Algebra, Geometry and Number Theory

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Flat spacetimes with compact hyperbolic Cauchy surfaces

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Introduction

In this thesis we study a paper of Bonsante [12], which is a generalization of a work done by Mess [21] in 1990 but published just in 2007. Mess studied constant curvature spacetimes in dimension $2+1$. Bonsante has generalized the flat case to all dimensions, similar results in positive curvature are obtained by Scannell in [27].

The interest for spacetimes, that are a particular type of Lorentzian manifolds, i.e manifolds endowed with a metric tensor of signature $(n, 1)$, comes from general relativity where they represent solutions to Einstein's equations. The flat case is a particular solution where gravity is not taken into account. In the mathematical literature closed (compact without boundary) flat spacetimes have been mainly studied, see for example [4]. However compact manifolds are regarded from a physical point of view as unrealistic, indeed they are never causal, this means that a compact spacetime always contains a closed causal curve. A causal curve is a curve whose tangent vectors have non-positive norm, it corresponds to the path of an observer moving at speed less or equal to the one of light. Furthermore there is an important notion in Lorentzian geometry, that again comes from a physical interest, that is global hyperbolicity, which is incompatible with compactness. Globally hyperbolic spacetimes are spacetimes admitting a Cauchy hypersurface, that is an hypersurface which is spacelike (the Lorentzian metric on it restricts to a Riemannian metric) and such that every inextendible causal curve intersects it exactly in one point. A Cauchy surface is regarded as a set of initial data that determines, at least locally, the future evolution of the spacetime. From a result of Geroch [16] a globally hyperbolic spacetime Y with Cauchy surface M decomposes in space and time, i.e. Y is diffeomorphic to $\mathbb{R} \times M$.

Bonsante focuses on globally hyperbolic flat spacetimes Y of dimension $n+1$ that admit a Cauchy surface diffeomorphic to a compact hyperbolic manifold M. An hyperbolic manifold is a manifold locally modelled on the hyperbolic space $\mathbb{H}^n$. Actually a classification of flat globally hyperbolic spacetimes that admit a Cauchy surface that is complete as a Riemannian manifold is done by T.Barbot in [3] and it turns out that in the compact case the situation studied by Bonsante is the 'generic' and most interesting case of the classification.

There are two possible approaches in order to describe such spacetimes.

The first one is a "cosmological" approach, where time functions are used. Indeed we will introduce a time function, that is regarded as a "canonical" one in this setting. It is called cosmological time and roughly speaking it associates to each point of the spacetime the length of time that this point has been in existence. This function will allow us to study the geometry of such spacetimes.

On the other hand we have a more "geometric" approach using the language of geometric structures. Indeed a flat spacetime may be regarded as a manifold locally modelled on Minkowski space $\mathbb{M}^{n+1}$ (that is $\mathbb{R}^{n+1}$ endowed with a bilinear form of signature $(n, 1)$). Generally speaking a geometric structure on a manifold is an atlas where the local charts are maps to open subsets of a model space (for example Minkowski space for flat Lorentzian manifold and hyperbolic space for hyperbolic manifolds) and the transition maps are equal to the restriction of an isometry of the model space. Associated to each flat Lorentzian structure on a manifold Y there is an important pair of objects (D, ρ), where $D : \widetilde{Y} \to \mathbb{M}^{n+1}$ is a local isometry, called the developing map, from the universal cover of Y to Minkowski space $\mathbb{M}^{n+1}$ and $\rho : \pi_1(Y) \to \mathrm{Iso}(\mathbb{M}^{n+1})$ is a group homomorphism, called the holonomy morphism, from the fundamental group of the manifold to the group of isometries of Minkowski space such that D is ρ-equivariant. This pair determines the flat Lorentzian structure on the manifold Y and in general both maps are needed in order to describe such structure. However we are going to see that when Y is a globally hyperbolic flat spacetime with a complete spacelike Cauchy surface the developing map D becomes injective and hence a global isometry with a subset of Minkowski space. Hence we can identify the universal cover of Y with some simply connected region of Minkowski space, namely $D(\widetilde{Y})$ and the holonomy morphism, that becomes injective, is sufficient to describe such structure. In fact the developing map gives an isometry between Y and $D(\widetilde{Y})/\rho(\pi_1(Y))$. Therefore from this more geometric point of view what becomes important are the holonomy groups (the image under the holonomy morphism of the fundamental group of the manifold) and their action on regions of Minkowski space. So the classification of such flat spacetimes appears as an extension of Bieberbach's theory about Crystallographic groups (discrete subgroups of the group of isometries of $\mathbb{R}^n$, with the standard Euclidean structure, that act freely and have compact fundamental domain) to the Lorentzian context.

What Bonsante proves is that if we fix a compact hyperbolic manifold M and a class, up to conjugacy, of group homomorphism $\rho : \pi_1(M) \to \mathrm{Iso}(\mathbb{M}^{n+1})$, there will be only two maximal flat Lorentzian structures on $\mathbb{R} \times M$ having ρ as holonomy morphism. These two structures are represented by two globally hyperbolic flat spacetimes Y_ρ^+, Y_ρ^-, one future complete and the other past complete. They are maximal in the sense that all other globally hyperbolic flat spacetimes with compact spacelike Cauchy surface sharing the same holonomy will embed isometrically in either Y_ρ^+ or Y_ρ^-. The way in [12] Bonsante proceeds in order to construct Y_ρ^+ (and in an analogous, time reversed, way Y_ρ^-) is to realize it as the quotient of a domain $\mathcal{D}_\rho$ of Minkowski space by the image, under the holonomy ρ, of the fundamental group of M. As we were discussing above from this result we obtain an important corollary about group actions on Minkowski space. Namely that the image of $\pi_1(M)$ under ρ does not act freely and properly discontinuously on the whole Minkowski space, in fact it does not act in such a way on the boundary of the domain $\mathcal{D}_\rho$ that is associated to ρ. Hence $\mathcal{D}_\rho$ is the biggest region of $\mathbb{M}^{n+1}$ over which the action of the holonomy group is free and properly discontinuous.

The region $\mathcal{D}_\rho$ of Minkowski space turns out to be what is called a future complete regular convex domain, i.e a convex proper subset of $\mathbb{M}^{n+1}$ that is the intersection of the future of at least two null support planes (a null plane is an affine plane where the quadratic form of Minkowski space restricts to a degenerate form). As an example one can think of the future cone of the origin. A nice feature of this class of domains is that the cosmological time function defined on them $\widetilde{T} : \mathcal{D}_\rho \to \mathbb{R}_+$ is regular, this means that it is finite and it goes to 0 on every inextendible causal curve. These properties imply in particular (see [2]) that $\widetilde{T}$ is a time function, in the sense that it is increasing on every future directed causal curve. Indeed Bonsante proves that on $\mathcal{D}_\rho$ the map $\widetilde{T}$ is a C^1-submersion and that the level surfaces $\widetilde{S}_a = \widetilde{T}^{-1}(a)$ are Cauchy surfaces for $\mathcal{D}_\rho$. From the classification of maximal flat globally hyperbolic spacetimes with compact Cauchy surfaces done by Barbot in [3] one realizes that the other simply connected maximal globally hyperbolic spacetimes with compact Cauchy surfaces do not have regular cosmological time, hence the study of regular domains and their quotients is equivalent to the study of globally hyperbolic flat spacetimes having a compact

Cauchy surface and regular cosmological time.

Together with the cosmological time, on the domain $\mathcal{D}_\rho$ are also defined two other maps. A map from $\mathcal{D}_\rho$ to its boundary $r : \mathcal{D}_\rho \to \partial\mathcal{D}_\rho$, called the retraction, whose image is called the singularity in the past and denoted by Σ_ρ. For a point $p \in \mathcal{D}_\rho$ the point $r(p)$ in the boundary is characterized as the point such that $\widetilde{T}(p) = d(r(p), p)$, where d is the Lorentzian distance function in $\mathcal{D}_\rho$. Since the cosmological time goes to 0 when we approach the initial singularity we can think of it as the beginning of everything: every particle came into existence at the initial singularity. It turns out that Σ_ρ is a contractible space. As the name suggests the initial singularity is generally not smooth, indeed for example in dimension $2+1$, it is a real tree (see [8]), that is a metric space where points are joined by a unique arc.

The other important map is a map from the regular domain to the hyperbolic space $N : \mathcal{D}_\rho \to \mathbb{H}^n$, called the normal field. The reason for the name is that $N(x)$ is the normal vector to the level surface $\tilde{S}_{\tilde{T}(x)}$ at x. Using the normal field we can associate to each future complete regular domain a geodesic stratification of the hyperbolic space $\mathbb{H}^n$, that for $n = 2$ is a geodesic lamination, that in a certain sense parametrizes such domains. A geodesic stratification of $\mathbb{H}^n$ is a decomposition of the hyperbolic space in convex subsets that are the convex hull of their boundary points in such a way that if they intersect then their intersection is contained in a face of them.

All these functions introduced for $\mathcal{D}_\rho$ are equivariant for the action of the holonomy group $\rho(\pi(M))$ and hence they induce a regular cosmological time, a retraction on the singularity in the past and a normal field on the quotient Y_ρ^+.

Chapter 1

Preliminaries

1.1 Minkowski Space

Definition 1.1.1. The $(n+1)$-dimensional *Minkowski spacetime*, $\mathbb{M}^{n+1}$, is the real vector space $\mathbb{R}^{n+1}$, with coordinates $(x_0, \ldots, x_n)$, endowed with a non degenerate, symmetric, bilinear form of signature $(n, 1)$

$$\eta = -dx_0^2 + dx_1^2 + \ldots + dx_n^2.$$

Using the orthonormal frame $\left(e_i = \frac{\partial}{\partial x_i}\right)_{i=0,\ldots,n}$ we may identify in a standard way the tangent space $(T_x\mathbb{M}^{n+1}, \eta_x)$ with $(\mathbb{R}^{n+1}, \langle \cdot, \cdot \rangle)$ where

$$\langle v, w \rangle = -v_0 w_0 + v_1 w_1 + \ldots + v_n w_n.$$

Definition 1.1.2. A non zero tangent vector v is classified as

1. *spacelike* if $\eta(v, v) > 0$
2. *timelike* if $\eta(v, v) < 0$
3. *null* if $\eta(v, v) = 0$.

Remark 1.1.3. Minkowski space is an orientable manifold. Let us put the standard orientation, for which the canonical basis $(e_0, \ldots, e_n)$ is positive oriented. Furthermore we give to it a time orientation: a timelike tangent vector is said *future directed* if $\langle v, e_0 \rangle < 0$ i.e. if we write $v = \sum_{i=0}^n v_i e_i$ where $(e_0, \ldots, e_n)$ is the canonical basis then v is future directed if $v_0 > 0$.

Definition 1.1.4. We call *orthonormal affine coordinates* a set $(y_0, \ldots, y_n)$ of affine coordinates on $\mathbb{M}^{n+1}$ such that the frame $\{\frac{\partial}{\partial y_i}\}$ is orthonormal and positive and the vector $\frac{\partial}{\partial y_0}$ is future directed.

Proposition 1.1.5. *Two non zero null vectors are orthogonal if and only if they are parallel i.e.* $\exists t \in \mathbb{R}$ *such that* $v = tw$.

Proof. Obviously if $\exists t \in \mathbb{R}$ such that $v = tw$ then $\langle v, w \rangle = t \langle w, w \rangle = 0$. On the other hand if $\langle v, w \rangle = 0$ then $v_0 w_0 = v_1 w_1 + \ldots + v_n w_n$ hence $(v_0 w_0)^2 = (v_1 w_1 + \ldots + v_n w_n)^2 \leq (v_1^2 + \ldots + v_n^2)(w_1^2 + \ldots + w_n^2) = v_0^2 w_0^2$ where the inequality comes from Cauchy-Schwartz inequality, and also since we have equality it implies that $(v_1, \ldots, v_n)$ and $(w_1, \ldots, w_n)$ are linearly dependent. So $\exists t \in \mathbb{R}$ such that $v_i = t w_i$ $i = 1, \ldots, n$ and furthermore $v_0^2 = t^2(w_1^2 + \ldots + w_n^2) = t^2 w_0^2$, hence $v_0 = \pm t w_0$. If by contradiction $v_0 = -t w_0$ then $v_0 w_0 = -t w_0^2 = -t(w_1^2 + \ldots + w_n^2)$ since $w_0^2 = w_1^2 + \ldots + w_n^2$, on the other hand we have $v_0 w_0 = v_1 w_1 + \ldots + v_n w_n = t(w_1 + \ldots + w_n)$ hence since, they are non zero, a contradiction. Then we must have $v_0 = t w_0$. $\square$

Proposition 1.1.6. *Let v and w be timelike or null vectors and in case they are both null let them be non parallel. Write $v = (v_0, \ldots, v_n)$ and $w = (w_0, \ldots, w_n)$ with respect to an orthonormal basis then either*

$$(i)\ v_0 w_0 > 0 \text{ in which case } \langle v, w\rangle < 0 \text{ or}$$

$$(ii)\ v_0 w_0 < 0 \text{ in which case } \langle v, w\rangle > 0.$$

Proof. Suppose v is timelike, then since $\langle v, v\rangle < 0$ and $\langle w, w\rangle \leq 0$ we have $v_0^2 > v_1^2 + \ldots + v_n^2$ and $w_0^2 \geq w_1^2 + \ldots + w_n^2$, thus $(v_0 w_0)^2 > (v_1^2 + \ldots + v_n^2)(w_1^2 + \ldots + w_n^2) \geq (v_1 w_1 + \ldots + v_n w_n)^2$ form Cauchy Schwartz. In case they are both null non parallel vectors we find $(v_0 w_0)^2 \geq (v_1^2 + \ldots + v_n^2)(w_1^2 + \ldots + w_n^2) > (v_1 w_1 + \ldots + v_n w_n)^2$, the second strict inequality comes from the fact that they are linearly independent. Hence in any case we find $|v_0 w_0| > |v_1 w_1 + \ldots + v_n w_n|$. In particular $v_0 w_0 \neq 0$ and $\langle v, w\rangle \neq 0$. Suppose $v_0 w_0 > 0$ then $v_0 w_0 = |v_0 w_0| > |v_1 w_1 + \ldots + v_n w_n| \geq v_1 w_1 + \ldots + v_n w_n$ hence $\langle v, w\rangle < 0$, on the other hand if $v_0 w_0 < 0$ then $\langle v, -w\rangle < 0$ hence $\langle v, w\rangle > 0$. □

Corollary 1.1.7. *If a non zero vector is orthogonal to a timelike vector then it is spacelike.*

Theorem 1.1.8 (Reverse Cauchy-Schwartz inequality)**.** *Let v and w be timelike vectors then*

$$\langle v, w\rangle^2 \geq \langle v, v\rangle \langle w, w\rangle$$

with equality if and only if they are linearly dependent.

Proof. Consider the vector $u = av - bw$ where $a = \langle v, w\rangle$ and $b = \langle v, v\rangle$. Observe that $\langle u, v\rangle = a\langle v, v\rangle - b\langle v, w\rangle = 0$. Since v is timelike u is either 0 or spacelike, thus $0 \leq \langle u, u\rangle = a^2\langle v, v\rangle + b^2\langle w, w\rangle - 2ab\langle v, w\rangle$ with equality only if $u = 0$. Consequently $2ab\langle v, w\rangle \leq a^2\langle v, v\rangle + b^2\langle w, w\rangle$ i.e. $2\langle v, v\rangle\langle v, w\rangle^2 \leq \langle v, v\rangle\langle v, w\rangle^2 + \langle v, v\rangle^2\langle w, w\rangle$ and $2\langle v, w\rangle^2 \geq \langle v, w\rangle^2 + \langle v, v\rangle\langle w, w\rangle$ so $\langle v, w\rangle^2 \geq \langle v, v\rangle\langle w, w\rangle$. Equality holds only if $u = av - bw = 0$ and hence since $a \neq 0$ this means that v and w are linearly dependent. Conversely if v and w are linearly dependent then one is multiple of the other and the equality clearly holds. □

Corollary 1.1.9. *If v,w are positive (negative) timelike vectors then $\langle v, w\rangle \leq \|v\|\|w\|$. With equality if and only if they are linearly dependent. (Here $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^{n+1}$).*

Proof. Consider v, w positive timelike vectors then form the Reverse Cauchy Schwartz inequality taking the square roots $|\langle v, w\rangle| \geq \sqrt{\langle v, v\rangle\langle w, w\rangle}$. Notice that it is always true that $\langle v, v\rangle \leq \|v\|^2$. Hence $\langle v, w\rangle \leq \|v\|\|w\|$. □

Theorem 1.1.10 (Reverse triangular inequality or Twin Paradox)**.** *Let v and w be timelike vectors with the same time orientation then*

$$\tau(v + w) \geq \tau(v) + \tau(w)$$

and equality holds if and only if v and w are linearly dependent. Here

$$\tau(v) = \sqrt{-\langle v, v\rangle}.$$

Proof. By Theorem 1.1.8, $\langle v, w\rangle^2 \geq \langle v, v\rangle\langle w, w\rangle = (-\langle v, v\rangle)(-\langle w, w\rangle)$ so this implies that $|\langle v, w\rangle| \geq \sqrt{-\langle v, v\rangle}\sqrt{-\langle w, w\rangle}$ But $\langle v, w\rangle < 0$ since they have the same time orientation, so we must have $\langle v, w\rangle \leq -\sqrt{-\langle v, v\rangle}\sqrt{-\langle w, w\rangle}$ and therefore $-2\langle v, w\rangle \geq 2\sqrt{-\langle v, v\rangle}\sqrt{-\langle w, w\rangle}$.

Notice that $v+w$ is timelike and we have $-\langle v+w, v+w\rangle = -\langle v,v\rangle - 2\langle v,w\rangle - \langle w,w\rangle \geq -\langle v,v\rangle + 2\sqrt{-\langle v,v\rangle}\sqrt{-\langle w,w\rangle} - \langle w,w\rangle$, thus

$$-\langle v+w, v+w\rangle \geq \left(\sqrt{-\langle v,v\rangle} + \sqrt{-\langle w,w\rangle}\right)^2$$

$$\sqrt{-\langle v+w, v+w\rangle} \geq \sqrt{-\langle v,v\rangle} + \sqrt{-\langle w,w\rangle}$$

$$\tau(v+w) \geq \tau(v) + \tau(w)$$

If the equality holds we obtain $-2\langle v,w\rangle = 2\sqrt{-\langle v,v\rangle}\sqrt{-\langle w,w\rangle}$ and therefore $\langle v,w\rangle^2 = \langle v,v\rangle\langle w,w\rangle$ so by Theorem 1.1.8 they are linearly dependent. □

Remark 1.1.11. The reason for the name is that the situation of the famous twin paradox in special relativity can be seen as a consequence of the reverse triangular inequality if we see curves as motion of particles in the universe and length of timelike curves as their proper time.

Lemma 1.1.12. *The sum of any finite number of vectors all of which are timelike or null and all future directed (past directed) is timelike and future directed (past directed) except when all of the vectors are null and parallel, in which case the sum is null and future directed (past directed).*

Proof. It is sufficient to prove it for future directed vectors, it is also clear that any sum of future directed vectors is future directed. Now observe that if v_1 and v_2 are future directed timelike or null (not parallel) vectors then $\langle v_1, v_1\rangle \leq 0$, $\langle v_2, v_2\rangle \leq 0$ and by Proposition 1.1.6 $\langle v_1, v_2\rangle < 0$ so $\langle v_1+v_2, v_1+v_2\rangle = \langle v_1,v_1\rangle + 2\langle v_1,v_2\rangle + \langle v_2,v_2\rangle < 0$ and therefore v_1+v_2 is timelike. If they are both null and future directed and parallel then the sum is obviously null. The proof follows by induction. □

Corollary 1.1.13. *Let $v_1, \ldots, v_n$ be timelike vectors, all with the same orientation, then*

$$\tau(v_1 + \ldots + v_n) \geq \tau(v_1) + \ldots + \tau(v_n)$$

and equality holds if and only if $v_1, \ldots, v_n$ are all parallel.

Proof. The inequality follows by Theorem 1.1.10 and the previous lemma. Now we show by induction on n that equality implies that $v_1, \ldots, v_n$ are parallel. For $n = 2$ this is just Theorem 1.1.10. Thus we may assume that the statement is true for sets of n vectors and consider a set of $n+1$ timelike future directed vectors such that $\tau(v_1 + \ldots + v_n + v_{n+1}) = \tau(v_1) + \ldots + \tau(v_n) + \tau(v_{n+1})$, since $v_1 + \ldots + v_n$ is timelike and future directed $\tau(v_1 + \ldots + v_n) + \tau(v_{n+1}) \leq \tau(v_1) + \ldots + \tau(v_n) + \tau(v_{n+1})$ we claim that, in fact, equality must hold here. Indeed otherwise we have $\tau(v_1 + \ldots + v_n) < \tau(v_1) + \ldots + \tau(v_n)$ and hence applying Theorem 1.1.10 again $\tau(v_1 + \ldots + v_{n-1}) < \tau(v_1) + \ldots + \tau(v_{n-1})$ continuing the process we eventually conclude that $\tau(v_1) < \tau(v_1)$ which is a contradiction and so we have equality and hence $\tau(v_1 + \ldots + v_n) = \tau(v_1) + \ldots + \tau(v_n)$, now the inductive hypothesis implies that $v_1, \ldots, v_n$ are parallel. Let $v = v_1 + \ldots + v_n$ then v is timelike and future directed thus $\tau(v + v_{n+1}) = \tau(v) + \tau(v_{n+1})$ and this implies that v_{n+1} is parallel to v and therefore to all $v_1, \ldots, v_n$. □

Remark 1.1.14. The geodesics in $\mathbb{M}^{n+1}$ are straight lines $L = \mathbb{R}\vec{u} + x$, where $\vec{u}, x \in \mathbb{R}^{n+1}$. We can classify the geodesics up to isometries as follows:

1. *spacelike* if $\eta_x(\vec{u}, \vec{u}) > 0$,
2. *timelike* if $\eta_x(\vec{u}, \vec{u}) < 0$,
3. *null* if $\eta_x(\vec{u}, \vec{u}) = 0$.

Remark 1.1.15. Affine k-planes in $\mathbb{M}^{n+1}$ are also classified up to isometries by the restriction of the Lorentzian form to them, let P be a k-plane:

1. P is *spacelike* if $\eta_{|P}$ is a flat Riemannian form,

2. P is *timelike* if $\eta_{|P}$ is a flat Lorentz form,

3. P is *null* if $\eta_{|P}$ is a degenerate form.

Remark 1.1.16. Notice that a k-plane that is spacelike will have no null line, one that is timelike will have at least two null lines, finally a null hyperplane will have only one isotropic line.

Remark 1.1.17. Recall that $\mathrm{O}(n,1)$ is the group of linear transformations of $\mathbb{R}^{n+1}$ that preserve the inner product $\langle\cdot,\cdot\rangle$.

Theorem 1.1.18. *Let* $\mathrm{Iso}(\mathbb{M}^{n+1})$ *be the group of isometries (i.e. diffeomorphisms preserving the Lorentzian form) of Minkowski space, then* $\mathrm{Iso}(\mathbb{M}^{n+1}) \cong \mathbb{R}^{n+1} \rtimes \mathrm{O}(n,1)$.

Proof. Let f be a diffeomorphism, then we have

$$\begin{aligned}
\partial_i(f^*\eta)_x(\partial_j,\partial_k) &= \partial_i\eta_{f(x)}(df(x)\partial_j, df(x)\partial_k) = \\
&= \eta_{f(x)}(\nabla_{\partial_i}df(x)\partial_j, df(x)\partial_k) + \eta_{f(x)}(df(x)\partial_j, \nabla_{\partial_i}df(x)\partial_k) = \\
&= \eta_{f(x)}(d^2f(x)\partial_i\partial_j, df(x)\partial_k) + \eta_{f(x)}(df(x)\partial_j, d^2f(x)\partial_i\partial_k) = \\
&= (f^*\eta)_x((df(x))^{-1}d^2f(x)\partial_i\partial_j,\partial_k) + (f^*\eta)_x(\partial_j,(df(x))^{-1}d^2f(x)\partial_i\partial_k).
\end{aligned}$$

Now if f is an isometry we have that $f^*\eta = \eta$ hence it follows that $\partial_i(f^*\eta)_x(\partial_j,\partial_k) = \partial_i\eta_x(\partial_j,\partial_k) = 0$ and if we permute the roles of i,j,k we find that also the following equations hold

$$\eta_x((df(x))^{-1}d^2f(x)\partial_j\partial_k,\partial_i) + (f^*\eta)_x(\partial_k,(df(x))^{-1}d^2f(x)\partial_j\partial_i) = 0$$
$$\eta_x((df(x))^{-1}d^2f(x)\partial_k\partial_i,\partial_j) + (f^*\eta)_x(\partial_i,(df(x))^{-1}d^2f(x)\partial_k\partial_j) = 0.$$

If we put the previous equations together we obtain $\eta_x((df(x))^{-1}d^2f(x)\partial_i\partial_j,\partial_k) = 0$, this implies $d^2f(x) = 0$. Hence $f(x) = f(0) + df(0)x$. Since f preserves the Lorentzian form $df(0)$ should preserve the inner product on the tangent space, this implies that $df(0)$ belongs to $O(n,1)$. It follow that the group of isometries is generated by $\mathrm{O}(n,1)$ and the group of translations $\mathbb{R}^{n+1}$. Furthermore $\mathbb{R}^{n+1}$ is a normal subgroup of $\mathrm{Iso}(\mathbb{M}^{n+1})$ since it is the kernel of the map $\mathrm{Iso}(\mathbb{M}^{n+1}) \ni f \to df(0) \in \mathrm{O}(n,1)$, so $\mathrm{Iso}(\mathbb{M}^{n+1})$ is isomorphic to $\mathbb{R}^{n+1} \rtimes \mathrm{O}(n,1)$. □

Remark 1.1.19. Notice that $\mathrm{O}(n,1)$ is the stabilizer of 0 in $\mathrm{Iso}(\mathbb{M}^{n+1})$. It is a semisimple Lie group of dimension $\frac{n(n-1)}{2}$. It has the following important subgroups:

1. $\mathrm{O}^+(n,1) = \{$ linear isometries which preserve time orientation $\}$.
   A transformation is said to preserve time orientation if it sends future directed timelike vectors to future directed timelike vectors. They are called *orthochronus* transformations. It is a subgroup of index two of $\mathrm{O}(n,1)$.

2. $\mathrm{SO}(n,1) = \{$ linear isometries which preserve the orientation of $\mathbb{M}^{n+1}$ $\}$.
   They are called *proper* transformations. It is a subgroup of $\mathrm{O}(n,1)$ of index 2.

3. $\mathrm{SO}^+(n,1) = \{$ linear isometries which preserve both orientation and time-orientation $\}$.
   It is a subgroup of index 2 of both $\mathrm{O}^+(n,1)$ and $\mathrm{SO}(n,1)$. It is called the *Lorentz group*.

$\mathrm{O}(n,1)$ is not compact and it has four connected components. The connected component of the identity is $\mathrm{SO}^+(n,1)$ and the set of connected components can be given a group structure as the quotient $\mathrm{O}(n,1)/\mathrm{SO}^+(n,1) = \{Id, P, T, PT\}$ where P represents a transformation that reverse the space orientation and preserve the time orientation and T viceversa.

1.2 Lorentzian Manifolds

Definition 1.2.1. A *Lorentzian (n+1)-manifold* is a pair (M, η) where M is a smooth $(n+1)-$manifold (we may assume it is metrizable and with a countable basis) and η is a symmetric non degenerate 2-form of signature $(n, 1)$.

Example 1.2.2. Minkowski $(n+1)$-dimensional spacetime $(\mathbb{M}^{n+1}, \eta)$ is a Lorentzian manifold.

Definition 1.2.3. As for Minkowski space we classify a non zero tangent vector as *spacelike*, *timelike* or *null*, depending on whether the form evaluate on it is positive, negative or null. We also call a vector *non-spacelike* if it is timelike or null.

Definition 1.2.4. Let M be a connected Lorentzian manifold. A continuous vector field X on it is said to be timelike if $\eta(X(p), X(p)) < 0$ for all $p \in M$. In general a Lorentzian manifold does not necessarily admit a globally defined timelike vector field, if it does then it is said to be *time-orientable.* In this situation the timelike vector field X divides all the non-spacelike vectors in the tangent bundle of M in two connected components. A time orientation is a choice of one timelike vector field.

Definition 1.2.5. A *spacetime* is a connected time-orientable Lorentzian manifold equipped with a time orientation.

For now on, since we will essentially be interested in spacetimes, let (M, η) be a spacetime.

Definition 1.2.6. A non-spacelike tangent vector $v \in T_pM$ is said to be *future directed* if $\eta_p(X(p), v) < 0$ and it is said to be *past directed* otherwise.

Definition 1.2.7. A C^1 curve is said to be

1. *chronological* (or *timelike*) if its tangent vectors are always timelike,
2. *causal* (or *non-spacelike*) if its tangent vectors are always non-spacelike.

   A causal curve is said to be

3. *future directed* if its tangent vectors are future directed,
4. *past directed* if its tangent vectors are past directed.

Definition 1.2.8. In a spacetime M we can define the causal structure, i.e. the causal relations between two points: given $p, q \in M$ we write

1. $p \ll q$ and we say p *chronologically precedes* q if there exists a smooth future directed chronological curve from p to q,
2. $p \leq q$ and we say p *causally precedes* q if there exists a smooth future directed causal curve from p to q .

Observe that these relations are transitive.

Now given $p \in M$ we can define

1. the *chronological future of* p, $I^+(p) = \{q \in M \mid p \ll q\}$,
2. the *chronological past of* p, $I^-(p) = \{q \in M \mid q \ll p\}$,
3. the *causal future of* p, $J^+(p) = \{q \in M \mid p \leq q\}$,

4. the *causal past of p*, $J^-(p) = \{q \in M \mid q \le p\}$.

For general subsets $S \subseteq M$ we may define in an analogous way the sets $I^+(S), I^-(S), J^+(S), J^-(S)$. For example $I^+(S) = \{q \in M \mid s \ll q \text{ for some } s \in S\} = \bigcup_{s\in S} I^+(s)$.

Lemma 1.2.9. *If p is a point of M the sets $I^+(p)$ and $I^-(p)$ are open subsets of M.*

Proof. [5, Lemma 3.5]. □

Remark 1.2.10. In general $J^+(p)$ and $J^-(p)$ are neither open nor closed, but for example in Minkowski spacetime they are both closed.

Definition 1.2.11. An (open) subset $F \subseteq M$ is said to be *future* if $F = I^+(F)$ and *past* if $F = I^-(F)$.

Proposition 1.2.12. *If F is a future set its topological closure is characterized as $\overline{F} = \{p \in M \mid I^+(p) \subseteq F\}$. If P is a past set $\overline{P} = \{p \in M \mid I^-(p) \subseteq P\}$.*

Proof. Suppose $p \in M$ such that $I^+(p) \subseteq F$ let $\{q_n\}_{n\in\mathbb{N}} \subseteq I^+(p)$ with $q_n \to p$ then since $\{q_n\} \subseteq F$ we have $p \in \overline{F}$. On the other hand let $p \in \overline{F}$ take any $q \in I^+(p)$ then $p \in I^-(q)$ which is open, and since $p \in \overline{F}$ there exists some $z \in I^-(q) \cap F$, hence $z \in F$ and $z \ll q$ implies, since F is a future set, $q \in F$ so $I^+(p) \subseteq F$. □

Corollary 1.2.13. *Let F be a future set, its topological boundary has the following characterization $\partial F = \{p \in M \mid p \notin F \text{ and } I^+(p) \subseteq F\}$. Let P be a past set then $\partial P = \{p \in M \mid p \notin P \text{ and } I^-(p) \subseteq P\}$.*

Remark 1.2.14. In particular, as $I^+(p)$ and $J^+(p)$ are future sets, we have $\overline{I^+(p)} = \overline{J^+(p)} = \{q \in M \mid I^+(q) \subseteq I^+(p)\}$ and $\partial I^+(p) = \{q \in M \mid q \notin I^+(p) \text{ and } I^+(q) \subseteq I^+(p)\}$. Similarly for the past set of a point.

Definition 1.2.15. Let $\gamma : [a, b) \to M$ be a curve, p is said to be an *endpoint* of γ corresponding to $t = b$ if $\lim_{t\to b^-} \gamma(t) = p$.
If γ is a future (past) directed causal curve with endpoint p corresponding to $t = b$ the point is called *future (past) endpoint of* γ. A causal curve is said to be *future inextendible* if it has no future endpoint. A causal curve is said to be *inextendible* if it is both future and past inextendible.

Definition 1.2.16. M is said to be *chronological* if $p \notin I^+(p)$ for all $p \in M$, i.e. M does not contain any closed timelike curve.

Definition 1.2.17. M is said to be *causal* if it contains no pair of distinct points $p, q \in M$ with $p \le q \le p$, i.e. M does not contain any closed causal curve.

Definition 1.2.18. M is said to be *strongly causal* if around every point there exist arbitrary small neighborhoods such that no causal curve that leaves one of these neighborhood ever returns.

Definition 1.2.19. A *Cauchy surface* S is an embedded topological hypersurface of M which every inextendible causal curve intersects exactly once.

Remark 1.2.20. From the point of view of physics a Cauchy surface is regarded as a set of initial data, that can be integrated a finite distance in the future and that determines the future evolution of the spacetime at least locally.

Remark 1.2.21. We can notice that a Cauchy surface is an *acausal* subset of M, that is: it does not contain a pair of points joined by a causal curve.

Example 1.2.22. In the $(n+1)$-dimensional Minkowski space $\mathbb{M}^{n+1}$ every spacelike hyperplane is a Cauchy surface.

Example 1.2.23. In the $(n+1)$-dimensional Minkowski space $\mathbb{M}^{n+1}$ let $\mathbb{H}^n = \{(x_0, \ldots, x_n) \in \mathbb{M}^{n+1} \mid -x_0^2 + x_1^2 + \ldots + x_n^2 = -1,\ x_0 > 0\}$ be the hyperboloid model of the n-dimensional hyperbolic space, then $\mathbb{H}^n$ is a Cauchy surface for $I^+(0)$, but it is not a Cauchy surface for the whole Minkowski space $\mathbb{M}^{n+1}$ where it is embedded.

Definition 1.2.24. Given an achronal set S (no two points of it may be joined by a timelike curve) the *future domain of dependence* of S, denoted by $D^+(S)$, consists of all the points p of the spacetime such that every past-directed, inextendible in the past, causal curve from p intersects S.
The *past domain of dependence* $D^-(S)$ is defined in a similar way changing the role of the past with the future. Finally the *domain of dependence* of S is $D(S) = D^+(S) \cup D^-(S)$.

Remark 1.2.25. Notice that the set S is always included in its domain of dependence.

Remark 1.2.26. Notice that if S is an acausal subset of a spacetime M, S is a Cauchy surface for M if and only if $D(S) = M$.

Example 1.2.27. In the $(n+1)$-dimensional Minkowski space $\mathbb{M}^{n+1}$ let $\mathbb{H}^n$ be the hyperboloid model of the n-dimensional hyperbolic space then $D^+(\mathbb{H}^n) = \{(x_0, \ldots, x_n) \in \mathbb{M}^{n+1} \mid -x_0^2 + x_1^2 + \ldots + x_n^2 \geq -1,\ x_0 > 0\}$ and $D^-(\mathbb{H}^n) = \{(x_0, \ldots, x_n) \in \mathbb{M}^{n+1} \mid 0 \leq -x_0^2 + x_1^2 + \ldots + x_n^2 \leq -1,\ x_0 > 0\}$ hence $D(\mathbb{H}^n) = I^+(0)$.

Definition 1.2.28. A strongly causal spacetime (M, η) is said to be *globally hyperbolic* if for all $p, q \in M$ we have that $J^+(p) \cap J^-(q)$ is compact.

Proposition 1.2.29. *In a globally hyperbolic spacetime M the causal future (past) of every point is closed.*

Proof. [5, Proposition 3.16]. □

Proposition 1.2.30. *In a globally hyperbolic spacetime $J^+(K) \cap J^-(K')$ is compact $\forall K, K' \subseteq M$ compact.*

Proof. First let us prove that if K is compact then $J^+(K)$, and, in an analogous way $J^-(K)$, is closed. Let $\{q_n\}_{n\in\mathbb{N}} \subseteq J^+(K)$ so $q_n \in J^+(p_n)$ with $p_n \in K$ suppose $q_n \to q$, by the compactness of K from $\{p_n\}_n$ we can extract a convergent subsequence $p_n \to p$. From a consequence of Ascoli-Arzela's Theorem [5, Proposition 3.31] the sequence of future directed causal curves $\{\gamma_n\}$ joining p_n with q_n admits a limiting curve γ passing through p that is still causal. Since γ is a limiting curve for the γ_n between p_n and q_n it will also pass through q hence $q \in J^+(p) \subseteq J^+(K)$. For any $p \in K$ take a point $q \in I^-(p)$ and notice that $J^+(p) \subseteq J^+(q)$. If we consider the open covering of K $\{I^+(q) \cap K\}_{q\in I^-(p), p\in K}$, by compactness we can extract a finite subcover $\{(I^+(q_1) \cap K), \ldots, (I^+(q_n) \cap K)\}$, hence $J^+(K) \subseteq \cup_{i=1}^n J^+(q_i)$. In the same way $J^-(K') \subseteq \bigcup_{j=1}^n J^-(q'_j)$ hence $J^+(K) \cap J^-(K') \subseteq \cup_{i=1}^n \cup_{j=1}^m (J^+(q_i) \cap J^-(q'_j))$. On the right side we have a finite union of compact hence a compact set and on the left side we have the intersection of two closed sets hence a close set. This implies that $J^+(K) \cap J^-(K')$ is compact. □

Theorem 1.2.31. *If S is a Cauchy surface for the spacetime M then there exists a diffeomorphism $f : M \to \mathbb{R} \times S$ such that $f^{-1}(\{*\} \times S)$ is a Cauchy surface for M.*

Proof. In [16, Property 7] Geroch proves only a topological splitting of M as a product of the form $\mathbb{R} \times S$ where S is a Cauchy surface for M. Recently in [11] it has been proved the smoothness of the splitting $M \cong \mathbb{R} \times S$. □

Remark 1.2.32. Actually what is proved in [11] is that every globally hyperbolic spacetime decomposes as $\mathbb{R} \times S$ where S is a smooth spacelike Cauchy hypersurface for M and every other $\{*\} \times S$ is a spacelike Cauchy hypersurface of M. So we can deduce that every globally hyperbolic spacetime contains a smooth spacelike hypersurface.

Remark 1.2.33. While proving the above theorem there is an important side result that is the following. Let M be a C^k-spacetime and S a C^r-Cauchy surface for M with $r \leq k$, then any further C^r-Cauchy surface for M will be C^r-diffeomorphic to S. This is done in [10, Lemma 2.2.] or [16, Property 7]. Essentially if T is a complete smooth timelike vector field on M let ϕ be its flow. Then it can be proved that $\psi : \mathbb{R} \times S \to M$ defined as $\psi(s, x) = \phi_s(x)$ is a C^r-diffeomorphism and that if we write $\psi^{-1}(z) = (s(z), \rho(z))$ and we let S' be another C^r-Cauchy surface for M the map $\rho_{|S'} : S' \to S$ is a C^r-diffeomorphism.

Theorem 1.2.34. *A spacetime is globally hyperbolic if and only if it admits a Cauchy surface.*

Proof. [16, Theorem 11]. □

Remark 1.2.35. From Theorem 1.2.31 and Theorem 1.2.34 we can see that a globally hyperbolic spacetime is never compact.

Remark 1.2.36. Minkowski spacetime is globally hyperbolic. Indeed it admits many Cauchy surfaces, see Example 1.2.22.

Proposition 1.2.37. *If S is a spacelike Cauchy surface for a spacetime M then its lifting $\tilde{S}$ to the isometric universal cover $\widetilde{M}$ of M is a Cauchy surface for $\widetilde{M}$. In particular if M is globally hyperbolic so is $\widetilde{M}$.*

Proof. Since $\widetilde{M}$ is isometric to M the lifting of S to $\widetilde{M}$ is still a spacelike hypersurface. Furthermore if $\widetilde{\gamma}$ is a causal curve in $\widetilde{M}$ and $\pi : \widetilde{M} \to M$ the covering projection then $\pi(\widetilde{\gamma})$ is a causal curve in M hence it intersects S. This implies that $\widetilde{\gamma}$ intersects $\widetilde{S}$ and since $\widetilde{S}$ is spacelike it intersects it exactly once. Finally if M is globally hyperbolic by Remark 1.2.32 it admits a spacelike Cauchy hypersurface, call it S. Then from the previous part $\widetilde{S}$ is a Cauchy hypersurface for $\widetilde{M}$ so by Theorem 1.2.34 $\widetilde{M}$ is globally hyperbolic. □

Definition 1.2.38. Let M be a globally hyperbolic spacetime and let S be a Cauchy hypersurface of it. Then a *S-embedding* is an isometric embedding $f : M \to M'$ where M' is another spacetime such that $f(S)$ is a Cauchy hypersurface for M'. This notion is independent from the choice of the Cauchy surface, i.e. if S' is another Cauchy surface for M then f is a S-embedding if and only if it is a S'-embedding. Therefore the map f is called a *Cauchy embedding*. A globally hyperbolic manifold is said to be *maximal* if any Cauchy embedding into another globally hyperbolic manifold is necessarily surjective.

Example 1.2.39. For instance $I^+(0)$ in $\mathbb{M}^{n+1}$ is a maximal globally hyperbolic spacetime even if it is embedded in a bigger globally hyperbolic spacetime, namely $\mathbb{M}^{n+1}$. This is because $\mathbb{H}^n$, the Cauchy hypersurface for $I^+(0)$, is not a Cauchy hypersurface for $\mathbb{M}^{n+1}$.

Theorem 1.2.40 (Choquet-Bruhat and Geroch)**.** *Every globally hyperbolic spacetime M admits a Cauchy embedding in a maximal globally hyperbolic spacetime. Moreover this maximal globally hyperbolic extension is unique up to isometries.*

Proof. [15, Theorem 3]. □

Definition 1.2.41. Let (M, η) a Lorentzian manifold, let $\gamma : I \to M$ a causal curve, the *Lorentzian length* of γ is

$$\mathcal{L}(\gamma) = \int_I \sqrt{-\eta(\dot{\gamma}(t), \dot{\gamma}(t))}dt.$$

Definition 1.2.42. The *Lorentzian distance* between $p, q \in M$ is defined as follows

$$d(p,q) = \begin{cases} \sup\{\mathcal{L}(\gamma) \mid \gamma \text{ is a future directed causal curve from p to q}\} \text{ if } p \leq q \\ 0 \text{ if } p \not\leq q \end{cases}$$

Remark 1.2.43. The Lorentzian distance in general Lorentzian manifolds does not satisfy the properties for being a distance, for instance

1. it needs not be finite: if $p \in I^+(p) \Rightarrow d(p,p) = \infty$,
2. it may fail to be non-degenerate: if $I^+(p) \neq M$ then if we take $q \neq p$ and $q \notin I^+(p)$ then $d(p,q) = 0$,
3. it tends to be non-symmetric: if $p \neq q$ and $d(p,q)$ and $d(q,p)$ are finite then $d(p,q) = 0$ or $d(q,p) = 0$,
4. however it always satisfies the reverse triangular inequality:

$$p \leq r \leq q \Rightarrow d(p,q) \geq d(p,r) + d(r,q),$$

5. when it is finite it is lower semicontinuous:

$$\text{if } d(p,q) < \infty \; p_n \to p, \; q_n \to q \Rightarrow d(p,q) \leq \liminf_n d(p_n, q_n)$$

see [5, Lemma 4.4].

Theorem 1.2.44. *In a globally hyperbolic spacetime (M, η) for any given points $p, q \in M$ with $q \in J^+(p)$ there is a maximal future directed non-spacelike geodesic segment γ from p to q with $\mathcal{L}(\gamma) = d(p,q)$.*

Proof. [5, Theorem 6.1.]. □

Definition 1.2.45. Let (M, η) be a spacetime define the *cosmological time* $\tau : M \to (0, \infty]$ as

$$\tau(p) = \sup_{q \leq p} d(q,p)$$

We can also write $\tau(p) = \sup\{\mathcal{L}(c) \mid c \text{ past directed causal curve starting at p}\}$.

Remark 1.2.46. In general spacetimes time functions (i.e. functions that are strictly increasing on every future directed causal curve) are defined in order to permit the decomposition into space and time, in particular Geroch [16] defines a time function for every globally hyperbolic spacetime in order to prove his theorem for the topological decomposition of the spacetime as $\mathbb{R} \times S$, where S is a Cauchy surface. However the choice of such functions is rather arbitrary. In [2] the cosmological time function is studied, which can be thought to be a canonical one in the cosmological setting.

Remark 1.2.47. In general the cosmological time function may not be nice. For instance in the case of Minkowski space $\tau \equiv \infty$. And also there are some examples where it is finite but discontinuous, see [2].

Definition 1.2.48. The cosmological time function τ on (M, η) is *regular* if and only if

1. $\tau(p) < \infty$ for all $p \in M$ and
2. $\tau \to 0$ on every past inextendible causal curve.

If τ is regular we call it a *canonical cosmological time* (CT).

Theorem 1.2.49. *Let (M, η) be a spacetime such that the cosmological time τ is regular, then the following properties hold*

1. *(M, η) is globally hyperbolic,*
2. *τ is a time function, i.e. it is continuous and strictly increasing along future directed causal curves,*
3. *for each $p \in M$ there is a future directed timelike ray $\gamma_p : (0, \tau(p)] \to M$ that realizes the distance from the "initial singularity" to p, that is, γ_p is a future directed timelike unit speed geodesic, which is maximal on each segment ($d(\gamma_p(t), \gamma_p(s)) = t - s$ for $0 < s < t \leq \tau(p)$), such that $\gamma_p(\tau(p)) = p$ and $\tau(\gamma_p(t)) = t$, $\forall t \in (0, \tau(p)]$,*
4. *τ is locally Lipschitz ant its first and second derivatives exist almost everywhere.*

Proof. [2, Theorem 1.2]. □

Corollary 1.2.50. *The level sets of a regular cosmological time $S_a = \tau^{-1}(a)$ are future Cauchy surfaces, i.e. each inextendible causal curve that intersects the future of the surface actually intersects the surface once.*

Proof. [2, Corollary 2.6.]. □

Example 1.2.51. Consider $\mathbb{M}^{2+1}$ with coordinates (t, x, y), consider the chronological future of the origin $I^+(0) = \{(t, x, y) \in \mathbb{M}^{2+1} \mid x^2 + y^2 - t^2 < 0,\ t > 0\}$. The cosmological time of $I^+(0)$ at $p \in I^+(0)$ equals the Lorentzian length of the timelike geodesic arc connecting p to 0

$$\begin{aligned} \tau : I^+(0) &\longrightarrow (0, \infty) \\ p &\to \sqrt{t^2 - x^2 - y^2}. \end{aligned}$$

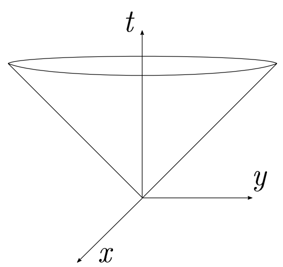

Figure 1.1: Future of the origin

In this case τ is a smooth submersion and the level surfaces are the upper part of the hyperboloids

$$\mathbb{H}^n(a) = \tau^{-1}(a) = \{(t, x, y) \in \mathbb{M}^{2+1} \mid x^2 + y^2 - t^2 = -a^2,\ t > 0\}.$$

Example 1.2.52. Consider $\mathbb{M}^{2+1}$ with coordinates (t, x, y), let $I^+(\Delta)$ be the chronological future of the line $\Delta = \{x = t = 0\}$

$$I^+(\Delta) = \{(t, x, y) \in \mathbb{M}^{2+1} \mid x^2 - t^2 < 0,\ t > 0\}.$$

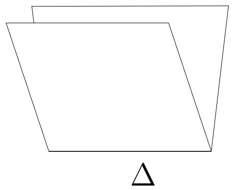

Figure 1.2: Future of a spacelike line

The cosmological time of $I^+(\Delta)$ at $p = (t, x, y) \in I^+(\Delta)$ equals the Lorentzian length of the timelike geodesic arc connecting p to $(0, 0, y)$

$$\tau : I^+(\Delta) \longrightarrow (0, \infty)$$
$$p \to \sqrt{t^2 - x^2}.$$

The level surfaces are

$$\mathbb{I}(\Delta, a) = \tau^{-1}(a) = \{(t, x, y) \in \mathbb{M}^{2+1} \mid x^2 - t^2 = -a^2,\ t > 0\}.$$

Remark 1.2.53. The problem of the cosmological time is that in general it has very low regularity: typically C^1 but not C^2.

Example 1.2.54. For example, using the same notation as in Example 1.2.52, if we fix $r \in \mathbb{R}$ and consider the domain $U = A \cup B \cup C$ where $A = I^+(0) \cap \{y \leq 0\}$, $B = I^+(\Delta) \cap \{0 \leq y \leq r\}$ and $C = I^+(0) \cap \{y \geq 0\} + r(0, 0, 1)$, then the cosmological time of the different pieces fits well together giving us a regular cosmological time on U but just with a C^1-regularity.

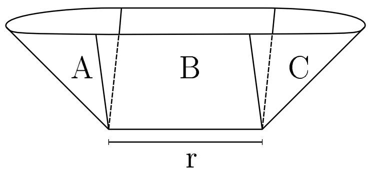

Figure 1.3: Future of a spacelike segment

1.3 Hyperbolic Space

Definition 1.3.1. We will identify the hyperbolic n-space with its hyperboloid model in Minkowski (n+1)-space. The *hyperbolic n-space* $\mathbb{H}^n$ is

$$\mathbb{H}^n = \{x \in \mathbb{M}^{n+1} \mid \langle x, x \rangle = -1,\ x_0 > 0\},$$

where as in section 1.1 $\langle \cdot, \cdot \rangle$ is the Lorentzian inner product on $\mathbb{M}^{n+1}$.

Proposition 1.3.2. *The pair $(\mathbb{H}^n, \eta_{|\mathbb{H}^n})$, where η in the Lorentzian form on $\mathbb{M}^{n+1}$, is an oriented differentiable manifold with a natural Riemannian structure.*

Proof. Since $\mathbb{H}^n$ is the inverse image of a regular value of a differentiable function it is a smooth submanifold of $\mathbb{R}^{n+1}$. From an easy computation we can see $T_x\mathbb{H}^n = x^\perp$, where the orthogonal complement is with respect to the Lorentzian inner product on $\mathbb{M}^{n+1}$. Since x is timelike, its orthogonal is a spacelike hyperplane, hence the Lorentzian form on $\mathbb{M}^{n+1}$ restrict to a positive definite form on $\mathbb{H}^n$. □

Remark 1.3.3. What we have defined as the hyperbolic space is often referred to as the hyperboloid model of hyperbolic space that in the general definition is the simply connected Riemannian manifold with constant sectional curvature equal to -1. There are other possible models for this space.

Disc model Let us consider the stereographic projection with center $-e_0$, where $(e_0, \dots, e_n)$ is the standard basis of $\mathbb{R}^{n+1}$,

$$\begin{aligned} \phi :\ \{x \in \mathbb{R}^{n+1} \mid x_0 > 0\} &\longrightarrow \mathbb{R}^n \cong \{0\} \times \mathbb{R}^n \\ x &\rightarrow \frac{(x_1, \dots, x_n)}{1 + x_0}. \end{aligned}$$

It restricts to a diffeomorphism of $\mathbb{H}^n$ with D^n the open unit ball in $\mathbb{R}^n$. We denote by $\mathbb{D}^n = \{x \in \mathbb{R}^n \mid x_1^2 + \ldots + x_n^2 < 1\}$ the manifold D^n endowed with the pullback metric, so that ϕ becomes an isometry. Namely

$$(\phi^{-1})^*(\eta)(x_1, \dots, x_n) = \frac{4}{(1 - \|x\|^2)^2}(dx_1^2 + \ldots + dx_n^2).$$

Here $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^n$.

Half space model Let $H_+^n = \{y \in \mathbb{R}^n \mid y_n > 0\}$ and let $(e_1, \dots, e_n)$ be the standard basis of $\mathbb{R}^n$. Consider the following map

$$\begin{aligned} \psi :\ \mathbb{D}^n &\longrightarrow H_+^n \\ x &\rightarrow 2\frac{x + e_n}{\|x + e_n\|} - e_n. \end{aligned}$$

It is a diffeomorphism and we will write $\mathbb{H}_+^n$ to denote the manifold H_+^n with the pullback metric

$$(\psi^{-1})^* \left(\frac{4}{1 - \|x\|}(dx_1^2 + \ldots + dx_n^2) \right)(y_1, \dots, y_n) = \frac{1}{y_n^2}(dy_1^2 + \ldots + dy_n^2).$$

Remark 1.3.4. If we consider the models of the disc and of the half-space we can see that $\mathbb{H}^n$ has a natural boundary, for instance in the disc model we see it is homeomorphic to S^{n-1}. Let us consider the projection $\pi : \mathbb{M}^{n+1} \setminus \{0\} \to \mathbb{P}^n$, then $\pi(\mathbb{H}^n)$ is diffeomorphic to the open ball of all the timelike lines. If we consider its closure the boundary is formed by the set of null lines. Hence define $\partial\mathbb{H}^n = \{\text{null lines}\}$ and define $\overline{\mathbb{H}^n} = \mathbb{H}^n \cup \partial\mathbb{H}^n$. It is the compactification of the hyperbolic space, and we give it the topology such that the map $\pi_{|\overline{\mathbb{H}^n}}$ becomes a homeomorphism onto its image.

Theorem 1.3.5. *Let $\gamma : [a, b] \to \mathbb{H}^n$ be a curve then the following are equivalent:*

1. *γ is a geodesic arc,*
2. *there exist two Lorentz orthogonal vectors $x, y \in \mathbb{R}^{n+1}$ such that $\langle x, x\rangle = -1$ and $\langle y, y\rangle = 1$ and*
$$\gamma(t) = (\cosh(t - a))x + (\sinh(t - b))y,$$
3. *the curve satisfies the differential equation $\gamma'' - \gamma = 0$.*

Proof. [26, Theorem 3.2.4]. □

Theorem 1.3.6. *A curve $\gamma : \mathbb{R} \to \mathbb{H}^n$ is a geodesic line if there are Lorentz orthogonal vectors $x, y \in \mathbb{R}^{n+1}$ such that $\langle x, x\rangle = -1$ and $\langle y, y\rangle = 1$ and*

$$\gamma(t) = \cosh(t)x + \sinh(t)y.$$

Proof. [26, Theorem 3.2.5.]. □

Corollary 1.3.7. *The geodesics in $\mathbb{H}^n$ are the intersections of $\mathbb{H}^n$ with a timelike 2-plane of $\mathbb{M}^{n+1}$ passing through the origin.*

Proof. [26, Corollary 4, § 3.2.]. □

Remark 1.3.8. We can see then that $\mathbb{H}^n$ is geodesically complete and hence by Hopf-Rinow theorem complete as a metric space.

Remark 1.3.9. We can also see that given any two distinct points in $\overline{\mathbb{H}^n}$ we have a unique geodesic containing them.

Definition 1.3.10. An *hyperbolic k-plane* is the intersection of $\mathbb{H}^n$ with a $(k+1)$-dimensional timelike linear subspace of $\mathbb{M}^{n+1}$.

Remark 1.3.11. Hyperbolic k-planes are totally geodesic submanifolds.

Remark 1.3.12. It follows that convex subsets of $\mathbb{H}^n$ are intersections of $\mathbb{H}^n$ with a convex cone with apex at 0.

Proposition 1.3.13. *The group of isometries of $\mathbb{H}^n$ (i.e. diffeomorphism preserving the Riemannian form) is isomorphic to $\mathrm{O}^+(n,1)$.*

Proof. [26, Corollary 3, § 3.2.]. □

Corollary 1.3.14. *The group of orientation preserving isometries of $\mathbb{H}^n$ is isomorphic to $\mathrm{SO}^+(n,1)$.*

Remark 1.3.15. Notice that the action of the group of isometries extends to an action on the whole $\overline{\mathbb{H}^n}$ by homeomorphisms.

Definition 1.3.16. From the previous remark and from Brower's fixed point Theorem we can see that every isometry of $\mathbb{H}^n$ fixes some point of $\overline{\mathbb{H}^n}$. The isometry $\phi \in \mathrm{O}^+(n,1)$ is said to be

1. *Elliptic* if it fixes a point in $\mathbb{H}^n$.
   In this case ϕ has a timelike eigenvector with eigenvalue 1.
2. *Parabolic* if it fixes no point in $\mathbb{H}^n$ and a unique point in $\partial\mathbb{H}^n$.
   In this case ϕ has a unique null eigenvector with eigenvalue 1.
3. *Hyperbolic* if it fixes no points in $\mathbb{H}^n$ and two points in $\partial\mathbb{H}^n$.
   In this case ϕ has two null eigenvectors with eigenvalues $\lambda > 1$ and λ^{-1}.

Remark 1.3.17. See [7, Prop. A.5.14] for the proof that these are the only possibilities.

Remark 1.3.18. It can be seen that this classification depends only on the conjugacy class of ϕ in the group of isometries.

Remark 1.3.19. When ϕ is an hyperbolic transformation there exists a unique ϕ invariant geodesic in $\mathbb{H}^n$, indeed it is the unique geodesic between the two fixed points of ϕ. Such a geodesic is called the *axis* of ϕ.

We now recall some facts about discrete subgroups of Iso($\mathbb{H}^n$) and their action on $\mathbb{H}^n$ since they will be important in the next section when we will talk about hyperbolic manifolds.

Remark 1.3.20. We can give a topology to Iso($\mathbb{H}^n$) viewing it as a subset of the set of continuous self maps $C(\mathbb{H}^n, \mathbb{H}^n)$ with the compact-open topology.

Definition 1.3.21. A topological group is discrete if all its points are open.

Remark 1.3.22. A *discrete* subgroup of Iso($\mathbb{H}^n$) is a subgroup that is discrete with the induced topology.

Definition 1.3.23. A group G acts *freely* on a space X if the stabilizer of every point is trivial.

Definition 1.3.24. A group G acts *properly discontinuously* on a locally compact Hausdorff topological space X if for every compact subset K of X the set of elements g in G such that $gK \cap K \neq \emptyset$ is finite.

The importance of discrete subgroups is the following theorem

Theorem 1.3.25. *Let X be a finitely compact metric space (all its closed balls are compact) then a group Γ of isometries of X is discrete if and only if it acts properly discontinuously on X.*

Proof. [26, Theorem 5.3.5.]. □

Remark 1.3.26. We remark that the implication Γ acts properly discontinuously on X then Γ is discrete in the group of isometries of X is always true without the assumption of X being a metric space.

Proposition 1.3.27. *A discrete torsion-free group of isometries of a finitely compact metric space X acts freely on X.*

Proof. Since Γ is discrete it acts properly discontinuously on X and hence the stabilizer of every point is finite. Being Γ torsion-free the stabilizer of every point is trivial. □

Theorem 1.3.28. *A discrete group of isometries Γ of $\mathbb{H}^n$ acts freely on $\mathbb{H}^n$ if and only if it is torsion free.*

Proof. [26, Theorem 8.2.1.] □

Definition 1.3.29. A subgroup Γ of Iso($\mathbb{H}^n$) is called *elementary* if it has a finite orbit in $\overline{\mathbb{H}^n}$.

Definition 1.3.30. Let Γ be a subgroup of Iso($\mathbb{H}^n$) then the *limit set* of Γ is

$$L(\Gamma) = \{p \in \partial\mathbb{H}^n \mid \exists q \in \mathbb{H}^n \text{ and } \exists \text{ a sequence } \{\gamma_i\}_{i=1}^{\infty} \subseteq \Gamma \text{ such that } \gamma_i q \to p\}.$$

Proposition 1.3.31. *If Γ is a discrete torsion-free cocompact subgroup of Iso($\mathbb{H}^n$) then $L(\Gamma) = \partial\mathbb{H}^n$*

Proof. Let us consider P a fundamental polyhedron for Γ containing 0, then from [26, Theorem 6.6.9], P is compact. Consider any $x \in \partial\mathbb{H}^n$ and for all $n \in \mathbb{N}$ consider an euclidean ball centered at x of radius $1/n$, since P is compact and since the euclidean diameter of gP goes to 0 when we are approaching the boundary of $\mathbb{H}^n$ there exists $g_n \in \Gamma$ such that $g_n P$ is all contained in $B(x, 1/n)$. Hence $\Gamma \cdot 0$ accumulates at x. □

Remark 1.3.32. From the previous theorem and from the classification of elementary subgroups of Iso($\mathbb{H}^n$) done in [26, § 5.5.] we can conclude that if Γ is a torsion-free discrete cocompact subgroup of Iso($\mathbb{H}^n$) then Γ is not elementary.

Remark 1.3.33. Furthermore we can also see that Γ does not leave invariant any m-plane of $\mathbb{H}^n$ with $m < n$ otherwise the limit set of Γ will not be the whole $\partial\mathbb{H}^n$. So finally from [26, Corollary 2, § 12.2.] the centralizer of Γ in Iso($\mathbb{H}^n$) is trivial.

1.4 Geometric Structures

Definition 1.4.1. A group G acting on a manifold X is said to act *analytically* if any element of G that acts as the identity on any non empty open subset of X is the identity of G.
Notice that this implies, in particular, that the action is faithful.

Definition 1.4.2. Let G be a Lie group acting smoothly, transitively and analytically on a manifold X, let M be a manifold of the same dimension as X.
A (G, X)-*atlas* on M is a pair $(\mathcal{U}, \Phi)$ where $\mathcal{U} = \{U_\alpha\}_{\alpha\in\mathcal{A}}$ is an open covering of M and $\Phi = \{\phi_\alpha : U_\alpha \to V_\alpha \subseteq X\}_{U_\alpha\in\mathcal{U}}$ is a collection of coordinate charts (i.e. homeomorphism to open subsets of X) such that the restriction of $\phi_\alpha \circ \phi_\beta^{-1}$ to each connected component of $\phi_\beta(U_\alpha \cap U_\beta)$ is the restriction of an element $g_{\alpha\beta} \in G$.

Definition 1.4.3. A (G, X)-*structure* on M is a maximal (G, X)-atlas on M. A (G, X)-*manifold* is a manifold M equipped with a (G, X)-structure.

Definition 1.4.4. If M and N are two (G,X)-manifolds and $f : M \to N$ a differentiable map, then f is a (G, X)-*map* if for each pair of charts $\phi_\alpha : U_\alpha \to V_\alpha$ and $\psi_\beta : U'_\beta \to V'_\beta$ for M and N respectively there exists $g_{\alpha\beta} \in G$ such that

$$\psi_\beta \circ f \circ \phi_\alpha^{-1}{}_{|V_\alpha\cap\phi_\alpha(f^{-1}(U'_\beta))} = g_{\alpha\beta|V_\alpha\cap\phi_\alpha(f^{-1}(U'_\beta))}.$$

Remark 1.4.5. Notice, in particular, that every (G,X)-map is a local diffeomorphism since G acts on X as a subgroup of diffeomorphism of X.

Remark 1.4.6. If $f : M \to N$ is a local diffeomorphism where M and N are smooth manifolds then for every (G, X)-structure on N there exists a unique (G, X)-structure on M such that f becomes a (G, X)-map. It is achieved by pulling back the structure via f.

Example 1.4.7. From Remark 1.4.6 it follows that in particular every covering space (hence also the universal cover) of a (G, X)-manifold has a natural (G, X)-structure.

Example 1.4.8. If $\Gamma \subseteq G$ is a discrete subgroup which acts properly discontinuously and freely on X then X/Γ is a (G, X)-manifold.

Theorem 1.4.9. *Let M be a (G, X)-manifold, let $\widetilde{M}$ be its universal cover endowed with the (G, X)-structure such that the covering map becomes a (G, X)-map, then there exists a (G, X)-morphism*

$$D : \widetilde{M} \to X$$

and a group homomorphism

$$\rho : \pi_1(M) \to G$$

such that

$$D \circ \gamma = \rho(\gamma) \circ D \qquad \forall\gamma \in \pi_1(M)$$

Such a pair (D, ρ) is uniquely determined up to the action of G: any other pair has the form $(g \circ D, g \circ \rho \circ g^{-1})$ for some $g \in G$.

Proof. [18, p. 174-176]. □

Definition 1.4.10. The pair (D, ρ) arising from Theorem 1.4.9 is called a *development pair*. The map D is called a *developing map* of M and the group homomorphism is called the *holonomy* of the (G, X)-structure.

Remark 1.4.11. The development pair completely determines the (G, X)-structure on M, in the sense that the (G, X)-structure on M coincide with the one we get in the following way. Using the developing map associated to it, that is a local diffeomorphism, we can define a (G, X)-structure on $\widetilde{M}$ and then using the universal cover $\pi : \widetilde{M} \to M$ we define the structure on M.

Remark 1.4.12. Generally the map D is just a local isometry neither injective nor surjective.

Definition 1.4.13. When D is a covering map then the (G, X)-structure on M is said to be *complete*.

Remark 1.4.14. Notice that in case the manifold X is simply connected then if M is a complete (G, X)-manifold, this implies that D is a global isometry, i.e. a (G, X)-map that is a diffeomorphism, so that we can identify X with the universal cover of M. In this situation the holonomy morphism is injective so that we may identify $\pi_1(M)$ with its image $\Gamma = \rho(\pi_1(M))$ in G, by the equivariance of the map D, Γ acts freely and properly discontinuously on X, hence D induces an isometry $M \cong X/\Gamma$.

Definition 1.4.15. An *hyperbolic manifold* is an $(\mathbb{H}^n, \mathrm{SO}^+(n, 1))$-manifold.

Remark 1.4.16. Notice that, by the previous remark, a complete hyperbolic manifold will be isometric to $\mathbb{H}^n/\Gamma$ where Γ is a discrete subgroup of $\mathrm{SO}^+(n, 1)$ which is isomorphic to $\pi_1(M)$ via the holonomy morphism.

Remark 1.4.17. Notice that we can endow an hyperbolic manifold with a Riemannian metric in the following way. Let α be the Riemannian metric on $\mathbb{H}^n$, let M be an hyperbolic manifold and let (D, ρ) the associated developing pair. Then since D is a local diffeomorphism we can pull back the metric α on $\widetilde{M}$ and define $\widetilde{\alpha} = D^*\alpha$, i.e.

$$\widetilde{\alpha}_x(v, w) = \alpha_{D(x)} \left(dD(x)v, dD(x)w\right)$$

noticing that this metric is $\pi_1(M)$-invariant we can conclude that it induces a well defined metric $\bar{\alpha}$ on M.

Proposition 1.4.18. *An hyperbolic manifold is complete as a Riemannian manifold if and only if it is* $(\mathbb{H}^n, \mathrm{SO}^+(n, 1))$*-complete.*

Proof. [26, Theorem. 8.5.7]. □

Remark 1.4.19. In particular, by Hopf-Rinow theorem, every compact hyperbolic manifold M will be complete as Riemannian manifold, hence complete as (G, X)-manifold. Hence we may identify compact hyperbolic manifolds with $\mathbb{H}^n/\Gamma$ where Γ as before is a discrete torsion-free cocompact subgroup of $\mathrm{SO}^+(n, 1)$ isomorphic to $\pi_1(M)$.

Proposition 1.4.20. *If M is a compact hyperbolic manifold, hence $M = \mathbb{H}^n/\Gamma$, then all elements of Γ act on $\mathbb{H}^n$ as hyperbolic isometries.*

Proof. [7, Lemma B.4.4]. □

Following [18] we may define for a fixed manifold M a "space of (G, X)-structures" on it.

Definition 1.4.21. Fix a manifold M and consider the following set

$$\mathcal{D}'_{(G,X)}(M) = \{(D, \rho) \mid \rho \in \mathrm{Hom}(\pi_1(M), G) \\ D : \widetilde{M} \to X \ \rho\text{-equivariant local diffeomorphism}\}.$$

We can also think of it as

$$\mathcal{D}'_{(G,X)}(M) = \{(\phi, S) \mid \phi : M \to S \text{ is a diffeomorphism} \\ S \text{ is a } (G,X)-\text{manifold}\}.$$

Let us put on it the C^∞-topology on the maps D and the compact-open topology on the maps ρ (that in case M is compact, so $\pi_1(M)$ is finitely generated, it coincides with the topology of pointwise convergence). We have a natural continuous map

$$hol' : \mathcal{D}'_{(G,X)}(M) \to \mathrm{Hom}(\pi_1(M), G)$$

that is the projection.

Remark 1.4.22. Notice that the group of homotopically trivial diffeomorphisms $\mathrm{Diffeo}_0(M)$ acts on $\mathcal{D}'_{(G,X)}(M)$ by pre-composition with $(\tilde{\psi}, \psi_*)$ where $\psi \in$Diffeo$_0(M)$, $\tilde{\psi}$ is its lift to the universal cover and ψ_* is the map induced on the fundamental group.
Since ψ is homotopically trivial we have that $\psi_* = id$, hence hol' is invariant under this action.

Definition 1.4.23. We may consider the quotient, equipped with the quotient topology

$$\mathcal{D}_{(G,X)}(M) = \mathcal{D}'_{(G,X)}(M)/\mathrm{Diffeo}_0(M),$$

The map hol' induces a well defined continuous map

$$hol : \ \mathcal{D}_{(G,X)}(M) \to \mathrm{Hom}(\pi_1(M), G)$$

Theorem 1.4.24 (Deformation Theorem or Thurston Theorem). *Let M be a compact manifold then the holonomy map hol is a local homeomorphism.*

Proof. [18]. □

Remark 1.4.25. The group G acts on $\mathcal{D}'_{(G,X)}(M)$ as $g(D, \rho) = (g \circ D, g\rho g^{-1})$ and hence on $\mathrm{Hom}(\pi_1(M), G)$ in the same way by conjugacy. The map hol' is obviously equivariant with respect to these actions by G.

Definition 1.4.26. We may define the *Teichmuller space* (*deformation space*) of (G,X)-structures on M to be

$$\mathcal{T}_{(G,X)}(M) = \mathcal{D}_{(G,X)}(M)/G$$

equipped with the quotient topology. Hence *hol* induces a continuous map

$$hol : \mathcal{T}_{(G,X)}(M) \to \mathrm{Hom}(\pi_1(M), G)/G.$$

Definition 1.4.27. Another example of (G,X)-manifolds are *flat spacetimes.* They are, by definition, $(\mathrm{Iso}(\mathbb{M}^{n+1}), \mathbb{M}^{n+1})$-manifolds.

Remark 1.4.28. Notice that if Y is an oriented flat spacetime (so it is also time oriented) then it has also a $(\mathrm{Iso}_0(\mathbb{M}^{n+1}), \mathbb{M}^{n+1})$-structure. Where $\mathrm{Iso}_0(\mathbb{M}^{n+1}) \cong \mathbb{R}^{n+1} \rtimes \mathrm{SO}^+(n,1)$.

Let us specialize to the case, that will be central in our study, where Y is a globally hyperbolic flat spacetime admitting a Cauchy surface diffeomorphic to a compact hyperbolic manifold $M = \mathbb{H}^n/\Gamma$, with Γ a discrete subgroup of $\mathrm{SO}^+(n,1)$.

For every group G let

$$\mathcal{R}_G = \mathrm{Hom}(\pi_1(M), G)/G$$

where the action of G is again given by conjugacy.
As we have already pointed out, since Y is globally hyperbolic with Cauchy surfaces diffeomorphic to M, it decomposes as $Y \cong \mathbb{R} \times M$ and hence $\pi_1(Y) \cong \pi_1(M)$. For simplicity let us set

$$\mathcal{T}_{\mathrm{Lor}}(M) = \{\text{ globally hyperbolic flat Lorentzian structures on } \mathbb{R} \times M \ \}/\mathrm{Diffeo}_0(\mathbb{R} \times M).$$

The holonomy map in this case is

$$hol : \mathcal{T}_{\mathrm{Lor}}(M) \to \mathcal{R}_{\mathrm{Iso}_0(\mathbb{M}^{n+1})}.$$

Let $L : \mathrm{Iso}_0(\mathbb{M}^{n+1}) \to \mathrm{SO}^+(n,1)$ be the projection to the linear part $\mathbb{R}^{n+1} \rtimes \mathrm{SO}^+(n,1) \ni (a, A) \mapsto A \in \mathrm{SO}^+(n,1)$ then we obtain

$$\begin{aligned} L \circ hol : \ \mathcal{T}_{\mathrm{Lor}}(M) &\longrightarrow \mathcal{R}_{\mathrm{SO}^+(n,1)} \\ [(D,\rho)] &\to [L \circ \rho]. \end{aligned}$$

We need the following, very useful lemma about immersed spacelike hypersurfaces in $\mathbb{M}^{n+1}$.

Proposition 1.4.29. *Let S be a connected manifold of dimension n and $f : S \to \mathbb{M}^{n+1}$ a C^r-immersion for $r \geq 1$ such that $f^*(\eta)$ is a complete Riemannian metric on S. Then f is an embedding. Moreover if we fix orthonormal affine coordinates $(y_0, \ldots, y_n)$, we get that $f(S)$ is the graph of a function defined over the horizontal plane $\{y_0 = 0\}$.*

Proof. Let $f(s) = (i_0(s), \ldots, i_n(s))$. Consider the canonical projection $\pi : f(S) \to \{y_0 = 0\}$, namely $\pi(i_0(s), \ldots, i_n(s)) = (0, i_1(s), \ldots, i_n(s))$. Notice that $\pi \circ f$ is distance increasing, in the sense

$$\langle d(\pi \circ f)(x)v, d(\pi \circ f)(x)v \rangle \geq (f^*\eta)(v,v)$$

In fact given $v \in T_sS$ let $df(s)v = (v_0, \ldots, v_n)$ then $d(\pi \circ f)(s)v = (0, v_1, \ldots, v_n)$, hence $f^*\eta(v,v) = \langle d(\pi \circ f)(s)v, d(\pi \circ f)(s)v \rangle - v_0^2$, and the inequality follows. Hence $d(\pi \circ f)(s)$ is an isomorphism for all $s \in S$. In fact let $v \neq w \in T_s(S)$ and assume by contradiction that $d(\pi \circ f)(s)v = d(\pi \circ f)(s)w$. Since f is an immersion we have that $df(s)v \neq df(s)w$ hence $f^*\eta(v-w, v-w) > 0$. On the other hand from the previous inequality we get

$$0 = \langle d(\pi \circ f)(s)(v-w), d(\pi \circ f)(s)(v-w) \rangle \geq f^*\eta(v-w, v-w)$$

hence a contradiction. This implies, since S and $\{y_0 = 0\}$ have the same dimension, that $d(\pi \circ f)$ is an isomorphism, hence $\pi \circ f$ is a local C^r-diffeomorphism. An isometric immersion $p : S^* \to S$ between connected Riemannian manifolds of the same dimension where S^* is complete is a covering map. The proof of this fact can be found in [20, IV, Theorem 4.6]. This implies that $\pi \circ f$ is a covering map, but $\{y_0 = 0\}$ is simply connected hence $\pi \circ f$ is a C^r-diffeomorphism. Hence f is an embedding and $f(S)$ is the graph of a real valued function defined over $\{y_0 = 0\}$. □

Remark 1.4.30. When S is a spacelike hypersurface of $\mathbb{M}^{n+1}$, since points on S are not chronologically related, the function $\varphi : \{y_0 = 0\} \to \mathbb{R}$ that arises from Proposition 1.4.29 such that S is the graph of φ is 1-Lipschitz. Indeed at every point $p \in S$ the graph of φ lies completely outside the future and past cone at p otherwise the tangent plane at p to S would be timelike or null. This implies that φ is 1-Lipschitz.

Recall for the following that $M = \mathbb{H}^n/\Gamma$ is a compact hyperbolic manifold.

Lemma 1.4.31. *Let $\rho : \Gamma \to Iso_0(\mathbb{M}^{n+1})$ be the holonomy morphism of a globally hyperbolic flat Lorentzian structure on $\mathbb{R} \times M$. Then $\rho(\Gamma)$ is isomorphic to Γ and it is a discrete subgroup of $Iso_0(\mathbb{M}^{n+1})$.*

Proof. Let Y be a globally hyperbolic flat spacetime representing the structure on $\mathbb{R} \times M$ and let N be a spacelike Cauchy surface for Y. Since $Y \cong \mathbb{R} \times N \cong \mathbb{R} \times M$ it follows that N and M have isomorphic fundamental groups that we shall identify both equal to Γ. Let $D : \widetilde{N} \to \mathbb{M}^{n+1}$ be the restriction of the developing map of Y to the universal cover of N. From Proposition 1.4.29 it follows that D is an embedding and hence, from the equivariance of D, that ρ is injective and $\rho(\Gamma)$ acts properly discontinuously on $D(\widetilde{N})$ since Γ acts properly discontinuously on $\widetilde{N}$. Hence $\rho(\Gamma)$ is a discrete subgroup of $\mathrm{Iso}_0(\mathbb{M}^{n+1})$. □

Lemma 1.4.32. *Let $\Gamma' = \rho(\Gamma)$, where ρ is the holonomy morphism $\rho : \Gamma \to Iso_0(\mathbb{M}^{n+1})$ of a globally hyperbolic flat Lorentzian structure on $\mathbb{R} \times M$. Let $L : Iso_0(\mathbb{M}^{n+1}) \to \mathrm{SO}^+(n,1)$ be the projection to the linear part then $L_{|\Gamma'} : \Gamma' \to L(\Gamma')$ is an isomorphism. Furthermore $L(\Gamma')$ is a discrete subgroup of $\mathrm{SO}^+(n,1)$.*

Proof. Let $T(\Gamma')$ be the kernel of L restricted to Γ'. As in the previous lemma let Y a globally hyperbolic flat spacetime representing the structure on $\mathbb{R} \times M$ and let N the spacelike Cauchy surface of Y with fundamental group identified with Γ. Since $D(\widetilde{N})$ is spacelike and Γ'-invariant $T(\Gamma')$ must consists of spacelike vectors. Since Γ' is discrete in $\mathrm{Iso}(\mathbb{M}^{n+1})$ and $T(\Gamma')$ is spacelike it follows that $T(\Gamma') = \mathbb{Z}^k$ for some $k \leq n$, see [26, Theorem 5.3.2.]. But $\Gamma' \cong \Gamma = \pi_1(M)$ and as we saw in Remark 1.3.32 Γ is not an elementary subgroup of $\mathrm{SO}^+(n,1)$ and hence by [26, Theorem 5.5.11.] Γ has no nontrivial normal subgroups $\mathbb{Z}^k$, and so $T(\Gamma') = 0$. Hence $L : \Gamma' \to L(\Gamma')$ is an isomorphism. Now we want to show that $L(\Gamma')$ is discrete is $\mathrm{SO}^+(n,1)$. Suppose it is not and let $\overline{L(\Gamma')}$ be its closure in $\mathrm{SO}(n,1)$. By a Theorem of Auslander [25, Theorem 8.24] the identity component of the closure $\overline{L(\Gamma')}_0$ is solvable. Then from [26, Theorem 5.5.10.] $\overline{L(\Gamma')}_0$ is elementary. From the classification of elementary groups done in [26, § 5.5] we see that $\overline{L(\Gamma')}_0$ either fixes a point in $\mathbb{H}^n$ or one or two points in $\partial\mathbb{H}^n$, call F the set of points fixed by $\overline{L(\Gamma')}_0$. Since $\overline{L(\Gamma')}_0$ is a normal subgroup of $\overline{L(\Gamma')}$ it follows that $L(\Gamma')$ normalizes it, then $L(\Gamma')$ leaves F invariant. In the first case $L(\Gamma')$ would be conjugate to a subgroup of $\mathrm{O}(n)$ (see [26, Theorem 5.5.1.]) in the second case, up to finite index, it would be conjugate to the stabilizer in $\mathrm{SO}^+(n,1)$ of a point at infinity which is isomorphic to the group of orientation preserving similarities of $\mathbb{R}^{n-1}$ by [26, Theorem 4.4.4.]. Both these groups are amenable, see [13, Theorem 2.1.3]. Hence also Γ' would be contained in an amenable group. As Γ' is discrete, by Tits' Theorem (see [13, Theorem 2.1.4.]) it implies that Γ' is virtually solvable. Contradicting the hypothesis that $\Gamma' \cong \Gamma$ cannot be virtually solvable otherwise Γ would be elementary. □

Lemma 1.4.33. *Let M and N be hyperbolic n-manifolds such that M is compact, N is complete and $\pi_1(M) \cong \pi_1(N)$ then N is compact as well.*

Proof. The first observation is that M and N are aspherical, i.e. $\pi_i(M) = \pi_i(N) = 0$ for all $i > 1$. This follows from the long exact sequence of homotopy groups associated to a fibration which implies that $\pi_i(M) \cong \pi_i(N) \cong \pi_i(\mathbb{H}^n)$ for $i > 1$ and then from Hadamar-Cartan Theorem we know that $\mathbb{H}^n$ is diffeomorphic to $\mathbb{R}^n$ hence contractible, hence $\pi_i(\mathbb{H}^n) = 0$ for all $i \geq 1$. Now a Theorem of Hurewicz ([19, Proposition 4.30.]) implies that since $\pi_1(M) \cong \pi_1(N)$ and M and N are aspherical then they are homotopy equivalent, hence in particular they have isomorphic n-th homology group $H_n(M;\mathbb{Z}) \cong H_n(N;\mathbb{Z})$. Hence if we assume by contradiction that N is not compact this implies ([19, Proposition 3.29.]) $H_n(N;\mathbb{Z}) = 0$. On the other hand since M is compact and connected we have $H_n(M;\mathbb{Z}) \cong H^0(M;\mathbb{Z}) = \mathbb{Z}$ and we get a contradiction. □

Proposition 1.4.34. *For $n \geq 3$ the image of the holonomy map*

$$hol : \mathcal{T}_{Lor}(M) \to \mathcal{R}_{Iso_0(\mathbb{M}^{n+1})}$$

is contained in $\mathcal{R}(\Gamma) = \{[\rho] \in \mathcal{R}_{Iso_0(\mathbb{M}^{n+1})} \mid L \circ \rho(\gamma) = \gamma \ \forall \gamma \in \Gamma\}$.

Proof. From Lemma 1.4.31 and Lemma 1.4.32 we know that every linear part of the holonomy morphism of a flat globally hyperbolic spacetime diffeomorphic to $\mathbb{R}\times M$ is faithful with discrete image. Hence $L\circ\rho : \pi_1(M)\to \mathrm{SO}^+(n,1)$ is an isomorphism onto its image $L(\rho(\pi_1(M)))$, i.e. $L(\rho(\pi_1(M)))\cong\pi_1(M)=\Gamma$. Notice that from Lemma 1.4.33 we have that $\mathbb{H}^n/L(\rho(\pi_1(M)))$ is a compact hyperbolic manifold hence for $n\geq 3$ Mostow Rigidity Theorem [26, Theorem 11.8.5] implies that $L(\rho(\pi_1(M)))$ coincide with Γ up to conjugacy. □

Proposition 1.4.35. *$\mathcal{R}(\Gamma)$ is naturally identified with $H^1(\Gamma,\mathbb{R}^{n+1})$.*

Proof. Let ρ be a representation of Γ into $\mathrm{Iso}(\mathbb{M}^{n+1})$ whose linear part is the identity, then $\rho(\gamma)=\gamma+\tau_\gamma\in\Gamma\ltimes\mathbb{R}^{n+1}$. Since ρ is a group homomorphism we have $\alpha\beta+\tau_{\alpha\beta}=\rho(\alpha\beta)=\rho(\alpha)\rho(\beta)=\rho(\alpha)(\beta+\tau_\beta)=\alpha\beta+\alpha\tau_\beta+\tau_\alpha$ for all $\alpha,\beta\in\Gamma$. Hence $\tau_{\alpha\beta}=\alpha\tau_\beta+\tau_\alpha$, this implies that $(\tau_\gamma)_{\gamma\in\Gamma}\in Z^1(\Gamma,\mathbb{R}^{n+1})$. Conversely the assignment $\Gamma\to\mathrm{Iso}(\mathbb{M}^{n+1})$ $\gamma\to\tau_\gamma+\gamma$ is a group homomorphism for the same equality as before. Hence we have a bijection between $Z^1(\Gamma,\mathbb{R}^{n+1})$ and the group homomorphisms $\Gamma\to\mathrm{Iso}(\mathbb{M}^{n+1})$ whose linear part is the identity. Now if ρ,ρ' are two such representations and $f\in\mathrm{Iso}_0(\mathbb{M}^{n+1})$ and $\rho'(\gamma)=f\rho(\gamma)f^{-1}$ we have $L(\rho'(\gamma))=\gamma=L(f)\circ\gamma\circ L(f)^{-1}$. This implies that the linear part of f commutes with Γ, but the centralizer of Γ in $\mathrm{SO}^+(n,1)$ is trivial (see Remark 1.3.33), hence f is just a translation by a vector $v\in\mathbb{R}^{n+1}$. So $\rho'(\gamma)(v)=\gamma v+\tau'_\gamma=f\circ\rho(\gamma)\circ f^{-1}(v)=f(\rho(\gamma)(0))=\tau_\gamma+v$, we obtain $\tau'_\gamma-\tau_\gamma=v-\gamma v$. Hence their difference is a coboundary. Conversely if $(\tau_\gamma)_{\gamma\in\Gamma}$ and $(\tau'_\gamma)_{\gamma\in\Gamma}$ are such that $\tau'_\gamma-\tau_\gamma=\gamma v-v$ for some $v\in\mathbb{R}^{n+1}$ the associated group homomorphisms are conjugated by $f=(id,v)$ for the same equality as before. Hence we obtain an identification between $\mathcal{R}(\Gamma)$ and $Z^1(\Gamma,\mathbb{R}^{n+1})/B^1(\Gamma,\mathbb{R}^{n+1})=H^1(\Gamma,\mathbb{R}^{n+1})$. □

Remark 1.4.36. Using this identification we shall denote by ρ_τ the group homomorphism associated to $\tau\in H^1(\Gamma,\mathbb{R}^{n+1})$ and by Γ_τ its image in $\mathrm{Iso}_0(\mathbb{M}^{n+1})$.

Example 1.4.37. One of the simplest example of flat spacetime with Cauchy surface homeomorphic to a compact manifold $M\cong\mathbb{H}^n/\Gamma$ is obtained as follows. Notice that Γ acts properly discontinuously and freely on the whole $I^+(0)$, the future cone at 0. To see why it is sufficient to prove that the action is properly discontinuous since Γ is torsion free. Let K be a compact subset of $I^+(0)$ and consider $C=(J^+(K)\cup J^-(K))\cap\mathbb{H}^n$, notice that since $I^+(0)$ is the domain of dependence of $\mathbb{H}^n$ we have that $\Gamma(K)=\{\gamma\in\Gamma\mid\gamma K\cap K\neq\emptyset\}$ is contained in $\Gamma(C)$ which is finite since C is compact and the action of Γ on $\mathbb{H}^n$ is properly discontinuous, hence this implies that also $\Gamma(K)$ is finite. We can then consider the manifold $\mathcal{C}^+(M)=I^+(0)/\Gamma$, it is called the *Minkowskian cone over M* or the *Minkowskian suspension of M*. We have seen that $I^+(0)$ has canonical cosmological time $\widetilde{T}$

$$\begin{aligned}\widetilde{T}: I^+(0)&\to\mathbb{R}_+\\ p&\to d(0,p)\end{aligned}$$

every level surface is $\widetilde{T}^{-1}(a)=\mathbb{H}^n(a)=\{x\in I^+(0)\mid -x_0^2+x_1^2+\ldots+x_n^2=-a^2\}$. So $I^+(0)$ is globally hyperbolic with Cauchy surfaces $\mathbb{H}^n(a)$. Notice that all the Cauchy surfaces in a globally hyperbolic spacetime are diffeomorphic by Remark 1.2.33, hence $\mathbb{H}^n(a)\cong\mathbb{H}^n(1)=\mathbb{H}^n$. Since $\widetilde{T}$ is Γ-invariant, indeed Γ is acting by isometries, it induces a canonical cosmological time on $\mathcal{C}^+(M)$ where the level surfaces will be $\mathbb{H}^n(a)/\Gamma\cong\mathbb{H}^n/\Gamma=M$. Hence $\mathcal{C}^+(M)$ is a globally hyperbolic flat spacetime with Cauchy surfaces diffeomorphic to M.

Chapter 2

Main Theorem

Now we state the main theorem of the paper of Bonsante [12] of which we are going to explain and articulate the proof in the following sections.
The first part of the theorem about the construction of a future complete flat spacetime associated to any affine deformation of Γ is a generalization of what was done by Mess [21, Proposition 3] in dimension $2+1$. The second part of the theorem where it is proved that the spacetime thus constructed has canonical cosmological time and where this function is used in order to recover some information about the geometry of the flat spacetime is related to a work of Benedetti and Guadagnini [6], where the role of the cosmological time function is emphasized.

Let us fix M a compact hyperbolic manifold, and Γ a discrete torsion-free cocompact subgroup of $\mathrm{SO}^+(n,1)$ such that $M = \mathbb{H}^n/\Gamma$. Recall that we have identified the space $\mathcal{R}(\Gamma)$ of group homomorphisms up to conjugacy $\Gamma \to \mathrm{Iso}(\mathbb{M}^{n+1})$ whose linear part that is the identity with $H^1(\Gamma, \mathbb{R}^{n+1})$ and for a fixed $[\tau] \in H^1(\Gamma, \mathbb{R}^{n+1})$ we denote by $\rho_\tau : \Gamma \to \mathrm{Iso}(\mathbb{M}^{n+1})$ the group homomorphsim associated to τ, i.e $\rho_\tau(\gamma) = \gamma + \tau_\gamma$ and by Γ_τ the image of Γ under ρ_τ. By $\mathcal{T}_{\mathrm{Lor}}(M)$ we mean the Teichmuller space of globally hyperbolic flat Lorentzian structures on $\mathbb{R}_+ \times M$ up to homotopically trivial diffeomorphisms.

Theorem 1. *For every $[\tau] \in H^1(\Gamma, \mathbb{R})$ there exists a unique $[Y_\tau] \in \mathcal{T}_{Lor}(M)$ represented by a maximal globally hyperbolic future complete spacetime Y_τ that admits a developing pair (D, ρ) where*

$$D : \widetilde{Y}_\tau \to \mathbb{M}^{n+1}$$

and

$$\rho : \pi_1(M) \to Iso(\mathbb{M}^{n+1})$$

such that

1. *$\rho = \rho_\tau$,*
2. *the developing map is a global isometry onto its image $\mathcal{D}_\tau$ that is a future complete regular convex proper domain of $\mathbb{M}^{n+1}$,*
3. *the action of $\Gamma_\tau = \rho(\pi_1(M))$ over $\mathcal{D}_\tau$ is free and properly discontinuous so that it induces an isometry between Y_τ and $\mathcal{D}_\tau/\Gamma_\tau$,*
4. *the spacetime $\mathcal{D}_\tau$ has a canonical cosmological time $\widetilde{T} : \mathcal{D}_\tau \to \mathbb{R}_+$ that is a C^1-submersion and such that its level surfaces $\widetilde{S}_a = \widetilde{T}^{-1}(a)$ are Cauchy surfaces for $\mathcal{D}_\tau$ and can be described as the graph of a proper C^1 convex function defined over the horizontal hyperplane $\{y_0 = 0\}$,*

5. *the map $\widetilde{T}$ is Γ_τ-invariant and induces a canonical cosmological time T on Y_τ, that is a proper C^1-submersion and such that every level surface $S_a = \widetilde{S}_a/\Gamma_\tau$ is C^1-diffeomorphic to M,*

6. *for every $p \in \mathcal{D}_\tau$ there exists a unique point $r(p) \in \partial\mathcal{D}_\tau \cap I^-(p)$ such that $\widetilde{T}(p) = d(p, r(p))$. The map $r : \mathcal{D}_\tau \to \partial\mathcal{D}_\tau$ is continuous and the image $\Sigma_\tau = r(\mathcal{D}_\tau)$ is called singularity in the past. Σ_τ is spacelike arc-connected, contractible and Γ_τ-invariant, since the map r is Γ_τ-equivariant.*

The map

$$\mathcal{R}(\Gamma) \to \mathcal{T}_{Lor}(M)$$
$$[\rho_\tau] \to [Y_\tau]$$

is a continuous section of the holonomy map.
The same statement holds if we replace future with past and denote Y_τ^- and $\mathcal{D}_\tau^-$ the corresponding spacetimes.
Every globally hyperbolic flat spacetime with compact spacelike Cauchy surface and holonomy morphism ρ_τ is diffeomorphic to $\mathbb{R}_+ \times M$ and isometrically embeds either in Y_τ or Y_τ^-.

2.1 Construction of $\mathcal{D}_\tau$

For now on we fix Γ torsion-free discrete cocompact subgroup of $\mathrm{SO}^+(n,1)$ and $M = \mathbb{H}^n/\Gamma$. We also fix $[\tau] \in H^1(\Gamma, \mathbb{R}^{n+1})$, and denote by Γ_τ the image of Γ under the holonomy morphism ρ_τ associated to τ. For $\gamma \in \Gamma$ we denote by $\gamma_\tau \in \Gamma_\tau$ the isometry of Minkowski space $\gamma_\tau(x) = \gamma(x) + \tau_\gamma$.

Remark 2.1.1. Note that the assumptions on Γ imply that the action of Γ on $\mathbb{H}^n$ is free and properly discontinuous so that $M = \mathbb{H}^n/\Gamma$ is a compact hyperbolic manifold.

Definition 2.1.2. A closed connected spacelike hypersurface S of Minkowski space $\mathbb{M}^{n+1}$ is said to be *future convex* if $I^+(S)$ is a convex set and $S = \partial I^+(S)$. It is said to be *future strictly convex* if moreover $I^+(S)$ is strictly convex. We can give the same definition, time reversed, for *past convex* and *past strictly convex.*

Example 2.1.3. The hyperbolic space $\mathbb{H}^n \subseteq \mathbb{M}^{n+1}$ is a spacelike future strictly convex hypersurface of $\mathbb{M}^{n+1}$.

The following theorem is the starting point of all the construction needed in order to prove Theorem 1. We will show that for every fixed $\tau \in Z^1(\Gamma, \mathbb{R}^{n+1})$ there exists a spacelike hypersurface in $\mathbb{M}^{n+1}$ that is Γ_τ-invariant, and such that the Γ_τ-action on it is free and properly discontinuous so that the quotient is diffeomorphic to M.

Theorem 2.1.4. *For a fixed $\tau \in Z^1(\Gamma, \mathbb{R}^{n+1})$ there exists a C^∞-embedded spacelike hypersurface $\widetilde{F}_\tau$ of $\mathbb{M}^{n+1}$ that is future strictly convex and Γ_τ-invariant such that the quotient $\widetilde{F}_\tau/\Gamma_\tau$ is diffeomorphic to M.*

Proof. Consider the flat Lorentzian spacetime $N_0 = \left[\frac{1}{2}, \frac{3}{2}\right] \times M$. The Lorentzian structure on it can be defined using the inclusion $N_0 \subseteq \mathcal{C}^+(M) \cong \mathbb{R}_+ \times M$ (see Example 1.4.37 for the definition of $\mathcal{C}^+(M)$). Its universal cover can be identified with $\widetilde{N}_0 = \{x \in \mathbb{M}^{n+1} \mid x \in I^+(0),\ d(0,x) \in \left[\frac{1}{2}, \frac{3}{2}\right]\}$. Notice that $\mathbb{H}^n \subseteq \widetilde{N}_0$. The developing pair associated to the flat Lorentzian structure on N_0 is the inclusion of $\widetilde{N}_0$ in $\mathbb{M}^{n+1}$ and the holonomy morphism associated to $0 \in Z^1(\Gamma, \mathbb{R}^{n+1})$. From the Deformation Theorem 1.4.24 and by our identification of the holonomy morphisms with the cocycles in $Z^1(\Gamma, \mathbb{R}^{n+1})$ we can find a neighborhood U' of $0 \in Z^1(\Gamma, \mathbb{R}^{n+1})$ such that for all $\sigma \in U'$ there exists a flat Lorentzian structure on N_0 with holonomy morphism ρ_σ, that has developing map close in the C^∞-topology to the inclusion of $\tilde{N}_0$ in $\mathbb{M}^{n+1}$. So we may see it as a C^∞-map

$$dev' : U' \times \widetilde{N}_0 \to \mathbb{M}^{n+1}$$

such that for all $\sigma \in U'$ $dev'_\sigma = dev'(\sigma, \cdot)$ is the developing map associated to ρ_σ and $dev'_0 = id_{|\tilde{N}_0}$. Since dev'_σ is close to the identity in the C^∞ topology, by definition they are uniformly close on every compact subset of $\widetilde{N}_0$. We may take a relatively compact covering of $\widetilde{N}_0$ and project it to N_0, by compactness of M we may extract a finite covering and via the universal covering map find a finite covering of $\mathbb{H}^n$. Hence if we choose σ sufficiently small $dev'_\sigma(\mathbb{H}^n)$ will be uniformly close to $\mathbb{H}^n$ hence it will still be a future convex spacelike hypersurface. Now fix a bounded neighborhood U of $0 \in Z^1(\Gamma, \mathbb{R}^{n+1})$ containing τ, then there exists a constant $K > 0$ such that $KU' \supset U$, fix such a constant and consider the map

$$\begin{aligned} dev : U \times \widetilde{N}_0 &\longrightarrow \mathbb{M}^{n+1} \\ (\sigma, x) &\to K dev'(\sigma/K, x). \end{aligned}$$

Then $dev_\sigma = dev(\sigma, \cdot)$ is a developing map whose holonomy morphism is ρ_σ since $dev(\sigma, \gamma x) = K dev'(\sigma/K, \gamma x) = K(\rho_{\sigma/K}(\gamma) dev'(\sigma/K, x)) = \rho_\sigma(\gamma) dev(\sigma, x)$ for all $\gamma \in \Gamma$. And for all $\sigma \in U$ $dev_\sigma(\mathbb{H}^n) = K dev'_{\sigma/K}(\mathbb{H}^n)$ is a future strictly convex spacelike hypersurface that is Γ_σ-invariant. So let $\widetilde{F}_\tau = dev_\tau(\mathbb{H}^n)$ as we just said it is a future strictly convex spacelike hypersurface that is Γ_τ-invariant. Since $\mathbb{H}^n$ is complete when we restrict dev_τ to it by Proposition 1.4.29 dev_τ becomes a diffeomorphism to its image, so the action of Γ_τ on $\widetilde{F}_\tau$ is free and properly discontinuous, since it is so for the action of Γ on $\mathbb{H}^n$, and finally the developing map dev_τ induces a diffeomorphism $\widetilde{F}_\tau/\Gamma_\tau \cong M$. □

Remark 2.1.5. In the same way as in the previous proposition we can obtain a Γ_τ-invariant past strictly convex spacelike hypersurface $\widetilde{F}_\tau^- = dev(\mathbb{H}^n_-)$ such that $\widetilde{F}_\tau^-/\Gamma_\tau \cong M$. Where $\mathbb{H}^n_-$ denotes the lower part of the hyperboloid, $\mathbb{H}^n_- = \{x \in \mathbb{M}^{n+1} \mid \langle x, x \rangle = -1,\ x_0 < 0\}$.

Remark 2.1.6. Notice that when S is a Γ_τ-invariant spacelike hypersurface of $\mathbb{M}^{n+1}$ such that the Γ_τ-action on it is free and properly discontinuous and such that S/Γ_τ is compact then S is a complete Riemannian manifold. In fact if we consider the canonical projection $\pi : S \to S/\Gamma_\tau$ it is a covering map and a local isometry hence if S/Γ_τ is compact it is complete by Hopf-Rinow theorem hence S is also complete since π is a local isometry, see [20, IV, Theorem 4.6].

Remark 2.1.7. Notice that since Γ_τ acts properly discontinuously on $\widetilde{F}_\tau$ then it is a discrete subgroup of $\mathrm{Iso}_0(\mathbb{M}^{n+1})$. Also since it is isomorphic to Γ it is torsion-free.

We have seen in Definition 1.2.24 that the domain of dependence of an achronal set S in a spacetime is defined as the set of points such that every inextendible causal curve passing through them intersects S. After showing some properties that hold for domains of dependence of spacelike hypersurfaces in Minkowski space we are going to see that if $\widetilde{F}$ is a Γ_τ-invariant complete spacelike hypersurface such that the Γ_τ-action on it is free and properly discontinuous then the same holds for its domain of dependence.

Proposition 2.1.8. *Let S be a spacelike hypersurface of $\mathbb{M}^{n+1}$ then the domain of dependence $D(S)$ of S is open.*

Proof. Let p be a point in $D(S)$, choose a causal direction δ through p. From the definition of $D(S)$ there exits $q \in \{p + \mathbb{R}\delta\} \cap S$. Notice that every submanifold of $\mathbb{R}^{n+1}$ is always locally the graph of a function φ hence in particular since S is a spacelike hypersurface it follows that it is locally the graph of a Lipschitz function. Then there exist U a neighboorhood of p and

V a neigborhood of δ in the set of causal directions inside the projective space such that any line with direction in V starting from a point in U will cut S. When δ varies it gives an open covering of the set of causal direction that is compact (homeomorphic to a closed ball). Take a finite cover $V_1, \ldots, V_k$ then every point in the neighborhood of p, defined as $U_1 \cap \cdots \cap U_k$, will stay in $D(S)$. □

Proposition 2.1.9. *If S is a complete spacelike hypersurface of $\mathbb{M}^{n+1}$ then a point $p \in \mathbb{M}^{n+1}$ lies in $D(S)$ if and only if each null line which passes through p intersects S.*

Proof. From Proposition 1.4.29 we know that S is the graph of a function $\varphi : \{y_0 = 0\} \to \mathbb{R}$, hence consider $p \in \mathbb{M}^{n+1}$, and assume $p \in I^+(S)$ (in case $p \in I^-(S)$ we can reason in the same, time reversed way) is such that every null line that passes through it intersects S, when we consider the intersection of the null cone at p with S and we project it onto $\{y_0 = 0\}$ it will be the boundary of a region B, homeomorphic to a n-dimensional ball. This region B is the projection of $J^-(p) \cap S$ onto $\{y_0 = 0\}$. So $J^-(p) \cap S$ will be the image under φ of B and hence all the inextendible causal curve that passes through p will intersect S. □

Proposition 2.1.10. *Let $\widetilde{F}$ be a complete spacelike C^1-hypersurface and suppose there exist a point $p \notin D(\widetilde{F})$, fix a null vector v such that $p + \mathbb{R}v$ does not intersect $\widetilde{F}$. Then the null plane $P = p + v^{\perp}$ does not intersect $\widetilde{F}$.*

Proof. Notice that the existence of the null vector v such that $p + \mathbb{R}v$ does not intersect $\widetilde{F}$ is guaranteed by Proposition 2.1.9. Suppose the intersection $S := P \cap \widetilde{F}$ is not empty. If x is a point in the intersection, the tangent plane at x to P is null and the tangent plane at x to $\widetilde{F}$ is spacelike, hence the intersection is transverse, it follows that S is an $(n-1)$-dimensional closed submanifold of $\widetilde{F}$ and so it is complete. Fix orthonormal affine coordinates $(y_0, \ldots, y_n)$ such that p is the origin and $P = \{y_0 = y_1\}$, so that the null vector v becomes $(1, 1, 0, \ldots, 0)$. Consider the projection

$$\begin{aligned} \pi : \quad S &\longrightarrow \{y_0 = y_1 = 0\} \\ (y_0, \ldots, y_n) &\to (0, 0, y_2, \ldots, y_n) \end{aligned}$$

As in the proof of Proposition 1.4.29 we can argue that π is a C^1-diffeomorphism. Thus there exists $s \in \mathbb{R}$ such that $q = (s, s, 0, \ldots, 0)$ belongs to S and hence to $\widetilde{F}$. But q lies in $p + \mathbb{R}v$ and this is a contradiction. □

Corollary 2.1.11. *Let $\widetilde{F}$ be a complete spacelike hypersurface then the domain of dependence $D(\widetilde{F})$ is a convex set. Moreover for every point p that does not belong to $D(\widetilde{F})$ a null support hyperplane through p exists.*

Proof. In order to conclude that $D(\widetilde{F})$ is convex it is enough to prove that its closure is convex and this reduces to prove (see [9, Proposition 11.5.4.]) that through each boundary point there exists a support hyperplane (i.e. an hyperplane P such that $D(\widetilde{F})$ is all contained in one of the two closed half spaces bounded by P). Notice that from Proposition 2.1.8 $D(\widetilde{F})$ is open hence if p is a boundary point of $\overline{D(\widetilde{F})}$ then $p \notin D(\widetilde{F})$. Furthermore if $p \notin D(\widetilde{F})$ from Propostion 2.1.9 there exists a null direction v such that $p + \mathbb{R}v$ does not intersect $\widetilde{F}$. Finally from Proposition 2.1.10 this implies that the hyperplane $P = p + v^{\perp}$ does not intersect $\widetilde{F}$ hence a fortiori it does not intersect $D(\widetilde{F})$. Hence P is a support hyperplane for $\overline{D(\widetilde{F})}$. □

We now investigate more closely some properties of the domain of dependence of complete Γ_τ-invariant spacelike hypersurfaces. They will be useful in the following chapter in order to show that the domain of dependence of a Γ_τ-invariant future convex complete spacelike hypersurface is what is called a future complete regular convex domain. Notice that from general facts about convex sets that can be found in [9, Proposition 11.5.5.] each closed convex set is the intersection of its supporting half spaces (the half spaces containing it bounded by the supporting hyperplanes through the boundary points).

Corollary 2.1.12. *Let $\widetilde{F}$ be a complete Γ_τ-invariant spacelike hypersurface and suppose $D(\widetilde{F})$ is not the whole $\mathbb{M}^{n+1}$ then either:*

$$D(\widetilde{F}) = \bigcap_{P\text{null plane},\ P\cap\widetilde{F}=\emptyset} I^+(P)$$

or

$$D(\widetilde{F}) = \bigcap_{P\text{null plane},\ P\cap\widetilde{F}=\emptyset} I^-(P)$$

thus $D(\widetilde{F})$ is either a future or a past set.

Proof. We want to show that $D(\widetilde{F})$ is contained either in the future or in the past of its null support planes. Suppose by contradiction that there exist two null support planes P and Q of $D(\widetilde{F})$ such that $D(\widetilde{F}) \subseteq I^-(P) \cap I^+(Q)$. First suppose P and Q are not parallel, then there exists a timelike support plane R. Indeed since P and Q are not parallel, let w_1 and w_2 be their null directions and let us take them so that one is future directed and the other past directed, then we can see that their sum is a spacelike vector. Let $R = p + (w_1 + w_2)^\perp$ be a timelike plane with $p \in P \cap Q$, then for every $x \in D(\widetilde{F})$ we have $\langle x, w_1 + w_1\rangle = \langle x, w_1\rangle + \langle x, w_2\rangle \leq \langle p, w_1\rangle + \langle p, w_2\rangle = \langle p, w_1 + w_2\rangle$, hence R is a support plane for $D(\widetilde{F})$. Fix orthonormal affine coordinates $(y_0, \ldots, y_n)$ such that $R = \{y_n = 0\}$. We know from Proposition 1.4.29 that $\widetilde{F}$ is the graph of a function defined over $\{y_0 = 0\}$. Then $\widetilde{F} \cap R \neq \emptyset$ and this is a contradiction. Now suppose we cannot chose non-parallel null supporting hyperplanes, then all null support hyperplanes are parallel. Let v be the null direction that is orthogonal to all the null support hyperplanes and let $[v]$ the corresponding point in $\partial\mathbb{H}^n$. Since Γ_τ acts on $\widetilde{F}$ and the null support hyperplanes are $p + v^\perp$ for $p \notin D(\widetilde{F})$ such that $p + \mathbb{R}v$ does not intersect $\widetilde{F}$ then Γ_τ permutes the null supporting hyperplanes and hence $\Gamma \cdot [v]$ should be the unique null vector orthogonal to all of them, i.e. $\Gamma \cdot [v] = [v]$. But if all elements of Γ would have a common fixed point in $\partial\mathbb{H}^n$ then Γ would be elementary contradicting Remark 1.3.32 about discrete cocompact subgroups of $\mathrm{SO}^+(n,1)$. □

Proposition 2.1.13. *Let $\widetilde{F}$ be a complete Γ_τ-invariant spacelike hypersurface such that the Γ_τ-action on it is free and properly discontinuous, then Γ_τ acts on $D(\widetilde{F})$ freely and properly discontinuously, moreover $D(\widetilde{F})/\Gamma_\tau$ is diffeomorphic to $\mathbb{R}_+ \times \widetilde{F}/\Gamma_\tau$.*

Proof. Since $\widetilde{F}$ is Γ_τ-invariant it is easy to see that also $D(\widetilde{F})$ is Γ_τ-invariant. Furthermore since Γ_τ is torsion-free, in order to conclude that the action on $D(\widetilde{F})$ is free it is sufficient to prove that the action is properly discontinuous. Take $K \subseteq D(\widetilde{F})$ a compact subset and consider $\Gamma(K) = \{\gamma \in \Gamma \mid \gamma_\tau(K) \cap K \neq \emptyset\}$. We want to show it is finite. Consider the set $C = (J^+(K) \cup J^-(K)) \cap \widetilde{F}$. We want to show it is compact. Since $\widetilde{F}$ is complete it is enough to say that this intersection is closed and bounded. We have already said that when K is compact $J^-(K)$ and $J^+(K)$ are closed in a globally hyperbolic manifold, see the proof of Proposition 1.2.30, hence the intersection with $\widetilde{F}$ is closed in $\widetilde{F}$. Furthermore it is bounded by the intersection of all the light rays starting at points in K with $\widetilde{F}$. In addition $\gamma_\tau(C) = (J^+(\gamma_\tau(K)) \cup J^-(\gamma_\tau(K))) \cap \widetilde{F}$. Hence $\Gamma(K)$ is contained in $\Gamma(C)$, but since the action of Γ_τ on $\widetilde{F}$ is properly discontinuous $\Gamma(C)$ is finite. So we have proved that the action of Γ_τ on $D(\widetilde{F})$ is properly discontinuous. Finally from Remark 1.2.26, noticing that $\widetilde{F}$ is an acausal subset of $\mathbb{M}^{n+1}$, it follows that $\widetilde{F}$ is a Cauchy surface for $D(\widetilde{F})$. This implies that $\widetilde{F}/\Gamma_\tau$ is a Cauchy surface for $D(\widetilde{F})/\Gamma_\tau$. In fact every causal curve c in $D(\widetilde{F})/\Gamma_\tau$ intersects $\widetilde{F}/\Gamma_\tau$ since its lift $\tilde{c}$ to $D(\widetilde{F})$ intersects $\widetilde{F}$. Moreover $\widetilde{F}/\Gamma_\tau$ is spacelike so a causal curve could intersect it more than once only if it would be a closed causal curve. But then it would lift to a causal path that intersects more than once $\widetilde{F}$ that being spacelike is not possible. Finally from the decomposition of a globally hyperbolic spacetime we have $D(\widetilde{F})/\Gamma_\tau \cong \mathbb{R}_+ \times \widetilde{F}/\Gamma_\tau$. □

Remark 2.1.14. Let $\widetilde{F}$ be a future convex spacelike hypersurface then $D(\widetilde{F})$ is future complete. In fact suppose there exists a future directed geodesic $\gamma : [a,b] \to D(\widetilde{F})$ that cannot be defined on the whole $[a,\infty)$ then there exists $t_0 > b$ such that $\gamma(t_0) \notin D(\widetilde{F})$. But γ may be extended in the past until it reaches $\widetilde{F}$ since it passes through points in $D(\widetilde{F})$, hence this contradicts the fact that $\widetilde{F}$ is future convex.

In Theorem 2.1.4 we have constructed $\widetilde{F}_\tau$ for every $\tau \in Z^1(\Gamma, \mathbb{R}^{n+1})$ and we have seen that it is a future strictly convex Γ_τ-invariant spacelike hypersurface such that the quotient $\widetilde{F}_\tau/\Gamma_\tau$ is diffeomorphic to M. Recall that this implies from Remark 2.1.6 that $\widetilde{F}_\tau$ is complete. Let us denote $\mathcal{D}_\tau$ the domain of dependence of the hypersurface $\widetilde{F}_\tau$. From Remark 2.1.14 we also know that $\mathcal{D}_\tau$ is future complete and from Corollary 2.1.11 we know it is a convex set. Furthermore from Proposition 2.1.13 $\mathcal{D}_\tau = D(\widetilde{F})$ is a Γ_τ-invariant set such that the action of Γ_τ on it is still free and properly discontinuous and such that $\mathcal{D}_\tau/\Gamma_\tau \cong \mathbb{R}_+ \times M$. If we denote by Y_τ the quotient $\mathcal{D}_\tau/\Gamma_\tau$ it is a future complete globally hyperbolic flat spacetime with Cauchy surfaces diffeomorphic to M. So we have constructed a family of globally hyperbolic flat spacetimes parametrized by all possible holonomies $\rho : \pi_1(M) \to \mathrm{Iso}(\mathbb{M}^{n+1})$ of globally hyperbolic flat Lorentzian structures on $\mathbb{R} \times M$.

Remark 2.1.15. Let us denote by $\mathcal{D}_\tau^- = D(\widetilde{F}_\tau^-)$, in the same way as for $\mathcal{D}_\tau$, it will be a past complete convex domain of $\mathbb{M}^{n+1}$.

In the final part we want to show that $\mathcal{D}_\tau$ is not the whole $\mathbb{M}^{n+1}$. In fact this is a necessary condition in order to have regular cosmological time (indeed $\mathbb{M}^{n+1}$ does not). Actually we will see in the next section that this is also a sufficient condition.

First let us prove a very useful lemma about proper convex subsets of $\mathbb{M}^{n+1}$.

Lemma 2.1.16. *Let Ω be a proper convex set of $\mathbb{M}^{n+1}$. If we fix set of orthonormal affine coordinates $(y_0, \dots, y_n)$ then Ω is a future convex set if and only if $\partial\Omega$ is the graph of a 1-Lipschitz convex function defined over $\{y_0 = 0\}$.*

Proof. If $\partial\Omega$ is the graph of a 1-Lipschitz convex function defined over $\{y_0 = 0\}$ then Ω is a convex set and since from the Lipschitz condition $\partial\Omega$ is achronal we can see that $\Omega = I^+(\partial\Omega)$ hence it is a future set. On the other hand if we prove that $\pi : \partial\Omega \to \{y_0 = 0\}$ is an homeomorphism then $\partial\Omega$ will be the graph of a function defined over $\{y_0 = 0\}$. Since Ω is convex its boundary is a topological manifold hence it is sufficient to prove that π is bijective. Since Ω is a future set $\partial\Omega = \partial I^+(\Omega)$ is an achronal set, hence the projection is injective. It remains to show that given $(a_1, \dots, a_n)$ there exists a_0 such that $(a_0, a_1, \dots, a_n) \in \partial\Omega$. Fix $p \in \partial\Omega$ then there exist a_0^+ and a_0^- such that $(a_0^+, a_1, \dots, a_n) \in I^+(p)$ and $(a_0^-, a_1, \dots, a_n) \in I^-(p)$. Since $I^+(p) \subseteq \Omega$ and $I^-(p) \cap \Omega = \emptyset$ there exists a_0 such that $(a_0, a_1, \dots, a_n) \in \partial\Omega$. Hence $\partial\Omega$ is the graph of a function f defined over $\{y_0 = 0\}$ and since Ω is convex f is convex and since two points on $\partial\Omega$ are not chronologically related f is 1-Lipschitz. □

We now state some properties about Γ_τ-invariant future convex sets. A future convex set is a set that is both convex and future (see Definition 1.2.11).

Lemma 2.1.17. *Let Ω be a Γ_τ-invariant proper future convex set, then for every $u \in \mathbb{H}^n$ there exists a plane $P = p + u^\perp$ such that $\Omega \subseteq I^+(P)$.*

Proof. Since Ω is a proper convex set there exists a support plane for Ω in $\mathbb{M}^{n+1}$. Let

$$K = \{v \in \mathbb{M}^{n+1} \mid v \text{ is orthogonal to some support planes for } \Omega\},$$

since if v is orthogonal to some support planes so is λv, for $\lambda \in \mathbb{R}$, K is a convex cone with apex at 0. Notice that Ω is future convex hence future complete, so the vectors in K are not spacelike, this is equivalent to say that Ω does not admit a timelike support plane. In fact suppose by contradiction that $\Omega \subseteq I^+(P)$ with P timelike support plane, let us fix a set of orthonormal affine coordinates $(y_0, \dots, y_n)$ such that $P = \{y_n = 0\}$ then from Lemma 2.1.16 $\partial\Omega$ is the graph of a 1-Lipschitz function defined over $\{y_0 = 0\}$ hence we see that P cannot be a support plane for Ω. So the projection $\mathbb{P}K$ of K in $\mathbb{P}^n$ is contained in $\overline{\mathbb{H}^n}$. On the other hand since Ω is Γ_τ-invariant, K is Γ-invariant, hence $\mathbb{P}K$ is a Γ-invariant convex subset of $\overline{\mathbb{H}^n}$. It follows that since K is non empty it should contain the whole $\mathbb{H}^n$, since otherwise the limit set of Γ would be contained in $\overline{\mathbb{P}K} \cap \partial\mathbb{H}^n$, contradicting the fact that when Γ is co-compact its limit set is $\partial\mathbb{H}^n$, see Proposition 1.3.31. □

Lemma 2.1.18. *Let Ω be a Γ_τ-invariant proper future convex set then each timelike coordinate on $\partial\Omega$ is proper.*

Proof. From Lemma 2.1.16 if we fix orthonormal affine coordinates $(y_0, \dots, y_n)$ then $\partial\Omega$ is the graph of $\varphi : \{y_0 = 0\} \to \mathbb{R}$. Hence if we show that φ is a proper function the statement follows. It is sufficient to show that

$$K_C = \{x \in \{y_0 = 0\} \mid \varphi(x) \le C\}$$

is compact for every $C \in \mathbb{R}$. Since φ is a convex function K_C is a closed convex subset of $\{y_0 = 0\}$. Suppose by contradiction that it is not bounded, then there exists $\bar{x} \in \{y_0 = 0\}$ and a vector $w \in \{y_0 = 0\}$, that we can suppose unitary $\langle w, w\rangle = 1$, such that the ray $\bar{x} + \mathbb{R}_{\ge 0} w$ is all contained in K_C. Let $v = \frac{\partial}{\partial y_0}$, it is a future directed timelike vector such that $\langle v, v\rangle = -1$ orthogonal to w and let $u = \sqrt{2}v + w$, then u is timelike future directed and $\langle u, u\rangle = -1$. Then from Lemma 2.1.17 there exists a spacelike support plane for Ω orthogonal to u, hence there exists $M \in \mathbb{R}$ such that $\langle p, u\rangle \le M$ for all $p \in \partial\Omega$. On the other hand consider $p_t = (\bar{x} + tw) + \varphi(\bar{x} + tw)v$ we have that $p_t \in \partial\Omega$ and

$$\begin{aligned}\langle p_t, u\rangle &= -\sqrt{2}\varphi(\bar{x} + tw) + \langle \bar{x} + tw, \bar{x} + tw\rangle \\ &\ge -\sqrt{2}C + \langle \bar{x} + tw, \bar{x} + tw\rangle\end{aligned}$$

Since $\langle \bar{x} + tw, \bar{x} + tw\rangle \to +\infty$ we have a contradiction. □

Proposition 2.1.19. *Let Ω a Γ_τ-invariant future complete convex proper subset of $\mathbb{M}^{n+1}$. Then there exists a null support plane for Ω.*

Proof. Take $p \in \partial\Omega$ and $v \in \mathbb{H}^n$ such that $P = p + v^\perp$ is a support plane for Ω at p. Recall that since Ω is a future convex set, if we fix orthonormal affine coordinates $(y_0, \dots, y_n)$, its boundary can be described as the graph of a convex function over the hyperplane $\{y_0 = 0\}$ so that for a fixed $p \in \partial\Omega$ and $v \in \mathbb{H}^n$ there exists a unique support plane at p for Ω orthogonal to v. Now for a fixed $\gamma \in \Gamma$ consider the sequence of support planes $P_k = \gamma_\tau^k(P)$. If this sequence does not escape to infinity then there is a convergent subsequence which converges to a support plane Q. The normal direction of Q is the limit of the normal directions of the P_k's that are $\gamma^k(v)$. In the projective space the sequence $[\gamma^k(v)]$ tends to a null vector, hence

Q will be a null support plane. Thus we have to prove that P_k does not escape to infinity. Set $v_k = |\langle v, \gamma^k v\rangle|^{-1}\gamma^k(v)$, we know that v_k converges to an attractor eigenvector of γ in $\mathbb{M}^{n+1}$. On the other hand we have

$$P_k = \left\{x \in \mathbb{M}^{n+1} \mid \langle x, v_k\rangle = \langle \gamma_\tau^k p, v_k\rangle\right\}.$$

Thus the sequence P_k does not escape to infinity if and only if the coefficients $C_k = \langle \gamma_\tau^k p, v_k\rangle$ are bounded. Since the sequence $\{v_k\}$ has compact closure it is sufficient to show that $C'_k = \langle \gamma_\tau^k p - p, v_k\rangle$ are bounded. For $\alpha \in \Gamma$ set $z(\alpha) = \alpha_\tau(p) - p$, we can see that z is a cocycle. Thus we have

$$C'_k = \left|\frac{\langle z(\gamma^k), \gamma^k v\rangle}{\langle \gamma^k v, v\rangle}\right| = \left|\frac{\langle \gamma^{-k} z(\gamma^k), v\rangle}{\langle \gamma^k v, v\rangle}\right| = \left|\frac{\langle z(\gamma^{-k}), v\rangle}{\langle \gamma^k v, v\rangle}\right|.$$

For the last equality notice that $\gamma^{-k}z(\gamma^k) = \gamma^{-k}(\gamma_\tau^k p - p) = p + \gamma^{-k}\tau_{\gamma^k} - \gamma^{-k}p = p - \tau_{\gamma^{-k}} - \gamma^{-k}p = -(\gamma^{-k}p - p)$. Now let $\lambda > 1$ be the maximum eigenvalue of γ. Then we have $\|\gamma^{-1}(x)\| \leq \lambda\|x\|$ for every $x \in \mathbb{R}^{n+1}$, where $\|\cdot\|$ denotes the Euclidean norm. Since

$$z(\gamma^{-k}) = -\sum_{i=1}^{k} \gamma^{-i}(z(\gamma))$$

it follows that $\|z(\gamma^{-k})\| \leq K\lambda^k$ for some $K > 0$. Thus we have

$$\left|\langle z(\gamma^{-k}), v\rangle\right| \leq K'\lambda^k.$$

On the other hand v can be decomposed as follows $v = x^+ + x^- + x^0$ where x^+ is an eigenvector for the eigenvalue λ, x^- is an eigenvector for λ^{-1} and x^0 is orthogonal to both x^+ and x^-. Since v is a future directed timelike vector it turns out that x^+ and x^- are future directed null vectors. Thus

$$\langle \gamma^k v, v\rangle = (\lambda^k + \lambda^{-k})\langle x^+, x^-\rangle + \langle x^0, \gamma^k x^0\rangle.$$

Now notice that $\mathrm{Span}(x^+, x^-)^\perp$ is Γ-invariant and spacelike. Hence $\langle x^0, \gamma^k x^0\rangle \leq \langle x^0, x^0\rangle$ so that there exists $M > 0$ such that $\left|\langle \gamma^k v, v\rangle\right| > M\lambda^k$. Thus $|C'_k| \leq K'/M$ and this concludes the proof. □

We can finally prove that the domain of dependence of any Γ_τ-invariant future convex spacelike hypersurface is not the whole $\mathbb{M}^{n+1}$ and in particular that $\mathcal{D}_\tau$ is a proper subset of $\mathbb{M}^{n+1}$.

Proposition 2.1.20. *Let $\widetilde{F}$ be a Γ_τ-invariant future convex spacelike hypersurface then $D(\widetilde{F}) \neq \mathbb{M}^{n+1}$.*

Proof. Take Ω to be $I^+(\widetilde{F})$. Then it is a Γ_τ-invariant future complete convex proper subset of $\mathbb{M}^{n+1}$. From Proposition 2.1.19 this implies that there exists P a null support plane for $I^+(\widetilde{F})$. In particular $P \cap \widetilde{F} = \emptyset$. Hence $D(\widetilde{F}) \neq \mathbb{M}^{n+1}$ since points on P do not belong to $D(\widetilde{F})$. □

2.2 Cosmological Time

We are going to define a class of domains that admits regular cosmological time.

Definition 2.2.1. Let $\Omega \subseteq \mathbb{M}^{n+1}$ be a non empty convex open subset, then we say that Ω is a *future complete regular domain* if it is the intersection of the future of at least two non-parallel null support planes.

Example 2.2.2. Examples of future complete regular domains are the future of a point $I^+(p)$, the future of a spacelike line $I^+(\Delta)$ and the future of a spacelike segment $I^+(\Sigma)$ as in Examples 1.2.51, 1.2.52 and 1.2.54.

Remark 2.2.3. The condition that there are at least two non-parallel null support planes guarantees that Ω is neither the whole $\mathbb{M}^{n+1}$ nor the future of just one null plane. In fact these domains do not have regular cosmological time, indeed the cosmological time for them is constantly equal to $+\infty$.

Remark 2.2.4. On the other hand future complete regular convex domains admit a spacelike support hyperplane, and this condition ensure that the cosmological time function is regular, see the following Theorem 2.2.8. To see why the existence of at least two non-parallel null support planes P, Q guarantees the existence of a spacelike support plane let R be the intersection of P and Q. Since P and Q are not parallel their intersection is an $(n-1)$-dimensional submanifold that is spacelike. Write $P = \mathbb{R}w_1 \oplus R$ and $Q = \mathbb{R}w_2 \oplus R$ where w_1, w_2 are null vectors. Now take a spacelike direction v orthogonal to R, the hyperplane $R \oplus \mathbb{R}v$ will be a spacelike supporting hyperplane for Ω.

Remark 2.2.5. Let $\widetilde{F}$ be a Γ_τ-invariant future convex complete spacelike hypersurface, from Proposition 2.1.20 we know that $D(\widetilde{F}) \neq \mathbb{M}^{n+1}$, then we have seen in Proposition 2.1.12 that $D(\widetilde{F})$ is the intersection of the future of its null support planes, and from Lemma 2.1.17 we see that there exists at least one spacelike support plane, hence $D(\widetilde{F})$ cannot be the future of just one null support plane. So $D(\widetilde{F})$ is a future complete regular domain. In particular $\mathcal{D}_\tau$ is so.

Remark 2.2.6. As the name suggests a future complete regular domain Ω is complete in the future. This means that the domain of definition of any future directed timelike or null geodesic $\gamma : [a, b] \to \Omega$ can be extended to all $[a, \infty)$.

In fact in order to have regular cosmological time it is sufficient to be a future complete convex set with at least one spacelike support hyperplane. Notice that a future complete regular domain satisfies these hypothesis.

Remark 2.2.7. Notice that if A is a future complete convex set with at least one spacelike support hyperplane then in particular it is a future set hence its boundary $S = \partial A$ is an achronal set and all its support hyperplanes are spacelike or null from Proposition 2.1.16.

Theorem 2.2.8. *Let A be a future complete convex subset of $\mathbb{M}^{n+1}$ and $S = \partial A$, suppose that A admits a spacelike support plane. Then for every $p \in A$ there exists a unique point $r(p) \in S$ which maximizes the Lorentzian distance from p in $\overline{A} \cap J^-(p)$. Moreover the map $A \ni p \mapsto r(p) \in S$ is continuous, we call it the retraction.*
The point $r = r(p)$ can be characterized as the unique point in S such that the plane $r + (p-r)^\perp$ is a support plane for A.
The cosmological time for A is expressed by the formula

$$T(p) = \sqrt{-\langle p - r(p), p - r(p)\rangle}.$$

T is a concave C^1-function.
The Lorentzian gradient of T is given by

$$\nabla_L(T)(p) = -\frac{1}{T(p)}(p - r(p)).$$

Proof. Since A is convex the Lorentzian distance in A is just the restriction of the Lorentzian distance in $\mathbb{M}^{n+1}$. In fact if p and q belong to $J^-(p) \cap A$ then we have that

$$d(p, q) = \sup\{\mathcal{L}(\gamma) \mid \gamma \text{ causal curve between p and q}\}$$

and since A is convex $\overrightarrow{pq}$ is contained in A and we know that $\sqrt{-\langle p-q, p-q\rangle}$ maximizes the distance among all causal curves between p and q in $\mathbb{M}^{n+1}$. Hence $d(p,q) = \sqrt{-\langle p-q, p-q\rangle}$ for all $p \in A, q \in J^-(q) \cap A$. Let us fix $p \in A$ and a spacelike support plane P of A. Since $J^-(p) \cap J^+(P)$ is compact and $J^-(p) \cap A \subseteq J^-(p) \cap J^+(P)$ there exists a point $r \in \overline{A} \cap J^-(p)$ which maximizes the Lorentzian distance from p. From the definition r should lie in the boundary S of A. Now we want to show that r is unique. Assume it is not and that $r' \in S$ different from r is another point such that $d(p,r) = d(p,r')$ maximal. Define $\mathbb{H}^-(p,\alpha) = \{q \in I^-(p) \mid d(p,q) = \alpha\}$, it is a past convex spacelike hypersurface. The segment (r,r') is contained in $I^-(\mathbb{H}^-(p, d(p,r)))$, hence $d(p,s) > d(p,r)$ for all $s \in (r,r')$. On the other hand $(r,r') \subseteq \overline{A}$, and this contradicts the choice of r.

We have to prove that the map $p \mapsto r(p)$ is continuous. Let $\{p_k\} \subseteq A$ be such that $p_k \to p$, set $r_k = r(p_k)$. First we want to show that $\{r_k\}$ is bounded. Notice that for a fixed $q \in I^+(p)$, there exists k_0 such that $p_k \in J^-(q)$ for every $k \geq k_0$ so $r_k \in J^-(q) \cap S$ for $k \geq k_0$. Since $J^-(q) \cap S$ is compact, it is sufficient to prove that if $r_k \to r$ then $r = r(p)$. If $q \in A$ then we have, by the definition of r_k,

$$\langle p_k - r_k, p_k - r_k\rangle \leq \langle p_k - q, p_k - q\rangle .$$

By passing to the limit we obtain that r maximizes the Lorentzian distance.

Take $p \in A$ and $P_p = r(p) + (p - r(p))^\perp$, we claim that it is a support plane for A. Notice that P_p is the tangent plane of $\mathbb{H}^-(p, d(p,r))$ at $r(p)$. Suppose by contradiction there exists $q \in A \cap I^-(P_p)$. We have, since A is convex, $(q,r) \subseteq A$. On the other hand there exists $q' \in (q,r) \cap I^-(\mathbb{H}^-(p, d(p,r)))$. Then $d(p,q') > d(p,r)$ and this is a contradiction. Conversely, let $s \in \overline{A}$ such that $P = s + (p-s)^\perp$ is a support plane for A. We want to show that $s = r(p)$. First we want to show that $s \in I^-(p)$. Assume by contradiction that s is not in the past of p then either $p - s$ is a null vector or $s \in I^+(p)$ (the support plane can only be null or spacelike). In the first case the support plane contains p hence it coincide with $p + (s-p)^\perp$, this implies that $p \in S$ and $p = r(p) = s$. In the second case since $p \in I^+(r(p))$ we get that $s \in I^+(r(p))$ contradicting the fact the points on the boundary of A are not chronologically related. So P is the tangent plane of $\mathbb{H}^-(p, d(p,s))$ at s, hence if by contradiction $s \neq r(p)$ then $d(p, r(p)) > d(p,s)$ implies $r(p) \in I^-(\mathbb{H}^-(p, d(p,s)))$ contradicting the fact that P is a support plane for A. Hence $s = r(p)$.

Now we want to show that the cosmological time T is C^1. We shall use the following fact from elementary analysis.

Let $\Omega \subseteq \mathbb{R}^N$ be an open set and $f : \Omega \to \mathbb{R}$ a continuous function. Suppose there exist $f_1, f_2 : \Omega \to \mathbb{R}$ such that

1. $f_1 \leq f \leq f_2$,
2. $f_1(x_0) = f_2(x_0) = f(x_0)$ and
3. f_1, f_2 are C^1 and $df_1(x_0) = df_2(x_0)$

then f is differentiable in x_0 and $df(x_0) = df_1(x_0)$.

Let us fix $p \in A$, $r = r(p)$ and orthonormal affine coordinates $(y_0, \ldots, y_n)$ such that $r(p)$ is the origin and P_p is the plane $\{y_0 = 0\}$, so $p = (\mu, 0, \ldots, 0)$ where $\mu = T(p)$. Consider the differentiable functions $f_1 f_2 : A \to \mathbb{R}$,

$$f_1(y) = y_0^2 - \sum_{i=1}^n y_i^2 \quad \text{and}$$

$$f_2(y) = y_0^2,$$

then $f_1 \leq T^2 \leq f_2$ and $f_1(p) = T^2(p) = f_2(p)$. Moreover $\nabla_L f_1(p) = -2\mu \frac{\partial}{\partial y_0} = \nabla_L f_2(p)$. Hence T^2 is differentiable at p and $\nabla_L(T^2)(p) = -2(p-r)$. Thus T is differentiable at p and

$$\nabla_L(T)(p) = -\frac{1}{T(p)}(p - r(p))$$

Finally we will show that T is concave. Let us set $\varphi(p) = -T(p)^2 = \langle p - r(p), p - r(p)\rangle$, then we have to prove that

$$-\varphi(tp + (1-t)q) \geq \left(t\sqrt{-\varphi(p)} + (1-t)\sqrt{-\varphi(p)}\right)^2$$

for all $p, q \in A, t \in [0,1]$. Since $r_t := tr(p) + (1-t)r(q) \in \overline{A}$ we have by definition of r that

$$\begin{aligned}
-\varphi(tp + (1-t)q) \geq& -\langle (tp + (1-t)q) - r_t, (tp + (1-t)q) - r_t\rangle \\
=& -\langle t(p - r(p)) + (1-t)(q - r(q)), t(p - r(p)) + (1-t)(q - r(q))\rangle \\
=& -(t^2\varphi(p) + (1-t)^2\varphi(q) + 2t(1-t)\langle p - r(p), q - r(q)\rangle)
\end{aligned}$$

Since $p-r(p)$ and $q-r(q)$ are future directed timelike vectors, we have, by the reverse Cauchy-Schwartz inequality 1.1.8, $\langle p - r(p), q - r(q)\rangle \leq -\sqrt{\varphi(p)\varphi(q)}$, so

$$\begin{aligned}
-\varphi(tp + (1-t)q) \geq& (t^2(-\varphi(p)) + (1-t)^2(-\varphi(q)) + 2t(1-t)\sqrt{\varphi(p)\varphi(q)}) \\
=& \left(t\sqrt{-\varphi(p)} + (1-t)\sqrt{-\varphi(q)}\right)^2.
\end{aligned}$$

□

Corollary 2.2.9. *Let A be a future complete convex subset of $\mathbb{M}^{n+1}$ that admits a spacelike support plane then the cosmological time is regular, i.e it goes to 0 on every inextendible past directed causal curve.*

Proof. We want to prove that for all $\{p_k\}_{k\in\mathbb{N}} \subseteq A$ such that $p_k \to p \in \partial A$ then $\lim_{k\to\infty} T(p_k) = 0$. This implies that if $\gamma : (a, \infty) \to A$ is a inextendible past directed causal curve, then $\lim_{t\to a} \gamma(t) \in \partial A$ and hence $\lim_{t\to a} T(\gamma(t)) = 0$. Fix $q \in I^+(p)$, then, as we have argued in Theorem 2.2.8 $p_k \in J^-(q)$ for all $k \gg 0$. Hence $r_k = r(p_k) \in J^-(q) \cap S$ for $k \gg 0$, where $S = \partial A$. Since $J^-(q) \cap S$ is compact we can deduce that $\{r_k\}$ is bounded, so up to passing to a subsequence we can conclude that $r_k \to r$. Since $p_k - r_k$ is a timelike vector, the limit $p - r$ is non-spacelike. On the other hand S is an achronal set, so $p - r$ is null. Since $T(p_k)^2 = -\langle p_k - r_k, p_k - r_k\rangle$ we have that $\lim_{k\to\infty} T^2(p_k) = -\langle p - r, p - r\rangle = 0$. □

Proposition 2.2.10. *With the notation as in Theorem 2.2.8, set $\widetilde{S}_a = T^{-1}(a)$ for $a > 0$. Then $\widetilde{S}_a$ is a future convex spacelike hypersurface and $T_p\widetilde{S}_a = (p - r(p))^\perp$ for all $p \in \widetilde{S}_a$. Furthermore $I^+(\widetilde{S}_a) = \bigcup_{b>a} \widetilde{S}_b$, and let $r_a : I^+(\widetilde{S}_a) \to \widetilde{S}_a$ be the retraction and $T_a : I^+(\widetilde{S}_a) \to \mathbb{R}_+$ be the cosmological time, then we have*

$$r_a(p) = \widetilde{S}_a \cap [p, r(p)]$$

$$T_a(p) = T(p) - a.$$

Proof. Notice that $\widetilde{S}_a$ is a future convex hypersurface since T is a concave function, indeed take $p, q \in I^+(\widetilde{S}_a)$ then both $T(p)$ and $T(q)$ are greater than a hence $T(t(p)+(1-t)q) \geq tT(p)+(1-t)T(q) > a$ so $tT(p)+(1-t)T(q) \in I^+(\widetilde{S}_a)$. Furthermore since $T_p\widetilde{S}_a = \nabla_L T(p)^\perp = (p-r(p))^\perp$, and $p-r(p)$ is timelike, $\widetilde{S}_a$ is a spacelike hypersurface. Notice that $I^+(\widetilde{S}_a)$ is a future complete convex subset of $\mathbb{M}^{n+1}$ that admits a spacelike support plane, namely $T_p\widetilde{S}_a$ with $p \in \widetilde{S}_a$, hence Theorem 2.2.8 applies and we can define for $I^+(\widetilde{S}_a)$ the retraction r_a and the cosmological time T_a that is regular. Obviously $r_a(p) = \widetilde{S}_a \cap [p, r(p)]$ since $r_a(p) \in \widetilde{S}_a$ is the point such that $r_a(p) + (p - r_a(p))^\perp$ is a support plane for $I^+(\widetilde{S}_a)$ hence $p - r_a(p)$ is a scalar multiple of $p - r(p)$ and then by definition $r(p) = r(r_a(p))$. Finally since $p, r_a(p)$ and $r(p)$ are collinear $T(p) = T_a(p) + a$. □

Let A be a future complete convex domain of $\mathbb{M}^{n+1}$ that admits a spacelike support plane, we have seen that it is provided with a map $r : A \to \partial A$ called the *retraction*, we denote its image by $\Sigma_A = r(A)$ and we call it the *singularity in the past*. The following is a useful characterization of points in the singularity in the past.

Corollary 2.2.11. *Let A be a future complete convex domain which has a spacelike support plane. Then $r_0 \in \Sigma_A$ if and only if there exists a timelike vector v such that the plane $r_0 + v^\perp$ is a spacelike support plane for A. Moreover*

$$r^{-1}(r_0) = \{r_0 + v \mid r_0 + v^\perp \textit{is a support plane for } A\}.$$

Proof. By Theorem 2.2.8 if $r_0 = r(p)$ then $r_0 + (p - r(p))^\perp$ is a spacelike support plane for A. Conversely if $r_0 + v^\perp$ is a support plane for A, then $p_\lambda = r_0 + \lambda v \in A$ for $\lambda > 0$ and then $r(r_0 + \lambda v) = r_0$, again by Theorem 2.2.8. □

Remark 2.2.12. The map $r : A \to \Sigma_A$ continuously extends to a retraction $r : A \cup \Sigma_A \to \Sigma_A$. This map is a deformation retraction where $r_t(p) = t(p - r(p)) + r(p)$ gives the homotopy between $i \circ r$ and $id_{A \cup \Sigma_A}$, where $i : \Sigma_A \to A \cup \Sigma_A$. Since $A \cup \Sigma_A$ is convex hence contractible so is Σ_A.

There is another map defined on A that will be useful for the study of future complete regular domains.

Definition 2.2.13. Define the *normal field* on A to be the map

$$\begin{aligned} N : A &\longrightarrow \mathbb{H}^n \\ p &\to \frac{p - r(p)}{T(p)}. \end{aligned}$$

Notice that it coincides, up to sign with the Lorentzian gradient of T, hence if $\widetilde{S}_a = T^{-1}(a)$ then $N_{|\widetilde{S}_a}$ is the normal field on $\widetilde{S}_a$.

Notice that the following identity holds:

$$p = r(p) + T(p)N(p) \qquad \text{for all } p \in A.$$

We call $r(p)$ the *singularity part of* p and $T(p)N(p)$ the *hyperbolic part.*

Here are inequalities that come from the fact that $r(p) + N(p)^\perp$ with $p \in A$ is a support plane for A.

Corollary 2.2.14. *With the above notation we have that*

$$\langle N(p), r(q) - r(p)\rangle \leq 0,$$

$$\langle q, p - r(p)\rangle < \langle r(p), p - r(p)\rangle\,,$$
$$\langle T(p)N(p) - T(q)N(q), r(p) - r(q)\rangle \geq 0$$

for all $p, q \in A$.

Proof. Notice that for all $p \in A$, $r(p) + N(p)^{\perp}$ is a support plane for A, hence, for all $q \in \overline{A}$ we have $\langle q, N(p)\rangle \leq \langle r(p), N(p)\rangle$, hence the first inequality follows. Writing $N(p) = (p - r(p))(T(p))^{-1}$ we get $\langle q, p - r(p)\rangle \leq \langle r(p), p - r(p)\rangle$. The equality holds if and only if q belongs to the plane $r(p) + N(p)^{\perp}$ hence only if it would belong to the boundary of A, but this is not the case, being A open, so we get a strict inequality.
Instead, again from $\langle q, N(p)\rangle \leq \langle r(p), N(p)\rangle$ we get $\langle r(q), p - r(p)\rangle \leq \langle r(p), p - r(p)\rangle$ hence $\langle p - r(p), r(p) - r(q)\rangle \geq 0$ hence $\langle (p - r(p)) - (q - r(q)), r(p) - r(q)\rangle \geq 0$. This proves the third inequality. □

Now le us go back to Ω future complete regular domain. Theorem 2.2.8 applies in this situation and let us set

- $T : \Omega \to \mathbb{R}_+$ the cosmological time on Ω and $\widetilde{S}_a = T^{-1}(a)$;
- $r : \Omega \to \partial\Omega$ the retraction onto the singularity in the past $\Sigma = r(\Omega)$;
- $N : \Omega \to \mathbb{H}^n$ the normal field.

Lemma 2.2.15. *The hypersurface $\widetilde{S}_a$ is a Cauchy surface for Ω, moreover Ω is the domain of dependence of $\widetilde{S}_a$.*

Proof. Notice that, being $\widetilde{S}_a$ a spacelike hypersurface, it is an acausal subset of $\mathbb{M}^{n+1}$, hence it is enough to show that $D(\widetilde{S}_a) = \Omega$ in order to conclude that $\widetilde{S}_a$ is a Cauchy surface for Ω.
Let $p \in D(\widetilde{S}_a)$ then every inextendible causal curve passing through p intersects $\widetilde{S}_a$. Consider a past directed inextendible timelike curve through p, it will intersect $\widetilde{S}_a \subseteq \Omega$ in a point q. But Ω is a future complete regular domain hence if $q \in \Omega$ and p is in the future of q then $p \in \Omega$. So, we have $D(\widetilde{S}_a) \subseteq \Omega$. Now let $p \in \Omega$ and v be a future directed non-spacelike vector. By definition $T(p + \lambda v)^2 \geq -\langle p + \lambda v - r(p), p + \lambda v - r(p)\rangle$, so there exists $\lambda > 0$ such that $T(p + \lambda v) > a$. On the other hand there exists $\mu < 0$ such that $p + \mu v \in \partial\Omega$, then $\lim_{t\to\mu} T(p + tv) = 0$, since by Corollary 2.2.9, T goes to 0 on every inextendible past directed causal curve. So there exists $\lambda' \in \mathbb{R}$ such that $T(p + \lambda' v) = a$, thus $\Omega \subseteq D(\widetilde{S}_a)$. □

Lemma 2.2.16. *If $c : [0, 1] \to \widetilde{S}_a$ is a Lipschitz path then the paths $N(t) = N(c(t))$ and $r(t) = r(c(t))$ are differentiable almost everywhere and we have that $\dot{N}(t)$ and $\dot{r}(t)$ lie in $T_{c(t)}\widetilde{S}_a$, so they are spacelike. Moreover $\left\langle \dot{N}(t), \dot{r}(t)\right\rangle \geq 0$ almost everywhere.*

Proof. In order to prove the first claim it is sufficient to prove that $N : \widetilde{S}_a \to \mathbb{H}^n$ and $r : \widetilde{S}_a \to \Sigma$ are locally Lipschitz with respect to the Euclidean distance d_E on $\mathbb{M}^{n+1}$. Furthermore since for $p \in \widetilde{S}_a$ we have the decomposition $p = r(p) + aN(p)$, it is sufficient to prove the claim for N. Fix a compact subset $K \subseteq \widetilde{S}_a$ and let $H = N(K) \subseteq \mathbb{H}^n$. Since H is compact there exists a constant $C \in \mathbb{R}$ such that $d_E(x, y) = \|x - y\| \leq C\sqrt{\langle x - y, x - y\rangle}$ for all $x, y \in H$. On the other hand we have for $p, q \in \widetilde{S}_a$

$$\langle p - q, p - q\rangle = a^2 \langle N(p) - N(q), N(p) - N(q)\rangle +$$

$$+2a \langle N(p) - N(q), r(p) - r(q)\rangle + \langle r(p) - r(q), r(p) - r(q)\rangle$$

and then by Corollary 2.2.14 we have that $\langle N(p) - N(q), r(p) - r(q)\rangle \geq 0$ and furthermore being $\Sigma \subseteq \partial\Omega$ achronal, since Ω is a future complete regular domain, no two points on it are chronologically related hence $\langle r(p) - r(q), r(p) - r(q)\rangle \geq 0$ and we get

$$\sqrt{\langle N(p) - N(q), N(p) - N(q)\rangle} \leq \frac{1}{a}\sqrt{\langle p - q, p - q\rangle}$$

for all $p, q \in \widetilde{S}_a$. Since $\langle p-q, p-q\rangle \leq \|p-q\|^2$ we can deduce that $\|N(p)-N(q)\| \leq \frac{C}{a}\|p-q\|$ for all $p, q \in K$. Finally notice that $N(t)$ is a path in $\mathbb{H}^n$ so $\dot{N}(t) \in T_{N(t)}\mathbb{H}^n = T_{c(t)}\widetilde{S}_a$. Where the last equality is because of Proposition 2.2.10. Since $\dot{c}(t) = \dot{r}(t) + a\dot{N}(t)$ then $\dot{r}(t) \in T_{c(t)}\widetilde{S}_a$ almost everywhere. Finally again by Corollary 2.2.14 we have that

$$\langle N(t+h) - N(t), r(t+h) - r(t)\rangle \geq 0$$

thus $\left\langle \dot{N}(t), \dot{r}(t) \right\rangle \geq 0$. □

Lemma 2.2.17. *Let Ω a future complete regular domain, then the level surfaces $\widetilde{S}_a = T^{-1}(a)$ of the cosmological time defined on Ω are complete.*

Proof. Since $\widetilde{S}_a$ is Riemannian all completness notions are equivalent. Let $t \mapsto c(t)$ be an incomplete geodesic in $\widetilde{S}_a$ defined on $[0, t_\infty)$ parametrized by unit length. Since from Lemma 2.2.16 N is $\frac{1}{a}$-Lipschitz on $\widetilde{S}_a$ it follows that the path $N(c(t))$ has finite length in $\mathbb{H}^n$ hence there exists $v \in \mathbb{H}^n$ limit point of $N(c(t))$. Let us fix $(y_0, \ldots, y_n)$ orthonormal affine coordinates such that $v = (1, 0, \ldots, 0)$. Since also the orthogonal projection of the geodesic c to the plane $\{y_0 = 0\}$ has finite length for the usual Euclidean metric it follows that the projection has a limit point c_∞ in $\{y_0 = 0\}$. Since $\partial\Omega$ is the graph of a convex function defined over $\{y_0 = 0\}$ the vertical line above c_∞ intersects Ω. Since $\widetilde{S}_a$ is a Cauchy surface for Ω the vertical line starting in c_∞ must intersect $\widetilde{S}_a$ in a unique point p_∞. Then the geodesic can be completed on $[0, t_\infty]$ by $c(t_\infty) = p_\infty$. □

Remark 2.2.18. When Ω is a Γ_τ-invariant future complete regular domain then T is a Γ_τ-invariant function, hence $\widetilde{S}_a$ are Γ_τ-invariant future convex spacelike hypersurfaces. Moreover r and N are Γ_τ-equivariant in the sense :

$$r \circ \gamma_\tau = \gamma_\tau \circ r \quad \text{and}$$

$$N \circ \gamma_\tau = \gamma \circ N.$$

Thus Σ is also a Γ_τ-invariant subset of $\partial\Omega$. From Lemma 2.2.15 it follows that $\widetilde{S}_a/\Gamma_\tau$ are Cauchy surfaces for Ω/Γ_τ. In particular when $\Omega = \mathcal{D}_\tau = D(\widetilde{F}_\tau)$ we have that $\widetilde{S}_a/\Gamma_\tau$ is C^1-diffeomorphic to $\widetilde{F}_\tau/\Gamma_\tau \cong M$, since two Cauchy surfaces are always diffeomorphic, see Remark 1.2.33.

Remark 2.2.19. We have seen that given a Γ_τ-invariant future convex complete spacelike hypersurface its domain of dependence is a Γ_τ-invariant future complete regular domain. On the other hand given a Γ_τ-invariant future complete regular domain Ω the level surfaces of the cosmological time are Γ_τ-invariant future convex complete spacelike hypersurfaces and Ω is the domain of dependence of them. So we have that Γ_τ-invariant future complete regular domains are domains of dependence of Γ_τ-invariant future convex complete spacelike hypersurfaces.

Now for Ω future complete regular convex domain we can give a another characterization of the points in the singularity in the past. First we need two lemmas.

Lemma 2.2.20. *Let Ω be a future complete regular convex domain then for all $p \in \partial\Omega$ there exists a null vector v such that $p + \mathbb{R}_+ v \subseteq \partial\Omega$. Furthermore*

$$\Omega = \bigcap \{I^+(p + v^\perp) \mid p \in \partial\Omega \text{ and } v \text{ is a null vector such that } p + \mathbb{R}_+ v \subseteq \partial\Omega\}.$$

Proof. Fix $p \in \partial\Omega$ then since Ω is a future complete regular convex domain it is the intersection of the future of its null support planes, therefore there exists a null vector v such that $p + v^\perp$ is a support plane for Ω. From $I^+(p + \mathbb{R}_+ v) \subseteq I^+(p) \subseteq \Omega$ and from the characterization of the boundary of a future set we have $p + \mathbb{R}_+ v \subseteq \partial\Omega$. Conversely suppose that the ray $R = p + \mathbb{R}_+ v$ is contained in $\partial\Omega$, then from Han-Banach Theorem [9, Théorème 11.4.1] there exists an hyperplane P such that Ω and R are contained in the opposite closed half-spaces bounded by P. Since then P is a support hyperplane for Ω it is not timelike, and also since R is contained in $\partial\Omega$ then it is contained in P. It follows that P is parallel to v so that $P = p + v^\perp$. $\square$

Lemma 2.2.21. *Let V be a finite dimensional vector space and $G \subseteq V^*$ a subset of the dual space. Consider the convex set $K = \{v \in V \mid g(v) \leq C_g \ \forall\ g \in G\}$. Suppose the following properties hold:*

1. *if $g \in G$ and $\lambda > 0$ then $\lambda g \in G$ and $C_{\lambda g} = \lambda C_g$,*
2. *if $g_n \to g$ and $C_{g_n} \to C$ then $g \in G$ and $C_g \leq C$,*

then for all $v \in \partial K$ the set $G_v = \{g \in G \mid g(v) = C_g\}$ is not empty. Moreover the plane $v + P$ is a support hyperplane for K in v if and only if $P = \ker h$ with h in the convex hull of G_v.

Proof. If $v \in \partial K$, let $B_{\frac{1}{n}}(v)$ a ball of radius $\frac{1}{n}$ around v, then $\forall\ n \in \mathbb{N}$ there exists $w_n \in B_{\frac{1}{n}}(v)$ such that $w_n \notin K$ and there exists $w'_n \in B_{\frac{1}{n}}(v)$ that belongs to K, hence there exists $g_n \in G$ such that $g_n(w'_n) \leq C_{g_n}$ and $g_n(w_n) > C_{g_n}$. This implies that there exists $x_n \in B_{\frac{1}{n}}(v)$ such that $g_n(x_n) = C_{g_n}$. By property 1 after rescaling g_n we may assume the g_n lie in a compact. Hence we may extract a convergent subsequence $g_n \to g$, then $C_{g_n} = g_n(x_n) \to g(v)$ hence $g \in G$ by Property 2 and $C_g \leq g(v)$, on the other hand if $g \in G$ we have $g(v) \leq C_g$ hence $g(v) = C_g$ and $g \in G_v$. So G_v is not empty. For the second part of the statement obviously if $P = \ker h$ with h in the convex hull of G_v then $v + P$ is a support hyperplane for K at v. Conversely let $v + P$ be a support plane for K at v. Notice that the dual of K, $K^* = \{h \in V^* \mid h(v) \leq C_h\ \forall v \in K\}$ is a closed convex and bounded set (actually it is bounded in the projectivization of V^* i.e. after rescaling the h to have unit norm), hence it is the convex hull of its extreme points. Notice furthermore that the elements in G_v are the extreme points of K^* [9, Théorème 11.6.8], hence if $v + \ker h$ is a support hyperplane for K then $h \in K^*$ and it is in the convex hull of G_v. $\square$

Proposition 2.2.22. *Let Ω be a future complete regular domain, then a point $p \in \partial\Omega$ lies in Σ if and only if there are at least two future directed null rays contained in $\partial\Omega$ starting from p. Moreover if $p \in \Sigma$ then $r^{-1}(p)$ is the intersection of Ω with the convex hull of the null rays contained in $\partial\Omega$ and starting from p.*

Proof. From Corollary 2.2.11 a point $p \in \partial\Omega$ is in the singularity in the past if and only if there exists a spacelike support plane for Ω at p. Since Ω is a future complete regular convex domain there always exists a null support plane for Ω at p, this implies that $p \in \partial\Omega$ admits a spacelike support plane if and only if there are at least two null support planes at p. Furthermore if v_1, v_2 are two future directed null vectors such that $p + v_1^\perp$ and $p + v_2^\perp$ are support planes for Ω then Lemma 2.2.20 implies that $p + \mathbb{R}_+ v_1$ and $p + \mathbb{R}_+ v_2$ are future directed null rays contained in $\partial\Omega$. Hence the first part of the proposition follows. Now let L be the family of null future directed vectors that are orthogonal to some null support planes for Ω. If $v \in L$ and $C_v = \sup_{r \in \Omega} \langle v, r \rangle$ then from Lemma 2.2.20 we have that $\Omega = \{x \in \mathbb{M}^{n+1} \mid \langle x, v \rangle \leq C_v\ \forall\ v \in L\}$. Let $p \in \Sigma$ and let $L(p)$ be the set of null future directed vectors v at p such that $p + v^\perp$ is a support plane for Ω. From Corollary 2.2.11 we know that $r^{-1}(p) = \{p + v \mid p + v^\perp$ is a support plane for $\Omega\}$. From Lemma 2.2.21 if v is a future directed non-spacelike vector such that $p + v^\perp$ is a support hyperplane for Ω in p then v belongs to the convex hull of $L(p)$. Then we get that $r^{-1}(p)$ is the intersection of Ω with the convex hull of $p + L(p)$. Finally notice that $v \in L(p)$ if and only if $p + \mathbb{R}_+ v$ is a null ray contained in $\partial\Omega$. $\square$

Using the characterization of Proposition 2.2.22 for the points belonging to the singularity in the past we are going to associate to each such point an ideal convex set (a convex set that is the convex hull of its boundary points) in $\mathbb{H}^n$. This will be used in the last chapter where we will talk about geodesic stratifications of $\mathbb{H}^n$.

Definition 2.2.23. For $p \in \Sigma$ let us define a subset of $\mathbb{H}^n$

$$\mathcal{F}(p) = N(r^{-1}(p)).$$

Corollary 2.2.24. *Let $p \in \Sigma$ and let $L(p)$ be the set of future directed null vectors at p such that $p + v^\perp$ is a support plane for Ω at p. Denote by $\hat{L}(p) = \{[v] \in \partial\mathbb{H}^n \mid v \in L(p)\}$. Then $\mathcal{F}(p)$ is the convex hull in $\mathbb{H}^n$ of $\hat{L}(p)$*

Definition 2.2.25. Given two convex sets C, C' in $\mathbb{H}^n$ we say that an hyperplane P *separates* C from C' if C and C' are contained in the opposite half-spaces bounded by P.

Proposition 2.2.26. *Let Ω be a future complete regular domain. For every $p, q \in \Sigma$ the plane $(p-q)^\perp$ separates $\mathcal{F}(p)$ from $\mathcal{F}(q)$. The segment $[p, q]$ is contained in Σ if and only if $\mathcal{F}(p) \cap \mathcal{F}(q) \neq \emptyset$. In this case for all $r \in (p, q)$ we have*

$$\mathcal{F}(r) = \mathcal{F}(p) \cap (p-q)^\perp = \mathcal{F}(q) \cap (p-q)^\perp = \mathcal{F}(p) \cap \mathcal{F}(q).$$

Proof. From the inequalities in Corollary 2.2.14 we have that $\langle tv, p-q \rangle \leq \langle sw, p-q \rangle$ for all $t, s \in \mathbb{R}_+$ and $v \in \mathcal{F}(p)$, $w \in \mathcal{F}(q)$. This implies that $\langle v, p-q \rangle \leq 0$ and $\langle w, p-q \rangle \geq 0$. This shows that $(p-q)^\perp$ separates $\mathcal{F}(p)$ from $\mathcal{F}(q)$. Suppose now that $\mathcal{F}(p) \cap \mathcal{F}(q) \neq \emptyset$, then $\mathcal{F}(p) \cap \mathcal{F}(q) \subset (p-q)^\perp$. Let $v \in \mathcal{F}(p) \cap \mathcal{F}(q)$ and let P_v be the unique support hyperplane orthogonal to v which intersects $\partial\Omega$. Then P_v passes through p and q, but since P_v is a support plane for Ω this implies $[p, q] \subseteq \partial\Omega$. Since P_v is a spacelike plane which passes through all $r \in (p, q)$ by the characterization of the points in the singularity in the past of Corollary 2.2.11 we have that $[p, q] \subseteq \Sigma$ and $\mathcal{F}(p) \cap \mathcal{F}(q) \subseteq \mathcal{F}(r)$. Conversely suppose $[p, q] \subseteq \Sigma$, take $r \in (p, q)$ and $v \in \mathcal{F}(r)$ then since $(p-r)^\perp$ separates $\mathcal{F}(p)$ from $\mathcal{F}(r)$ we have $\langle v, p-r \rangle \leq 0$ and similarly $\langle v, r-q \rangle \geq 0$. But since $p-r$ and $r-q$ have the same direction we have $\langle v, p-r \rangle = 0$ and $\langle v, r-q \rangle = 0$, hence $v \in \mathcal{F}(p) \cap \mathcal{F}(q)$. In order to conclude the proof we need to show that $\mathcal{F}(r) \supseteq \mathcal{F}(p) \cap (p-q)^\perp$. We know that $\mathcal{F}(p) \cap (p-q)^\perp$ is the convex hull of $\hat{L}(p) \cap (p-q)^\perp$, thus it is sufficient to show that $L(r) \supseteq L(p) \cap (p-q)^\perp$. Fix $v \in L(p) \cap (p-q)^\perp$ and consider the plane $P = p + v^\perp$, then the intersection of this plane with $\overline{\Omega}$ includes the ray $p + \mathbb{R}_+ v$ and hence the segment $[p, q]$. Since this intersection is convex we have that the ray $r + \mathbb{R}_+ v$ is contained in $P \cap \overline{\Omega}$ and thus $v \in L(r)$ from Lemma 2.2.20. □

Here is a nice property of regular domains with surjective normal field.

Lemma 2.2.27. *Let Ω be a future complete regular domain of $\mathbb{M}^{n+1}$ such that the normal field $N : \Omega \to \mathbb{H}^n$ is surjective, then the restriction of N to the level surfaces $\widetilde{S}_a$ is a proper map.*

Proof. Suppose by contradiction that $\{p_n\}_{n\in\mathbb{N}}$ is a divergent sequence in $\widetilde{S}_a$ such that $N(p_n) \to x \in \mathbb{H}^n$. Since N is surjective there exists $p'_\infty \in \Omega$ such that $N(p'_\infty) = x$. But, in fact, taking $p_\infty = p'_\infty - (T(p'_\infty) - a)x$ we have that there exists $p_\infty \in \widetilde{S}_a$ such that $N(p_\infty) = x$. Consider the sequence of segments $R_n = [p_\infty, p_n]$. Since the sequence $\{R_n/\|R_n\|\}$ is bounded, up to passing to a subsequence it converges to some direction w, and, since the sequence $\{p_n\}_n$ diverges, the sequence of segments R_n converges, up to passing to a subsequence, to a ray $R = p_\infty + \mathbb{R}_{\geq 0} w$. Now since the planes $p_n + N(p_n)^\perp$ and $p_\infty + x_\infty^\perp$ are supporting planes of $I^+(\widetilde{S}_a)$ (see Proposition 2.2.10) the Euclidean angle between R_n and $p_\infty + x_\infty^\perp$ is less than $\pi/2$ and we can see that the Euclidean angle between R_n and $p_\infty + x_\infty^\perp$ is less than the Euclidean angle between $p_\infty + x_\infty^\perp$

and $p_n + N(p_n)^\perp$. But since $N(p_n)^\perp \to x_\infty^\perp$ we can deduce that R is contained in the plane $p_\infty + x_\infty^\perp$, hence $R \subseteq N^{-1}(x_\infty)$ and $\langle w, x_\infty\rangle = 0$. Since $p_\infty + x_\infty^\perp$ is a supporting plane of $I^+(\widetilde{S}_a)$ and since $R \subseteq \overline{I^+(\widetilde{S}_a)}$ we have that $R \subseteq \widetilde{S}_a = \partial I^+(\widetilde{S}_a)$, hence the direction of R is spacelike. Now take $y \in \mathbb{H}^n$ such that $\langle w, y\rangle > 0$ and $q \in \widetilde{S}_a$ such that $N(q) = y$. This implies that $q + y^\perp$ is a support plane for $I^+(\widetilde{S}_a)$, hence for all $r \in I^+(\widetilde{S}_a)$ we have $\langle r, y\rangle \le C$ for some constant $C > 0$, but $\langle p_\infty + tw, y\rangle \to \infty$ as $t \to \infty$, hence we get a contradiction. □

Remark 2.2.28. We remark that the regular domain $\mathcal{D}_\tau$ has surjective normal field by Lemma 2.1.17. And, in fact, N is surjective when restricted to each level surface $\widetilde{S}_a$.

Now as a consequence of the study of future complete regular domains we get that the action of the affine deformation Γ_τ of Γ on $\partial\mathcal{D}_\tau$ is not free and properly discontinuous; in particular it is not free and properly discontinuous on the whole $\mathbb{M}^{n+1}$. First we need a lemma.

Lemma 2.2.29. *Let Ω be a future complete regular domain. Suppose Σ is closed in $\partial\Omega$, then the retraction $r : \Omega \to \Sigma$ extends to a deformation retraction $\bar r : \overline{\Omega} \to \Sigma$. Furthermore every point of $\partial\Omega \setminus \Sigma$ belongs to a unique null ray contained in $\partial\Omega$ with starting point in Σ.*

Proof. Consider a sequence of elements $\{x_n\}_{n\in\mathbb{N}} \subseteq \Omega$ such that $x_n \to p \in \partial\Omega \setminus \Sigma$. Then the sequence $\{r(x_n)\}_n \subseteq \Sigma$ converges to some r that belongs to Σ since Σ is closed in $\partial\Omega$. Define $\bar r(p) := r$. By definition this is a continuous extension of $r : \Omega \to \Sigma$. The sequence of timelike vectors $x_n - r(x_n)$ converges to $p - r$. Hence $p - r$ is either a null or timelike vector, but, since $\partial\Omega$ is an achronal set, $p - r$ is a null vector. Furthermore since the planes $r(x_n) + (x_n - r(x_n))^\perp$ are supporting planes of Ω so is $r + (p - r)^\perp$. This implies that $r + \mathbb{R}_{\ge 0}(p - r)$ is a null ray contained in $\partial\Omega$. □

In the setting of the previous lemma, in order to prove next proposition, we have to explain, when Σ is a closed subset of $\partial\Omega$, how to construct the boundary of $X := \partial\Omega$ and how to give to $\overline{X} := X \cup \partial X$ the structure of manifold with boundary. From Lemma 2.1.16 we know that X is the graph of a convex Lipschitz function φ. From the previous lemma it follows that at all points in $X \setminus \Sigma$ there is exactly one supporting hyperplane for Ω, this implies that the points in $X \setminus \Sigma$ correspond to the set of points where φ is differentiable, hence $X \setminus \Sigma$ is a C^1-manifold. Also from the previous lemma $X \setminus \Sigma$ is foliated by null rays with starting points in Σ. For $p \in X \setminus \Sigma$ let $R(p)$ be the null ray of the foliation which passes through p. As we saw the retraction on X is defined as follows $r(p) = p$ if $p \in \Sigma$ and $r(p)$ is the initial point of $R(p)$ if $p \in X \setminus \Sigma$. We hence define

$$\partial X := \{R \mid R \text{ is a ray of the foliation }\}$$

Let us define a topology on $\overline{X} = X \cup \partial X$ such that it agrees with the natural topology on X and such that it makes $\overline{X}$ a topological manifold with boundary ∂X. Let $R \in \partial X$ and fix a C^1-embedded closed $(n-1)$-ball D which intersects the foliation transversely and passes through R and define

$$U(D, R) = \{p \in X \setminus \Sigma \mid R(p) \cap \operatorname{int}(D) \neq \emptyset \text{ separates } p \text{ from } r(p)\} \cup$$

$$\cup \{S \in \partial X \mid S \cap \operatorname{int} D \neq \emptyset\}$$

We consider the topology on $\overline{X}$ that agrees with the natural topology on X and such that for every $R \in \partial X$ has $U(D, R)$ as fundamental system of neighborhoods. With this topology $\overline{X}$ is a Hausdorff space. Indeed if $p \in X$ and $R = R(p) \in \partial X$ it is sufficient to take D such that p and $r(p)$ are on the same side of D so that $p \notin U(D, R)$ and we can separate p and R.
Now we want to construct an atlas for $\overline{X}$. For $p \in X \setminus \Sigma$ let $v(p)$ be a future directed null

vector tangent to $R(p)$ such that $y_0(v(p)) = 1$, where y_0 is the timelike coordinate of an affine orthonormal coordinate system of $\mathbb{M}^{n+1}$. For all D closed $(n-1)$-ball as above define the map $\mu_D : D \times (0, \infty] \to U(D, R)$ defined as follows

$$\mu_D(x,t) = \begin{cases} x + tv(x) & \text{if } t < \infty \\ R(x) & \text{if } t = \infty \end{cases}$$

These maps are local charts that make $\overline{X}$ a manifold with boundary. Now the retraction $r : X \to \Sigma$ uniquely extends to a retraction $r : \overline{X} \to \Sigma$, where if $R \in \partial X$ then $r(R)$ is the starting point of the ray. This retraction is a proper map. In fact suppose we have a divergent sequence $\{p_n\}_n \subseteq X \setminus \Sigma$ and suppose that $r(p_n) \to r(p) \in \Sigma$, for some $p \in \Omega$. Since each $r(p_n)$ is the starting point of a null ray of the foliation $r(p)$ will be as well the starting point of a null ray of the foliation, call it R. Then since $r(p_n)$ tends to $r(p)$ and $\{p_n\}$ diverges, for every embedded $(n-1)$-ball D that passes through R we have $p_n \in U(D, R)$ for infinitely many p_n. But then $\mu_D^{-1}(p_n) \to (R \cap D, \infty)$. Hence $p_n \to R \in \overline{X}$.

Proposition 2.2.30. *The action of Γ_τ on $\partial\mathcal{D}_\tau$ is not free and properly discontinuous. Hence the affine group Γ_τ does not act freely and properly discontinuously on the whole $\mathbb{M}^{n+1}$.*

Proof. Suppose by contradiction that Γ_τ acts on $\partial\mathcal{D}_\tau$ freely and properly discontinuously. Set

$$X = \partial\mathcal{D}_\tau, \;\; M' = X/\Gamma_\tau, \;\; K = \Sigma/\Gamma_\tau$$

and $\hat{r} : \mathcal{D}_\tau/\Gamma_\tau \to K$ be the surjective map which is induced by the retraction $r : \mathcal{D}_\tau \to \Sigma$. Notice that if $p \in \mathcal{D}_\tau$ and $r(p) \in \Sigma$ then $r(p) + N(p) \in \widetilde{S}_1$ and $r(r(p) + N(p)) = r(p)$, hence $\Sigma = r(\widetilde{S}_1)$. Since $\widetilde{S}_1/\Gamma_\tau$ is compact, being homeomorphic to M we have that K is compact. Since M' is an Hausdorff space K is closed in M' and hence Σ is closed in $\partial\mathcal{D}_\tau$. Thus we can construct the boundary ∂X of X as above. The action of Γ_τ on X uniquely extends to an action on $\overline{X}$. The map $r : \overline{X} \to \Sigma$ is Γ_τ-equivariant hence the action of Γ_τ on $\overline{X}$ is free and properly discontinuous. Indeed if $K \subseteq \overline{X}$ is a compact subset then $\Gamma_\tau(K) = \{\gamma_\tau \in \Gamma_\tau \mid \gamma_\tau K \cap K \neq \emptyset\}$ is contained in $\Gamma_\tau(r(K))$ which is finite since $r(K)$ is compact. Then we can construct the manifold with boundary $\overline{M}' = \overline{X}/\Gamma_\tau$ of which M' is the interior. Since the retraction $r : \overline{X} \to \Sigma$ is a Γ_τ-equivariant map and a proper map it induces a proper map $\bar{r} : \overline{M}' \to K$. Since K is compact $\overline{M}'$ is a compact manifold with boundary. Furthermore $\bar{r} : \overline{M}' \to K$ is a deformation retraction so

$$H_n(K) \cong H_n(\overline{M}').$$

Now by Lefschetz duality [19, Theorem 3.43.] we have that

$$H_n(\overline{M}') \cong H^0(\overline{M}', \partial\overline{M}') = 0.$$

On the other hand we have $Y_\tau = \mathcal{D}_\tau/\Gamma_\tau$ and $\overline{Y}_\tau = \overline{\mathcal{D}}_\tau/\Gamma_\tau$. The map $r : \overline{\mathcal{D}}_\tau \to \Sigma$ induces a deformation retraction $\overline{Y}_\tau \to K$, so

$$H_n(K) \cong H_n(\overline{Y}_\tau) \cong H_n(Y_\tau).$$

For the last isomorphism see [17, XI, Théorème 3.7.]. But we know that $Y_\tau \cong \mathbb{R} \times M$ hence $H_n(Y_\tau) \cong H_n(M) = \mathbb{Z}$. Hence a contradiction. □

As a consequence we get that the domain of dependence of any Γ_τ-invariant complete spacelike hypersurface, such that the action on it is free and properly discontinuous, is a future (or past) complete regular convex domain.

Corollary 2.2.31. *If $\widetilde{F}$ is a Γ_τ-invariant complete spacelike hypersurface such that the Γ_τ-action on it is free and properly discontinuous then $D(\widetilde{F})$ is a regular domain, either future or past complete.*

Proof. If $\widetilde{F}$ is a complete spacelike hypersurface such that the Γ_τ-action on it is free and properly discontinuous then from Proposition 2.1.13 the Γ_τ-action on $D(\widetilde{F})$ is as well free and properly discontinuous, hence $D(\widetilde{F})$ is not the whole $\mathbb{M}^{n+1}$. By Corollary 2.1.12 $D(\widetilde{F})$ is the intersection of either the future or the past of its null support planes. By lemma 2.1.17 we know that $D(\widetilde{F})$ admits a spacelike support plane, hence this condition ensures that $D(\widetilde{F})$ is the intersection of the future or the past of at least two null support planes, hence it is a regular domain. □

2.3 Uniqueness of the domain of dependence

In this section we want to show that $\mathcal{D}_\tau = D(\widetilde{F}_\tau)$ (and respectively $\mathcal{D}_\tau^- = D(\widetilde{F}_\tau^-)$), the regular domain we have associated to each $\tau \in Z^1(\Gamma, \mathbb{R}^{n+1})$, is the unique Γ_τ-invariant future (past) complete regular domain. This will allow us to deduce that any Γ_τ-invariant complete spacelike hypersurface is contained in either $\mathcal{D}_\tau$ or $\mathcal{D}_\tau^-$ and it is indeed a Cauchy surface of it. Finally we will show that $Y_\tau = \mathcal{D}_\tau/\Gamma_\tau$ and $Y_\tau^- = \mathcal{D}_\tau^-/\Gamma_\tau$ are the only maximal globally hyperbolic flat spacetimes with compact spacelike Cauchy surface and holonomy group Γ_τ.

Theorem 2.3.1. *$\mathcal{D}_\tau$ is the unique Γ_τ-invariant future complete regular domain.*

Proof. For the proof see [12, Theorem 5.1.]. We shall just briefly summarize the principal arguments. Given a Γ_τ-invariant future complete regular domain Ω, we want to show that $\Omega = \mathcal{D}_\tau$. Let T_Ω be the regular cosmological time associated to Ω and T the one associated to $\mathcal{D}_\tau$. Furthermore for $a > 0$ let $\widetilde{S}_a^\Omega = T_\Omega^{-1}(a)$ and $\widetilde{S}_a = T^{-1}(a)$ be the level surfaces of T_Ω and T respectively. Hence by Lemma 2.2.15 we have $\Omega = D(\widetilde{S}_a^\Omega)$ and $\mathcal{D}_\tau = D(\widetilde{S}_a)$. It is sufficient then to prove that $\widetilde{S}_a \subseteq \Omega$ and $\widetilde{S}_a^\Omega \subseteq \mathcal{D}_\tau$ for a sufficiently large. Because then this implies that $\mathcal{D}_\tau \subseteq \Omega$ since if $p \in \mathcal{D}_\tau$ take any inextendible causal curve passing through p, since $\mathcal{D}_\tau = D(\widetilde{S}_a)$ this curve will intersect $\widetilde{S}_a$ in a unique point q, but since $\widetilde{S}_a \subseteq \Omega$ for $a >> 0$ we have $q \in \Omega = D(\widetilde{S}_a^\Omega)$ hence this curve should also intersect $\widetilde{S}_a^\Omega$, this implies that $p \in \Omega$. Analogously $\Omega \subseteq \mathcal{D}_\tau$. Now the proof is declined in four steps.

1. The first step shows that $\Omega \cap \mathcal{D}_\tau \neq \emptyset$. This follows from the fact that they are both future complete and hence if $p \in \Omega$ and $q \in \mathcal{D}_\tau$ then $I^+(p) \cap I^+(q) \subseteq \Omega \cap \mathcal{D}_\tau$. Then notice that intersection of the future of two points in $\mathbb{M}^{n+1}$ is not empty.

2. In the second step let us fix a point $p_0 \in \Omega \cap \mathcal{D}_\tau$ and call C the closure of the convex hull of the Γ_τ orbit of p_0, then we claim that C is a future complete convex set.
From a general lemma about closed convex sets in $\mathbb{M}^{n+1}$ [12, Lemma 5.2.] this reduces to prove that the interior of C is not empty, C is not of the form $\{x \in \mathbb{M}^{n+1} \mid \alpha_1 \leq \langle x, v\rangle \leq \alpha_2\}$ for some v non-spacelike vector and that C has not a timelike support hyperplane. In order to say that the interior of C is not empty this is equivalent to say that the dimension of C is $n+1$ [9, Proposition 11.2.7], so supposing by contradiction that $\dim C = k < n+1$ we would get that Γ_τ leaves invariant the k-plane P that is the affine hull of C, hence Γ leaves invariant the tangent plane of P contradicting the fact that if Γ is cocompact then from Remark 1.3.33 we know that Γ does not leave invariant any hyperplane of $\mathbb{H}^n$. In order to exclude the other case what is used is that Γ is a cocompact subgroup of $SO^+(n,1)$ therefore all its elements are hyperbolic isometries of the hyperbolic space hence they have an attractor null eigenvector and a repulsive one and that, as already mentioned, the limit set of Γ is the whole $\partial\mathbb{H}^n$.

3. If we call $\Delta = \partial C$ then the third step shows that Δ/Γ_τ is compact. In order to do so we notice that being Δ the boundary of a Γ_τ-invariant convex set then it is Γ_τ-invariant as well and a topological manifold. Moreover if $T(p_0) = a_0$ then $\Delta \subseteq \overline{I^+(\widetilde{S}_{a_0})}$. If $r : \mathcal{D}_\tau \to \partial\mathcal{D}_\tau$ is the retraction map, then we can define a Γ_τ-equivariant map

$$f : \Delta \longrightarrow \widetilde{S}_{a_0}$$
$$p \to r(p) + \frac{a_0}{T(p)}(p - r(p))$$

that induces a map $\bar{f} : \Delta/\Gamma_\tau \to \widetilde{S}_{a_0}/\Gamma_\tau$ that turns out to be an homeomorphism, hence $\Delta/\Gamma_\tau \cong M$ is compact.

4. The final step shows that if $a > \sup_{q\in\Delta} T_\Omega(q) \vee \sup_{q\in\Delta} T(q)$ then $\widetilde{S}^\Omega_a \subseteq \mathcal{D}_\tau$ and $\widetilde{S}_a \subseteq \Omega$. For this we notice that $T : \Delta \to \mathbb{R}$ and $T_\Omega : \Delta \to \mathbb{R}$ are Γ_τ-equivariant map and since Δ/Γ_τ is compact there exists $a > 0$ such that $T(x) < a$ and $T_\Omega(x) < a$ for all $x \in \Delta$. Finally in order to conclude we have to show that $\widetilde{S}_a$ and $\widetilde{S}^\Omega_a$ are contained in C. Let $y \in \mathcal{D}_\tau$ and suppose $y \notin C$ and $y \in I^-(\Delta)$ then there exists $y' \in \Delta \cap I^+(y)$, so we have $T(y) < T(y') < a$. It follows that $\widetilde{S}_a \subseteq C$. Analogously $\widetilde{S}^\Omega_a \subseteq C$. Hence $\widetilde{S}_a \subseteq \Omega$ and $\widetilde{S}^\Omega_a \subseteq \mathcal{D}_\tau$.

□

Remark 2.3.2. The same theorem holds replacing $\mathcal{D}_\tau$ with $\mathcal{D}^-_\tau$ and future complete with past complete.

Corollary 2.3.3. *If τ and σ are elements of $Z^1(\Gamma, \mathbb{R}^{n+1})$ that differ by a coboundary then $\mathcal{D}_\tau$ and $\mathcal{D}_\sigma$ differ by a translation. Moreover $\mathcal{D}_{-\tau} = -(\mathcal{D}^-_\tau)$.*

Proof. Suppose $\tau_\gamma - \sigma_\gamma = \gamma(x) - x$ for some $x \in \mathbb{R}^{n+1}$. Then $\mathcal{D}_\tau + x$ is a Γ_σ-invariant future complete regular domain, hence the statement follows by the unicity of $\mathcal{D}_\sigma$. On the other hand $-(\mathcal{D}^-_\tau)$ is a future complete regular domain that is invariant under the action of $\Gamma_{-\tau}$. □

Using the previous corollary we can notice that Y_τ and Y_σ are isometric if and only if τ and σ differ by a coboundary. In fact if $f : Y_\tau \to Y_\sigma$ is an isometry, let $\widetilde{f} : \mathcal{D}_\tau \to \mathcal{D}_\sigma$ be a lift of f, then $\gamma_\sigma \circ \widetilde{f} = \widetilde{f} \circ \gamma_\tau$ and this implies that σ and τ differ by a coboundary. So the isometric class of the globally hyperbolic flat spacetime Y_τ depends only on the cohomology class $[\tau] \in H^1(\Gamma, \mathbb{R}^{n+1})$. Hence we have a well defined map

$$H^1(\Gamma, \mathbb{R}^{n+1}) \to \mathcal{T}_{\mathrm{Lor}}(M)$$
$$[\tau] \to [Y_\tau]$$

Also notice that since $\mathcal{D}_0 = D(\mathbb{H}^n) = I^+(0)$, then Y_0 is the Minkowskian cone $\mathcal{C}^+(M)$. Furthermore a time-orientation reversing isometry between $Y_{-\tau}$ and Y^-_τ exists.

Now we can prove that every Γ_τ-invariant complete spacelike hypersurface, over which the action is free and properly discontinuous, is contained in either $\mathcal{D}_\tau$ or $\mathcal{D}^-_\tau$ and it is a Cauchy surface of it.

Remark 2.3.4. We recall that the degree of a map is defined as follows (see [17, XIV, 8.]). Let M and N two compact, connected, oriented without boundary differentiable manifolds of dimension n, and let $f : M \to N$ a continuous map between them. Notice that f induces a group homomorphism between the highest cohomology groups of M and N, $f^* : H^n(M) \to H^n(N)$. Furthermore since M and N are compact oriented manifold we have that $H^n(M) \cong$

$H^n(N) \cong \mathbb{Z}$ and a generator for them is represented by the fundamental class of the manifold $[M]$ and $[N]$. Hence there exists a number, called the *degree* of f, denoted by $deg(f)$, such that $f^*([M]) = deg(f)[N]$. It is an homotopy invariant. A degree 1 map is a map that induce the identity map on the level of the highest cohomology groups.

Corollary 2.3.5. *Let $\widetilde{F}$ be a Γ_τ-invariant complete spacelike hypersurface on which the action of Γ_τ is free and properly discontinuous then $\widetilde{F}$ is contained in either $\mathcal{D}_\tau$ or $\mathcal{D}_\tau^-$. In particular every timelike coordinate on $\widetilde{F}$ is proper. Furthermore $\widetilde{F}/\Gamma_\tau$ is diffeomorphic to M and the Gauss map induces on $\widetilde{F}/\Gamma_\tau$ a map $\bar{N} : \widetilde{F}/\Gamma_\tau \to M$ which has degree 1.*

Proof. From Corollary 2.2.31 we know that $D(\widetilde{F})$, the domain of dependence of $\widetilde{F}$, is a Γ_τ-invariant either future or past complete regular domain. Hence by Theorem 2.3.1 we get that either $D(\widetilde{F}) = \mathcal{D}_\tau$ or $D(\widetilde{F}) = \mathcal{D}_\tau^-$. Thus $\widetilde{F}$ is contained in either $\mathcal{D}_\tau$ or $\mathcal{D}_\tau^-$ and it is indeed a Cauchy surface of it. Hence $\widetilde{F}/\Gamma_\tau$ is a Cauchy surface for $\mathcal{D}_\tau/\Gamma_\tau$ or $\mathcal{D}_\tau^-/\Gamma_\tau$ and this implies that $\widetilde{F}/\Gamma_\tau$ is diffeomorphic to M. Suppose $\widetilde{F} \subseteq \mathcal{D}_\tau$. First of all we show that any timelike coordinate on $\widetilde{F}$ is proper. In fact since $\widetilde{F}$ is a complete spacelike hypersurface, if we fix a set of affine orthonormal coordinates $(y_0, \ldots, y_n)$, by Proposition 1.4.29 $\widetilde{F}$ is the graph of a function φ defined over $\{y_0 = 0\}$. So in order to show that y_0 is a proper function on $\widetilde{F}$ it is sufficient to show that φ is a proper function. From Lemma 2.1.16 it follows that $\partial\mathcal{D}_\tau$ is also the graph of a function ψ defined over $\{y_0 = 0\}$, since $\widetilde{F} \subseteq \mathcal{D}_\tau$ we have that $\psi \leq \varphi$. But from Lemma 2.1.18 we know that ψ is a proper map, so if we assume by contradiction that $\{p_n\} \subseteq \{y_0 = 0\}$ is a divergent sequence such that $\varphi(p_n)$ converges, then we have that $\psi(p_n)$ diverges and since $\psi \leq \phi$ we get an absurd. So φ is a proper map.

Finally since $\widetilde{F}$ is contained in $\mathcal{D}_\tau$ we can consider the Gauss map $N : \mathcal{D}_\tau \to \mathbb{H}^n$ of $\mathcal{D}_\tau$ restricted to $\widetilde{F}$. Since it is Γ_τ-equivariant it induces a map $\bar{N} : \widetilde{F}/\Gamma_\tau \to M$. This map is homotopic to the identity, hence it has degree 1. In fact consider the family of scaled cocycles $t\tau \in Z^1(\Gamma, \mathbb{R}^{n+1})$ with $t \in [0, 1]$ and consider the associated Gauss maps $N_{t\tau}$. When $t = 0$ we get $N_0 = id : \mathbb{H}^n/\Gamma \to \mathbb{H}^n/\Gamma$ and when $t = 1$ we get our induced Gauss map $\bar{N}$. Hence $\bar{N}$ is homotopic to the identity. □

Now it remains to show that Y_τ and Y_τ^- are the only maximal globally hyperbolic flat spacetimes with compact spacelike Cauchy surface and holonomy group Γ_τ. We start with a remark.

Corollary 2.3.6. *For every $\tau \in Z^1(\Gamma, \mathbb{R}^{n+1})$, the intersection $\mathcal{D}_\tau \cap \mathcal{D}_\tau^-$ is empty.*

Proof. The intersection $\mathcal{D}_\tau \cap \mathcal{D}_\tau^-$ is a Γ_τ-invariant convex set that is also bounded since it is limited from above and below by the graphs of the functions φ and φ' defining $\partial\mathcal{D}_\tau$ and $\partial\mathcal{D}_\tau^-$ respectively. Hence if the intersection is not empty its barycentre p is fixed by Γ_τ. Hence $I^+(p)$ and $I^-(p)$ are respectively a future complete Γ_τ-invariant regular domain and a past complete one. Hence $I^+(p) = \mathcal{D}_\tau$ and $I^-(p) = \mathcal{D}_\tau^-$. So their intersection is empty. □

Lemma 2.3.7. *The developing map $D : \widetilde{Y} \to \mathbb{M}^{n+1}$ of a flat globally hyperbolic spacetime Y with a complete spacelike Cauchy surface S is injective.*

Proof. Fix a timelike direction v in $\mathbb{M}^{n+1}$ and for $\widetilde{x} \in \widetilde{Y}$ let $\delta(\widetilde{x})$ be the timelike line in $\mathbb{M}^{n+1}$ passing through $D(\widetilde{x})$ with direction v. Let $d(\widetilde{x})$ be the connected part of $D^{-1}(\delta(\widetilde{x}))$ containing $\widetilde{x}$. Then $d(\widetilde{x})$ is a timelike geodesic in $\widetilde{Y}$. Since $\widetilde{S}$ is a Cauchy surface for $\widetilde{Y}$ $d(\widetilde{x})$ intersects $\widetilde{S}$ in exactly one point $p(\widetilde{x})$. Suppose now that $D(\widetilde{x}) = D(\widetilde{y})$, then $\delta(x) = \delta(y) = \delta$ and hence $D(p(\widetilde{x})) = \delta \cap D(\widetilde{S}) = D(p(\widetilde{y}))$. But the restriction of D to the hypersurface $\widetilde{S}$ is an embedding from lemma 1.4.29 hence $p(\widetilde{x}) = p(\widetilde{y})$. This implies that $d(\widetilde{x}) = d(\widetilde{y})$. But D restricted to $d(\widetilde{x})$ is a local homeomorphism from a topological line to $\mathbb{R}$ hence it is injective. We obtain $\widetilde{x} = \widetilde{y}$. □

Proposition 2.3.8. *Every globally hyperbolic flat spacetime with compact spacelike Cauchy surface and holonomy group Γ_τ is diffeomorphic to $\mathbb{R} \times M$ and isometrically embeds in either Y_τ or Y_τ^-. Hence Y_τ and Y_τ^- are the unique maximal globally hyperbolic flat spacetimes with compact spacelike Cauchy surface and holonomy group Γ_τ.*

Proof. Let Y be a globally hyperbolic flat spacetime with compact spacelike Cauchy surface S and holonomy group Γ_τ. We have to show that Y isometrically embeds in either Y_τ or Y_τ^-. It is sufficient to show that the developing map $D : \widetilde{Y} \to \mathbb{M}^{n+1}$ of Y is an embedding with image contained either in $\mathcal{D}_\tau$ or in $\mathcal{D}_\tau^-$. From Lemma 2.3.7 D is injective, then $D(\widetilde{Y})$ is isometric to $\widetilde{Y}$ and hence the image of $\widetilde{S}$ is a Γ_τ-invariant spacelike hypersurface such that the action of Γ_τ on it is free and properly discontinuous, by the equivariance of D. Hence from Corollary 2.3.5 we have that the image is a Cauchy surface of either $\mathcal{D}_\tau$ or $\mathcal{D}_\tau^-$. It follows that S is homeomorphic to M and hence Y is homeomorphic to $\mathbb{R} \times M$. Since $\widetilde{Y} \cong \mathbb{R} \times \widetilde{S}$ and since $\forall\, t \in \mathbb{R}$ we have that $D_{|\{t\}\times\widetilde{S}}$ is contained in either $\mathcal{D}_\tau$ or $\mathcal{D}_\tau^-$ it follows that $D(\widetilde{Y}) \subseteq \mathcal{D}_\tau \cup \mathcal{D}_\tau^-$. Since these domains are disjoint it follows that $D(\widetilde{Y})$ is contained in one of them, let us say $\mathcal{D}_\tau$. Hence $D : \widetilde{Y} \to \mathbb{M}^{n+1}$ is an isometric embedding into $\mathcal{D}_\tau$, this induces on the quotient an embedding of Y in Y_τ. Hence any isometric embedding of Y_τ and Y_τ^- in another globally hyperbolic flat spacetime with compact Cauchy surfaces and holonomy Γ_τ is surjective. □

Remark 2.3.9. Actually we can conclude that Y_τ and Y_τ^- are maximal in the sense that every isometric embedding in another globally hyperbolic flat spacetime is surjective. To see this let Y be a globally hyperbolic spacetime and $\varphi : Y_\tau \to Y$ an isometric embedding. Let N be a spacelike Cauchy surface of Y_τ diffeomorphic to M, then $\varphi(N)$ is a Cauchy surface for Y since from [16, Property 6] a compact spacelike hypersurface in a globally hyperbolic spacetime is automatically a Cauchy surface. So Y has as well compact Cauchy surfaces diffeomorphic to M, this implies that the holonomy group of Y is Γ_σ for some cocycle $\sigma \in Z^1(\Gamma, \mathbb{R}^{n+1})$. If we consider $\tilde{\varphi} : \mathcal{D}_\tau \to \widetilde{Y}$ a lift of φ to the universal covering spaces and compose it with the developing map of Y, which is an isometric embedding of $\widetilde{Y}$ in $\mathcal{D}_\sigma$ from Proposition 2.3.8, we can deduce that $D(\tilde{\varphi}(\mathcal{D}_\tau))$ is a Γ_σ-invariant future complete regular domain and then by uniqueness of Theorem 2.3.1 conclude that D is an isometry between $\widetilde{Y}$ and $\mathcal{D}_\sigma$ so that Y is isometric to Y_σ. Then σ and τ differ by a coboundary and hence Y is isometric to Y_τ.

Remark 2.3.10. This result agrees with the Theorem of Choquet-Bruhat and Geroch 1.2.40 that states that every globally hyperbolic spacetime Y admits a Cauchy embedding , see Definition 1.2.38, in a maximal globally hyperbolic spacetime. Moreover this maximal globally hyperbolic extension is unique up to isometries. Furthermore the maximal globally hyperbolic extension of a flat globally hyperbolic spacetime is flat. By maximal in this contest we mean a globally hyperbolic spacetime such that every Cauchy-embedding in another globally hyperbolic spacetime is surjective. Nevertheless as we have already stated a compact spacelike hypersurface in a globally hyperbolic spacetime is automatically a Cauchy surface. Hence for flat globally hyperbolic spacetime with compact Cauchy surface an embedding is automatically a Cauchy-embedding.

2.4 Continuous family of domains of dependence

Recall that Γ is a discrete, torsion-free, cocompact subgroup of $\mathrm{SO}^+(n,1)$, $M = \mathbb{H}^n/\Gamma$ is a compact hyperbolic manifold and $Y_\tau = \mathcal{D}_\tau/\Gamma_\tau$ is the unique future complete globally hyperbolic flat spacetime with compact spacelike Cauchy surfaces homeomorphic to M and holonomy group Γ_τ, that is an affine deformation of Γ for a fixed $[\tau] \in H^1(\Gamma, \mathbb{R}^{n+1})$.

In this section we will show that the map defined in the previous section

$$H^1(\Gamma, \mathbb{R}^{n+1}) \to \mathcal{T}_{\mathrm{Lor}}(M)$$
$$[\tau] \to [Y_\tau]$$

is continuous with respect to the topologies we have put on these sets in the section about Geometric structures 1.4. More precisely under the identification $H^1(\Gamma, \mathbb{R}^{n+1}) \cong \mathcal{R}(\Gamma)$ we put on $H^1(\Gamma, \mathbb{R}^{n+1})$ the topology of pointwise convergence of the group homomorphisms ρ_τ associated to each $\tau \in Z^1(\Gamma, \mathbb{R}^{n+1})$. Instead on $\mathcal{T}_{\mathrm{Lor}}(M)$ we put the compact-open topology on the associated developing maps, notice that we are asking less regularity than in the general contest of section 1.4, this is because the maps we have associated to every future complete regular domain and that we are going to use in order to define the developing map associated to every τ, namely the cosmological time, the retraction and the normal field a priori are just continuous. So the continuity statement reduces to prove that given any bounded neighborhood U of $0 \in Z^1(\Gamma, \mathbb{R}^{n+1})$ there exists a continuous map

$$dev: \; U \times (\mathbb{R}_+ \times \widetilde{M}) \to \mathbb{M}^{n+1}$$

such that for all $\tau \in U$ the map $dev_\tau = dev(\tau, \cdot)$ is the developing map associated to the flat spacetime Y_τ.
In Theorem 2.1.4 for any bounded neighborhood U of 0 we have constructed a map $dev : U \times \widetilde{N}_0 \to \mathbb{M}^{n+1}$, noticing that $\widetilde{M} \subseteq \widetilde{N}_0$ let us call dev^0 the restriction of dev to $\widetilde{M} = \mathbb{H}^n$. Hence we obtain a C^∞-map

$$dev^0 : U \times \widetilde{M} \to \mathbb{M}^{n+1}$$

This map is such that for all $\tau \in U$ the map $dev^0_\tau = dev^0(\tau, \cdot)$ is an embedding onto a future strictly convex spacelike hypersurface that is Γ_τ-invariant. Furthermore dev^0_τ is Γ_τ-equivariant in the following sense $dev^0_\tau(\gamma x) = \gamma_\tau dev^0_\tau(x)$ for all $\gamma \in \Gamma$. Recall that for every $\tau \in U$ we let $\widetilde{F}_\tau$ be the image under dev^0_τ of $\widetilde{M}$. It is a future convex complete spacelike hypersurface hence from Proposition 1.4.29 we know that if we fix a set of orthonormal affine coordinates $(y_0, \ldots, y_n)$ then $\widetilde{F}_\tau$ is the graph of a 1-Lipschitz convex function $\varphi_\tau : \{y_0 = 0\} \to \mathbb{R}$. First we remark that the dependence of φ_τ on τ is continuous in the sense that if $\tau_k \to \tau$ in U then $\varphi_{\tau_k} \to \varphi_\tau$ in the compact open topology. This is because $dev^0_{\tau_k} \to dev^0_\tau$ in the C^∞-topology, i.e. they converge uniformly on each compact subset of $\widetilde{M}$ hence $\widetilde{F}_{\tau_k} = dev^0_{\tau_k}(\widetilde{M})$ is uniformly close to $\widetilde{F}_\tau$ on each compact, this implies that the maps φ_{τ_k} and φ_τ converge uniformly on each compact subset of $\{y_0 = 0\}$.

Since it will be the basic tool for all the following proofs let us recall the statement of Ascoli-Arzelà Theorem in his more general version.

Theorem 2.4.1 (Ascoli-Arzelà)**.** *Let X be a topological space and (Y, d) a metric space. Give $C(X, Y)$ the topology of compact convergence (uniform convergence on each compact subset) and let $\mathcal{F}$ be a subset of $C(X, Y)$. If $\mathcal{F}$ is equicontinuous under d and the set $\mathcal{F}_a = \{f(a) \mid f \in \mathcal{F}\}$ has compact closure for every $a \in X$ then $\mathcal{F}$ is contained in a compact subspace of $C(X, Y)$. The converse holds if X is locally compact Hausdorff.*

Proof. [22, Theorem 47.1] □

Now recall that $\mathcal{D}_\tau$ for each $\tau \in U$ is the domain of dependence of $\widetilde{F}_\tau$ and it is a future complete regular convex domain hence $\partial\mathcal{D}_\tau$ is also defined, from Lemma 2.1.16, as the graph of a 1-Lipschitz convex function $\psi_\tau : \{y_0 = 0\} \to \mathbb{R}$. The following result shows that also the map ψ_τ depends continuously on τ.

For the next propositions fix U a bounded neighborhood of 0 in $Z^1(\Gamma, \mathbb{R}^{n+1})$.

Proposition 2.4.2. *Let $\{\tau_k\}_{k\in\mathbb{N}}$ be a sequence of cocycles in U which converges to $\tau \in U$, then $\psi_{\tau_k} \to \psi_\tau$ in the compact-open topology.*

Proof. First we will show that the hypothesis of Ascoli-Arzelà Theorem are satisfied in order to conclude that we can extract a subsequence $\{\psi_{\tau_k}\}$ that converges uniformly on every compact subset to some ψ_∞. Since the maps ψ_{τ_k} are all 1-Lipschitz they form an equicontinuous family. Now we have to prove that they are locally bounded. Since $\widetilde{F}_{\tau_k} \subseteq \mathcal{D}_{\tau_k}$ we have that $\psi_{\tau_k} \leq \varphi_{\tau_k}$ for all τ_k, on the other hand we can consider a family of past strictly convex spacelike hypersurfaces $\{\widetilde{F}^-_\tau\}_{\tau\in U}$ and let $\varphi^-_{\tau_k} : \{y_0 = 0\} \to \mathbb{R}$ be such that $\widetilde{F}^-_{\tau_k}$ is the graph of $\varphi^-_{\tau_k}$, we have as well that $\widetilde{F}^-_{\tau_k} \subseteq \mathcal{D}^-_{\tau_k}$ and since $\mathcal{D}_{\tau_k} \cap \mathcal{D}^-_{\tau_k} = \emptyset$ it follows that $\varphi^-_{\tau_k} \leq \psi_{\tau_k} \leq \varphi_{\tau_k}$. Since $\{\varphi^-_{\tau_k}\}_k$ and $\{\varphi_{\tau_k}\}_k$ are convergent and hence locally bounded ψ_{τ_k} is locally bounded as well. So we can apply Ascoli-Arzelà Theorem and extract a convergent subsequence $\psi_{\tau_k} \to \psi_\infty$. It remains to prove that $\psi_\infty = \psi_\tau$. Since ψ_∞ is the limit of convex functions it is convex as well and if S is the graph of ψ_∞ it is Γ_τ-invariant. In fact if $p \in S$, write $p = \lim_k p_k$ with $p_k \in \partial\mathcal{D}_{\tau_k}$ then $\gamma_\tau p = \lim_k \gamma_{\tau_k} p_k$ and since $\gamma_{\tau_k} p_k \in \partial\mathcal{D}_{\tau_k}$ we have that $\gamma_\tau p \in S$. Furthermore since ψ_∞ is 1-Lipschitz the epigraph of ψ_∞ has no timelike support hyperplane. Hence $I^+(S)$ coincide with the epigraph of ψ_∞ and thus it is a future convex set. Actually $I^+(S)$ is a Γ_τ-invariant future complete regular domain. In fact each $\mathcal{D}_{\tau_k}$ is the intersection of the future of at least two null support hyperplanes and since locally we have uniform convergence of the boundary $\partial\mathcal{D}_{\tau_k}$ to S, then also $I^+(S)$ has at least two null support hyperplanes. Finally from the uniqueness of $\mathcal{D}_\tau$, see Theorem 2.3.1, we have that $I^+(S) = \mathcal{D}_\tau$ and hence $\psi_\infty = \psi_\tau$. □

Let again $\{\tau_k\}$ be a sequence in U that converges to $\tau \in U$. If we fix a compact set $K \subseteq \mathcal{D}_\tau$, since we have proved that $\psi_{\tau_k} \to \psi_\tau$ in the compact-open topology, where ψ_{τ_k} and ψ_τ define the boundary of $\mathcal{D}_{\tau_k}$ and of $\mathcal{D}_\tau$ respectively, this implies that for k big enough $K \subseteq \mathcal{D}_{\tau_k}$. Hence we may suppose that $K \subseteq \mathcal{D}_{\tau_k}$ for all $k \in \mathbb{N}$. Notice that associated to each $\mathcal{D}_{\tau_k}$ we have the cosmological time $T_{\tau_k} : \mathcal{D}_{\tau_k} \to \mathbb{R}_+$, the retraction $r_{\tau_k} : \mathcal{D}_{\tau_k} \to \partial\mathcal{D}_{\tau_k}$ and the normal field $N_{\tau_k} : \mathcal{D}_{\tau_k} \to \mathbb{H}^n$. In the following propositions we want to show that these maps converge in the compact-open topology to the respective maps defined on $\mathcal{D}_\tau$ as $k \to \infty$.

First we need a technical lemma.

Lemma 2.4.3. *Let $\{\tau_k\}_{k\in\mathbb{N}}$ a sequence in U which converges to $\tau \in U$. For $C \in \mathbb{R}$ and for any cocycle σ let*

$$K_C(\sigma) = \{x \in \{y_0 = 0\} \mid \psi_\sigma(x) \leq C\}$$

where ψ_σ is the function $\psi_\sigma : \{y_0 = 0\} \to \mathbb{R}$ such that $\partial\mathcal{D}_\sigma$ is the graph of ψ_σ. Then for every $C \in \mathbb{R}$ and for every $\varepsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that

$$K_{C-\varepsilon}(\tau) \subseteq K_C(\tau_k) \subseteq K_{C+\varepsilon}(\tau) \qquad \forall k \geq k_0$$

For every cocycle σ let $M(\sigma)$ be the minimum of the function ψ_σ. Then $\{M(\tau_k)\}_k$ converges to $M(\tau)$.

Proof. Since ψ_σ is a convex function $K_C(\sigma)$ is a closed convex subset of $\{y_0 = 0\}$. We also claim that it is compact. Indeed we can argue as in the proof of Lemma 2.1.18. Moreover if $C > M(\sigma)$ then $K_C(\sigma)$ has non empty interior and $\partial K_C(\sigma) = \{x \in \{y_0 = 0\} \mid \psi_\sigma(x) = C\}$. Now set $M = M(\tau)$. First let us suppose that $C > M$. Fix $\varepsilon > 0$ and let $k_0 \in \mathbb{N}$ such that

$$\|\psi_\tau - \psi_{\tau_k}\|_{\infty, K_{C+\varepsilon}(\tau)} < \frac{\varepsilon}{2} \quad \text{for all } k \geq k_0.$$

Then $K_{C-\varepsilon}(\tau) \subseteq K_C(\tau_k)$ for all $k \geq k_0$. Now let $x \notin K_{C+\varepsilon}(\tau)$ we claim that $\psi_{\tau_k}(x) \geq C + \frac{\varepsilon}{2}$ for all $k \geq k_0$ and this proves the other inclusion. To see why let $k \geq k_0$ and $x_0 \in \{y_0 = 0\}$ such that

$\psi_\tau(x_0) = M$ and consider the map $c(t) = \psi_{\tau_k}(x_0 + t(x - x_0))$ for $t \in [0,1]$. Since $x_0 \in K_{C+\varepsilon}(\tau)$ but by assumption $x \notin K_{C+\varepsilon}(\tau)$ there exists $t_0 \in (0,1)$ such that $x_0 + t_0(x - x_0) \in \partial K_{C+\varepsilon}(\tau)$. Then

$$c(0) = \psi_{\tau_k}(x_0) \leq M + \frac{\varepsilon}{2} \qquad \text{and}$$

$$c(t_0) = \psi_{\tau_k}(x_0 + t(x - x_0)) \geq \psi_\tau(x_0 + t_0(x - x_0)) - \frac{\varepsilon}{2} = C + \frac{\varepsilon}{2}.$$

Since $c(t_0) = \psi_{\tau_k}(x_0 + t_0(x - x_0)) \leq (1 - t_0)\psi_{\tau_k}(x_0) + t_0\psi_{\tau_k}(x) = (1 - t_0)c(0) + t_0 c(1)$ we have that $C + \frac{\varepsilon}{2} \leq (1 - t_0)(M + \frac{\varepsilon}{2}) + t_0 c(1) < (1 - t_0)(C + \frac{\varepsilon}{2}) + t_0 c(1)$ hence $\psi_{\tau_k}(x) = c(1) \geq C + \frac{\varepsilon}{2}$. Now suppose $C < M$ fix k_0 such that

$$K_{M+1}(\tau_k) \subseteq K_{M+2}(\tau) \quad \text{and}$$

$$\|\psi_\tau - \psi_{\tau_k}\|_{\infty, K_{M+2}(\tau)} < \frac{M - C}{2} \qquad \text{for all } k > k_0.$$

Then combining the two conditions we get that $K_C(\tau_k) = \emptyset$ for all $k > k_0$. Hence $M(\tau) \leq M(\tau_k)$ for all $k > k_0$ and this implies that $M(\tau) \leq \liminf_k M(\tau_k)$, furthermore since $\psi_{\tau_k} \to \psi_\tau$ we have that $M(\tau) \geq \limsup_k M(\tau_k)$. □

Now we can prove the continuous dependence of the cosmological time T_τ on τ.

Proposition 2.4.4. *Let $\{\tau_k\}_{k \in \mathbb{N}}$ be a sequence in U which converges to $\tau \in U$, set $T_k = T_{\tau_k}$ and let T be the cosmological time on $\mathcal{D}_\tau$. Then T_k converges uniformly on each compact $K \subseteq \mathcal{D}_\tau$ to $T_{|K}$.*

Proof. Let M be the minimum of ψ_τ, by Lemma 2.4.3 there exists k_0 such that $\psi_{\tau_k}(x) \geq M - 1$ for all $x \in \{y_0 = 0\}$ and for all $k \geq k_0$. Notice that for a compact subset $K \subseteq \mathcal{D}_\tau$ the set $J^-(K) \cap \{y_0 \geq M - 1\}$ is compact and let H be the projection onto $\{y_0 = 0\}$. Fix $\varepsilon > 0$ and $k(\varepsilon)$ such that $\|\psi_\tau - \psi_{\tau_k}\|_{\infty, H} < \frac{\varepsilon}{2}$ for all $k \geq k(\varepsilon)$. Let $p \in K$ and $r = r(p) \in \partial\mathcal{D}_\tau$, where r is the retraction on $\mathcal{D}_\tau$. Notice that $r \in J^-(K) \cap \{y_0 \geq M - 1\}$. If $k \geq k(\varepsilon)$ we have that $r + \varepsilon\frac{\partial}{\partial y_0} \in \mathcal{D}_{\tau_k}$ and by the definition of $T_k(p)$

$$T_k(p) > \sqrt{-\left\langle p - (r + \varepsilon\frac{\partial}{\partial y_0}), p - (r + \varepsilon\frac{\partial}{\partial y_0})\right\rangle}$$

notice that

$$\left\langle p - (r + \varepsilon\frac{\partial}{\partial y_0}), p - (r + \varepsilon\frac{\partial}{\partial y_0})\right\rangle = -T(p)^2 - \varepsilon^2 + 2\varepsilon(p - r)_0.$$

Since $J^-(K) \cap \{y_0 \geq M - 1\}$ is compact there exists $C \in \mathbb{R}$ such that $(p - r)_0 \leq C$, hence

$$T_k(p) > \sqrt{T(p)^2 + \varepsilon^2 - 2\varepsilon C},$$

since the right hand side tends to $T(p)$ for every fixed $\eta > 0$ if we call $\alpha(\varepsilon) = -(\varepsilon^2 - 2\varepsilon C)$ we have that there exists a δ such that if $|\alpha(\varepsilon)| < \delta$ we have $\sqrt{T(p)^2 - \alpha(\varepsilon)} > T(p) - \eta$ hence if we take ε small enough such that $|\alpha(\varepsilon)| < \delta$ we get that $T_k(p) > \sqrt{T(p)^2 - \alpha(\varepsilon)} > T(p) - \eta$ for all $k \geq k(\varepsilon)$ and for all $p \in K$. On the other hand all the projections $r_k(p)$ belong to $J^-(K) \cap \{y_0 \geq M - 1\}$ so the same argument shows that $T(p) > T_k(p) - \eta$ for all $k \geq k(\varepsilon)$ and all $p \in K$. □

Now for every $\tau \in U$ consider the level surfaces of the cosmological time T_τ and denote them by $\widetilde{S}_a(\tau) = T_\tau^{-1}(a)$. We know from Proposition 1.4.29 that they are also defined as the graph of functions $\psi_\tau^a : \{y_0 = 0\} \to \mathbb{R}$. As a Corollary of Proposition 2.4.4 we get that also ψ_τ^a are continuous functions of τ for every fixed $a \in \mathbb{R}_+$.

Corollary 2.4.5. *Let $\{\tau_k\}$ be a sequence of cocycles in U that converges to τ, then $\psi^a_{\tau_k} \to \psi^a_\tau$ in the compact-open topology.*

Proof. We can argue that every $\psi^a_{\tau_k}$ is 1-Lipschitz and hence they constitute an equicontinuous family. On the other hand let ψ_{τ_k} be the map that defines $\partial\mathcal{D}_{\tau_k}$ then $\psi_{\tau_k} < \psi^a_{\tau_k} < \psi_{\tau_k} + a$ so since $\{\psi_{\tau_k}\}$ is a locally bounded family so is $\{\psi^a_{\tau_k}\}_k$. So we can apply again Ascoli-Arzelà Theorem and extract a convergent subsequence and since $T_{\tau_k} \to T_\tau$ the limit is ψ^a_τ. □

Now we can prove that both the retraction r_τ and the normal field N_τ are continuous functions of τ.

Proposition 2.4.6. *Let again $\{\tau_k\}$ be a convergent sequence in U, $\tau_k \to \tau$. Let r_k, N_k the retraction and the normal field defined on $\mathcal{D}_{\tau_k}$. Fix K a compact subset of $\mathcal{D}_\tau$, then the maps $r_{k|K}$ and $N_{k|K}$ converge uniformly to $r_{|K}$ and $N_{|K}$, where r and N are the retraction and the normal field on $\mathcal{D}_\tau$.*

Proof. Let M be the minimum of ψ_τ and fix k_0 such that $\psi_{\tau_k} \geq M - 1$ for all $k \geq k_0$. In particular $r_k(p) \in J^-(K) \cap \{y_0 \geq M - 1\}$ for all $p \in K$ and $k \geq k_0$. Since $J^-(K) \cap \{y_0 = 0\}$ is compact there exists C such that $\|p - r(p)\| \leq C$ for all $p \in K$ and $k \geq k_0$, where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^{n+1}$. On the other hand since from Proposition 2.4.4 the cosmological time $T_k = T_{\tau_k}$ tends to T_τ on K there exists $k_1 > k_0$ such that $\beta > T_k(p) \geq \alpha > 0$ for all $p \in K$ and all $k \geq k_1$. Hence the image of N_k restricted to K in $\mathbb{H}^n$ is contained in $\{v \in \mathbb{H}^n \mid \|v\| \leq \frac{C}{\alpha}\}$ for all $k \geq k_1$. Since this is a compact set in $\mathbb{H}^n$ the family of functions $\{N_{k_{|K}}\}_k$ is bounded. In order to show that $N_{k|K} \to N_{|K}$ uniformly it is sufficient to show that if $p_k \to p$ in K then $N_k(p_k) \to N(p)$. Since $N_k(p_k)$ runs in a compact set we can suppose it tends to a timelike vector v. Set $a = T(p)$, in order to show that $N(p) = v$ it is sufficient to show that $p + v^\perp$ is a support plane for $\widetilde{S}_a = T^{-1}(a)$. This is equivalent to prove that $\langle q, v\rangle \leq \langle p, v\rangle$ for all $q \in \widetilde{S}_a$. Fix $q \in \widetilde{S}_a$ and put $q = (\psi^a_\tau(y), y)$ set $a_k = T_k(p_k)$ and consider the sequences $q_k = (\psi^{a_k}_{\tau_k}(y), y)$ and $q'_k = (\psi^a_{\tau_k}(y), y)$. From Corollary 2.4.5 we know that $q'_k \to q$. On the other hand $\|q_k - q'_k\| \leq |a_k - a|$ since $\|q_k - q'_k\|^2 = (\psi^{a_k}_{\tau_k}(y) - \psi^a_{\tau_k}(y))^2 \leq (a_k - a)^2$, so $q_k \to q$ as $a_k \to a$. We know that $\langle q_k, N_k(p_k)\rangle \leq \langle p_k, N_k(p_k)\rangle$ from Corollary 2.2.14 and hence passing to the limit we get $\langle q, v\rangle \leq \langle p, v\rangle$. Finally since $r_k + T_k N_k = id$ also $r_{k|K}$ converges uniformly to $r_{|K}$. □

Finally we can prove a stronger convergence for T_k.

Corollary 2.4.7. *Let again $\{\tau_k\}$ be a convergent sequence in U, $\tau_k \to \tau$, fix K a compact subset of $\mathcal{D}_\tau$ then T_k converge in the C^1-topology to T_τ*

Proof. In fact $\nabla_L T_k = -N_k$ and N_k converges uniformly on K to $N = -\nabla_L T$ □

Finally we can prove what stated at the beginning of the section, namely the continuity of $H^1(\Gamma, \mathbb{R}^{n+1}) \ni [\tau] \to [Y_\tau] \in \mathcal{T}_{\mathrm{Lor}}(M)$.

Theorem 2.4.8. *For every bounded neighborhood U of 0 in $Z^1(\Gamma, \mathbb{R}^{n+1})$ there exists a continuous map*

$$dev : U \times (\mathbb{R}_+ \times \widetilde{M}) \to \mathbb{M}^{n+1}$$

such that $dev_\tau = dev(\tau, \cdot)$ is the (continuous) developing map of Y_τ for every $\tau \in U$.

Proof. As in the discussion at the beginning of the section let $dev^0 : U \times \mathbb{H}^n \to \mathbb{M}^{n+1}$ be the C^∞-map coming from Theorem 2.1.4. Fix $\tau \in U$, $x \in \mathbb{H}^n$ and $t > 0$, consider the timelike geodesic γ in $\mathcal{D}_\tau$ which passes through $dev^0_\tau(x)$ and has the direction of the normal field at $dev^0_\tau(x)$. Let $dev(\tau, t, x)$ be the point on γ with cosmological time t, i.e.

$$dev(\tau, t, x) = r_\tau(dev^0_\tau(x)) + tN_\tau(dev^0_\tau(x))$$

where r_τ and N_τ are the retraction and the normal field defined on $\mathcal{D}_\tau$. Since the maps dev_τ^0, r_τ, N_τ are Γ_τ-equivariant so is the map dev_τ just defined. Furthermore dev_τ is an homeomorphism onto $\mathcal{D}_\tau$, in fact from Proposition 2.4.6 and from the continuity of dev_τ^0, the map dev_τ is continuous and since $\mathcal{D}_\tau$ is the domain of dependence of $\widetilde{F}_\tau = dev_\tau^0(\mathbb{H}^n)$, for every $z \in \mathcal{D}_\tau$ there exists a unique $y = dev_\tau^0(x) \in \widetilde{F}_\tau$ such that the timelike geodesic γ which passes through z and has direction x intersects $\widetilde{F}_\tau$ in y, so $dev(\tau, t, x) = r_\tau(y) + tN_\tau(y) = z$, hence dev_τ is a homeomorphism onto $\mathcal{D}_\tau$. Hence we can deduce that dev_τ is a developing map of Y_τ. □

Remark 2.4.9. In general the developing map has a C^∞-regularity. In this case it is just continuous hence it gives the topological structure on Y_τ.

2.5 Proof of the Main Theorem

We now summarize all the steps that have been done in the previous sections in order to prove Theorem 1.

Proof. For every $\tau \in Z^1(\Gamma, \mathbb{R}^{n+1})$ we have constructed Y_τ as the quotient $\mathcal{D}_\tau/\Gamma_\tau$, where $\mathcal{D}_\tau$ is the domain of dependence of the future convex Γ_τ-invariant spacelike hypersurface $\widetilde{F}_\tau$ constructed in Theorem 2.1.4 such that $\widetilde{F}_\tau/\Gamma_\tau$ is diffeomorphic to M. We have seen in Proposition 2.1.13 that $\mathcal{D}_\tau/\Gamma_\tau$ is a globally hyperbolic flat spacetime with Cauchy surfaces diffeomorphic to M hence $Y_\tau = \mathcal{D}_\tau/\Gamma_\tau \cong \mathbb{R}_+ \times M$, and since from Remark 2.1.14 $\mathcal{D}_\tau$ is future complete so is Y_τ. From Remark 2.2.5 the domain $\mathcal{D}_\tau$ is a future complete regular domain. Moreover from Proposition 2.3.8 Y_τ is the unique maximal globally hyperbolic future complete spacetime with holonomy group $\Gamma_\tau = \rho_\tau(\Gamma)$ and compact spacelike Cauchy surfaces and every other such spacetime isometrically embeds in either Y_τ or Y_τ^-. From Theorem 2.2.8 and Corollary 2.2.9 $\mathcal{D}_\tau$ has regular cosmological time $\widetilde{T} : \mathcal{D}_\tau \to \mathbb{R}_+$ which has C^1-regularity and whose gradient is, up to sign, the normal field $\widetilde{N}$ that is surjective from Remark 2.2.28 and hence $\widetilde{T}$ is a C^1-submersion. Since from Proposition 2.2.10 and Lemma 2.2.17 every level surface $\widetilde{S}_a$ of $\widetilde{T}$ is a complete spacelike C^1-hypersurface of $\mathbb{M}^{n+1}$, from Lemma 1.4.29 $\widetilde{S}_a$ is the graph of a C^1-convex function defined over $\{y_0 = 0\}$. Moreover this function is proper from Corollary 2.3.5. And since $\widetilde{T}$ is Γ_τ-invariant, see Remark 2.2.18, it induces a canonical cosmological time on Y_τ such that the level surfaces are C^1-diffeomorphic to M. In Theorem 2.2.8 we have also defined a continuous map $r : \mathcal{D}_\tau \to \partial\mathcal{D}_\tau$ such that for all $p \in \mathcal{D}_\tau$ we have $\widetilde{T}(p) = d(p, r(p))$. The image of $\mathcal{D}_\tau$ under r is denoted by Σ_τ and it is called the singularity in the past. Since, again from Remark 2.2.18, r is Γ_τ-equivarinat, Σ_τ is Γ_τ-invariant. Furthermore Σ_τ is contractible from Remark 2.2.12 and is connected by spacelike Lipschitz paths by Lemma 2.2.16. Finally the map $\mathcal{R}(\Gamma) \ni [\rho_\tau] \to [Y_\tau] \in \mathcal{T}_{\mathrm{Lor}}(M)$ is continuous from Theorem 2.4.8. □

Chapter 3

Geodesic stratification

In his work Mess [21, Proposition 12 and Proposition 13] gave a bijection between domains of dependence of closed spacelike hyperbolic surfaces F and measured geodesic laminations on F. A more explicit and general construction of such bijection can be found in [8]. We first try to give an idea of the construction through a simple example without giving all the proofs that can be found in the references. Then we spend some words on the generalizations done by Bonsante. What we saw in the previous chapter, Corollary 2.2.24 and Proposition 2.2.26, is that in any dimension associated to a future complete regular domain with surjective normal field there is a geodesic stratification (that for $n = 2$ is a geodesic lamination) of $\mathbb{H}^n$. Equipping a geodesic stratification $\mathcal{C}$ with a transverse measure enable us to construct a future complete regular domain whose stratification coincide with $\mathcal{C}$. For $n = 2$ this correspondence is a bijection, in higher dimensions for some technical reason in [12] Bonsante is not able to extend the argument. However for a particular class of domains, namely the one with simplicial singularity, the bijection is recovered.

3.1 Geodesic lamination

First of all we recall the definition of geodesic lamination on a complete hyperbolic surface and we refer to [14] for a good explanation of the topic.

Definition 3.1.1. Given a complete hyperbolic surface F, a *geodesic* is the image of a complete geodesic in $\mathbb{H}^2$ under the universal covering map, identifying $\mathbb{H}^2$ with the universal cover of F. A geodesic in F is *simple* if it has no self transverse intersection.

Definition 3.1.2. A *geodesic lamination* on a complete hyperbolic surface F is a non empty closed subset of F which is a disjoint union of simple geodesics. The geodesics that form the lamination are called the *leaves* of the lamination. So each leaf is either a simple closed geodesic or an isometric copy of $\mathbb{R}$ embedded in F.

Example 3.1.3. The simplest example of geodesic lamination is a finite union of disjoint simple closed geodesics.

Remark 3.1.4. Notice that if $F = \mathbb{H}^2/\Gamma$ is a complete hyperbolic surface then a geodesic lamination $\mathcal{L}$ on F lifts to a Γ-invariant geodesic lamination $\widetilde{\mathcal{L}}$ on $\mathbb{H}^2$.

Definition 3.1.5. A *measured geodesic lamination* on a complete hyperbolic surface F is a couple $(\mathcal{L}, \mu)$ where $\mathcal{L}$ is geodesic lamination on F and μ is a transverse measure on $\mathcal{L}$. A rectificable arc c on F is transverse to the lamination $\mathcal{L}$ if for every point $p \in c$ there exists a neighborhood U of p in F such that $U \cap c$ intersects each component of $U \cap \mathcal{L}$ in at most one point and the intersection of U with each connected component of $F \setminus \mathcal{L}$ is a connected set. Thus a transverse measure μ on $\mathcal{L}$ is the assignment of a positive measure μ_c on every rectificable path c transverse to $\mathcal{L}$. This means that μ_c assigns a non-negative number to every Borel subset of the arc in such a way that:

1. the support of μ_c is $\mathcal{L} \cap c$,
2. if $c' \subseteq c$ then $\mu_{c'} = \mu_{c|c'}$,
3. if c and c' are arcs that are homotopic through arcs that are transverse to the leaves of $\mathcal{L}$ keeping the endpoints either on the same leaf or on the same connected component of $F \setminus \mathcal{L}$ then the homotopy sends the measure μ_c to $\mu_{c'}$.

Example 3.1.6. The simplest example of measured geodesic lamination $(\mathcal{L}, \mu)$ on $\mathbb{H}^2$ is given by a finite family of disjoint geodesic lines each endowed with a real positive weight. This example is called a weighted multi-curve.

Definition 3.1.7. A leaf c of a lamination $\mathcal{L}$ on a complete hyperbolic surface F is *isolated* if for each $x \in c$ there exists a neighborhood U of x such that $(U, U \cap \mathcal{L})$ is homeomorphic to (disk, diameter). The *simplicial part* of $\mathcal{L}$ is the union of the isolated leaves. We denote it by $\mathcal{L}_S$.

Remark 3.1.8. If $\mathcal{L}$ is a lamination on a compact hyperbolic surface F and if all its leaves are isolated then $\mathcal{L}$ is a finite disjoint union of simple closed geodesics.

Definition 3.1.9. A leaf l of a measured geodesic lamination $(\mathcal{L}, \mu)$ is called *weighted* if there exists a transverse arc c such that $\mu_c \cap l$ is an atom of μ_c. By property 3 of the definition of measured geodesic laminations, for every transverse arc c the intersection of c with l consists of atoms of μ_c whose masses are all equal to a positive number called the weight of the leaf. The *weighted part* of $\mathcal{L}$ is the union of all the weighted leaves. We denote it by $\mathcal{L}_W = \mathcal{L}_W(\mu)$.

Remark 3.1.10. We remark that in general both $\mathcal{L}_S$ and $\mathcal{L}_W$ are not sublaminations of $\mathcal{L}$. Furthermore since the support of the measure μ is the whole lamination $\mathcal{L}$ we have that $\mathcal{L}_S \subseteq \mathcal{L}_W$. However in general this inclusion is strict. Consider for instance a lamination $\mathcal{L}$ of $\mathbb{H}^2$ given by all the geodesics with a fixed starting point, if we take the upper half plane model for $\mathbb{H}^2$ and let the starting point be ∞ then we see that the geodesics are parametrized by $\mathbb{R}$. Hence we may choose a dense sequence $\{q_n\}_{n \in \mathbb{N}}$ in $\mathbb{R}$ and construct a measure on $\mathcal{L}$ such that l_{q_n} is endowed with the weight 2^{-n}. Then $\mathcal{L}_W$ is a dense subset of $\mathbb{H}^2$.

Remark 3.1.11. However when the surface F is compact the situation of the previous remark is not possible and indeed for a geodesic lamination $\mathcal{L}$ on F we have $\mathcal{L}_S = \mathcal{L}_W$ and $\mathcal{L}_S$ is the maximal weighted multi-curve sublamination of $\mathcal{L}$. By a weighted multi-curve me wean the union of a finite number of disjoint simple closed geodesics on F each endowed with a strictly positive real weight. Hence $\mathcal{L} = \mathcal{L}_S \cup \mathcal{L}_1$, where $\mathcal{L}_1$ is a sublamination with no closed leaves.

Let us denote by

- $\mathcal{R}$ the set of regular domains Ω of $\mathbb{M}^{2+1}$ with surjective normal field $N : \Omega \to \mathbb{H}^2$ and
- $\mathcal{ML}$ the set of measured geodesic laminations on $\mathbb{H}^2$.

Note that there is a natural left action of $\mathrm{SO}^+(2,1)$ on $\mathcal{ML}$ and of $\mathrm{Iso}_0(\mathbb{M}^{2+1})$ on $\mathcal{R}$. We will now summarize the construction, given in [8], of a map $\mathcal{ML} \to \mathcal{R}$ that induces a bijection between $\mathcal{ML}$ and $\mathcal{R}/\mathbb{R}^3$, where $\mathbb{R}^3$ acts on $\mathcal{R}$ by translations, and a bijection between $\mathcal{ML}/\mathrm{SO}^+(2,1)$ and $\mathcal{R}/\mathrm{Iso}_0(\mathbb{M}^{2+1})$.

From geodesic laminations to regular domains: we now construct a map

$$\begin{aligned} \Omega^0: \quad & \mathcal{ML} \longrightarrow \mathcal{R} \\ \lambda = & (\mathcal{L}, \mu) \rightarrow \Omega^0_\lambda \end{aligned}$$

Fix a geodesic lamination $\lambda = (\mathcal{L}, \mu)$ on $\mathbb{H}^2$ and fix a base point $x_0 \in \mathbb{H}^2 \setminus \mathcal{L}_W$. For every $x \in \mathbb{H}^2 \setminus \mathcal{L}_W$ let c be an arc transverse to $\mathcal{L}$ between x_0 and x. For $t \in c \cap \mathcal{L}$ let $v(t) \in \mathbb{R}^3$ be the vector tangent to $\mathbb{H}^2$ and orthogonal to the leaf through t pointing towards x. For $t \in c \setminus \mathcal{L}$ set $v(t) = 0$. Hence we have defined a function $v : c \rightarrow \mathbb{R}^3$ that is continuous on the support of μ. Define

$$\rho(x) = \int_c v(t) d\mu(t).$$

We can notice that the definition of ρ is independent from the choice of the path c between x_0 and x. Indeed another arc from x_0 to x will be homotopic to c and hence by Property 3 of the definition of measured geodesic laminations we will have that the two line integrals coincide. Moreover ρ is constant on each connected component of $\mathbb{H}^2 \setminus \mathcal{L}$ and it is a continuous function $\rho : \mathbb{H}^2 \setminus \mathcal{L}_W \rightarrow \mathbb{M}^{2+1}$. Hence we define the domain Ω^0_λ as follows

$$\Omega^0_\lambda = \bigcap_{x \in \mathbb{H}^2 \setminus \mathcal{L}_w} I^+(\rho(x) + x^\perp)$$

Example 3.1.12. We now see an easy example of the above construction. Consider the geodesic lamination of $\mathbb{H}^2$ made by one geodesic γ with weight a. Where $\gamma = \mathbb{H}^2 \cap \{y = 0\}$. Fix the base point x_0 in $\mathbb{H}^2 \cap \{y < 0\}$. Let $C_1 = \mathbb{H}^2 \cap \{y < 0\}$ and $C_2 = \mathbb{H}^2 \cap \{y > 0\}$ the two connected components of $\mathbb{H}^2 \setminus \mathcal{L}$. Then the map $\rho : \mathbb{H}^2 \setminus \{\gamma\} \rightarrow \mathbb{M}^{2+1}$ becomes $\rho(x) = 0$ if $x \in C_1$ and $\rho(x) = a\vec{v}$ if $x \in C_2$. Where $\vec{v} = (0, 0, 1)$. Hence looking at the definition of $\Omega^0_{(\mathcal{L},\mu)}$ we get the regular domain showed in Figure 3.1.

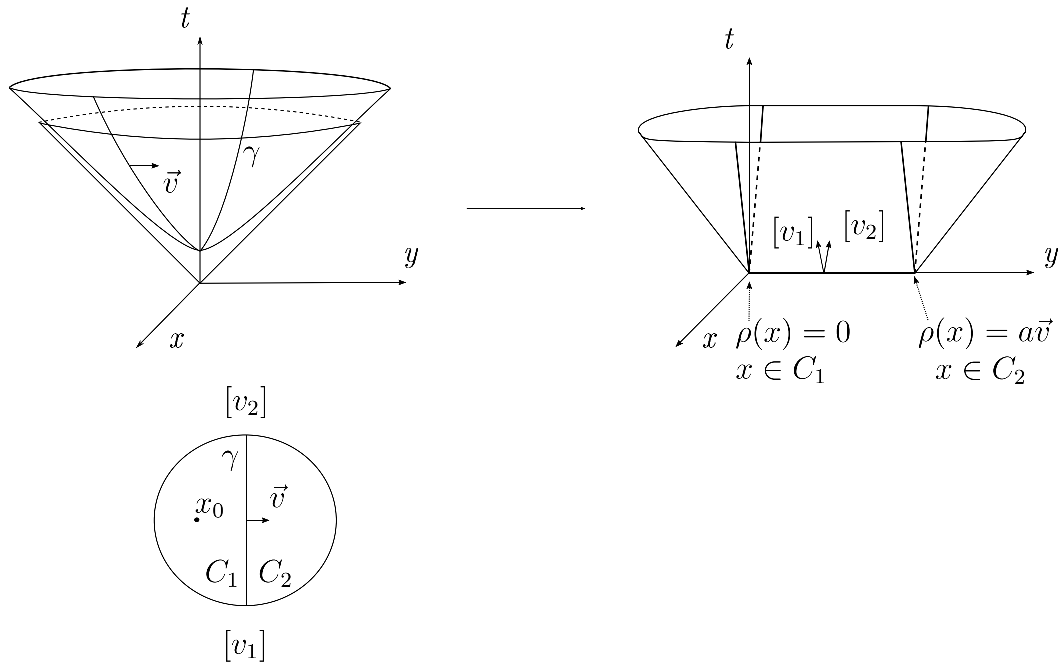

Figure 3.1: Construction of a regular domain from a geodesic lamination.

Let us now prove that the domain arising from the above construction is a regular domain with surjective normal field.

Proposition 3.1.13. *Ω^0_λ is a regular domain.*

Proof. If x belongs to a geodesic l of the lamination $\mathcal{L}$ denote by $M(x)$ such geodesic, otherwise if x belongs to $\mathbb{H}^2 \setminus \mathcal{L}$ denote by $M(x)$ the connected component of $\mathbb{H}^2 \setminus \mathcal{L}$ that contains x. Notice that $M(x)$ is a convex subset of $\mathbb{H}^2$ that is the convex hull of its boundary points. If we show that

$$\Omega^0_\lambda = \bigcap_{x \in \mathbb{H}^2 \setminus \mathcal{L}_w,\ [v] \in M(x) \cap \partial \mathbb{H}^2} I^+(\rho(x) + v^\perp)$$

then Ω^0_λ will be a regular domain. Since ρ is constant on each $M(x)$ we have by definition that $\rho(x) + y^\perp$ is a support plane for Ω^0_λ for all $y \in M(x)$ hence if v is a null vector such that $[v]$ is a point on the boundary of $M(x)$ then $\rho(x) + v^\perp$ will be a support plane for Ω^0_λ thus we have one inclusion, namely $\Omega^0_\lambda \subseteq \bigcap_{x \in \mathbb{H}^2 \setminus \mathcal{L}_w,\ [v] \in M(x) \cap \partial \mathbb{H}^2} I^+(\rho(x) + v^\perp)$. For the other inclusion suppose that $p \in \mathbb{M}^{2+1}$ and $\langle p - \rho(x), v \rangle < 0$ for every $x \in \mathbb{H}^2 \setminus \mathcal{L}_W$ and $[v] \in M(x) \cap \partial \mathbb{H}^2$. Notice that every $x \in \mathbb{H}^2 \setminus \mathcal{L}$ is a convex linear combination of some null vectors representing points on the boundary of $M(x)$ it follows that $\langle p - \rho(x), x \rangle < 0$ so $p \in \Omega^0_\lambda$. □

Proposition 3.1.14. *The normal field $N : \Omega^0_\lambda \to \mathbb{H}^2$ associated to the regular domain Ω^0_λ constructed above is surjective.*

Proof. From the definition of Ω^0_λ and from the fact that points in the image of the normal field are characterized as points in $\mathbb{H}^2$ such that there is a support plane for Ω^0_λ orthogonal to them it follows that $\mathbb{H}^2 \setminus \mathcal{L}_W$ is contained in the image of N. Suppose now that $x \in \mathbb{H}^2$ belongs to a weighted leaf l of $\mathcal{L}$ we can consider the geodesic arc c passing through x_0 and x, then there exist $\rho_-(x) = \lim_{t \to x^-} \rho(t)$ and $\rho_+(x) = \lim_{t \to x^+} \rho(t)$ and the difference $\rho_+(x) - \rho_-(x)$ is a spacelike vector orthogonal to x. The plane passing through $\rho_-(x)$ and orthogonal to x is a support plane for Ω^0_λ and indeed it contains also $\rho_+(x)$. Hence x is in the image of N. □

From regular domains to measured geodesic laminations: let us now construct a map in the other direction

$$\mathcal{R} \to \mathcal{ML}.$$

Fix Ω a regular domain and $N : \Omega \to \mathbb{H}^2$ the normal map associated to Ω. As in Definition 2.2.23 we set $\mathcal{F}(p) = N(r^{-1}(p))$ for $p \in \Sigma$. From Corollary 2.2.24 and Proposition 2.2.26 we know that $\mathcal{F}(p)$ is an ideal convex set of $\mathbb{H}^2$ and that $\mathcal{F}(p)$ and $\mathcal{F}(q)$ do not meet transversally, thus geodesics that are either boundary components of some $\mathcal{F}(p)$ or that coincide with some $\mathcal{F}(p)$ are pairwise disjoint and hence they form a geodesic lamination of $\mathbb{H}^2$. Thus

$$\mathcal{L} = \bigcup_{\dim \mathcal{F}(r)=1} \mathcal{F}(r) \bigcup_{\dim \mathcal{F}(r)=2} \partial \mathcal{F}(r)$$

is a geodesic lamination of $\mathbb{H}^2$.

Example 3.1.15. For example let Ω be the regular domain we have constructed in Example 3.1.12, then in Figure 3.2 we see the associated geodesic lamination of $\mathbb{H}^2$. The lamination is made by one geodesic, that corresponds to points that are internal to the closed segment $[p_1, p_2]$, and that divides $\mathbb{H}^2$ into two ideal convex sets that correspond to the endpoints of the segment.

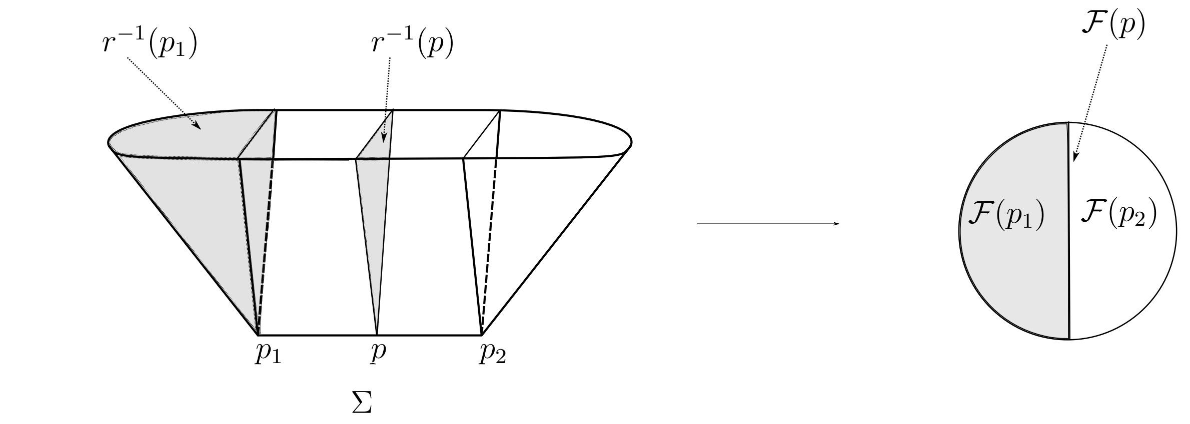

Figure 3.2: Geodesic lamination associated to a future complete regular domain.

We want to put on $\mathcal{L}$ a transverse measure such that $\Omega = \Omega^0_\lambda$. Set

$$Y = \{x \in \mathbb{H}^2 \mid \# N^{-1}(x) \cap \widetilde{S}_1 > 1\},$$

where $\widetilde{S}_1$ is the level surface of the cosmological time of Ω at time 1. Let $c : [0,1] \to \mathbb{H}^2$ be a geodesic segment transverse to $\mathcal{L}$ with no endpoints on Y. Consider the inverse image $\tilde{c}$ of c on $\widetilde{S}_1$, i.e. $N(\tilde{c}) = c$. It is a Lipschitz hence rectificable path, ([8, Proposition 3.23]). Let $r(t) = r(\tilde{c}(t))$ and $N(t) = N(\tilde{c}(t))$, then $\tilde{c}(t) = r(t) + N(t)$. As we have proved in Proposition 2.2.16 $r(t)$ is locally Lipschitz hence it is differentiable almost everywhere with spacelike derivatives and $\dot{r}(t) \in T_{\tilde{c}(t)}\widetilde{S}_1 = T_{N(t)}\mathbb{H}^2$. Hence we may define a measure $\tilde{\mu}$ on each Borel set E of $[0,1]$ as follows

$$\tilde{\mu}(E) = \int_E |\dot{r}(t)| dt$$

where $|\dot{r}| = \sqrt{\langle \dot{r}(t), \dot{r}(t) \rangle}$. So we may define a transverse measure μ_c as

$$\mu_c = N_*(\tilde{\mu}).$$

Moreover from Proposition 2.2.26 we have that $\langle r(t+h), x\rangle \geq 0$ and $\langle r(t+h), y\rangle \leq 0$ for $x \in \mathcal{F}(r(t+h))$ and $y \in \mathcal{F}(r(t))$ hence $\dot{r}(t)$ is 0 unless $N(t)$ belongs to a leaf of the lamination and in this situation $\dot{r}(t) \in T_{N(t)}\mathcal{F}(r(t))^\perp$. So we have that $\dot{r}(t) = |\dot{r}(t)| v(N(t))$ where $v(N(t))$ is either the unit vector orthogonal to the leaf through $N(t)$ or 0. It follows that

$$r(N^{-1}(c(1))) - r(N^{-1}(c(0))) = \int_{\tilde{c}} \dot{r}(t) dt = \int_{\tilde{c}} |\dot{r}(t)| v(N(t)) dt = \int_c v(y) d\mu_c(t).$$

Then since for $\Omega_{(\mathcal{L},\mu)}$ we have that if $c(t)$ is a geodesic arc that does not meet the weighted part then $\rho(t) = r(t)$ and by construction $\rho(t) - \rho(t_0) = \int_{[c(t_0), c(t)]} v(t) d\mu_c(t)$ this implies that $\Omega = \Omega_{(\mathcal{L},\mu)}$.

3.2 Generalization to all dimensions

Now we spend some words on how Bonsante attempts to generalize these ideas. First of all we give a definition.

Definition 3.2.1. A *geodesic stratification* of $\mathbb{H}^n$ is a family $\mathcal{C} = \{C_i\}_{i \in I}$ such that

1. C_i is an ideal convex set of $\mathbb{H}^n$. By an *ideal* convex set of $\mathbb{H}^n$ we mean a convex set that is the convex hull of its boundary points,
2. $\mathbb{H}^n = \bigcup_{i\in I} C_i$,
3. for every $i \neq j \in I$ there exists a support plane $P_{i,j}$ which separates C_i from C_j and such that $C_i \cap C_j = C_i \cap P_{i,j} = C_j \cap P_{i,j}$.

Every C_i is called a *piece* of the stratification.

Remark 3.2.2. For $n = 2$ geodesic stratifications of $\mathbb{H}^2$ are in fact geodesic laminations.

Definition 3.2.3. If C is an ideal convex set, then we say that a point $p \in C$ is *internal* if all the support planes passing through p contain C. Let us denote by bC the set of points of C that are not internal. Note that unless C has non-empty interior bC is not the topological boundary of C. If C has dimension k then bC has a natural decomposition in convex pieces that are ideal convex sets of dimension strictly less than k. If $\mathcal{C}$ is a geodesic stratification of $\mathbb{H}^n$, we can add to it the pieces of the decomposition of bC_i for all $C_i \in \mathcal{C}$. In this way we obtain a new geodesic stratification $\overline{\mathcal{C}}$ called the *completion* of $\mathcal{C}$. Notice that $\overline{\overline{\mathcal{C}}} = \overline{\mathcal{C}}$. A stratification is said to be *complete* if $\overline{\mathcal{C}} = \mathcal{C}$.

Definition 3.2.4. For $k = 1, \ldots, n-1$ a *k-stratum* of a geodesic stratification $\mathcal{C}$ of $\mathbb{H}^n$ is defined as the set

$$X_{(k)} = \bigcup \{F \in \overline{\mathcal{C}} \mid \dim F \leq k\}.$$

Remark 3.2.5. Recall that the *Hausdorff distance* between two compact subsets X, Y of a metric space (M, d) is defined as follows

$$d_H(X, Y) = \inf\{\varepsilon > 0 \mid X \subseteq Y_\varepsilon \text{ and } Y \subseteq X_\varepsilon\}$$

where $X_\varepsilon = \bigcup_{x\in X}\{z \in M \mid d(x, z) \leq \varepsilon\}$. Or equivalently defined as

$$d_H(X, Y) = \max\{\sup_{x\in X} \inf_{y\in Y} d(x, y), \sup_{y\in Y} \inf_{x\in X} d(x, y)\}.$$

We call Hausdorff topology the topology defined by the Hausdorff distance.

Remark 3.2.6. For $n = 2$ stratifications are continuous in the following sense: if $\{x_k\}_{k\in\mathbb{N}} \subseteq \mathbb{H}^2$ with $x_k \in C_k$ and $x_k \to x \in \mathbb{H}^2$ then there exists a piece C of the stratification such that $x \in C$ and C_k tends in the Hausdorff topology to C. Unfortunately for $n > 2$ we do not have such continuity but however we can define a weaker notion that will be satisfied by geodesic stratifications arising from regular domains with surjective normal field.

Definition 3.2.7. A geodesic stratification $\mathcal{C}$ is *weakly continuous* if the following property holds. Suppose $\{x_k\}_{k\in\mathbb{N}}$ is a convergent sequence of $\mathbb{H}^n$ and $\lim_k x_k = x$. Let C_k be a piece of the stratification which contains x_k and suppose $C_k \to C$ in the Hausdorff topology. Then there exists a piece $G \in \overline{\mathcal{C}}$ such that $C \subseteq G$.

From Corollary 2.2.24 and Proposition 2.2.26 we know that every regular domain Ω with surjective normal field N produces a geodesic stratification $\{\mathcal{F}(p)\}_{p\in\Sigma}$ of $\mathbb{H}^n$. We now show that this geodesic stratification is weakly continuous.

Proposition 3.2.8. *Let Ω be a future complete regular domain with surjective normal field N. Then the geodesic stratification $\mathcal{C}$ associated with it is weakly continuous.*

Proof. Let $\{x_k\}_{k\in\mathbb{N}}$ be a convergent sequence in $\mathbb{H}^n$, with $x_k \to x$. Let C_k be the piece of the stratification which contains x_k and suppose $C_k \to C$. We have to prove that C is contained in a piece G of $\overline{\mathcal{C}}$. Let us take $r_k \in \Sigma$ such that $\mathcal{F}(r_k) = C_k$. Notice that $p_k = r_k + x_k \in \widetilde{S}_1$. Since from Proposition 2.2.27 the map $N_{|\widetilde{S}_1}$ is a proper map there exists a convergent subsequence $\{p_{k(j)}\}$. Set $p = \lim p_{k(j)}$ and $r = r(p)$. We want to show that C is contained in $\mathcal{F}(r)$. Notice that C is the convex hull of $\hat{L}_C = C \cap \partial\mathbb{H}^n$, hence it is sufficient to show that $\hat{L}_C \subseteq \hat{L}(r)$. Now let $[v] \in \hat{L}_C$. From the convergence $C_k \to C$ we know that there exists a sequence $[v_n] \in \hat{L}(r_n)$ such that $[v_n] \to [v] \in \partial\mathbb{H}^n$. Hence we have that $r_n + \mathbb{R}_+ v_n \subseteq \partial\Omega$. Since $\partial\Omega$ is closed this implies that $r + \mathbb{R}_+ v \subseteq \partial\Omega$. Thus we can conclude that $[v] \in \hat{L}(r)$. □

Now we may give the notion of transverse measure on a stratification which will be a generalization of the one of transverse measure on a geodesic lamination. Fix a complete weakly continuous geodesic stratification $\mathcal{C}$ of $\mathbb{H}^n$. For $p \in \mathbb{H}^n$ let $C(p)$ be the piece in $\mathcal{C}$ which contains p and has minimum dimension. First of all we define the notion of transverse measure on a piece-wise geodesic path.

Definition 3.2.9. Let $c : [0,1] \to \mathbb{H}^n$ be a piece-wise geodesic path. Then a *transverse measure* on it is a $\mathbb{R}^{n+1}$-valued measure μ_c on $[0,1]$ such that

1. there exists a finite positive measure $|\mu_c|$ such that μ_c is $|\mu_c|$-absolutely continuous, (i.e. $|\mu_c|(A) = 0$ implies $\mu_c(A) = 0$ for all A Borelian subset of $[0,1]$) and supp$|\mu_c|$ is the topological closure of the set $\{t \in (0,1) \mid \dot{c}(t) \notin T_{c(t)}C(c(t))\}$,

2. let $v_c = \frac{d\mu_c}{d|\mu_c|}$ be the $|\mu_c|$-density of μ_c (i.e. $\mu_c((a,b)) = \int_a^b v_c(t)d|\mu_c|$) then

$$v_c(t) \in T_{c(t)}\mathbb{H}^n \cap T_{c(t)}C(c(t))^\perp$$

$$\langle v_c(t), v_c(t)\rangle = 1$$

$$\langle v_c(t), \dot{c}(t)\rangle > 0 \qquad |\mu_c| - a.e.,$$

3. the endpoints of c are not atoms of the measure $|\mu_c|$, (i.e $|\mu_c|(0) = |\mu_c|(1) = 0$).

In order to define a transverse measure on a geodesic stratification we need the following definition.

Definition 3.2.10. Let $\varphi_s : [0,1] \to \mathbb{H}^n$ be an homotopy between φ_0 and φ_1. We say that φ is $\mathcal{C}$*-preserving* if $C(\varphi_s(t)) = C(\varphi_0(t))$ for all $t, s \in [0,1]$.

Now we give the definition of transverse measure on a geodesic stratification.

Definition 3.2.11. Let $\mathcal{C}$ be a weakly continuous stratification and let us fix a subset Y of $\mathbb{H}^n$ which is a union of pieces of $\mathcal{C}$ such that the Lebesgue measure of Y is 0. By a $(\mathcal{C}, Y)$*-admissible path* we mean any piece-wise geodesic path $c : [0,1] \to \mathbb{H}^n$ such that every maximal geodesic subsegment of c has no endpoint on Y.
A *transverse measure* on $(\mathcal{C}, Y)$ is the assignment of a transverse measure μ_c to every admissible path $c : [0,1] \to \mathbb{H}^n$ such that

1. if there exists a $\mathcal{C}$-preserving homotopy between two paths c and d then $\mu_c = \mu_d$ (so this implies in particular that the measure μ is constant on each piece of the stratification),

2. for every admissible path c and every parametrization $s : [0,1] \to [0,1]$ of an admissible sub-arc of c we have that $\mu_{c\circ s} = s^*(\mu_c)$, that means that if $c' = c \circ s$ we have that $\mu_{c'}(E) = s^*(\mu_c)(E) = (s^{-1})_*(\mu_c)(E) = \mu_c(s(E))$,

3. the atoms of $|\mu_c|$ are contained in $c^{-1}(Y)$ and for every $y \in Y$ there exists an admissible path c such that $|\mu_c|$ has some atoms on $c^{-1}(y)$, (this implies in particular that $\text{supp}|\mu_c| \subseteq c^{-1}(Y)$),

4. $\mu_c(c) = 0$ for every closed admissible path, (this implies in particular that $\mu_{\bar{c}}(\bar{c}) = -\mu_c(c)$ where $\bar{c}$ is the inverse path),

5. for all sequences $\{x_k\}_{k\in\mathbb{N}}$ such that $x_k \in \mathbb{H}^n \setminus Y$ and $x = \lim_k x_k \in \mathbb{H}^n \setminus Y$, we have that $\mu_{c_k}(c_k) \to 0$ where c_k is the admissible arc $[x_k, x]$.

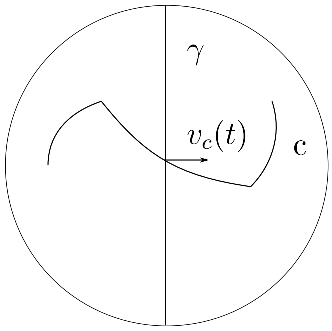

Figure 3.3: Admissible path c with $Y = \{\gamma\}$.

Definition 3.2.12. A *measured geodesic stratification* is given by a weakly continuous geodesic stratification $\mathcal{C}$, a subset Y as above and a transverse measure μ on $(\mathcal{C}, Y)$.

As in the first part of the construction for $n = 2$ we can associate to each measured geodesic stratification $(\mathcal{C}, Y, \mu)$ a future complete regular domain with surjective normal field and such that the stratification associated to it coincide with $\mathcal{C}$ on $\mathbb{H}^n \setminus Y$. Indeed we can fix a base point $x_0 \in \mathbb{H}^n \setminus Y$ and define for every $x \in \mathbb{H}^n \setminus Y$

$$\rho(x) = \mu_{c_x}(c_x)$$

where c_x is an admissible path between x_0 and x. We can notice that this definition is independent of the chosen path and that

$$\rho(y) = \rho(x) + \mu_{c_{x,y}}(c_{x,y})$$

where $c_{x,y}$ is an admissible path between x and y. As before, using property 5 of Definition 3.2.11, ρ defines a continuous function $\rho : \mathbb{H}^n \setminus Y \to \mathbb{M}^{n+1}$ and we define the future complete regular domain associated to $(\mathcal{C}, Y, \mu)$ as follows

$$\Omega = \bigcap_{x \in \mathbb{H}^n \setminus Y} I^+(\rho(x) + x^\perp)$$

In [12, Theorem 8.6 and Proposition 8.8] Bonsante proves, as in Proposition 3.1.13, that Ω is a future complete regular domain with surjective normal field and that the geodesic stratification associated to Ω coincide with $\mathcal{C}$ at least on $\mathbb{H}^n \setminus Y$. However for some technicalities the inverse construction cannot be carried in any dimension. Nevertheless for a particular class of domains the bijection is recovered. They are domains associated to simplicial stratifications that are defined as follows.

Definition 3.2.13. A geodesic stratification $\mathcal{C}$ of $\mathbb{H}^n$ is called *simplicial* if any $p \in \mathbb{H}^n$ admits a neighborhood U intersecting only a finite number of pieces of $\mathcal{C}$.

Remark 3.2.14. For $n = 2$ a simplicial stratification coincide with a simplicial lamination that is the union of the isolated leaves of the lamination.

For convenience we specialize to the case $n = 3$, however the results are general.

Lemma 3.2.15. *If $(\mathcal{C}, Y, \mu)$ is a measured geodesic stratification with simplicial support then $Y = X_{(2)}$, the 2-stratum of $\mathcal{C}$.*

Proof. Since Y is union of pieces of $\mathcal{C}$ of dimension 2 or less then $Y \subseteq X$. On the other hand let c be a geodesic path with no endpoint in X transverse to X, hence it is admissible. Then $\text{supp}|\mu_c|$ is $c^{-1}(X)$, since this set is finite $|\mu_c|$ has an atom on every point of $c^{-1}(X)$. The atoms of $|\mu_c|$ are contained in $c^{-1}(Y)$, thus $X \subseteq Y$. □

Definition 3.2.16. Let $\mathcal{C}$ be a simplicial stratification of $\mathbb{H}^3$. A family of positive constants $a = \{a(P)\}_P$ parametrized by the set of 2-pieces of $\mathcal{C}$ is called a family of *weights* for the stratification if it satisfies the following equation for every l 1-piece of the stratification

$$\sum_{l \subseteq P} a(P)w(P) = 0$$

where $w(P)$ is the unitary vector of $l^\perp$ tangential to P and pointing inward.

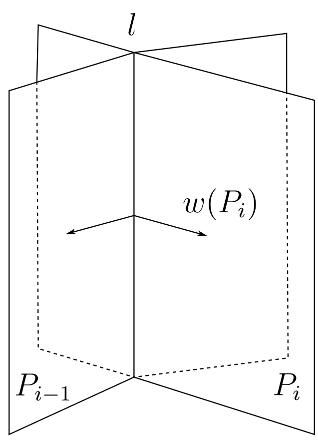

Figure 3.4: Neighborhood of a 1-piece of a simplicial stratification.

Proposition 3.2.17. *Let $\mathcal{C}$ be a simplicial stratification then the families of weights on $\mathcal{C}$ parametrize the transverse measures on $\mathcal{C}$.*

Proof. [12, Proposition 9.1]. □

Remark 3.2.18. We just remark that if we have a set of weights $\{a(P)\}_P$ on a simplicial geodesic stratification $\mathcal{C}$ then we can define a transverse measure on it in the following way. If c is an admissible geodesic path which does not intersect any geodesic of the stratification then we can define

$$\mu_c = \sum_P a(P)w(P)\delta_{c^{-1}(P)},$$

where $w(P)$ is the normal vector to P pointing in the direction of c and δ_x is the Dirac measure centered at x. If c intersects only one geodesic l, let $P_1, \ldots, P_k$ and $\Delta_1, \ldots, \Delta_k$ be respectively the two and three pieces that incide on l. If we choose a numeration as in Figure 3.5 and suppose that c comes from Δ_1 and goes to Δ_j then we can define

$$\mu_c = \sum_{i=1}^{j-1} a(P_i)w(P_i)\delta_{c^{-1}(l)}.$$

Then for every admissible path c we may choose a decomposition in geodesic admissible paths $c = c_1 * \ldots * c_k$ such that each c_i intersects either only one geodesic of the stratification or only one 2-piece.

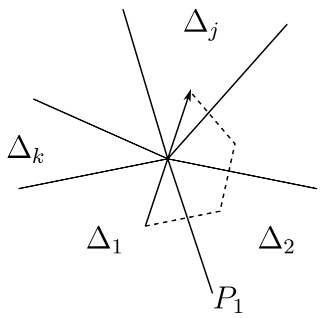

Figure 3.5: Definition of a transverse measure on a geodesic which passes through the 1-stratum.

Proposition 3.2.19. *Let Ω be the future complete regular domain associated to a measured geodesic stratification $(\mathcal{C}, X_{(2)}, \mu)$ whose support is simplicial, then the singularity in the past Σ of Ω has a natural cellular decomposition in the following sense. For $i = 0, 1, 2$ let*

$$\Sigma_i = \{p \in \Sigma \mid \dim \mathcal{F}(p) = 3 - i\}$$

then Σ_0 is a numerable set. Every component of Σ_1 is an open segment, moreover the closure of such segment has endpoints contained in Σ_0, every component of Σ_2 is an open 2-cell, moreover the closure of such cell is a finite sided polygon with vertices in Σ_0 and edges in Σ_1.

Proof. [12, Corollary 9.4.]. □

Definition 3.2.20. Let Σ be the singularity in the past associated to a future complete regular domain Ω. A point $p \in \Sigma$ is called a *vertex* if there exists a spacelike support plane at p which intersects $\overline{\Omega}$ only at p.
We say that Σ is *simplicial* if the set of vertices Σ_0 is discrete and Σ is a cellular complex with cellularization $\Sigma_0 \subseteq \Sigma_1 \subseteq \Sigma_2$ such that every component of $\Sigma_1 \setminus \Sigma_0$ is a straight segment with endpoints in Σ_0 and every component of $\Sigma_2 \setminus \Sigma_1$ is a finite-sided polygon with vertices in Σ_0 and edges in Σ_1.

Proposition 3.2.21. *Let Ω be a regular domain with surjective normal field and simplicial singularity. The stratification associated to Ω is simplicial. Moreover there exists a unique measure μ on $\mathcal{C}$ such that Ω is equal up to translations to the domain associated to $(\mathcal{C}, X_{(2)}, \mu)$.*

Proof. [12, Proposition 9.9.]. □

Remark 3.2.22. We just point out how we can define a measure on the stratification $\mathcal{C}$ associated to a regular domain with surjective normal field and simplicial singularity. From Proposition 3.2.17 it is sufficient to give a set of weights on $\mathcal{C}$. Given a 3-piece Δ of $\mathcal{C}$ then from Proposition 2.2.26 there exists a vertex $v(\Delta)$ of Σ such that $\Delta = \mathcal{F}(v(\Delta))$. Now if P is a 2-piece there exist 3-pieces Δ_1 and Δ_2 such that P is a face of them. Then $r(N^{-1}(P))$ is the spacelike segment $[v(\Delta_1), v(\Delta_2)]$, thus we can define

$$a(P) = (\langle v(\Delta_2) - v(\Delta_1), v(\Delta_2) - v(\Delta_1) \rangle)^{\frac{1}{2}} .$$

3.3 Equivariant construction

Fix Γ discrete torsion-free cocompact subgroup of $\mathrm{SO}^+(n,1)$.

Definition 3.3.1. A measured geodesic stratification $(\mathcal{C}, Y, \mu)$ is Γ-invariant if $\mathcal{C}$ is Γ-invariant, Y is Γ-invariant and we have

$$\mu_{\gamma \circ c}(E) = \gamma(\mu_c(E))$$

for all admissible paths $c : [0,1] \to \mathbb{H}^n$, Borel set $E \subseteq [0,1]$ and $\gamma \in \Gamma$.

Proposition 3.3.2. *Let $(\mathcal{C}, Y, \mu)$ be a Γ-invariant measured geodesic stratification of $\mathbb{H}^n$. Fix a basepoint $x_0 \notin Y$ and set $\tau_\gamma = \rho(\gamma(x_0))$. Then $\tau \in Z^1(\Gamma, \mathbb{R}^{n+1})$. Let Ω be the domain associated to $(\mathcal{C}, Y, \mu)$ then we have $\Omega = \mathcal{D}_\tau$.*

Proof. Since μ is Γ-invariant we have

$$\rho(\gamma(x)) = \gamma\rho(x) + \rho(\gamma(x_0))$$

Indeed $\rho(\gamma(x)) = \mu_{c_{\gamma(x)}}(c_{\gamma(x)})$ where $c_{\gamma(x)}$ is an admissible path between x_0 and $\gamma(x)$. If c_x is an admissible path between x_0 and x, then $\gamma(c_x)$ is an admissible path between $\gamma(x_0)$ and $\gamma(x)$ and $\mu_{\gamma(c_x)}(\gamma(c_x)) = \gamma\mu_{c_x}(c_x) = \gamma(\rho(x))$. So $\rho(\gamma(x)) = \rho(\gamma(x_0)) + \mu_{c_{\gamma(x_0),\gamma(x)}}(c_{\gamma(x_0),\gamma(x)}) = \rho(\gamma(x_0)) + \gamma(\rho(x))$. Thus if $\tau_\gamma = \rho(\gamma(x_0))$ we have that $\tau_{\alpha\beta} = \alpha\tau_\beta + \tau_\alpha$ hence $\tau_\gamma \in Z^1(\Gamma, \mathbb{R}^{n+1})$. Furthermore let Ω be the regular domain associated to $(\mathcal{C}, Y, \mu)$ then we can notice that it is a Γ_τ-invariant regular domain. In fact recall that Ω is defined as follows $\bigcap_{x \in \mathbb{H}^n \setminus Y} I^+(\rho(x) + x^\perp)$ then since $\gamma_\tau(\rho(x)+x^\perp) = \gamma(\rho(x))+\gamma(x)^\perp+\tau_\gamma = \rho(\gamma(x))+\gamma(x)^\perp$ we have that Ω is Γ_τ-invariant. Hence by Theorem 2.3.1 $\Omega = \mathcal{D}_\tau$. □

Remark 3.3.3. If Ω is a Γ_τ-invariant future complete regular domain then by Lemma 2.1.17 the normal field is surjective and it is evident that the geodesic stratification $\{\mathcal{F}(p)\}_{p \in \Sigma}$ is Γ-invariant.

Now restricting to the case of Γ_τ-invariant regular domains with simplicial singularity we get a complete bijection.

Lemma 3.3.4. *Let $\mathcal{C}$ be a Γ-invariant simplicial geodesic stratification then the set of measures on it is parametrized by a finite number of positive numbers satisfying a finite set of linear equations.*

Proof. [12, Proposition 9.11 and following discussion.]. □

Proposition 3.3.5. *There exists a bijective correspondence between Γ-invariant measured simplicial stratifications of $\mathbb{H}^3$ and future complete regular domains which are invariant for some affine deformation Γ_τ of Γ and have a simplicial singularity.*

Proof. We know that if Ω is a Γ_τ-invariant regular domain with simplicial singularity then it has surjective normal field so that it has an associated geodesic stratification $\mathcal{C}$ which is Γ-invariant and from Proposition 3.2.21 we know that $\mathcal{C}$ is simplicial and furthermore we can put on it a unique measure μ such that $\Omega = \Omega_{(\mathcal{C},\mu)}$. Notice that from the definition of the set of weights in Remark 3.2.22 and the definition of the associated measure μ in Remark 3.2.18 we can see that $(\mathcal{C}, X_{(2)}, \mu)$ will satisfy the condition in order to be Γ-invariant. On the other side if we start with a Γ-invariant measured simplicial stratification we can construct by Proposition 3.3.2 a regular domain that is Γ_τ-invariant for some τ and from Proposition 3.2.19 the singularity in the past of such domain has a cellular decomposition. So what is left to prove is that Γ-invariant measured simplicial stratifications give domains of dependence with simplicial singularity, i.e. the set of vertices Σ_0 is discrete. Fix $(\mathcal{C}, X_{(2)}, \mu)$ a Γ-invariant measured simplicial stratification of $\mathbb{H}^3$ and let $\{a(P)\}$ be the family of weights associated with it. By Lemma 3.3.4 there exists an $a > 0$ such that $a \leq a(P)$ for all P 2-piece of the stratification. Now for a 3-piece Δ let ρ_Δ be the corresponding point on Σ, by Proposition 3.2.19 we have that $\Sigma_0 = \{\rho_\Delta \mid \Delta \text{ is a 3-piece}\}$. On the other hand from Remark 3.2.22 we know that $\langle \rho_\Delta - \rho_{\Delta'}, \rho_\Delta - \rho_{\Delta'} \rangle = a(P)^2 \geq a^2$ so Σ_0 is discrete. □

Remark 3.3.6. We can remark that this result holds in every dimension.

In dimension $n = 2$ the above theorem has a simpler formulation without the need of the simplicial assumption.

Proposition 3.3.7. *There is a bijection between Γ-invariant measured geodesic laminations of $\mathbb{H}^2$, up to the action of $SO^+(2,1)$ and Γ_τ-invariant future complete regular domains, up to the action of $Iso_0(\mathbb{M}^{2+1})$,*

Proof. [8, Theorem 1.5.] □

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