\pi}] \end{array}$$

sending $\frac{E^i(u)}{i!}$ to $\frac{E^i([\tilde{\pi}])}{i!}$. We have the following result due to Faltings.

LEMMA 6.0.1.13. *Let G be a p-divisible group over $\mathscr{O}_K$. There is a canonical injection of G_{K_∞}-modules*

$$T_p(G) \hookrightarrow \mathrm{Hom}_{S,Fil,\varphi}(\mathcal{M}(G), A_{cris}),$$

where $\mathcal{M}$ is the functor $\mathrm{BT}(\mathscr{O}_K) \to \mathrm{BT}^{\varphi}_{/S}$. This map is an isomorphism if $p > 2$, and has cokernel killed by p when $p = 2$.

PROOF. We define the map

$$T_p(G) \longrightarrow \mathrm{Hom}_{S,Fil,\varphi}(\mathcal{M}(G), A_{cris}).$$

Since A_{cris} is the p-adic completion of the P.D. envelope of W(R), we get a diagram

$$\begin{tikzcd} W(R) \arrow[r] & A_{cris} \arrow[r, two heads, "\theta"] & \mathscr{O}_{\overline{K}} \\ \mathfrak{S} \arrow[u] \arrow[r] & S \arrow[u] \arrow[r, two heads] & \mathscr{O}_K \arrow[u] \end{tikzcd}$$

By the rigidity condition on the Dieudonné crystal we get that given a p-divisible group over K

$$\mathbb{D}(G \otimes_{\mathscr{O}_K} \mathscr{O}_{\widehat{\overline{K}}})(A_{cris}) \simeq \mathbb{D}(G) \otimes_S A_{cris} = \mathcal{M}(G) \otimes_S A_{cris}.$$

Applying these considerations to the Tate module of G, we get, since $\mathbb{D}$ is contravariant,

$$\begin{array}{rcl} T_p(G) := \mathrm{Hom}_{\mathscr{O}_{\overline{K}}}(\mathbb{Q}_p/\mathbb{Z}_p, G \otimes \mathscr{O}_{\overline{K}}) & \xrightarrow{\text{apply } \mathbb{D}} & \mathrm{Hom}_{S,\varphi,\mathrm{Fil}}(\mathbb{D}(G \otimes_{\mathscr{O}_K} \mathscr{O}_{\overline{K}})(A_{cris}), \mathbb{D}(\mathbb{Q}_p/\mathbb{Z}_p)(A_{cris})) \\ & \simeq & \mathrm{Hom}_{S,\varphi}(\mathcal{M}(G), A_{cris}). \end{array}$$

This last isomorphism comes to the fact that $\mathbb{D}(\mathbb{Q}_p/\mathbb{Z}_p) \simeq S$ ((3.0.4.1)). Note that by the definition of the functor $\mathcal{M}$, given $f \in T_p(G)$, the obtained map $\mathcal{M}(G) \to A_{cris}$ respects the Frobenius and the filtration, that is, we have obtained a map

$$T_p(G) \to \mathrm{Hom}_{S,Fil,\varphi}(\mathcal{M}, A_{cris}).$$

□

COROLLARY 6.0.1.14. *For $\mathfrak{M} \in \mathrm{BT}^{\varphi}_{/\mathfrak{S}}$, $\mathcal{M} := \mathfrak{M} \otimes_{\mathfrak{S},\varphi} S$, there is a natural map*

$$\mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \mathfrak{S}^{ur}) \longrightarrow \mathrm{Hom}_{S,\mathrm{Fil},\varphi}(\mathcal{M}, A_{cris}).$$

This is an isomorphism in case $p > 2$.

PROOF. Since by definition $\mathfrak{S}^{nr} \subset W(R)$, there is an inclusion $\mathfrak{S}^{nr} \hookrightarrow A_{cris}$. Hence we define the map

$$\begin{array}{ccc} \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \mathfrak{S}^{ur}) & \longrightarrow & \mathrm{Hom}_{S,\mathrm{Fil},\varphi}(\mathcal{M}, A_{cris}) \\ \{\mathfrak{M} \xrightarrow{f} \mathfrak{S}^{nr}\} & \longmapsto & g \end{array}$$

where g is the extension to $\mathcal{M}$ of $\{\mathfrak{S}^{nr} \hookrightarrow A_{cris}\} \circ f$ by S-linearity. The map is hence injective and by Lemma (6.0.1.13) and Corollary (5.1.1.3) the two homomorphisms groups have the same rank. It follows that this is an isomorphism after inverting p. By Breuil ([**Br2**]) the map is an isomorphism for $p > 2$. □

PROOF OF THEOREM (6.0.1.10). We have seen that we have naturally a functor

$$G : \mathrm{BT}^{\varphi}_{/\mathfrak{S}} \longrightarrow \mathrm{BT}(\mathscr{O}_K).$$

We define an inverse

$$\mathfrak{M} : \mathrm{BT}(\mathscr{O}_K) \longrightarrow \mathrm{BT}^{\varphi}_{/\mathfrak{S}}.$$

Consider G a p-divisible group over $\mathscr{O}_K$ and consider its Tate module $T_p(G) := \mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, G \otimes_{\mathscr{O}_K} \mathscr{O}_{\overline{K}})$. This is a $\mathbb{Z}_p$-lattice under the action of $G_K = \mathrm{Gal}(\overline{K}/K)$ of the G_K-representation $V_p(G) := T_p(G) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Note that$V_p(G)$ is crystalline. Indeed, by Lemma (6.0.1.13)., we have

$$V_p(G) := T_p(G) \otimes \mathbb{Q}_p \xrightarrow{\simeq} \mathrm{Hom}_{S,Fil,\varphi}(\mathcal{M}(G), A_{cris}) \otimes \mathbb{Q}_p \xrightarrow{\simeq} \mathrm{Hom}_{S,Fil,\varphi}(\mathcal{M}(G), B^+_{cris}) \in Rep^{cris}_{G_K}.$$

Moreover $V_p(G)$ has Hodge-Tate weights in $\{0,1\}$ (this fact is proved in [**Tate**]). By the results introduced in (5.1.0.1) we get a module of BT-type $\mathcal{M}$ through the map

$$\{\ Rep^{cris}_{\mathbb{Q}_p}(G_K) \text{ of Hodge-Tate weights in } \{0,1\}\ \} \xrightarrow{D_{cris}} \{\text{admissible } \varphi\text{-modules of BT-type}\}$$

We find hence *uniquely* a module $\mathfrak{M} \in \mathrm{BT}^{\varphi}_{/\mathfrak{S}} \otimes \mathbb{Q}_p$, through the equivalence

$$\{\text{admissible } \varphi\text{-modules of BT-type}\} \xrightarrow{\simeq} \mathrm{BT}^{\varphi}_{/\mathfrak{S}} \otimes \mathbb{Q}_p.$$

We apply finally Lemma (5.1.1.9) to $\mathfrak{M}$ (of course $V_p(G) \in Rep_{\mathbb{Q}_p}(G_{K_\infty})$). This gives us namely a bijection between $\mathbb{Z}_p$-lattices of $V_p(G)$ and the modules $\mathfrak{N} \in \mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ such that $\mathfrak{N} \subset \mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{E}$, $\mathfrak{N} \otimes_{\mathfrak{S}} \mathcal{E} = \mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{E}$ and $\mathfrak{N}/\varphi^*\mathfrak{N}$ of finite E-height. We take the $\mathbb{Z}_p$-lattice $T_p(G)$: this corresponds bijectively to such a module $\mathfrak{N}$, and we put $\mathfrak{N} =: \mathfrak{M}(G)$.

We have hence built the functor $\mathfrak{M}$. We would like now to show that we have an equivalence of categories.

Suppose fist that $p > 2$. We have by Lemma (6.0.1.13) and Corollary (6.0.1.14) that

$$T_p(G) \xrightarrow{\simeq} \mathrm{Hom}_{S,\mathrm{Fil},\varphi}(\mathcal{M} = \mathfrak{M} \otimes_{\mathfrak{S}} S, A_{cris}) \xrightarrow{\simeq} \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \mathfrak{S}^{nr}).$$

This is a G_{K_∞}-equivariant map and hence a G_K-map of crystalline representations, by Breuil's conjecture (Corollary (5.1.1.8)). By the fact that $G(\mathfrak{M}) = G(\mathcal{M})$ and by the construction of the functor $\mathfrak{M}$ we conclude that $\mathfrak{M}(G(\mathfrak{M})) \simeq \mathfrak{M}$. On the other hand, for G a p-divisible group over $\mathscr{O}_K$, the same results give us

$$T_p(G(\mathfrak{M}(G))) \xrightarrow{\simeq} \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}(G), \mathfrak{S}^{nr}) \xrightarrow{\simeq} T_p(G),$$

where the first isomorphism comes from the construction of $\mathfrak{M}$. By Tate's theorem (2.2.0.9) we obtain an isomorphism on the fibers and hence we obtain an isomorhism

$$G(\mathfrak{M}(G)) \simeq G.$$

For $p = 2$, the same results are true after inverting p, that is, up to isogeny. □

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